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A minimization approach to hyperbolic Cauchy problems

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Abstract. Developing an original idea of De Giorgi, we introduce a new and purely variational approach to the Cauchy problem for a wide class of defocusing hyperbolic equations. The main novel feature is that the solutions are obtained as limits of functions that minimize suitable functionals in spacetime (where the initial data of the Cauchy problem serve as prescribed *boundary* conditions). This opens up the way to new connections between the hyperbolic world and that of the calculus of variations. Also dissipative equations can be treated. Finally, we discuss several examples of equations that fit into this framework, including nonlocal equations, in particular equations with the fractional Laplacian.

Keywords. Nonlinear hyperbolic equations, minimization, a priori estimates

1. Introduction

In this paper we introduce a new and purely variational approach to the Cauchy problem for a wide class of defocusing hyperbolic PDEs having the formal structure

$$w''(t,x) = -\nabla\mathcal{W}(w(t,\cdot))(x), \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \tag{1.1}$$

with prescribed initial conditions

$$w(0,x) = w_0(x), \qquad w'(0,x) = w_1(x) \tag{1.2}$$

(for notation, see the remark at the end of this section). While a precise setting with all formal details and our main results are given in Section 2, here we confine ourselves to a rather informal description of our approach, focusing on the main ideas that lie behind it and on the possible new perspectives that it opens up, especially some new connections between the variational world and hyperbolic PDEs of the kind (1.1).

In (1.1), $\nabla\mathcal{W}$ is the Gâteaux derivative of a functional (e.g. one from the calculus of variations) $\mathcal{W} : W \to [0,\infty)$, where W is some Banach space of functions in $\mathbb{R}^n$, typically a Sobolev space. If, for instance, $\mathcal{W}(u) = \frac{1}{2}\int |\nabla u|^2\,dx$ is the Dirichlet integral and $W = H^1(\mathbb{R}^n)$ then, formally, $-\nabla\mathcal{W}(u) = \Delta u$, and (1.1) reduces to the wave equation

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$w'' = \Delta w$, much in the same spirit as the heat equation $u' = \Delta u$ is the *gradient flow* of the Dirichlet integral. Thus, in a sense, (1.1) can be considered as a "second order gradient flow" for the functional $\mathcal{W}$.

Our aim is to initiate and try to develop a rather general program, suggested by De Giorgi in [2] (see also [3]), that offers a new, purely variational approach to equations of the kind (1.1), possibly with the addition of a dissipative term (see below). We alert the reader that in this paper the term "variational" refers, in the spirit of De Giorgi, to *minimization*, rather than critical point theory.

The main idea, the abstract counterpart to a specific conjecture stated in [2] and proved in [10] (see also [11] for a related partial result), is to associate with the abstract evolution equation (1.1) the functional

$$F_\varepsilon(w) = \frac{\varepsilon^2}{2}\int_0^\infty \int_{\mathbb{R}^n} e^{-t/\varepsilon}|w''(t,x)|^2\,dx\,dt + \int_0^\infty e^{-t/\varepsilon}\mathcal{W}(w(t,\cdot))\,dt. \tag{1.3}$$

This functional is to be minimized, for fixed $\varepsilon > 0$, over all functions $w(t,x)$ in spacetime $\mathbb{R}^+ \times \mathbb{R}^n$ subject to the constraints (1.2), which now play the role of *boundary* conditions. Assuming the existence of an absolute minimizer w_ε, the Euler–Lagrange equation of (1.3) formally reads

$$\varepsilon^2(e^{-t/\varepsilon}w_\varepsilon'')'' + e^{-t/\varepsilon}\nabla\mathcal{W}(w_\varepsilon(t,\cdot))(x) = 0,$$

that is, the fourth order in time equation

$$\varepsilon^2 w_\varepsilon'''' - 2\varepsilon w_\varepsilon''' + w_\varepsilon'' + \nabla\mathcal{W}(w_\varepsilon(t,\cdot))(x) = 0. \tag{1.4}$$

The connection with (1.1) is clear: letting $\varepsilon \downarrow 0$, one formally obtains (1.1) in the limit. This motivates the following

Problem 1 (De Giorgi, [2, 3]). Let w_ε be a minimizer of F_ε in (1.3) subject to the boundary conditions (1.2). Investigate the existence of the limit function

$$w(t,x) = \lim_{\varepsilon\to 0^+} w_\varepsilon(t,x), \tag{1.5}$$

and see if it solves the Cauchy problem (1.1)&(1.2).

In its generality, as long as the structure of the functional $\mathcal{W}$ is unknown, this may sound a little vague. In fact, De Giorgi [2] raised this general question taking cue from a precise conjecture in a particular case, namely when

$$\mathcal{W}(w) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla w(x)|^2\,dx + \frac{1}{p}\int_{\mathbb{R}^n} |w(x)|^p\,dx \quad (p \ge 2)$$

and (1.1) becomes the nonlinear wave equation

$$w'' = \Delta w - w|w|^{p-2} \quad (p \ge 2).$$

In this case, Problem 1 has an affirmative answer [10]. As we will show, however, much can be said on Problem 1 under very mild assumptions on $\mathcal{W}$, and a robust theory can be built that provides several a priori estimates on the minimizers w_ε. In some cases, basically when $\mathcal{W}(w)$ is quadratic in the highest order derivatives of w, Problem 1 can be completely solved without any other assumption. In *all* cases, however, up to subsequences the limit (1.5) always exists and estimates on w_ε entail the fulfillment of (1.2). When (1.1) is highly nonlinear, the general estimates still apply, but additional work is needed to get stronger compactness on w_ε and possibly obtain (1.1) in the limit (of course such further estimates, if any, will depend on the particular structure of $\mathcal{W}(w)$, and should be obtained *ad hoc* on a case-by-case basis).

The variational approach suggested by Problem 1 is by genuine minimization, a completely new and unconventional feature, when it comes to *hyperbolic* equations. The typical case is when $\mathcal{W}$ is a convex (lower semicontinuous, etc.) functional of the calculus of variations (possibly depending on x, w and some of its *spatial* derivatives): in this case F_ε in (1.3) inherits the good properties of $\mathcal{W}$, and the existence of w_ε (a minimizer of F_ε subject to (1.2)) is not an issue. Moreover, one may try to exploit several powerful techniques such as the theory of regularity for minimizers to get strong compactness on w_ε and pass to the limit in (1.5).

We believe that these features are a major point of interest of the present work. Indeed, on the one hand our results provide a new, general starting point for the investigation of a wide class of hyperbolic problems, and on the other they allow one to use methods (coming from the elliptic theory) that have never been applied before in this context. Thus, our framework might shed a new light on several long-standing open problems in the theory of nonlinear hyperbolic equations.

We also point out that although the fourth order equation (1.4) has the structure of a singularly perturbed equation, this fact is never used in our results, which are simply based on the properties of minimizers of the functional F_ε. For instance, no estimates on the third and fourth order derivatives are required.

Our approach also works with an extra (dissipative) term on the right hand side of (1.1), namely

$$w''(t,x) = -\nabla\mathcal{W}(w(t,\cdot))(x) - \nabla\mathcal{H}(w'(t,\cdot))(x), \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \tag{1.6}$$

where $\mathcal{H} : H \to [0,\infty)$ is a Gâteaux differentiable functional, defined on a suitable Hilbert space $H \hookrightarrow L^2(\mathbb{R}^n)$. For simplicity, in contrast to $\mathcal{W}$, we will assume that $\mathcal{H}$ is a *quadratic* form on H. Note that while $\nabla\mathcal{W}$ is computed at w, $\nabla\mathcal{H}$ is computed at w'; if, for instance, both $\mathcal{W}$ and $\mathcal{H}$ are the Dirichlet integral, then (1.6) reduces to the strongly damped wave equation $w'' = \Delta w + \Delta w'$. The reader is invited to look at Section 7, where we discuss several examples of hyperbolic problems (with or without dissipative terms) that fit into our scheme.

For equations with dissipative terms the counterpart to Problem 1 is

Problem 2 (Dissipative case). Let w_ε be a minimizer of the functional

$$\frac{\varepsilon^2}{2}\int_0^\infty \int_{\mathbb{R}^n} e^{-t/\varepsilon}|w''(t,x)|^2\,dx\,dt + \int_0^\infty e^{-t/\varepsilon}\{\mathcal{W}(w(t,\cdot)) + \varepsilon\mathcal{H}(w'(t,\cdot))\}\,dt \tag{1.7}$$

subject to the boundary conditions (1.2). Investigate the existence of a limit for w_ε as in (1.5), and see if it solves the Cauchy problem (1.6)&(1.2).

As before, the functional (1.7) relates to (1.6) via its Euler–Lagrange equation

$$\varepsilon^2(e^{-t/\varepsilon}w_\varepsilon'')'' + e^{-t/\varepsilon}\nabla\mathcal{W}(w_\varepsilon(t,\cdot))(x) - \varepsilon\big(e^{-t/\varepsilon}\nabla\mathcal{H}(w_\varepsilon'(t,\cdot))(x)\big)' = 0,$$

namely,

$$\varepsilon^2 w_\varepsilon'''' - 2\varepsilon w_\varepsilon''' + w_\varepsilon'' + \nabla\mathcal{W}(w_\varepsilon(t,\cdot))(x) + \nabla\mathcal{H}(w_\varepsilon'(t,\cdot))(x) - \varepsilon\big(\nabla\mathcal{H}(w_\varepsilon'(t,\cdot))(x)\big)' = 0,$$

which formally reduces to (1.6) when $\varepsilon \downarrow 0$.

Also in the dissipative cases our results provide estimates for the minimizers w_ε, existence of a limit w, and in general all the properties described above.

A further point of interest is that, as is well known, the energy

$$\mathcal{E}(t) = \frac{1}{2}\int_{\mathbb{R}^n} |w'(t,x)|^2\,dx + \mathcal{W}(w(t,\cdot))$$

is formally preserved by the solutions of equation (1.1), while for equation (1.6) the presence of dissipative terms entails that the preserved quantity is

$$\mathcal{E}(t) + 2\int_0^t \mathcal{H}(w'(s,\cdot))\,ds.$$

Generally, however, energy conservation is purely formal, since weak solutions are not regular enough to justify the computations needed in its proof. Our solutions are no exception, but in all cases they satisfy the "energy inequalities"

$$\mathcal{E}(t) \le \mathcal{E}(0) \quad \text{and} \quad \mathcal{E}(t) + 2\int_0^t \mathcal{H}(w'(s,\cdot))\,ds \le \mathcal{E}(0)$$

for equations (1.1) and (1.6) respectively.

Finally, we point out that our results are stated for functions defined in the whole of $\mathbb{R}^n$. This choice is motivated by the fact that this is a model case of particular interest. However, our results hold, without significant changes, also in different contexts, for instance for functions defined on an open subset Ω of $\mathbb{R}^n$ with Dirichlet or Neumann conditions imposed on $\partial\Omega$.

The paper is organized as follows. The main results are stated in Section 2 and proved in Sections 5 and 6. Section 3 contains preliminary results and Section 4 is devoted to the key argument for the construction of the a priori estimates. Finally, several examples are reported in Section 7.

Remark on notation. Throughout the paper, a prime as in v', v'' etc. denotes partial differentiation with respect to the time variable t. For functions defined in spacetime we will write freely $u(t,x)$ or $u(t)$. So if $u(t,\cdot)$ is an element of a space X, and $\mathcal{G}$ is a functional on X, we will write indiscriminately $\mathcal{G}(u(t,\cdot))$ or $\mathcal{G}(u(t))$. Moreover, through the rest of the paper symbols like $\int v\,dx$ will always denote spatial integrals extended to the whole of $\mathbb{R}^n$, and short forms such as L^2, H^1 etc. will denote $L^2(\mathbb{R}^n)$, $H^1(\mathbb{R}^n)$ etc. Finally, $\langle\cdot,\cdot\rangle$ will denote the duality pairing between a Banach space X and its dual X', the space X being clear from the context.

2. Functional setting and main results

The functional $F_\varepsilon(w)$ to be minimized, subject to the boundary conditions (1.2), is defined by (1.3) in the nondissipative case, and by (1.7) in the dissipative case. We shall treat the two cases simultaneously, by letting

$$F_\varepsilon(w) = \frac{\varepsilon^2}{2}\int_0^\infty \int_{\mathbb{R}^n} e^{-t/\varepsilon}|w''(t,x)|^2\,dx\,dt + \int_0^\infty e^{-t/\varepsilon}\{\mathcal{W}(w(t,\cdot)) + \kappa\varepsilon\mathcal{H}(w'(t,\cdot))\}\,dt \quad (\kappa \in \{0,1\}), \tag{2.1}$$

where the parameter $\kappa \in \{0,1\}$ plays the role of an on/off variable. Dealing with Problem 2 (dissipative case) one should let $\kappa = 1$, while dealing with Problem 1 (non-dissipative case) one should let $\kappa = 0$ and ignore the functional $\mathcal{H}$.

Concerning the functionals $\mathcal{W}$ and $\mathcal{H}$, we make the following assumptions:

(H1) The functional $\mathcal{W} : L^2 \to [0,\infty]$ is lower semicontinuous in the weak topology, i.e.,

$$\mathcal{W}(v) \le \liminf_{k\to\infty} \mathcal{W}(v_k) \quad \text{whenever } v_k \rightharpoonup v \text{ in } L^2. \tag{2.2}$$

Moreover we assume that $\mathcal{W}(v) < \infty \Leftrightarrow v \in W$, a Banach space with

$$C_0^\infty \hookrightarrow W \hookrightarrow L^2 \quad \text{(dense and continuous inclusions).} \tag{2.3}$$

We also assume that $\mathcal{W}$ is Gâteaux differentiable on W, and that its derivative $\nabla\mathcal{W} : W \to W'$ satisfies the estimate

$$\|\nabla\mathcal{W}(v)\|_{W'} \le C(1 + \mathcal{W}(v)^\theta), \quad C \ge 0,\ \theta \in (0,1), \quad \forall v \in W. \tag{2.4}$$

(H2) If $\kappa = 1$, we assume that $\mathcal{H} : L^2 \to [0,\infty]$ is a quadratic functional

$$\mathcal{H}(v) = \begin{cases} \frac{1}{2}B(v,v) & \text{if } v \in H, \\ \infty & \text{if } v \in L^2 \setminus H, \end{cases} \tag{2.5}$$

where $B : H \times H \to \mathbb{R}$ is a symmetric, bounded, nonnegative bilinear form on a Hilbert space H with the norm $\|v\|_H^2 = \|v\|_{L^2}^2 + 2\mathcal{H}(v)$, and such that

$$C_0^\infty \hookrightarrow H \hookrightarrow L^2 \quad \text{(dense and continuous inclusions).} \tag{2.6}$$

If $\kappa = 0$, for definiteness we set $\mathcal{H} \equiv 0$ and $H = L^2$.

Remark 2.1. If $\nabla^k v$ denotes the tensor of all k-th partial derivatives of v, the Dirichlet-like functional

$$\mathcal{W}(v) = \frac{1}{p}\int_{\mathbb{R}^n} |\nabla^k v(x)|^p\,dx \quad (p > 1)$$

satisfies assumption (H1) with W the Banach space of all L^2 functions v such that $\nabla^k v \in L^p$, endowed with its natural norm. Since

$$\langle \nabla \mathcal{W}(v), \eta \rangle = \int_{\mathbb{R}^n} |\nabla^k v(x)|^{p-2} \nabla^k v(x) \cdot \nabla^k \eta(x)\, dx, \qquad v, \eta \in W, \tag{2.7}$$

we see that (2.4) holds with $\theta = 1 - 1/p$. In view of the embeddings (2.3), the term $\nabla \mathcal{W}(w(t, \cdot))$ in equations (1.1) and (1.6), as a distribution (note that $W' \hookrightarrow \mathcal{D}'$ by (2.3)), acts as a differential operator (linear when $p = 2$) of order $2k$. Note also that the functional $\mathcal{W}$ need not be convex.

Remark 2.2. A typical functional $\mathcal{H}$ fulfilling (H2) has the form

$$\mathcal{H}(v) = \frac{1}{2} \sum_{j \in S} \int_{\mathbb{R}^n} |\partial^j v|^2\, dx \tag{2.8}$$

where $S \subset \mathbb{N}^n$ is any finite set of multi-indices and ∂^j denotes partial differentiation. Here H is the space of those $v \in L^2$ such that $\mathcal{H}(v) < \infty$, and $\nabla H(v)$, as a distribution (note that H' is a space of distributions by (2.6)), is the differential operator $\sum_{j \in S} (-1)^{|j|} \partial^{2j}$.

Remark 2.3. Assumptions (H1) and (H2) are *additively stable*. More precisely, if $\mathcal{W}_i : L^2 \to [0, \infty]$ ($i = 1, 2$) are two functionals each satisfying (H1) (with Banach spaces W_i, constants θ_i etc.), then the sum $\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2$ still satisfies (H1), now with $W = W_1 \cap W_2$ normed by $\|\cdot\|_W = \|\cdot\|_{W_1} + \|\cdot\|_{W_2}$ (this makes sense, in view of (2.3)). In particular, by the Young inequality, (2.4) will hold true with $\theta = \max\{\theta_1, \theta_2\}$.

Finally, a similar argument applies to (H2).

Theorem 2.4 (nondissipative case). *Given $w_0, w_1 \in W$ and $\varepsilon \in (0, 1)$, under assumption* (H1) *the functional F_ε defined in* (1.3) *has a minimizer w_ε in the space $H^2_{\mathrm{loc}}([0, \infty); L^2)$ subject to* (1.2)*. Moreover:*

(a) (Estimates) *There exists a constant C, independent of ε, such that*

$$\int_\tau^{\tau+T} \mathcal{W}(w_\varepsilon(t, \cdot))\, dt \le CT \qquad \forall \tau \ge 0,\ \forall T \ge \varepsilon, \tag{2.9}$$

$$\int_{\mathbb{R}^n} |w'_\varepsilon(t, x)|^2\, dx \le C, \qquad \int_{\mathbb{R}^n} |w_\varepsilon(t, x)|^2\, dx \le C(1 + t^2) \qquad \forall t \ge 0, \tag{2.10}$$

$$\|w''_\varepsilon\|_{L^\infty(\mathbb{R}^+; W')} \le C. \tag{2.11}$$

(b) (Convergence) *Every sequence w_{ε_i} (with $\varepsilon_i \downarrow 0$) admits a subsequence which is convergent, in the weak topology of $H^1((0, T); L^2)$ for every $T > 0$, to a function w such that*

$$w \in H^1_{\mathrm{loc}}([0, \infty); L^2), \qquad w' \in L^\infty(\mathbb{R}^+; L^2), \qquad w'' \in L^\infty(\mathbb{R}^+; W'). \tag{2.12}$$

Moreover, w satisfies the initial conditions (1.2)*.*

(c) (Energy inequality) *Let*

$$\mathcal{E}(t) = \frac{1}{2}\int_{\mathbb{R}^n} |w'(t,x)|^2\,dx + \mathcal{W}(w(t,\cdot)). \tag{2.13}$$

Then the function $w(t,x)$ satisfies the energy inequality

$$\mathcal{E}(t) \le \mathcal{E}(0) = \frac{1}{2}\int_{\mathbb{R}^n} |w_1(x)|^2\,dx + \mathcal{W}(w_0) \quad \textit{for a.e. } t > 0. \tag{2.14}$$

Theorem 2.5 (dissipative case). *Given $w_0 \in W$, $w_1 \in W \cap H$ and $\varepsilon \in (0,1)$, under assumptions* (H1) *and* (H2) *the functional F_ε defined in* (1.7) *has a minimizer w_ε in the space $H^2_{\mathrm{loc}}([0,\infty); L^2)$ subject to* (1.2)*. Moreover, all claims of Theorem* 2.4 *apply, with the following extensions and modifications:*

(a) *The additional estimate*

$$\int_0^\infty \mathcal{H}(w'_\varepsilon(t))\,dt \le C \tag{2.15}$$

holds true, while (2.11) *should be replaced with*

$$\|w''_\varepsilon\|_{L^\infty(\mathbb{R}^+;W')+L^2(\mathbb{R}^+;H')} \le C. \tag{2.16}$$

(b) *The part on w'' in* (2.12) *should be replaced with*

$$w'' \in L^\infty(\mathbb{R}^+;W') + L^2(\mathbb{R}^+;H'). \tag{2.17}$$

Moreover, the convergence $w'_\varepsilon \to w'$ holds in a stronger sense, namely

$$w'_\varepsilon \rightharpoonup w' \quad \textit{weakly in } L^2((0,T);H), \textit{ for every } T > 0. \tag{2.18}$$

(c) *With the same $\mathcal{E}(t)$, the inequality* (2.14) *is replaced with*

$$\mathcal{E}(t) + 2\int_0^t \mathcal{H}(w'(t,\cdot))\,dt \le \mathcal{E}(0) \quad \textit{for a.e. } t > 0. \tag{2.19}$$

Observe that, under so general assumptions as in Theorem 2.4 (or 2.5), we do not claim that the limit function w satisfies (1.1) (or (1.6)). On the other hand, to our knowledge there are no counterexamples that rule out this possibility. Of course, to perform this step (by which one would completely solve Problem 1 or 2) one should obtain extra estimates exploiting the particular structure of the functional $\mathcal{W}$, on a case-by-case basis. In some cases, however, the estimates of Theorem 2.4 (or 2.5 if $\kappa = 1$) are enough to pass to the limit in the main equation, as the following result illustrates.

Theorem 2.6. *Assume that, for some real number $m > 0$,*

$$\mathcal{W}(v) = \frac{1}{2}\|v\|^2_{\dot{H}^m} + \sum_{0\le k<m} \frac{\lambda_k}{p_k}\int_{\mathbb{R}^n} |\nabla^k v(x)|^{p_k}\,dx \quad (\lambda_k \ge 0,\ p_k > 1). \tag{2.20}$$

Then assumption (H1) *is fulfilled if W is the space of those $v \in H^m$ with $\nabla^k v \in L^{p_k}$ $(0 \le k < m)$, endowed with its natural norm.*

Moreover, the limit function w obtained via Theorem 2.4 (*or* 2.5 *if* $\kappa = 1$) *solves, in the sense of distributions, the hyperbolic equation* (1.1) (*or* (1.6) *if* $\kappa = 1$).

Remark 2.7. In (2.20), as usual, $\|v\|_{\dot{H}^m}$ is the L^2 norm of $|\xi|^m \hat{v}(\xi)$, where $\hat{v}$ is the Fourier transform of v. The typical case is when m is an integer, so that $\|v\|^2_{\dot{H}^m}$ reduces to $\|\nabla^m v\|^2_{L^2}$. In this case (see Remark 2.1) the first term in (2.20) gives rise to a differential operator of order $2m$ in equations (1.1) and (1.6).

On the other hand, in (2.20), m may fail to be an integer. In this case, however, one can interpret the distribution $\nabla\mathcal{W}(w)$ in 1.1 or (1.6) as a *fractional* differential operator; this enables us to treat, for instance, equations with the *fractional Laplacian* (see Example 7 in Section 7).

Several variants are possible in the same spirit. For instance, one may introduce nonconstant coefficients in (2.20) (and possibly exploit Gårding-type inequalities to make $\mathcal{W}(v)$ coercive), or consider more general lower order terms with suitable convexity and growth assumptions (e.g. powers of single partial derivatives as in (2.8)). Indeed, the central assumption is that $\mathcal{W}$ be quadratic (and coercive) in the highest order terms, which makes the hyperbolic PDEs (1.1) and (1.6) quasilinear.

We end this section by discussing some consequences of assumptions (H1) and (H2) which will be used in the following. First, (2.4) implies the linear control

$$\|\nabla\mathcal{W}(v)\|_{W'} \le C(1+\mathcal{W}(v)), \qquad C \ge 0, \qquad \forall v \in W. \tag{2.21}$$

Moreover, (2.4) entails Lipschitz continuity of $\mathcal{W}$ along rays, as follows. Given $a, \overline{b} \in W$ with $\|\overline{b}\|_W = 1$, the function $f(\lambda) = \mathcal{W}(a+\lambda\overline{b})$ is differentiable and estimate (2.4) gives $|f'| \le C(1+f^\theta)$. From well known variants of the Gronwall Lemma, one finds that $f(\lambda) \le C(1+f(0)+\lambda^{1/(1-\theta)})$ and so

$$\sup_{[a,a+b]} \mathcal{W} \le C(1+\mathcal{W}(a)+\|b\|_W^{1/(1-\theta)}), \qquad \forall a, b \in W, \tag{2.22}$$

where $[a, a+b]$ is the segment in W from a to $a+b$. Combining this with (2.21) yields

$$\sup_{[a,a+b]} \|\nabla\mathcal{W}\|_{W'} \le C(1+\mathcal{W}(a)+\|b\|_W^{1/(1-\theta)}), \qquad a, b \in W. \tag{2.23}$$

Then, from the Lagrange mean value theorem, for every $\delta \ne 0$,

$$\left|\frac{\mathcal{W}(a+\delta b)-\mathcal{W}(a)}{\delta}\right| \le \|b\|_W \sup_{[a,a+\delta b]} \|\nabla\mathcal{W}\|_{W'},$$

and together with (2.23) this gives

$$\left|\frac{\mathcal{W}(a+\delta b)-\mathcal{W}(a)}{\delta}\right| \le C\|b\|_W\big(1+\mathcal{W}(a)+\delta^{1/(1-\theta)}\|b\|_W^{1/(1-\theta)}\big), \tag{2.24}$$

a quantitative bound for the Lipschitz constant of $\mathcal{W}$. Thus, in particular,

$$\mathcal{W}(a+\delta b) \le \mathcal{W}(a) + C\delta\|b\|_W\big(1+\mathcal{W}(a)+\delta^{1/(1-\theta)}\|b\|_W^{1/(1-\theta)}\big). \tag{2.25}$$

Finally, assumption (H2) entails that $\mathcal{H}$ is differentiable in H, with

$$\langle\nabla\mathcal{H}(v), \eta\rangle = B(v,\eta), \qquad \|\nabla\mathcal{H}(v)\|_{H'} \le \sqrt{2\mathcal{H}(v)}, \qquad v, \eta \in H. \tag{2.26}$$

Moreover, $\mathcal{H}$ is a fortiori weakly lower semicontinuous in L^2, namely

$$\mathcal{H}(v) \le \liminf_{k\to\infty} \mathcal{H}(v_k) \qquad \text{whenever } v_k \rightharpoonup v \text{ in } L^2. \tag{2.27}$$

3. Existence of minimizers and preliminary estimates

Since the space $H^2_{\text{loc}}([0,\infty); L^2)$ is invariant under time dilations $t \mapsto \varepsilon t$, it is convenient to introduce the simpler functional

$$J_\varepsilon(u) = \int_0^\infty e^{-t}\left(\int \frac{|u''(t,x)|^2}{2\varepsilon^2}\,dx + \mathcal{W}(u(t)) + \frac{\kappa}{\varepsilon}\mathcal{H}(u'(t))\right)dt, \tag{3.1}$$

equivalent to F_ε in (2.1) in that $F_\varepsilon(w) = \varepsilon J_\varepsilon(u)$ whenever $u, w \in H^2_{\text{loc}}([0,\infty); L^2)$ are related by the change of variable $u(t,x) = w(\varepsilon t, x)$. Of course, the boundary conditions in (1.2) must be scaled accordingly, namely as in (3.3) below.

The existence of minimizers w_ε for F_ε (as claimed in Theorems 2.4 and 2.5) then follows from the existence of minimizers u_ε for J_ε and

$$u_\varepsilon(t,x) = w_\varepsilon(\varepsilon t, x), \qquad t \geq 0,\ x \in \mathbb{R}^n. \tag{3.2}$$

Lemma 3.1. *Given $\varepsilon \in (0,1)$ and $w_0, w_1 \in W$ (with $w_1 \in W \cap H$ if $\kappa = 1$) the functional J_ε has an absolute minimizer u_ε in the class of those functions $u \in H^2_{\text{loc}}([0,\infty); L^2)$ satisfying the boundary conditions*

$$u(0) = w_0, \qquad u'(0) = \varepsilon w_1. \tag{3.3}$$

Moreover,

$$J_\varepsilon(u_\varepsilon) \leq \mathcal{W}(w_0) + C\varepsilon. \tag{3.4}$$

Remark 3.2. Throughout, the symbol C will always denote (possibly different) constants that are *independent* of ε (but may depend on all the other data, including the initial conditions w_0, w_1).

Proof of Lemma 3.1. The function $\psi(t,x) = w_0(x) + \varepsilon t w_1(x)$ satisfies the boundary conditions (3.3). We also deduce from (2.25), applied with $a = w_0$, $b = w_1$ and $\delta = \varepsilon t$, that

$$\mathcal{W}(w_0 + \varepsilon t w_1) \leq \mathcal{W}(w_0) + C\varepsilon t(1 + \mathcal{W}(w_0) + (\varepsilon t)^{1/(1-\theta)})$$

with $\|w_1\|_W$ absorbed into C. Multiplying by e^{-t} and integrating, we find that

$$\int_0^\infty e^{-t}\mathcal{W}(\psi(t))\,dt \leq \mathcal{W}(w_0) + C\varepsilon.$$

Moreover, if $\kappa = 1$, since $\psi' = \varepsilon w_1$ and $w_1 \in H$, from (2.5) we see that

$$\frac{\kappa}{\varepsilon}\int_0^\infty e^{-t}\mathcal{H}(\psi'(t))\,dt = \frac{\varepsilon}{2}B(w_1, w_1)\int_0^\infty e^{-t}\,dt \leq C\varepsilon.$$

Summing up, we obtain $J_\varepsilon(\psi) \leq \mathcal{W}(w_0) + C\varepsilon$: in particular, J_ε has a finite infimum and (3.4) follows as soon as J_ε has an absolute minimizer u_ε. To show this, consider a minimizing sequence u_k and fix $T > 0$. Combining the estimate

$$\int_0^T \|u_k''(t)\|^2_{L^2}\,dt \leq e^T\int_0^T e^{-t}\|u_k''(t)\|^2_{L^2}\,dt \leq 2\varepsilon^2 e^T J_\varepsilon(u_k)$$

with the initial conditions (3.3) satisfied by u_k, we see that $\{u_k\}$ is bounded in the space $H^2_{\mathrm{loc}}([0,\infty); L^2)$, whence, up to subsequences, $u_k(t) \rightharpoonup u(t)$ and $u_k'(t) \rightharpoonup u'(t)$ in L^2 for every $t \geq 0$, for some $u \in H^2_{\mathrm{loc}}([0,\infty); L^2)$ that fulfills (3.3). Now the term involving u'' in (3.1) is lower semicontinuous, and the same is true of the other two terms by Fatou's Lemma and weak convergence in L^2 of $u_k(t)$ and $u_k'(t)$ for fixed t, using (2.2) and (2.27). This shows that $J_\varepsilon(u) \leq \liminf J_\varepsilon(u_k)$, hence $u = u_\varepsilon$ is a global minimizer. □

In some cases, a weaker version of (3.4) will be used, namely

$$J_\varepsilon(u_\varepsilon) \leq C. \tag{3.5}$$

Remark 3.3. To simplify notation, given a minimizer u_ε, we define, for $t \geq 0$,

$$\mathcal{W}_\varepsilon(t) := \mathcal{W}(u_\varepsilon(t,\cdot)) \quad \text{and} \quad \mathcal{H}_\varepsilon(t) := \mathcal{H}(u_\varepsilon'(t,\cdot)). \tag{3.6}$$

We also set

$$D_\varepsilon(t) := \frac{1}{2\varepsilon^2}\int |u_\varepsilon''(t,x)|^2\,dx \qquad \text{for a.e. } t > 0, \tag{3.7}$$

so that we write

$$L_\varepsilon(t) := D_\varepsilon(t) + \mathcal{W}_\varepsilon(t) + \frac{\kappa}{\varepsilon}\mathcal{H}_\varepsilon(t) \tag{3.8}$$

for the locally integrable "Lagrangian". Finally, we introduce the kinetic energy function

$$K_\varepsilon(t) := \frac{1}{2\varepsilon^2}\int |u_\varepsilon'(t,x)|^2\,dx \qquad \forall t \geq 0.$$

The notation just introduced will be used systematically in what follows.

Note that, due to Lemma 3.4 below, $K_\varepsilon \in W^{1,1}(0,T)$ for all $T > 0$ and

$$K_\varepsilon'(t) = \frac{1}{\varepsilon^2}\int u_\varepsilon'(t,x)u_\varepsilon''(t,x)\,dx \qquad \text{for a.e. } t > 0. \tag{3.9}$$

Lemma 3.4. *The minimizers u_ε defined by Lemma 3.1 satisfy*

$$\int_0^\infty e^{-t} D_\varepsilon(t)\,dt = \int_0^\infty e^{-t}\int \frac{|u_\varepsilon''|^2}{2\varepsilon^2}\,dx\,dt \leq C, \tag{3.10}$$

$$\int_0^\infty e^{-t} K_\varepsilon(t)\,dt = \int_0^\infty e^{-t}\int \frac{|u_\varepsilon'|^2}{2\varepsilon^2}\,dx\,dt \leq C. \tag{3.11}$$

Proof. Estimate (3.10) follows immediately from (3.5). The inequality (see [10])

$$\int_0^\infty\int e^{-t}|v(t,x)|^2\,dx\,dt \leq 2\int |v(0,x)|^2\,dx + 4\int_0^\infty\int e^{-t}|v'(t,x)|^2\,dx\,dt,$$

applied with $v(t,x) = u_\varepsilon'(t,x)$, shows, using (3.3) and (3.10), that

$$\int_0^\infty\int e^{-t}|u_\varepsilon'|^2\,dx\,dt \leq 2\varepsilon^2\int |w_1(x)|^2\,dx + C\varepsilon^2,$$

and (3.11) is established since $w_1 \in L^2$ by (2.3). □

4. The approximate energy

Since integrals with an exponential weight play a major role in our investigation, it is convenient to introduce the following *average operator*.

Definition 4.1. If $f : \mathbb{R}^+ \to [0, \infty]$ is measurable, we let

$$\mathcal{A}f\,(s) := \int_s^\infty e^{-(t-s)} f(t)\,dt, \qquad s \geq 0.$$

Note that $\mathcal{A}f$ is well defined (possibly ∞) as $f \geq 0$. However, since

$$\mathcal{A}f\,(0) = \int_0^\infty e^{-t} f(t)\,dt, \tag{4.1}$$

if $\mathcal{A}f\,(0) < \infty$ then $\mathcal{A}f$ is absolutely continuous on intervals $[0, T]$, and

$$(\mathcal{A}f)' = \mathcal{A}f - f. \tag{4.2}$$

In any case, since $\mathcal{A}f \geq 0$, starting from $f \geq 0$ one can iterate $\mathcal{A}$, and a simple computation gives

$$\mathcal{A}^2 f\,(s) = \int_s^\infty e^{-(t-s)}(t-s) f(t)\,dt, \tag{4.3}$$

and in particular

$$\mathcal{A}^2 f\,(0) = \int_0^\infty e^{-t} t f(t)\,dt. \tag{4.4}$$

We now introduce a fundamental quantity for our approach.

Definition 4.2. Let u_ε be a minimizer of J_ε. The *approximate energy* is the function

$$E_\varepsilon := K_\varepsilon + \mathcal{A}^2 \mathcal{W}_\varepsilon, \tag{4.5}$$

or, more explicitly,

$$E_\varepsilon(s) = K_\varepsilon(s) + \int_s^\infty e^{-(t-s)}(t-s)\mathcal{W}(u_\varepsilon(t))\,dt, \qquad s \geq 0. \tag{4.6}$$

Remark 4.3. In (4.6), the kinetic energy K_ε is evaluated pointwise at time s, while the potential energy $\mathcal{W}_\varepsilon$ is *averaged* over times $t \geq s$ via the *probability kernel* $e^{-(t-s)}(t-s)$. However, recalling the time scaling $t \mapsto \varepsilon t$ that links the functionals F_ε and J_ε, in the original time scale the probability kernel in (4.6) concentrates close to s as $\varepsilon \to 0$. Thus, heuristically, from (3.2) one expects that $E_\varepsilon(t/\varepsilon) \approx \mathcal{E}(t)$ where $\mathcal{E}$ is the physical energy defined in (2.13).

Observe that, from (3.8) and (4.1), we have

$$\mathcal{A}\mathcal{W}_\varepsilon\,(0) \leq \mathcal{A}L_\varepsilon\,(0) = J_\varepsilon(u_\varepsilon) \leq C,$$

and so $\mathcal{A}\mathcal{W}_\varepsilon$ is well defined. But since $\mathcal{A}$ is iterated twice in (4.5), it is not even clear why $E_\varepsilon(s)$ should be finite. In fact, as we will show, $E_\varepsilon(s)$ is finite and *decreasing*, and this monotonicity will be the key to our estimates.

The monotonicity of E_ε will be deduced from the following proposition.

Proposition 4.4. *Let u_ε be a minimizer of J_ε. For every $g \in C^2([0,\infty))$ such that $g(0) = 0$ and $g(t)$ is constant for large t,*

$$\int_0^\infty e^{-s}(g'(s) - g(s))L_\varepsilon(s)\,ds - \int_0^\infty e^{-s}\Big(4D_\varepsilon(s)g'(s) + K_\varepsilon'(s)g''(s) + \frac{2\kappa}{\varepsilon}\mathcal{H}_\varepsilon(s)g'(s)\Big)\,ds = g'(0)R(u_\varepsilon), \tag{4.7}$$

where

$$R(u_\varepsilon) = -\varepsilon\int_0^\infty e^{-s}s\langle\nabla\mathcal{W}(u_\varepsilon(s)), w_1\rangle\,ds - \kappa\int_0^\infty e^{-s}\langle\nabla\mathcal{H}(u_\varepsilon'(s)), w_1\rangle\,ds. \tag{4.8}$$

The quantity $R(u_\varepsilon)$ is finite, and satisfies the estimate

$$|R(u_\varepsilon)| \le C(\varepsilon + \kappa\sqrt{\varepsilon}) \le C\sqrt{\varepsilon}. \tag{4.9}$$

Proof. For every $\delta \in \mathbb{R}$ with $|\delta|$ small enough, the function

$$\varphi(t) = \varphi(t,\delta) = t - \delta g(t) \tag{4.10}$$

is a diffeomorphism of $\mathbb{R}^+$ of class C^2. We denote by ψ its inverse,

$$\psi(s) = \varphi^{-1}(s), \qquad s \ge 0$$

(the dependence on δ, which is fixed, is omitted to simplify the notation). For small δ, we consider the competitor

$$U(t) = u_\varepsilon(\varphi(t)) + t\delta\varepsilon g'(0)w_1,$$

which satisfies the boundary conditions $U(0) = w_0$ and $U'(0) = \varepsilon w_1$, because $\varphi(0) = 0$ and $\varphi'(0) = 1 - \delta g'(0)$. We have

$$U'(t) = u_\varepsilon'(\varphi(t))\varphi'(t) + \delta\varepsilon g'(0)w_1,$$
$$U''(t) = u_\varepsilon''(\varphi(t))|\varphi'(t)|^2 + u_\varepsilon'(\varphi(t))\varphi''(t),$$

and hence

$$J_\varepsilon(U) = \int_0^\infty e^{-t}\Big\{\frac{1}{2\varepsilon^2}\big\|u_\varepsilon''(\varphi(t))|\varphi'(t)|^2 + u_\varepsilon'(\varphi(t))\varphi''(t)\big\|_{L^2}^2 + \mathcal{W}\big(u_\varepsilon(\varphi(t)) + t\delta\varepsilon g'(0)w_1\big) + \frac{\kappa}{\varepsilon}\mathcal{H}\big(u_\varepsilon'(\varphi(t))\varphi'(t) + \delta\varepsilon g'(0)w_1\big)\Big\}\,dt.$$

Changing variable in the integral letting $t = \psi(s)$, that is, $s = \varphi(t)$, we obtain

$$J_\varepsilon(U) = \int_0^\infty \psi'(s)e^{-\psi(s)}\bigg\{\frac{1}{2\varepsilon^2}\big\|u_\varepsilon''(s)|\varphi'(\psi(s))|^2 + u_\varepsilon'(s)\varphi''(\psi(s))\big\|_{L^2}^2 + \mathcal{W}\big(u_\varepsilon(s) + \delta\varepsilon g'(0)w_1\psi(s)\big) + \frac{\kappa}{\varepsilon}\mathcal{H}\big(u_\varepsilon'(s)\varphi'(\psi(s)) + \delta\varepsilon g'(0)w_1\big)\bigg\}\,ds. \tag{4.11}$$

Note that, from (4.10), $s = \varphi(\psi(s)) = \psi(s) - \delta g(\psi(s))$, that is,

$$\psi(s) = s + \delta g(\psi(s)). \tag{4.12}$$

In view of the assumptions on g, we have $\psi(s) \geq s - \delta\|g\|_\infty$, and hence $e^{-\psi(s)} \leq e^{\delta\|g\|_\infty}e^{-s}$. Furthermore, by (2.22) and (2.5),

$$\mathcal{W}\big(u_\varepsilon(s) + \delta\varepsilon g'(0)w_1\psi(s)\big) \leq C\big(1 + \mathcal{W}(u_\varepsilon(s)) + \psi(s)^{1/(1-\theta)}\big)$$

and

$$\mathcal{H}\big(u_\varepsilon'(s)\varphi'(\psi(s)) + \delta\varepsilon g'(0)w_1\big) \leq 2\varphi'(\psi(s))^2\mathcal{H}(u_\varepsilon'(s)) + C\mathcal{H}(w_1).$$

These inequalities, together with (3.10), (3.11) and the finiteness of $\|\varphi'\|_\infty$ and $\|\varphi''\|_\infty$, show that $J_\varepsilon(U)$ is finite and hence U is an admissible competitor.

Since $U(t)$ reduces to $u_\varepsilon(t)$ when $\delta = 0$, the minimality of u_ε implies that

$$\frac{d}{d\delta}J_\varepsilon(U)\bigg|_{\delta=0} = 0. \tag{4.13}$$

To compute this derivative, we differentiate under the integral sign in (4.11) (reasoning as above for the finiteness of $J_\varepsilon(U)$, it is easy to prove that this is possible). From (4.12),

$$\frac{\partial}{\partial\delta}\big(\psi'(s)e^{-\psi(s)}\big)\bigg|_{\delta=0} = g'(s)e^{-s} - g(s)e^{-s}.$$

Moreover, elementary computations give

$$\frac{\partial}{\partial\delta}|\varphi'(\psi(s))|^2\bigg|_{\delta=0} = -2g'(s), \qquad \frac{\partial}{\partial\delta}\varphi''(\psi(s))\bigg|_{\delta=0} = -g''(s).$$

Denoting by $\Theta(s)$ the function within braces in (4.11), and recalling (3.8), we have

$$\Theta(s)|_{\delta=0} = \frac{1}{2\varepsilon^2}\|u_\varepsilon''(s)\|_{L^2}^2 + \mathcal{W}_\varepsilon(s) + \frac{\kappa}{\varepsilon}\mathcal{H}_\varepsilon(s) = L_\varepsilon(s)$$

and, recalling (3.7) and (3.9),

$$\begin{aligned}\frac{\partial}{\partial\delta}\Theta(s)\bigg|_{\delta=0} &= -\frac{1}{\varepsilon^2}\langle u_\varepsilon''(s), 2u_\varepsilon''(s)g'(s) + u_\varepsilon'(s)g''(s)\rangle_{L^2} - \frac{2\kappa}{\varepsilon}g'(s)\mathcal{H}_\varepsilon(s)\\ &\quad + \varepsilon g'(0)s\langle\nabla\mathcal{W}(u_\varepsilon(s)), w_1\rangle + \kappa g'(0)\langle\nabla\mathcal{H}(u_\varepsilon'(s)), w_1\rangle\\ &= -4D_\varepsilon(s)g'(s) - K_\varepsilon'(s)g''(s) - \frac{2\kappa}{\varepsilon}g'(s)\mathcal{H}_\varepsilon(s)\\ &\quad + \varepsilon g'(0)s\langle\nabla\mathcal{W}(u_\varepsilon(s)), w_1\rangle + \kappa g'(0)\langle\nabla\mathcal{H}(u_\varepsilon'(s)), w_1\rangle.\end{aligned}$$

Combining these facts, we obtain

$$\begin{aligned}\frac{\partial}{\partial\delta}(\psi'(s)e^{-\psi(s)}\Theta(s))\Big|_{\delta=0} &= e^{-s}(g'(s)-g(s))L_\varepsilon(s) \\ &\quad - e^{-s}\left(4D_\varepsilon(s)g'(s)+K_\varepsilon'(s)g''(s)+\frac{2\kappa}{\varepsilon}g'(s)\mathcal{H}_\varepsilon(s)\right) \\ &\quad + e^{-s}\big(\varepsilon g'(0)s\langle\nabla\mathcal{W}(u_\varepsilon(s)),w_1\rangle+\kappa g'(0)\langle\nabla\mathcal{H}(u_\varepsilon'(s)),w_1\rangle\big).\end{aligned}$$

Finally, integrating in s we see that (4.13) reduces to (4.7).

We now prove estimate (4.9). For the first integral in (4.8), from (2.4) and the Young inequality we have

$$\begin{aligned}\left|\int_0^\infty e^{-s}s\langle\nabla\mathcal{W}(u_\varepsilon(s)),w_1\rangle\,ds\right| &\le \|w_1\|_W\int_0^\infty e^{-s}s\|\nabla\mathcal{W}(u_\varepsilon(s))\|_{W'}\,ds \\ &\le C\int_0^\infty e^{-s}s(1+\mathcal{W}_\varepsilon(s)^\theta)\,ds = C+C\int_0^\infty e^{-s}s\mathcal{W}_\varepsilon(s)^\theta\,ds \\ &\le C+C\int_0^\infty e^{-s}s^{1/(1-\theta)}\,ds+\int_0^\infty e^{-s}\mathcal{W}_\varepsilon(s)\,ds \le C+J_\varepsilon(u_\varepsilon)\le C,\end{aligned}$$

where we have used (3.5), and thus $|R(u_\varepsilon)|\le C\varepsilon$ when $\kappa=0$. If, on the other hand, $\kappa=1$, we also estimate the second integral in (4.8):

$$\begin{aligned}\left|\int_0^\infty e^{-s}\langle\nabla\mathcal{H}(u_\varepsilon'(s)),w_1\rangle\,ds\right| &\le \|w_1\|_H\int_0^\infty e^{-s}\|\nabla\mathcal{H}(u_\varepsilon'(s))\|_{H'}\,ds \\ &\le C\int_0^\infty e^{-s}\sqrt{\mathcal{H}_\varepsilon(s)}\,ds \le C\left(\int_0^\infty e^{-s}\mathcal{H}_\varepsilon(s)\,ds\right)^{1/2} \le C(\varepsilon J_\varepsilon(u_\varepsilon))^{1/2}\le C\sqrt{\varepsilon},\end{aligned}$$

where we have used (2.26), the Jensen inequality and (3.5). □

Corollary 4.5. *If $g\ge 0$ is of class $C^{1,1}$, satisfies $g(0)=0$ and is affine for large t, then* (4.7) *remains true* (*all integrals being finite*). *In particular, when $g(t)=t$,*

$$\mathcal{A}^2L_\varepsilon\,(0)+\frac{2\kappa}{\varepsilon}\mathcal{A}\mathcal{H}_\varepsilon\,(0)+4\mathcal{A}D_\varepsilon\,(0)=\mathcal{A}L_\varepsilon\,(0)-R(u_\varepsilon). \tag{4.14}$$

Remark 4.6. Since $L_\varepsilon(t)\ge\mathcal{W}_\varepsilon(t)$, the finiteness of $\mathcal{A}^2L_\varepsilon\,(0)$ in (4.14) entails that the approximate energy $E_\varepsilon(s)$ is finite for *every* $s\ge 0$ (in fact, it is absolutely continuous on intervals $[0,T]$, see the discussion after (4.1)).

Proof of Corollary 4.5. By smoothing a truncation of g, one can find an increasing sequence g_k of C^2 functions, each eventually constant, such that as $k\to\infty$,

$$g_k(t)\uparrow g(t),\quad g_k'(t)\uparrow g'(t),\quad g_k''(t)\to g''(t)\qquad\text{for almost every } t\ge 0,$$

with g_k' and g_k'' uniformly bounded. We now write (4.7) for g_k and let $k\to\infty$. Since the functions

$$e^{-t}L_\varepsilon(t),\qquad e^{-t}D_\varepsilon(t),\qquad e^{-t}|K_\varepsilon'(t)|,\qquad e^{-t}\mathcal{H}_\varepsilon(t)$$

are all in $L^1(\mathbb{R}^+)$ (either by the finiteness of $J_\varepsilon(u_\varepsilon)$ or by Lemma 3.4) and $g_k'(0)R(u_\varepsilon)$ does not depend on k, all integrals pass to the limit, except for the integral of $e^{-t}g_k(t)L_\varepsilon(t)$ because the g_k are not uniformly bounded. For this term, however, one can use *monotone* convergence, and the integral of $e^{-t}g(t)L_\varepsilon(t)$ in the limit is finite, by finiteness of all other terms. In particular, one can let $g(t) = t$ in (4.7), which (after recalling (4.1) and (4.4)) yields (4.14). □

Corollary 4.7. *For almost every $T > 0$,*

$$\mathcal{A}^2 L_\varepsilon(T) - \mathcal{A}L_\varepsilon(T) + K_\varepsilon'(T) = -4\mathcal{A}D_\varepsilon(T) - \frac{2\kappa}{\varepsilon}\mathcal{A}\mathcal{H}_\varepsilon(T). \tag{4.15}$$

Proof. Consider the function $g \in C^{1,1}(\mathbb{R})$ defined as

$$g(t) = \begin{cases} 0 & \text{if } t \le 0,\\ t^2/2 & \text{if } t \in (0,1),\\ t - 1/2 & \text{if } t \ge 1,\end{cases}$$

and for $T > 0$ and $\delta > 0$ (we will let $\delta \downarrow 0$), set

$$g_\delta(t) = \delta g((t-T)/\delta). \tag{4.16}$$

Each g_δ satisfies the assumptions of Corollary 4.5, and $g_\delta''(t) = \frac{1}{\delta}\chi_{(T,T+\delta)}$. Letting $g = g_\delta$ in (4.7) and rearranging terms, gives

$$\int_T^\infty e^{-t}(g_\delta(t) - g_\delta'(t))L_\varepsilon(t)\,dt + \frac{1}{\delta}\int_T^{T+\delta} e^{-t}K_\varepsilon'(t)\,dt = -\int_T^\infty e^{-t}\left(4D_\varepsilon(t)g_\delta'(t) + \frac{2\kappa}{\varepsilon}\mathcal{H}_\varepsilon(t)g_\delta'(t)\right)dt.$$

Note that, as $\delta \to 0$, $g_\delta(t) \to (t-T)^+$ while $g_\delta'(t) \to \chi_{(T,\infty)}$, with bounds $|g_\delta(t)| \le (t-T)^+$ and $|g_\delta'(t)| \le 1$. By dominated convergence we can let $\delta \downarrow 0$, thus obtaining, for a.e. T,

$$\int_T^\infty e^{-t}(t-T)L_\varepsilon(t)\,dt - \int_T^\infty e^{-t}L_\varepsilon(t)\,dt + e^{-T}K_\varepsilon'(T) = -\int_T^\infty e^{-t}\left(4D_\varepsilon(t) + \frac{2\kappa}{\varepsilon}\mathcal{H}_\varepsilon(t)\right)dt,$$

and multiplying by e^T one obtains (4.15). □

Theorem 4.8. *The function E_ε is finite and decreasing. More precisely,*

$$E_\varepsilon'(T) \le -\frac{\kappa}{\varepsilon}\big(\mathcal{A}\mathcal{H}_\varepsilon(T) + \mathcal{A}^2\mathcal{H}_\varepsilon(T)\big), \tag{4.17}$$

and

$$E_\varepsilon(T) + \frac{2\kappa}{\varepsilon}\int_0^T \mathcal{H}_\varepsilon(t)\,dt \le \tfrac{1}{2}\|w_1\|_{L^2}^2 + \mathcal{W}(w_0) + C\varepsilon + C\kappa\sqrt{\varepsilon}, \qquad \forall T \ge 0. \tag{4.18}$$

Proof. From Remark 4.6 we know that E_ε is absolutely continuous on intervals $[0, T]$. Hence, differentiating (4.5) and using (4.2) written with $f = \mathcal{A}\mathcal{W}_\varepsilon$ yields

$$E_\varepsilon' = K_\varepsilon' - \mathcal{A}\mathcal{W}_\varepsilon + \mathcal{A}^2\mathcal{W}_\varepsilon.$$

But since $\mathcal{W}_\varepsilon = L_\varepsilon - D_\varepsilon - \frac{\kappa}{\varepsilon}\mathcal{H}_\varepsilon$, using (4.15) we obtain

$$E_\varepsilon' = -3\mathcal{A}D_\varepsilon - \mathcal{A}^2 D_\varepsilon - \frac{\kappa}{\varepsilon}\mathcal{A}\mathcal{H}_\varepsilon - \frac{\kappa}{\varepsilon}\mathcal{A}^2\mathcal{H}_\varepsilon,$$

and (4.17) follows. Choose now $f = \frac{\kappa}{\varepsilon}\mathcal{H}_\varepsilon$, so that (4.17) reads $E_\varepsilon' + \mathcal{A}f + \mathcal{A}^2 f \le 0$. Integrating we find

$$E_\varepsilon(T) + \int_0^T \mathcal{A}f\,dt + \int_0^T \mathcal{A}^2 f\,dt \le E_\varepsilon(0). \tag{4.19}$$

For the former integral, using (4.2) we have

$$\int_0^T \mathcal{A}f\,dt = \int_0^T f\,dt + \mathcal{A}f(T) - \mathcal{A}f(0).$$

For the latter, iterating twice the same argument gives

$$\int_0^T \mathcal{A}^2 f\,dt = \int_0^T f\,dt + \mathcal{A}^2 f(T) + \mathcal{A}f(T) - \mathcal{A}^2 f(0) - \mathcal{A}f(0),$$

so that (4.19) in particular yields

$$\begin{aligned} E_\varepsilon(T) + 2\int_0^T f(t)\,dt &\le E_\varepsilon(0) + \mathcal{A}^2 f(0) + 2\mathcal{A}f(0) \\ &= K_\varepsilon(0) + \mathcal{A}^2\mathcal{W}_\varepsilon(0) + \mathcal{A}^2 f(0) + 2\mathcal{A}f(0) \le \tfrac{1}{2}\|w_1\|_{L^2}^2 + \mathcal{A}^2 L_\varepsilon(0) + \frac{2\kappa}{\varepsilon}\mathcal{A}H(0). \end{aligned}$$

Therefore, since $4D_\varepsilon(t) \ge 0$, using (4.14) we find that

$$E_\varepsilon(T) + 2\int_0^T f(t)\,dt \le \tfrac{1}{2}\|w_1\|_{L^2}^2 + \mathcal{A}L_\varepsilon(0) - R(u_\varepsilon),$$

and since $\mathcal{A}L_\varepsilon(0) = J_\varepsilon(u_\varepsilon)$, from (3.4) we see that (4.18) follows from (4.9). □

5. Proof of the a priori estimates

In this section we prove part (a) of Theorems 2.4 and 2.5.

As discussed at the beginning of Section 3, the minimizers w_ε of F_ε in (2.1) (subject to (1.2)) are related to the minimizers u_ε of J_ε in (3.1) (subject to (3.3)) by the change of variable (3.2), and in particular the functions w_ε satisfy the boundary conditions

$$w_\varepsilon(0, x) = w_0(x), \qquad w_\varepsilon'(0, x) = w_1(x). \tag{5.1}$$

So the estimates on w_ε will follow from analogous estimates on u_ε by scaling.

Proof of (2.10). Scaling as in (3.2) and using (4.18) and (4.6) yields

$$\frac{1}{2}\int |w_\varepsilon'(t,x)|^2\,dx = K_\varepsilon(t/\varepsilon) \le C,$$

which proves the first estimate in (2.10). The second estimate follows immediately from the first and the boundary condition in (5.1), since $w_0 \in W \hookrightarrow L^2$. □

Proof of (2.15). When $\kappa = 1$, observe that (4.18) gives

$$\int_0^\infty \mathcal{H}_\varepsilon(t)\,dt = \int_0^\infty \mathcal{H}(u_\varepsilon'(t))\,dt \le C\varepsilon, \tag{5.2}$$

and (2.15) follows from (3.2) and scaling, with the use of (2.5). □

Proof of (2.9). Since $L_\varepsilon \ge 0$, from (3.5) we have

$$e^{-2}\int_0^2 \mathcal{W}_\varepsilon(t)\,dt \le \int_0^2 e^{-t}L_\varepsilon(t)\,dt \le J_\varepsilon(u_\varepsilon) \le C. \tag{5.3}$$

In the same spirit, for every $s \ge 0$, we have

$$e^{-2}\int_{s+1}^{s+2} \mathcal{W}_\varepsilon(t)\,dt \le \int_{s+1}^{s+2} (t-s)e^{-(t-s)}\mathcal{W}_\varepsilon(t)\,dt \le \mathcal{A}^2\mathcal{W}_\varepsilon\,(s) \le E_\varepsilon(s) \le C,$$

which combined with (5.3) yields

$$\int_s^{s+1} \mathcal{W}_\varepsilon(t)\,dt \le C \quad \forall s \ge 0. \tag{5.4}$$

Writing $s = \tau/\varepsilon$ and scaling, recalling (3.6) we obtain

$$\int_\tau^{\tau+\varepsilon} \mathcal{W}(w_\varepsilon(z))\,dz \le C\varepsilon \quad \forall \tau \ge 0. \tag{5.5}$$

Now, if $\tau \ge 0$ and $T \ge \varepsilon$ as in (2.9), by covering $[\tau, \tau+T]$ with consecutive intervals of length ε and using (5.5) in each interval, one obtains (2.9). □

In the next lemma we are going to use the inequality

$$\int_t^{t+1} \|\nabla\mathcal{W}(u_\varepsilon(s))\|_{W'}^{1/\theta}\,ds \le C \quad \forall t \ge 0, \tag{5.6}$$

which follows immediately on combining (2.4) and (5.4).

Lemma 5.1 (Euler–Lagrange equation). *Suppose that* $\eta(t,x) = \varphi(t)h(x)$ *with* $\varphi \in C^{1,1}([0,\infty))$, $\varphi(0) = \varphi'(0) = 0$ *and* $h \in W \cap H$. *Then*

$$\int_0^\infty e^{-t}\left(\frac{1}{\varepsilon^2}\langle u_\varepsilon'', \eta''\rangle_{L^2} + \langle \nabla\mathcal{W}(u_\varepsilon(t)), \eta\rangle + \frac{\kappa}{\varepsilon}\langle \nabla\mathcal{H}(u_\varepsilon'(t)), \eta'\rangle\right)dt = 0. \tag{5.7}$$

Moreover, the same conclusion holds if $\eta \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$.

Proof. The Euler–Lagrange equation (5.7) corresponds to the condition $f'(0) = 0$ where $f(\delta) = J_\varepsilon(u_\varepsilon + \delta\eta)$; it is enough to justify differentiation under the integral sign in (3.1) in the term involving $\mathcal{W}$ (the term with $\mathcal{H}$ is quadratic due to (2.5)).

First consider the case where $\eta = \varphi(t)h(x)$, and set $v = u_\varepsilon + \delta\eta$ with, say, $|\delta| \le 1$. As $\varphi \in C^{1,1}$, $\varphi(t)$ grows at most quadratically as $t \to \infty$; applying (2.25) with $a = u_\varepsilon(t)$ and $b = \varphi(t)h$, multiplying by e^{-t} and integrating, one sees that $J_\varepsilon(v)$ is finite (and v satisfies the boundary conditions (3.3)). For a.e. $t > 0$, we have

$$\frac{d}{d\delta}\mathcal{W}(u_\varepsilon(t) + \delta\eta(t))\bigg|_{\delta=0} = \langle \nabla\mathcal{W}(u_\varepsilon(t)), \eta(t)\rangle = \varphi(t)\langle \nabla\mathcal{W}(u_\varepsilon(t)), h\rangle,$$

and this function, multiplied by e^{-t}, is integrable on $\mathbb{R}^+$ due to (5.6). Indeed, one can easily check that differentiation in δ under the integral sign is justified, now using (2.24) with a and b as before.

Now consider a generic test function $\eta \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$. Due to (5.6) and (2.3), the left hand side of (5.7) defines a distribution on $\mathbb{R}^+ \times \mathbb{R}^n$. If $\eta = \varphi(t)h(x)$ with $\varphi \in C_0^\infty(\mathbb{R}^+)$ and $h \in C_0^\infty(\mathbb{R}^n)$, then in particular $\varphi \in C^{1,1}([0,\infty))$ and (5.7) has just been established. The general case then follows from the fact that test functions of the form $\varphi(t)h(x)$ are dense in $C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$ (see [9, Chap. IV, and in particular Thm. III]). □

Proof of (2.11), (2.16). These estimates will follow from the following representation formula (proved below) for u_ε'', valid for a.e. $T > 0$:

$$\frac{1}{\varepsilon^2}\langle u_\varepsilon''(T), h\rangle_{L^2} = -\mathcal{A}^2 f_1(T) - \frac{\kappa}{\varepsilon}\mathcal{A} f_2(T), \qquad h \in \begin{cases} W & \text{if } \kappa = 0, \\ W \cap H & \text{if } \kappa = 1, \end{cases} \tag{5.8}$$

where

$$f_1(t) = \langle \nabla\mathcal{W}(u_\varepsilon(t)), h\rangle, \qquad f_2(t) = \langle \nabla\mathcal{H}(u_\varepsilon'(t)), h\rangle. \tag{5.9}$$

Note that in view of (2.21),

$$|f_1(t)| \le \|h\|_W \|\nabla\mathcal{W}(u_\varepsilon(t))\|_{W'} \le C\|h\|_W(1 + \mathcal{W}_\varepsilon(t)).$$

But since $\mathcal{A}^2 1 = 1$ by (4.3), and $\mathcal{A}^2\mathcal{W}_\varepsilon \le E_\varepsilon \le C$ by (4.5) and (4.18), we have

$$|\mathcal{A}^2 f_1(T)| \le \mathcal{A}^2|f_1|(T) \le C\|h\|_W \qquad \forall T \ge 0. \tag{5.10}$$

Thus, if $\kappa = 0$, (5.8) can be seen (via the second inclusion in (2.3)) as a representation formula for $u_\varepsilon''(T)$ as an element of W', and the last estimate gives

$$\frac{1}{\varepsilon^2}\|u_\varepsilon''(T)\|_{W'} \le C \qquad \text{for a.e. } T \ge 0.$$

Then, scaling according to (3.2), one obtains (2.11).

In addition, if $\kappa = 1$ and $h \in W \cap H$, using (2.26) we have

$$|f_2(t)| \le \|h\|_H \|\nabla\mathcal{H}(u_\varepsilon'(t))\|_{H'} \le C\|h\|_H\sqrt{\mathcal{H}_\varepsilon(t)},$$

and thus, by (5.2), $\|f_2\|_{L^2(\mathbb{R}^+)} \leq C\sqrt{\varepsilon}\|h\|_H$. Therefore, since the operator $\mathcal{A}$ maps $L^2(\mathbb{R}^+)$ continuously into itself, we find that

$$\|\mathcal{A}f_2\|_{L^2(\mathbb{R}^+)} \leq C\sqrt{\varepsilon}\|h\|_H.$$

Then, recalling (2.6), formula (5.8) can be written as $u_\varepsilon''/\varepsilon^2 = \Phi_1 + \Phi_2$, with the bounds $\|\Phi_1\|_{L^\infty(\mathbb{R}^+;W')} \leq C$ by (5.10), and $\|\Phi_2\|_{L^2(\mathbb{R}^+;H')} \leq C/\sqrt{\varepsilon}$ by the previous inequality. On scalling according to (3.2), this means that $w_\varepsilon''(t) = \Phi_1(t/\varepsilon) + \Phi_2(t/\varepsilon)$, and (2.16) follows since $\|\Phi_2(t/\varepsilon)\|_{L^2} = \sqrt{\varepsilon}\|\Phi_2\|_{L^2}$.

It remains to prove (5.8). For $T, \delta > 0$, we take the $C^{1,1}$ function g_δ defined in (4.16). Given h as in (5.8), we set $\eta(t,x) = g_\delta(t)h(x)$ and we apply Lemma 5.1.

As $g_\delta''(t) = \delta^{-1}\chi_{(T,T+\delta)}(t)$, (5.7) multiplied by e^T reads

$$\frac{e^T}{\varepsilon^2\delta}\int_T^{T+\delta} e^{-t}\langle u_\varepsilon''(t), h\rangle_{L^2}\,dt = -\int_T^\infty e^{-(t-T)}\left(g_\delta(t)f_1(t) + \frac{\kappa}{\varepsilon}g_\delta'(t)f_2(t)\right)dt$$

with f_1, f_2 as in (5.9). Since $|g_\delta(t)| \leq (t-T)^+$ and $|g_\delta'(t)| \leq \chi_{(T,\infty)}$, one can dominate the integrands as done above for f_1 and f_2. Finally, letting $\delta \downarrow 0$, since $g_\delta \to (t-T)^+$ and $g_\delta' \to \chi_{(T,\infty)}$, one obtains (5.8) for a.e. T. □

6. Proof of convergence and energy inequality

In this section we first prove parts (b) and (c) of Theorems 2.4 and 2.5. Then, we prove Theorem 2.6.

Below, we deal with a sequence of minimizers w_{ε_i} as in (b) of Theorem 2.4, and we will *tacitly* extract several subsequences. For ease of notation, however, we will denote by w_ε the original sequence, as well as the subsequences we extract.

Proof of part (b): passage to the limit. Regardless of $\kappa \in \{0,1\}$, (2.10) shows that the w_ε are equibounded in $H^1_{\text{loc}}([0,\infty); L^2)$. More precisely, for every $T > 0$ there exists a constant C_T such that

$$\|w_\varepsilon\|^2_{H^1((0,T);L^2)} = \int_0^T (\|w_\varepsilon'(t)\|_{L^2} + \|w_\varepsilon(t)\|_{L^2})\,dt \leq C_T. \tag{6.1}$$

Thus there exists a function $w \in H^1_{\text{loc}}([0,\infty); L^2)$ such that

$$w_\varepsilon \rightharpoonup w \quad \text{in } H^1_{\text{loc}}([0,\infty); L^2) \quad \text{and} \quad w_\varepsilon(t) \rightharpoonup w(t) \quad \text{in } L^2 \ \ \forall t \geq 0 \tag{6.2}$$

as $\varepsilon \to 0$. Clearly, the claims on w', w'' in (2.12) and (2.17) follow from the uniform bounds in (2.10), (2.11) and (2.16). Moreover, when $\kappa = 1$, since H is normed by $\|v\|_H^2 = \|v\|_{L^2}^2 + 2\mathcal{H}(v)$, (2.15) combined with (6.1) provides a uniform bound for w_ε' in $L^2((0,T); H)$ for every $T > 0$, whence (2.18).

To prove that w satisfies conditions (1.2), we recall that they are satisfied, by assumption, by each w_ε; then the first condition for w follows easily from the second part of (6.2), considering $t = 0$.

For the second condition, if $\kappa = 0$ then (2.11) and (2.10) (combined with $L^2 \hookrightarrow W'$, which follows from (2.3)) yield a uniform bound for w'_ε in $W^{1,\infty}(\mathbb{R}^+; W')$, which guarantees the maintenance, in the limit, of $w'_\varepsilon(0) = w_1$ (now viewed as an equality in W'). If $\kappa = 1$ then the argument is similar; since $W \cap H \hookrightarrow L^2$ densely by (2.3) and (2.6), in particular (2.16) yields a uniform bound for w''_ε in $L^2((0, 1); (W \cap H)')$, hence a bound for w'_ε in $H^1((0, 1); (W \cap H)')$, sufficient to guarantee that $w'(0) = w_1$ (now seen as an equality in $(W \cap H)'$). □

Proof of part (c): energy inequality. To obtain (2.14) and (2.19) we need the following lemma, proved in [10] and reformulated here in terms of the operator $\mathcal{A}$.

Lemma 6.1. *Let $l(t)$, $m(t)$ be nonnegative functions in L^1_{loc} such that*

$$(\mathcal{A}^2 l)(t) \le m(t) \qquad \text{for a.e. } t > 0. \tag{6.3}$$

Then, for any $a > 0$ and $\delta \in (0, 1)$,

$$\left(\int_0^{\delta a} s e^{-s}\, ds\right) \int_{T+\delta a}^{T+a} l(t)\, dt \le \int_T^{T+a} m(t)\, dt \qquad \forall T \ge 0.$$

Recalling (4.5) and (4.18), we can apply Lemma 6.1 with $l(t) = \mathcal{W}_\varepsilon(t)$ and

$$m(t) = -K_\varepsilon(t) - \frac{2\kappa}{\varepsilon} \int_0^t \mathcal{H}_\varepsilon(s)\, ds + \tfrac{1}{2}\|w_1\|^2_{L^2} + \mathcal{W}(w_0) + C\sqrt{\varepsilon}$$

(assumption (6.3) corresponds to (4.18) via (4.5)). This gives, for every $T \ge 0$, every $a > 0$ and every $\delta \in (0, 1)$,

$$Y(\delta a) \int_{T+\delta a}^{T+a} \mathcal{W}_\varepsilon(t)\, dt \le -\int_T^{T+a} \left(K_\varepsilon(t) + \frac{2\kappa}{\varepsilon} \int_0^t \mathcal{H}_\varepsilon(s)\, ds\right) dt + a\mathcal{E}(0) + aC\sqrt{\varepsilon},$$

where, for simplicity, $Y(z) = \int_0^z s e^{-s}\, ds$ and $\mathcal{E}(0)$ is defined as in (2.14). Now, recalling (3.2), we want to rewrite this estimate in terms of w_ε instead of u_ε; in view of this, it is convenient to first replace T with T/ε and a with a/ε, and then change variable in the integrals according to (3.2), thus obtaining, after rearranging terms,

$$\begin{aligned} Y(\delta a/\varepsilon) \int_{T+\delta a}^{T+a} \mathcal{W}(w_\varepsilon(t))\, dt + \int_T^{T+a} \left(\tfrac{1}{2}\|w'_\varepsilon(t)\|^2_{L^2} + 2\kappa \int_0^t \mathcal{H}(w'_\varepsilon(s))\, ds\right) dt \\ \le a\mathcal{E}(0) + aC\sqrt{\varepsilon}, \qquad \forall T \ge 0,\ \forall a > 0,\ \forall \delta \in (0, 1). \end{aligned}$$

Now, for fixed T, a, δ, recalling (6.2) we let $\varepsilon \to 0$ in the previous estimate. Since $Y(\delta a/\varepsilon) \to 1$, by semicontinuity we obtain

$$\int_{T+\delta a}^{T+a} \mathcal{W}(w(t))\, dt + \int_T^{T+a} \left(\tfrac{1}{2}\|w'(t)\|^2_{L^2} + 2\kappa \int_0^t \mathcal{H}(w'(s))\, ds\right) dt \le a\mathcal{E}(0)$$

(for the integral involving $\mathcal{W}$ one uses Fatou's Lemma, (6.2) and (2.2), while the double integral with $\mathcal{H}$ is a convex and strongly continuous function of w'_ε in $L^2((0, T+a); H)$, and one may use (2.18)). Now we let $\delta \to 0^+$ (with T and a fixed), then we divide by a and finally we let $a \to 0^+$, to obtain

$$\mathcal{W}(w(T)) + \frac{1}{2}\|w'(T)\|_{L^2}^2 + 2\kappa \int_0^T \mathcal{H}(w'(s))\, ds \le \mathcal{E}(0) \qquad \text{for a.e. } T \ge 0.$$

When $\kappa = 0$ this reduces to (2.14), while when $\kappa = 1$ it reduces to (2.19). □

Lemma 6.2. *For every $\eta \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$,*

$$\int_0^\infty \langle w_\varepsilon'(\tau), \varepsilon^2\eta'''(\tau) + 2\varepsilon\eta''(\tau) + \eta'(\tau)\rangle_{L^2}\, d\tau \\ = \int_0^\infty \big(\langle \nabla\mathcal{W}(w_\varepsilon(\tau)), \eta(\tau)\rangle + \kappa\langle \nabla\mathcal{H}(w_\varepsilon'(\tau)), \eta(\tau) + \varepsilon\eta'(\tau)\rangle\big)\, d\tau. \tag{6.4}$$

Proof. Let $\psi \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$. Choosing $\eta = e^t\psi$ in (5.7) gives

$$\int_0^\infty \frac{1}{\varepsilon^2}\langle u_\varepsilon''(t), \psi''(t) + 2\psi'(t) + \psi(t)\rangle_{L^2}\, dt \\ = -\int_0^\infty \left(\langle \nabla\mathcal{W}(u_\varepsilon(t)), \psi(t)\rangle + \frac{\kappa}{\varepsilon}\langle \nabla\mathcal{H}(u_\varepsilon'(t)), \psi(t) + \psi'(t)\rangle\right) dt.$$

Now we replace u_ε with w_ε using (3.2) and, accordingly, we take ψ of the form $\psi(t,x) = \eta(\varepsilon t, x)$ for an arbitrary test function η. Plugging into the last equation and changing variable $\tau = \varepsilon t$ in each integral, one obtains (6.4) after integrating by parts the first term. □

Proof of Theorem 2.6. The functional $v \mapsto \frac{1}{2}\|v\|_{\dot H^m}^2$ clearly satisfies assumption (H1), on letting $W = H^m$ and $\theta = 1/2$; then the first part of the claim follows on combining Remarks 2.1 and 2.3. Thus, one may apply Theorem 2.4 (or 2.5, if $\kappa = 1$). We wish to pass to the limit in (6.4), in particular in the nonlinear term involving $\nabla\mathcal{W}$, namely we wish to prove that (up to subsequences)

$$\lim_{\varepsilon\to 0}\int_0^\infty \langle \nabla\mathcal{W}(w_\varepsilon(\tau)), \eta(\tau)\rangle\, d\tau = \int_0^\infty \langle \nabla\mathcal{W}(w(\tau)), \eta(\tau)\rangle\, d\tau, \tag{6.5}$$

where w is the limit function obtained by Theorem 2.4 (or 2.5). Due to (2.20) (see also Remark 2.1 and (2.7)), we have

$$\int_0^\infty \langle \nabla\mathcal{W}(w_\varepsilon(\tau)), \eta(\tau)\rangle\, d\tau = \int_0^\infty \langle w_\varepsilon(\tau), \eta(\tau)\rangle_{\dot H^m}\, d\tau \\ + \sum_{0\le k<m} \lambda_k \int_0^\infty \int |\nabla^k w_\varepsilon(\tau)|^{p_k-2}\nabla^k w_\varepsilon(\tau)\cdot\nabla^k\eta(\tau)\, dx\, d\tau.$$

Thus, to prove (6.5), we need *strong* convergence of $|\nabla^k w_\varepsilon|^{p_k-2}\nabla^k w_\varepsilon$ ($k < m$) in $L^1(Q)$, for every cylinder $Q = (0,T) \times B$ (B being a ball in $\mathbb{R}^n$), and *weak* convergence of w_ε in $L^2((0,T); H^m)$.

Now, due to (2.20), the bounds in (2.9) (with $\tau = 0$) take the concrete form

$$\int_0^T \left(\|w_\varepsilon(t)\|^2_{\dot{H}^m} + \sum_{0 \le k < m} \lambda_k \int_{\mathbb{R}^n} |\nabla^k w_\varepsilon(t,x)|^{p_k}\, dx \right) dt \le C_T \qquad (T \ge 1),$$

so that, in view of the second part of (2.10), w_ε is *weakly compact* in $L^2((0,T); H^m)$, while $\nabla^k w_\varepsilon$ is *bounded* in $L^{p_k}(Q)$ (we focus on those k for which $\lambda_k > 0$). Thus, to conclude, it suffices to check the *strong* convergence of $\nabla^k w_\varepsilon$ in $L^2(Q)$ (this condition is even stronger than necessary if $p_k < 3$). Now fix a cylinder $Q = (0,T) \times B$. If $0 \le k < m$ we have $H^m(B) \hookrightarrow H^k(B) \hookrightarrow L^2(B)$, and the first injection is compact; thus, combining the bound for w_ε in $L^2((0,T); H^m)$ with the bound for w'_ε in $L^2((0,T); L^2(B))$ (from 2.10), we obtain the *strong compactness* of w_ε in $L^2((0,T); H^k(B))$ (see e.g. [7, Thm. 5.1]), whence $\nabla^k w_\varepsilon$ converges strongly in $L^2(Q)$.

The terms in (6.4) other than $\nabla\mathcal{W}(w_\varepsilon)$ are *linear* in w_ε, and using Theorem 2.4(b) one can pass to the limit in (6.4) when $\kappa = 0$. Finally, if $\kappa = 1$, recalling (2.5) and (2.26), also the term involving $\nabla\mathcal{H}(w_\varepsilon)$ passes to the limit in (6.4), using (2.18) and (2.26). In either case, taking the limit in (6.4) one obtains

$$\int_0^\infty \langle w'(\tau), \eta'(\tau)\rangle_{L^2}\, d\tau = \int_0^\infty \big(\langle \nabla\mathcal{W}(w(\tau)), \eta(\tau)\rangle + \kappa \langle \nabla\mathcal{H}(w'(\tau)), \eta(\tau)\rangle \big)\, d\tau,$$

that is, w is a weak solution of (1.1) (or (1.6) if $\kappa = 1$). □

7. Examples

In this section we show how several concrete problems fit into the general scheme described above. Let us begin with equations without dissipative terms.

1. Linear equations. These are obtained when $\mathcal{W}$ is a quadratic functional, e.g.

$$\mathcal{W}(v) = \frac{1}{2} \sum_{j \in \mathcal{R}} \int |\partial^j v|^2\, dx,$$

where $\mathcal{R} \subset \mathbb{N}^n$ is a finite set of multi-indices and ∂^j denotes partial differentiation. In this case the natural choice for the domain of $\mathcal{W}$ is $W = \{v \in L^2 \mid \partial^j v \in L^2,\ \forall j \in \mathcal{R}\}$, and assumption (H1) is fulfilled (in particular, (2.4) is satisfied with $\theta = 1/2$).

Reasoning as in Remark 2.1, we see that the hyperbolic equation corresponding to (1.1) is

$$w'' = -\sum_{j \in \mathcal{R}} (-1)^{|j|} \partial^{2j} w.$$

In this case Theorem 2.6 applies ($\kappa = 0$), and Problem 1 can be completely solved. Concrete instances are the linear wave equation $w'' = \Delta w$, the Klein–Gordon equation $w'' = \Delta w - w$, or the biharmonic wave equation $w'' = -\Delta^2 w$.

2. Defocusing NLW. This matches De Giorgi's original conjecture in [2], and has been dealt with in [10]. It corresponds to the choice

$$\mathcal{W}(v) = \int \left(\frac{1}{2} |\nabla v|^2 + \frac{1}{p} |v|^p \right) dx$$

in (1.3), for some $p > 2$. Here, by Remark 2.1, (1.1) takes the concrete form

$$w'' = \Delta w - |w|^{p-2} w.$$

If we let $W = H^1 \cap L^p$, assumption (H1) is satisfied (with $\theta = 1 - 1/p$ in (2.4)), and all the results in [10] are recovered as an application of Theorem 2.6.

3. Sine-Gordon equation. If we let

$$W(v) = \int \big(\tfrac{1}{2} |\nabla v|^2 + 1 - \cos v \big)\, dx$$

with domain $W = H^1$, then (1.1) becomes the sine-Gordon equation

$$w'' = \Delta w - \sin w.$$

Then (H1) is fulfilled with $\theta = 1/2$, and (a slight variant of) Theorem 2.6 applies. Note that the functional $\mathcal{W}$ associated with this problem is not convex.

4. Quasilinear wave equations. Powers other than 2 on the gradient term in $\mathcal{W}$ give rise to quasilinear wave equations. For example

$$\mathcal{W}(v) = \frac{1}{p} \int |\nabla v|^p \, dx \quad \text{or} \quad \mathcal{W}(v) = \int \left(\frac{1}{p} |\nabla v|^p + \frac{1}{q} |v|^q \right) dx \qquad (p, q > 1)$$

correspond, respectively, to the quasilinear wave equations

$$w'' = \Delta_p w \quad \text{and} \quad w'' = \Delta_p w - |w|^{q-2} w,$$

where Δ_p is the p-laplacian. Assumption (H1) is satisfied, for the former equation, if we let $W = \{v \in L^2 \mid \nabla v \in L^p\}$ and $\theta = 1 - 1/p$, while for the latter one may set $W = \{v \in L^2 \mid \nabla v \in L^p,\ v \in L^q\}$ and $\theta = 1 - 1/\max\{p, q\}$. In both cases Theorem 2.4 applies, while Theorem 2.6 *cannot* be applied (unless $p = 2$). It is an *open problem*, however, to establish if the last claim of Theorem 2.6 (passage to the limit in the equation) still applies when $p \neq 2$. A positive answer would settle the long-standing open question of *global existence* of weak solutions for this kind of equations (see [1]).

5. Higher order nonlinear equations. Just to give an example (see for instance [8]), consider

$$\mathcal{W}(v) = \int \left(\frac{1}{2} |\Delta v|^2 + \frac{1}{p} |\nabla v|^p + \frac{1}{q} |v|^q \right) dx \qquad (p, q > 1).$$

Then (1.1) becomes the nonlinear vibrating-beam equation

$$w'' = -\Delta^2 w + \Delta_p w - |w|^{q-2} w,$$

where Δ^2 is the biharmonic operator. Here $W = \{v \in H^2 \mid \nabla v \in L^p,\ v \in L^q\}$, while $\theta = 1 - 1/\max\{2, p, q\}$. Here Theorem 2.6 applies, and provides global existence.

6. Kirchhoff equations. The general scheme presented in this paper allows one to treat also *nonlocal* problems. A typical example is the Kirchhoff equation

$$w'' = \left(\int |\nabla w|^2\,dx\right)\Delta w.$$

Here one chooses

$$\mathcal{W}(v) = \frac{1}{4}\left(\int |\nabla v|^2\,dx\right)^2, \qquad W = H^1$$

(note that (2.4) holds with $\theta = 3/4$), and Theorem 2.4 applies (while it is an *open problem* whether the last claim of Theorem 2.6 is true in this case). More generally, if $\mathcal{W}(v) = \frac{1}{2}\Phi(\int |\nabla v|^2\,dx)$ for some appropriate function Φ, one formally obtains the equation

$$w'' = \Phi'\left(\int |\nabla w|^2\,dx\right)\Delta w$$

(the appropriate constant θ in (2.4) will depend on Φ).

7. Wave equations with the fractional Laplacian. Given $s \in (0, 1)$, we may consider the nonlocal energy

$$\mathcal{W}(v) = c\iint \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}}\,dx dy + \frac{\lambda}{p}\int |v|^p\,dx \qquad (c > 0,\ \lambda \geq 0,\ p > 1),$$

with domain $W = H^s \cap L^p$ (or simply H^s if $\lambda = 0$). It is well known (see e.g. [4]) that the first integral is the natural energy associated with the *fractional Laplacian* $(-\Delta)^s$, so that (for a proper choice of c, see [4]) (1.1) becomes

$$w'' = (-\Delta)^s w - \lambda|w|^{p-2}w,$$

a (nonlinear if $\lambda > 0$) wave equation with the fractional Laplacian. One may check that assumption (H1) is satisfied with $\theta = 1/2$ (or $\theta = 1 - 1/\max\{2, p\}$ if $\lambda > 0$) in (2.4). Here one may apply Theorem 2.6, thus obtaining global existence.

The next examples concern *dissipative* equations with a structure as in (1.6); these are related to the functional in (1.7), as stated in Problem 2. We will mainly focus on the choice of the functional $\mathcal{H}$, thus obtaining dissipative variants of the previous examples.

8. Telegraph type equations. These are obtained by letting

$$\mathcal{H}(v) = \frac{1}{2}\int |v|^2\,dx \qquad \text{(with domain } H = L^2\text{)},$$

thus fulfilling assumption (H2). Since $\langle \nabla\mathcal{H}(v), \cdot\rangle = \langle v, \cdot\rangle_{L^2}$, by Remark 2.2 this choice of $\mathcal{H}$ generates the term $-w'$ on the right hand side of (1.6).

If, for instance, $\mathcal{W}$ is as in Example 2, then we can obtain the nonlinear telegraph equation

$$w'' = \Delta w - |w|^{p-2}w - w'.$$

In this case one can solve Problem 2 completely, since Theorem 2.6 can now be applied with $\kappa = 1$. See also [6] for related results on bounded domains.

If, on the other hand, $\mathcal{W}$ is as in Example 4, then one obtains

$$w'' = \Delta_p w - |w|^{q-2} w - w'$$

and so on. In fact, in practice, the term $-w'$ can be inserted in any of the above examples (Theorem 2.5 can always be applied, while Theorem 2.6 should now be applied with $\kappa = 1$, when possible).

9. Strongly damped wave equations. The term "strongly damped" usually denotes the presence of $\Delta w'$ in the equation (see e.g. [5]). We can treat this case by letting

$$\mathcal{H}(v) = \frac{1}{2} \int |\nabla v|^2 \, dx \qquad \text{(with domain } H = H^1\text{)},$$

so that $\nabla \mathcal{H}(v)$ corresponds to $-\Delta v$ by Remark 2.2. Then, building on Example 2, we may consider

$$w'' = \Delta w - |w|^{p-2} w + \Delta w'$$

(for which Theorem 2.6 applies with $\kappa = 1$), or quasilinear versions such as

$$w'' = \Delta_p w - |w|^{q-2} w + \Delta w'.$$

The last equation does *not* satisfy the assumptions of Theorem 2.6 (unless $p = 2$). In a forthcoming paper, however, we will show that the *claim* of Theorem 2.6 is *in fact true* for every $p, q > 1$.

10. Other damped equations. In each of Examples 1–3 one can add several dissipative terms. For example, by Remark 2.2, the choice

$$\mathcal{H}(v) = \frac{1}{2} \int (|\Delta v|^2 + |\nabla v|^2 + |v|^2) \, dx$$

would introduce, in any given equation, the term $-\Delta^2 w' + \Delta w' - w'$.

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