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Noncritical holomorphic functions on Stein spaces

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Abstract. We prove that every reduced Stein space admits a holomorphic function without critical points. Furthermore, every closed discrete subset of a reduced Stein space X is the critical locus of a holomorphic function on X. We also show that for every complex analytic stratification with nonsingular strata on a reduced Stein space there exists a holomorphic function whose restriction to every stratum is noncritical. These results provide some information on critical loci of holomorphic functions on desingularizations of Stein spaces. In particular, every 1-convex manifold admits a holomorphic function that is noncritical outside the exceptional variety.

Keywords. Holomorphic functions, critical points, Stein manifolds, Stein spaces, 1-convex manifolds, stratifications

1. Introduction

Every Stein manifold X admits a holomorphic function $f \in \mathcal{O}(X)$ without critical points: see Gunning and Narasimhan [26] for the case of open Riemann surfaces and [14] for the general case. In the algebraic category this fails on any compact Riemann surface of genus $g \geq 1$ with a puncture (every algebraic function on such a surface has a critical point, as follows from the Riemann–Hurwitz theorem), but it holds for holomorphic functions of finite order [17]. Noncritical holomorphic functions are of interest in particular since they define nonsingular holomorphic hypersurface foliations; results on this topic can be found in [14, 15].

In this paper we prove that, somewhat surprisingly, the same holds on Stein spaces. The following is a special case of our main result, Theorem 1.3.

Theorem 1.1. *Every reduced Stein space admits a holomorphic function without critical points.*

We begin by recalling the relevant notions. All complex spaces are assumed to be paracompact and reduced. For the theory of Stein spaces we refer to Grauert and Remmert [25].

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Let X be a complex space. Denote by $\mathcal{O}_{X,x}$ the ring of germs of holomorphic functions at $x \in X$ and by $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_{X,x}$, so $\mathcal{O}_{X,x}/\mathfrak{m}_x \cong \mathbb{C}$. Given $f \in \mathcal{O}_{X,x}$ we denote by $f - f(x) \in \mathfrak{m}_x$ the germ obtained by subtracting from f its value $f(x) \in \mathbb{C}$.

Definition 1.2. Assume that x is a nonisolated point of a complex space X.

(a) A germ $f \in \mathcal{O}_{X,x}$ at x is said to be *critical* (and x is a *critical point* of f) if $f - f(x) \in \mathfrak{m}_x^2$, and is *noncritical* if $f - f(x) \in \mathfrak{m}_x \setminus \mathfrak{m}_x^2$.
(b) A germ $f \in \mathcal{O}_{X,x}$ is *strongly noncritical at* x if the germ at x of the restriction $f|_V$ to any local irreducible component V of X is noncritical.

Any function is considered (strongly) noncritical at any isolated point of X.

One can characterize these notions by the (non)vanishing of the differential df_x on the *Zariski tangent space* T_xX. Recall that T_xX is isomorphic to $(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$, the dual of $\mathfrak{m}_x/\mathfrak{m}_x^2$, the latter being the cotangent space T_x^*X (cf. [10, p. 78] or [30, p. 111]). The number $\dim_{\mathbb{C}} T_xX$ is the embedding dimension of the germ X_x of X at x. The differential $df_x\colon T_xX \to \mathbb{C}$ of $f \in \mathcal{O}_{X,x}$ is determined by the class $f - f(x) \in \mathfrak{m}_x/\mathfrak{m}_x^2 = T_x^*X$, so f is critical at x if and only if $df_x = 0$. If $X_x = \bigcup_{j=1}^k V_j$ is a decomposition into local irreducible components, then f is strongly noncritical at x if $df_x\colon T_xV_j \to \mathbb{C}$ is nonvanishing for every $j = 1, \dots, k$.

At a regular point $x \in X_{\mathrm{reg}}$ these notions coincide with the usual one: x is a critical point of f if and only if in some (hence in any) local holomorphic coordinates $z = (z_1, \dots, z_n)$ on a neighborhood of x, with $z(x) = 0$ and $n = \dim_x X$, we have $\frac{\partial f}{\partial z_j}(0) = 0$ for $j = 1, \dots, n$. Hence the set $\mathrm{Crit}(f)$ of all critical points of a holomorphic function on a complex manifold X is a closed complex subvariety of X. On a Stein manifold X this set is discrete for a generic choice of $f \in \mathcal{O}(X)$.

The following is our main result.

Theorem 1.3. *On every reduced Stein space X there exists a holomorphic function which is strongly noncritical at every point. Furthermore, given a closed discrete set $P = \{p_1, p_2, \dots\}$ in X, germs $f_k \in \mathcal{O}_{X,p_k}$ and integers $n_k \in \mathbb{N}$, there exists a holomorphic function $F \in \mathcal{O}(X)$ which is strongly noncritical on $X \setminus P$ and agrees with the germ f_k to order n_k at $p_k \in P$ (i.e., $F_{p_k} - f_k \in \mathfrak{m}_{p_k}^{n_k}$) for every $k = 1, 2, \dots$.*

The hypothesis on P in Theorem 1.3 is a natural one since the critical locus a generic holomorphic function on a Stein space is discrete (see Corollary 2.11).

We also prove the following result (cf. Theorem 5.1 and Corollary 5.2). Given a closed complex subvariety X' of a reduced Stein space X and a function $f \in \mathcal{O}(X')$, there exists $F \in \mathcal{O}(X)$ such that $F|_{X'} = f$ and F is strongly noncritical on $X \setminus X'$, or it has critical points in a prescribed discrete set contained in $X \setminus X'$.

The proof of these results for Stein manifolds in [14] relies on two main ingredients:

(i) the Runge approximation theorem for noncritical holomorphic functions on a polynomially convex subset of $\mathbb{C}^n$ by entire noncritical functions (cf. [14, Theorem 3.1] or [16, Theorem 8.11.1, p. 381]), and
(ii) a splitting lemma for biholomorphic maps close to the identity on a Cartan pair (cf. [14, Theorem 4.1] or [16, Theorem 8.7.2)].

These tools do not apply directly at singular points of X. In addition, the following two phenomena make the analysis very delicate.

Firstly, the critical locus of a holomorphic function $f \in \mathcal{O}(X)$ need not be a closed complex subvariety of X near a singularity. A simple example is $X = \{zw = 0\} \subset \mathbb{C}^2_{(z,w)}$ and $f(z,w) = z$ with $\mathrm{Crit}(f) = \{(0,w)\colon w \neq 0\}$; for another example on an irreducible isolated surface singularity see Example 2.2 in §2. However, we will show that $\mathrm{Crit}(f|X_{\mathrm{reg}}) \cup X_{\mathrm{sing}}$ is a closed complex subvariety of X (cf. Lemma 2.4).

Secondly, the class of noncritical (or strongly noncritical) functions is not stable under small perturbations on compact sets which include singular points of X: see Example 2.3.

The key idea used in this paper stems from the following observation:

($*$) *If $S \subset X$ is a local complex submanifold of positive dimension at a point $x \in S$, and if the restriction of a function $f \in \mathcal{O}(X)$ to S is noncritical at x, then f is noncritical at x (as a function on X). If such an S is contained in every local irreducible component of X at x, then f is strongly noncritical at x.*

This observation naturally leads one to consider complex analytic stratifications of a Stein space and to construct holomorphic functions that are noncritical on every stratum.

Recall that a (complex analytic) *stratification* $\Sigma = \{S_j\}$ of a complex space X is a subdivision of X into the union $\bigcup_j S_j$ of at most countably many pairwise disjoint connected complex manifolds S_j, called the *strata* of Σ, such that

- every compact set in X intersects at most finitely many strata, and
- $bS = \overline{S} \setminus S$ is a union of lower-dimensional strata for every $S \in \Sigma$.

Such a pair (X, Σ) is called a *stratified complex space*.

Every complex analytic space admits a stratification (cf. Whitney [40, 41]). An example is obtained by taking $X = X_0 \supset X_1 \supset \cdots$, where $X_{j+1} = (X_j)_{\mathrm{sing}}$ for every j, and decomposing the smooth differences $X_j \setminus X_{j+1}$ into connected components. This chain of subvarieties is stationary on each compact subset of X.

Definition 1.4. Let (X, Σ) be a stratified complex space. A function $f \in \mathcal{O}(X)$ is said to be a *stratified noncritical holomorphic function* on (X, Σ), or a Σ*-noncritical function*, if the restriction $f|_S$ to any stratum $S \in \Sigma$ of positive dimension is noncritical on S.

Clearly the critical locus of a Σ-noncritical function on (X, Σ) is contained in the union X_0 of all zero-dimensional strata of Σ; note that X_0 is a discrete subset of X.

Theorem 1.5. *On every stratified Stein space (X, Σ) there exists a Σ-noncritical holomorphic function $F \in \mathcal{O}(X)$. Furthermore, F can be chosen to agree to a given order $n_x \in \mathbb{N}$ with a given germ $f_x \in \mathcal{O}_{X,x}$ at any zero-dimensional stratum $\{x\} \in \Sigma$.*

Theorem 1.5 is proved in §5. Assuming it for the moment we give

Proof of Theorems 1.1 and 1.3. We may assume that X has no isolated points. Choose a complex analytic stratification Σ of X containing a given discrete set $P \subset X$ in the union $X_0 = \{p_1, p_2, \ldots\}$ of its zero-dimensional strata. For every $i = 1, 2, \ldots$ let X_i

denote the union of all strata of dimension at most i (the *i-skeleton* of Σ). Note that X_i is a closed complex subvariety of X (since the boundary of each stratum is a union of lower-dimensional strata), the difference $X_i \setminus X_{i-1}$ is either empty or a complex manifold of dimension i, and

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset \bigcup_{i=0}^{\infty} X_i = X. \tag{1.1}$$

Given germs $f_k \in \mathcal{O}_{X,p_k}$ ($p_k \in X_0$) and integers $n_k \in \mathbb{N}$, Theorem 1.5 furnishes a Σ-noncritical function $F \in \mathcal{O}(X)$ such that $F_{p_k} - f_{p_k} \in \mathfrak{m}_{p_k}^{n_k}$ for every $p_k \in X_0$. We claim that F is strongly noncritical on $X \setminus X_0$. Indeed, given a point $x \in X \setminus X_0$, pick the smallest $i \in \mathbb{N}$ such that $x \in X_i$, so $x \in X_i \setminus X_{i-1}$, which is a complex manifold of dimension i. Let $S_i \subset X_i \setminus X_{i-1}$ be the connected component containing x. Then the germ of S_i at x is contained in every local irreducible component of X at x. Since x is a noncritical point of $F|_{S_i}$, it follows from $(*)$ that F is strongly noncritical at x, thereby proving the claim. By choosing f_k to be strongly noncritical at $p_k \in X_0$ and $n_k \geq 2$ for every k we obtain a function $F \in \mathcal{O}(X)$ that is strongly noncritical on X. (To get a strongly noncritical function at a point $p \in X$, we can embed X_p as a local complex subvariety of the Zariski tangent space $T_pX \cong \mathbb{C}^N$ and choose a linear function on T_pX which is nondegenerate on the tangent space to every local irreducible component of X.) □

The proof of Theorem 1.5 (cf. §5) proceeds by induction on the skeleta X_i in (1.1). The main induction step is furnished by Theorem 4.1, which provides holomorphic functions on a Stein space that have no critical points in the regular locus. When passing from X_{i-1} to X_i, we first apply the transversality theorem to show that the critical locus of a generic holomorphic extension of a given function on X_{i-1} is discrete and does not accumulate on X_{i-1} (cf. Lemma 2.9). We then extend the function to X_i without creating any critical points in $X_i \setminus X_{i-1}$, keeping it fixed to a high order along X_{i-1}. To this end we adjust one of the main tools from [14], namely *the splitting lemma for biholomorphic maps close to the identity* on a Cartan pair [14, Theorem 4.1], to the setting of Stein spaces (see Theorem 3.2 below).

Besides its original use, this splitting lemma from [14] has found a variety of applications. In particular, it was used for exposing boundary points of certain classes of pseudoconvex domains, a technique applied in the constructions of proper holomorphic embeddings of open Riemann surfaces to $\mathbb{C}^2$ [20, 21, 32], in the construction of complete bounded complex curves in $\mathbb{C}^n$ and minimal surfaces in $\mathbb{R}^3$ [2, 3], and in the study of the *holomorphic squeezing function* of domains in $\mathbb{C}^n$ [6, 7, 11]. We hope that Theorem 3.2 of this paper will also prove useful for other purposes. As explained in Remark 3.9, Theorem 3.2 and its proof can be generalized to the case when the biholomorphic map to be decomposed, and possibly also the underlying Cartan pair, depend on some additional parameters.

We mention a couple of immediate corollaries of Theorem 1.5.

Corollary 1.6. *Let (X, Σ) be a stratified Stein space. Given a closed discrete set P in X, there exists a holomorphic function $F \in \mathcal{O}(X)$ such that for any stratum $S \in \Sigma$ with* $\dim S > 0$ *we have* $\mathrm{Crit}(F|_S) = P \cap S$.

This follows from Theorem 1.5 applied to a substratification Σ' of Σ which contains the given discrete set P in the zero-dimensional skeleton.

By considering the level sets of a function satisfying the conclusion of Theorem 1.5 we obtain the following.

Corollary 1.7. *Every stratified Stein space (X, Σ) admits a holomorphic foliation $\mathcal{L} = \{L_a\}_{a\in A}$ with closed leaves such that for every stratum $S \in \Sigma$ the restricted foliation $\mathcal{F}|_S = \{L_a \cap S\}_{a\in A}$ is a nonsingular hypersurface foliation on S.*

In the remainder of this introduction we indicate how Theorems 1.1, 1.3, and 1.5 imply results concerning critical loci of holomorphic functions on complex manifolds which are obtained by desingularizing Stein spaces.

The simplest example of this type is obtained by desingularizing a Stein space Y with isolated singular points, $Y_{\rm sing} = \{p_1, p_2, \ldots\}$. Let $\pi : X \to Y$ be a desingularization (cf. [4, 5, 28]). The fiber $E_j = \pi^{-1}(p_j)$ over any singular point of Y is a connected compact complex subvariety of X of positive dimension with negative normal bundle in the sense of Grauert [22]. (A local strongly plurisubharmonic function near $p_j \in Y$ pulls back to a function that is strongly plurisubharmonic on a deleted neighborhood of E_j.) The set $\mathcal{E} = \pi^{-1}(Y_{\rm sing}) = \bigcup_j E_j$ is a complex subvariety of X with compact irreducible components of positive dimension, and $\mathcal{E}$ contains any compact complex subvariety of X without zero-dimensional components. Furthermore, we have $\pi_*\mathcal{O}_X = \mathcal{O}_Y$ and $\pi : X \to Y$ is the *Remmert reduction* of X [22, 35]. If Y has only finitely many singular points then the manifold X is 1-*convex* and $\mathcal{E}$ is the *exceptional variety* of X [23]. By choosing a noncritical function $g \in \mathcal{O}(Y)$ furnished by Theorem 1.1, the function $f = g\circ\pi \in \mathcal{O}(X)$ clearly satisfies $\mathrm{Crit}(f) \subset \mathcal{E}$. Similarly, if A is a discrete set in X then $\pi(A)$ is discrete in Y, and by choosing $g \in \mathcal{O}(Y)$ with $\mathrm{Crit}(g) = \pi(A)$ we get a function $f = g \circ \pi \in \mathcal{O}(X)$ with $\mathrm{Crit}(f) \setminus \mathcal{E} = A \setminus \mathcal{E}$. If A intersects every connected component of $\mathcal{E}$, we have $\mathrm{Crit}(f) = A \cup \mathcal{E}$. This gives the following corollary.

Corollary 1.8. *A* 1*-convex manifold X with the exceptional variety $\mathcal{E}$ admits a holomorphic function $f \in \mathcal{O}(X)$ with* $\mathrm{Crit}(f) \subset \mathcal{E}$*. Furthermore, given a closed discrete set A in X, there exists $f \in \mathcal{O}(X)$ with* $\mathrm{Crit}(f) = A \cup \mathcal{E}$*.*

In general we cannot find a holomorphic function f on a 1-convex manifold X that is noncritical at every point of the exceptional variety $\mathcal{E}$ of X. Indeed, assume that E is a smooth component of $\mathcal{E}$. Since E is compact, $f|_E$ is constant, so the differential of f vanishes along E in the directions tangent to E. Hence, if $df_x \neq 0$ for all $x \in E$, the differential defines a nowhere vanishing section of the conormal bundle of E in X, a nontrivial condition which does not always hold, as is seen in the following example.

Example 1.9. Fix an integer $n > 1$. Let X be $\mathbb{C}^n$ blown up at the origin, and let $\pi : X \to \mathbb{C}^n$ denote the base point projection. The exceptional variety is $\mathcal{E} = \pi^{-1}(0) \cong \mathbb{CP}^{n-1}$. The conormal bundle of $\mathcal{E}$ is the line bundle $\mathcal{O}_{\mathbb{CP}^{n-1}}(+1)$ which does not admit any nonvanishing sections, so X does not admit any noncritical holomorphic functions. On the other hand, the function $g(z) = z_1^2 + \cdots + z_n^2$ on $\mathbb{C}^n$, with $\mathrm{Crit}(g) = \{0\}$, pulls back to a holomorphic function $f = g \circ \pi \in \mathcal{O}(X)$ with $\mathrm{Crit}(f) = \mathcal{E}$. Similarly, the coordinate function z_j on $\mathbb{C}^n$ pulls back to a holomorphic function $z_j \circ \pi = \pi_j$ which is noncritical on $X \setminus \mathcal{E} \cong \mathbb{C}^n \setminus \{0\}$, and

$$\mathrm{Crit}(\pi_j) = \{[z_1 \colon \cdots \colon z_n] \in \mathcal{E} \colon z_j = 0\} \cong \mathbb{CP}^{n-2}.$$

Hence the critical locus may be a proper subvariety of the exceptional variety.

Problem 1.10. Let X be a 1-convex manifold. Which closed analytic subsets of its exceptional variety $\mathcal{E}$ are critical loci of holomorphic functions on X?

Going a step further, recall that a complex space X is said to be *holomorphically convex* if for any compact set $K \subset X$ its $\mathcal{O}(X)$-convex hull

$$\widehat{K}_{\mathcal{O}(X)} = \Big\{x \in X \colon |f(x)| \le \sup_K |f| \ \ \forall f \in \mathcal{O}(X)\Big\}$$

is also compact. This class contains all 1-convex spaces, but many more. For example, the total space of any holomorphic fiber bundle $X \to Y$ with a compact fiber over a Stein space Y is holomorphically convex. By Remmert [35], every holomorphically convex space X admits a proper holomorphic surjection $\pi \colon X \to Y$ onto a Stein space Y such that the (compact) fibers of π are connected, $\pi_* \mathcal{O}_X = \mathcal{O}_Y$, the map $f \mapsto f \circ \pi$ is an isomorphism of $\mathcal{O}(Y)$ onto $\mathcal{O}(X)$, and every holomorphic map $X \to S$ to a Stein space S factors through π. If $g \in \mathcal{O}(Y)$ is a noncritical function on Y furnished by Theorem 1.1, then $f = g \circ \pi \in \mathcal{O}(X)$ is noncritical on the set where π is a submersion. What else could be said?

Another possible line of investigation is the following. In [14] we proved that on any Stein manifold X of dimension n there exist $q = [(n+1)/2]$ holomorphic functions $f_1, \ldots, f_q \in \mathcal{O}(X)$ with pointwise independent differentials, i.e., such that $df_1 \wedge \cdots \wedge df_q$ is a nowhere vanishing holomorphic $(q,0)$-form on X, and this number q is maximal in general for topological reasons. Furthermore, we have the h-principle for holomorphic submersions $X \to \mathbb{C}^q$ with any $q < n = \dim X$, saying that every q-tuple of pointwise linearly independent continuous $(1,0)$-forms can be deformed to a q-tuple of linearly independent holomorphic differentials $df_1, \ldots, df_q$. What could be said regarding this problem on Stein spaces? For example:

Problem 1.11. Assume that X is a pure n-dimensional Stein space and let q be as above. Do there exist $f_1, \ldots, f_q \in \mathcal{O}(X)$ such that $df_1 \wedge \cdots \wedge df_q$ is nowhere vanishing on X_{reg}? What is the answer if X has only isolated singularities?

Our methods strongly rely on the fact that the critical locus of a generic holomorphic function on a Stein space is discrete (see §2). If $q > 1$ then the set $df_1 \wedge \cdots \wedge df_q = 0$ (if nonempty) is a subvariety of complex dimension $\ge q - 1 > 0$, and we do not know how to ensure nonvanishing of this form on a deleted neighborhood of a subvariety of X as in the case $q = 1$. The problem seems nontrivial even for an isolated singular point of X.

2. Critical points of a holomorphic function on a complex space

We begin by recalling certain basic facts of complex analytic geometry.

Let $(X, \mathcal{O}_X)$ be a reduced complex space. Following standard practice we shall simply write X below. We denote by $\mathcal{O}(X) \cong \Gamma(X, \mathcal{O}_X)$ the algebra of all holomorphic functions on X. Given a holomorphic function f on an open set $U \subset X$, we denote by $f_p \in \mathcal{O}_{X,p}$ the germ of f at a point $p \in U$. Similarly, X_p stands for the germ of X at $p \in X$.

By $\mathfrak{m}_p = \mathfrak{m}_{X,p}$ we denote the maximal ideal of the local ring $\mathcal{O}_{X,p}$, so $\mathcal{O}_{X,p}/\mathfrak{m}_p \cong \mathbb{C}$. We say that $f \in \mathcal{O}_{X,p}$ *vanishes to order* $k \in \mathbb{N}$ *at* p if $f \in \mathfrak{m}_p^k$. The quotient ring $\mathcal{O}_{X,p}/\mathfrak{m}_p^k \cong \mathbb{C} \oplus \mathfrak{m}_p/\mathfrak{m}_p^k$ is a finite-dimensional complex vector space, called the *space of* $(k-1)$*-jets of holomorphic functions on* X *at* p. Recall that $\mathfrak{m}_p/\mathfrak{m}_p^2 \cong T_p^* X$ is the Zariski cotangent space and its dual $(\mathfrak{m}_p/\mathfrak{m}_p^2)^* \cong T_p X$ is the Zariski tangent space of X at p.

If X' is a complex subvariety of X and $p \in X'$, then the maximal ideal $\mathfrak{m}_{X',p}$ of the ring $\mathcal{O}_{X',p}$ consists of all germs at p of restrictions $f|_{X'}$ with $f \in \mathfrak{m}_{X,p}$.

Lemma 2.1. *Let X' be a closed complex subvariety of a complex space X and $p \in X'$. If $f \in \mathcal{O}_{X',p}$ and $h \in \mathcal{O}_{X,p}$ are such that $f - (h|_{X'})_p \in \mathfrak{m}_{X',p}^k$ for some $k \in \mathbb{N}$, then there exists $\tilde{h} \in \mathcal{O}_{X,p}$ such that $\tilde{h} - h \in \mathfrak{m}_{X,p}^k$ and $(\tilde{h}|_{X'})_p = f \in \mathcal{O}_{X',p}$.*

Proof. The conditions imply that $f = (h|_{X'})_p + \sum_j \xi_{j,1} \cdots \xi_{j,k}$ where $\xi_{j,i} \in \mathfrak{m}_{X',p}$ for all i and j. Then $\xi_{j,i} = \tilde{\xi}_{j,i}|_{X'}$ for some $\tilde{\xi}_{j,i} \in \mathfrak{m}_{X,p}$, and the germ $\tilde{h} = h + \sum_j \tilde{\xi}_{j,1} \cdots \tilde{\xi}_{j,k} \in \mathcal{O}_{X,p}$ has the stated properties. □

Given $f \in \mathcal{O}(X)$, the collection of its differentials $df_x \colon T_x X \to \mathbb{C}$ at all $x \in X$ defines the *tangent map* $Tf \colon TX \to X \times \mathbb{C}$ on the *tangent space* $TX = \bigcup_{x \in X} T_x X$. Recall that TX carries the structure of a not necessarily reduced *linear space* over X such that the tangent map Tf is holomorphic. Here is a local decription of TX (see e.g. [10, Chapter 2]). Assume that X is a closed complex subvariety of an open set $U \subset \mathbb{C}^N$, defined by holomorphic functions $h_1, \ldots, h_m \in \mathcal{O}(U)$ which generate the sheaf of ideals $\mathcal{J}_X$ of X (hence $\mathcal{O}_X \cong (\mathcal{O}_U/\mathcal{J}_X)|_X$). Let $(z_1, \ldots, z_N, \xi_1, \ldots, \xi_N)$ be complex coordinates on $U \times \mathbb{C}^N$. Then TX is the closed complex subspace of $U \times \mathbb{C}^N$ generated by the functions

$$h_1, \ldots, h_m \quad \text{and} \quad \frac{\partial h_i}{\partial z_1}\xi_1 + \cdots + \frac{\partial h_i}{\partial z_N}\xi_N \quad \text{for } i = 1, \ldots, m. \tag{2.1}$$

This means that TX is the common zero set of the above functions and its structure sheaf $\mathcal{O}_{TX}$ is the quotient of $\mathcal{O}_{U \times \mathbb{C}^N}$ by the ideal generated by them. The projection $TX \to X$ is the restriction of the projection $U \times \mathbb{C}^N \to U$, $(z, \xi) \mapsto z$. Different local representations of X give isomorphic representations of TX. If X is a complex manifold then TX is the usual tangent bundle of X; this holds in particular over the regular locus X_{reg} of any complex space.

Since the critical locus $\mathrm{Crit}(f)$ of a holomorphic function $f \in \mathcal{O}(X)$ is the set of points $x \in X$ at which the differential $df_x \colon T_x X \to \mathbb{C}$ vanishes, one might expect that $\mathrm{Crit}(f)$ is a closed complex subvariety of X. This is clearly true if X is a complex manifold (in particular, it holds on the regular locus X_{reg} of any complex space), but it fails in general near singularities. Furthermore, unlike in the smooth case, the set of (strongly) noncritical holomorphic functions is not stable under small deformations. The following example illustrates these phenomena in a simple setting of an irreducible quadratic surface singularity in $\mathbb{C}^3$.

Example 2.2. Let A be the subvariety of $\mathbb{C}^3$ given by

$$A = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \colon h(z) = z_1^2 + z_2^2 + z_3^2 = 0\}. \tag{2.2}$$

(In the theory of minimal surfaces this is called the *null quadric*, and a complex curve in $\mathbb{C}^3$ whose derivative belongs to $A^* = A \setminus \{(0,0,0)\}$ is said to be an (immersed) *null holomorphic curve*. Such curves are related to conformally immersed minimal surfaces in $\mathbb{R}^3$. See e.g. [34] for a classical survey of this subject and [3] for some recent results.) Clearly $A_{\mathrm{sing}} = \{(0,0,0)\}$, A is locally and globally irreducible, and $T_{(0,0,0)}A = \mathbb{C}^3$. For any $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3 \setminus \{(0,0,0)\}$ the linear function

$$f_\lambda(z_1, z_2, z_3) = \lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3$$

restricted to A is strongly noncritical at $(0,0,0)$. Clearly df_λ is colinear with $dh = 2(z_1 dz_1 + z_2 dz_2 + z_3 dz_3)$ precisely along the complex line $\Lambda = \mathbb{C}\lambda = \{t\lambda \colon t \in \mathbb{C}\}$. If $\lambda \in A^*$, it follows that $\mathrm{Crit}(f_\lambda|_A) = \Lambda \setminus \{0\}$, which is not closed. An explicit example is obtained by taking

$$\lambda = (1, \imath, 0) \in A^*, \qquad f(z) = z_1 + \imath z_2.$$

(Here $\imath = \sqrt{-1}$.)

Let us now show on the same example that the set of (strongly) noncritical functions fails to be stable under small deformations.

Example 2.3. Let A be the quadric (2.2). Consider the family of functions

$$f_\epsilon(z_1, z_2, z_3) = z_1 + z_1(z_1 - 2\epsilon) + \imath z_2, \qquad \epsilon \in \mathbb{C}.$$

Since $(df_\epsilon)_0 = (1 - 2\epsilon)dz_1 + \imath dz_2$, $f_\epsilon|_A$ is (strongly) noncritical at the origin for any $\epsilon \in \mathbb{C}$. A calculation shows that for $\epsilon \neq 1/2$ the differentials df_ϵ and dh (considered on the tangent bundle $T\mathbb{C}^3$) are colinear precisely at points of the complex curve

$$C_\epsilon = \{(z_1, z_2, 0) \in \mathbb{C}^3 \colon z_2 = \imath z_1/(2z_1 - 2\epsilon + 1)\}.$$

This curve intersects the quadric A in the following four points:

$$A \cap C_\epsilon = \{(0,0,0), (\epsilon, \imath\epsilon, 0), (\epsilon - 1, -\imath(\epsilon - 1), 0)\}.$$

Hence the second and the third of these points are the critical points of $f_\epsilon|_A$ when $\epsilon \notin \{0, 1\}$. For ϵ close to 0 the point $(\epsilon, \imath\epsilon, 0)$ lies close to the origin, while the third point is close to $(-1, \imath, 0)$. Hence $f_0|_A$ is noncritical on the intersection of A with the ball of radius $1/2$ around the origin in $\mathbb{C}^3$, but $f_\epsilon|_A$ for small $\epsilon \neq 0$ is close to f_0 and has a critical point $(\epsilon, \imath\epsilon, 0) \in A$ near the origin.

Although we have seen in Example 2.2 that $\mathrm{Crit}(f)$ need not be a closed complex subvariety near singular points of a complex space, we still have the following result.

Lemma 2.4. *Let f be a holomorphic function on a complex space X. If $X' \subset X$ is a closed complex subvariety of X containing the singular locus X_{sing} of X, then the set*

$$C_{X'}(f) := \{x \in X_{\mathrm{reg}} : df_x = 0\} \cup X'$$

is a closed complex subvariety of X.

Proof. By the desingularization theorem [4, 5, 28] there are a complex manifold M and a proper holomorphic surjection $\pi : M \to X$ such that $\pi : M \setminus \pi^{-1}(X_{\mathrm{sing}}) \to X \setminus X_{\mathrm{sing}}$ is a biholomorphism and $\pi^{-1}(X_{\mathrm{sing}})$ is a compact complex hypersurface in M. Given $f \in \mathcal{O}(X)$, consider $F = f \circ \pi \in \mathcal{O}(M)$ and the subvariety $M' = \pi^{-1}(X')$ of M. Since M is a complex manifold, $\mathrm{Crit}(F) \subset M$ is a closed complex subvariety of M, and hence so is $C_{M'}(F) = \mathrm{Crit}(F) \cup M'$. As π is proper, $\pi(C_{M'}(F))$ is a closed complex subvariety of X according to Remmert [36]. Since π is biholomorphic over X_{reg}, we have $\pi(C_{M'}(F)) = C_{X'}(f)$, which proves the result. □

In spite of the lack of stability of noncritical functions, illustrated by Example 2.3, we shall obtain a certain stability result (Lemma 2.7 below) which will be used in the construction of stratified noncritical holomorphic functions on Stein spaces.

Given a compact set K in a complex space X, we denote by $\mathcal{O}(K)$ the space of all functions f that are holomorphic on an open neighborhood $U_f \subset X$ of K (depending on the function), identifying two functions that agree on some neighborhood of K. We denote by $\mathring{K}$ the topological interior of a set K.

For any coherent analytic sheaf $\mathcal{F}$ on a complex space X the $\mathcal{O}(X)$-module $\mathcal{F}(X) = \Gamma(X, \mathcal{F})$ of all global sections of $\mathcal{F}$ over X can be endowed with a Fréchet space topology (the topology of uniform convergence on compacts in X) such that for every $x \in X$ the natural restriction map $\mathcal{F}(X) \mapsto \mathcal{F}_x$ is continuous (see [25, Theorem 5, p. 167]). The topology on the stalks $\mathcal{F}_x$ is the *sequence topology* (cf. [24, p. 86ff]). Thus every set of the second category in $\mathcal{F}(X)$ (an intersection of at most countably many open dense sets) is dense in $\mathcal{F}(X)$. The expression *generic holomorphic function* on X will always mean a function in a certain set of the second category in $\mathcal{O}(X)$, and likewise for $\mathcal{F}(X)$.

If $\mathcal{S}$ is a coherent subsheaf of a coherent sheaf $\mathcal{F}$ over X then $\mathcal{S}(X)$ is a closed submodule of $\mathcal{F}(X)$ (the Closedness Theorem [25, p. 169]). Since every $\mathcal{O}_{X,x}$-submodule M of $\mathcal{F}_x$ is closed in the sequence topology, it follows that $\{f \in \mathcal{F}(X) : f_x \in M\}$ is a closed subspace of $\mathcal{F}(X)$, hence a Fréchet space.

In particular, if X' is a closed complex subvariety of a complex space X, and $\mathcal{J}_{X'}$ is the sheaf of ideals of X' (a coherent subsheaf of $\mathcal{O}_X$), then

$$\mathcal{J}(X') := \Gamma(X, \mathcal{J}_{X'}) = \{f \in \mathcal{O}(X) : f|_{X'} = 0\}$$

is a closed (hence Fréchet) ideal in $\mathcal{O}(X)$. Given a function $g \in \mathcal{O}(X')$ on a closed complex subvariety $X' \subset X$, the set

$$\mathcal{O}_{X',g}(X) = \{f \in \mathcal{O}(X) : f|_{X'} = g\} \tag{2.3}$$

is a closed affine subspace of $\mathcal{O}(X)$ and hence a Baire space.

The Closedness Theorem [25, p. 169] shows that for any $x \in X$ and $k \in \mathbb{N}$ the set

$$\{f \in \mathcal{O}(X) \colon f_x - f(x) \in \mathfrak{m}_x^k\}$$

is closed in $\mathcal{O}(X)$. For $k = 2$ this is the set of functions with a critical point at x.

Let $X_x = \bigcup_{j=1}^m V_j$ be a decomposition into local irreducible components at $x \in X$. According to Definition 1.2, $f \in \mathcal{O}_{X,x}$ fails to be strongly noncritical at x if there is a $j \in \{1, \dots, m\}$ such that $(f|_{V_j})_x - f(x) \in \mathfrak{m}^2_{V_j,x}$. This defines a closed subset of $\mathcal{O}_{X,x}$, so the set of all strongly noncritical germs is open in $\mathcal{O}_{X,x}$. Since the restriction maps in the space of sections of a coherent sheaf are continuous, we get the following conclusion.

Lemma 2.5. *The set of all functions $f \in \mathcal{O}(X)$ which are noncritical* (*or strongly noncritical*) *at a certain point $x \in X$ is open in $\mathcal{O}(X)$.*

However, Example 2.3 above shows that the set of functions $f \in \mathcal{O}(X)$ that are noncritical (or strongly noncritical) on a certain compact set $K \subset X$ may fail to be open in $\mathcal{O}(X)$, unless K is contained in the regular locus X_{reg}.

The following result is [19, Lemma 3.1, p. 52] in the case that X is a Stein manifold; we shall need it also when X is a Stein space. (We correct a misprint in the original source.)

Lemma 2.6 (Bounded extension operator). *Let X be a Stein space, X' be a closed complex subvariety of X, and $\Omega \Subset X$ be a Stein domain in X. For any relatively compact subdomain $D \Subset \Omega$ there exists a bounded linear extension operator $T \colon \mathcal{H}^\infty(\Omega \cap X') \to \mathcal{H}^\infty(D)$ such that*

$$(Tf)(x) = f(x) \quad \forall f \in \mathcal{H}^\infty(\Omega \cap X'),\ \forall x \in D \cap X'.$$

Proof. Choose a Stein neighborhood $W \Subset X$ of the compact set $\overline{\Omega}$ and embed it as a closed complex subvariety (still denoted W) of some Euclidean space $\mathbb{C}^N$ (see [16, Theorem 2.2.8] and the references therein). By Siu's theorem [38] there is a Stein domain $\Omega' \Subset \mathbb{C}^N$ such that $\Omega = \Omega' \cap W$. Also choose a domain D' in $\mathbb{C}^N$ such that $D \subset D'$ and $\overline{D'} \subset \Omega'$. By [19, Lemma 3.1], applied to the subvariety $X' \cap W$ of the Stein manifold $\mathbb{C}^N$ and domains $D' \Subset \Omega' \Subset \mathbb{C}^N$, there exists a bounded linear extension operator $T' \colon \mathcal{H}^\infty(\Omega' \cap X') \to \mathcal{H}^\infty(D')$. Since $\Omega' \cap W = \Omega$, by restricting the resulting function $T'f$ to $D \subset W \cap D'$ we obtain a bounded extension operator T as in the lemma. □

Lemma 2.7 (The Stability Lemma). *Assume that X is a complex space, $X' \subset X$ is a closed complex subvariety containing X_{sing}, and $K \subset L$ are compact subsets of X with $K \subset \mathring{L}$. Assume that $f \in \mathcal{O}(X)$ is noncritical on $L \setminus X'$. Then there exist $r \in \mathbb{N}$ and $\epsilon > 0$ such that the following holds. If $g \in \mathcal{O}(L)$ satisfies*

(i) *$f - g \in \Gamma(L, \mathcal{J}^r_{X'})$, where $\mathcal{J}^r_{X'}$ is the r-th power of the ideal sheaf $\mathcal{J}_{X'}$, and*
(ii) *$\|f - g\|_L := \sup_{x \in L} |f(x) - g(x)| < \epsilon$,*

then g has no critical points on $K \setminus X'$.

Proof. The result holds on compact subsets of $X \setminus X' \subset X_{\mathrm{reg}}$ in view of Lemma 2.5, so it suffices to consider the behavior of g near $K \cap X'$.

Fix $p \in K \cap X'$ and embed an open neighborhood $U \subset X$ of p as a closed complex subvariety (still denoted U) of an open ball $B \subset \mathbb{C}^N$, with p corresponding to the origin $0 \in B$. We choose U small enough such that $U \subset \mathring{L}$. Pick a slightly smaller ball $B' \Subset B$ and set $U' := B' \cap U$. Lemma 2.6 (applied with $\Omega = B$ in $X = \mathbb{C}^N$, $X' = U$, and $D = B' \Subset B$) furnishes a bounded linear extension operator T mapping bounded holomorphic functions on U to bounded holomorphic functions on B'. In the embedded picture, $x \in U' \setminus X' \subset B'$ is a critical point of f if and only if the differential $d\tilde{f}_x \colon T_x\mathbb{C}^N \to \mathbb{C}$ of the extended function $\tilde{f} = Tf \in \mathcal{O}(B')$ annihilates the Zariski tangent space T_xU. The latter condition is expressed by a finite number of holomorphic equations $F_j(\tilde{f}) = 0$ on B' $(j = 1, \dots, k)$ involving the first order partial derivatives of $\tilde{f}$ and of some fixed holomorphic defining functions $h_1, \dots, h_m$ for the subvariety U in B. (These equations express the fact that the differential of $\tilde{f}$ is contained in the linear span of the differentials of the functions $h_1, \dots, h_m$; compare with the local description (2.1) of TX.) By the assumption this system of equations has no solutions on $U \setminus X'$.

Fix $r \in \mathbb{N}$. Choose holomorphic functions $\xi_1, \dots, \xi_l \in \mathcal{O}(B)$ that vanish to order r on the subvariety $X' \cap B$ and generate the sheaf $\mathcal{J}^r_{X' \cap B}$. Pick a ball $B_0 \subset \mathbb{C}^N$ with $B' \Subset B_0 \Subset B$ and set $U_0 := U \cap B_0$. For every function $g \in \mathcal{O}(U)$ that agrees with f to order r along the subvariety $X' \cap U$ we have $g = f + \sum_{j=1}^{l} g_j \xi_j$ where $g_j \in \mathcal{O}(U)$. Furthermore, by the open mapping theorem we can choose the functions g_j to satisfy the estimates $\|g_j\|_{U_0} \le C\|f - g\|_U$ for some constant $C > 0$. By Lemma 2.6 there is an extension operator T_0 mapping bounded holomorphic functions on U_0 to bounded holomorphic functions on B'. The bounded holomorphic function $\tilde{g} := \tilde{f} + \sum_{j=1}^{l} T_0(g_j)\xi_j \in \mathcal{O}(B')$ then agrees with $\tilde{f}$ to order r along the subvariety $U' \cap X'$, and $\tilde{g}$ is arbitrarily close to $\tilde{f}$ on B' if g is close enough to f on U. It follows that the corresponding functions $F_j(\tilde{g})$ for $j = 1, \dots, k$ agree with $F_j(\tilde{f})$ to order $r - 1$ along $U' \cap X'$. By choosing $r \in \mathbb{N}$ sufficiently large and ϵ bounded from above by some fixed $\epsilon_0 > 0$, we can ensure that for any g satisfying conditions (i) and (ii) the system of holomorphic equations on $B' \subset \mathbb{C}^N$,

$$h_1 = 0, \dots, h_m = 0, \quad F_1(\tilde{g}) = 0, \dots, F_k(\tilde{g}) = 0, \tag{2.4}$$

has no solutions in $W \setminus X'$, where $W \subset U'$ is a neighborhood of p whose size depends on r and ϵ_0. (This follows from the Łojasiewicz inequality; see e.g. [29]. The details of this argument can also be found in [13, proof of Theorem 1.3]; see in particular pp. 507–509. In fact, looking at the common zero set of (2.4) as the inverse image of the origin $0 \in \mathbb{C}^{m+k}$ by the holomorphic map $B' \to \mathbb{C}^{m+k}$ whose components are the functions in (2.4), the local aspect of the cited result from [13] applies verbatim.) Since finitely many open sets U' of this kind cover $K \cap X'$, we see that (2.4) has no solutions on a deleted neighborhood of $K \cap X'$ in K. By choosing $\epsilon > 0$ small enough we can also ensure in view of Lemma 2.5 that there are no solutions on the rest of K. □

Lemma 2.7 fails in general without the interpolation condition as shown by Example 2.3. Here is an even simpler example on the cusp curve, showing that being *stratified non-*

critical (see Definition 1.4) is not a stable property even on complex curves if we allow critical points in the zero-dimensional skeleton.

Example 2.8. The cusp curve $X = \{(z, w) \in \mathbb{C}^2 \colon z^2 = w^3\}$ has a singularity at $(0, 0) \in \mathbb{C}^2$ and is smooth elsewhere. It is desingularized by the map $\pi \colon \mathbb{C} \to X$, $\pi(t) = (t^3, t^2)$. The function $f(z, w) = zw$ on X pulls back to the function $h(t) = f(\pi(t)) = t^5$ with the only critical point at $t = 0$, so $f|_X$ is stratified noncritical with respect to $\{(0, 0)\} \subset X$. The perturbation of h given by

$$h_\epsilon(t) = t^3(t-\epsilon)^2 = t^5 - 2\epsilon t^4 + \epsilon^2 t^3 = zw - 2\epsilon w^2 + \epsilon^2 z$$

induces a holomorphic function $f_\epsilon \colon X \to \mathbb{C}$ with a critical point at (ϵ^3, ϵ^2), so f_ϵ is not stratified noncritical on $\{(0, 0)\} \subset X$ if $\epsilon \neq 0$.

Lemma 2.9 (The Genericity Lemma). *Let X be a Stein space.*

(i) *For a generic $f \in \mathcal{O}(X)$ the set $A(f) := \mathrm{Crit}(f|_{X_{\mathrm{reg}}})$ is discrete in X.*

(ii) *If $X' \subset X$ is a closed complex subvariety containing X_{sing} and $g \in \mathcal{O}(X')$, then for a generic $f \in \mathcal{O}_{X',g}(X)$ the set $\mathrm{Crit}(f|_{X\setminus X'})$ is discrete in X. In particular, a generic holomorphic extension of g is noncritical on a deleted neighborhood of X' in X.*

(iii) *If g is a holomorphic function on an open neighborhood of X' in X and $r \in \mathbb{N}$, then the conclusion of part (ii) holds for a generic extension $f \in \mathcal{O}(X)$ of $g|_{X'}$ which agrees with g to order r along X'.*

Proof. We begin by proving (i). A point $x \in X_{\mathrm{reg}}$ is a critical point of $f \in \mathcal{O}(X)$ if and only if the partial derivatives $\partial f/\partial z_j$ in any system of local holomorphic coordinates $z = (z_1, \ldots, z_n)$ on an open neighborhood $U \subset X_{\mathrm{reg}}$ of x vanish at $z(x)$. (Here $n = \dim_x X$.) This gives n independent holomorphic equations on the 1-jet extension $j^1 f$ of f, so the jet transversality theorem for holomorphic maps $X \to \mathbb{C}$ (cf. [12] or [16, §7.8]) implies that every $x \in A(f)$ is an isolated point of $A(f)$ for a generic $f \in \mathcal{O}(X)$. (The argument goes as follows: write $X_{\mathrm{reg}} = \bigcup_{j=1}^\infty U_j$ where $U_j \subset X_{\mathrm{reg}}$ is a compact connected coordinate neighborhood for every j. The set of all $f \in \mathcal{O}(X)$ whose 1-jet extension $U_j \ni x \mapsto j^1_x f \in \mathbb{C}^{n_j}$ (with $n_j = \dim U_j$) is transverse to $0 \in \mathbb{C}^{n_j}$ on U_j is open and dense in $\mathcal{O}(X)$. Taking the countable intersection of these sets over all j gives the statement.)

For any f as above the set $A(f)$ is discrete in X_{reg}, and we claim that $A(f)$ is then also discrete in X. If not, there are $x_0 \in X_{\mathrm{sing}}$ and $x_j \in A(f)$ with $\lim_{j\to\infty} x_j = x_0$. By Lemma 2.4 the set $C(f) = A(f) \cup X_{\mathrm{sing}}$ is a closed complex subvariety of X. Pick a compact neighborhood $K \subset X$ of x_0. Each x_j which belongs to K is an isolated point of $C(f)$, hence an irreducible component of $C(f)$. Thus the compact subset $K \cap C(f)$ of the complex space $C(f)$ contains infinitely many irreducible components of $C(f)$, a contradiction. This proves (i).

Part (ii) follows similarly by applying the jet transversality theorem in the Baire space $\mathcal{O}_{X',g}(X) = \{f \in \mathcal{O}(X) \colon f|_{X'} = g\}$.

Finally, let g be as in (iii). Consider the short exact sequence of coherent analytic sheaves $0 \to \mathcal{J}^r_{X'} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{J}^r_{X'} \to 0$. The sheaf $\mathcal{O}_X/\mathcal{J}^r_{X'}$ is supported on X', and hence g determines a section of it. Since $H^1(X; \mathcal{J}^r_{X'}) = 0$ by Cartan's Theorem B, the same section is determined by a function $G \in \mathcal{O}(X)$. This means that $G - g$ vanishes to order r along X'. To conclude the proof, it suffices to apply the transversality theorem in the Baire space $G + \mathcal{J}^r_{X'}(X) \subset \mathcal{O}(X)$; the details are similar to those in (i). □

Proposition 2.10. *If (X, Σ) is a stratified Stein space, then the set $\bigcup_{S \in \Sigma} \mathrm{Crit}(f|_S)$ is discrete in X for a generic $f \in \mathcal{O}(X)$.*

Proof. Let $\Sigma = \{S_j\}_j$ where S_j are (smooth) strata. Each stratum S_j of positive dimension $n_j > 0$ is a union $\bigcup_k U_{j,k}$ of countably many compact coordinate sets $U_{j,k}$. The same argument as in the proof of Lemma 2.9 shows that the set $\mathcal{U}_{j,k} \subset \mathcal{O}(X)$, consisting of all $f \in \mathcal{O}(X)$ such that the 1-jet extension map $U_{j,k} \ni x \mapsto j^1_x f \in \mathbb{C}^{n_j}$ is transverse to $0 \in \mathbb{C}^{n_j}$ on $U_{j,k}$, is open and dense in $\mathcal{O}(X)$. Every $f \in \bigcap_{j,k} \mathcal{U}_{j,k}$ satisfies the conclusion of the proposition. □

Since every complex space admits a stratification, Proposition 2.10 implies

Corollary 2.11. *A generic holomorphic function on a Stein space has discrete critical locus.*

We also have the following result in which X is not necessarily Stein.

Corollary 2.12. *Let X be a complex space and $X' \subset X$ a closed Stein subvariety containing X_{sing}. Given $g \in \mathcal{O}(X')$, there are an open neighborhood $U \subset X$ of X' and a function $f \in \mathcal{O}(U)$ such that $f|_{X'} = g$ and f has no critical points in $U \setminus X'$.*

In particular, an isolated sigular point p of a complex space X admits a holomorphic function on a neighborhood U of p which is noncritical on $U \setminus \{p\}$.

Proof. According to Siu [38] (see also [16, §3.1] and the additional references therein) a Stein subvariety X' in any complex space X admits an open Stein neighborhood $\Omega \subset X$ containing X' as a closed complex subvariety. The conclusion then follows from Lemma 2.9 applied to the Stein space Ω. □

In the proof of Theorem 3.2 we shall also need the following result. This is well known when X is a complex manifold (i.e., without singularities), and we shall reduce the proof to this particular case.

Lemma 2.13. *Let X be a reduced complex space and $U \Subset U'$ open relatively compact sets in X. Fix a distance function* dist *on X inducing the standard topology. There is an $\epsilon > 0$ such that for any holomorphic map $f : U' \to X$ satisfying* $\sup_{x \in U'} \mathrm{dist}(x, f(x)) < \epsilon$ *the restriction $f|_U : U \to f(U) \subset X$ is biholomorphic onto its image.*

Proof. We first prove the lemma when U' is Stein and its (compact) closure $\overline{U'}$ admits a Stein neighborhood W in X. Assuming as we may that W is relatively compact, it embeds as a closed complex subvariety of a Euclidean space $\mathbb{C}^N$ (cf. [16, Theorem 2.2.8]). Since U' is Stein, Siu's theorem [38] provides a bounded Stein domain $D' \Subset \mathbb{C}^N$ such that $D' \cap W = U'$. Choose a pair of domains $D_0 \Subset D$ in $\mathbb{C}^N$ such that $\overline{U} \subset D_0 \cap W$ and $\overline{D} \subset D'$. Let T be a bounded linear extension operator furnished by Lemma 2.6, mapping bounded holomorphic functions on U' to bounded holomorphic functions on D

and satisfying

$$Tg|_{D\cap U'} = g|_{D\cap U'} \quad \text{and} \quad \|Tg\|_D \le C\|g\|_{U'}$$

for some constant $C > 0$ independent of g.

Consider a holomorphic map $f: U' \to X$ close to the identity. We may assume that $f(U') \subset W \subset \mathbb{C}^N$. Write $f(x) = x + g(x)$ for $x \in U'$, where $g : U' \to \mathbb{C}^N$ is close to zero. Applying the operator T to each component of g we get a holomorphic map $F = \mathrm{Id} + Tg: D \to \mathbb{C}^N$ which is close to the identity in the sup norm on D. Hence F is biholomorphic on the smaller domain D_0 provided that f is close enough to the identity on U'. Since $U \subset D_0$ and $F|_U = \mathrm{Id}_U + g|_U = f|_U$, we infer that $f: U \to f(U) \subset W$ is biholomorphic as well. Furthermore, the inverse map $F^{-1}: F(D_0) \to D_0$, restricted to $F(D_0) \cap W$, has range in W, as is easily seen by considering the situation on W_{reg} and applying the identity principle. This completes the proof in the special case.

The general case follows as in the standard manifold situation. By compactness of $\overline{U}$ we can choose finitely many triples of open sets $V_j \Subset U_j \Subset U'_j$ in X ($j = 1, \dots, m$) such that

(i) $\overline{U} \subset \bigcup_{j=1}^m V_j$ and $\bigcup_{j=1}^m U'_j \subset U'$, and
(ii) U'_j is Stein and $\overline{U'_j}$ has a Stein neighborhood in X for every $j = 1, \dots, m$.

Pick $\epsilon_0 > 0$ such that $\mathrm{dist}(V_j, X \setminus U_j) > 2\epsilon_0$ for every $j = 1, \dots, m$. By the special case proved above, applied to the pair $U_j \Subset U'_j$, we can find $\epsilon \in (0, \epsilon_0)$ such that $f|_{U_j}: U_j \to f(U_j)$ is biholomorphic for every j provided that $\mathrm{dist}(x, f(x)) < \epsilon$ for all $x \in U'$. Since $U \subset \bigcup_{j=1}^m U_j$, it follows that $f|_U : U \to f(U)$ is biholomorphic as long as it is injective. Suppose that $f(x) = f(y)$ for some $x \neq y$ in U. Since the sets V_j cover U, we have $x \in V_j$ for some j. As f is injective on U_j, it follows that $y \in U \setminus U_j$ and hence $\mathrm{dist}(x, y) > 2\epsilon_0$. The triangle inequality and the choice of ϵ then give

$$\mathrm{dist}(f(x), f(y)) \ge \mathrm{dist}(x, y) - \mathrm{dist}(x, f(x)) - \mathrm{dist}(y, f(y)) > 2\epsilon_0 - 2\epsilon > 0,$$

a contradiction to $f(x) = f(y)$. Thus f is injective on U. □

3. A splitting lemma for biholomorphic maps on complex spaces

In this section we prove a splitting lemma for biholomorphic maps close to the identity on Cartan pairs in complex spaces (see Theorem 3.2 below). This result is the key to the proof of our main theorems; it will be used for gluing pairs of holomorphic functions with control of their critical loci. The nonsingular case is given by [14, Theorem 4.1].

Recall that a compact set K in a complex space X is said to be a *Stein compact* if K admits a basis of open Stein neighborhoods in X. We recall the following notion.

Definition 3.1 ([16, p. 209]).

(I) A pair (A, B) of compact subsets in a complex space X is a *Cartan pair* if it satisfies the following conditions:

  (i) $A, B, D = A \cup B$ and $C = A \cap B$ are Stein compacts, and
  (ii) A, B are *separated* in the sense that $\overline{A \setminus B} \cap \overline{B \setminus A} = \emptyset$.

(II) A pair (A, B) of open sets in a complex manifold X is a *strongly pseudoconvex Cartan pair of class* $\mathcal{C}^\ell$ $(\ell \geq 2)$ if $(\bar{A}, \bar{B})$ is a Cartan pair in the sense of (I) and the sets $A, B, D = A \cup B$ and $C = A \cap B$ are Stein domains with strongly pseudoconvex boundaries of class $\mathcal{C}^\ell$.

We shall use the following properties of Cartan pairs:

(a) Let (A, B) be a Cartain pair in X. If X is a complex subspace of another complex space $\widetilde{X}$, then (A, B) is also a Cartan pair in $\widetilde{X}$ (cf. [16, Lemma 5.7.2, p. 210]).
(b) Every Cartan pair (A, B) in a complex manifold X can be approximated from outside by smooth strongly pseudoconvex Cartain pairs (cf. [16, Proposition 5.7.3, p. 210]).
(c) One can solve any Cousin-I problem with sup-norm bounds on a strongly pseudoconvex Cartan pair (cf. [16, Lemma 5.8.2, p. 212]).

We denote by dist a distance function on X which induces its standard complex space topology. (The precise choice will not be important.) Given a compact set $K \subset X$ and continuous maps $f, g \colon K \to X$, we shall write

$$\mathrm{dist}_K(f, g) = \sup_{x \in K} \mathrm{dist}(f(x), g(x)).$$

By Id we denote the identity map; its domain will always be clear from the context.

Theorem 3.2. *Assume that X is a complex space and X' is a closed complex subvariety of X containing its singular locus X_{sing}. Let (A, B) be a Cartan pair in X such that $C := A \cap B \subset X \setminus X'$. For any open set $\widetilde{C} \subset X$ containing C there exist open sets $A' \supset A$, $B' \supset B$, $C' \supset C$ in X, with $C' \subset A' \cap B' \subset \widetilde{C}$, satisfying the following property. Given $\eta > 0$, there exists $\epsilon_\eta > 0$ such that for each holomorphic map $\gamma \colon \widetilde{C} \to X$ with $\mathrm{dist}_{\widetilde{C}}(\gamma, \mathrm{Id}) < \epsilon_\eta$ there exist biholomorphic maps $\alpha = \alpha_\gamma \colon A' \to \alpha(A') \subset X$ and $\beta = \beta_\gamma \colon B' \to \beta(B') \subset X$ with the following properties:*

(a) *$\gamma \circ \alpha = \beta$ on C',*
(b) *$\mathrm{dist}_{A'}(\alpha, \mathrm{Id}) < \eta$ and $\mathrm{dist}_{B'}(\beta, \mathrm{Id}) < \eta$, and*
(c) *α and β are tangent to the identity map to any given finite order along the subvariety X' intersected with their respective domains.*

In view of Lemma 2.13 we can shrink the set $\widetilde{C}$ if necessary and assume that γ is biholomorphic onto its image. The crucial property (a) then furnishes a compositional splitting of γ. As in [14], the proof will also show that the maps α_γ and β_γ can be chosen to depend smoothly on γ such that $\alpha_{\mathrm{Id}} = \mathrm{Id}$ and $\beta_{\mathrm{Id}} = \mathrm{Id}$.

The proof follows in spirit that of [14, Theorem 4.1], but is technically more involved. We embed a Stein neighborhood of the Stein compact $D = A \cup B$ in X as a closed complex subvariety of $\mathbb{C}^N$. We then use a holomorphic retraction on a neighborhood $\Omega \subset \mathbb{C}^N$ of the Stein compact $C = A \cap B \subset X_{\mathrm{reg}}$ in order to transport the linearized splitting problem to a suitable 1-parameter family of Cartan pairs in $\mathbb{C}^N$ (see Lemma 3.4). (We have been unable to apply [14, Theorem 4.1] directly since the resulting biholomorphic maps α, β need not map the subvariety X to itself.) From this point on we perform an iteration, similar to the one in [14], in which the domains of maps shrink by a controlled amount at every step and the error term converges to zero quadratically.

Proof of Theorem 3.2. Replacing X by an open Stein neighborhood of D we may assume that X is a closed complex subvariety of a Euclidean space $\mathbb{C}^N$ [16, Theorem 2.2.8]. The pair (A, B) is then also a Cartan pair in $\mathbb{C}^N$ [16, Lemma 5.7.2, p. 210]. We shall assume that the distance function dist on X is induced by the Euclidean distance on $\mathbb{C}^N$.

By Cartan's Theorem A there exist entire functions $h_1, \ldots, h_l \in \mathcal{O}(\mathbb{C}^N)$ such that

$$X = \{z \in \mathbb{C}^N : h_i(z) = 0,\ i = 1, \ldots, l\}$$

and $h_1, \ldots, h_l$ generate the ideal sheaf $\mathcal{J}_X$ of X. (We shall only need finite ideal generation on compact subsets of $\mathbb{C}^N$, but in our case this actually holds globally since X is a relatively compact subset of the original Stein space.) Consider the analytic subsheaf $\mathcal{T}_X \subset \mathcal{O}^N_{\mathbb{C}^N}$ whose stalk $\mathcal{T}_{X,p}$ at any $p \in \mathbb{C}^N$ consists of all N-tuples $(g_1, \ldots, g_N) \in \mathcal{O}^N_{\mathbb{C}^N,p}$ satisfying the system of equations

$$\sum_{j=1}^N g_j \frac{\partial h_i}{\partial z_j} \in \mathcal{J}_{X,p}, \qquad i = 1, \ldots, l.$$

The condition is void when $p \notin X$, while at $p \in X$ it means that the vector $V(p) = (g_1(p), \ldots, g_N(p)) \in \mathbb{C}^N \cong T_p\mathbb{C}^N$ is Zariski tangent to X. Observe that $\mathcal{T}_X$ is the preimage of the coherent subsheaf $(\mathcal{J}_X)^l \subset \mathcal{O}^l_{\mathbb{C}^N}$ under the homomorphism $\sigma : \mathcal{O}^N_{\mathbb{C}^N} \to \mathcal{O}^l_{\mathbb{C}^N}$ whose i-th component equals $\sigma_i(g_1, \ldots, g_N) = \sum_{j=1}^N g_j \frac{\partial h_i}{\partial z_j}$. Therefore $\mathcal{T}_X$ is a coherent analytic subsheaf of $\mathcal{O}^N_{\mathbb{C}^N}$. Sections of $\mathcal{T}_X$ are holomorphic vector fields on $\mathbb{C}^N$ which are tangent to X along X. (Note that the quotient $\mathcal{T}_X / \mathcal{J}_X \mathcal{T}_X$, restricted to X, is the *tangent sheaf* of X [10].)

Denote by $\mathcal{J}_{X'}$ the sheaf of ideals of the subvariety $X' \subset X$. Fix $n_0 \in \mathbb{N}$ and consider the coherent analytic sheaf $\mathcal{E} := \mathcal{J}^{n_0}_{X'} \mathcal{T}_X$ on $\mathbb{C}^N$. By Cartan's Theorem A there exist sections $V_1, \ldots, V_m$ of $\mathcal{E}$ that generate $\mathcal{E}$ over the compact set $C = A \cap B \subset X \setminus X'$. These sections are holomorphic vector fields on $\mathbb{C}^N$ which are tangent to X and vanish to order n_0 on X'. Furthermore, as C is contained in X_{reg} and TX_{reg} is the usual tangent bundle, the vectors $V_1(p), \ldots, V_m(p) \in T_p\mathbb{C}^N$ span the tangent space $T_pX \subset T_p\mathbb{C}^N$ at every point $p \in X$ in a neighborhood of C.

Denote by ϕ^j_t the local holomorphic flow of the vector field V_j for a complex value of time t. For each $z \in \mathbb{C}^N$ the flow $\phi^j_t(z)$ is defined for t in a neighborhood of $0 \in \mathbb{C}$. Let $t = (t_1, \ldots, t_m)$ be holomorphic coordinates on $\mathbb{C}^m$. The map

$$s(z, t) = s(z, t_1, \ldots, t_m) = \phi^1_{t_1} \circ \cdots \circ \phi^m_{t_m}(z), \qquad z \in \mathbb{C}^N, \tag{3.1}$$

is defined and holomorphic on an open neighborhood of $\mathbb{C}^N \times \{0\}^m$ in $\mathbb{C}^N \times \mathbb{C}^m$ and assumes values in $\mathbb{C}^N$. Since the vector fields V_j are tangent to X, we have $s(z, t) \in X$ for all t whenever $z \in X$. For any $z \in X$ we denote by

$$Vd(s)_z = \frac{\partial}{\partial t}\Big|_{t=0} s(z, t) : \mathbb{C}^m \to T_zX \tag{3.2}$$

the partial differential of s at z in the fiber direction; we call $Vd(s)$ the *vertical derivative* of s over the subvariety X. The definition of the flow of a vector field implies

$$\left.\frac{\partial s(z,t)}{\partial t_j}\right|_{t=0} = V_j(z), \quad j = 1, \dots, m.$$

Since the vectors $V_1(z), \dots, V_m(z)$ span $T_z X$ at every point $z \in C$, the vertical derivative $Vd(s)$ (3.2) is surjective over a neighborhood of C. Thus s is a *local holomorphic spray* on $\mathbb{C}^N$, and the restriction of s to a neighborhood of C in X is a *dominating spray* (see [16, p. 203] for these notions).

Fix an open Stein set $U_0 \Subset X \setminus X' \subset X_{\rm reg}$ such that $C \subset U_0$ and $Vd(s)$ is surjective over $\overline{U}_0$. It follows that $U_0 \times \mathbb{C}^m = E \oplus E'$, where $E' = \ker Vd(s)|_{U_0}$ and E is a holomorphic vector subbundle of $U_0 \times \mathbb{C}^m$ complementary to E'. (Such an E exists since U_0 is Stein and hence every holomorphic vector subbundle splits the bundle.) Then $Vd(s)\colon E|_{U_0} \to TX|_{U_0} = TU_0$ is an isomorphism of holomorphic vector bundles. By the inverse mapping theorem, the restriction of s to the fiber E_z for any $z \in U_0$ maps an open neighborhood of the origin in E_z biholomorphically onto an open neighborhood of z in X. Shrinking U_0 slightly around C we get an open set $U_1 \supset C$ in X such that the following holds (cf. [14, Lemma 4.4]).

Lemma 3.3. *There are a neighborhood $U_1 \subset X \setminus X'$ of C and constants $\epsilon_1 > 0$ and $M_1 \geq 1$ such that for every open set $U \subset U_1$ and every holomorphic map $\gamma\colon U \to \gamma(U) \subset X$ satisfying* $\mathrm{dist}_U(\gamma, \mathrm{Id}) < \epsilon_1$ *there exists a unique holomorphic section $c\colon U \to E|_U$ satisfying*

$$\gamma(z) = s(z, c(z)) \quad \forall z \in U, \qquad M_1^{-1}\,\mathrm{dist}_U(\gamma, \mathrm{Id}) \leq \|c\|_U \leq M_1\,\mathrm{dist}_U(\gamma, \mathrm{Id}).$$

Since E is a subbundle of the trivial bundle $U \times \mathbb{C}^m$, we may consider any section c in Lemma 3.3 as a holomorphic map $U \to \mathbb{C}^m$, and $\|c\|_U$ denotes the sup norm of c on U measured with respect to the Euclidean metric on $\mathbb{C}^m$. As $C = A \cap B \subset X_{\rm reg}$ is a Stein compact, the Docquier–Grauert theorem [8] (see also [16, Theorem 3.3.3, p. 67]) furnishes an open Stein neighborhood $\Omega \Subset \mathbb{C}^N$ of C and a holomorphic retraction

$$\rho\colon \Omega \to \Omega \cap X \Subset X_{\rm reg}. \tag{3.3}$$

The map

$$T\colon \mathcal{O}(\Omega \cap X) \to \mathcal{O}(\Omega), \quad c \mapsto Tc = c \circ \rho, \tag{3.4}$$

is then a bounded extension operator satisfying $\|Tc\|_\Omega = \|c\|_{\Omega\cap X}$. By choosing Ω small enough we may assume that $\overline{\Omega} \cap X \subset \widetilde{C}$, where $\widetilde{C}$ is as in the statement of Theorem 3.2. We fix the domain Ω, the retraction ρ, and the extension operator T for the rest of the proof.

The following lemma provides the key geometric ingredient in the proof of Theorem 3.2.

Lemma 3.4. *Assume that X is a closed complex subvariety of $\mathbb{C}^N$. Let (A, B) be a Cartan pair in X such that $C := A \cap B \subset X_{\mathrm{reg}}$. Let $\Omega \Subset \mathbb{C}^N$ be an open Stein neighborhood of C and $\rho\colon \Omega \to \Omega \cap X$ be a holomorphic retraction* (3.3)*. Let U_A, U_B be open sets in $\mathbb{C}^N$ such that $A \subset U_A$, $B \subset U_B$, and $U_A \cap U_B \Subset \Omega$. Then there exists a family of smoothly bounded strongly pseudoconvex Cartan pairs (A_t, B_t) in $\mathbb{C}^N$, depending smoothly on the parameter $t \in [0, t_0]$ for some $t_0 > 0$, satisfying the following properties:*

(i) *For any numbers t, τ such that $0 \le t < \tau \le t_0$ we have $A \subset A_t \subset A_\tau \Subset U_A$, $B \subset B_t \subset B_\tau \Subset U_B$, and*

$$\Big(\bigcup_{t\in[0,t_0]} \overline{A_t \setminus B_t}\Big) \cap \Big(\bigcup_{t\in[0,t_0]} \overline{B_t \setminus A_t}\Big) = \emptyset.$$

(ii) *Set $C_t := A_t \cap B_t$. For any numbers t, τ such that $0 \le t < \tau \le t_0$ the distances* $\mathrm{dist}(A_t, \mathbb{C}^N \setminus A_\tau)$, $\mathrm{dist}(B_t, \mathbb{C}^N \setminus B_\tau)$, *and* $\mathrm{dist}(C_t, \mathbb{C}^N \setminus C_\tau)$ *are* $\ge \tau - t > 0$.

(iii) *For every $t \in [0, t_0]$ the boundaries bA_t, bB_t, and bC_t intersect X transversely at any intersection point belonging to $\Omega \cap X$.*

(iv) *$\rho(C_t) = C_t \cap X$ for every $t \in [0, t_0]$.*

Proof. We shall modify the proof of Proposition 5.7.3 in [16, p. 210] so as to also obtain property (iv) which will be crucial. (For a similar result see [39, Proposition 4.4].)

We shall use the function $\mathrm{rmax}\{x, y\}$ on $\mathbb{R}^2$, the *regularized maximum* of x and y (see e.g. [16, p. 61]). It depends on a positive parameter which will be chosen as close to zero as needed at each application. We have $\max\{x, y\} \le \mathrm{rmax}\{x, y\}$, the two functions are equal outside a small conical neighborhood of the diagonal $\{x = y\}$, and they can be made arbitrarily close by choosing the parameter small enough. The rmax of two smooth strongly plurisubharmonic functions is still smooth and strongly plurisubharmonic. The domain $\{\mathrm{rmax}\{\phi, \psi\} < 0\}$ is obtained by smoothing the corners of the intersection $\{\phi < 0\} \cap \{\psi < 0\}$.

Since $C = A \cap B$ is a Stein compact, there is a smoothly bounded strongly pseudoconvex domain $V \Subset U_A \cap U_B$ such that $C \subset V$ and bV intersects X transversely. Let $\theta\colon V_0 \to \mathbb{R}$ be a smooth strongly plurisubharmonic defining function for V, where $V_0 \subset U_A \cap U_B$ is an open neighborhood of $\overline{V}$. Given a subset $A \subset \mathbb{C}^N$ and $r > 0$, we set

$$A(r) = \{z \in \mathbb{C}^N : |z - p| < r \text{ for some } p \in A\}.$$

It is elementary that $(A \cup B)(r) = A(r) \cup B(r)$ and $(A \cap B)(r) \subset A(r) \cap B(r)$. Since A and B are compact and separated, we also have

$$(A \cap B)(r) = A(r) \cap B(r), \qquad \overline{A(r) \setminus B(r)} \cap \overline{B(r) \setminus A(r)} = \emptyset$$

for all sufficiently small $r > 0$ (cf. [16, Lemma 5.7.4]). By choosing $r > 0$ small enough we can also ensure that

$$C(r) = A(r) \cap B(r) \Subset V. \tag{3.5}$$

Fix such an r. Since $A \cup B$ is a Stein compact, there is a smoothly bounded strongly pseudoconvex Stein domain $\Omega_0 \subset \mathbb{C}^N$ such that

$$A \cup B \subset \Omega_0 \Subset A(r) \cup B(r).$$

Pick a smooth strongly plurisubharmonic function $\phi\colon \Omega_0' \to \mathbb{R}$ on an open set $\Omega_0' \supset \overline{\Omega}_0$ such that $\Omega_0 = \{z \in \Omega_0' \colon \phi(z) < 0\}$ and $d\phi \neq 0$ on $b\Omega_0 = \{\phi = 0\}$. We may assume that $\Omega_0' \Subset A(r) \cup B(r)$. Choose $\epsilon_0 > 0$ such that

$$\Omega_1' := \{z \in \Omega_0' \colon \phi(z) < 3\epsilon_0\} \Subset \Omega_0'. \tag{3.6}$$

By [37, Lemma 2.2] there exists a smooth function $\psi \geq 0$ on $\mathbb{C}^N$ such that $\{\psi = 0\} = X$, ψ is strongly plurisubharmonic on $\mathbb{C}^N \setminus X = \{\psi > 0\}$, and the Levi form of ψ at any $z \in X$ is positive except on the tangent space $T_z X$. Choose a smooth function $\chi : \mathbb{C}^N \to [0, 1]$ which equals 0 on a neighborhood of $\overline{U}_A \cap \overline{U}_B \subset \Omega$ and equals 1 on a neighborhood of $\mathbb{C}^N \setminus \Omega$. Since ϕ is strongly plurisubharmonic on Ω_0', there is an $\epsilon \in (0, \epsilon_0)$ such that $\phi - 2\epsilon\chi$ and $\phi + \epsilon\chi$ are strongly plurisubharmonic on Ω_1'; fix such an ϵ. Given constants $M, M' > 0$ (to be determined later) we consider the functions

$$\Phi_1 = (\phi - 2\epsilon\chi)\circ\rho + M\psi, \qquad \Phi_2 = \phi - 2\epsilon + \epsilon\chi + M'\psi, \qquad \Phi = \mathrm{rmax}\big\{\Phi_1, \Phi_2\big\}. \tag{3.7}$$

The function Φ_1 is defined on $\Omega \cap \Omega_0'$ while Φ_2 is defined on Ω_0'. We shall see that for suitable choices of $M, M' > 0$ the function Φ is well defined, smooth and strongly plurisubharmonic on the domain $\Omega_1' = \{\phi < 3\epsilon_0\}$ of (3.6), and for all $t \in \mathbb{R}$ sufficiently close to 0 we have

$$A \cup B \subset D_t := \{z \in \Omega_1' \colon \Phi(z) < t\} \Subset \Omega_1'. \tag{3.8}$$

Since $\phi + \epsilon\chi$ is strongly plurisubharmonic on Ω_1' and ψ is plurisubharmonic on $\mathbb{C}^N$, the function Φ_2 is strongly plurisubharmonic on Ω_1' for any choice of $M' > 0$.

Consider now Φ_1. By the choice of ϵ the function $\phi - 2\epsilon\chi$ is strongly plurisubharmonic on Ω_1', whence Φ_1 is strongly plurisubharmonic on $\Omega \cap \Omega_1'$. Indeed, the first summand $(\phi - 2\epsilon\chi) \circ \rho$ is plurisubharmonic and its Levi form at any point $z \in \Omega \cap X$ is positive definite in the directions tangent to X. The second summand $M\psi$ is strongly plurisubhamonic on $\mathbb{C}^N \setminus X$, and its Levi form at points of X is positive in directions that are not tangent to X. Hence the Levi form of the sum is positive everywhere. Observe that $\Phi_1 > 0$ near $b\Omega_1' \cap \Omega \cap X$ by the definition (3.6) of Ω_1' and the fact that $2\epsilon\chi \leq 2\epsilon < 2\epsilon_0$. By choosing $M > 0$ sufficiently large we can thus ensure that $\Phi_1 > 0$ near $b\Omega_1' \cap \Omega$. We fix such an M for the rest of the proof.

Next we show that $\Phi = \mathrm{rmax}\big\{\Phi_1, \Phi_2\big\}$ is well defined if the constant $M' > 0$ in Φ_2 is chosen large enough. We need to ensure that $\Phi_1 < \Phi_2$ on the domain of Φ_1 near the boundary of Ω, so Φ_2 takes over in rmax before we exit the domain of Φ_1. On $X = \{\psi = 0\}$ this is clear since near $b\Omega$ we have $\chi = 1$ and hence $\Phi_1 = \phi - 2\epsilon < \phi - \epsilon = \Phi_2$. By choosing $M' > 0$ sufficiently large we get $\Phi_1 < \Phi_2$ on a neighborhood of $b\Omega \cap \Omega_1'$; hence $\mathrm{rmax}\{\Phi_1, \Phi_2\}$ is well defined on Ω_1'. By increasing M' we can also ensure that $\Phi > 0$ near $b\Omega_1'$, so the domains $D_t = \{\Phi < t\}$ for t close to 0 (say $|t| \leq t_1$ for some $t_1 > 0$) satisfy (3.8). By Sard's theorem and compactness of the level sets of Φ we can find a nontrivial interval $I \subset [-t_1, t_1]$ which contains no critical values of Φ or $\Phi|_{X\cap\overline{\Omega}}$. Hence D_t for $t \in I$ are smoothly bounded strongly pseudoconvex domains intersecting X transversely within Ω.

On the intersection of the domain of Φ_1 with $\{\chi = 0\}$ (in particular, on $U_A \cap U_B \cap \Omega'_1$) we have $\Phi_1 = \phi\circ\rho + M\psi \geq \phi\circ\rho$, so on this set the retraction ρ of (3.3) projects $\{\Phi_1 < t\}$ to $\{\Phi_1 < t\} \cap X$. Furthermore, on $X \cap \{\chi = 0\}$ we have $\Phi_1 = \phi > \phi - 2\epsilon = \Phi_2$. This shows that the domain $D_t = \{\Phi < t\}$ agrees with $\{\Phi_1 < t\}$ near $X \cap \{\chi = 0\}$. It follows that

$$\rho\,(D_t \cap \{\chi = 0\}) = D_t \cap X \cap \{\chi = 0\}. \tag{3.9}$$

It remains to find a Cartan pair decomposition (A_t, B_t) of D_t. Recall that $\overline{C(r)} \subset V = \{\theta < 0\}$ by (3.5). Replacing θ by $c\theta$ for a suitably chosen constant $c > 0$ we may therefore assume that $\theta < \Phi$ on $C(r) \cap \Omega'_1$. Perturbing θ and V slightly we can ensure that the real hypersurfaces bV and $bD_0 = \{\Phi = 0\}$ intersect transversely. The function

$$\phi_C = \mathrm{rmax}\{\phi, \theta\} \colon \Omega'_1 \cap V_0 \to \mathbb{R}$$

is smooth and strongly plurisubharmonic. For every $t \in I$ the set

$$C'_t := \{z \in \Omega'_1 \cap V_0 \cap X \colon \phi_C(z) < t\} \subset X_{\mathrm{reg}}$$

is a smoothly bounded strongly pseudoconvex domain. We have $C(r) \cap X \subset C'_t$ in view of (3.5) and $C'_t \subset D'_t := D_t \cap X$ since $\phi_C \geq \phi$. As $\theta < \phi$ on $\overline{C(r)} \cap \Omega'_1 \cap X$, we have $\phi_C = \phi$ there, so the boundaries bC'_t and bD'_t coincide along their intersection with the compact set $\overline{C(r)} \cap X$. Hence C'_t separates D'_t in the sense of a Cartan pair, i.e., $D'_t = A'_t \cup B'_t$ and $A'_t \cap B'_t = C'_t$. Set

$$\Theta = \mathrm{rmax}\{\phi_C \circ \rho + M\psi, \Phi\}, \qquad C_t = \{\Theta < t\},$$

where $\Phi = \mathrm{rmax}\{\Phi_1, \Phi_2\}$ (see (3.7)) and M is the constant in the definition of Φ_1. One easily verifies that C_t is a strongly pseudoconvex domain which separates D_t into a Cartan pair (A_t, B_t) with $D_t = A_t \cap B_t$ and $C_t = A_t \cap B_t$. It follows from (3.9) that C_t satisfies Lemma 3.4(iv). By decreasing the parameter interval I we ensure that Θ and $\Theta|_X$ have no critical values in I, so the boundaries bC_t for $t \in I$ are smooth and intersect X transversely. The same is then true for the domains A_t and B_t since their boundaries are contained in $bD_t \cup bC_t$. Reparametrizing the t variable and changing the functions Φ and Θ by an additive constant we may assume that $I = [0, t_0]$ for some $t_0 > 0$ and property (ii) holds. The remaining properties of A_t, B_t and C_t follow directly from the construction. □

Given an open set $U \subset X$ and a number $\delta > 0$, we shall use the notation

$$U(\delta) = \{z \in X \colon \mathrm{dist}(z, U) < \delta\}.$$

Recall that s is the spray (3.1) and M_1 is the constant from Lemma 3.3.

The following lemma is a special case of [14, Lemma 4.5].

Lemma 3.5. *Let $U_1 \subset X \setminus X'$ be the open set from Lemma 3.3. There exist constants $\delta_0 > 0$ (small) and $M_2 > 0$ (large) with the following property. Let $0 < \delta < \delta_0$ and $0 < 4\epsilon < \delta$. Let U be an open set in X such that $U(\delta) \subset U_1$. Assume $\alpha, \beta, \gamma \colon U(\delta) \to X$*

are holomorphic maps which are ϵ*-close to the identity on* $U(\delta)$*. Then* $\tilde{\gamma} := \beta^{-1} \circ \gamma \circ \alpha \colon U \to X$ *is a well defined holomorphic map. Write*

$$\alpha(z) = s(z, a(z)), \quad \beta(z) = s(z, b(z)),$$
$$\gamma(z) = s(z, c(z)), \quad \tilde{\gamma}(x) = s(z, \tilde{c}(z)),$$

where a, b, c *are holomorphic sections of the vector bundle* $E|_{U(\delta)}$ *and* $\tilde{c}$ *is a holomorphic section of* $E|_U$ *furnished by Lemma 3.3. If* $c = b - a$ *on* $U(\delta)$*, then*

$$\|\tilde{c}\|_U \le M_2 \delta^{-1} \epsilon^2 \quad \textit{and} \quad \mathrm{dist}_U(\tilde{\gamma}, \mathrm{Id}) \le M_1 M_2 \delta^{-1} \epsilon^2.$$

The next lemma provides a solution of the Cousin-I problem with bounds on the family of strongly pseudoconvex domains $D_t = A_t \cup B_t$ from Lemma 3.4, intersected with the subvariety X. We denote by $\mathcal{H}^\infty(D)$ the Banach space of all bounded holomorphic functions on D.

Lemma 3.6. *Let* (A_t, B_t) $(t \in [0, t_0])$ *be a family of strongly pseudoconvex Cartan pairs furnished by Lemma 3.4. There is a constant* $M_3 > 0$ *with the following property. For every* $t \in [0, t_0]$ *and* $c \in \mathcal{H}^\infty(C_t \cap X)$ *there exist* $a \in \mathcal{H}^\infty(A_t)$ *and* $b \in \mathcal{H}^\infty(B_t)$ *such that*

$$c = b - a \quad \textit{on } C_t \cap X, \qquad \|a\|_{A_t} \le M_3 \|c\|_{C_t \cap X}, \qquad \|b\|_{B_t} \le M_3 \|c\|_{C_t \cap X}.$$

The functions a *and* b *are given by bounded linear operators applied to* c*.*

Proof. We begin by finding a constant $M_3 > 0$ independent of $t \in [0, t_0]$ and linear operators

$$\mathcal{A}_t \colon \mathcal{H}^\infty(C_t) \to \mathcal{H}^\infty(A_t), \quad \mathcal{B}_t \colon \mathcal{H}^\infty(C_t) \to \mathcal{H}^\infty(B_t)$$

such that for every $g \in \mathcal{H}^\infty(C_t)$ $(t \in [0, t_0])$ we have

$$g = \mathcal{A}_t(g) - \mathcal{B}_t(g) \quad \text{on } C_t \tag{3.10}$$

and the estimates

$$\|\mathcal{A}_t(g)\|_{A_t} \le M_3 \|g\|_{C_t}, \qquad \|\mathcal{B}_t(g)\|_{B_t} \le M_3 \|g\|_{C_t}. \tag{3.11}$$

The proof is similar to that of [16, Lemma 5.8.2, p. 212] and uses standard techniques; we include it for completeness.

In view of Lemma 3.4(i) there is a smooth function $\xi \colon \mathbb{C}^N \to [0, 1]$ such that $\xi = 0$ on $\bigcup_{t \in [0,1]} \overline{A_t \setminus B_t}$ and $\xi = 1$ on $\bigcup_{t \in [0,1]} \overline{B_t \setminus A_t}$. For any $g \in \mathcal{H}^\infty(C_t)$ the product ξg extends to a bounded smooth function on A_t that vanishes on $A_t \setminus B_t$, and $(\xi - 1)g$ extends to a bounded smooth function on B_t that vanishes on $B_t \setminus A_t$. Furthermore, $\overline{\partial}(\xi g) = \overline{\partial}((\xi - 1)g) = g\,\overline{\partial}\xi$ is a smooth bounded $(0, 1)$-form on the strongly pseudoconvex domain $D_t = A_t \cup B_t$ with support contained in $C_t = A_t \cap B_t$.

Let S_t be a sup-norm bounded linear solution operator to the $\overline{\partial}$-equation on D_t at the level of $(0, 1)$-forms. (Such an S_t can be found as a Henkin–Ramírez integral kernel operator; see the monographs by Henkin and Leiterer [27] or Lieb and Michel [31]. The operators S_t can be chosen to depend smoothly on $t \in [0, 1]$. For small perturbations of a given strongly pseudoconvex domain this is evident from the construction and is stated explicitly in the cited sources; for compact families of strongly pseudoconvex domains the result follows by applying a smooth partition of unity on the parameter space.) Given $g \in \mathcal{H}^\infty(C_t)$, set

$$\mathcal{A}_t(g) = \xi g - S_t(g\overline{\partial}\xi) \in \mathcal{H}^\infty(A_t), \qquad \mathcal{B}_t(g) = (\xi - 1)g - S_t(g\overline{\partial}\xi) \in \mathcal{H}^\infty(B_t).$$

It is immediate that these operators have the stated properties.

By Lemma 3.4(iv) the map $c \mapsto T(c) = c \circ \rho$ of (3.4) induces a linear extension operator $\mathcal{O}(C_t \cap X) \to \mathcal{O}(C_t)$ satisfying $\|Tc\|_{C_t} = \|c\|_{C_t \cap X}$. The compositions

$$\mathcal{A}_t \circ T\colon \mathcal{H}^\infty(C_t \cap X) \to \mathcal{H}^\infty(A_t), \qquad \mathcal{B}_t \circ T\colon \mathcal{H}^\infty(C_t \cap X) \to \mathcal{H}^\infty(B_t)$$

are then bounded linear operators satisfying the conclusion of Lemma 3.6. □

Lemma 3.7. *Let $(A_t, B_t) = (A(t), B(t))$ $(t \in [0, t_0])$ be strongly pseudoconvex Cartan pairs furnished by Lemma 3.4. Set $C(t) = A(t) \cap B(t)$. Let $\delta_0 > 0$ be chosen as in Lemma 3.5. Then there are constants $M_4, M_5 > 0$ satisfying the following property. Let $0 \le t < t + \delta \le t_0$ and $0 < \delta < \delta_0$. For every holomorphic map $\gamma\colon C(t+\delta) \cap X \to X$ satisfying*

$$\epsilon := \mathrm{dist}_{C(t+\delta)\cap X}(\gamma, \mathrm{Id}) < \frac{\delta}{4M_4}$$

there exist holomorphic maps $\alpha\colon A(t+\delta) \to \mathbb{C}^N$ and $\beta\colon B(t+\delta) \to \mathbb{C}^N$, tangent to the identity map to order n_0 along the subvariety X' and satisfying

$$\mathrm{dist}_{A(t+\delta)}(\alpha, \mathrm{Id}) < M_4\epsilon, \quad \mathrm{dist}_{B(t+\delta)}(\beta, \mathrm{Id}) < M_4\epsilon, \tag{3.12}$$

such that

$$\tilde{\gamma} = \beta^{-1} \circ \gamma \circ \alpha\colon C(t) \cap X \to X$$

is a well defined holomorphic map satisfying

$$\mathrm{dist}_{C(t)\cap X}(\tilde{\gamma}, \mathrm{Id}) < M_5\delta^{-1}\mathrm{dist}_{C(t+\delta)\cap X}(\gamma, \mathrm{Id})^2 = M_5\delta^{-1}\epsilon^2. \tag{3.13}$$

Proof. On the Banach space $\mathcal{H}^\infty(D)^N$ we use the norm $\|g\| = \sum_{j=1}^N \|g_j\|$, where $\|g_j\|$ is the sup norm on D. By Lemma 3.3 there is a holomorphic section $c\colon C(t+\delta)\cap X \to E$ of the holomorphic vector bundle $E \to U_1$ such that $\gamma(z) = s(z, c(z))$ for $z \in C(t+\delta) \cap X$ and $\|c\|_{C(t+\delta)\cap X} \le M_1\epsilon$. (Here M_1 is the constant from Lemma 3.3.) By Lemma 3.6 we have $c = b - a$, where $a \in \mathcal{H}^\infty(A_t)^N$ and $b \in \mathcal{H}^\infty(B_t)^N$ satisfy the estimates

$$\|a\|_{A_t} \le NM_1M_3\epsilon, \qquad \|b\|_{B_t} \le NM_1M_3\epsilon.$$

Let s be the spray (3.1). Set

$$\alpha(z) = s(z, a(z)), \qquad z \in A(t+\delta),$$
$$\beta(z) = s(z, b(z)), \qquad z \in B(t+\delta).$$

The maps α and β are tangent to the identity to order n_0 along the subvariety X' and satisfy $\alpha(A_t \cap X) \subset X$ and $\beta(B_t \cap X) \subset X$. By Lemma 3.3 we have

$$\mathrm{dist}_{A(t+\delta)}(\alpha, \mathrm{Id}) < N M_1^2 M_3\, \epsilon, \quad \mathrm{dist}_{B(t+\delta)}(\beta, \mathrm{Id}) < N M_1^2 M_3\, \epsilon.$$

Setting $M_4 = N M_1^2 M_3$ we get the estimates (3.12). If the number $\epsilon = \mathrm{dist}_{C(t+\delta)}(\gamma, \mathrm{Id})$ satisfies $4M_4\epsilon < \delta$, then by Lemma 3.5 the composition $\tilde{\gamma} = \beta^{-1} \circ \gamma \circ \alpha$ is a well defined holomorphic map on $C(t) \cap X$ satisfying (3.13) with $M_5 = M_2 M_4^2$. □

We now complete the proof of Theorem 3.2.by a recursive process, using Lemma 3.7 at every step. The initial map γ is defined on $\widetilde{C} \supset C(t_0) \cap X$. For each $k \in \mathbb{Z}_+$ we set

$$t_k = t_0 \prod_{j=1}^{k} (1 - 2^{-j}) \quad \text{and} \quad \delta_k = t_k - t_{k+1} = t_k 2^{-k-1}.$$

The sequence $t_k > 0$ is decreasing, $t^* = \lim_{k\to\infty} t_k > 0$, $\delta_k > t^* 2^{-k-1}$ for all k, and $\sum_{k=0}^{\infty} \delta_k = t_0 - t^*$. Set $A_k = A(t_k)$, $B_k = B(t_k)$, $C_k = C(t_k) = A_k \cap B_k$ and observe that

$$\bigcap_{k=0}^{\infty} A_k = \overline{A(t^*)}, \quad \bigcap_{k=0}^{\infty} B_k = \overline{B(t^*)}, \quad \bigcap_{k=0}^{\infty} C_k = \overline{C(t^*)}.$$

To begin the induction, pick $\epsilon_0 > 0$ such that $4M_4\epsilon_0 < \delta_0 = t_0/2$. Set $\gamma_0 = \gamma$ and assume that $\mathrm{dist}_{C_0 \cap X}(\gamma_0, \mathrm{Id}) \le \epsilon_0$. Lemma 3.7 furnishes holomorphic maps $\alpha_1 \colon A_1 \to \mathbb{C}^N$ and $\beta_1 \colon B_1 \to \mathbb{C}^N$ satisfying

$$\mathrm{dist}_{A_1}(\alpha_1, \mathrm{Id}) < M_4\, \epsilon_0, \quad \mathrm{dist}_{B_1}(\beta_1, \mathrm{Id}) < M_4\, \epsilon_0$$

(see (3.12)) such that $\gamma_1 = \beta_1^{-1} \circ \gamma_0 \circ \alpha_1 \colon C_1 \cap X \to X$ is a well defined holomorphic map satisfying (cf. (3.13))

$$\epsilon_1 := \mathrm{dist}_{C_1 \cap X}(\gamma_1, \mathrm{Id}) < M_5 \delta_0^{-1} \epsilon_0^2 < 2M\epsilon_0^2$$

where $M = M_5/t^*$. If we assume that $4M_4\epsilon_1 < \delta_1$ (which holds if $\epsilon_0 > 0$ is small enough), Lemma 3.7 furnishes holomorphic maps $\alpha_2 \colon A_2 \to \mathbb{C}^N$ and $\beta_2 \colon B_2 \to \mathbb{C}^N$ satisfying

$$\mathrm{dist}_{A_2}(\alpha_2, \mathrm{Id}) < M_4\, \epsilon_1, \quad \mathrm{dist}_{B_2}(\beta_2, \mathrm{Id}) < M_4\, \epsilon_1$$

and such that $\gamma_2 = \beta_2^{-1} \circ \gamma_1 \circ \alpha_2 \colon C_2 \cap X \to X$ is a well defined holomorphic map satisfying

$$\epsilon_2 := \mathrm{dist}_{C_2 \cap X}(\gamma_2, \mathrm{Id}) < M_5 \delta_1^{-1} \epsilon_1^2 < 2^2 M \epsilon_1^2.$$

Proceeding inductively, we obtain sequences of holomorphic maps

$$\alpha_k \colon A_k \to \mathbb{C}^N, \quad \beta_k \colon B_k \to \mathbb{C}^N, \quad \gamma_k \colon C_k \cap X \to X$$

such that the following conditions hold for every $k = 0, 1, 2, \ldots$:

$$\gamma_{k+1} = \beta_{k+1}^{-1} \circ \gamma_k \circ \alpha_{k+1} \colon C_{k+1} \cap X \to X,$$

$$\mathrm{dist}_{A_{k+1}}(\alpha_{k+1}, \mathrm{Id}) < M_4\, \epsilon_k, \qquad \mathrm{dist}_{B_{k+1}}(\beta_{k+1}, \mathrm{Id}) < M_4\, \epsilon_k, \tag{3.14}$$

$$\epsilon_{k+1} := \mathrm{dist}_{C_{k+1}\cap X}(\gamma_{k+1}, \mathrm{Id}) < M_5 \delta_k^{-1} \epsilon_k^2 < 2^{k+1} M \epsilon_k^2. \tag{3.15}$$

The necessary condition for the induction to proceed is that $4M_4\epsilon_k < \delta_k$ for each $k = 0, 1, 2, \ldots$. By [14, Lemma 4.8] this holds as long as the initial number $\epsilon_0 > 0$ is chosen small enough, and the resulting sequence $\epsilon_k > 0$, defined recursively by (3.15), then converges to zero very rapidly. In particular, we can ensure that

$$M_4 \sum_{j=0}^{\infty} \epsilon_j < \eta \tag{3.16}$$

where $\eta > 0$ is as in the statement of the theorem.

The estimates (3.14) imply that the compositions

$$\tilde\alpha_k = \alpha_1 \circ \cdots \circ \alpha_k \colon A_k \to \mathbb{C}^N, \qquad \tilde\beta_k = \beta_1 \circ \cdots \circ \beta_k \colon B_k \to \mathbb{C}^N \tag{3.17}$$

are well defined holomorphic maps for $k = 1, 2, \ldots$ satisfying

$$\tilde\beta_k \circ \gamma_k = \gamma \circ \tilde\alpha_k \quad \text{ on } C_k \cap X. \tag{3.18}$$

As $k \to \infty$, the estimates (3.15) and (3.16) show that γ_k converges to the identity map uniformly on $C(t^*) \cap X$.

Consider now the sequences $\tilde\alpha_k$ and $\tilde\beta_k$. From (3.14) and (3.16) we clearly get

$$\mathrm{dist}_{A_k}(\tilde\alpha_k, \mathrm{Id}) < M_4 \sum_{j=0}^{k-1} \epsilon_j < \eta, \qquad \mathrm{dist}_{B_k}(\tilde\beta_k, \mathrm{Id}) < M_4 \sum_{j=0}^{k-1} \epsilon_j < \eta. \tag{3.19}$$

Hence $\tilde\alpha_k$ and $\tilde\beta_k$ are normal families on $A(t^*)$ and $B(t^*)$, respectively. Passing to convergent subsequences we get holomorphic maps $\alpha \colon A(t^*) \to \mathbb{C}^N$ and $\beta \colon B(t^*) \to \mathbb{C}^N$ satisfying

$$\mathrm{dist}_{A(t^*)}(\alpha, \mathrm{Id}) < \eta, \qquad \mathrm{dist}_{B(t^*)}(\beta, \mathrm{Id}) < \eta, \qquad \gamma \circ \alpha = \beta \quad \text{ on } C(t^*) \cap X.$$

The last equation follows from (3.18). Assuming that $\epsilon_0, \eta > 0$ are small enough, the maps α, β and γ are biholomorphic on slightly smaller domains in view of Lemma 2.13. Finally, since all α_k and β_k are tangent to the identity to order n_0 along X', the same is true for α and β. □

Remark 3.8. 1. The sequences $\tilde\alpha_k$ and $\tilde\beta_k$ in (3.17) actually converge uniformly on any compact subset $K_A \subset A(t^*)$ and $K_B \subset B(t^*)$, respectively. Indeed, choose open domains A' and B' in $\mathbb{C}^N$ such that $K_A \subset A' \Subset A(t^*)$ and $K_B \subset B' \Subset B(t^*)$. From (3.19) and the Cauchy estimates we infer that these sequences are uniformly Lipschitz on A' and B', respectively. From this and (3.19) we easily see that the sequences are uniformly Cauchy, and hence uniformly convergent, on K_A and K_B, respectively.

2. If X is a Stein manifold with the *density property* in the sense of Varolin [16, §4.10] and (A, B) is a Cartan pair in X such that the set $C = A \cap B$ is $\mathcal{O}(X)$-convex, then the conclusion of Theorem 3.2 holds for any biholomorphic map γ on a neighborhood $U \subset X$ of C which is isotopic to the identity map on C through a smooth 1-parameter family of biholomorphic maps $\gamma_t \colon U \to \gamma_t(U) \subset X$ $(t \in [0,1])$ such that $\gamma_0 = \mathrm{Id}$, $\gamma_1 = \gamma$, and $\gamma_t(C)$ is $\mathcal{O}(X)$-convex for every $t \in [0,1]$. The main result of Andersén–Lempert theory (cf. [16, Theorem 4.9.2] for $X = \mathbb{C}^n$ and [16, Theorem 4.10.6] for the general case) implies that γ can be approximated uniformly on a neighborhood of C by holomorphic automorphisms $\phi \in \mathrm{Aut}(X)$. This allows us to write $\gamma = \phi \circ \tilde{\gamma}$, where $\tilde{\gamma}$ is a biholomorphic map close to the identity on a neighborhood of C. Applying Theorem 3.2 gives $\tilde{\gamma} = \tilde{\beta} \circ \alpha^{-1}$, where α and $\tilde{\beta}$ are biholomorphic maps close to the identity near A and B, respectively. Setting $\beta = \phi \circ \tilde{\beta}$ gives $\gamma = \beta \circ \alpha^{-1}$.

Remark 3.9. Theorem 3.2 and its proof are amenable to various generalizations. In particular, since all steps in the proof are obtained by using (possibly nonlinear) operators on various function spaces, it immediately generalizes to the parametric case when the data (in particular, the map γ to be decomposed in the form $\gamma = \beta \circ \alpha^{-1}$) depend on parameters. A more ambitious generalization would amount to also letting the domains of these maps depend on parameters; this may be applicable in various constructions. A first step in this direction has already been made in [7].

4. Functions without critical points in the top-dimensional strata

In this section we construct holomorphic functions that have no critical points in the regular locus of a Stein space. The following result is the main inductive step in the proof of Theorem 1.5 given in the following section.

Theorem 4.1. *Assume that X is a Stein space, $X' \subset X$ is a closed complex subvariety of X containing X_{sing}, $P = \{p_1, p_2, \ldots\}$ is a closed discrete set in X', K is a compact $\mathcal{O}(X)$-convex set in X (possibly empty), and f is a holomorphic function on a neighborhood of $K \cup X'$ such that $\mathrm{Crit}(f|_{U\setminus X'}) = \emptyset$ for some neighborhood $U \subset X$ of K. Then for any $\epsilon > 0$ and $r, n_k \in \mathbb{N}$ $(k = 1, 2, \ldots)$ there exists $F \in \mathcal{O}(X)$ satisfying the following conditions:*

(i) *$F - f$ vanishes to order r along X',*
(ii) *$F - f$ vanishes to order n_k at $p_k \in P$ for every $k = 1, 2, \ldots$,*
(iii) *$\|F - f\|_K < \epsilon$, and*
(iv) *F has no critical points in $X \setminus X'$.*

Applying Theorem 4.1 with $X' = X_{\mathrm{sing}}$ we find holomorphic functions on X that have no critical points in the smooth part X_{reg}.

Remark 4.2. Theorem 4.1 implies at no cost the following result. Let $A = \{a_j\}$ be a closed discrete set in X contained in $X \setminus (K \cup X')$. Then there exists $F \in \mathcal{O}(X)$ satisfying (i)–(iii) and also

(iv′) $\mathrm{Crit}(F|_{X\setminus X'}) = A$.

Indeed, choose any germs $g_j \in \mathcal{O}_{X,a_j}$ at $a_j \in A$ and apply Theorem 4.1 with the subvariety $X'_0 = A \cup X'$, the discrete set $P_0 = A \cup P$, and the function f extended by g_j to a small neighborhood of $a_j \in A$.

We begin with a couple lemmas.

Lemma 4.3. (Assumptions as in Theorem 4.1.) *Let L be a compact $\mathcal{O}(L)$-convex set in X such that $K \subset \mathring{L}$. Then there exists $\tilde f \in \mathcal{O}(X)$ with the following properties:*

(a) *$\tilde f - f$ vanishes to order r along X',*
(b) *$\tilde f - f$ vanishes to order n_k at $p_k \in P$ for every $k = 1, 2, \ldots,$*
(c) *$\|\tilde f - f\|_K < \epsilon$, and*
(d) *there is a neighborhood $W \subset X$ of the compact set $K \cup (L \cap X')$ such that $\tilde f$ has no critical points in $W \setminus X'$.*

Proof. Let $\mathcal{E} \subset \mathcal{O}_X$ be the coherent sheaf of ideals whose stalk at any $p_k \in P$ equals $\mathfrak{m}_{p_k}^{n_k}$ and $\mathcal{E}_x = \mathcal{O}_{X,x}$ for every $x \in X \setminus P$. The product

$$\widetilde{\mathcal{E}} := \mathcal{E}\mathcal{J}^r_{X'} \subset \mathcal{O}_X \tag{4.1}$$

of $\mathcal{E}$ and the r-th power of the ideal sheaf $\mathcal{J}_{X'}$ is a coherent sheaf of ideals in $\mathcal{O}_X$. Consider the short exact sequence of sheaf homomorphisms

$$0 \to \widetilde{\mathcal{E}} \to \mathcal{O}_X \to \mathcal{O}_X/\widetilde{\mathcal{E}} \to 0.$$

Since the quotient sheaf $\mathcal{O}_X/\widetilde{\mathcal{E}}$ is supported on X', the function f determines a section of $\mathcal{O}_X/\widetilde{\mathcal{E}}$. Since $H^1(X;\widetilde{\mathcal{E}}) = 0$ by Cartan's Theorem B, the same section is induced by a function $g \in \mathcal{O}(X)$. Clearly g satisfies conditions (a) and (b) of the lemma (with g in place of $\tilde f$). To get (c) we proceed as follows. Cartan's Theorem A furnishes sections $\xi_1, \ldots, \xi_m \in \Gamma(X, \widetilde{\mathcal{E}})$ which generate the sheaf $\widetilde{\mathcal{E}}$ over the compact set K. By the choice of g the difference $f - g$ is a section of $\widetilde{\mathcal{E}}$ over a neighborhood of K. Applying Theorem B to the epimorphism of coherent analytic sheaves $\mathcal{O}_X^m \to \widetilde{\mathcal{E}} \to 0$, $(h_1, \ldots, h_m) \mapsto \sum_{i=1}^m h_i\xi_i$, we obtain $f = g + \sum_{i=1}^m h_i\xi_i$ on a neighborhood of K for some $h_i \in \mathcal{O}(K)$. By the Oka–Weil theorem we can approximate the h_i's uniformly on K by some $\tilde h_i \in \mathcal{O}(X)$. The function $\tilde f = g + \sum_{i=1}^m \tilde h_i\xi_i \in \mathcal{O}(X)$ satisfies (a)–(c). By the Stability Lemma 2.7 and the Genericity Lemma 2.9 we can also satisfy (d) by choosing $\tilde f$ generic and taking $\epsilon > 0$ small enough. ◻

The next lemma is the main step in the proof of Theorem 4.1; here we use the splitting lemma furnished by Theorem 3.2. Another key ingredient is the Runge approximation theorem for noncritical holomorphic functions on $\mathbb{C}^n$, furnished by Theorem 3.1 in [14].

Lemma 4.4. (Assumptions as in Theorem 4.1.) *Let $L \subset X$ be a compact $\mathcal{O}(X)$-convex set such that $K \subset \mathring{L}$. Then there exists $F \in \mathcal{O}(X)$ which satisfies conditions* (i)–(iii) *in Theorem 4.1 and also the following condition:*

(iv′) $\mathrm{Crit}(F|_{U' \setminus X'}) = \emptyset$, *where $U' \subset X$ is an open neighborhood of L.*

Proof. To simplify the exposition, we replace the number r by the maximum of r and the numbers $n_k \in \mathbb{N}$ over all points $p_k \in P \cap L$ (a finite set). If we choose F to satisfy condition (i) for this new r, it will also satisfy (ii) at $p_k \in P \cap L$.

Let $\tilde{f} \in \mathcal{O}(X)$ be a function satisfying the conclusion of Lemma 4.3. To simplify the notation we replace f by $\tilde{f}$ and drop the tilde in what follows. Let W be the set from Lemma 4.3(d). By [18, Lemma 8.4, p. 662] there exist finitely many compact $\mathcal{O}(X)$-convex sets $A_0 \subset A_1 \subset \cdots \subset A_m = L$ such that $K \cup (L \cap X') \subset A_0 \subset W$ and for every $j = 0, 1, \ldots, m-1$ we have $A_{j+1} = A_j \cup B_j$, where (A_j, B_j) is a Cartan pair (Definition 3.1) and $B_j \subset L \setminus X' \subset X_{\mathrm{reg}}$. Furthermore, the construction in [19] gives for every $j = 0, 1, \ldots, m-1$ an open set $U_j \subset X_{\mathrm{reg}}$ containing B_j and a biholomorphic map $\phi_j \colon U_j \to U'_j \subset \mathbb{C}^n$ onto an open subset of $\mathbb{C}^n$ (where n is the dimension of X at the points of B_j) such that $\phi_j(C_j)$ is polynomially convex in $\mathbb{C}^n$. (Here $C_j = A_j \cap B_j$.) The proof of Lemma 8.4 in [18] is written in the case when X is a Stein manifold, but it also applies in the present situation, for example, by embedding a relatively compact neighborhood of $L \subset X$ as a closed complex subvariety in $\mathbb{C}^N$.

We shall first find a function $\widetilde{F}$ that is holomorphic on a neighborhood of L and satisfies the conclusion of the lemma there. This is accomplished by a finite induction, starting with $F_0 = f$ which by the assumption satisfies these properties on the open set $W \supset A_0$. We provide an outline and refer to [14] for further details.

By the assumption F_0 is noncritical on a neighborhood of the set $C_0 = A_0 \cap B_0$. Since C_0 is polynomially convex in a certain holomorphic coordinate system on a neighborhood of B_0 in $X \setminus X' \subset X_{\mathrm{reg}}$, Theorem 3.1 in [14, p. 154] furnishes a noncritical holomorphic function G_0 on a neighborhood of B_0 in $X \setminus X'$ such that G_0 approximates F_0 as closely as desired uniformly on a neighborhood of C_0. Assuming that the approximation is close enough, we can apply Theorem 3.2 to glue F_0 and G_0 into a new function F_1 that is holomorphic on a neighborhood of $A_0 \cup B_0 = A_1$ and has no critical points, except perhaps on X'. The gluing of F_0 and G_0 is accomplished by first finding a biholomorphic map γ close to the identity on a neighborhood of the attaching set C_0 such that

$$F_0 = G_0 \circ \gamma \quad \text{on a neighborhood of } C_0.$$

Since C_0 is a Stein compact in the complex manifold $X \setminus X'$, such a γ is furnished by [14, Lemma 5.1, p. 167]. If γ is close enough to Id (which holds if G_0 is chosen sufficiently uniformly close to F_0 on a neighborhood of C_0), then Theorem 3.2 furnishes a decomposition

$$\gamma \circ \alpha = \beta,$$

where α is a biholomorphic map close to the identity on a neighborhood of A_0 in X, and β is a map with the analogous properties on a neighborhood of B_0 in X. By Theorem 3.2 we can ensure in addition that α is tangent to the identity to order r along the subvariety X' intersected with its domain. (The domain of β does not intersect X'.) Then

$$F_0 \circ \alpha = G_0 \circ \beta \quad \text{on a neighborhood of } C_0,$$

so the two sides amalgamate into a holomorphic function F_1 on a neighborhood of $A_0 \cup B_0 = A_1$. By the construction, F_1 approximates F_0 on a neighborhood of A_0, $F_1 - F_0$ vanishes to order r along X', and F_1 is noncritical except perhaps on X'. The last property holds because the maps α and β are biholomorphic on their respective domains and $\alpha|_{X'}$ is the identity.

Repeating the same construction with F_1 we get the next function F_2 on a neighborhood of A_2, etc. In m steps of this kind we find a function $\widetilde{F} = F_m$ on a neighborhood of the set $A_m = L$ satisfying the stated properties.

It remains to replace $\widetilde{F}$ by a function $F \in \mathcal{O}(X)$ satisfying the same properties. This is done as in Lemma 4.3 above. Let $\widetilde{\mathcal{E}}$ be the sheaf (4.1). Pick $\xi_1, \ldots, \xi_m \in \Gamma(X, \widetilde{\mathcal{E}})$ which generate $\widetilde{\mathcal{E}}$ over the compact set L. By the construction of $\widetilde{F}$, the difference $\widetilde{F} - f$ is a section of $\widetilde{\mathcal{E}}$ over a neighborhood of L. Hence Cartan's Theorem B furnishes holomorphic functions $\tilde{h}_1, \ldots, \tilde{h}_m \in \mathcal{O}(U')$ on an open set $U' \supset L$ such that

$$\widetilde{F} = f + \sum_{i=1}^{m} \tilde{h}_i \xi_i \quad \text{on } U'.$$

Choose a compact $\mathcal{O}(X)$-convex set L' such that $L \subset \mathring{L}' \subset L' \subset U'$. Approximating each $\tilde{h}_i$ uniformly on L' by a function $h_i \in \mathcal{O}(X)$ and setting

$$F = f + \sum_{i=1}^{m} h_i \xi_i \in \mathcal{O}(X)$$

we get a function F satisfying (i)–(iii). By Lemma 2.7 the function F also satisfies (iv′) provided that the differences $\|h_i - \tilde{h}_i\|_{L'}$ for $i = 1, \ldots, m$ are small enough. □

Proof of Theorem 4.1. In view of Lemma 4.3 we may assume that $f \in \mathcal{O}(X)$. Choose an increasing sequence $K = K_0 \subset K_1 \subset \cdots \subset \bigcup_{i=0}^{\infty} K_i = X$ of compact $\mathcal{O}(X)$-convex sets satisfying $K_i \subset \mathring{K}_{i+1}$ for every $i = 0, 1, \ldots$. Set $F_0 = f$, $\epsilon_0 = \epsilon/2$, and $r_0 = r$. We inductively construct functions $F_i \in \mathcal{O}(X)$ and numbers $\epsilon_i > 0$, $r_i \in \mathbb{N}$ such that the following conditions hold for every $i = 0, 1, 2, \ldots$:

(a) $\mathrm{Crit}(F_i|_{U_i \setminus X'}) = \emptyset$ for an open neighborhood $U_i \supset K_i$,
(b) $\|F_i - F_{i-1}\|_{K_{i-1}} < \epsilon_{i-1}$,
(c) $F_i - F_{i-1}$ vanishes to order r_{i-1} along X',
(d) $F_i - F_{i-1}$ vanishes to order n_k at each $p_k \in P$,
(e) $0 < \epsilon_i < \epsilon_{i-1}/2$ and $r_i \geq r_{i-1}$, and
(f) if $F \in \mathcal{O}(X)$ is such that $\|F - F_i\|_{K_i} < 2\epsilon_i$ and $F - F_i$ vanishes to order r_i along X', then $\mathrm{Crit}(F|_{U \setminus X'}) = \emptyset$ for an open neighborhood $U \supset K_{i-1}$.

Assume that we have already found these quantities up to index $i-1$ for some $i \in \mathbb{N}$. (For $i=0$ the function $F_0 = f$ satisfies condition (a) and the remaining conditions are void.) Lemma 4.4 furnishes the next map $F_i \in \mathcal{O}(X)$ which satisfies (a)–(d). For this F_i we then pick the next $\epsilon_i > 0$ and $n_i \in \mathbb{N}$ such that (e) and (f) hold. In view of the Stability Lemma 2.7, condition (f) holds as soon as $\epsilon_i > 0$ is chosen small enough and $r_i \in \mathbb{N}$ is large enough. This completes the induction step.

It is straightforward to verify that the sequence F_i converges uniformly on compacts in X and the limit function $F = \lim_{i\to\infty} F_i \in \mathcal{O}(X)$ satisfies the conclusion of Theorem 4.1. □

In the proof of Theorem 1.5 (see §5) we shall combine Theorem 4.1 with the following lemma which provides extension from a subvariety and jet interpolation on a discrete set.

Lemma 4.5 (Extension with jet interpolation). *Let X be a Stein space, X' a closed complex subvariety of X, and $P = \{p_1, p_2, \ldots\}$ a closed discrete subset of X'. Given $f \in \mathcal{O}(X')$ and germs $f_k \in \mathcal{O}_{X,p_k}$ for each $p_k \in P$ such that $f_{p_k} - (f_k|_{X'})_{p_k} \in \mathfrak{m}^{n_k}_{X',p_k}$ for some $n_k \in \mathbb{N}$, there exists $F \in \mathcal{O}(X)$ such that $F|_{X'} = f$ and $F_{p_k} - f_k \in \mathfrak{m}^{n_k}_{X,p_k}$ for every $p_k \in P$.*

Proof. Let $\mathcal{J}_{X'}$ denote the sheaf of ideals of X'. By Lemma 2.1 there exists for every $p_k \in P$ a germ $g_k \in \mathcal{O}_{X,p_k}$ such that $f_k - g_k \in \mathfrak{m}^{n_k}_{X,p_k}$ and $(g_k|_{X'})_{p_k} = f_{p_k} \in \mathcal{O}_{X',p_k}$. Pick $\tilde f \in \mathcal{O}(X)$ with $\tilde f|_{X'} = f$; then $\tilde f_{p_k} - g_k \in \mathcal{J}_{X',p_k}$. Let $\mathcal{E} \subset \mathcal{O}_X$ be the coherent sheaf of ideals whose stalk at any $p_k \in P$ equals $\mathfrak{m}^{n_k}_{p_k}$ and $\mathcal{E}_x = \mathcal{O}_{X,x}$ for every $x \in X \setminus P$. Consider the following short exact sequence of coherent analytic sheaves on X:

$$0 \to \mathcal{E}\mathcal{J}_{X'} \to \mathcal{J}_{X'} \to \mathcal{J}_{X'}/(\mathcal{E}\mathcal{J}_{X'}) \to 0.$$

The quotient sheaf $\mathcal{J}_{X'}/(\mathcal{E}\mathcal{J}_{X'})$ is supported on the discrete set P, and hence the collection of germs $\tilde f_{p_k} - g_k \in \mathcal{J}_{X',p_k}$ determines a section of this sheaf. Since $H^1(X; \mathcal{E}\mathcal{J}_{X'}) = 0$ by Theorem B, this section lifts to a section h of $\mathcal{J}_{X'}$.

Set $F := \tilde f - h \in \mathcal{O}(X)$. We have $F|_{X'} = \tilde f|_{X'} = f$. Furthermore, for every $p_k \in P$ the following identities hold in the ring $\mathcal{O}_{X,p_k}/\mathfrak{m}^{n_k}_{X,p_k}$ of $(n_k - 1)$-jets at p_k:

$$F_{p_k} = \tilde f_{p_k} - h_{p_k} = \tilde f_{p_k} - (\tilde f_{p_k} - g_k) = g_k = f_k \quad \mod \mathfrak{m}^{n_k}_{X,p_k}.$$

Thus F satisfies the conclusion of the lemma. □

Remark 4.6. Lemma 4.5 gives a version of Theorem 4.1 in which f is assumed to be defined and holomorphic only on the subvariety $X' \subset X$ and on a neighborhood of a compact $\mathcal{O}(X)$-convex set $K \subset X$. Furthermore, we are given germs $f_k \in \mathcal{O}_{X,p_k}$ at $p_k \in P$ such that the conditions of Lemma 4.5 hold. Then for any $n_k \in \mathbb{N}$ there exists $F \in \mathcal{O}(X)$ satisfying the conclusion of Theorem 4.1, except that conditions (i) and (ii) are replaced by

(i′) $F|_{X'} = f$, and
(ii′) $F - f_k$ vanishes to order n_k at each $p_k \in P$.

5. Stratified noncritical functions on Stein spaces

In this section we prove Theorem 1.5 on the existence of stratified noncritical holomorphic functions. As shown in the Introduction, this will also prove Theorems 1.1 and 1.3.

Proof of Theorem 1.5. Let (X, Σ) be a stratified Stein space (see §1). For every $i \in \mathbb{Z}_+$ we let Σ_i denote the collection of all strata of dimension at most i in Σ, and let X_i denote the union of all strata in Σ_i (the i-skeleton of Σ). Since the boundary of any stratum is a union of lower-dimensional strata, X_i is a closed complex subvariety of X of dimension $\le i$ for every $i \in \mathbb{Z}_+$. Clearly $\dim X_i = i$ precisely when Σ contains at least one i-dimensional stratum; otherwise $X_i = X_{i-1}$. We have $X_0 \subset X_1 \subset \cdots \subset \bigcup_{i=0}^{\infty} X_i = X$, the sequence X_i is stationary on any compact subset of X, and (X_i, Σ_i) is a stratified Stein subspace of (X, Σ) for every i. Note that $X_0 = \{p_1, p_2, \ldots\}$ is a discrete subset of X.

By the assumption of Theorem 1.5 we are given for each $p_k \in X_0$ a germ $f_k \in \mathcal{O}_{X,p_k}$. Our task is to find a Σ-noncritical $F \in \mathcal{O}(X)$ which agrees with f_k at $p_k \in X_0$ to order $n_k \in \mathbb{N}$. If the germs $f_k \in \mathcal{O}_{X,p_k}$ $(p_k \in X_0)$ are chosen (strongly) noncritical and $n_k \ge 2$ for each k, then the resulting F will be (strongly) noncritical on X.

Let $F_0 \colon X_0 \to \mathbb{C}$ be defined by $F_0(p_k) = f_k(p_k)$ for every $p_k \in X_0$. We shall inductively construct $F_i \in \mathcal{O}(X_i)$ satisfying the following conditions for $i = 1, 2, \ldots$:

(i) $F_i|_{X_{i-1}} = F_{i-1}$,
(ii) $F_i - f_k|_{X_i}$ vanishes to order n_k at every $p_k \in X_0$, and
(iii) F_i is a stratified noncritical function on the stratified Stein space (X_i, Σ_i).

Assuming that we have found $F_1, \ldots, F_{i-1}$ with these properties, we now explain how to find F_i. If $X_i = X_{i-1}$ then we can simply take $F_i = F_{i-1}$. If this is not the case, then $X_i \setminus X_{i-1}$ is a complex manifold of dimension i. Apply Lemma 4.5 with $X = X_i$, $X' = X_{i-1}$ and $f = F_{i-1}$ to find $G_i \in \mathcal{O}(X)$ (called F in the lemma) which satisfies

(i′) $G_i|_{X_{i-1}} = F_{i-1}$, and
(ii′) $(G_i)_{p_k} - (f_k|_{X_i})_{p_k} \in \mathfrak{m}^{n_k}_{X_i,p_k}$ at every $p_k \in X_0$.

Now G_i satisfies the hypotheses of Theorem 4.1 (with $X = X_i$, $X' = X_{i-1}$ and $f = G_i$), so we get $F_i \in \mathcal{O}(X)$ which agrees with G_i on X_{i-1}, agrees with G_i (and hence with f_k) to order n_k at $p_k \in X_0$ for each k, and is noncritical on $X_i \setminus X_{i-1}$. Hence F_i satisfies (i)–(iii) and the induction may proceed.

Since the sequence of subvarieties $X_0 \subset X_1 \subset \cdots \subset \bigcup_{i=0}^{\infty} X_i = X$ is stationary on any compact subset of X, the sequence of functions $F_i \in \mathcal{O}(X_i)$ obtained in this way determines a holomorphic function $F \in \mathcal{O}(X)$ by setting $F = F_i$ on X_i for any $i \in \mathbb{N}$. It is immediate that F satisfies the conclusion of Theorem 1.5. □

A similar construction yields the following result.

Theorem 5.1. *Given a reduced Stein space X, a closed complex subvariety X' of X and a function $f \in \mathcal{O}(X')$, there exists $F \in \mathcal{O}(X)$ such that $F|_{X'} = f$ and F is strongly noncritical on $X \setminus X'$. Alternatively, we can choose F to have critical points at a prescribed discrete set P in X which is contained in $X \setminus X'$.*

Proof. We can stratify $X \setminus X'$ into a union $\bigcup_j S_j$ of pairwise disjoint connected complex manifolds (strata) such that

- the boundary $bS_j = \overline{S}_j \setminus S_j$ of any stratum is contained in the union of X' and of lower-dimensional strata,
- every point of P is a zero-dimensional stratum, and
- every compact set in X intersects at most finitely many strata.

Consider the increasing chain $X' \subset X_0 \subset X_1 \subset \cdots \subset \bigcup_{i=1}^{\infty} X_i = X$ of closed complex subvarieties, where X_i is the union of X' and all strata S_j of dimension at most i. In particular, we have $P \cup X' \subset X_0$. Then $X_i \setminus X_{i-1}$ is either empty or a disjoint union of i-dimensional complex manifolds contained in $X \setminus X'$.

Let $P = \{p_1, p_2, \ldots\}$, and assume that we are given germs $f_k \in \mathcal{O}_{X,p_k}$ and integers $n_k \in \mathbb{N}$. We start with the function $F_0 \in \mathcal{O}(X_0)$ which agrees with f on X' and satisfies $F(p_k) = f_k(p_k)$ for every $p_k \in P$. By Lemma 4.5 we can find $G_1 \in \mathcal{O}(X)$ which agrees with F_0 on X_0 and satisfies $(G_1)_{p_k} - f_k \in \mathfrak{m}_{X,p_k}^{n_k}$ at each $p_k \in P$. Theorem 4.1, applied to the Stein pair $X_0 \subset X_1$ and the function G_1 (denoted by f in the theorem), furnishes $F_1 \in \mathcal{O}(X_1)$ which agrees with G_1 (and hence with F_0) on X_0 and satisfies $(F_1)_{p_k} - (f_k|_{X_1})_{p_k} \in \mathfrak{m}_{X_1,p_k}^{n_k}$ for every $p_k \in P$. This completes the first step of the induction. By using again Lemma 4.5 and then Theorem 4.1 we find $F_2 \in \mathcal{O}(X_2)$ such that $F_2|_{X_1} = F_1$ and $(F_2)_{p_k} - (f_k|_{X_2})_{p_k} \in \mathfrak{m}_{X_2,p_k}^{n_k}$ at every point $p_k \in P$. Clearly this process can be continued inductively. We obtain a sequence $F_i \in \mathcal{O}(X_i)$ for $i = 1, 2, \ldots$ such that the function $F \in \mathcal{O}(X)$ defined by $F|_{X_i} = F_i$ for every $i = 1, 2, \ldots$ satisfies the conclusion of Theorem 5.1. □

Corollary 5.2. *Let X be a reduced Stein space, X' a closed complex subvariety of X without isolated points, and $f \in \mathcal{O}(X')$ a noncritical holomorphic function. Then there exists a noncritical $F \in \mathcal{O}(X)$ such that $F|_{X'} = f$.*

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