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Overconvergent subanalytic subsets in the framework of Berkovich spaces

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Abstract. We study the class of overconvergent subanalytic subsets of a k-affinoid space X when k is a non-archimedean field. These are the images along the projection $X \times \mathbb{B}^n \to X$ of subsets defined by inequalities between functions on $X \times \mathbb{B}^n$ which are overconvergent in the variables of $\mathbb{B}^n$. In particular, we study the local nature, with respect to X, of overconvergent subanalytic subsets. We show that they behave well with respect to the Berkovich topology, but not the G-topology. This gives counterexamples to previous results on the subject, and a way to correct them. Moreover, we study the case $\dim(X) = 2$, for which a simpler characterisation of overconvergent subanalytic subsets is proven.

Keywords. Berkovich spaces, semianalytic sets, subanalytic sets, overconvergent

Contents

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Introduction

Motivations

Let us consider a complete non-trivially valued non-archimedean field k (assumed to be algebraically closed in this introduction for simplicity). Since non-archimedean fields are totally disconnected, one cannot define the notion of analytic spaces over k as easily as in the case of $\mathbb{R}$ or $\mathbb{C}$. Tate [Tat71] developed such a theory, and called his spaces rigid spaces, whose building blocks are affinoid spaces. However, these spaces are not endowed with a classical topology, but with a Grothendieck topology (the G-topology). Afterwards, V. Berkovich developed another viewpoint for k-analytic geometry [Ber90, Ber93]. His spaces, called k-analytic spaces, or Berkovich spaces, have more points than the corresponding rigid spaces and are equipped with a topology which is locally arcwise connected. Moreover, in this theory, affinoid spaces are compact. R. Huber also developed another viewpoint, in the setting of adic spaces [Hub96], and there also exists an approach, initiated by M. Raynaud, using formal geometry (see [BL93] for instance).

If X, Y are k-analytic spaces and $\varphi : Y \to X$ is an analytic map, it is natural to wonder what is the shape of $\varphi(Y)$. By analogy with Chevalley's theorem and the Tarski–Seidenberg theorem, one would like to be able to describe such images $\varphi(Y)$ using only functions on X.

Without assumptions on φ, this is impossible: one needs some kind of *compactness* at some point. One reasonable restriction is to consider analytic maps $\varphi : Y \to X$ where X and Y are affinoid spaces.

In this context the first natural approach is to define a *semianalytic subset* of a k-affinoid space as a finite boolean combination of sets defined by inequalities $|f| \leq |g|$ between analytic functions. But the class of semianalytic sets is not big enough: there exist morphisms $\varphi : Y \to X$ of affinoid spaces such that $\varphi(Y)$ is not semianalytic.

To overcome this problem, one has to consider more functions on an affinoid space X than the analytic ones. In the framework of $\mathbb{Z}_p^n$, Jan Denef and Lou van den Dries [DvdD88] have given a good description of images of analytic maps $\varphi : \mathbb{Z}_p^m \to \mathbb{Z}_p^n$. Their main idea is to allow division of functions. In the framework of rigid geometry, where $\mathbb{Q}_p$ has to be replaced by some non-archimedean algebraically closed field k, this idea of allowing divisions has been developed in two ways.

The first one is due to Leonard Lipshitz [Lip93, LR00b, Lip88, LR96] and rests on the introduction of an algebra $S_{m,n}$ of restricted analytic functions on products of closed and open balls. This allowed L. Lipshitz to define for each affinoid space X the class of subanalytic subsets on $X(k)$ (in terms of analytic functions on X, division and composition with $S_{m,n}$), and to prove that subanalytic sets are stable under analytic maps between affinoid spaces.

A second approach has been developed by Hans Schoutens [Sch94a]. This leads to the definition of overconvergent subanalytic sets of $X(k)$. Namely, the overconvergent subanalytic subsets of $X(k)$ form a subclass of the subanalytic sets as defined by L. Lipshitz. Overconvergent subanalytic sets are only stable under overconvergent analytic maps between affinoid spaces. For instance, if $\varphi : \mathbb{B}^n \to X$ is an analytic map which can be analytically extended to a polydisc of radius $r > 1$, then $\varphi(\mathbb{B}^n)$ is an overconvergent subanalytic set of X.

Overconvergent subanalytic sets

Hans Schoutens used the language of rigid geometry. We now summarize his results. First, let $D : k^2 \to k$ be defined by

$$D(x, y) = \begin{cases} x/y & \text{if } |x| \leq |y| \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{A}$ be an affinoid algebra and X its affinoid space. The algebra $\mathcal{A}\langle\langle D\rangle\rangle$ is defined as the smallest k-algebra of functions $f : X(k) \to k$ such that

- $\mathcal{A}\langle\langle D\rangle\rangle$ contains all functions induced by $\mathcal{A}$.
- If $f, g \in \mathcal{A}\langle\langle D\rangle\rangle$, then $D(f, g) \in \mathcal{A}\langle\langle D\rangle\rangle$.
- If $f \in \mathcal{A}\langle Y_1, \dots, Y_n\rangle$ is overconvergent in the variables Y_i, and $g_1, \dots, g_n \in \mathcal{A}\langle\langle D\rangle\rangle$ satisfy $|g_i|_{\sup} \leq 1$, then $f(g_1, \dots, g_n) \in \mathcal{A}\langle\langle D\rangle\rangle$.

Stability under overconvergent maps is contained in the following result (we denote by $\mathbb{B}$ the closed unit disc).

Theorem ([Sch94a]). *For a subset $S \subset X(k)$ the following are equivalent:*

- *There exists $n \in \mathbb{N}$ and a semianalytic subset T of $X \times \mathbb{B}^n(k)$ defined by inequalities $|f| \leq |g|$ where f and g are overconvergent with respect to the variables of $\mathbb{B}^n$ such that $S = \pi(T)$ where $\pi : X \times \mathbb{B}^n(k) \to X(k)$ is the first projection. We call such sets overconvergent subanalytic sets.*
- *S is defined by a boolean combination of inequalities $|f| \leq |g|$ where $f, g \in \mathcal{A}\langle\langle D\rangle\rangle$.*

For instance, if $\varphi : \mathbb{B}^n \to X$ is an overconvergent map (in the sense that it can be extended to a polydisc of radius greater than 1), then $\varphi(\mathbb{B}^n)$ is overconvergent subanalytic (take for T the graph of φ).

Results of this article

In this article we explain how Berkovich spaces are well suited to study overconvergent subanalytic sets. Indeed, the definitions that we have given above (semianalytic, overconvergent subanalytic) can be given in the framework of Berkovich spaces. For instance if we consider $X = \mathbb{B}^2$ with coordinate functions T_1, T_2, the inequality $|T_1| \leq |T_2|$ naturally defines two sets

$$S_{\text{rig}} = \{(t_1, t_2) \in (k^\circ)^2 \mid |t_1| \leq |t_2|\},$$
$$S_{\text{Berko}} = \{x \in \mathcal{M}(k\{T_1, T_2\}) \mid |T_1(x)| \leq |T_2(x)|\}.$$

Of course $S_{\text{rig}} \subset S_{\text{Berko}}$. More precisely, S_{rig} is the set of rigid points of S_{Berko}.

This new approach with Berkovich spaces allows us to simplify the proof of the theorem of [Sch94a] mentioned above. Part 2 of [Sch94a], *A combinatorial lemma*, is replaced by a simple compactness argument in Berkovich spaces.

If X is an affinoid space, recall that *affinoid domains* in X are some subsets $S \subset X$ satisfying some universal property with respect to morphisms $f : Y \to X$ of affinoid spaces such that $\mathrm{im}(f) \subset S$. See [BGR84, 7.2.2.2] or [Ber90, 2.2.1] for a precise definition. Weierstrass (resp. rational) domains are examples of affinoid domains which are defined by inequalities of the form $|f| \leq 1$ (resp. $|f| \leq |g|$) where f and g are analytic functions on X. Then we consider the local behaviour of overconvergent subanalytic sets. If X is an affinoid space there are two ways to consider local behaviour on X.

1. The G-topology, where a covering of X is a finite covering $\{X_i\}$ by affinoid domains.
2. The Berkovich topology [Ber90, Ber93] on X seen as a Berkovich space, which is a real topology.

If S is an overconvergent subanalytic set of X and U an affinoid domain in X, it is easy to see that $S \cap U$ is an overconvergent subanalytic set of U. It is then natural to wonder if overconvergent subanalytic sets fit well with one of these topologies. We give the following answers.

Proposition (see Proposition 2.4). *There exists an affinoid space X, a subset $S \subset X$, and a finite covering $\{X_i\}$ of X by affinoid domains such that for all i, $S \cap X_i$ is overconvergent subanalytic in X_i, but S is not overconvergent subanalytic in X.*

In other words, being overconvergent subanalytic is not local with respect to the G-topology. This contradicts some results of [Sch94a], for instance [Sch94a, QE theorem p. 270, Proposition 4.2, Theorem 5.2].

We prove however that the Berkovich topology corrects this. If X is an affinoid space seen as a Berkovich space, and $x \in X$, we say that V is an *affinoid neighbourhood* of x if V is an affinoid domain in X and in addition V is a neighbourhood of X with respect to the Berkovich topology.[1]

Theorem (see Theorem 1.42). *A subset $S \subset X$ is overconvergent subanalytic if and only if for every $x \in X$ (viewed as a Berkovich space), there exists an affinoid neighbourhood V of x such that $S \cap V$ is overconvergent subanalytic in V.*

In other words, being overconvergent subanalytic is a local property, but with respect to the Berkovich topology.

The mistake in [Sch94a] which we point out in Proposition 2.4 led to other mistakes in further work of H. Schoutens [Sch94c, Sch94b]. In particular [Sch94b], which relies on the false results of [Sch94a], claims that if k is algebraically closed of characteristic 0,

[1] When x is a rigid point, any affinoid domain containing x is an affinoid neighbourhood. But this is not true in general. For instance, in the unit disc with coordinate T, the rational domain defined by $|T| = 1$ is an affinoid domain which contains the Gauss point, but it is not a neighbourhood of the Gauss point.

then a subset of the unit bidisc is overconvergent subanalytic if and only if it is rigid-semianalytic (i.e. semianalytic locally for the G-topology). But the counterexample we give in Proposition 2.4 proves that this equivalence does not hold. Anyway, the proofs of [Sch94b] rely on some false equivalences of [Sch94a].

We show that the Berkovich topology allows one to correct the results of [Sch94b]. A k-analytic space is said to be *good* [Ber93, 1.2.16] if any point has an affinoid neighbourhood. For instance affinoid spaces are good k-analytic spaces. A k-analytic space X is said to be *quasi-smooth*[2] if X is geometrically regular [Duc11, Section 5]. When k is algebraically closed, this is equivalent to saying that for all $x \in X$, the local ring $\mathcal{O}_{X,x}$ is regular. When k is algebraically closed and X is a strictly k-analytic space, this is even equivalent to saying that for all rigid points $x \in X$, the local ring $\mathcal{O}_{X,x}$ is regular (this follows for instance from [Ber90, 2.3.4]).

Theorem (see Theorem 3.12). *Assume that k is algebraically closed. Let X be a good quasi-smooth strictly k-analytic space of dimension* 2*. Then a subset S of X is overconvergent subanalytic if and only if it is locally semianalytic.*

Here, we say that S is *locally semianalytic* if for every $x \in X$, there is an affinoid neighbourhood V of x such that $S \cap V$ is semianalytic in V.

Ideas behind the proofs

We want to point out that the two equivalent characterizations of overconvergent subanalytic sets which were given in [Sch94a] and which we have recalled on page 2407 are not very manageable. In particular, it is hard to prove that some set is not overconvergent subanalytic using these characterizations, whereas we have much more tools to decide whether a subset is semianalytic or not. In order to overcome this difficulty, we have introduced a third characterization of overconvergent subanalytic sets, which is more geometric. We remark that the quotient of two analytic functions f and g is not analytic any more, but becomes analytic if one blows up (f, g). With this in mind, in order to describe a subset of X defined by inequalities $|f| \leq |g|$ with $f, g \in \mathcal{A}\langle\langle D\rangle\rangle$ we can consider some finite sequences of blow-ups $\tilde{X} \to X$ and project some semianalytic sets of $\tilde{X}$ outside the exceptional locus (with some extra condition for the overconvergence condition). We call such subsets overconvergent constructible (see 1.8 for a precise definition). The idea of looking at analytic functions above some blow-up of X had already appeared in [LR00a, 2.3(iv)].

With this in mind we would like to restate the results of this paper more precisely.

First, we prove Theorem 1.35 which asserts that if X is an affinoid space, then $S \subset X$ is overconvergent subanalytic if and only if it is overconvergent constructible, using at some point the compactness of the Berkovich space X.

Then, according to the definition of an overconvergent constructible set, it is easy to prove that overconvergent subanalytic sets are local for the Berkovich topology (Proposition 1.42).

[2] The termin "rig-smooth" is also used by some other authors.

To justify our counterexample in Proposition 2.4, we use the more geometric approach of overconvergent constructible sets which allows one to use results on semianalytic sets. Ultimately, our argument relies on the study of some Gauss point in an embedded curve in the polydisc, which strengthens our feeling that Berkovich spaces are well suited to study overconvergent subanalytic sets.

Finally, we want to mention one more benefit of overconvergent constructible sets. In the author's thesis it is proved [Mar13, Proposition 2.4.1] that if k is algebraically closed, S a locally closed overconvergent subanalytic set of the compact k-analytic space X, and if we consider a prime number $\ell \neq \mathrm{char}(\tilde{k})$, then the étale cohomology groups with compact support of the germ (S, X) (see [Ber93, 3.4,5.1]),

$$H_c^i((S, X), \mathbb{Q}_\ell),$$

are finite-dimensional $\mathbb{Q}_\ell$-vector spaces. Here again the idea is that (thanks to the presentation of S as an overconvergent constructible set) we can reduce to the case where S is semianalytic, and in that case, the finiteness result is proved in [Mar13, Proposition 2.2.3] (which ultimately relies on a finiteness result for affinoid spaces proved by V. Berkovich).

Organisation of the paper

In Section 1, we define *constructible data* of X, in order to define overconvergent constructible subsets. Note that unlike [Sch94a] we do not assume that k is algebraically closed. In Section 1.2 we introduce overconvergent subanalytic subsets. In Section 1.3 we carefully treat Weierstrass division, trying to be as general as possible (namely our results hold for an arbitrary ultrametric Banach algebra, and an arbitrary radius of convergence). In Section 1.4 we prove that overconvergent constructible and overconvergent subanalytic subsets are the same. The proof of this result which appears in [Sch94a] is here simplified by the use of Berkovich spaces; in particular, the quite technical Section 2 of [Sch94a] is replaced by a simple compactness argument (see Theorem 1.35). In 1.5 we try to handle the following problem: how to pass from a definition that works only for k-affinoid spaces to a more local definition, with the hope that in the affinoid case the local and the global definitions would coincide. As we said earlier, trying to do this with the G-topology will not work. If however we do this with the Berkovich topology, the definitions will be compatible. In Section 1.6, we explain how these results can be extended to k-affinoid spaces (as opposed to strictly k-affinoid spaces). In addition, in that case, we can allow the field k to be trivially valued.

In Section 2, we give some counterexamples to erroneous statements of [Sch94a]. Precisely, in [Sch94a] five classes of subsets were defined: globally strongly subanalytic, globally strongly **D**-semianalytic, strongly subanalytic, locally strongly subanalytic and strongly **D**-semianalytic subsets. The last three classes were defined from the first two ones by adding "G-local" at some point. In [Sch94a] it was claimed that these five classes coincide. We explain that this is not the case: namely, of these five classes, the first two indeed coincide, but not the last three, which are larger (see Figure 1, p. 2438). The main idea is that if one replaces "G-locally" by "locally for the Berkovich topology", the results of [Sch94a], for instance [Sch94a, Quantifier Elimination Theorem p. 270], become true. Let us give one of the counterexamples that we study:

Example 0.1. Let $X = \mathbb{B}^2$ be the closed bidisc, $0 < r < 1$ with $r \in \sqrt{|k^\times|}$, and $f \in k\{r^{-1}x\}$ an analytic function whose radius of convergence is exactly r and such that $\|f\| < 1$. We define

$$S = \{(x, y) \in \mathbb{B}^2 \mid |x| < r \text{ and } y = f(x)\}.$$

Then (see Proposition 2.4) S is rigid-semianalytic, but not overconvergent subanalytic. The Berkovich approach is here helpful since to prove this, we use a point η of the Berkovich bidisc which is not a rigid point, and some properties of its local ring $\mathcal{O}_{X,\eta}$.

Finally, in Section 3 we correct the proof of [Sch94b] (which rested on the erroneous results of [Sch94a], and [Sch94c]) and restrict the hypothesis of it. Namely, we prove that when k is algebraically closed, and X is a good quasi-smooth strictly k-analytic space of dimension 2, then overconvergent subanalytic subsets are in fact locally semianalytic. Not only do we give a correct proof of this theorem, but moreover this result is more general than the result of [Sch94b], where X was the bidisc and where it was assumed that the characteristic of k was 0.

Contribution of this article

We want to stress the fact that Section 1 is highly inspired by the work of H. Schoutens. In particular, the definition we give of a constructible datum, and the resulting definition of an overconvergent constructible subset, is a *geometric* formulation of what is done in [Sch94a] concerning **D**-strongly semianalytic subsets. In particular, the proof of Theorem 1.35 is very close to that of [Sch94a, Th. 5.2]. We have however decided to include a proof of Theorem 1.35 for three reasons. First, the compactness argument that we use in Theorem 1.35 seems to us enlightening, and a way to see that Berkovich spaces are relevant in this context.[3] Secondly, we have the impression that replacing the strongly **D**-semianalytic subsets of [Sch94a] by our overconvergent constructible subsets is more geometric and gives a better understanding of the situation. Finally, the mistakes in [Sch94a], which we explain in Section 2, result in some invalid statements. For instance, [Sch94a, Theorem 5.2] is false, as we prove in Section 2. In this context it seemed to us relevant to write Section 1.

The same remarks hold for Section 3. A statement analogous to Theorem 3.12 was claimed in [Sch94b]. However, in that article it was assumed, and used in the proofs, that the five classes of subsets introduced in [Sch94a] were the same; since we prove that this is not the case, the proofs of [Sch94b] are erroneous.

Finally, let us mention that another proof of Theorem 1.35 has also been given in [CL11, 4.4.10].

[3] However, it has to be noted that we could have written this proof in the context of adic spaces, and used a similar argument of quasi-compactness.

1. Overconvergent constructible subsets

With a few exceptions that will be specified, k will be a non-trivially valued non-archimedean field, $\mathcal{A}$ will be a strictly k-affinoid algebra, and X the strictly k-affinoid space $\mathcal{M}(\mathcal{A})$.

1.1. Constructible data

Definition 1.1. Let X be a k-affinoid space whose k-affinoid algebra is $\mathcal{A}$. A subset S of X is called *semianalytic* if it is a finite boolean combination of sets of the form $\{x \in X \mid |f(x)| \leq |g(x)|\}$ where $f, g \in \mathcal{A}$ (by finite boolean combination, we mean finitely many uses of the set-theoretical operators $\cap$, $\cup$ and ^{c}). A subset of the form $\{x \in X \mid |f_i(x)| \diamondsuit_i |g_i(x)|,\ i = 1, \dots, n\}$ with $f_i, g_i \in \mathcal{A}$, and $\diamondsuit_i \in \{\leq, <\}$ will be called *basic semianalytic*.

Remark 1.2. With a repeated use of the rule $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ one can show that any semianalytic subset of X is a finite union of basic semianalytic subsets.

Definition 1.3. Let (X, S) be a *k-germ* in the sense of [Ber93, 3.4]; this just means that S is a subset of X. Let $f, g \in \mathcal{A}$ and let $r, s \in \mathbb{R}$ be such that $r > s > 0$ and $s \in \sqrt{|k^\times|}$. Let

$$Y = \mathcal{M}(\mathcal{A}\{r^{-1}t\}/(f - tg)) \xrightarrow{\varphi} \mathcal{M}(\mathcal{A}) = X$$

and let $R \subseteq Y$ be a semianalytic subset of Y. Set

$$T := \varphi^{-1}(S) \cap \{y \in R \mid g(y) \neq 0 \text{ and } |f(y)| \leq s|g(y)|\}.$$

Then $(Y, T) \xrightarrow{\varphi} (X, S)$ is an *elementary constructible datum* of (X, S). If $\psi : (Y', T') \simeq (Y, T)$ is an isomorphism of k-germs and $(Y, T) \xrightarrow{\varphi} (X, S)$ is an elementary constructible datum, and if we set $\varphi' = \varphi \circ \psi$, then we will also say that $(Y', T') \xrightarrow{\varphi'} (X, S)$ is an elementary constructible datum.

Remark 1.4. If $(Y, T) \xrightarrow{\varphi} (X, S)$ is an elementary constructible datum, then $\varphi(T) \subset S$, and φ realizes a homeomorphism between T and its image $\varphi(T)$. Moreover

$$\{y \in Y \mid |f(y)| \leq s|g(y)| \neq 0\}$$

is an analytic domain in Y, and can be identified through φ with the analytic domain in X,

$$\{x \in X \mid |f(x)| \leq s|g(x)| \neq 0\}.$$

Definition 1.5. Let (X, S) be a k-germ. A *constructible datum* is a sequence

$$(Y, T) = (X_n, S_n) \xrightarrow{\varphi_n} (X_{n-1}, S_{n-1}) \to \cdots \to (X_1, S_1) \xrightarrow{\varphi_1} (X_0, S_0) = (X, S)$$

where for $i = 1, \dots, n$, $(X_i, S_i) \xrightarrow{\varphi_i} (X_{i-1}, S_{i-1})$ is an elementary constructible datum. Let $\varphi = \varphi_1 \circ \cdots \circ \varphi_n$. Then we will denote this constructible datum by

$$(Y, T) \overset{\varphi}{\dashrightarrow} (X, S).$$

We will say that the *complexity* of φ is n.

In the particular case $S = X$, i.e. $(X, S) = (X, X)$, we will denote the constructible datum by

$$(Y, T) \overset{\varphi}{\dashrightarrow} X,$$

and we will call it a *constructible datum of* X. This is actually the case that will mainly interest us, but for technical reasons we have chosen to use k-germs.

Remark 1.6. If $(Y, T) \overset{\varphi}{\dashrightarrow} X$ is a constructible datum, it follows easily from the above definitions that T is a semianalytic subset of Y.

Remark 1.4 implies that if $(Y, T) \overset{\varphi}{\dashrightarrow} (X, S)$ is a constructible datum, then $\varphi_{|T} : T \to S$ induces a homeomorphism between T and $\varphi(T)$. It is also clear that if $(Z, U) \overset{\psi}{\dashrightarrow} (Y, T)$ is a constructible datum and $(Y, T) \overset{\varphi}{\dashrightarrow} (X, S)$ is another one, then $(Z, U) \overset{\varphi\circ\psi}{\dashrightarrow} (X, S)$ is also a constructible datum.

We want to point out that in the definition of a constructible datum, n cannot be recovered from φ alone.

Definition 1.7. Let $(X_i, S_i) \overset{\varphi_i}{\dashrightarrow} (X, S)$, $i = 1, \dots, m$, be m constructible data of the k-germ (X, S). They form a *constructible covering* of (X, S) if $\bigcup_{i=1}^{m} \varphi_i(S_i) = S$.

Definition 1.8. Let X be a k-affinoid space. A subset C of X is said to be an *overconvergent constructible subset* of X if there exist m constructible data $(X_i, S_i) \overset{\varphi_i}{\dashrightarrow} X$ for $i = 1, \dots, m$ such that $\bigcup_{i=1}^{m} \varphi_i(S_i) = C$.

Remark 1.9. Using the notation of Definition 1.3, when $(Y, T) \overset{\varphi}{\to} (X, S)$ is an elementary constructible datum with $Y = \mathcal{M}(\mathcal{A}\{r^{-1}t\}/(f - tg))$, then T (and hence $\varphi(T)$) are defined by the function t which mimics the function f/g when it makes sense, and its norm is $\leq s$. In addition the condition $r > s$ is here to make sure that the new functions of $\mathcal{B}$ are overconvergent in $t = f/g$, which we see as a function on the analytic domain $\{x \in X \mid |f(x)| \leq s|g(x)| \neq 0\}$.

The following three results are formal consequences of the previous definitions.

Lemma 1.10. *If $(Y, T) \overset{\varphi}{\to} (X, S)$ is an elementary constructible datum and $(Z, U) \overset{\psi}{\to} (X, S)$ is a morphism of k-germs, consider the cartesian product of k-germs*

$$\begin{array}{ccc} (Y, T) & \xrightarrow{\quad\varphi\quad} & (X, S) \\ {\scriptstyle \psi'}\uparrow & & \uparrow{\scriptstyle \psi} \\ (Y, T) \times_{(X,S)} (Z, U) & \xrightarrow{\varphi'} & (Z, U) \end{array}$$

Then $(Y, T) \times_{(X,S)} (Z, U) \overset{\varphi'}{\to} (Z, U)$ is an elementary constructible datum. Moreover if

$$(Y, T) \times_{(X,S)} (Z, U) =: (Y', T')$$

then $(\varphi \circ \psi')(T') = \varphi(T) \cap \psi(U)$.

Corollary 1.11. *Let* $(Y,T) \overset{\varphi}{\dashrightarrow} (X,S)$ *be a constructible datum*

$$(Y,T) = (X_n, S_n) \xrightarrow{\varphi_n} \cdots \xrightarrow{\varphi_1} (X_0, S_0) = (X,S)$$

and let $(X',S') \xrightarrow{\psi} (X,S)$ *be a morphism of* k*-germs. Consider the cartesian product*

$$\begin{array}{ccc} (Y,T) & \overset{\varphi}{\dashrightarrow} & (X,S) \\ {\scriptstyle\psi'}\uparrow & & \uparrow{\scriptstyle\psi} \\ (Y',T') & \overset{\varphi'}{\dashrightarrow} & (X',S') \end{array}$$

Then $(Y',T') \overset{\varphi'}{\dashrightarrow} (X',S')$ *is a constructible datum and* $(\psi \circ \varphi')(T') = \varphi(T) \cap \psi(S')$.

Corollary 1.12. *Let* $(X_1,T_1) \overset{\varphi}{\dashrightarrow} (X,S)$ *and* $(X_2,T_2) \overset{\psi}{\dashrightarrow} (X,S)$ *be two constructible data (with the same target). Consider the fibred product*

$$\begin{array}{ccc} (X_1,T_1) & \overset{\varphi}{\dashrightarrow} & (X,S) \\ {\scriptstyle\psi'}\uparrow & & \uparrow{\scriptstyle\psi} \\ (Z,U) & \overset{\varphi'}{\dashrightarrow} & (X_2,T_2) \end{array}$$

Then $(Z,U) \overset{\psi'}{\dashrightarrow} (X_1,T_1)$ *and* $(Z,U) \overset{\varphi'}{\dashrightarrow} (Y_2,T_2)$ *are constructible data. Moreover* $(\varphi \circ \psi')(U) = (\psi \circ \varphi')(U) = \varphi(T_1) \cap \psi(T_2)$.

Proof. Lemma 1.10 is a direct consequence of Definition 1.3. Corollary 1.11 is then proved by induction on the complexity of φ using Lemma 1.10. Similarly, Corollary 1.12 is proved by induction on the complexity of ψ using Corollary 1.11. □

Proposition 1.13. (1) *If* T *is a semianalytic subset of* X *then* T *is an overconvergent constructible subset of* X.

(2) *Let* $C \subseteq T$ *be an overconvergent constructible subset of* Y *and let* $(Y,T) \overset{\varphi}{\dashrightarrow} X$ *be a constructible datum. Then* $\varphi(C)$ *is an overconvergent constructible subset of* X.

(3) *The class of overconvergent constructible subsets of* X *is stable under finite boolean combinations.*

Proof. (1) Consider the elementary constructible datum $(X,T) \xrightarrow{\text{id}} X$.

(2) By definition, there exist constructible data $(Y_i,T_i) \overset{\varphi_i}{\dashrightarrow} Y$, for $i = 1, \dots, m$, such that $C = \bigcup_{i=1}^m \varphi_i(T_i)$. Now if we define $\psi_i := \varphi \circ \varphi_i$, then $(Y_i,T_i) \overset{\psi_i}{\dashrightarrow} X$ are m constructible data, and $\varphi(C) = \varphi(\bigcup_{i=1}^m \varphi_i(Y_i)) = \bigcup_{i=1}^m \psi_i(T_i)$, so it is an overconvergent constructible subset of (X,S).

(3) Stability under finite unions is a direct consequence of Definition 1.8, while the same for intersections is a consequence of Corollary 1.12. Let us show that if $C \subseteq X$ is an overconvergent constructible subset of X, then $X \setminus C$ is also overconvergent constructible.

By definition, $C = \bigcup_{i=1}^{m} \varphi_i(S_i)$ where $(X_i, S_i) \overset{\varphi_i}{\dashrightarrow} X$ are constructible data. We use induction on c, the maximum of the complexities of the φ_i's.

If $c = 0$, then C is a semianalytic subset of X, so $X \setminus C$ is semianalytic, hence overconvergent constructible.

If $c > 0$ and we assume the result holds for $c' < c$, then

$$X \setminus C = X \setminus \bigcup_{i=1}^{m} \varphi_i(S_i) = \bigcap_{i=1}^{m} (X \setminus \varphi_i(S_i)),$$

so we can assume that $m = 1$, that is, that $C = \varphi(T)$ where $(Y, T) \overset{\varphi}{\dashrightarrow} X$ is a constructible datum of complexity c. Then

$$\varphi = \psi \circ \varphi' : (Y, T) \overset{\varphi'}{\dashrightarrow} (Y', T') \overset{\psi}{\to} X$$

where the complexity of φ' is $c - 1$ and ψ is an elementary constructible datum. Now

$$X \setminus \varphi(T) = \psi(T' \setminus \varphi'(T)) \cup (X \setminus \psi(T'))$$

because $\varphi'_{|T}$ and $\psi_{|T'}$ are injective maps. By induction hypothesis,

$$T' \setminus \varphi'(T) = T' \cap (Y' \setminus \varphi'(T))$$

is an overconvergent constructible subset of Y', thus according to (1), so is $\psi(T' \setminus \varphi'(T))$.

Finally, if the elementary constructible datum ψ is associated with f, g, r and s, then by definition,

$$T' = \{y \in R \mid |f(y)| \leq s|g(y)| \neq 0\}$$

for some semianalytic subset R of Y'. And if we define

$$\tilde{T} = \{y \in Y' \setminus R \mid |f(y)| \leq s|g(y)| \neq 0\},$$

then

$$X \setminus \psi(T') = \psi(\tilde{T}) \cup \{y \in X \mid |f(y)| > s|g(y)|\} \cup \{y \in X \mid g(y) = 0\}.$$

Thus, it is also overconvergent constructible in X. □

Let $x \in X$, and let U be an affinoid neighbourhood of x. Shrinking U if necessary, we can assume [Ber90, 2.5.15] that U is a rational domain of the form $X\left(\underline{r}^{-1}\frac{f}{g}\right) = \{p \in X \mid |f_i(x)| \leq r_i|g(x)|\}$ such that $X\left((\underline{r}/2)^{-1}\frac{f}{g}\right)$ still contains x. For each i, we pick a real number s_i such that $r_i/2 < s_i < r_i$ and $s_i \in \sqrt{|k^\times|}$. For each i, we consider the elementary constructible datum $(X_i, S_i) \overset{\varphi_i}{\to} X$ defined by $X_i = \mathcal{A}\{r_i^{-1}t_i\}/(f_i - t_i g)$, and $S_i = \{p \in X_i \mid |f_i(p)| \leq s_i|g(p)| \text{ and } g(p) \neq 0\}$. One checks that $\varphi_i(S_i)$ is a neighbourhood of x. Now if we take the fibred product of all these elementary constructible data, we obtain (using Corollary 1.12) the following constructible datum:

$$\left(X\left(\underline{r}^{-1}\frac{f}{g}\right), X\left(\underline{s}^{-1}\frac{f}{g}\right)\right) \overset{\varphi}{\dashrightarrow} X.$$

Here φ just corresponds to the embedding of the affinoid domain $X\big(\underline{r}^{-1}\frac{f}{g}\big)$. Moreover $\varphi\big(X\big(\underline{s}^{-1}\frac{f}{g}\big)\big)$, which we might identify with $X\big(\underline{s}^{-1}\frac{f}{g}\big)$, is a neighbourhood of x. We can sum this up in the following lemma:

Lemma 1.14. *Let X be a strictly k-affinoid space. Let $x \in X$ and let U be an affinoid neighbourhood of x. Then there exists a constructible datum $(Y, T) \overset{\varphi}{\dashrightarrow} X$ such that T is an affinoid domain in Y, φ is the embedding of an affinoid domain $Y \to X$ such that Y is in fact an affinoid subdomain of U, and $\varphi(T)$ is an affinoid neighbourhood of x.*

Corollary 1.15. *Let X be a strictly k-affinoid space. Being overconvergent constructible in X is a local property.*

Proof. First, if $S \subset X$ is overconvergent constructible, and U is an affinoid domain of X, then $S \cap U$ is overconvergent constructible.

On the other hand, assume that locally for the Berkovich topology, S is overconvergent constructible, that is, for all $x \in X$ there exists an affinoid neighbourhood U of x such that $S \cap U$ is overconvergent constructible. Then according to Lemma 1.14, there exists a constructible datum $(Y, T) \overset{\varphi}{\dashrightarrow} X$ such that $Y \overset{\varphi}{\to} X$ is the embedding of an affinoid domain, $Y \subset U$, and T is an affinoid neighbourhood of x. Since $T \subset U$, $\varphi^{-1}(S) \cap T$ is overconvergent constructible in T, and so $\varphi(T) \cap S$ is overconvergent constructible in X (see Proposition 1.13(2)). But since $\varphi(T)$ is an affinoid neighbourhood of x, by compactness of X we conclude that S is overconvergent constructible. □

1.2. Overconvergent subanalytic subsets

We will denote by $\mathbb{B}$ (resp. $\mathbb{B}_r$ for $r > 0$) the closed disc of radius 1 (resp. r), and if n is an integer, $\mathbb{B}^n$ and $\mathbb{B}^n_r$ will denote the corresponding closed polydiscs.

More generally, if $\underline{r} = (r_1, \dots, r_n) \in (\mathbb{R}^*_+)^n$ is a polyradius, we will denote by

$$\mathbb{B}_{\underline{r}} = \mathcal{M}(k\{\underline{r}^{-1}T\}) = \mathcal{M}(k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\})$$

the polydisc of radius $\underline{r}$, and $\mathring{\mathbb{B}}(\underline{r})$ the corresponding open polydisc. When the number n is clear from the context, we will write $\underline{1}$ for $(1, \dots, 1) \in \mathbb{R}^n$, and $\underline{0}$ or 0 for $(0, \dots, 0) \in \mathbb{R}^n$. Finally, $\underline{\rho} > \underline{r}$ will mean that $\rho_i > r_i$ for $i = 1, \dots, n$.

Definition 1.16. Let X be a strictly k-affinoid space. A subset $S \subset X$ is said to be an *overconvergent subanalytic subset* of X if there exist $n \in \mathbb{N}$, $r > 1$, and a semianalytic subset $T \subseteq X \times \mathbb{B}^n_r$ such that $S = \pi(T \cap (X \times \mathbb{B}^n))$ where $\pi : X \times \mathbb{B}^n_r \to X$ is the natural projection.

Lemma 1.17. *Let $f : Y \to X$ be a morphism of strictly k-affinoid spaces and S an overconvergent subanalytic subset of X. Then $f^{-1}(S)$ is an overconvergent subanalytic subset of Y. In particular, if V is a strictly affinoid domain in X, and S an overconvergent subanalytic subset of X, then $S \cap V$ is an overconvergent subanalytic subset of V.*

Proof. Let $r > 1$ and let $T \subseteq X \times \mathbb{B}^n_r$ be a semianalytic subset such that $S = \pi(T \cap (X \times \mathbb{B}^n))$. Consider the cartesian diagram

$$\begin{array}{ccc} Y \times \mathbb{B}^n_r & \xrightarrow{f'} & X \times \mathbb{B}^n_r \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{f} & X \end{array} \tag{1.1}$$

Then $f^{-1}(S) = f^{-1}(\pi(T \cap (X \times \mathbb{B}^n))) = \pi'(f'^{-1}(T \cap (X \times \mathbb{B}^n)))$. The last equality holds because (1.1) is a cartesian diagram. Now $\pi'(f'^{-1}(T \cap (X \times \mathbb{B}^n))) = \pi'(f'^{-1}(T) \cap (Y \times \mathbb{B}^n)) = \pi'^{-1}(T' \cap (Y \times \mathbb{B}^n))$ where $T' = f'^{-1}(T)$ is a semianalytic subset of $Y \times \mathbb{B}^n_r$. Hence $f^{-1}(S) = \pi'(T' \cap (Y \times \mathbb{B}^n))$ is an overconvergent subanalytic subset of Y. □

Lemma 1.18. *Let X and Y be strictly k-affinoid spaces, and let $\varphi : X \to Y$ be a closed immersion.*

(1) *If S is a semianalytic subset of X, then $\varphi(S)$ is a semianalytic subset of Y.*
(2) *Let S be an overconvergent subanalytic subset of X. Then $\varphi(S)$ is an overconvergent subanalytic subset of Y.*

Proof. (1) Write $Y = \mathcal{M}(\mathcal{A})$ and $X = \mathcal{M}(\mathcal{A}/\mathcal{I})$ where $\mathcal{I} = (a_1, \dots, a_m)$ is an ideal of $\mathcal{A}$. Then, if $S = \{x \in X \mid |f_i(x)| \diamondsuit_i |g_i(x)|,\ i = 1, \dots, n\}$ with $f_i, g_i \in \mathcal{A}/\mathcal{I}$, we can find functions $F_i, G_i \in \mathcal{A}$ such that $\overline{F_i} = f_i$ and $\overline{G_i} = g_i$. In that case one checks that

$$\varphi(S) = \{y \in Y \mid |F_i(y)| \diamondsuit_i |G_i(y)|,\ i = 1, \dots, n\} \cap \{y \in Y \mid a_j(y) = 0,\ j = 1, \dots, m\},$$

which is indeed semianalytic.

(2) By definition there exists a semianalytic subset $T \subseteq X \times \mathbb{B}^n_r$ for some $r > 1$ such that $S = \pi(T \cap (X \times \mathbb{B}^n))$. We then consider the cartesian diagram

$$\begin{array}{ccc} X \times \mathbb{B}^n_r & \xrightarrow{\varphi'} & Y \times \mathbb{B}^n_r \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\varphi} & Y \end{array}$$

But φ' is also a closed immersion, so according to (1), $T' = \varphi'(T)$ is a semianalytic subset of $Y \times \mathbb{B}^n_r$. Then one checks that

$$\begin{aligned} \pi(T' \cap (Y \times \mathbb{B}^n)) &= \pi(\varphi'(T) \cap (Y \times \mathbb{B}^n)) = \pi(\varphi'(T \cap (X \times \mathbb{B}^n))) \\ &= \varphi(\pi'(T \cap (X \times \mathbb{B}^n))) = \varphi(S). \end{aligned}$$ □

Lemma 1.19. *Assume that $\underline{s} \in \sqrt{|k^\times|}^n$. Then $k\{\underline{s}^{-1}T\}$ is a strictly k-affinoid algebra (see* [Ber90, 2.1.1] *and* [BGR84, 6.1.5.4]*). For the same reasons, if $\underline{r} > \underline{s}$, and $S \subseteq X \times \mathbb{B}_{\underline{r}}$ is a semianalytic subset, then $\pi(S \cap (X \times \mathbb{B}_{\underline{s}}))$ is an overconvergent subanalytic subset of X.*

Proof. Let $\underline{s} \in \sqrt{|k^\times|}^n$ and $\underline{r} \in \mathbb{R}^n$ with $\underline{s} < \underline{r}$, and let $S \subseteq X \times \mathbb{B}_{\underline{r}}$ be a semianalytic subset of $X \times \mathbb{B}_{\underline{r}}$. Let us show that $\pi(S \cap (X \times \mathbb{B}_{\underline{s}})$ is overconvergent subanalytic in the sense of Definition 1.16. To avoid complications, we assume that $n = 1$ (but the proof is similar for an arbitrary n). So let $s \in \sqrt{|k^\times|}$ and $r > s$. Up to multiplication by some $\mu \in k^\times$ small enough, we can assume that $s \leq 1$. Since $s \in \sqrt{|k^\times|}$, there exist $\lambda \in k^\times$ and $m \in \mathbb{N}$ such that $s^m = |\lambda|$. Then in

$$\mathbb{B}_{(r,(r/s)^m)} = \mathcal{M}\big(k\{r^{-1}y, ((r/s)^m)^{-1}t\}\big)$$

consider the Zariski closed subset defined by $y^m = \lambda t$, i.e. $V(y^m - \lambda t)$. Then the map

$$\mathbb{B}_r \to \mathbb{B}_{(r,(r/s)^m)}, \qquad x \mapsto (x, x^m/\lambda),$$

identifies $\mathbb{B}_r$ with the Zariski closed subset $V(y^m - \lambda t)$, and moreover, since $s \leq 1$,

$$\mathbb{B}_s \to \mathbb{B}^2, \qquad x \mapsto (x, x^m/\lambda),$$

identifies $\mathbb{B}_s$ with the Zariski closed subset $V(y^m - \lambda t)$ of $\mathbb{B}^2$. Taking the fibre product with X we then obtain

$$\begin{array}{ccccc}
X \times \mathbb{B}_r & \xrightarrow{\simeq} & V(y^m - \lambda t) & \xhookrightarrow{\alpha} & X \times \mathbb{B}_{(r,(\frac{r}{s})^m)} \\
\uparrow & & \uparrow & & \uparrow \\
X \times \mathbb{B}_s & \xrightarrow{\simeq} & V(y^m - \lambda t) & \xhookrightarrow{\beta} & X \times \mathbb{B}^2 \\
 & \searrow^{\pi} & & \swarrow_{\pi} & \\
 & & X & &
\end{array}$$

Hence if $S \subseteq X \times \mathbb{B}_r$ is semianalytic, then $S' := \alpha(S)$ is also semianalytic in $X \times \mathbb{B}_{(r,(r/s)^m)}$ and $\alpha(S) \cap (X \times \mathbb{B}^2) = \beta(S \cap (X \times \mathbb{B}_s))$. So $\pi(S \cap (X \times \mathbb{B}_s)) = \pi(S' \cap (X \times \mathbb{B}^2)$ is overconvergent subanalytic in the sense of Definition 1.16. □

1.3. Weierstrass preparation

In this section, A will be an ultrametric complete normed ring, i.e. it satisfies the inequality $\|ab\| \leq \|a\|\,\|b\|$ and $\|a + b\| \leq \max(\|a\|, \|b\|)$ [BGR84, 1.2.1.1].

If $r > 0$, on $A\{r^{-1}T\}$ we will consider the following norm: if $g = \sum_{n \in \mathbb{N}} a_n T^n \in A\{r^{-1}X\}$ then $\|g\| = \max_{n \geq 0} \|a_n\| r^n$.

If $m \in \mathbb{N}$, we will denote by $A_m[T]$ the subset of $A[T]$ made of the polynomials of degree less than or equal to m.

Definition 1.20. An element $u \in A$ is a *multiplicative unit* if u is invertible and for all $a \in A$, $\|ua\| = \|u\|\,\|a\|$.

Note that if u and v are multiplicative units, so is uv.

Lemma 1.21. *An element $u \in A$ is a multiplicative unit if and only if $u \in A^*$ and $\|u^{-1}\| = \|u\|^{-1}$.*

Proof. If u is a multiplicative unit, then $1 = \|uu^{-1}\| = \|u\|\|u^{-1}\|$, so $\|u^{-1}\| = \|u\|^{-1}$.

Conversely, assume that u is invertible and $\|u^{-1}\| = \|u\|^{-1}$. Let $a \in A$. Then

$$\|a\| = \|u^{-1}(ua)\| \leq \|u^{-1}\| \, \|ua\| = \|u\|^{-1}\|ua\|.$$

So $\|ua\| \geq \|u\| \, \|a\|$. Since the reverse inequality always holds, we conclude that $\|ua\| = \|u\|\|a\|$. □

Remark 1.22. As a consequence, if $u \in A$ and $\|u\| < 1$, then $1 + u$ is a multiplicative unit because

$$\|1 + u\| = 1 = \left\| \sum_{n \geq 0} (-u)^n \right\| = \|(1 + u)^{-1}\|$$

Also note that if u is a multiplicative unit, then $|u(x)| = \|u\|$ for all $x \in \mathcal{M}(A)$. Indeed, the definition of $\mathcal{M}(\mathcal{A})$ implies that

$$|u(x)| \leq \|u\|, \tag{1.2}$$

hence $1 = |u(x)| \, |u^{-1}(x)| \leq \|u\| \, \|u^{-1}\| = 1$. So the inequality (1.2) could not be strict, thus $|u(x)| = \|u\|$.

Remark 1.23. If $\varphi : A \to B$ is a contractive morphism of normed rings (i.e. $\|\varphi(a)\| \leq \|a\|$ for all a in A), then φ sends multiplicative units to multiplicative units. Indeed,

$$1 = \|\varphi(u)\varphi(u)^{-1}\| \leq \|\varphi(u)\| \, \|\varphi(u^{-1})\| \leq \|u\| \, \|u^{-1}\| = 1,$$

so these are equalities and $\varphi(u)$ is a multiplicative unit because $\|\varphi(u)\| = \|u\|$, and $\|\varphi(u)^{-1}\| = \|u^{-1}\| = \|u\|^{-1} = \|\varphi(u)\|^{-1}$.

This remark will apply in the following context: $\mathcal{A}$ is a strictly k-affinoid algebra and we look at a morphism $\varphi : \mathcal{A} \to \mathcal{B} = \mathcal{A}\{r^{-1}T\}/I$ with I any ideal, and $\mathcal{B}$ is equipped with the quotient norm inherited from $\mathcal{A}\{r^{-1}T\}$. In this situation, φ is contractive. This is the case when φ is the morphism of a constructible datum $(Y, S) \overset{\varphi}{\dashrightarrow} X$.

Note that if φ is not contractive, multiplicative units are not necessarily preserved. For instance, consider $\mathcal{A} = k\{t\}$ and $\mathcal{B} = k\{2^{-1}x, y\}/(y - x^2)$ that we equip with the residue norm. These k-affinoid algebras are isomorphic through $\varphi : t \mapsto x$, and if we choose $\pi \in k$ such that $1/2 < |\pi| < 1$, then $u := 1 + \pi t$ is a multiplicative unit of $\mathcal{A}$, but $\varphi(u)$ is not.

Note however that if the field k is stable (for instance in our situation, where k is a non-archimedean complete field, k is stable if $\mathrm{char}(\tilde{k}) = 0$, or if it is algebraically closed, or a discrete valuation field [BGR84, 3.6.2]), then for a suitable choice of norm, any morphism of reduced affinoid algebras is contractive. Indeed, if k is stable, and $\mathcal{A}$ is a reduced affinoid algebra, then it is a distinguished affinoid algebra [BGR84, 6.4.3], i.e. the supremum seminorm is a residue norm on $\mathcal{A}$. If $\mathcal{B}$ is reduced, then for the same reason, the supremum seminorm is an admissible norm on it. So if we equip $\mathcal{A}$ and $\mathcal{B}$ with the supremum norm, then any morphism $\varphi : \mathcal{A} \to \mathcal{B}$ of affinoid algebras is contractive.

Definition 1.24. Let $r > 0$ be a real number and $s \in \mathbb{N}$. An element $g = \sum_{n\geq 0} g_n T^n$ of $A\{r^{-1}T\}$ is called *T-distinguished* of order s if g_s is a multiplicative unit and $\|g_s\| r^s = \|g\|$ $\|g_n\| r^n < \|g_s\| r^s$ for all $n > s$. Note that in that case, g is necessarily a non-zero element since $g_s \neq 0$.

Remark 1.25. We can extend the previous remark by saying that if $\varphi : A \to B$ is a contractive morphism and $g = \sum_{n\in\mathbb{N}} g_n T^n \in A\{r^{-1}T\}$ is T-distinguished of order s, then $\varphi(g) = \sum_{n\in\mathbb{N}} \varphi(g_n) T^n \in B\{r^{-1}T\}$ and it is a T-distinguished element of $B\{r^{-1}T\}$ of order s. This applies in particular when φ is the morphism of a constructible datum $(Y, S) \overset{\varphi}{\dashrightarrow} X$.

Lemma 1.26. *Let $g = \sum_{m\in\mathbb{N}} g_m T^m \in A\{r^{-1}T\}$ be T-distinguished of order s.*

(1) $\|gq\| = \|g\| \|q\|$ *for all* $q = \sum_{k\in\mathbb{N}} q_k T^k \in A\{r^{-1}T\}$.
(2) *Set* $gq = \sum_{l\in\mathbb{N}} c_l T^l$, *and assume that* $q \neq 0$. *Denote by k_0 the greatest rank such that* $\|q_{k_0}\| r^{k_0} = \|q\|$. *Then* $\|gq\| = \|c_{s+k_0}\| r^{s+k_0}$ *and* $\|c_{s+k_0}\| = \|g_s\| \|q_{k_0}\|$.

Proof. First, without any hypothesis,

$$\|gq\| \leq \|g\| \|q\|. \tag{1.3}$$

Conversely, by definition,

$$c_{s+k_0} = \sum_{m+k=s+k_0} g_m q_k. \tag{1.4}$$

So let m and k be two integers such that $m + k = s + k_0$.

If $k > k_0$, then $\|q_k\| r^k < \|q_{k_0}\| r^{k_0}$ by definition of k_0. So, since g_s is a multiplicative unit,

$$\|g_m q_k\| r^{s+k_0} = \|g_m q_k\| r^{m+k} \leq \|g_m\| r^m \|q_k\| r^k < \|g_s\| r^s \|q_{k_0}\| r^{k_0} = \|g_s q_{k_0}\| r^{s+k_0}.$$

Thus,

$$\|g_m q_k\| < \|g_s q_{k_0}\|. \tag{1.5}$$

If $k < k_0$, then $m > s$, and since $\|g_m\| r^m < \|g_s\| r^s$ (because g is T-distinguished of order s), the same reasoning yields

$$\|g_m q_k\| < \|g_s q_{k_0}\|. \tag{1.6}$$

Thus, (1.4)–(1.6) and the ultrametric inequality imply that $\|c_{s+k_0}\| = \|g_s q_{k_0}\|$. And since g_s is a multiplicative unit, $\|g_s q_{k_0}\| = \|g_s\| \|q_{k_0}\|$.

Finally, $\|gq\| \geq \|g_s\| r^s \|q_{k_0}\| r^{k_0} = \|q\| \|g\|$, which with (1.3) ends the proof. □

Proposition 1.27 (Weierstrass division). *Let $g \in A\{r^{-1}T\}$ be T-distinguished of order s. If $f = \sum_{n\in\mathbb{N}} f_n T^n \in A\{r^{-1}T\}$, then there exists a unique couple $(q, R) \in A\{r^{-1}T\} \times A_{s-1}[T]$ such that*

$$f = gq + R. \tag{1.7}$$

Moreover

$$\|f\| = \max(\|g\| \|q\|, \|R\|). \tag{1.8}$$

Proof. First, let us show that if a couple (q, R) satisfies (1.7), then it must satisfy (1.8). By the ultrametric inequality, $\|f\| \leq \max(\|g\|\,\|q\|, \|R\|)$. For the reverse inequality, we distinguish two cases.

If $\|gq\| \neq \|R\|$, then $\|f\| = \max(\|gq\|, \|R\|) = \max(\|g\|\,\|q\|, \|R\|)$ according to Lemma 1.26.

Otherwise $\|gq\| = \|g\|\,\|q\| = \|R\|$, and we again use Lemma 1.26 and its notation (so $gq = \sum_{l\in\mathbb{N}} c_l T^l$). We get $\|gq\| = \|c_{s+k_0}\|r^{s+k_0}$. Since R is a polynomial of degree d with $d < s$, and since $f = gq + R$ and $d < s + k_0$, the coefficient f_{s+k_0} of f is c_{s+k_0}, hence $\|f\| \geq \|c_{s+k_0}\|r^{s+k_0} = \|g\|\,\|q\|$.

This finally proves that $\|f\| = \max(\|g\|\,\|q\|, \|R\|)$.

From this we conclude that the couple (q, R) is unique because if $f = gq' + R'$ is another decomposition, we have $0 = g(q - q') + (R - R')$ and since $\|g\| \neq 0$, $\|q - q'\| = \|R - R'\| = 0$, i.e. $R = R'$ and $q = q'$.

Let us now show the existence of such a decomposition. Set

$$g' := \sum_{m=0}^{s} g_m T^m.$$

In particular, $\|g\| = \|g'\|$ because g is T-distinguished of degree s. Set

$$\kappa := \frac{\max_{m>s} \|g_m\|r^m}{\|g_s\|r^s} = \frac{\max_{m>s} \|g_m\|r^m}{\|g\|}.$$

Since g is T-distinguished of order s, we have $\kappa < 1$. Actually, if $\kappa = 0$ (which would mean that $g = g'$), replace κ by $1/2$. In any case $\|g - g'\| \leq \kappa\|g\|$ and $\kappa \in]0, 1[$.

Next, let $N \in \mathbb{N}$ and set

$$f' := \sum_{k=0}^{N} f_k T^k.$$

Assume that N is so large that $\|f - f'\| \leq \kappa\|f\|$. In particular, $\|f'\| = \|f\|$.

By definition and hypothesis, $g' \in A[T]$ is of degree s and possesses an invertible dominant coefficient, which is g_s. Hence in $A[T]$, one can carry out euclidean division by g' [Lan02, 4.1.1], which gives $f' = g'q + R$ with $R \in A_{s-1}[T]$ and $q \in A[T]$. We can then apply the norm equality (1.8) that we have shown in the first part of the proof (because g' is also T-distinguished of order s): $\|f'\| = \max(\|g'\|\,\|q\|, \|R\|)$. In particular $\|q\| \leq \|f'\|/\|g'\| = \|f\|/\|g\|$ so that

$$\|g\|\,\|q\| \leq \|f\|.$$

Moreover $\|R\| \leq \|f'\| = \|f\|$. Thus

$$f = f' + (f - f') = g'q + R + (f - f') = gq + R + (f - f') + (g' - g)q.$$

By definition of g' and of κ, $\|g' - g\| \leq \kappa\|g\|$, so

$$\|(g' - g)q\| \leq \|g\|\,\|q\|\kappa \leq \kappa\|f\|. \tag{1.9}$$

In addition, by hypothesis,

$$\|f - f'\| \leq \kappa \|f\|. \tag{1.10}$$

Hence if we set

$$h := f - f' + (g' - g)q = f - (gq + R),$$

then according to (1.9) and (1.10), we obtain $\|h\| \leq \kappa \|f\|$.

To sum up, we have found some $\kappa \in]0, 1[$ such that

$$\forall f \in A\{r^{-1}T\},\ \exists q' \in A\{r^{-1}T\},\ \exists R' \in A_{s-1}[T],\ \|f - (gq' + R')\| \leq \kappa \|f\|. \tag{1.11}$$

This allows us to define by induction two Cauchy sequences $(q^i) \in A\{r^{-1}T\}$ and $(R^i) \in A_{s-1}[T]$ such that $\|f - (gq^i + R^i)\| \leq \kappa^i \|f\|$ in the following way.

We start with $(q^0, R^0) = (0, 0)$.

To perform the induction step, let $i > 0$ and assume that (q^i, R^i) is defined. We set $h^i := f - (gq^i + R^i)$, which by induction hypothesis fulfils $\|h^i\| \leq \kappa^i \|f\|$. According to (1.11), we can define $q' \in A\{r^{-1}T\}$ and $R' \in A_{s-1}[T]$ such that $h^i = gq' + R' + h'$ with $\|q'\| \leq \|h^i\|/\|g\| \leq \kappa^i \|f\|/\|g\|$, and $\|R'\| \leq \|h^i\| \leq \kappa^i \|f\|$ and $\|h'\| \leq \kappa \|h^i\| \leq \kappa^{i+1} \|f\|$. Then we set $q^{i+1} := q^i + q'$ and $R^{i+1} := R^i + R'$.

Then $\|f - (gq^{i+1} + R_{i+1})\| = \|h^i - (gq + R)\| = \|h'\| \leq \kappa^{i+1} \|f\|$. By construction $\|q^{i+1} - q^i\| = \|q'\| \leq \kappa^i \|f\|/\|g\|$ and $\|R^{i+1} - R^i\| = \|R'\| \leq \kappa^i \|f\|$, so these sequences are Cauchy sequences. This ends our induction.

Now, by completeness of $A\{r^{-1}T\}$ and $A_{s-1}[T]$ the sequences (q^i) and (R^i) have limits, which we denote by $q \in A\{r^{-1}T\}$ and $R \in A_{s-1}[T]$, which satisfy $f = gq + R$ as desired. □

Corollary 1.28 (Weierstrass preparation). *Let $g \in A\{r^{-1}T\}$ be a T-distinguished element of order s. There exists a unique couple $(w, e) \in A_s[T] \times A\{r^{-1}T\}$ such that w is a monic polynomial of degree s, e is a multiplicative unit of $A\{r^{-1}T\}$, and $g = ew$.*

Proof. Using Weierstrass division, we can write $T^s = gq + R$ with $\|T^s\| = \max(\|g\|\ \|q\|, \|R\|)$ and $R \in A[T]_{s-1}$. Set

$$w := T^s - R = gq.$$

So $w \in A_s[T]$ is a monic polynomial. Since g is T-distinguished of order s, according to Lemma 1.26, and if we denote by k_0 the greatest index such that $\|q_{k_0}\| r^{k_0} = \|q\|$ and $w = \sum_{l=0}^{s} w_l T^l$, we obtain

$$\|w\| = \|gq\| = \|(gq)_{s+k_0}\| r^{s+k_0} = \|w_{s+k_0}\| r^{s+k_0}.$$

But since $w \in A_s[T]$, necessarily $s + k_0 = s$ and $k_0 = 0$. Hence, by definition of k_0, we have $\|q_0\| > \|q_k\| r^k$ for all $k > 0$.

The coefficient of degree s in gq being 1 (because $gq = T^s - R$), we have

$$1 = g_0 q_s + g_1 q_{s-1} + \cdots + g_s q_0,$$

and since $k_0 = 0$, and g is T-distinguished of order s, we obtain, with the same reasoning used in the proof of Lemma 1.26, $\|g_s q_0\| > \|g_{s-i} q_i\|$ for $i = 1, \ldots, s$. So $\|g_s q_0\| =$

$\|1\| = 1$, and $g_s q_0 = 1 - (g_{s-1}q_1 + \cdots + g_0 q_s)$, with $\|g_s q_1 + \cdots + g_0 q_s\| < 1$. Thus, $g_s q_0$ is a multiplicative unit. Moreover, since g_s is also a multiplicative unit, so is q_0, and $\|q_0\| = \|g_s\|^{-1}$. Hence

$$q = q_0\left(1 + \frac{q_1}{q_0}T + \cdots + \frac{q_k}{q_0}T^k + \cdots\right), \tag{1.12}$$

and since $k_0 = 0$ (so $\|q_i\|r^i < \|q_0\|$ for $i > 0$) and q_0 is a multiplicative unit, we find that $\|q_i/q_0\|r^i < 1$ for all $i > 0$. Hence

$$1 + \frac{q_1}{q_0}T + \cdots + \frac{q_k}{q_0}T^k + \cdots$$

is a multiplicative unit of $A\{r^{-1}T\}$, and according to (1.12), q is also a multiplicative unit. So $g = q^{-1}(T^s - R)$, with q^{-1} a multiplicative unit and $T^s - R$ a monic polynomial of degree s. So if we set $e := q^{-1}$ and $w = T^s - R$, we have the expected result: $g = ew$.

As for the uniqueness of this decomposition, if $g = ew$, e and w being as in the statement of the corollary, then $w = T^s + R$ with $R \in A_{s-1}[T]$, and $T^s = w - R = e^{-1}g + (-R)$, which is the Weierstrass division of T^s by g. Hence e and R are unique, and w too because $w = T^s + R$. □

Assume that $\mathcal{A}$ is a k-affinoid algebra, let $(r_1, \ldots, r_n)$ be a polyradius, and set $A := \mathcal{A}\{r_1^{-1}T_1, \ldots, r_{n-1}^{-1}T_{n-1}\}$. Then with $r = r_n$, we have $\mathcal{A}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} = A\{r^{-1}T\}$, and we can introduce the notion of a T-distinguished element. We apply Weierstrass theory to them, which corresponds to the classical one, especially if $\mathcal{A} = k$, where we find the classical Tate algebra $k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$.

Now we state a result that we will need in the next section.

Lemma 1.29. *Let $\varepsilon > 0$ be given and $\underline{r} > 0$ be a polyradius. Assume that A is Noetherian, and consider*

$$f = \sum_{\nu \in \mathbb{N}^n} f_\nu T^\nu \in A\{\underline{r}^{-1}T\}.$$

Then there exists a finite subset $J \subseteq \mathbb{N}^n$, and for all $\nu \in J$, a series $\phi_\nu \in A\{\underline{r}^{-1}T\}$ satisfying $\|\phi_\nu\| < \varepsilon$, such that

$$f = \sum_{\nu \in J} f_\nu (T^\nu + \phi_\nu)$$

and no terms T^μ with $\mu \in J$ appear in the ϕ_ν's. Moreover, if we fix some $\mu \in \mathbb{N}^n$, we can assume that $\mu \in J$.

Proof. Let $\mathcal{I}$ be the ideal generated by the family $\{f_\nu\}_{\nu\in\mathbb{N}^n}$. Since A is Noetherian, there exists a finite subset J of $\mathbb{N}^n$ such that $\mathcal{I} = A.(f_\nu)_{\nu\in J}$. So for all $\mu \in \mathbb{N}^n \setminus J$ one can find a decomposition $f_\mu = \sum_{\nu\in J} f_\nu a_\mu^\nu$ with $a_\mu^\nu \in A$. In fact, using [BGR84, 3.7.3], we can even assume[4] that there exists a real constant $C > 0$ such that

$$\forall \mu \in \mathbb{N}^n,\ \forall \nu \in J, \qquad \|a_\mu^\nu\| \le C\|f_\mu\|. \tag{1.13}$$

[4] Indeed, consider $\psi : A^J \to \mathcal{I}$, $(a_\nu)_{\nu\in J} \mapsto \sum_{\nu\in J} a_\nu f_\nu$. According to [BGR84, 3.7.3.1], $\mathcal{I}$ is a complete normed A-module, and ψ is a continuous map of normed A-modules. Hence there exists a constant C such that $\|\psi(x)\| \le C\|x\|$ for all $x \in A^J$.

Then we define, for $\nu \in J$,

$$\phi_\nu = \sum_{\mu \in \mathbb{N}^n \setminus J} a_\mu^\nu T^\mu.$$

Since $\|a_\mu^\nu\| \leq C\|f_\mu\|$, we have $\phi_\nu \in \mathcal{A}\{\underline{r}^{-1}T\}$. Hence, in $A\{\underline{r}^{-1}T\}$, the following equality is satisfied:

$$f = \sum_{\nu \in J} f_\nu\Big(T^\nu + \sum_{\mu \mathbb{N}^n \setminus J} a_\mu^\nu T^\mu\Big) = \sum_{\nu \in J} f_\nu(T^\nu + \phi_\nu). \tag{1.14}$$

Now, if $\nu_0 \notin J$ we set $J' = J \cup \{\nu_0\}$, $\phi'_{\nu_0} := 0$, and for $\nu \in J$, $\phi'_\nu := \sum_{\mu \in \mathbb{N}^n \setminus J'} a_\mu^\nu T^\mu$. One checks that the properties mentioned above still hold, namely $\|a_\nu^\mu\| \leq C\|f_\mu\|$, where the constant C has not been changed, and

$$f = \sum_{\nu \in J'} f_\nu(T^\nu + \phi'_\nu).$$

Moreover,

$$C\|f_\mu\|\underline{r}^\mu \xrightarrow[|\mu|\to\infty]{} 0,$$

so there exists a finite set $K \subset \mathbb{N}^n$ such that

$$\forall \nu \in J,\ \forall \mu \in \mathbb{N}^n \setminus K, \quad \|a_\mu^\nu\| < \varepsilon.$$

Hence if we increase J by adding the elements of $K \setminus J$ to J, we will obtain a decomposition

$$f = \sum_{\nu \in J} f_\nu(T^\nu + \phi_\nu)$$

such that $\|\phi_\nu\| < \varepsilon$ for all $\nu \in J$. □

1.4. Equivalence of the two notions

From now on, $\mathcal{A}$ will be a k-affinoid algebra, and $\underline{r} \in (\mathbb{R}_+^*)^n$ a polyradius such that $\underline{r} > \underline{1}$, and we set $\mathcal{A}\{\underline{r}^{-1}T\} = \mathcal{A}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$. If $\nu \in \mathbb{N}^n$ we set

$$T^\nu := T_1^{\nu_1} \ldots T_n^{\nu_n}, \qquad |\nu|_\infty = \max_{i=1,\ldots,n} \nu_i, \qquad \underline{r}^\nu = \prod_{i=1}^n r_i^{\nu_i}.$$

When $\mu, \nu \in \mathbb{N}^n$, we write $\mu <_{\text{lex}} \nu$ when μ is smaller than ν with respect to the lexicographic order, that is, there exists an index m such that $\mu_m < \nu_m$ and $\mu_{m-1} = \nu_{m-1}, \ldots, \mu_1 = \nu_1$.

We will use the following notation. If $\mathcal{A}$ is a k-affinoid algebra, $f = \sum_{n\in\mathbb{N}} a_n T^n \in \mathcal{A}\{r^{-1}T\}$ and $x \in \mathcal{M}(\mathcal{A})$, we will denote by f_x the element of $\mathcal{H}(x)\{r^{-1}T\}$ defined by

$$f_x = \sum_{n\in\mathbb{N}} a_n(x) T^n.$$

Since $\mathcal{A}$ is Noetherian, we can apply Lemma 1.29 to it.

Proposition 1.30. *Let $f = \sum_{\nu\in\mathbb{N}^n} f_\nu T^\nu \in \mathcal{A}\{\underline{r}^{-1}T\}$. There exists a constructible covering of X, $(X_i, S_i) \overset{\varphi_i}{\dashrightarrow} X$, $i = 0, \dots, N$, such that if we consider the cartesian diagrams*

$$\begin{array}{ccc} (X_i, S_i) & \overset{\varphi_i}{\dashrightarrow} & X \\ {\scriptstyle\pi_i}\uparrow & & \uparrow{\scriptstyle\pi} \\ X_i \times \mathbb{B}_{\underline{r}} & \overset{\varphi'_i}{\dashrightarrow} & X \times \mathbb{B}_{\underline{r}} \end{array}$$

and if we denote by $\mathcal{A}_i$ the k-affinoid algebra of X_i, then for all $i = 1, \dots, N$ there exist $a_i \in \mathcal{A}_i$ and a function

$$g_i = \sum_{\nu\in\mathbb{N}^n} g_{i,\nu} T^\nu \in \mathcal{A}_i\{\underline{r}^{-1}T\}$$

such that

- *For all i, the family $\{g_{i,\nu}\}_{\nu\in\mathbb{N}^n}$ generates the unit ideal in $\mathcal{A}_i$.*
- *For all i, $\varphi_i'^*(f)_{|\pi_i^{-1}(S_i)} = (a_i g_i)_{|\pi_i^{-1}(S_i)}$.*

Proof. By Lemma 1.29 (here we will not use the extra condition $\|\varphi_\nu\| < \varepsilon$ of that lemma), we can find a finite subset $J \subseteq \mathbb{N}^n$, and for $\nu \in J$ some $\phi_\nu \in \mathcal{A}\{\underline{r}^{-1}T\}$, such that

$$f = \sum_{\nu\in J} f_\nu (T^\nu + \phi_\nu).$$

Fix any $r > 1$, and for each $\nu \in J$ consider the constructible datum $(X_\nu, S_\nu) \overset{\varphi_\nu}{\dashrightarrow} X$ where the affinoid algebra of X_ν is $\mathcal{A}\{r^{-1}t_\mu\}_{\mu\in J\setminus\{\nu\}}/(f_\mu - t_\mu f_\nu)$, and

$$S_\nu := \{x \in X_\nu \mid |f_\kappa(x)| \leq |f_\nu(x)| \ \forall\kappa \in J \setminus \{\nu\} \text{ and } f_\nu(x) \neq 0\}.$$

This gives rise to the cartesian diagrams

$$\begin{array}{ccc} (X_\nu, S_\nu) & \overset{\varphi_\nu}{\dashrightarrow} & X \\ {\scriptstyle\pi'}\uparrow & & \uparrow{\scriptstyle\pi} \\ X_\nu \times \mathbb{B}_{\underline{r}} & \overset{\varphi'_\nu}{\dashrightarrow} & X \times \mathbb{B}_{\underline{r}} \end{array}$$

Now,

$$\varphi_\nu'^*(f) = f_\nu\Big(T^\nu + \phi_\nu + \sum_{\mu\in J\setminus\{\nu\}} t_\mu (T^\mu + \phi_\mu)\Big) =: f_\nu g_\nu.$$

Moreover, if we write $g_\nu = \sum_{\mu\in\mathbb{N}^n} g_{\nu,\mu} T^\mu$, then by Lemma 1.29 the coefficient $g_{\nu,\nu}$ is 1, so the coefficients of g_ν generate the unit ideal. Finally, denote by $\mathcal{I}$ the ideal of $\mathcal{A}$ generated by $(a_\nu)_{\nu\in J}$. By construction, $\mathcal{I}$ also equals the ideal generated by $(a_\nu)_{\nu\in\mathbb{N}^n}$. Then, according to the definition of the S_ν's,

$$\bigcup_{\nu\in J} \phi_\nu(S_\nu) = \{x \in X \mid \exists \nu \in J,\ f_\nu(x) \neq 0\} = X \setminus V(\mathcal{I}).$$

Thus, if we set $S_0 = V(\mathcal{I})$, then $(X, S_0) \xrightarrow{\text{id}} X$ is an elementary constructible datum and $\text{id}^*(f)_{|S_0} = f_{|S_0} = 0$.

Now if we combine the constructible data $(X_\nu, S_\nu) \overset{\varphi_\nu}{\dashrightarrow} X$ for $\nu \in J$ with $(X, S_0) \xrightarrow{\varphi} X$, we obtain the desired constructible covering. □

Definition 1.31. Let $\underline{r} \in (\mathbb{R}_+^*)^n$ be a polyradius and $d_1, \dots, d_{n-1}$ some integers such that

$$\forall i = 1, \dots, n-1, \quad r_n^{d_i} \leq r_i. \tag{1.15}$$

Then

$$\sigma : \begin{cases} T_i \mapsto T_i + T_n^{d_i} & \text{for } 1 \leq i \leq n-1, \\ T_n \mapsto T_n \end{cases}$$

is an automorphism of $\mathcal{A}\{\underline{r}^{-1}T\}$. We will call such an automorphism (as well as the automorphism it induces on the k-analytic space $\mathbb{B}_{\underline{r}}$) a *Weierstrass automorphism*.

Remark 1.32. If $\underline{r} > \underline{1}$, we will use the fact that σ induces a "classical" Weierstrass automorphism of $\mathcal{A}\{T_1, \dots, T_n\}$, hence of $X \times \mathbb{B}^n$.

Recall the following classical result. If $f \in k\{T_1, \dots, T_n\}$, then there exists a Weierstrass automorphism σ of $k\{T_1, \dots, T_n\}$ such that $\sigma(f)$ is T_n-distinguished. Roughly speaking, the next lemma says that if $\mathcal{A}$ is a k-affinoid algebra and $f \in \mathcal{A}\{T_1, \dots, T_n\}$ is overconvergent, then locally on $X = \mathcal{M}(\mathcal{A})$, we can obtain an analogous result.

Proposition 1.33. *Let $\mathcal{A}$ be a k-affinoid algebra. Let $X = \mathcal{M}(\mathcal{A})$ and $x \in X$. Let $\underline{r} \in \mathbb{R}^n$ be a polyradius such that $\underline{r} > 1$.*

(1) *Let $f \in \mathcal{A}\{\underline{r}^{-1}T\}$ be such that $f_x \neq 0$. Then there exist an affinoid neighbourhood $V = \mathcal{M}(\mathcal{B})$ of x, a polyradius $\underline{\rho}$ such that $1 < \underline{\rho} \leq \underline{r}$, and a Weierstrass automorphism σ of $\mathcal{B}\{\underline{\rho}^{-1}T\}$ such that in $\mathcal{B}\{\underline{\rho}^{-1}T\}$,*

$$\sigma(f) = ag$$

where $a \in \mathcal{B}$ and $g \in \mathcal{B}\{\underline{\rho}^{-1}T\}$ is T_n-distinguished.

(2) *More generally, consider m functions $f_1, \dots, f_m \in \mathcal{A}\{\underline{r}^{-1}T\}$ such that $(f_i)_x \neq 0$ for all i. Then there exist an affinoid neighbourhood $V = \mathcal{M}(\mathcal{B})$ of x, a polyradius $\underline{\rho}$ such that $1 < \underline{\rho} \leq \underline{r}$, and a Weierstrass automorphism σ of $\mathcal{B}\{\underline{\rho}^{-1}T\}$ such that for all i,*

$$\sigma(f_i) = a_i g_i$$

where $a_i \in \mathcal{B}$ and $g_i \in \mathcal{B}\{\underline{\rho}^{-1}T\}$ is T_n-distinguished.

Proof. We first prove (1).

Step 1. Let us write

$$f = \sum_{\nu \in \mathbb{N}^n} f_\nu T^\nu \in \mathcal{A}\{\underline{r}^{-1}T\}.$$

Let $\mu \in \mathbb{N}^n$ be the greatest index with respect to the lexicographic order such that

$$\max_{\nu \in \mathbb{N}^n} |f_\nu(x)| = |f_\mu(x)|.$$

Since by assumption $f_x \neq 0$, we have $f_\mu(x) \neq 0$. According to Lemma 1.29, there exists a finite set $J \subset \mathbb{N}^n$ such that $\mu \in J$, and for each $\nu \in J$ a series $\phi_\nu \in \mathcal{A}\{\underline{r}^{-1}T\}$ which satisfies $\|\phi_\nu\|_{\mathcal{A}\{\underline{r}^{-1}T\}} < 1$ such that

$$f = \sum_{\nu \in J} f_\nu(T^\nu + \phi_\nu). \tag{1.16}$$

Step 2. Let $\nu \in J$ and assume that $|f_\nu(x)| < |f_\mu(x)|$. Then we pick some $a, b \in \mathbb{R}$ such that

$$|f_\nu(x)| < a < b < |f_\mu(x)|.$$

Next, we introduce the following affinoid domain in X:

$$W := \{z \in X \mid |f_\nu(z)| \leq a < b \leq |f_\mu(z)|\} = \mathcal{M}(\mathcal{B}).$$

By construction, W is an affinoid neighbourhood of x, f_μ is invertible in $\mathcal{B}$ and

$$\left\| \frac{f_\nu}{f_\mu} \right\|_{\mathcal{B}} \leq \frac{a}{b} < 1.$$

So we can write

$$f_\nu(T^\nu + \phi_\nu) = f_\mu\left(\frac{f_\nu}{f_\mu}(T^\nu + \phi_\nu)\right).$$

Next we consider some polyradius $\underline{1} < \underline{\rho} \leq \underline{r}$. Clearly

$$\underline{\rho}^\nu \xrightarrow[\underline{\rho} \to 1]{} 1.$$

So we can choose some $\underline{\rho}$ close enough to $\underline{1}$ such that

$$\left\| \frac{f_\nu}{f_\mu} T^\nu \right\|_{\mathcal{B}\{\underline{\rho}^{-1}T\}} < 1.$$

Since we already know that $\|\phi_\nu\|_{\mathcal{B}\{\underline{\rho}^{-1}T\}} < 1$, it follows that

$$\left\| \frac{f_\nu}{f_\mu}(T^\nu + \phi_\nu) \right\|_{\mathcal{B}\{\underline{\rho}^{-1}T\}} < 1.$$

But since

$$f_\nu(T^\nu + \phi_\nu) = f_\mu\left(\frac{f_\nu}{f_\mu}(T^\nu + \phi_\nu)\right),$$

if we set

$$\phi'_\mu := \phi_\mu + \frac{f_\nu}{f_\mu}(T^\nu + \phi_\nu)$$

we still have $\|\phi'_\mu\|_{\mathcal{B}\{\underline{\rho}^{-1}T\}} < 1$ and

$$f_\mu(T^\mu + \phi_\mu) + f_\nu(T^\nu + \phi_\nu) = f_\mu(T^\mu + \phi'_\mu).$$

Hence we can remove ν from J and replace ϕ_μ by ϕ'_μ. The equality (1.16) will still be satisfied.

If we repeat this process for each $\nu \in J$ such that $|f_\nu(x)| < |f_\mu(x)|$, we can assume that

$$\forall \nu \in J, \quad |f_\nu(x)| = |f_\mu(x)|.$$

According to the definition of μ, this implies that μ is the greatest index in J with respect to the lexicographic order.

Step 3. We set

$$d := 1 + \max_{\nu \in J} |\nu|.$$

Since by assumption $\underline{1} < \underline{r}$, we fix $s > 1$ close enough to 1 so that

$$\underline{1} < (s^{d^{n-1}}, s^{d^{n-2}}, \ldots, s^d, s) \leq \underline{r}, \tag{1.17}$$

and we set

$$\underline{\rho} := (s^{d^{n-1}}, s^{d^{n-2}}, \ldots, s^d, s). \tag{1.18}$$

It is easy to check that $\underline{\rho}$ satisfies condition (1.15) of Definition 1.31, so

$$\sigma : \begin{cases} T_1 \mapsto T_1 + T_n^{d^{n-1}}, \\ \quad\vdots \qquad \vdots \\ T_i \mapsto T_i + T_n^{d^{n-i}}, \\ \quad\vdots \qquad \vdots \\ T_{n-1} \mapsto T_{n-1} + T_n^d, \\ T_n \mapsto T_n \end{cases}$$

defines a Weierstrass automorphism of $\mathcal{B}\{\underline{\rho}^{-1}T\}$. Then, for $\nu \in J \setminus \{\mu\}$,

$$\sigma(f_\nu(T^\nu + \phi_\nu)) = f_\nu(\sigma(T^\nu) + \sigma(\phi_\nu)) = f_\mu\left(\frac{f_\nu}{f_\mu}(\sigma(T^\nu) + \sigma(\phi_\nu))\right).$$

Since $\|\sigma(\phi_\nu)\|_{\mathcal{B}\{\underline{\rho}^{-1}T\}} = \|\phi_\nu\|_{\mathcal{B}\{\underline{\rho}^{-1}T\}} < 1$, we can choose s so close to 1 that

$$s\|\phi_\nu\| < 1. \tag{1.19}$$

Then we make the following calculation. If $\nu \in J$,

$$\|\sigma(T^\nu)\|_{\mathcal{B}\{\underline{\rho}^{-1}T\}} = \|T^\nu\|_{\mathcal{B}\{\underline{\rho}^{-1}T\}} = \prod_{k=1}^{n} (s^{d^{n-k}})^{\nu_k} = s^{\sum_{k=1}^{n} \nu_k d^{n-k}}. \tag{1.20}$$

Note that $\sum_{k=1}^{n} \nu_k d^{n-k}$ is nothing other than the integer encoded by ν in base d. Since by assumption, for all $\nu \in J \setminus \{\mu\}$ we have $\nu <_{\text{lex}} \mu$, it follows that for $\nu \in J \setminus \{\mu\}$,

$$\sum_{k=1}^{n} \nu_k d^{n-k} + 1 \leq \sum_{k=1}^{n} \mu_k d^{n-k}.$$

As a corollary,

$$s\|\sigma(T^\nu)\|_{\mathcal{B}\{\underline{\rho}^{-1}T\}} \leq \|\sigma(T^\mu)\|_{\mathcal{B}\{\underline{\rho}^{-1}T\}}. \tag{1.21}$$

Now consider some $s' \in \mathbb{R}$ such that $1 < s' < s$ and set

$$V := \{z \in X \mid \forall \nu \in J \setminus \{\mu\},\ |f_\nu(z)| \leq s'|f_\mu(z)|\}.$$

Then by construction, V is an affinoid neighbourhood of x. Let us replace $\mathcal{B}$ by the affinoid algebra of V. Then by construction, for all $\nu \in J \setminus \{\mu\}$,

$$\|f_\nu/f_\mu\|_{\mathcal{B}} \leq s' < s.$$

So according to (1.21),

$$\left\|\frac{f_\nu}{f_\mu}\sigma(T^\nu)\right\|_{\mathcal{B}\{\underline{\rho}^{-1}T\}} < s\|\sigma(T^\nu)\|_{\mathcal{B}\{\underline{\rho}^{-1}T\}} \leq \|\sigma(T^\mu)\|_{\mathcal{B}\{\underline{\rho}^{-1}T\}}.$$

Hence by (1.19), we can assume that

$$\left\|\frac{f_\nu}{f_\mu}\sigma(\phi_\nu)\right\|_{\mathcal{B}\{\underline{\rho}^{-1}T\}} \leq s'\|\sigma(\phi_\nu)\|_{\mathcal{B}\{\underline{\rho}^{-1}T\}} = s'\|\phi_\nu\|_{\mathcal{B}\{\underline{\rho}^{-1}T\}} < 1 \leq \|\sigma(T^\nu)\|.$$

Thus

$$\sigma(f_\nu(T^\nu + \phi_\nu)) = f_\mu\left(\frac{f_\nu}{f_\mu}(\sigma(T^\nu) + \sigma(\phi_\nu))\right)$$

where

$$\left\|\frac{f_\nu}{f_\mu}(\sigma(T^\nu) + \sigma(\phi_\nu))\right\|_{\mathcal{B}\{\underline{\rho}^{-1}T\}} < \|\sigma(T^\mu)\|_{\mathcal{B}\{\underline{\rho}^{-1}T\}}.$$

Step 4. We have

$$\sigma(f) = f_\mu\left(\sigma(T^\mu) + \sigma(\phi_\mu) + \sum_{\nu \in J\setminus\{\mu\}} \frac{f_\nu}{f_\mu}(\sigma(T^\nu) + \sigma(\phi_\nu))\right).$$

Hence if we set

$$\phi = \sigma(\phi_\mu) + \sum_{\nu \in J\setminus\{\mu\}} \frac{f_\nu}{f_\mu}(\sigma(T^\mu) + \sigma(\phi_\nu)),$$

the preceding inequalities imply that $\|\phi\|_{\mathcal{B}\{\underline{\rho}^{-1}T\}} < \|\sigma(T^\mu)\|_{\mathcal{B}\{\underline{\rho}^{-1}T\}}$, and by construction

$$\sigma(f) = f_\mu(\sigma(T^\mu) + \phi).$$

Hence $\sigma(T^\mu)+\phi$ is T_n-distinguished of order $\sum_{k=1}^{n} \mu_k d^{n-k}$, which ends the proof of (1).

For the proof of (2), it suffices to remark that we could have carried out the proof of (1) simultaneously for all the f_i's, the main point being that in Step 3, we have to take some d large enough that works for all f_i's simultaneously. □

Lemma 1.34. *If S is an overconvergent constructible subset of X, then S is an overconvergent subanalytic subset of X.*

Proof. It is sufficient to prove that if $(Y, T) \overset{\varphi}{\dashrightarrow} X$ is a constructible datum, then $\varphi(T)$ is overconvergent subanalytic in X.

We claim that if φ is a constructible datum of complexity n, there exist some polyradii $\underline{s}, \underline{r} \in \mathbb{R}^n$ such that $\underline{s} \in \sqrt{|k^\times|}^n$ and $0 < \underline{s} < \underline{r}$, and some closed immersion ι,

$$\begin{array}{ccc} Y & \overset{\iota}{\hookrightarrow} & X \times \mathbb{B}_{\underline{r}} \\ & \searrow^{\varphi} & \downarrow \pi \\ & & X \end{array}$$

such that $\iota(T) \subset X \times \mathbb{B}_{\underline{s}}$. Indeed, this follows from the definition of a constructible datum, and is proved easily by induction on the complexity of φ.

Hence $\varphi(T) = \pi(\iota(T))$, and since $\iota(T)$ is a semianalytic subset of $X \times \mathbb{B}_{\underline{r}}$ contained in $X \times \mathbb{B}_{\underline{s}}$, it follows that $\pi(\iota(T))$ is an overconvergent subanalytic subset of X. □

Theorem 1.35. *Let $S \subset X$. If S is overconvergent subanalytic, then it is also overconvergent constructible.*

Proof. Let S be an overconvergent subanalytic subset of X. By definition, there exist $r > 1$ and a semianalytic subset R of $X \times \mathbb{B}_{\underline{r}}$ such that $S = \pi(R \cap (X \times \mathbb{B}^n))$. We will show by induction on n that S is overconvergent constructible.

If $n = 0$, there is nothing to prove since in that case, S is a semianalytic subset of X, in particular it is an overconvergent constructible subset.

Let now $n > 0$ and assume that the assertion holds for integers $< n$. We can assume that R is a basic semianalytic subset (see Remark 1.2), i.e. there are $2m$ functions $f_1, \ldots, f_m, g_1, \ldots, g_m \in \mathcal{A}\{\underline{r}^{-1}T\}$ and $\Diamond_j \in \{\leq, <\}$ for $j = 1, \ldots, m$ such that

$$R = \{x \in X \times \mathbb{B}^n_r \mid |f_j(x)| \Diamond_j |g_j(x)|,\ j = 1, \ldots, m\}. \tag{1.22}$$

Step 1. According to Proposition 1.30 we can find a constructible covering $(X_i, S_i) \overset{\varphi_i}{\dashrightarrow} X$ where $X_i = \mathcal{M}(\mathcal{B}_i)$ which induces the cartesian diagram

$$\begin{array}{ccc} X_i \times \mathbb{B}^n_{\underline{r}} & \xrightarrow{\varphi'_i} & X \times \mathbb{B}^n_{\underline{r}} \\ \pi_i \downarrow & & \downarrow \pi \\ X_i & \overset{\varphi_i}{\dashrightarrow} & X \end{array}$$

such that for all $j = 1, \ldots, m$,

$$\varphi_i'^*(f_j)_{|\pi_i^{-1}(S_i)} = (a^i_j F^i_j)_{|\pi_i^{-1}(S_i)}, \tag{1.23}$$

$$\varphi_i'^*(g_j)_{|\pi_i^{-1}(S_i)} = (b^i_j G^i_j)_{|\pi_i^{-1}(S_i)}, \tag{1.24}$$

where $a^i_j, b^i_j \in \mathcal{B}_i$, $F^i_j, G^i_j \in \mathcal{B}_i\{\underline{r}^{-1}T\}$, and the coefficients of F^i_j (resp. of G^i_j) generate the unit ideal in $\mathcal{B}_i$. Then for each i we set

$$R_i := \{x \in X_i \times \mathbb{B}^n_r \mid |a^i_j F^i_j(x)| \lozenge_j |b^i_j G^i_j(x)|,\ j = 1, \ldots, m\}.$$

So (1.23) and (1.24) imply precisely that

$$R_i \cap \pi_i^{-1}(S_i) = \varphi_i'^{-1}(R) \cap \pi_i^{-1}(S_i).$$

Thus if we set

$$U_i := \pi_i(R_i \cap (X_i \times \mathbb{B}^n))$$

then $\varphi_i(S_i \cap U_i) = \varphi_i(S_i) \cap S$, hence since the $\varphi_i(S_i)$ form a covering of X,

$$S = \bigcup_{i=1}^{n} \varphi(S_i \cap U_i).$$

So if we prove that $\varphi_i(S_i \cap U_i)$ is overconvergent constructible, we are done.

But actually, since each S_i is overconvergent constructible in X_i (it is even semianalytic, see Remark 1.6), if we prove that U_i is an overconvergent constructible subset of X_i, then it will follow that $S_i \cap U_i$ is an overconvergent constructible subset of X_i, and then according to Proposition 1.13(2), $\varphi_i(S_i \cap U_i)$ will be overconvergent constructible in X. Thus, it remains to prove that U_i is overconvergent constructible in X_i.

Step 2. We can now replace X by one of the X_i's and assume that R is defined by

$$R = \{x \in X \times \mathbb{B}^n_r \mid |a_j f_j(x)| \lozenge_j |b_j g_j(x)|,\ j = 1, \ldots, m\} \tag{1.25}$$

with $a_j, b_j \in \mathcal{A}$ and $f_j, g_j \in \mathcal{A}\{\underline{r}^{-1}T\}$ such that for all j, the coefficients of f_j (resp. of g_j) generate the unit ideal of $\mathcal{A}$. In this situation we must show that S is overconvergent constructible in X where

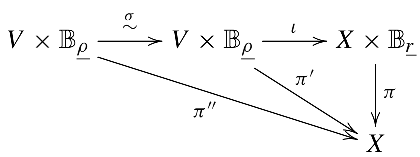
$$S = \pi(R \cap (X \times \mathbb{B}^n)).$$

Let $x \in X$. The above property of the f_j's and g_j's implies that $(f_j)_x \neq 0$ and $(g_j)_x \neq 0$. So we can apply Proposition 1.33 to them. Thus there exist an affinoid neighbourhood $V = \mathcal{M}(\mathcal{B})$ of x, some polyradius $\underline{1} < \underline{\rho} \leq \underline{r}$ and some Weierstrass automorphism σ of $\mathcal{B}\{\underline{\rho}^{-1}T\}$ such that for each j,

$$\sigma(f_j) = \alpha_j F_j, \tag{1.26}$$

$$\sigma(g_j) = \beta_j G_j, \tag{1.27}$$

where $\alpha_j, \beta_j \in \mathcal{B}$ and F_j, G_j are T_n-distinguished elements of $\mathcal{B}\{\underline{\rho}^{-1}T\}$. Consider the commutative diagram



where ι is the embedding of the affinoid domain $V \times \mathbb{B}_{\underline{\rho}}$ in $X \times \mathbb{B}_{\underline{\rho}}$. Then set

$$R' := \iota^{-1}(R), \qquad R'' := \sigma^{-1}(\iota^{-1}(R)).$$

First it is clear that

$$\begin{aligned} S \cap V &= \pi(R \cap (X \times \mathbb{B}^n)) = \pi(R \cap (V \times \mathbb{B}^n)) \\ &= \pi'(R' \cap (V \times \mathbb{B}^n)) = \pi''(R'' \cap (V \times \mathbb{B}^n)). \end{aligned} \tag{1.28}$$

For the last equality in (1.28), we use the fact that the Weierstrass automorphism σ induces an isomorphism of $V \times \mathbb{B}^n$ as noticed in Remark 1.32.

But since we know that being overconvergent constructible is a local property (see Corollary 1.15), if we prove that $S \cap V$ is overconvergent constructible, then since x has been taken arbitrarily, and since V is an affinoid neighbourhood of x, this will conclude the proof. So it suffices to prove that $\pi''(R'' \cap (V \times \mathbb{B}^n))$ is overconvergent constructible in V. Now according to (1.25)–(1.27), R'' is a semianalytic subset of $V \times \mathbb{B}_{\underline{\rho}}$ defined by inequalities between functions $a_j\alpha_j F_j$, $b_j\beta_j G_j$, where $a_j, \alpha_j, b_j, \beta_j \in \mathcal{B}$ and $F_j, G_j \in \mathcal{B}\{\underline{\rho}^{-1}T\}$ are T_n-distinguished.

Step 3. Replacing X by V, R by R'', $a_j\alpha_j$ by a_j, $b_j\beta_j$ by b_j, F_j by f_j and G_j by g_j, we can assume that

$$R = \{x \in X \times \mathbb{B}^n_{\underline{r}} \mid |a_j f_j(x)| \diamondsuit_j |b_j g_j(x)|,\ j = 1, \ldots, m\} \tag{1.29}$$

where $a_j, b_j \in \mathcal{A}$ and $F_j, G_j \in \mathcal{A}\{\underline{r}^{-1}T\}$ are T_n-distinguished in $\mathcal{A}\{\underline{r}^{-1}T\}$. Then we apply the Weierstrass Preparation Theorem 1.28 to f_j and g_j. Consequently, there exist multiplicative units $e_j, e'_j \in \mathcal{A}\{\underline{r}^{-1}T\}$ and monic polynomials $w_j, w'_j \in \mathcal{A}\{r_1^{-1}T_1, \ldots, (r_{n-1})^{-1}T_{n-1}\}[T_n]$ such that

$$f_j = e_j w_j, \qquad g_j = e'_j w'_j.$$

Thus if we set

$$P_j := a_j w_j, \qquad Q_j := b_j w'_j,$$

we have $P_j, Q_j \in \mathcal{A}\{r_1^{-1}T_1, \ldots, (r_{n-1})^{-1}T_{n-1}\}[T_n]$. In addition, since e_j, e'_j are multiplicative units, we have $|e_j(x)| = \|e_j\| \in \sqrt{|k^\times|}$ and $|e'_j(x)| = \|e'_j\| \in \sqrt{|k^\times|}$ for all $x \in X \times \mathbb{B}_{\underline{r}}$. So we finally obtain

$$\begin{aligned} R &= \{x \in X \times \mathbb{B}^n_{\underline{r}} \mid |a_j f_j(x)| \diamondsuit_j |b_j g_j(x)|,\ j = 1, \ldots, m\} \\ &= \{x \in X \times \mathbb{B}^n_{\underline{r}} \mid \|e_j\|\,|P_j(x)| \diamondsuit_j \|e'_j\|\,|Q_j(x)|,\ j = 1, \ldots, m\}. \end{aligned} \tag{1.30}$$

Consider the projection along the last coordinate of $\mathbb{B}_{\underline{r}}$,

$$X \times \mathbb{B}_{\underline{r}} \xrightarrow{\pi_1} X \times \mathbb{B}_{(r_1,\ldots,r_{n-1})} \xrightarrow{\pi_2} X.$$

According to [Duc03, 2.5], $\pi_1(R \cap (X \times \mathbb{B}^n)$ is a semianalytic subset of $X \times \mathbb{B}_{(r_1,\ldots,r_{n-1})}$. So by induction hypothesis, $\pi_2(\pi_1(R \cap (X \times \mathbb{B}^n)))$ is overconvergent constructible in X. Since $\pi_2 \circ \pi_1 = \pi$, this proves that S is overconvergent constructible and ends the proof. □

We have thus proved

Theorem 1.36. *Let X be a strictly k-affinoid space and $S \subset X$. Then S is overconvergent subanalytic if and only if it is overconvergent constructible.*

Thanks to this theorem we can use some obvious properties of overconvergent subanalytic (resp. constructible) subsets to prove less obvious results about overconvergent constructible (resp. subanalytic) subsets. For instance we can obtain a non-trivial result concerning overconvergent subanalytic subsets:

Proposition 1.37. *Let X be a strictly k-affinoid space. The class of overconvergent subanalytic subsets of X is stable under finite boolean combinations.*[5]

Proof. This was proven for overconvergent constructible subsets in Proposition 1.13. □

In the same way, we obtain a non-obvious stability property for overconvergent constructible subsets:

Corollary 1.38. *Let $\underline{r} \in \mathbb{R}^n$ be a polyradius such that $\underline{r} > \underline{1}$, and $S \subseteq X \times \mathbb{B}_{\underline{r}}$ be an overconvergent subanalytic (or constructible) subset of $X \times \mathbb{B}_{\underline{r}}$. Then $\pi(S \cap (X \times \mathbb{B}^n))$ is an overconvergent subanalytic (or constructible) subset of X.*

Proof. If S is an overconvergent subanalytic subset of $X \times \mathbb{B}_{\underline{r}}$, then by definition, there exists $s > 1$, an integer m and a semianalytic subset T of $X \times \mathbb{B}_{\underline{r}} \times \mathbb{B}^m_s$ such that $S = \pi_2(T \cap ((X \times \mathbb{B}_{\underline{r}}) \times \mathbb{B}^m))$ where $\pi_2 : (X \times \mathbb{B}_{\underline{r}}) \times \mathbb{B}^m_s \to X \times \mathbb{B}_{\underline{r}}$ is the natural projection. Hence $\pi(S \cap (X \times \mathbb{B}^n)) = \pi_2(T \cap ((X \times \mathbb{B}^n) \times \mathbb{B}^m)) = \pi_2(T \cap (X \times \mathbb{B}^{n+m}))$ where $\pi_2 : X \times \mathbb{B}_{\underline{r}} \times \mathbb{B}^m_s \to X$ is the natural projection (so $\pi_2 = \pi \circ \pi_1$). Hence S is an overconvergent subanalytic subset of X. □

1.5. From a global to a local definition

Definition 1.39. Let $\mathcal{P}$ be the data, for each k-affinoid space X, of a family $\mathcal{P}_X$ of subsets of X. If S is a subset of a k-affinoid space X, we will say that S satisfies $\mathcal{P}$ if $S \in \mathcal{P}_X$. We will say that:

- $\mathcal{P}$ is a *G-local property* if for every k-affinoid space X and any subset S of X, S satisfies $\mathcal{P}$ if and only if for all finite affinoid coverings $\{X_i\}$ of X, $S \cap X_i$ satisfies $\mathcal{P}$ relative to X_i (i.e. $S \cap X_i \in \mathcal{P}_{X_i}$).
- $\mathcal{P}$ is a *local property* if for every affinoid space X and any subset S of X, $S \in \mathcal{P}_X$ if and only if for all $x \in X$, there exists an affinoid neighbourhood U of x such that $S \cap U \in \mathcal{P}_U$.

If S is a subset of a topological space X, we will denote by $\mathring{S}$ the topological interior of S. Note that by the compactness of affinoid spaces, saying that $\mathcal{P}$ is a local property is equivalent to requiring that for all k-affinoid spaces X and any $S \subseteq X$, S satisfies $\mathcal{P}$ if and only if $S \cap X_i \in \mathcal{P}_{X_i}$ for any finite affinoid covering $\{X_i\}$ of X such that $\{\mathring{X}_i\}$ is also a covering of X. As a consequence, if $\mathcal{P}$ is a G-local property, then it is also a local property.

[5] In fact, the only non-trivial result is that overconvergent subanalytic subsets are stable under taking complements.

Example 1.40. A consequence of Kiehl's theorem [BGR84, 9.4.3] is that the class of Zariski closed subsets of affinoid spaces defines a class which is G-local.

Definition 1.41. Let X be a good k-analytic space. A *wide covering* of X is a covering $\{X_i\}$ such that the X_i's are affinoid domains in X and $\{\mathring{X}_i\}$ is also a covering of X.

Proposition 1.42. *Let X be a strictly k-affinoid space, and S a subset of X. The following assertions are equivalent:*

(1) *S is an overconvergent subanalytic subset of X.*
(2) *For every wide covering $\{X_i\}$ of X, $X_i \cap S$ is an overconvergent subanalytic subset of X_i.*
(3) *There exists a wide covering $\{X_i\}$ of X such that $X_i \cap S$ is overconvergent subanalytic in X_i for all i.*
(4) *For all $x \in X$ there exists an affinoid neighbourhood V of x such that $V \cap S$ is overconvergent subanalytic in V.*

Property (4) *implies that the class of overconvergent subanalytic subsets is local in the sense of Definition* 1.39.

Proof. (1)⇒(2) is obvious and is a consequence of Lemma 1.17; (2)⇒(3) and (3)⇔(4) are clear; and (4)⇒(1) follows from the analogous statement for overconvergent constructible subsets (Corollary 1.15) and Theorem 1.36. □

Definition 1.43. Let X be a good strictly k-analytic space. A subset $S \subset X$ is called *overconvergent subanalytic* if for all $x \in X$ there exists a strictly affinoid neighbourhood V of x such that $S \cap V$ is overconvergent subanalytic in V (according to Definition 1.16).

According to the last proposition, when X is a k-affinoid space, this definition is compatible with Definition 1.16.

Definition 1.44. Let X be a good strictly k-analytic space. A subset S of X is called *locally semianalytic* if for every $x \in X$ there exists a strictly affinoid neighbourhood V of x such that $V \cap S$ is semianalytic in V.

Corollary 1.45. *Let X be a good strictly k-analytic space. The class of locally semianalytic subsets of X is contained in the class of overconvergent constructible subsets of X.*

Corollary 1.46. *Let X be a strictly k-affinoid space and $S \subset X$. Then S is an overconvergent subanalytic subset of X if and only if there exist $r > 1$, an integer n, and a locally semianalytic subset $T \subseteq X \times \mathbb{B}^n_r$ such that $S = \pi(T \cap (X \times \mathbb{B}^n))$.*

Proof. The direct implication is true because a semianalytic subset of $X \times \mathbb{B}^n_r$ is in particular a locally semianalytic subset of $X \times \mathbb{B}^n_r$.

Conversely, if $S = \pi(T \cap (X \times \mathbb{B}^n))$ where T is a locally semianalytic subset of $X \times \mathbb{B}^n_r$, then according to Corollary 1.45, T is overconvergent subanalytic in $X \times \mathbb{B}^n_r$, so by Corollary 1.38, $\pi(T \cap (X \times \mathbb{B}^n))$ is also overconvergent subanalytic. □

Lemma 1.47. *Let $\varphi : Y \to X$ be a morphism of good strictly k-analytic spaces, and $S \subseteq X$ be a locally semianalytic subset of X. Then $\varphi^{-1}(S)$ is a locally semianalytic subset of Y.*

Proof. Let $y \in Y$ and $x = \varphi(y)$. There exists an affinoid neighbourhood V of x such that $V \cap S$ is semianalytic in V. Let W be an affinoid neighbourhood of y in $\varphi^{-1}(V)$. Then $W \cap \varphi^{-1}(S)$ is semianalytic in W. □

If $\varphi : Y \to X$ is a morphism of k-analytic spaces, one can define the *relative interior* of φ, denoted by $\mathrm{Int}(Y/X)$, which is a subset of Y. We refer to [Ber90, 2.5.7] for the definition. The complementary set of $\mathrm{Int}(Y/X)$ in Y is called the *relative boundary* of φ and denoted by $\partial(Y/X)$. For these sets, the non-rigid points are essential. For instance, if $\varphi : \mathbb{B} \to \mathcal{M}(k)$ is the structural morphism, $\partial(\mathbb{B}/\mathcal{M}(k))$ is simply the Gauss point.

Theorem 1.48. *Let $\varphi : Y \to X$ be a morphism of strictly k-affinoid spaces, and U an affinoid domain in Y such that $U \subseteq \mathrm{Int}(Y/X)$. If S is an overconvergent subanalytic subset of Y then $\varphi(U \cap S)$ is an overconvergent subanalytic subset of X.*

Proof. According to [Ber90, Prop. 2.5.9] there exist $\underline{r} > \underline{s} > 0$ and an admissible epimorphism $\mathcal{A}\{\underline{r}^{-1}T\} \to \mathcal{B}$ which identifies Y with a Zariski closed subset of $X \times \mathbb{B}_{\underline{r}}$, such that under this identification, $U \subseteq X \times \mathbb{B}_{\underline{s}}$. We can assume that $\underline{s} \in \sqrt{|k^\times|}^n$. If we denote by $\Gamma(\varphi)$ the graph of φ, this induces a Zariski closed embedding $Y \simeq \Gamma(\varphi) \xrightarrow{i} X \times \mathbb{B}_{\underline{r}}$. Now since S is an overconvergent subanalytic subset of Y, according to Lemma 1.18, $i(S)$ is an overconvergent subanalytic subset of $X \times \mathbb{B}_{\underline{r}}$. Finally, U is a semianalytic subset of Y (by the Gerritzen–Grauert theorem), so $i(U)$ is also semianalytic in $X \times \mathbb{B}_{\underline{r}}$, and by assumption, $i(U) \subseteq X \times \mathbb{B}_{\underline{s}}$, so $i(U \cap S) \subseteq X \times \mathbb{B}_{\underline{r}}$ is an overconvergent constructible subset of $X \times \mathbb{B}_{\underline{r}}$, and according to Corollary 1.38, $\pi(i(U \cap S))$ is an overconvergent subanalytic subset of X. But this set is precisely $\varphi(U \cap S)$. □

As in algebraic geometry, the notion of a proper morphism of k-analytic spaces is a little subtle. If $\varphi : Y \to X$ is a morphism of k-analytic spaces, let $|Y| \to |X|$ denote the associated map of topological spaces. Then φ is said to be *compact* [Ber90, p. 50] if the map $|Y| \to |X|$ is proper (in the topological sense). Finally, φ is said to be *proper* [Ber90, p. 50] if φ is compact and $\partial(Y/X) = \emptyset$.

Proposition 1.49. *Let $\varphi : Y \to X$ be a morphism of good strictly k-analytic spaces, and S an overconvergent subanalytic subset of Y such that the map $\overline{S} \to |X|$ of topological spaces is proper and $\overline{S} \subseteq \mathrm{Int}(Y/X)$. Then $\varphi(S)$ is an overconvergent subanalytic subset of X.*

Proof. If X' is an affinoid domain in X and if we consider the cartesian diagram

$$\begin{array}{ccc} S \subseteq Y & \xrightarrow{\varphi} & X \\ {\scriptstyle \psi'}\uparrow & & \uparrow{\scriptstyle \psi} \\ S' \subseteq Y' & \xrightarrow{\varphi'} & X' \end{array}$$

then $\psi'^{-1}(\overline{S})$ is closed in Y' and contains $\psi'^{-1}(S) = S'$, so $S' \subseteq \overline{S'} \subseteq \psi'^{-1}(\overline{S})$; furthermore, since properness is stable under base change, $\psi'^{-1}(\overline{S}) \to |X'|$ is proper, and since $\overline{S'}$ is closed, $\overline{S'} \to |X'|$ is proper. Moreover, $\psi'^{-1}(\mathrm{Int}(Y/X)) \subseteq \mathrm{Int}(Y'/X')$ [Ber90, 3.1.3(iii)], so $\overline{S'} \subseteq \psi'^{-1}(\overline{S}) \subseteq \mathrm{Int}(Y'/X')$. Thus S' and φ' fulfil the hypotheses of the proposition. Hence, since the property we want to check is local on X, we can assume that X is a k-affinoid space, hence that $\overline{S}$ is compact.

Now for every $y \in \overline{S}$ we can find an affinoid neighbourhood U such that $U \subseteq \mathrm{Int}(Y/X)$, because $\mathrm{Int}(Y/X)$ is open [Ber90, 2.5.7]. Then $\varphi(U \cap S)$ is an overconvergent subanalytic subset of X according to Theorem 1.48. Since $\overline{S}$ is compact, we can extract from this a finite covering of $\overline{S}$, which finishes the proof that $\varphi(S)$ is overconvergent subanalytic. □

Corollary 1.50. *Let $\varphi : Y \to X$ be a proper morphism of good strictly k-analytic spaces. Let S be an overconvergent subanalytic subset of Y. Then $\varphi(S)$ is an overconvergent subanalytic subset of X.*

Definition 1.51. A morphism $\varphi : Y \to X$ of good k-analytic spaces is *locally extendible without boundary* if, for all $y \in Y$, there exists an affinoid neighbourhood U of y, a k-affinoid space Y' that contains U as an affinoid domain, and $\psi : Y' \to X$ that extends $\varphi_{|U}$ such that $U \subseteq \mathrm{Int}(Y'/X)$.

Note that again by [Ber90, 3.1.3(iii)], this property is stable under base change.

Proposition 1.52. *Let $\varphi : Y \to X$ be a compact morphism of good strictly k-analytic spaces which is locally extendible without boundary. Then $\varphi(Y)$ is an overconvergent subanalytic subset of X.*

Proof. We can assume that X is a k-affinoid space, so Y is compact. Then for all $y \in Y$ we can find an affinoid neighbourhood U of y, a k-affinoid space Y' that contains U, and $\psi : Y' \to X$ that extends $\varphi_{|U}$ such that $U \subseteq \mathrm{Int}(Y'/X)$. Then, by Theorem 1.48, $\varphi(U)$ is an overconvergent subanalytic subset of X (take $S = Y'$). Hence by compactness of Y, $\varphi(Y)$ is overconvergent subanalytic. □

1.6. The non-strict case

In this section, k will be an arbitrary non-archimedean field (possibly trivially valued).

One of the advantages of Berkovich's approach is the possibility to use arbitrary $\lambda \in \mathbb{R}_+$ to define inequalities. It is then natural to give the following definitions:

Definition 1.53. Let $\mathcal{A}$ be a k-affinoid algebra, and set $X = \mathcal{M}(\mathcal{A})$.

- A subset $S \subset X$ is called *non-strictly semianalytic* if it is a boolean combination of subsets
$$\{x \in X \mid |f(x)| \leq \lambda |g(x)|\}$$
where $f, g \in \mathcal{A}$ and $\lambda \in \mathbb{R}_+$.

- A subset $S \subset X$ is called *non-strictly overconvergent subanalytic* if there exist $n \in \mathbb{N}$, a real number $r > 1$, and a non-strictly semianalytic set $T \subset X \times \mathbb{B}^n_r$ such that $S = \pi(T \cap (X \times \mathbb{B}^n))$ where $\pi : X \times \mathbb{B}^n_r \to X$ is the first projection.

Remark 1.54. Let X be a strictly k-affinoid space and let $S \subset X$. The following implication holds:

$$S \text{ is semianalytic } \Rightarrow S \text{ is non-strictly semianalytic.}$$

However, if $\sqrt{|k^\times|} \subsetneq \mathbb{R}^*_+$, the converse implication is false. Indeed, let $r \in]0, 1[$ be such that $r \notin \sqrt{|k^\times|}$, let $X = \mathbb{B}^1 = \mathcal{M}(k\{T\})$ and let $S = \{x \in \mathbb{B} \mid |T(x)| = r\}$. By definition, S is a non-strictly semianalytic set in $\mathbb{B}^1$, but we claim that it is not semianalytic. Indeed, we will see in 2.14 that semianalytic sets are entirely determined by their rigid points, that is, if S_1 and S_2 are semianalytic subsets of X, then $S_1 = S_2$ if and only if $S_1 \cap X_{\text{rig}} = S_2 \cap X_{\text{rig}}$. Since in our example, $S \cap X_{\text{rig}} = \emptyset$, if S were semianalytic, it would be empty, which it is not: actually, $S = \{\eta_r\}$.

Definition 1.55. Let X be a k-affinoid space. Let (X, S) be a k-germ, $f, g \in \mathcal{A}$, $0 < s < r$ where $r, s \in \mathbb{R}$, and

$$Y = \mathcal{M}(\mathcal{A}\{r^{-1}t\}/(f - tg)) \xrightarrow{\varphi} X$$

and $T = \varphi^{-1}(S) \cap R \cap \{y \in Y \mid |f(y)| \leq s|g(y)| \neq 0\}$ where R is a non-strictly semianalytic subset of Y. Then we say that $(Y, T) \xrightarrow{\varphi} (X, R)$ is a *non-strictly elementary constructible datum.*

The only difference from Definition 1.3 is that we do not assume any more that $s \in \sqrt{|k^\times|}$, and that R is allowed to be non-strictly semianalytic, that is, defined with inequalities involving some arbitrary $\lambda \in \mathbb{R}$.

Then we mimic Definition 1.5, and say that a *non-strictly constructible datum* $(Y, T) \overset{\varphi}{\dashrightarrow} (X, S)$ is a composite $\varphi = \varphi_1 \circ \cdots \circ \varphi_n$ where each φ_i is a non-strictly elementary constructible datum. Finally, if $(X_i, S_i) \overset{\varphi_i}{\dashrightarrow} X$, $i = 1, \ldots, n$, are n non-strictly constructible data, we say that $S := \bigcup_{i=1}^n \varphi_i(S_i)$ is a *non-strictly overconvergent constructible set.*

We claim that all results we have proven in this section for overconvergent subanalytic (resp. constructible) sets remain valid for non-strictly overconvergent subanalytic (resp. constructible) sets. For instance:

Theorem 1.56. *Let X be a k-affinoid space. Then $S \subset X$ is non-strictly overconvergent subanalytic if and only if it is non-strictly overconvergent constructible.*

In this context, we want to stress that for instance Propositions 1.42 and 1.49 also remain true.

2. Study of various classes

2.1. Many families

In this section $X = \mathcal{M}(\mathcal{A})$ will be a strictly k-affinoid space. The aim of this section is to first recall the definitions of the various classes of *rigid/locally/strongly/D-semianalytic/subanalytic* subsets of X that are defined in [Sch94a].

We now give the following definitions. A subset $S \subseteq X$ is called:

(a) *semianalytic* if it is a boolean combination of subsets of the form $\{x \in X \mid |f(x)| \leq |g(x)|\}$ with $f, g \in \mathcal{A}$;
(b) *locally semianalytic* if for all $x \in X$ there exists an affinoid neighbourhood V of x such that $S \cap V$ is semianalytic in V;
(c) *rigid-semianalytic* if there is a finite affinoid covering[6] $\{X_i\}_{i=1}^n$ such that $S \cap X_i$ is semianalytic in X_i for all i.
(d) *overconvergent subanalytic* if it is as defined in Definition 1.16 (as we proved in the previous section, this also corresponds to overconvergent constructible subsets; moreover, our definition of overconvergent subanalytic subset is the same as the definition of globally strongly subanalytic subset in [Sch94a, 1.3.8.1]; this is equivalent to being globally strongly **D**-semianalytic [Sch94a, 1.3.2]);
(e) *G-overconvergent subanalytic* if there exists a finite affinoid covering $\{X_i\}$ of X such that $S \cap X_i$ is overconvergent constructible in X_i for all i (this corresponds to the notion of *strongly* **D***-semianalytic* subset in [Sch94a, 1.3.7.1]);
(f) *strongly subanalytic* if there exist $n \in \mathbb{N}$, a real number $r > 1$, and a subset $T \subseteq X \times \mathbb{B}_r^n$ which is rigid-semianalytic, such that $S = \pi(T \cap (X \times \mathbb{B}^n))$ (this definition comes from [Sch94a, 1.3.8.1], and we will give an equivalent definition in Proposition 2.8);
(g) *locally strongly subanalytic* if there exists a finite affinoid covering $\{X_i\}$ of X such that $S \cap X_i$ is strongly subanalytic in X_i for all i (this definition comes from [Sch94a, 1.3.8.2]).

In [Sch94a] it is stated that (d)–(g) are equivalent (equivalence of (e)–(g) is stated in [Sch94a, Prop. 4.2], and the equivalence of (d) and (f) is stated in [Sch94a, Th. 5.2]). These results rest on [Sch94a, Lemma 4.1] which is false, and we will show indeed that (d), (e) and (f) correspond in general to three different classes. More precisely the aim of this section is to show that these classes satisfy the following relations:

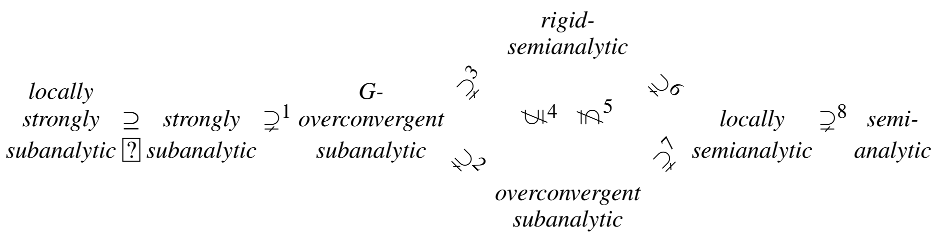


Fig. 1. The hierarchy. In this figure, A ⊋ B means that the class A *properly* contains the class B, and A ⊉ B means that the class A does not contain the class B.

In this diagram, all the inclusions are clear from the definitions, except inclusion 7 which states that the class of overconvergent subanalytic subsets contains the class of locally semianalytic subsets. But this is precisely the content of Corollary 1.42. In com-

[6] If X is an affinoid space we say that $\{X_i\}_{i=1}^n$ is a *finite affinoid covering* if X_i is an affinoid domain in X for all i and $X = \bigcup_{i=1}^n X_i$.

parison with what was stated in [Sch94a], the most striking relation is probably $\not\subseteq^5$ which asserts that rigid-semianalytic subsets are not necessarily overconvergent subanalytic subsets, contrary to [Sch94a, Th. 5.2]. In other words, when you project overconvergent semianalytic subsets, you obtain a class which is not G-local (but local for the Berkovich topology).

In this section we will show that the inclusions in Figure 1 are all proper in general (in the next section we will explain that if X is regular of dimension 2, overconvergent subanalytic subsets correspond to locally semianalytic subsets). We do not know if the inclusion on the left,

locally strongly subanalytic $\supseteq$ *strongly subanalytic*,

is proper.

2.2. *Rigid-semianalytic subsets are not necessarily overconvergent subanalytic*

Here we prove $\not\subseteq^5$.

Lemma 2.1. *Let $\eta \in X$ be such that $\mathcal{O}_{X,\eta}$ is a field, and $S \subset X$ a semianalytic subset. If $\eta \in \overline{S}$, then $\mathring{S}$ is non-empty.*

Proof. Since $\bigcup_{i=1}^n \overline{S_i} = \overline{\bigcup_{i=1}^n S_i}$ we can assume that S is a basic semianalytic subset, i.e. is of the form

$$S = \Big(\bigcap_{i=1}^{m} \{x \in X \mid |f_i(x)| \leq |g_i(x)|\}\Big) \cap \Big(\bigcap_{j=1}^{n} \{x \in X \mid |F_j(x)| < |G_j(x)|\}\Big).$$

We use the decomposition

$$\{x \in X \mid |f_i(x)| \leq |g_i(x)|\} = \{x \in X \mid f_i(x) = g_i(x) = 0\} \\ \cup \{x \in X \mid \ |f_i(x)| \leq |g_i(x)| \neq 0\}$$

and using again the fact that closure is stable under finite unions, we can assume that $\eta \in \overline{S}$ and that S is of the form

$$S = \bigcap_{i=1}^{l} \{x \in X \mid h_i(x) = 0\} \cap \bigcap_{j=1}^{m} \{x \in X \mid \ |f_j(x)| \leq |g_j(x)| \neq 0\} \\ \cap \bigcap_{k=1}^{n} \{x \in X \mid |F_k(x)| < |G_k(x)|\}.$$

Since the subsets $\{x \in X \mid h_i(x) = 0\}$ are closed and contain S, and $\eta \in \overline{S}$, it follows that $h_i(\eta) = 0$.

Since $\mathcal{O}_{X,\eta}$ is a field, we can find an affinoid neighbourhood V of η such that $h_{i|V} = 0$ for all i. Hence $V \cap S \neq \emptyset$ (because $\eta \in \overline{S}$) and we can remove the h_i's, and assume that

$$V \cap S = \bigcap_{j=1}^{m} \{x \in V \mid |f_j(x)| \leq |g_j(x)| \neq 0\} \cap \bigcap_{k=1}^{n} \{x \in V \mid |F_k(x)| < |G_k(x)|\}.$$

This defines a strictly k-analytic domain of X, which is non-empty, so its interior is also non-empty, for instance the interior contains some rigid points. $\square$

Lemma 2.2. *Let $\eta \in X$ and assume that $\mathcal{O}_{X,\eta}$ is a field. Let $(Y, T) \xrightarrow{\varphi} (X, S)$ be an elementary constructible datum with $Y = \mathcal{M}(\mathcal{A}\{r^{-1}t\}/(f - tg))$ where $T = \varphi^{-1}(S) \cap \{y \in R \mid |f(y)| \leq s|g(y)| \neq 0\}$ with $0 < s < r$, $s \in \sqrt{|k^\times|}$ and R a semianalytic subset of Y. Assume that $\eta \in \overline{\varphi(T)}$. Then*

(a) $g(\eta) \neq 0$.
(b) $|f(\eta)| \leq s|g(\eta)|$.
(c) *There exists a neighbourhood U of η such that $\varphi^{-1}(U) \xrightarrow{\varphi_{|\varphi^{-1}(U)}} U$ is an isomorphism. If is η' the only point of $\varphi^{-1}(U)$ such that $\varphi(\eta') = \eta$, then $\eta' \in \overline{T}$ and $\mathcal{O}_{Y,\eta'}$ is a field.*

Proof. (a) If we had $g(\eta) = 0$, since $\mathcal{O}_{X,\eta}$ is a field, there would exist an affinoid neighbourhood V of η such that $g_{|V} = 0$. Since $g(\varphi(p)) \neq 0$ for $p \in T$, we should have $\varphi(T) \cap V = \emptyset$, which is impossible since $\eta \in \overline{\varphi(T)}$.

(b) The set $\{x \in X \mid |f(x)| \leq s|g(x)|\}$ is a closed subset of X which contains $\varphi(T)$, hence by assumption also contains η.

(c) Set $U = \{y \in Y \mid g(y) \neq 0\}$. Then $\varphi_{|U}$ is an isomorphism of U onto $\varphi(U) = \{x \in X \mid |f(x)| \leq r|g(x)| \neq 0\}$ which is an analytic domain in X, and a neighbourhood of η according to the preceding two points. So $\eta \in \varphi(U)$, say $\eta = \varphi(\eta')$ with $\eta' \in U$. Now, $\mathcal{O}_{Y,\eta'} \simeq \mathcal{O}_{X,\eta}$ is a field and $\eta' \in \overline{T}$. $\square$

Corollary 2.3. *Let $\eta \in X$ be such that $\mathcal{O}_{X,\eta}$ is a field, and let U be an overconvergent subanalytic subset of X. If $\eta \in \overline{U}$, then $\mathring{U} \neq \emptyset$.*

Proof. First, according to Theorem 1.35, we can assume that U is an overconvergent constructible subset. Then, using similar arguments to those beginning the proof of Lemma 2.1, we can assume that $U = \varphi(T)$ where $(Y, T) \overset{\varphi}{\dashrightarrow} X$ is a constructible datum. Hence T is a semianalytic subset of Y. A repeated use of Lemma 2.2 furnishes an open neighbourhood U of η such that $\varphi_{|\varphi^{-1}(U)} : \varphi^{-1}(U) \to U$ is an isomorphism. Thanks to Lemma 2.2 again, we can introduce η', the only point of $\varphi^{-1}(U)$ such that $\varphi(\eta') = \eta$, and assert that $\mathcal{O}_{Y,\eta'}$ is a field and that $\eta' \in \overline{T}$. Now if V is a strictly affinoid neighbourhood of η' contained in $\varphi^{-1}(U)$, then $\eta' \in \overline{T \cap V}$ (closure in V). Now, $T \cap V$ is a semianalytic subset of V, so according to Lemma 2.1, $T \cap V$ has non-empty interior in V. We can then deduce that T has non-empty interior in X, whence $\varphi(T)$ also has non-empty interior. $\square$

Let $f = \sum_{n\in\mathbb{N}} a_n T^n$ be a series and $r \in \mathbb{R}_+^*$. We will say that the *radius of convergence of f is exactly r* when $|a_n|r^n \to 0$ as $n \to \infty$, and r is maximum with this property.

Proposition 2.4. *Let $X = \mathbb{B}^2 = \mathcal{M}(k\{T_1, T_2\})$ be the closed bidisc, let $0 < r < 1$ with $r \in |k^\times|$, say $r = |\varepsilon|$ for some $\varepsilon \in k$, and let $f \in k\{r^{-1}u\}$ be some function whose radius of convergence is exactly r, with $\|f\| < 1$. Define*

$$S = \{x \in X \mid |T_1(x)| < r \text{ and } T_2(x) = f(T_1(x))\}.$$

Then S is rigid-semianalytic but not overconvergent subanalytic. As a consequence, the class of overconvergent subanalytic subsets is not G-local.

Proof. In more concrete terms, S is the set of points of the *curve* whose equation is $T_2 = f(T_1)$, restricted to the subset $\{|T_1| < r\}$. Let

$$\psi : \mathbb{B} \to X, \quad u \mapsto (\varepsilon u, f(\varepsilon u)),$$

and set $\eta = \psi(g)$ where g is the Gauss point of $\mathbb{B}$. Then $S \subseteq \psi(\mathbb{B})$ and $\eta \in \overline{S}$. According to [Duc11, Prop. 4.4.6], $\mathcal{O}_{X,\eta}$ is a field. Furthermore $\mathring{S} = \emptyset$ because $S \subseteq Z := \{x \in \mathbb{B}_{(r,1)} \mid T_2(x) = f(T_1(x))\}$, which is a Zariski closed subset of dimension 1 of $\mathbb{B}_{(r,1)}$, which itself is of pure dimension 2, so Z is nowhere dense in $\mathbb{B}_{(r,1)}$ [Ber90, 2.3.7]. Hence according to Corollary 2.3, S is not overconvergent subanalytic. However, if we consider the covering of X given by $X_1 = \{x \in X \mid |T_1(x)| \le r\}$ and $X_2 = \{x \in X \mid |T_1(x)| \ge r\}$, then $S \cap X_1$ is indeed semianalytic in X_1 and $S \cap X_2 = \emptyset$, so S is rigid-semianalytic.

Now since the class of overconvergent subanalytic subsets contains the class of semianalytic subsets, if the former class were G-local, it would contain the class of rigid-semianalytic subsets, but we have shown that this is not the case. Hence the class of overconvergent subanalytic subsets is not G-local. ◻

Remark 2.5. Actually, this example gives a direct counterexample to [Sch94a, Lemma 4.1], which in our feeling is the source of mistakes in [Sch94a].

As a corollary, we obtain:

Proposition 2.6. *Let $0 < s < r < 1$ with $s \in \sqrt{|k^\times|}$, let $f \in k\{r^{-1}u\}$ with $\|f\| < 1$ have radius of convergence exactly r, and set $\mathbb{B}^2 = \mathcal{M}(k\{T_1, T_2\})$. Define*

$$S = \{x \in \mathbb{B}^2 \mid |T_1(x)| \le s,\ T_2(x) = f(T_1(x))\}.$$

Then S is a locally semianalytic subset of $\mathbb{B}^2$ which is not a semianalytic subset of $\mathbb{B}^2$.

Proof. If S were a semianalytic subset of $\mathbb{B}^2$, we could find $T \subseteq S$ which contains infinitely many points of S such that T is a basic semianalytic subset, and even a finite intersection of sets of the form $\{x \in \mathbb{B}^2 \mid |g_1(x)| < |g_2(x)|\}$, $\{x \in \mathbb{B}^2 \mid |g_1(x)| \le |g_2(x)| \neq 0\}$ and $\{x \in \mathbb{B}^2 \mid h(x) = 0\}$. Since an intersection of sets of the first two kinds is a strictly analytic domain, and $T \subseteq S$, and $\mathring{S} = \emptyset$, in this intersection, there must be a non-trivial set of the form $\{x \in \mathbb{B}^2 \mid h(x) = 0\}$. Now, consider in $\mathbb{B}_{(r,1)} = \mathcal{M}(k\{r^{-1}T_1, T_2\})$ the Zariski closed subset $Z = V(T_2 - f(T_1), h)$. By assumption, it is infinite. Moreover, since $\|f\| < 1$, $T_2 - f(T_1)$ is irreducible (see the lemma above) in $\mathcal{M}(k\{r^{-1}T_1, T_2\})$, so for dimensional reasons, in $\mathcal{M}(k\{r^{-1}T_1, T_2\})$, $V(T_2 - f(T_1)) \subseteq V(h)$. But now we introduce (as in the preceding proof)

$$\psi : \mathbb{B}_r \to \mathbb{B}^2, \quad u \mapsto (u, f(u)),$$

and $\eta = \psi(g)$ where g is the Gauss point of $\mathbb{B}_r$. Then $\eta \in V(h)$ (where we now view $V(h)$ as a Zariski closed subset of $\mathbb{B}^2$), $\mathcal{O}_{\mathbb{B}^2,\eta}$ is a field, but $\mathring{V(h)} = \emptyset$, and since $V(h)$ is a semianalytic (so overconvergent subanalytic) subset of $\mathbb{B}_{(r,1)}$, this contradicts Lemma 2.1.

Let us now show that S is a locally semianalytic subset of $\mathbb{B}^2$. Indeed, take $0 < s < t < r$ with $t, r \in \sqrt{|k^\times|}$, and consider $X_1 = \{x \in \mathbb{B}^2 \mid |T_1(x)| \le r\}$ and $X_2 = \{x \in \mathbb{B}^2 \mid$

$|T_1(x)| \geq t\}$. They define a wide covering of $\mathbb{B}^2$, and $X_1 \cap S$ (resp. $X_2 \cap S$) is semianalytic in X_1 (resp. X_2), so S is locally semianalytic in $\mathbb{B}^2$. □

We have implicitly used:

Lemma 2.7. *If $f \in k\{r^{-1}x\}$ and $\|f\| \leq 1$, then $F(x, y) := y - f(x)$ is irreducible in $k\{r^{-1}x, y\}$.*

Proof. As we have already seen, $V(f)$ is isomorphic to $\mathbb{B}_r$, so is irreducible. □

2.3. *The other inequalities*

We will now explain the other proper inclusions appearing in Figure 1. The following proposition will be implicitly used in the rest of this section. In addition, it illustrates that the mixture of overconvergence and rigid-semianalytic subsets (which is a G-local property) is somehow too strong, in the sense that in Proposition 2.8 below, the overconvergence condition seems to have disappeared.

Proposition 2.8. *Let $S \subseteq X$. The following properties are equivalent:*

(1) *S is strongly subanalytic.*
(2) *There exist $n \in \mathbb{N}$ and a rigid-semianalytic subset $T \subseteq X \times \mathbb{B}^n$ such that*

$$S = \pi(T \cap (X \times (\mathring{\mathbb{B}})^n))$$

where $\pi : X \times \mathbb{B}^n \to X$ is the natural projection.

Proof. Let us show that (1)⇒(2). Let S be a strongly subanalytic subset of X, so there exists $r > 1$ and a rigid-semianalytic subset $T \subseteq X \times \mathbb{B}^n_r$ such that $S = \pi(T \cap (X \times \mathbb{B}^n))$. Decreasing r if necessary, we can assume that $|r| \in \sqrt{|k^\times|}$. In fact, using similar arguments to the one given in Remark 1.19, we can even assume that $r \in |k|$. Then if we consider the homothety, which is an isomorphism $h : X \times \mathbb{B}^n_r \to X \times \mathbb{B}^n$, which can be defined as multiplication of each coordinate of $\mathbb{B}^n_r$ by $1/\lambda$, this gives the following commutative diagram:

$$\begin{array}{ccc} X \times \mathbb{B}^n_r & \xrightarrow{h} & X \times \mathbb{B}^n \\ & \searrow^{\pi} & \downarrow{\scriptstyle \pi'} \\ & & X \end{array}$$

and $S = \pi(T \cap (X \times \mathbb{B}^n)) = \pi'(h(T) \cap (X \times \mathbb{B}^n_{1/r}))$. Now $T' := h(T) \cap (X \times \mathbb{B}^n_{1/r})$ is a rigid-semianalytic subset of $X \times \mathbb{B}^n$ such that $T' \subseteq X \times (\mathring{\mathbb{B}})^n$ and $S = \pi'(T') = \pi'(T' \cap (X \times (\mathring{\mathbb{B}})^n))$.

Conversely, let $T \subseteq X \times (\mathring{\mathbb{B}})^n$ be a rigid-semianalytic subset of $X \times \mathbb{B}^n$ and $S = \pi(T)$. For any $r > 1$, we can define $X_0 = X \times \mathbb{B}^n$, and for $i = 1, \ldots, n$, let $X_i = \{(x, t_1, \ldots, t_n) \in X \times \mathbb{B}^n_r \mid |t_i| \geq 1\}$. So $\{X_i\}_{i=0}^n$ is an admissible covering of $X \times \mathbb{B}^n_r$. By assumption, $T \cap X_0$ is rigid-semianalytic, and $T \cap X_i = \emptyset$ for $i = 1, \ldots, n$. So T is rigid-semianalytic in $X \times \mathbb{B}^n_r$. We set $\pi : X \times \mathbb{B}^n_r \to X$. We then get $S = \pi(T)$, so S is strongly subanalytic. □

Proposition 2.9. *There exist strongly subanalytic subsets which are not G-overconvergent subanalytic.*

Proof. Let $r > 1$, $X = \mathcal{M}(k\{x, y, z\}) = \mathbb{B}^3$, and $Y = \mathcal{M}(k\{x, y, z, t\})$, and let $\pi : Y = \mathcal{M}(k\{x, y, z, t\}) \to X = \mathcal{M}(k\{x, y, z\})$ be the natural projection. We now choose $f \in k\{t\}$ whose radius of convergence is exactly 1, and such that $\|f\| \leq 1$, and $T = \{(x, y, z, t) \in Y \mid |t| < 1, x = yt, z = yf(t)\}$. It is a rigid-semianalytic subset of Y, and $S = \pi(T)$ is a strongly subanalytic subset of X according to the previous proposition. Since the family of closed balls with centre the origin is a fundamental system of neighbourhoods of the origin, if S were G-overconvergent subanalytic, for some $1 \geq |\mu| = \varepsilon > 0$ small enough, $S' := S \cap \mathbb{B}^3_\varepsilon$ would be overconvergent subanalytic in $\mathbb{B}^3_\varepsilon$. We then fix a $y_0 \in k^\times$ such that $0 < |y_0| < \varepsilon$, i.e. $|y_0|/|\mu| < 1$ and define $X' := \{(x, y, z) \in \mathbb{B}^3_\varepsilon \mid y = y_0\}$. Now X' is isomorphic to the bidisc $\mathbb{B}^2_\varepsilon = \{(x, y) \mid |x|, |y| \leq \varepsilon\}$, and $S'' := S \cap X'$ would be overconvergent constructible in X' thanks to Lemma 1.18(2). If we make a dilatation of X' by $1/\mu$, it becomes the bidisc of radius 1: the new coordinates are x', z' defined by $x = \mu x'$ and $z = \mu z'$. Now, in these new coordinates,

$$S'' = \{(x', z') \in \mathbb{B}^2 \mid |x'| < |y_0|/|\mu| \text{ and } z' = (y_0/\mu) f(x'\mu/y_0)\}$$

would be overconvergent subanalytic in $\mathbb{B}^2$. If we set $r := |y_0|/|\mu| < 1$ and $g(x') = (y_0/\mu) f(x'\mu/y_0)$, then the radius of convergence of g is precisely r, and $\|g\| < 1$, so $S'' = \{(x', z') \in \mathbb{B}^2 \mid |x'| < r \text{ and } z' = g(x')\}$, and S'' would be overconvergent subanalytic in $\mathbb{B}^2$, contrary to Proposition 2.4. □

Proposition 2.10. *There exist overconvergent subanalytic subsets which are not rigid-semianalytic.*

Proof. Let $1 < r = |\lambda|$, and let $f \in k\{r^{-1}X\}$ have radius of convergence exactly r, and $\|f\| < 1$. We set $X = \mathbb{B}^3 = \mathcal{M}(k\{x, y, z\})$, $Y = \mathcal{M}(k\{x, y, z, r^{-1}t\})$,

$$T = \{(x, y, z, t) \in Y \mid x = yt,\ z = yf(t),\ |t| \leq 1\}$$

and $S = \pi(T)$, where $\pi : \mathcal{M}(k\{x, y, z, r^{-1}t\}) \to \mathcal{M}(k\{x, y, z\})$ is the natural projection. Then S is overconvergent subanalytic. If S were rigid-semianalytic, there would exist $\mu \in k$ with $0 < \varepsilon := |\mu| < 1$ such that $S' = S \cap \mathbb{B}^3$ is semianalytic in $\mathbb{B}^3_\varepsilon$ (we *again* use the fact that if V is an affinoid domain in $\mathbb{B}^3$ that contains the origin, then there exists $\varepsilon > 0$ such that $\mathbb{B}^3_\varepsilon \subseteq V$). Let $y_0 \in k^\times$ be such that $0 < |y_0| < \varepsilon/r$. In particular $|y_0|/\varepsilon = |y_0/\mu| < 1/r$. Then $X' = \{(x, y, z) \in \mathbb{B}^3_\varepsilon \mid y = y_0\}$ is a Zariski closed subset of $\mathbb{B}^3_\varepsilon$, isomorphic to a bidisc $\mathbb{B}^2$. Now, $S'' := S \cap X'$ is defined by

$$S'' = \{(x, z) \in \mathbb{B}^2_\varepsilon \mid |x/y_0| \leq 1 \text{ and } z = y_0 f(x/y_0)\}.$$

As we said, X' is isomorphic to $\mathbb{B}^2$ with coordinates (x', z') where $x = \mu x'$ and $z = \mu z'$. In these new coordinates, $S'' = \{(x', z') \in \mathbb{B}^2 \mid |x'\mu/y_0| \leq 1 \text{ and } z'\mu = y_0 f(x'\mu/y_0)\}$. If we define $g(x') = (y_0/\mu) f(x'\mu/y_0)$ and $s = |y_0|/\varepsilon = |y_0/\mu| < 1/r$, then g has radius of convergence exactly ρ where $s < \rho = |y_0/\mu| r < 1$, and $\|g\| < \|f\| < 1$, so $S'' = \{(x', z') \in \mathbb{B}^2 \mid |x'| \leq s \text{ and } z' = g(x')\}$ would be semianalytic, which is not the case (see Proposition 2.6). □

Remark 2.11. The example given in the above proposition is very close to the so called Osgood example [Osg16, Theorem 1]. This example asserts that that the subset of $\mathbb{C}^3$ parametrized by

$$x = u, \quad y = uv, \quad z = uve^v$$

does not satisfy any relation of the form $F(x, y, z) = 0$ where F is a germ of analytic function around the origin. See also [BM00, 2.3].

The non-archimedean analogue of this fact holds (see the introduction of [LR99] for instance). The example studied in the above proposition amounts to considering the set parametrized by

$$x = uv, \quad y = v, \quad z = vf(v),$$

where f is transcendental. Osgood's original argument would have equally worked here, but let us stress that our argument is different.

From this one can deduce:

Corollary 2.12. *Let X be a strictly k-analytic space which contains a closed ball of dimension ≥ 3. Then there are overconvergent subanalytic subsets of X which are not rigid-semianalytic. In particular, the class of overconvergent subanalytic subsets of X properly contains the class of locally semianalytic subsets of X.*

In conclusion, in Figure 1, we have shown non-equalities 1, 4, 5 and 8. Now 2, 3, 6, 7 are set-theoretical consequences of 4, 5 and of the inclusions from left to right.

2.4. *Berkovich points versus rigid points*

Let $X = \mathcal{M}(\mathcal{A})$ be a strictly k-affinoid space. We denote by X_{rig} the set of rigid points of X. When one deals with semianalytic or overconvergent subanalytic subsets S of X, one can wonder if things change if we restrict to $S_{\text{rig}} = S \cap X_{\text{rig}}$. Actually the following two propositions show that it makes no difference whether one works with Berkovich spaces or rigid spaces.

To be precise, let $\mathcal{B}$ be the free boolean algebra whose set of variables consists of the set of *formal inequalities* $\{|f| \leq |g|\}$, $\{|f| < |g|\}$ and $\{f = 0\}$, for $f, g \in \mathcal{A}$. We denote by SA_{rig} the class of semianalytic subsets of X_{rig} and by SA_{Ber} the class of semianalytic subsets of the Berkovich space X. Then we define natural maps $\alpha : \mathcal{B} \to SA_{\text{Ber}}$ and $\beta : \mathcal{B} \to SA_{\text{rig}}$ where for instance $\alpha(\{|f| \leq |g|\}) = \{x \in X \mid |f(x)| \leq |g(x)|\}$ and $\beta(\{|f| \leq |g|\}) = \{x \in X_{\text{rig}} \mid |f(x)| \leq |g(x)|\}$. In addition we consider the forgetful map $\iota : SA_{\text{Ber}} \to SA_{\text{rig}}$: if $S \in SA_{\text{Ber}}$ is a semianalytic set, then $\iota(S) = S \cap X_{\text{rig}}$. We then obtain the commutative diagram

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\alpha} & SA_{\text{Ber}} \\ & \underset{\beta}{\searrow} & \downarrow \iota \\ & & SA_{\text{rig}} \end{array}$$

Proposition 2.13. *The map ι is bijective.*

Proof. First, ι is surjective by definition.

Now if $\iota(S_1) = \iota(S_2)$, we must show that $S_1 = S_2$. Considering $S_1 \setminus S_2$ and $S_2 \setminus S_1$, it suffices to show that if $S \in SA_{\mathrm{Ber}}$ and $\iota(S) = \emptyset$, then $S = \emptyset$. According to what has been previously done, we can assume that $S \in SA_{\mathrm{Ber}}$ is a finite intersection of subsets of the form $\{x \in X \mid |f(x)| \leq |g(x)| \neq 0\}$, $\{x \in X \mid |f(x)| < |g(x)|\}$ or $\{x \in X \mid h(x) = 0\}$, and that $\iota(S) = S \cap X_{\mathrm{rig}} = \emptyset$. Passing to $Y = \mathcal{M}(\mathcal{A}/\mathcal{I})$ where $\mathcal{I}$ is the ideal generated by the functions h appearing in the third case ($h(x) = 0$), we can assume that S is a finite intersection of subsets of the form $\{x \in X \mid |f(x)| \leq |g(x)| \neq 0\}$ or $\{x \in X \mid |f(x)| < |g(x)|\}$. But then it forms a non-empty strictly analytic domain in X, so $S \cap X_{\mathrm{rig}} \neq \emptyset$. □

If we denote by CD the family of finite subsets of constructible data of X, by OC the family of overconvergent constructible subsets of X, and by OC_{rig} the family of subsets of X_{rig} which are intersections of elements of OC with X_{rig}, then we can define as above the following commutative diagram:

$$\begin{array}{ccc} CD & \xrightarrow{\alpha} & OC \\ & \searrow^{\beta} & \downarrow \iota \\ & & OC_{\mathrm{rig}} \end{array}$$

To be precise, if $\mathcal{D} \in CD$ is the set of the constructible data $(X_i, T_i) \overset{\varphi_i}{\dashrightarrow} X$, then

$$\alpha(\mathcal{D}) = \bigcup_{i=1}^{n} \varphi_i(T_i).$$

Proposition 2.14. *In the above diagram, ι is a bijection.*

Proof. Since we showed that OC (and OC_{rig}) is stable under complements, it suffices to show that if $S \in OC$ is such that $\iota(S) = S \cap X_{\mathrm{rig}} = \emptyset$, then $S = \emptyset$. To show this we can even assume that $S = \varphi(T)$, where $(Y, T) \overset{\varphi}{\dashrightarrow} X$ is a constructible datum. But, if T is a non-empty semianalytic subset of Y, then by Proposition 2.13, $T_{\mathrm{rig}} \neq \emptyset$, so since φ preserves the rigid points, $\varphi(T)_{\mathrm{rig}} = S_{\mathrm{rig}}$ is non-empty. □

3. Overconvergent subanalytic subsets when $\dim(X) = 2$

In this section, k will be a non-archimedean algebraically closed field. In that case, a k-analytic space X is said to be *quasi-smooth* [Duc11, Section 5] if for all $x \in X$ the local ring $\mathcal{O}_{X,x}$ is regular.

3.1. *Algebraisation of functions*

Proposition 3.1. *Let X, Y be two k-affinoid spaces, so that we can consider the cartesian diagram*

$$\begin{array}{ccc} & X \times Y & \\ {\scriptstyle \pi_1} \swarrow & & \searrow {\scriptstyle \pi_2} \\ X & & Y \end{array}$$

Let $z \in X \times Y$, and set $z_1 = \pi_1(z)$ and $z_2 = \pi_2(z)$. Assume that $z_2 \in Y(k) = Y_{\mathrm{rig}}$.

(a) *Let V be an affinoid domain in $X \times Y$ such that $z \in V$. There exists an affinoid domain U in X (which contains z_1) such that if W is a small enough affinoid neighbourhood of z_2, then $V \cap (X \times W) = U \times W$.*

(b) *Let $\mathcal{V}$ be a neighbourhood of z. There exists an affinoid neighbourhood U of z_1 (resp. W of z_2) such that $\mathcal{V} \supseteq U \times W$.*

Proof. (a) [Sch94b, 2.2] Set $X = \mathcal{M}(\mathcal{A})$ and $Y = \mathcal{M}(\mathcal{B})$. First, using the Gerritzen–Grauert theorem, we can assume that V is a rational domain in $X \times Y$ defined by

$$V = \{x \in X \times Y \mid |f_i(x)| \leq |g(x)|,\ i = 1, \dots, n,\ |g(x)| \geq r\}$$

where $f_i, g \in \mathcal{A} \widehat{\otimes}_k \mathcal{B}$ and $r > 0$. Since we assume that $z_2 \in Y(k)$, it makes sense to evaluate the functions f_i, g at z_2, and we will denote by $f_{i\,z_2}$, g_{z_2} the corresponding functions, which we view as elements of $\mathcal{A}$ and of $\mathcal{A} \widehat{\otimes}_k \mathcal{B}$. In addition, since z_2 is a rigid point of Y, there exists an affinoid neighbourhood T of z_2 in Y such that

$$\forall i \sup_{x \in X \times T} |(f_i - f_{i\,z_2})(x)| < r, \tag{3.1}$$

$$\sup_{x \in X \times T} |(g - g_{z_2})(x)| < r. \tag{3.2}$$

Since $g = g_{z_2} + (g - g_{z_2})$, we conclude from (3.2) that if $x \in X \times T$, then

$$|g(x)| \geq r \ \Leftrightarrow\ |g_{z_2}(x)| \geq r. \tag{3.3}$$

Since also $g_i = g_{i\,z_2} + (g_i - g_{i\,z_2})$, from (3.1)–(3.3), we deduce that if $x \in X \times T$, then

$$\big(|g(x)| \geq r,\ |f_i(x) \leq |g(x)|\big) \ \Leftrightarrow\ \big(|g_{z_2}(x)| \geq r,\ |f_{i\,z_2}(x)| \leq |g_{z_2}(x)|\big).$$

Hence, if we set

$$U = \{x \in X \mid |(f_i)_{z_2}(x)| \leq |g_{z_2}(x)|,\ |g_{z_2}(x)| \geq r\},$$

then $V \cap (X \times T) = U \times T$. It then follows that if W is an affinoid domain in Y such that $W \subset T$, then $V \cap (X \times W) = U \times W$.

(b) We can assume that $\mathcal{V} = V$ is an affinoid neighbourhood of z. In (a), $V \cap (X \times W)$ is still a neighbourhood of z, since W is an affinoid neighbourhood of z_2 (because z_2 is a rigid point). If we denote by $s_{z_2} : X \to X \times Y$ the section of π_1 defined by $s_{z_2}(t) = (t, z_2)$, then

$$s_{z_2}^{-1}((V \cap (X \times W)) = s_{z_2}^{-1}(U \times W) = U$$

is an affinoid neighbourhood of x (since $s_{z_2}(x) = z$). Thus U is also an affinoid neighbourhood of z_1. □

Remark 3.2. Without the assumption that $z_2 \in Y(k)$ the previous corollary would be false. Take for instance $X = \mathcal{M}(k\{x\})$ and $Y = \mathcal{M}(k\{y\})$, and let $\varphi : \mathcal{M}(k\{t\}) \to X \times Y$ be defined by $\varphi(t) = (t, -t)$. Let η be the Gauss point of $\mathcal{M}(k\{t\})$ and $z := \varphi(\eta)$. Let $V = \{p \in \mathcal{M}(k\{x, y\}) \mid |(x + y)(p)| \leq 1/2\}$. It is a neighbourhood of z. However, $\pi_1(z)$ (resp. $\pi_2(z)$) is the Gauss point $z_1 = \eta_X$ of $\mathcal{M}(k\{x\})$ (resp. $z_2 = \eta_Y$ of $\mathcal{M}(k\{y\})$). It is then easy to see, according to the description of an affinoid domain in the unit disc as a Swiss cheese, that there does not exist an affinoid neighbourhood U (resp. W) of η_X (resp. η_Y) such that $V \supseteq U \times W$, for instance because in U there would necessarily exist a rigid point $x_0 \in \{x \in k \mid |x| \leq 1\}$ such that $\overline{x_0} = \overline{0}$ and in W a rigid point y_0 such that $\overline{y_0} = \overline{1}$ but $(x_0, y_0) \notin V$ (where $\overline{x}$ corresponds to the reduction of x in $\tilde{k}$).

Lemma 3.3. *Let $x \in X = \mathcal{M}(\mathcal{A})$, and let $f = \sum_{n \in \mathbb{N}} a_n T^n \in \mathcal{A}\{r^{-1}T\}$. Assume that $f_x \neq 0$. Then there exists an affinoid domain $V = \mathcal{M}(\mathcal{B})$ in X which contains x, $P \in \mathcal{B}[T]$, and a multiplicative unit $u \in \mathcal{B}\{r^{-1}T\}$ such that $f_{|V \times \mathbb{B}_r} = uP$.*

Proof. Since $f_x = \sum_{n \in \mathbb{N}} a_n(x) T^n \neq 0$, this series is distinguished of some order $s \geq 0$. We recall that this means that $|a_s(x)| r^s = \| f_x \|$ and that s is the greatest with this property.

We now use Lemma 1.29 in our specific situation where the polyradius $\underline{r}$ is in fact the real number r. Hence we can introduce a finite subset $J \subseteq \mathbb{N}$ such that $s \in J$ and some series $\phi_n \in \mathcal{A}\{r^{-1}T\}$ for $n \in J$ satisfying $\|\phi_n\| < 1$ such that $f = \sum_{n \in J} a_n(X^n + \phi_n)$.

We then define V as the rational domain:

$$V = \left\{ z \in X \;\middle|\; |a_s(z)| = |a_s(x)| \text{ and } |a_i(z)| r^i \leq |a_s(x)| r^s \text{ for } i \in J \setminus \{s\} \right\}$$

and denote by $\mathcal{B}$ the affinoid algebra of V. It is then true that $x \in V$. Moreover, on $V = \mathcal{M}(\mathcal{B})$, one checks that a_s is a multiplicative unit, and that on $\mathcal{B}\{r^{-1}T\}$, f is distinguished of order s. One can then apply Weierstrass preparation (Corollary 1.28) to conclude. □

Remark 3.4. The previous result (Lemma 3.3) is false if we remove the assumption $f_x \neq 0$.

Indeed, let $0 < r < 1$, let $f \in k\{r^{-1}x\}$ be a function whose radius of convergence is exactly r, and assume that $\|f\| < 1$. Let $\mathcal{A} = k\{y, t\}$, $X = \mathcal{M}(\mathcal{A})$ the unit bidisc, p the rigid point of X corresponding to the origin, and consider

$$F(y, t, x) = y - tf(x) \in k\{y, t\}\{r^{-1}x\} = \mathcal{A}\{r^{-1}X\}.$$

Then we claim that there does not exist an affinoid domain $V = \mathcal{M}(\mathcal{B})$ in X containing p such that $F_{|V \times \mathbb{B}_r} = uP$ where u is a multiplicative unit of $\mathcal{B}\{r^{-1}T\}$ and $P \in \mathcal{B}[t]$.

Indeed, otherwise there would exist some closed bidisc V of radius $s = |\lambda| \in |k^\times|$ where $\lambda \in k^\times$, and some $P \in k\{s^{-1}y, s^{-1}t\}[x]$ and a multiplicative unit $u \in k\{s^{-1}y, s^{-1}t\}\{r^{-1}x\}$ such that

$$F_{|V\times\mathbb{B}_r} = uP. \tag{3.4}$$

Fix $t = \lambda$. Then we consider

$$G(y, x) = F(y, \lambda, x) = y - \lambda f(x) \in k\{y, r^{-1}x\}.$$

According to (3.4), $G_{\mathbb{B}_s\times\mathbb{B}_r} = u(y, \lambda, t)P(y, \lambda, t)$. Replacing y by y/λ and f by λf, we then obtain

$$G(y, x) = y - f(x) \in k\{y, r^{-1}x\}, \quad G = uP,$$

where $u \in k\{y, r^{-1}x\}$ is a multiplicative unit, $P \in k\{y\}[x]$ and f with $\|f\| < 1$ has radius of convergence exactly $r < 1$. This implies that if we set

$$S := \{(x, y) \in \mathbb{B}^2 \mid |x| \leq r \text{ and } y = f(x)\}$$

then

$$S = \{(x, y) \in \mathbb{B}^2 \mid |x| \leq r \text{ and } P(x, y) = 0\},$$

so S would be semianalytic in $\mathbb{B}^2$, but in Section 2, we exploited many times the fact that this is not the case. □

Lemma 3.5 (Local algebraisation of a function in a family of rings). *Let n be an integer, let $a_0, \ldots, a_n \in \{x \in k \mid |x| \leq 1\}$ and let $r_0, \ldots, r_n$ be positive real numbers. Let $Y \subseteq \mathcal{M}(k\{T\}) = \mathbb{B}$ be the Laurent domain defined by*

$$Y = \{y \in \mathcal{M}(k\{T\}) \mid |(T - a_0)(y)| \leq r_0 \text{ and } |(T - a_i)(y)| \geq r_i,\ i = 1, \ldots, n\},$$

and let $X = \mathcal{M}(\mathcal{A})$ be a k-affinoid space. Let $f \in \mathcal{O}(X \times Y)$, let $z \in X \times Y$ be such that $\pi_1(z) = x \in X(k)$, and set $y := \pi_2(z)$. Assume that $f_x \in \mathcal{H}(x) \otimes \mathcal{O}(Y) \simeq \mathcal{O}(Y)$ is non-zero.[7] *Then there exists an affinoid neighbourhood $V = \mathcal{M}(\mathcal{B})$ of x, and $Y' \subset Y$ defined by*

$$Y' = \{y \in \mathcal{M}(k\{T\}) \mid |(T - b_0)(y)| \leq s_0 \text{ and } |(T - b_i)(y)| \geq s_i,\ i = 1, \ldots, m\},$$

which is an affinoid neighbourhood of y such that

$$f_{|V\times Y'} = (uP)_{|V\times Y'}$$

where the s_i's are positive real numbers, $b_i \in k^\circ$, u is a multiplicative unit of $V \times Y'$ and $P \in \mathcal{B}[T, (T - b_1)^{-1}, \ldots, (T - b_m)^{-1}]$.

Remark 3.6. Let us mention that in the proof we distinguish two very different cases.

1. If y is a rigid point then Y' can in fact be chosen to be a closed ball, i.e. $m = 0$.
2. Otherwise, if y is not a rigid point, then in fact $s_0 = r_0$, that is, we do not have to decrease the radius of the ambient closed ball, but we may have to remove some open balls.

[7] Here $\mathcal{H}(x) \simeq k$ because $x \in X(k)$.

Proof of Lemma 3.5. If y is a rigid point, we can indeed find a closed disc Y' which contains y, and the result follows from Lemma 3.3.

If y is not a rigid point, then $f_x \in \mathcal{H}(x) \otimes \mathcal{O}(Y) \simeq \mathcal{O}(Y)$. Hence by classical results on factorization of functions on rational domains in the closed disc (cf. [FvdP04, 2.2.9]), there exist $\alpha_1, \ldots, \alpha_N \in k$, $d_1, \ldots, d_N \in \mathbb{N}$, and an invertible function $g \in \mathcal{O}(Y)$ such that

$$f_x = \prod_{i=1}^{N} (T - \alpha_i)^{d_i} g. \tag{3.5}$$

We then set $m = n + N$, $b_i = a_i$ and $s_i = r_i$ for $i = 0, \ldots, n$, and $b_{n+j} = \alpha_j$ for $j = 1, \ldots, N$, and we take s_{n+j} small enough so that $\{z \in Y \mid |T - \alpha_j|(z) \geq s_{n+j}\}$ is a neighbourhood of y (this is possible because y is not a rigid point). Then we define

$$Y' := \{y \in \mathcal{M}(k\{T\}) \mid |(T - b_0)(y)| \leq s_0 \text{ and } |(T - b_i)(y)| \geq s_i,\ i = 1, \ldots, m\}.$$

Next, we set

$$G = f \prod_{i=1}^{N} (T - \alpha_i)^{-d_i} \in \mathcal{O}(X \times Y').$$

Then, according to (3.5), $G_x = g$, which does not vanish on Y'_x. So there exists an affinoid neighbourhood $V = \mathcal{M}(\mathcal{B})$ of x such that G is invertible on $V \times Y'$, because the locus of points x where G_x is invertible is open. Now using the explicit description of $\mathcal{O}(V \times Y')$, we can write

$$G = \sum_{\nu = (\nu_0, \ldots, \nu_m) \in \mathbb{N}^{m+1}} b_\nu (T - b_0)^{\nu_0} (T - b_1)^{-\nu_1} \ldots (T - b_m)^{-\nu_m}.$$

Now for $M \geq 0$ set

$$G_M = \sum_{|\nu| \leq M} b_\nu (T - b_0)^{\nu_0} (T - b_1)^{-\nu_1} \ldots (T - b_m)^{-\nu_m}.$$

By definition, $G_M \in \mathcal{B}[T, (T - b_1)^{-1}, \ldots, (T - b_m)^{-1}]$. In addition, $G_M \to G$ as $M \to \infty$, so G_M is invertible for M large enough. For such an M,

$$G = G_M + (G - G_M) = G_M(1 + G_M^{-1}(G - G_M)). \tag{3.6}$$

Moreover, if we take M still larger, we can assume that $\|G_M^{-1}\| = \|G^{-1}\|$, and so

$$\|G_M^{-1}(G - G_M)\| \xrightarrow[M \to \infty]{} 0.$$

Thus, for M large enough, if we set

$$u_M = 1 + G_M^{-1}(G - G_M),$$

then u_M is a multiplicative unit, and according to (3.6),

$$f = G_M u_M \prod_{i=1}^{N} (T - \alpha_i)^{d_i}.$$

We then set $u := u_M$ and $P := G_M \prod_{i=1}^{N} (T - \alpha_i)^{d_i}$ to conclude. $\square$

3.2. Blowing up

From now on, X will be a quasi-smooth k-analytic space of dimension 2. We now make two simple remarks that we will use in the proof of Theorem 3.12 below.

Lemma 3.7. *Let $\mathcal{A}$ be a k-affinoid algebra, $X = \mathcal{M}(\mathcal{A})$, $0 < r < s$ and $h \in \mathcal{A}$.*

(1) *Consider the Weierstrass domain*

$$V = \{x \in X \mid |h(x)| \leq s\}$$

in X and let S be a locally semianalytic subset of V such that

$$S \subseteq \{x \in X \mid |h(x)| \leq r\}.$$

Then S is also a locally semianalytic subset of X.

(2) *Consider the Laurent domain*

$$V = \{x \in X \mid |h(x)| \geq r\}$$

in X and let S be a locally semianalytic subset of V such that

$$S \subseteq \{x \in X \mid |h(x)| \geq s\}.$$

Then S is also a locally semianalytic subset of X.

Proof. Choose a real number t such that $r < t < s$.

(1) Set $W = \{x \in X \mid t \leq |h(x)|\}$. Then $\{V, W\}$ is a wide covering of X, and $S \cap V$ is by hypothesis locally semianalytic in V, and by assumption, $S \cap W = \emptyset$, so it is also locally semianalytic in W, hence S is locally semianalytic in X.

(2) Likewise, set $W = \{x \in X \mid |h(x)| \leq t\}$. Then $\{V, W\}$ is a wide covering of X, $S \cap V$ is locally semianalytic in V, and $S \cap W = \emptyset$, so S is locally semianalytic in X. □

This lemma will be used jointly with the following remark:

Remark 3.8. Let $X = \mathcal{M}(\mathcal{A})$ be a k-affinoid space, $f, g \in \mathcal{A}$, $0 < s < r$ and

$$(Z, S) \xrightarrow{\varphi} X$$

the elementary constructible datum given by $Z = \mathcal{M}(\mathcal{B})$ where $\mathcal{B} = \mathcal{A}\{r^{-1}t\}/(f - tg)$ and

$$S = \{z \in Z \mid |f(z)| \leq s|g(z)| \neq 0\}.$$

Moreover, let $(Y, U) \overset{\psi}{\dashrightarrow} (Z, S)$ be a constructible datum.

Case A. Assume $g \mid f$, so there exists $h \in \mathcal{A}$ such that $f = gh$. Let $\mathcal{C} = \mathcal{A}\{r^{-1}t\}/(h - t)$ and $V = \mathcal{M}(\mathcal{C})$. Note that V is the Weierstrass domain in X defined by

$$V = \{x \in X \mid |h(x)| \leq r\}.$$

Let β be the immersion of the affinoid domain V in X, and let

$$T = \{x \in V \mid |h(x)| \leq s,\ g(x) \neq 0\}.$$

Since $f - tg = g(h - t)$, we have $(h - t) \mid (f - tg)$, and there is a closed immersion $V \xrightarrow{\alpha} Z$. Moreover, $\alpha(T) = S$.

Indeed, $\alpha(T) \subseteq S$ by their respective definitions. Conversely, if $z \in S$, then $(f - tg)(z) = 0 = g(z)(h - t)(z)$, but since $g(z) \neq 0$, we have $(h - t)(z) = 0$, which implies that $z \in V$, and by the definition of S, it follows that $z \in \alpha(T)$.

Consider the cartesian diagram of k-germs

$$\begin{array}{ccccc} (Y, U) & \overset{\psi}{\dashrightarrow} & (Z, S) & \overset{\varphi}{\longrightarrow} & X \\ \alpha' \uparrow & & \alpha \uparrow & \nearrow \beta & \\ (Y', U') & \overset{\psi'}{\dashrightarrow} & (V, T) & & \end{array}$$

Here, $(Y', U') \overset{\psi'}{\dashrightarrow} (V, T)$ is still a constructible datum according to Corollary 1.12. Since $\alpha(T) = S$, it follows that $\alpha(\psi'(U')) = \psi(U)$, so

$$\varphi(\psi(U)) = \varphi(\alpha(\psi'(U'))) = \beta(\psi'(U')). \tag{3.7}$$

Roughly speaking, we started with the constructible datum

$$(Y, U) \overset{\psi}{\dashrightarrow} (Z, S) \xrightarrow{\varphi} X$$

such that the elementary constructible datum of φ was defined with functions f and g such that $g \mid f$. And we have been able to replace φ by the constructible datum

$$(Y', U') \overset{\psi'}{\dashrightarrow} (V, T) \xrightarrow{\beta} X$$

where V is a Weierstrass domain. Note moreover that T and so also $\psi'(U')$ satisfy the hypothesis of Lemma 3.7(1).

Case B. If $f \mid g$, there exists $h \in \mathcal{A}$ such that $g = fh$. Let $\mathcal{C} = \mathcal{A}\{r^{-1}t\}/(1 - th)$ and $V = \mathcal{M}(\mathcal{C})$. Note that V is the Laurent domain in X defined by

$$V = \{x \in X \mid |h(x)| \geq 1/r\}.$$

Let β be the immersion of the Laurent domain V in X, and let

$$T = \{x \in V \mid |h(x)| \geq 1/s,\ g(x) \neq 0\}.$$

Since $(1 - th) \mid (f - tg)$, there is a closed immersion $V \xrightarrow{\alpha} Z$. Moreover, $\alpha(T) = S$.

We then consider the cartesian diagram of k-germs

$$\begin{array}{ccccc} (Y, U) & \overset{\psi}{\dashrightarrow} & (Z, S) & \overset{\varphi}{\longrightarrow} & X \\ \uparrow{\scriptstyle \alpha'} & & \uparrow{\scriptstyle \alpha} & \nearrow{\scriptstyle \beta} & \\ (Y', U') & \overset{\psi'}{\dashrightarrow} & (V, T) & & \end{array}$$

Here, $(Y', U') \overset{\psi'}{\dashrightarrow} (V, T)$ is still a constructible datum. Since $\alpha(T) = S$, it follows that $\alpha(\psi'(U')) = \psi(U)$, so

$$\varphi(\psi(U)) = \varphi(\alpha(\psi'(U'))) = \beta(\psi'(U')). \tag{3.8}$$

In this case, we started with the constructible datum $(Y, U) \overset{\psi}{\dashrightarrow} (Z, S) \overset{\varphi}{\to} X$ such that $f \mid g$, and we have been able to replace it by the constructible datum $(Y', U') \overset{\psi'}{\dashrightarrow} (V, T) \overset{\beta}{\to} X$ where V is a Laurent domain in X. Note moreover that T and so also $\psi'(U')$ satisfy the hypothesis of Lemma 3.7(2).

Remark 3.9. We are going to use some blowing-up of k-analytic spaces in the following context: X will be a quasi-smooth k-analytic space of dimension 2, and we will blow up a rigid point p of X. In particular, the resulting blowing-up $\tilde{X}$ will still be quasi-smooth. To give a precise description of the situation, since k is algebraically closed, we can assume that $X = \mathbb{B}^2$ and p is the origin. The blowing-up can then be described with two charts as follows. We consider

$$\pi_1 : X_1 = \mathcal{M}(k\{x, t_1\}) \to \mathbb{B}^2 = \mathcal{M}(k\{x, y\}), \quad (x, t_1) \mapsto (x, t_1 x),$$
$$\pi_2 : X_2 = \mathcal{M}(k\{y, t_2\}) \to \mathbb{B}^2 = \mathcal{M}(k\{x, y\}), \quad (y, t_2) \mapsto (t_2 y, y).$$

Then $\widetilde{\mathbb{B}^2}$ is obtained by gluing X_1 and X_2 along the domains $U_1 = \{z \in X_1 \mid t_1(z) \neq 0\}$ and $U_2 = \{z \in X_2 \mid t_2(z) \neq 0\}$ via the isomorphism

$$U_1 \to U_2, \quad (x, t_1) \mapsto (x t_1, t_1^{-1}).$$

Proposition 3.10. *Let $X = \mathcal{M}(\mathcal{A})$ be a quasi-smooth k-affinoid space of dimension 2 and let $f, g \in \mathcal{A}$. Then there exists a succession of blowing-ups of rigid points $\pi : \tilde{X} \to X$ such that for all $x \in \tilde{X}$, $f_x \mid g_x$ or $g_x \mid f_x$. Note that $\tilde{X}$ is still quasi-smooth.*

Proof. We may assume that X is irreducible. If $f = 0$ or $g = 0$, there is nothing to prove, so we may assume that $f \neq 0$ and $g \neq 0$. Likewise, if $f = g$, there is nothing to do, so we may also assume that $f - g \neq 0$.

Let $h = fg(f - g)$. Hence, $h \neq 0$. We can find a succession of blowing-ups of rigid points $\pi : \tilde{X} \to X$ such that $\pi^*(h)$ is a normal crossing divisor. Indeed, the classical proof (see [Kol07, 1.8]) that works in the algebraic case, or the complex analytic case, can be translated verbatim in our context, and since we are dealing with a compact space, the local procedure of [Kol07, 1.8] needs only to be applied to a finite number of points. So let $x \in \tilde{X}$.

If x is not a rigid point, $\mathcal{O}_{\tilde{X},x}$ is a field or a discrete valuation ring and the result is clear.

Otherwise, if x is a rigid point, its local ring is a regular local ring of dimension 2. By assumption, $h = fg(f-g)$ is a normal crossing divisor, thus it can be written in $\mathcal{O}_{\tilde{X},x}$ as

$$(fg(f-g))_x = u\xi_1^n \xi_2^m \tag{3.9}$$

where ξ_1, ξ_2 is a system of local parameters around x and u is a unit in $\mathcal{O}_{\tilde{X},x}$. Dividing by the common divisor of f_x and g_x in $\mathcal{O}_{\tilde{X},x}$, we can assume for instance that $f_x = v\xi_1^p$ and $g_x = w\xi_2^q$ and $f_x - g_x = z\xi_1^a \xi_2^b$ where v, w and z are units of $\mathcal{O}_{\tilde{X},x}$.

If $p > 0$ then modulo ξ_1 we obtain $f = 0$, so $f_x - g_x = w\xi_2^q$ modulo ξ_1. This implies that $a = 0$ and that $b = q$. So $f_x = (f_x - g_x) + g_x$ is divisible by ξ_2^q, and this implies that $q = 0$. So g_x is invertible and $g_x \mid f_x$.

And if $p = 0$, then f_x is invertible, so $f_x \mid g_x$. □

Lemma 3.11. *Let X be a good quasi-smooth strictly k-analytic space of dimension* 2.

(1) *Let $q \in X_{\mathrm{rig}}$ and $\pi : \tilde{X} \to X$ the blowing-up of X at q, and let $S \subseteq \tilde{X}$ be a locally semianalytic subset. Then $\pi(S)$ is locally semianalytic.*
(2) *If $\pi : \tilde{X} \to X$ is a succession of blowing-ups of rigid points, and $S \subseteq \tilde{X}$ is locally semianalytic, then $\pi(S)$ is also locally semianalytic.*

Proof. (2) is a consequence of (1) so we only have to show (1).

The problem is local on X, and since outside q, π is a local isomorphism, we can restrict to an affinoid neighbourhood of q; moreover, since X is regular at q, we can assume that $X = \mathbb{B}^2$ and q is the origin.

Then $\pi : \tilde{X} \to X$ can be described with two charts, one of them being

$$\pi_1 : X_1 = \mathcal{M}(k\{x,t\}) \to X = \mathcal{M}(k\{x,y\}), \qquad (x,t) \mapsto (x,tx).$$

The other chart being analogous, we only consider π_1. Now, changing S to $S \cap X_1$, it suffices to show that if S is locally semianalytic in X_1, so is $\pi_1(S)$. Since π_1 induces an isomorphism between $X_1 \setminus V(x)$ and $\{p \in \mathbb{B}^2 \mid |y(p)| \leq |x(p)| \neq 0\}$, we only have to show that $\pi_1(S)$ is semianalytic around q, the origin of $\mathbb{B}^2$.

Now if for each $p \in \mathbf{E} := V(x) \subseteq X_1$ we can find an affinoid neighbourhood V_p of p and $\varepsilon_p > 0$ such that $\pi_1(V_p \cap S) \cap \mathbb{B}^2_{\varepsilon_p}$ is semianalytic in $\mathbb{B}^2_{\varepsilon_p} \subseteq X$, then by compactness of $\mathbf{E}$, we can extract a finite covering $V_1, \dots, V_n$ of $\mathbf{E}$ and $\varepsilon > 0$ such that

$$\bigcup_{i=1}^{n} (\pi_1(V_i \cap S)) \cap \mathbb{B}^2_\varepsilon = \pi_1(S) \cap \mathbb{B}^2_\varepsilon$$

is semianalytic in $\mathbb{B}^2_\varepsilon$. So we fix $p \in \mathbf{E} = V(x)$ and try to find an affinoid neighbourhood V of p and $\varepsilon > 0$ such that $\pi_1(V \cap S) \cap \mathbb{B}^2_\varepsilon$ is semianalytic in $\mathbb{B}^2_\varepsilon$.

Since S is locally semianalytic in X_1, we can find an affinoid neighbourhood V of p such that $V \cap S$ is semianalytic in V. According to Corollary 3.1, we can assume that[8] $V = \mathbb{B}_\varepsilon \times W$ where

$$W = \{w \in \mathcal{M}(k\{t\}) \mid |(t-a_0)(w)| \leq r_0 \text{ and } |(t-a_i)(w)| \geq r_i,\ i = 1, \dots, n\}$$

for some $a_0, \dots, a_n \in k^\circ$ and $r_0, \dots, r_n \in \mathbb{R}_+$.

To simplify the notation, we can also assume that the semianalytic subset S of V we are dealing with is of the following form:

$$S = \bigcap_{j=1}^{m} \{v \in V \mid |f_j(v)| \diamondsuit_j |g_j(v)|\}.$$

Now recall that $V = \mathbb{B}_\varepsilon \times W$ with $\mathbb{B}_\varepsilon = \mathcal{M}(k\{\varepsilon^{-1}x\})$. So we can factor each f_j and g_j by the greatest power of x which is a factor, hence introduce some integers b_j, c_j such that

$$S = \bigcap_{j=1}^{m} \{v \in V \mid |x^{b_j} \tilde{f}_j(v)| \diamondsuit_j |x^{c_j} \tilde{g}_j(v)|\}$$

where the series $\tilde{f}_j(0,t)$ and $\tilde{g}_j(0,t)$ are non-zero, and $f_j = x^{b_j}\tilde{f}_j$, $g_j = x^{c_j}\tilde{g}_j$. To simplify the notation, we will use f_j (resp. g_j) instead of $\tilde{f}_j$ (resp. $\tilde{g}_j$), so that

$$S = \bigcap_{j=1}^{m} \{v \in V \mid |x^{b_j} f_j(v)| \diamondsuit_j |x^{c_j} g_j(v)|\}$$

where the series $f_j(0,t)$ and $g_j(0,t)$ are non-zero.

Then according to Lemma 3.5 we can decrease ε and W so that for each $f_j, g_j \in \{f_1, \dots, f_m, g_1, \dots, g_m\}$, $f_j = u_j P_j$ (resp. $g_j = v_j Q_j$) where u_j (resp. v_j) is a multiplicative unit, and P_j (resp. Q_j) $\in k\{\varepsilon^{-1}x\}[t, (t-a_1)^{-1}, \dots, (t-a_n)^{-1}]$.

In other words, and with different notation, there exists an integer N such that $f_j = u_j \cdot P_j/((t-a_1)\dots(t-a_n))^N$ where u_j is a multiplicative unit and $P_j \in k\{\varepsilon^{-1}x\}[t]$ (and resp. $g_j = v_j \cdot Q_j/((t-a_1)\dots(t-a_n))^N$). Hence

$$\begin{aligned} & |f_j(v)| \diamondsuit_j |g_j(v)| \\ \Leftrightarrow\ & \left| u_j(v) \frac{P_j(v)}{((t-a_1)\dots(t-a_n))^N(v)} \right| \diamondsuit_j \left| v_j(v) \frac{Q_j(v)}{((t-a_1)\dots(t-a_n))^N(v)} \right| \\ \Leftrightarrow\ & |P_j(v)| \diamondsuit_j \lambda_j |Q_j(v)| \end{aligned}$$

where

$$\lambda_j = \|v_j\| / \|u_j\| \in |k^\times|.$$

Moreover,

$$S \cap V = (S \cap \{v \in V \mid x(v) = 0\}) \cup (S \cap \{v \in V \mid x(v) \neq 0\})$$

and $\pi_1(\{v \in V \mid x(v) = 0\}) = q$, the origin of $\mathbb{B}^2$.

[8] Here we use the explicit description of affinoid domains in $\mathbb{B}$.

So, by adding if necessary the origin to $\pi_1(S \cap \{v \in V \mid v(x) \neq 0\})$ (which will not change the fact that it is semianalytic), it suffices to show that $\pi_1(S \cap \{v \in V \mid v(x) \neq 0\})$ is semianalytic around the origin. Moreover, since on $\{v \in V \mid v(x) \neq 0\}$, π_1 is bijective, the following holds:

$$\pi_1\Big(\bigcap_{j=1}^{m}\{v \in V \mid |x^{b_j} f_j(v)| \lozenge_j |x^{c_j} g_j(v)|\} \cap \{v \in V \mid x(v) \neq 0\}\Big)$$
$$= \bigcap_{j=1}^{m} \pi_1\big(\{v \in V \mid |x^{b_j} f_j(v)| \lozenge_j |x^{c_j} g_j(v)|\}\big) \cap \{v \in V \mid x(v) \neq 0\}.$$

Now since $y = tx$ and $P_j \in k\{\varepsilon^{-1}x\}[t]$ there exists an integer $M \geq 0$ such that $x^M P_j(x,t) \in k\{\varepsilon^{-1}x\}[tx] = k\{\varepsilon^{-1}x\}[y]$, i.e. $x^M P_j(x,t) = \pi^*(\tilde{P}_j(x,y))$ for some $\tilde{P}_j(x,y) \in k\{\varepsilon^{-1}x\}[y]$, and such that $x^M Q_j(x,t) \in k\{\varepsilon^{-1}x\}[y]$, i.e. $x^M Q_j(x,t) = \pi^*(\tilde{Q}_j(x,y))$ for some $\tilde{Q}_j(x,y) \in k\{\varepsilon^{-1}x\}[y]$.

Now on $\{v \in V \mid v(x) \neq 0\}$,

$$|x^{b_j} f_j(v)| \lozenge_j |x^{c_j} g_j(v)| \Leftrightarrow |x^{M+b_j} f_j(v)| \lozenge_j |x^{M+c_j} g_j(v)|$$
$$\Leftrightarrow |x^{b_j} \tilde{P}_j(\pi_1(v))| \lozenge_j \lambda_j |x^{c_j} \tilde{Q}_j(\pi_1(v))|.$$

From this we conclude that

$$z \in \pi_1\Big(\bigcap_{j=1}^{m}\{v \in V \mid |x^{b_j} f_j(v)| \lozenge_j |x^{c_j} g_j(v)|\} \cap \{v \in V \mid x(v) \neq 0\}\Big)$$
$$\Leftrightarrow z \in \bigcap_{j=1}^{m}\{z \in \pi_1(V) \mid |x^{b_j} \tilde{P}_j(z)| \lozenge_j |x^{c_j} \tilde{Q}_j(z)|\} \cap \{z \in X \mid x(z) \neq 0\}.$$

Since $\pi_1(\mathbb{B}^2_\varepsilon)$ is contained in $\mathbb{B}^2_\varepsilon$ and is semianalytic in $\mathbb{B}^2_\varepsilon$, we conclude that the set $\pi_1(S \cap \{v \in V \mid v(x) \neq 0\})$ is semianalytic in $\mathbb{B}^2_\varepsilon$, which ends the proof. □

Theorem 3.12. *Let X be a good quasi-smooth strictly k-analytic space of dimension 2 with k algebraically closed, and $S \subseteq X$. Then S is overconvergent subanalytic if and only if it is locally semianalytic.*

Proof. Since the problem is local, we can assume that X is affinoid and that $S = \varphi(U)$ where $(Y,U) \overset{\varphi}{\dashrightarrow} X$ is a constructible datum, and just check that S is locally semianalytic. We do it by induction on the complexity of φ. So let $(Y,U) \overset{\varphi}{\dashrightarrow} X$ be a constructible datum, which we decompose as

$$\varphi = (Y,U) \overset{\psi}{\dashrightarrow} Z \overset{\chi}{\to} X$$

where χ is an elementary constructible datum, and ψ a constructible datum whose complexity is one less than that of φ. So we can introduce $f, g \in \mathcal{A}$ and $0 < s < r$ such that $Z = \mathcal{M}(\mathcal{A}\{r^{-1}t\}/(f - tg))$. According to Proposition 3.10, we can find a succession of blowing-ups of rigid points $\pi : \tilde{X} \to X$ such that for all $x \in \tilde{X}$, $f_x \mid g_x$ or $g_x \mid f_x$.

According to Remark 3.9, $\tilde{X}$ is still quasi-smooth. This gives us the cartesian diagram

$$\begin{array}{ccc} (Y,U) & \overset{\varphi}{\dashrightarrow} & X \\ \uparrow{\scriptstyle \pi'} & & \uparrow{\scriptstyle \pi} \\ (Y',U') & \overset{\varphi'}{\dashrightarrow} & \tilde{X} \end{array}$$

Then $\varphi(U) = \pi(\varphi'(U'))$. Moreover, since $\tilde{X}$ is compact, we can find a finite wide covering $\{X_i\}_{i=1}^n$ of $\tilde{X}$ by affinoid domains such that for all i, $f_{|X_i} \mid g_{|X_i}$ or $g_{|X_i} \mid f_{|X_i}$. We denote by $\pi_i : X_i \to X$ the composition of the embedding of the affinoid domain $X_i \to \tilde{X}$ with $\pi : \tilde{X} \to X$. This gives the cartesian diagrams

$$\begin{array}{ccccc} (Y,U) & \overset{\psi}{\dashrightarrow} & Z & \overset{\chi}{\longrightarrow} & X \\ \uparrow{\scriptstyle \pi_i''} & & \uparrow{\scriptstyle \pi_i'} & & \uparrow{\scriptstyle \pi_i} \\ (Y_i,U_i) & \overset{\psi_i}{\dashrightarrow} & Z_i & \overset{\chi_i}{\longrightarrow} & X_i \end{array}$$

Then

$$\varphi(U) = \pi(\varphi'(U')) = \pi\Big(\bigcup_{i=1}^n \chi_i(\psi_i(U_i))\Big).$$

But $(Y_i, U_i) \overset{\psi_i}{\dashrightarrow} Z_i$ is a constructible datum of lower complexity than that of φ, so that we would like to use our induction hypothesis, and claim that $\psi_i(U_i)$ is locally semianalytic. However, Z_i is not necessarily quasi-smooth so we cannot do that. But since $f_{|X_i} \mid g_{|X_i}$ or $g_{|X_i} \mid f_{|X_i}$, according to Remark 3.8, we can in fact replace Z_i by a Weierstrass (or a Laurent) domain in X_i, and hence assume that Z_i is quasi-smooth. Thus by induction hypothesis, $\psi_i(U_i)$ is locally semianalytic in Z_i.

Next we use Lemma 3.7 to assert that $\chi_i(\psi_i(U_i))$ is locally semianalytic in X_i. So

$$\varphi'(U') = \bigcup_{i=1}^n \chi_i(\psi_i(U_i))$$

is locally semianalytic in $\tilde{X}$, since $\{X_i\}$ was a wide covering of $\tilde{X}$. Finally, according to Lemma 3.11, $\pi(\varphi'(U')) = S$ is also locally semianalytic. □

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