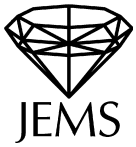
J. Eur. Math. Soc. 12, 2733–2784

DOI 10.4171/JEMS/652



Adrian Ioana

Orbit equivalence and Borel reducibility rigidity for profinite actions with spectral gap

Received December 2, 2013 and in revised form June 19, 2015

Abstract. We study equivalence relations $\mathcal{R}(\Gamma \curvearrowright G)$ that arise from left translation actions of countable groups on their profinite completions. Under the assumption that the action $\Gamma \curvearrowright G$ is free and has spectral gap, we describe precisely when $\mathcal{R}(\Gamma \curvearrowright G)$ is orbit equivalent or Borel reducible to another such equivalence relation $\mathcal{R}(\Lambda \curvearrowright H)$. As a consequence, we provide explicit uncountable families of free ergodic probability measure preserving (p.m.p.) profinite actions of $\mathrm{SL}_2(\mathbb{Z})$ and its non-amenable subgroups (e.g. $\mathbb{F}_n$ with $2 \leq n \leq \infty$) whose orbit equivalence relations are not mutually orbit equivalent or Borel reducible to each other. In particular, we show that if S and T are distinct sets of primes, then the orbit equivalence relations associated to the actions $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \prod_{p\in S} \mathrm{SL}_2(\mathbb{Z}_p)$ and $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \prod_{p\in T} \mathrm{SL}_2(\mathbb{Z}_p)$ are neither orbit equivalent nor Borel reducible to each other. This settles a conjecture of S. Thomas [Th01, Th06]. Other applications include the first calculations of outer automorphism groups for concrete treeable p.m.p. equivalence relations, and the first concrete examples of free ergodic p.m.p. actions of $\mathbb{F}_\infty$ whose orbit equivalence relations have trivial fundamental group.

Keywords. Spectral gap, rigidity, orbit equivalence, Borel reducibility, equivalence relations, profinite actions, outer automorphism group, II_1 factor

1. Introduction and statement of main results

1.1. Introduction

The general goal of this paper is to establish new rigidity results for countable equivalence relations, in both the measure-theoretic and Borel contexts. Our main technical result (Theorem A) gives necessary and sufficient conditions for equivalence relations, which are associated to "translation profinite" actions $\Gamma \curvearrowright G$ with spectral gap, to be orbit equivalent or Borel reducible to each other. The novelty of this theorem lies in that there are no assumptions on the groups, but instead, all the assumptions are imposed on their actions. In particular, our result applies to many natural families of translation profinite actions of $\mathrm{SL}_2(\mathbb{Z})$ and the free groups $\mathbb{F}_n$ that are known to have spectral gap as a consequence of Selberg's theorem and its recent generalizations [BG05, BV10].

As an application, we provide explicit uncountable families of free ergodic p.m.p. actions of $\mathrm{SL}_2(\mathbb{Z})$ and the free groups whose orbit equivalence relations are not pairwise

A. Ioana: Department of Mathematics, University of California San Diego, 9500 Gilman Drive, La Jolla, CA 92093, USA; e-mail: aioana@ucsd.edu

Mathematics Subject Classification (2010): Primary 37A20; Secondary 46L36

orbit equivalent or Borel reducible to each other. Additionally, we compute the outer automorphism and fundamental groups of the orbit equivalence relations arising from several of these actions. This improves on several results from [GP03, OP07, PV08, Ga08, Hj08] which showed the existence of such families of actions.

In order to state our results in detail, we first need to review a few concepts concerning countable equivalence relations. Recall that a *standard probability space* (X, μ) is a Polish space X equipped with its σ-algebra of Borel sets and a Borel probability measure μ.

Countable p.m.p. and countable Borel equivalence relations are fundamental objects of study in orbit equivalence theory and descriptive set theory. If $\Gamma \curvearrowright (X, \mu)$ is a p.m.p. (respectively, Borel) action of a countable group Γ on a standard probability space (X, μ), then the *orbit equivalence relation* $\mathcal{R}(\Gamma \curvearrowright X) := \{(x, y) \in X^2 \mid \Gamma \cdot x = \Gamma \cdot y\}$ is a countable p.m.p. (respectively, Borel) equivalence relation. Conversely, it was shown in [FM77, Theorem 1] that any countable p.m.p. and any countable Borel equivalence relation can be realized in this way.

The study of countable equivalence relations is organized using the notions of orbit equivalence and Borel reducibility. Firstly, let $\mathcal{R}, \mathcal{S}$ be countable p.m.p. equivalence relations on standard probability spaces (X, μ), (Y, ν). Then $\mathcal{R}$ is said to be *orbit equivalent* to $\mathcal{S}$ if there exists an isomorphism of probability spaces $\theta : (X, \mu) \to (Y, \nu)$ such that $(\theta \times \theta)(\mathcal{R}) = \mathcal{S}$. Moreover, we say that $\mathcal{R}, \mathcal{S}$ are *stably orbit equivalent* if there exist Borel subsets $A \subset X$, $B \subset Y$ of positive measure such that the restrictions $\mathcal{R}_{|A}, \mathcal{S}_{|B}$ are orbit equivalent. Two p.m.p. actions $\Gamma \curvearrowright (X, \mu)$, $\Lambda \curvearrowright (Y, \nu)$ are [stably] orbit equivalent if their orbit equivalence relations are [stably] orbit equivalent.

Secondly, let $\mathcal{R}, \mathcal{S}$ be countable Borel equivalence relations on standard Borel spaces X, Y. Then $\mathcal{R}$ is *Borel reducible* to $\mathcal{S}$ if there exists a Borel map $\theta : X \to Y$ such that $(x, y) \in \mathcal{R}$ iff $(\theta(x), \theta(y)) \in \mathcal{S}$. The condition that $\mathcal{R}$ is Borel reducible to $\mathcal{S}$ is usually interpreted to mean that the classification problem associated to $\mathcal{R}$ is at most as complicated as the classification problem associated to $\mathcal{S}$.

1.2. Orbit equivalence rigidity and Borel reducibility rigidity

We are now ready to state the main technical result of this paper. Recall that an ergodic p.m.p. action $\Gamma \curvearrowright (X, \mu)$ is said to have *spectral gap* if the Koopman representation of Γ on $L^2(X, \mu) \ominus \mathbb{C}1$ has no almost invariant vectors. Let Γ be a residually finite group and $G = \varprojlim \Gamma/\Gamma_n$ be its profinite completion with respect to a descending chain $\{\Gamma_n\}_n$ of finite index, normal subgroups with trivial intersection, $\bigcap_n \Gamma_n = \{e\}$. Let $\rho : \Gamma \hookrightarrow G$ be the embedding given by $\rho(g) = (g\Gamma_n)_n$. Then the *left translation* action $\Gamma \curvearrowright G$ defined by $g \cdot x = \rho(g)x$ is free, ergodic, and preserves the Haar measure of G. Moreover, this action is *profinite*, i.e. it is an inverse limit of actions of Γ on finite probability spaces.

Theorem A. *Let Γ, Λ be residually finite groups. Let $G = \varprojlim \Gamma/\Gamma_n$, $H = \varprojlim \Lambda/\Lambda_n$ be profinite completions of Γ, Λ with respect to descending chains $\{\Gamma_n\}_n$, $\{\Lambda_n\}_n$ of finite index normal subgroups with trivial intersection. Denote by m_G and m_H the Haar measures of G and H. Assume that the left translation action $\Gamma \curvearrowright (G, m_G)$ has spectral gap. Then:*

(1) $\Gamma \curvearrowright (G, m_G)$ *is stably orbit equivalent* (*respectively, orbit equivalent*) *to* $\Lambda \curvearrowright (H, m_H)$ *if and only if we can find open subgroups* $G_0 < G$ *and* $H_0 < H$ *and a continuous isomorphism* $\delta : G_0 \to H_0$ *such that* $\delta(\Gamma \cap G_0) = \Lambda \cap H_0$ (*and, respectively,* $[G : G_0] = [H : H_0]$).
(2) $\mathcal{R}(\Gamma \curvearrowright G)$ *is Borel reducible to* $\mathcal{R}(\Lambda \curvearrowright H)$ *if and only if we can find an open subgroup* $G_0 < G$*, a closed subgroup* $H_0 < H$*, and a continuous isomorphism* $\delta : G_0 \to H_0$ *such that* $\delta(\Gamma \cap G_0) = \Lambda \cap H_0$.

Next, let us make a few comments on the statement of Theorem A and discuss some classes of actions to which it applies.

Assume that Γ has property (T) of Kazhdan (e.g. take $\Gamma = \mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$). Then every ergodic p.m.p. action $\Gamma \curvearrowright (X, \mu)$ has spectral gap. In particular, the first part of Theorem A holds for any left translation action of Γ on one of its profinite completions G. Let us point out that in this case much more can be said. Indeed, if Γ has property (T), we showed that the action $\Gamma \curvearrowright G$ is *orbit equivalent superrigid*, in the sense that any free ergodic p.m.p. action $\Lambda \curvearrowright (Y, \nu)$ that is orbit equivalent to $\Gamma \curvearrowright G$ is virtually conjugate to it (see [Io08, Theorem A]). This means that we can find a finite index subgroup $\Lambda_0 < \Lambda$ and an open subgroup $G_0 < G$ such that an ergodic component of the action $\Lambda_0 \curvearrowright (Y, \nu)$ is conjugate to the action $\Gamma \cap G_0 \curvearrowright G_0$. Moreover, if Γ has property (T), then the action $\Gamma \curvearrowright G$ satisfies a cocycle superrigidity theorem [Io08, Theorem B] which can be used to deduce the second part of Theorem A.

In this paper, however, we are interested in studying actions of groups such as $\mathrm{SL}_2(\mathbb{Z})$ and the free groups $\mathbb{F}_n$, for which property (T) fails. Nevertheless, these groups still satisfy a weak form of property (T). Recall that a countable group Γ has *property* (τ) *with respect to a family* $\{\Gamma_n\}_n$ *of subgroups* if the unitary representation of Γ on $\bigoplus_n \ell^2_0(\Gamma/\Gamma_n)$ has no almost invariant vectors, where $\ell^2_0(\Gamma/\Gamma_n) = \ell^2(\Gamma/\Gamma_n) \ominus \mathbb{C}1$ (see [LZ03, Lu12]).

Selberg's famous theorem implies that $\mathrm{SL}_2(\mathbb{Z})$ has property (τ) with respect to the family of congruence subgroups $\Gamma(n) = \ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z}))$. Recently, Selberg's theorem has been vastly generalized starting with the breakthrough work of J. Bourgain and A. Gamburd [BG05]. In particular, J. Bourgain and P. Varjú have shown that any non-amenable subgroup $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ has property (τ) with respect to the family $\{\Gamma \cap \Gamma(n)\}_{n \geq 1}$ of subgroups (see [BV10, Theorem 1]). Note that a translation action $\Gamma \curvearrowright \varprojlim \Gamma/\Gamma_n$ has spectral gap if and only if Γ has property (τ) with respect to $\{\Gamma_n\}_n$. Thus, the results mentioned above can be used to construct many interesting families of translation actions with spectral gap. We single out two such families that we will use subsequently:

Examples. Consider the profinite groups $G_S = \prod_{p \in S} \mathrm{SL}_2(\mathbb{F}_p)$ and $K_S = \prod_{p \in S} \mathrm{SL}_2(\mathbb{Z}_p)$ for a set S of primes. Whenever S is infinite, we view $\mathrm{SL}_2(\mathbb{Z})$ as a subgroup of G_S, via the diagonal embedding. We also embed $\mathrm{SL}_2(\mathbb{Z})$ diagonally into K_S, for any set S. By the Strong Approximation Theorem (see e.g. [LS03]) both of these embeddings are dense.

Given a subgroup $\Gamma < \mathrm{SL}_2(\mathbb{Z})$, we denote by $G_{\Gamma,S}$ and $K_{\Gamma,S}$ its closures in G_S and K_S, respectively. If Γ is non-amenable, then the translation actions $\Gamma \curvearrowright G_{\Gamma,S}$, $\Gamma \curvearrowright K_{\Gamma,S}$ have spectral gap [BV10]. Moreover, the Strong Approximation Theorem implies that $G_{\Gamma,S} < G_S$ and $K_{\Gamma,S} < K_S$ are open.

Let us point out that the closures $G_{\Gamma,S}$ and $K_{\Gamma,S}$ of Γ are, at least in principle, explicit. In general, if $G = \varprojlim \Gamma/\Gamma_n$ is a profinite completion of a countable group Γ and $\Gamma_0 < \Gamma$ is a subgroup, then the closure of Γ_0 in G is isomorphic, as a topological group, to the profinite group $\varprojlim \Gamma_0/(\Gamma_0 \cap \Gamma_n)$.

Remarks. • The conclusion of the first part of Theorem A is optimal, in the sense that the translation action $\Lambda \curvearrowright (H, m_H)$ cannot be replaced with an arbitrary free ergodic p.m.p. action $\Lambda \curvearrowright (Y, \nu)$. In other words, the action $\Gamma \curvearrowright (G, m_G)$ is not necessarily orbit equivalent superrigid. This is easy to see if Γ is a free group. Indeed, by [MS02, Theorem 2.27], any free ergodic p.m.p. action of a free group is orbit equivalent to actions of uncountably many non-isomorphic groups, and hence cannot be orbit equivalent superrigid.

- If Γ is the product of two groups having property (T) and the action $\Gamma \curvearrowright G$ satisfies a certain growth condition, then the conclusion of Theorem A can be deduced from the cocycle rigidity result of [OP08, Theorem C].
- An analogous result to the first part of Theorem A, where orbit equivalence is replaced with weak equivalence of actions, was recently obtained by M. Abért and G. Elek [AE10, Theorem 2]. Their result in particular shows that, under the assumptions of Theorem A, the actions $\Gamma \curvearrowright (G, m_G)$, $\Lambda \curvearrowright (H, m_H)$ are weakly equivalent if and only if they are conjugate.
- The notion of "spectral gap rigidity" has been introduced by S. Popa in the context of von Neumann algebras and used to great effect starting with [Po06a].
- Several rigidity results were recently proven in [Po09] for II_1 factors M that are an inductive limit of a sequence $\{M_n\}$ of subfactors with spectral gap. If Γ is not inner amenable and $\Gamma \curvearrowright X = \varprojlim X_n$ is a profinite action with spectral gap, then the II_1 factor $M = L^\infty(X) \rtimes \Gamma$ has this property, where $M_n = L^\infty(X_n) \rtimes \Gamma$. Note, however, that one cannot directly apply [Po09, Theorem 3.5] as the relative commutant condition $(M_n' \cap M)' \cap M = M_n$ fails.
- We do not know to what extent Theorem A can be extended to arbitrary ergodic compact (or profinite) actions. Recall that these are actions of the form $\Gamma \curvearrowright G/K$, where G is a compact (respectively, compact profinite) group in which Γ embeds densely, and $K < G$ is a closed subgroup. However, the proof of Theorem A relies on a cocycle rigidity result (Theorem 3.1) for translation profinite actions which admits an analogue in the case of translation actions $\Gamma \curvearrowright G$ on compact connected groups G with finite fundamental group (Theorem 3.2). As a consequence, we are able to prove an analogue of Theorem A in this case (see Corollary 4.7 and Theorem 6.1). Moreover, Theorem 6.1 establishes an analogue of Theorem A for a fairly general class of compact actions.

1.3. Orbit inequivalent and Borel incomparable actions

In the rest of the introduction, we discuss several applications of Theorem A and of its method of proof. Our first applications provide examples of actions of the free groups that are orbit inequivalent and Borel incomparable. We begin by giving some context and motivation.

In the last 15 years, remarkable progress has been made in the study of countable equivalence relations, in both the measure-theoretic and Borel contexts (see the surveys [Po06b,Fu09,Ga10] and [Th06,TS07]). In particular, considerable effort has been devoted to the investigation of equivalence relations that arise from actions of free groups and, more generally, of *treeable* equivalence relations (i.e. equivalence relations whose classes are the connected components of a Borel acyclic graph).

On the orbit equivalence side, D. Gaboriau proved that free ergodic p.m.p. actions $\mathbb{F}_n \curvearrowright X$, $\mathbb{F}_m \curvearrowright Y$ of free groups of different ranks ($n \neq m$) are never orbit equivalent [Ga99,Ga01]. This result, however, offered little insight on how to distinguish between actions of the same free group. It was not until the work of D. Gaboriau and S. Popa [GP03] that every non-abelian free group $\mathbb{F}_n$ ($2 \leq n \leq \infty$) was shown to admit a continuum of non-orbit equivalent free ergodic p.m.p. actions. However, [GP03] only demonstrates the existence of such a continuum of actions. This motivated our work [Io06], where we found an explicit list of uncountably many orbit inequivalent free ergodic p.m.p. actions of $\mathbb{F}_n$. Note that finding natural classes of orbit inequivalent actions of $\mathbb{F}_n$ is a difficult task, since some obvious candidates turn out to be orbit equivalent. Indeed, L. Bowen proved that any two Bernoulli actions of $\mathbb{F}_n$ are orbit equivalent if $2 \leq n \leq \infty$, and any two Bernoulli actions of $\mathbb{F}_n, \mathbb{F}_m$ are stably orbit equivalent whenever $2 \leq n, m < \infty$ [Bo09a, Bo09b] (see also [MRV11]).

In descriptive set theory, the investigation of treeable equivalence relations started in [JKL01] where it was proved that the orbit equivalence relation $E_{\infty\mathcal{T}}$ of the free part of the Bernoulli action $\mathbb{F}_\infty \curvearrowright \{0,1\}^{\mathbb{F}_\infty}$ is maximal among treeable countable Borel equivalence relations, with respect to Borel reducibility. In the same paper, the authors asked whether there exist infinitely many treeable countable Borel equivalence relations, up to Borel reducibility. After providing a first example of a non-hyperfinite treeable equivalence relation that lies strictly below $E_{\infty\mathcal{T}}$ [Hj03], G. Hjorth answered this question in the affirmative in [Hj08]. More precisely, he proved that there exist uncountably many treeable countable Borel equivalence relations that are mutually incomparable with respect to Borel reducibility. Since the proof of [Hj08] uses a separability argument, it only provides an existence result, leaving open the problem of finding specific treeable equivalence relations that are Borel incomparable. In fact, at the time of the writing, not a single example of a pair of treeable countable Borel equivalence relations such that neither is Borel reducible to the other was known.

As a first application of Theorem A, we exhibit the first concrete uncountable families of actions of $\mathrm{SL}_2(\mathbb{Z})$ and the free groups $\mathbb{F}_n$ that are neither orbit equivalent nor Borel reducible.

Corollary B. *Let S, T be infinite sets of primes, and $\Gamma, \Lambda < \mathrm{SL}_2(\mathbb{Z})$ non-amenable subgroups.*

(1) *If $S \neq T$, then the actions $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright G_S$ and $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright G_T$ are not stably orbit equivalent, and the equivalence relations $\mathcal{R}(\mathrm{SL}_2(\mathbb{Z}) \curvearrowright G_S)$ and $\mathcal{R}(\mathrm{SL}_2(\mathbb{Z}) \curvearrowright G_T)$ are not comparable with respect to Borel reducibility.*

(2) *If $|S \vartriangle T| = \infty$, then the actions $\Gamma \curvearrowright G_{\Gamma,S}$ and $\Lambda \curvearrowright G_{\Lambda,T}$ are not stably orbit equivalent, and the equivalence relations $\mathcal{R}(\Gamma \curvearrowright G_{\Gamma,S})$ and $\mathcal{R}(\Lambda \curvearrowright G_{\Lambda,T})$ are not comparable with respect to Borel reducibility.*

Corollary B improves on a result of N. Ozawa and S. Popa who were the first to show the existence of uncountably many orbit inequivalent free ergodic profinite actions of $\mathbb{F}_n$ [OP07, Theorem 5.4]. More precisely, in the context of Corollary B, they proved that if $\{S_i\}_{i\in I}$ is an uncountable family of infinite sets of primes such that $|S_i \cap S_j| < \infty$ for all $i \neq j$, then among the actions $\{\Gamma \curvearrowright G_{\Gamma,S_i}\}_{i\in I}$ there are uncountably many orbit equivalence classes.

In [Th01, Conjecture 5.7] and [Th06, Conjecture 2.14], S. Thomas proposed a scenario for providing an explicit uncountable family of treeable countable Borel equivalence relations that are pairwise incomparable with respect to Borel reducibility. Specifically, he conjectured that if S, T are distinct non-empty sets of primes, then the orbit equivalence relations of the actions of $\mathrm{SL}_2(\mathbb{Z})$ on K_S, K_T are neither Borel reducible nor stably orbit equivalent. Note that the analogous results with SL_2 replaced by SL_n, for some $n \geq 3$, were shown to hold in [Th01] and [GG88], respectively.

As a consequence of Theorem A, we settle this conjecture and, more generally, show that:

Corollary C. *Let S, T be distinct non-empty sets of primes, and $\Gamma, \Lambda < \mathrm{SL}_2(\mathbb{Z})$ non-amenable subgroups. Then the actions $\Gamma \curvearrowright K_{\Gamma,S}$ and $\Lambda \curvearrowright K_{\Lambda,T}$ are not stably orbit equivalent, and the equivalence relations $\mathcal{R}(\Gamma \curvearrowright K_{\Gamma,S})$ and $\mathcal{R}(\Lambda \curvearrowright K_{\Lambda,T})$ are not comparable with respect to Borel reducibility.*

Remarks. • Since the matrices $a = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ and $b = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ generate an isomorphic copy of $\mathbb{F}_2$ inside $\mathrm{SL}_2(\mathbb{Z})$, we have an embedding of $\mathbb{F}_n$ into $\mathrm{SL}_2(\mathbb{Z})$ for any $2 \leq n \leq \infty$. Therefore, both Corollaries B and C yield concrete uncountable families of pairwise orbit inequivalent and Borel incomparable free ergodic p.m.p. actions of $\mathbb{F}_n$.

- The orbit inequivalent actions of $\mathbb{F}_n$ provided by Corollaries B and C are different from the ones found in [GP03, Io06]. Indeed, the latter actions admit quotients that have the relative property (T) [Po01], and hence are not orbit equivalent to profinite actions.
- By a well-known result from [OP07], if $\Gamma \curvearrowright (X, \mu)$ is a free ergodic p.m.p. profinite action of a non-amenable subgroup of $\mathrm{SL}_2(\mathbb{Z})$, then the II_1 factor $L^\infty(X) \rtimes \Gamma$ has a unique Cartan subalgebra, up to unitary conjugacy. This implies that the orbit inequivalent actions given by Corollaries B and C have in fact non-isomorphic II_1 factors.

Our techniques also allow us to confirm a second conjecture of S. Thomas [Th01, Conjecture 6.10]. This conjecture asserts that the orbit equivalence relations arising from the natural actions of $\mathrm{SL}_2(\mathbb{Z})$ on the projective lines over the various p-adic fields are pairwise incomparable with respect to Borel reducibility. More generally, we prove:

Corollary D. *For a prime p, let $\mathrm{PG}(1, \mathbb{Q}_p) = \mathbb{Q}_p \cup \{\infty\}$ be the projective line over the field $\mathbb{Q}_p$ of p-adic numbers. Consider the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $\mathrm{PG}(1, \mathbb{Q}_p)$ by linear fractional transformations. Let $\Gamma < \mathrm{GL}_2(\mathbb{Q}_p)$ and $\Lambda < \mathrm{GL}_2(\mathbb{Q}_q)$ be countable subgroups for some primes p and q. If $p \neq q$ and $\Gamma \cap \mathrm{SL}_2(\mathbb{Z})$ is non-amenable, then $\mathcal{R}(\Gamma \curvearrowright \mathrm{PG}(1, \mathbb{Q}_p))$ is not Borel reducible to $\mathcal{R}(\Lambda \curvearrowright \mathrm{PG}(1, \mathbb{Q}_q))$.*

Corollary D strengthens and offers a different approach to a theorem of G. Hjorth and S. Thomas. Namely, the case $\Gamma = \Lambda = \mathrm{GL}_2(\mathbb{Q})$ of Corollary D recovers the main result

of [HT04]. The original motivation for this result came from the classification problem for torsion-free abelian groups of finite rank (for more on this, see the survey [Th06]). More precisely, by [Th00, Theorem 5.7], the main result of [HT04] can be reformulated as follows: if $p \neq q$ are primes, then the classification problems for the p-local and q-local torsion-free abelian groups of rank 2 are incomparable with respect to Borel reducibility.

1.4. Calculations of outer automorphism groups and fundamental groups

The second direction in which our techniques apply is the calculation of the *fundamental group* $\mathcal{F}(\mathcal{R})$ and the *outer automorphism group* $\mathrm{Out}(\mathcal{R})$ of orbit equivalence relations $\mathcal{R}$ associated to actions of free groups (see Section 2.1 for the definitions). Let us briefly recall known results along these lines.

First, it was shown in [Ga99, Ga01] that if $2 \leq n < \infty$, then the fundamental group of $\mathcal{R}(\mathbb{F}_n \curvearrowright X)$ is trivial for any free ergodic p.m.p. action of $\mathbb{F}_n$. On the other hand, no calculations of fundamental groups of equivalence relations arising from actions of $\mathbb{F}_\infty$ or outer automorphism groups of equivalence relations arising from actions of $\mathbb{F}_n$ ($2 \leq n \leq \infty$) were available until recently. S. Popa and S. Vaes proved that the fundamental group of $\mathcal{R}(\mathbb{F}_\infty \curvearrowright X)$ can be equal to any countable subgroup as well as many uncountable subgroups of $\mathbb{R}_+^*$ [PV08]. Moreover, they showed the existence of free ergodic p.m.p. actions $\mathbb{F}_\infty \curvearrowright X$ such that $\mathcal{R}(\mathbb{F}_\infty \curvearrowright X)$ has trivial outer automorphism group. Shortly after, D. Gaboriau proved that the outer automorphism group of $\mathcal{R}(\mathbb{F}_n \curvearrowright X)$ can in fact be trivial for any $2 \leq n \leq \infty$ [Ga08]. Nevertheless, the proofs of [PV08, Ga08] did not provide a single explicit example of an action of $\mathbb{F}_\infty$ (respectively, of $\mathbb{F}_n$ with $2 \leq n \leq \infty$) for which the fundamental group (respectively, the outer automorphism group) of the orbit equivalence relation could be computed.

We address this problem here by exhibiting the first examples of free ergodic p.m.p. actions of $\mathbb{F}_\infty$ whose orbit equivalence relation has trivial fundamental group, and the first examples of actions of $\mathbb{F}_2$ for which the outer automorphism group of the orbit equivalence relation can be calculated.

Corollary E. *Let S be an infinite set of primes and $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ a non-amenable subgroup. Then:*

(1) $\mathrm{Out}(\mathcal{R}(\mathrm{SL}_2(\mathbb{Z}) \curvearrowright G_S)) \cong (G_S/Z) \rtimes \mathbb{Z}/2\mathbb{Z}$. *Here, $Z = \{\pm I\}$ denotes the center of G_S and the action of $\mathbb{Z}/2\mathbb{Z}$ on G_S is given by conjugation with $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$.*

(2) *The equivalence relation $\mathcal{R}(\Gamma \curvearrowright G_{\Gamma,S})$ and the associated II_1 factor $L^\infty(G_{\Gamma,S}) \rtimes \Gamma$ have trivial fundamental groups, i.e. $\mathcal{F}(\mathcal{R}(\Gamma \curvearrowright G_{\Gamma,S})) = \mathcal{F}(L^\infty(G_{\Gamma,S}) \rtimes \Gamma) = \{1\}$.*

Remarks. • Consider a fixed embedding $\mathbb{F}_\infty \subset \mathrm{SL}_2(\mathbb{Z})$. Then the combination of Corollaries C and E yields a continuum of pairwise non-stably orbit equivalent free ergodic p.m.p. actions of $\mathbb{F}_\infty$ whose equivalence relations and II_1 factors have trivial fundamental groups.

• Let us explain how Corollary E also leads to examples of orbit equivalence relations of free ergodic p.m.p. actions of $\mathbb{F}_2$ whose outer automorphism group can be explicitly computed. Let S be an infinite set of primes containing 2. Denote by $G_S(2)$ the kernel

of the natural homomorphism $G_S \to \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$. Note that $Z \subset G_S(2)$ and $\mathrm{SL}_2(\mathbb{Z}) \cap G_S(2)$ is the congruence subgroup $\Gamma(2)$. If $A \subset G_S(2)$ is a fundamental domain for the left translation action $Z \curvearrowright G_S(2)$, it is easy to see that $\mathcal{R}(\mathrm{SL}_2(\mathbb{Z}) \curvearrowright G_S)_{|A} \cong \mathcal{R}(\Gamma(2)/Z \curvearrowright G_S(2)/Z)$. Since the quotient $\Gamma(2)/Z$ is isomorphic to $\mathbb{F}_2$ and the outer automorphism group is insensitive to taking restrictions (i.e. $\mathrm{Out}(\mathcal{R}_{|A}) \cong \mathrm{Out}(\mathcal{R})$ for any ergodic countable p.m.p. equivalence relation $\mathcal{R}$), we get examples of actions of $\mathbb{F}_2$ with the desired property.

- Let n be a natural number of the form $n = p_1 \dots p_k$, where $3 \leq p_1 < \cdots < p_k$ are prime numbers. Let S be an infinite set of primes containing $p_1, \dots, p_k$. Denote by $G_S(n)$ the kernel of the natural homomorphism $G_S \to \prod_{i=1}^k \mathrm{SL}_2(\mathbb{Z}/p_i\mathbb{Z})$. Then $\mathrm{SL}_2(\mathbb{Z}) \cap G_S(n) = \Gamma(n)$ and $\mathcal{R}(\mathrm{SL}_2(\mathbb{Z}) \curvearrowright G_S)_{|G_S(n)} = \mathcal{R}(\Gamma(n) \curvearrowright G_S(n))$. Since $\Gamma(n)$ is a free group (of rank $1 + \frac{1}{12}\prod_{i=1}^k p_i(p_i^2 - 1)$), by Corollary E we get more examples of actions of free groups with the same property as above.
- Let Γ be free group and $\Gamma \curvearrowright^{\sigma} (X, \mu)$ a free ergodic p.m.p. profinite action. Denote $\mathcal{R} = \mathcal{R}(\Gamma \curvearrowright X)$ and $M = L^\infty(X) \rtimes \Gamma$. Then the main result of [OP07] implies that $L^\infty(X)$ is the unique Cartan subalgebra of M, up to unitary conjugacy. As a consequence, $\mathrm{Out}(M) \cong H^1(\sigma) \rtimes \mathrm{Out}(\mathcal{R})$, where $H^1(\sigma)$ denotes the first cohomology group of σ with values in $\mathbb{T}$. Thus, if σ is any of the actions from the previous two remarks, then the outer automorphism group of its II_1 factor M can be computed explicitly in terms of $H^1(\sigma)$.

Our next result provides further computations of fundamental groups and outer automorphism groups.

Corollary F. *Let p be a prime number and $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ a non-amenable subgroup. Then:*

(1) $\mathrm{Out}(\mathcal{R}(\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathrm{SL}_2(\mathbb{Z}_p)))$ *is a* $(\mathbb{Z}/2\mathbb{Z})^2$*-extension of* $\mathrm{PSL}_2(\mathbb{Q}_p)$*, i.e. there is an (explicit) exact sequence* $1 \to \mathrm{PSL}_2(\mathbb{Q}_p) \to \mathrm{Out}(\mathcal{R}(\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathrm{SL}_2(\mathbb{Z}_p))) \to (\mathbb{Z}/2\mathbb{Z})^2 \to 1$.

(2) $\mathcal{F}(\mathcal{R}(\Gamma \curvearrowright K_\Gamma)) = \mathcal{F}(L^\infty(K_\Gamma) \rtimes \Gamma) = \{1\}$*, where K_Γ denotes the closure of Γ in* $\mathrm{SL}_2(\mathbb{Z}_p)$.

Note that a statement similar to the first part of Corollary F, where SL_2 is replaced with SL_n for some $n \geq 3$, was obtained by A. Furman [Fu03, Theorem 1.6].

1.5. Treeable equivalence relations with trivial outer automorphism group

Corollaries E and F leave open the problem of finding concrete actions of $\mathbb{F}_n$ for which $\mathcal{R}(\mathbb{F}_n \curvearrowright X)$ has trivial outer automorphism group. Furthermore, they do not provide any examples of treeable equivalence relations with trivial outer automorphism groups. To approach these problems, we show that, under fairly general assumptions on a countable subgroup $\Gamma < \mathrm{SO}(3)$, the outer automorphism groups of the equivalence relations associated to the natural actions of Γ on $\mathrm{SO}(3)$, S^2 and $P^2(\mathbb{R})$ can be explicitly expressed in terms of the normalizer of Γ in $\mathrm{SO}(3)$.

Corollary G. *Let Γ be a countable icc subgroup of $G = \mathrm{SO}(3)$. Assume that Γ contains matrices $g_1, \ldots, g_k$ which have algebraic entries and generate a dense subgroup of G. Consider the free ergodic p.m.p. actions $\Gamma \curvearrowright (G, m)$, $\Gamma \curvearrowright (S^2, \lambda)$ and $\Gamma \curvearrowright (P^2(\mathbb{R}), \mu)$, where m denotes the Haar measure of G, while λ and μ denote the Lebesgue measures of the 2-dimensional sphere S^2 and the 2-dimensional real projective space $P^2(\mathbb{R})$. Then*

$$\mathrm{Out}(\mathcal{R}(\Gamma \curvearrowright G)) = N_G(\Gamma)/\Gamma \times G, \quad \mathrm{Out}(\mathcal{R}(\Gamma \curvearrowright S^2)) \cong N_G(\Gamma)/\Gamma \times (\mathbb{Z}/2\mathbb{Z}),$$
$$\mathrm{Out}(\mathcal{R}(\Gamma \curvearrowright P^2(\mathbb{R}))) \cong N_G(\Gamma)/\Gamma,$$

where $N_G(\Gamma)$ denotes the normalizer of Γ in G. Moreover,

$$\mathcal{F}(\mathcal{R}(\Gamma \curvearrowright G)) = \mathcal{F}(\mathcal{R}(\Gamma \curvearrowright S^2)) = \mathcal{F}(\mathcal{R}(\Gamma \curvearrowright P^2(\mathbb{R}))) = \{1\}.$$

The work of J. Bourgain and A. Gamburd [BG06] implies that, under the assumptions imposed on Γ, the left translation action $\Gamma \curvearrowright (G, m_G)$ has spectral gap. This fact will be a key ingredient in the proof of Corollary G.

In view of Corollary G, in order to give examples of actions of $\Gamma = \mathbb{F}_n$ whose equivalence relations have trivial outer automorphism group, it suffices to a find a copy of Γ inside G which is generated by matrices with algebraic entries and has trivial normalizer. We were unable, however, to compute the normalizer of Γ for any of the known "algebraic" embeddings of Γ into G (see Remark 10.6).

Nevertheless, we managed to calculate the normalizer of certain countable subgroups $\Gamma < G$ (see Corollary 10.4). These groups Γ, although not free, are treeable, in the sense that any equivalence relation $\mathcal{R}(\Gamma \curvearrowright X)$ arising from a free p.m.p. action of Γ is treeable. Moreover, for some of these groups we showed that $N_G(\Gamma) = \Gamma$. This leads to the first concrete examples of treeable countable p.m.p. equivalence relations which have trivial outer automorphism group.

Corollary H. *Let $p \geq 4$, $q \geq 6$ be even integers such that $p \neq q$ and $q = 2s$ with s odd. Denote by Γ the subgroup of $G = \mathrm{SO}(3)$ generated by the rotation about the x-axis by angle $2\pi/p$ and the rotation about the z-axis by angle $2\pi/q$. Then*

$$\mathrm{Out}(\mathcal{R}(\Gamma \curvearrowright G)) \cong G, \quad \mathrm{Out}(\mathcal{R}(\Gamma \curvearrowright S^2)) \cong \mathbb{Z}/2\mathbb{Z}, \quad \mathrm{Out}(\mathcal{R}(\Gamma \curvearrowright P^2(\mathbb{R}))) \cong \{e\}.$$

Moreover,

$$\mathcal{F}(\mathcal{R}(\Gamma \curvearrowright G)) = \mathcal{F}(\mathcal{R}(\Gamma \curvearrowright S^2)) = \mathcal{F}(\mathcal{R}(\Gamma \curvearrowright P^2(\mathbb{R}))) = \{1\}.$$

C. Radin and L. Sadun [RS98] showed that Γ is isomorphic to an amalgamated free product of the form $D_p *_{D_2} D_q$, where $D_n = \mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ denotes the dihedral group of order $2n$. By a result of D. Gaboriau [Ga99] it follows that Γ is treeable, showing that the equivalence relations from Corollary H are indeed treeable.

1.6. Comments on the proofs

Since all the results stated above are derived from either Theorem A or variations of it, let us give an outline of its proof. There are two main ingredients in that proof.

The first is a criterion for untwisting cocycles of translation profinite actions (Theorem 3.1). As we explain in the next paragraphs, this criterion shows that any cocycle satisfying a certain local condition is essentially cohomologous to a homomorphism.

In [Io08, Theorem B], we proved a cocycle superrigidity theorem for profinite actions. In [Fu09, Theorem 5.21], A. Furman provided an alternative proof. His proof applies to the wider class of compact actions and has partially inspired our approach.

The main result of [Io08], in the formulation given in [Fu09], shows that if Γ has property (T) and $\Gamma \curvearrowright G$ is a translation profinite action, then any cocycle $w : \Gamma \times G \to \Lambda$ taking values in a countable group Λ is "virtually cohomologous to a homomorphism". More precisely, we can find an open subgroup $G_0 < G$ and a homomorphism $\delta : \Gamma \cap G_0 \to \Lambda$ such that the restriction of w to $(\Gamma \cap G_0) \times G_0$ is cohomologous to δ.

In both [Io08] and [Fu09], one combines property (T) with S. Popa's deformation/rigidity approach to deduce that w satisfies a "local uniformity" condition, which is then exploited to conclude that w is virtually cohomologous to a homomorphism. This condition, in the form which appears in [Fu09], amounts to the existence of a neighborhood V of the identity in G and of a constant $C \in (31/32, 1)$ such that $m_G(\{x \in G \mid w(g, xt) = w(g, x)\}) \geq C$ for all $g \in \Gamma$ and every t in V.

The second ingredient in the proof of Theorem A is an elementary lemma which, roughly speaking, asserts that if $\Gamma \curvearrowright (X, \mu)$ is a p.m.p. action with spectral gap, then any "almost Γ-invariant" Borel map $\rho : X \to Y$ into a Polish space Y is "almost constant" (Lemma 2.5). In precise terms, this means that for any $\varepsilon > 0$, we can find a finite subset $F \subset \Gamma$ and $\delta > 0$ such that whenever a Borel map $\rho : X \to Y$ satisfies $\mu(\{x \in X \mid \rho(gx) = \rho(x)\}) \geq 1 - \delta$ for all $g \in F$, there exists $y \in Y$ with $\mu(\{x \in X \mid \rho(x) = y\}) \geq 1 - \varepsilon$. In particular, for any such map ρ we have $\mu(\{x \in X \mid \rho(gx) = \rho(x)\}) \geq 1 - 2\varepsilon$ for all $g \in \Gamma$.

To explain how the above ingredients are combined to prove Theorem A, assume that the translation action $\Gamma \curvearrowright (G, m_G)$ has spectral gap and let $\theta : G \to H$ be a Borel map such that $\theta(\Gamma x) \subset \Lambda\theta(x)$ for almost every $x \in G$. Consider the cocycle $w : \Gamma \times G \to \Lambda$ defined by $\theta(gx) = w(g, x)\theta(x)$.

The key idea of the proof is to show that w satisfies the local uniformity condition defined above. Once this is achieved, the first ingredient of the proof implies that w is virtually cohomologous to a homomorphism, and the conclusion of Theorem A follows easily by using standard arguments.

To prove the local uniformity condition, for every $t \in G$ we introduce a Borel map $\rho_t : G \to H$ given by $\rho_t(x) = \theta(x)^{-1}\theta(xt)$. It is clear that if we fix $g \in \Gamma$, then $\theta_t(gx) = \theta_t(x) \Leftrightarrow w(g, xt) = w(g, x)$. Since w takes values in a countable group, it follows that $m_G(\{x \in G \mid \theta_t(gx) = \theta_t(x)\}) \to 1$ as t approaches the identity in G. Since $g \in \Gamma$ is arbitrary, the second ingredient of the proof implies that we must have $\inf_{g \in \Gamma} m_G(\{x \in G \mid \theta_t(gx) = \theta_t(x)\}) \to 1$ as t approaches the identity. This implies that w indeed satisfies the local uniformity condition.

1.7. Approximately trivial cocycles

As explained in the previous subsection, the proofs of our main results ultimately rely on studying cocycles. As a byproduct of this study, we obtain two results which give

instances of when "approximately trivial" cocycles are cohomologous to the trivial homomorphism. I am grateful to one of the referees for pointing out that these results are of independent interest, and suggesting that they be included in the introduction.

Lemma I. *Let $\Gamma \curvearrowright (X, \mu)$ be a strongly ergodic p.m.p. action, H a Polish group, and $w : \Gamma \times X \to H$ a cocycle. Assume that there exists a sequence $\{\phi_n : X \to H\}_{n\geq 1}$ of Borel maps such that for all $g \in \Gamma$ we have $\mu(\{x \in X \mid w(g, x) = \phi_n(gx)\phi_n(x)^{-1}\}) \to 1$ as $n \to \infty$. Then w is cohomologous to the trivial homomorphism from Γ to H, i.e. there exists a Borel map $\psi : X \to H$ such that $w(g, x) = \psi(gx)\psi(x)^{-1}$ for all $g \in \Gamma$ and almost every $x \in X$.*

For the definition of strong ergodicity, see Subsection 2.3. If H is a compact Polish group, then the proof of [Sc80, Proposition 2.3] implies the following stronger version of Lemma I.

Lemma J ([Sc80]). *Let $\Gamma \curvearrowright (X, \mu)$ be a strongly ergodic p.m.p. action, H a compact Polish group, and $w : \Gamma \times X \to H$ a cocycle. Assume that there exists a sequence $\{\phi_n : X \to H\}_{n\geq 1}$ of Borel maps such that $\lim_{n\to\infty} \phi_n(gx)\phi_n(x)^{-1} = w(g, x)$ for all $g \in \Gamma$ and almost every $x \in X$. Then w is cohomologous to the trivial homomorphism from Γ to H.*

1.8. Organization of the paper

In Sections 2 and 3, we collect a number of results that are needed in the rest of the paper, including the two main ingredients of the proof of Theorem A. In Sections 4–6, we prove several rigidity results for homomorphisms between equivalence relations associated to compact actions. In particular, Theorem A is proven in Section 4. Finally, the last four sections are devoted to the proofs of the corollaries presented in the introduction.

2. Preliminaries

In this section we collect several basic notions and results that we will use throughout the paper.

Recall that a *standard Borel space* is a Polish space X (i.e. a separable complete metrizable topological space) endowed with its σ-algebra of Borel subsets. A *standard probability space* (X, μ) is a Polish space X endowed with a Borel probability measure μ.

Given a compact group G and a closed subgroup $K < G$, we denote by m_G the Haar measure of G and by $m_{G/K}$ the unique G-invariant Borel probability measure on G/K.

2.1. Countable equivalence relations

We continue by recalling several notions about countable equivalence relations.

An equivalence relation $\mathcal{R}$ on a standard Borel space X is called *countable Borel* if

- the equivalence class $[x]_{\mathcal{R}} := \{y \in X \mid (x, y) \in \mathcal{R}\}$ is countable for every $x \in X$, and
- $\mathcal{R}$ is a Borel subset of $X \times X$.

If $\Gamma \curvearrowright X$ is a Borel action of a countable group Γ, then the *orbit equivalence relation*

$$\mathcal{R}(\Gamma \curvearrowright X) := \{(x, y) \in X \times X \mid \Gamma x = \Gamma y\}$$

is countable Borel. Conversely, J. Feldman and C. C. Moore proved that every countable Borel equivalence relation arises in this way [FM77, Theorem 1].

Let $\mathcal{R}$ and $\mathcal{S}$ be countable Borel equivalence relations on standard Borel spaces X and Y. We say that $\mathcal{R}$ is *Borel reducible* to $\mathcal{S}$, and write $\mathcal{R} \leq_B \mathcal{S}$, if there exists a Borel map $\theta : X \to Y$ such that $(x, y) \in \mathcal{R}$ if and only if $(\theta(x), \theta(y)) \in \mathcal{S}$. In this case, θ is called a *reduction from* $\mathcal{R}$ *to* $\mathcal{S}$.

A Borel map $\theta : X \to Y$ is called a *homomorphism from* $\mathcal{R}$ *to* $\mathcal{S}$ if $(\theta(x), \theta(y)) \in \mathcal{S}$ for all $(x, y) \in \mathcal{R}$. If X is endowed with a Borel probability measure μ, then we say that a homomorphism $\theta : X \to Y$ from $\mathcal{R}$ to $\mathcal{S}$ is *trivial* if there is some $y \in Y$ such that $\theta(x) \in [y]_{\mathcal{S}}$ for μ-almost every $x \in X$.

A countable Borel equivalence relation $\mathcal{R}$ on a standard probability space (X, μ) is called *probability measure preserving* (abbreviated *p.m.p.*) if every Borel automorphism $\theta : X \to X$ that satisfies $\theta(x) \in [x]_{\mathcal{R}}$ for all $x \in X$ preserves μ.

The *full group* of $\mathcal{R}$, denoted $[\mathcal{R}]$, is the group of automorphisms of (X, μ) such that $\theta(x) \in [x]_{\mathcal{R}}$ for μ-almost every $x \in X$. The *full pseudogroup* of $\mathcal{R}$, denoted $[[\mathcal{R}]]$, is the set of μ-preserving bijections $\theta : A \to B$ between Borel subsets of X satisfying $\theta(x) \in [x]_{\mathcal{R}}$ for μ-almost every $x \in A$.

The *automorphism group* of $\mathcal{R}$, denoted $\mathrm{Aut}(\mathcal{R})$, is the group of automorphisms θ of (X, μ) such that $(x, y) \in \mathcal{R}$ if and only if $(\theta(x), \theta(y)) \in \mathcal{R}$. Then $[\mathcal{R}]$ is a normal subgroup of $\mathrm{Aut}(\mathcal{R})$, and the *outer automorphism group* of $\mathcal{R}$ is defined as $\mathrm{Out}(\mathcal{R}) = \mathrm{Aut}(\mathcal{R})/[\mathcal{R}]$.

Let $\mathcal{R}$ and $\mathcal{S}$ be countable p.m.p. equivalence relations on standard probability space (X, μ) and (Y, ν). We say that $\mathcal{R}$ and $\mathcal{S}$ are *orbit equivalent*, and write $\mathcal{R} \sim_{\mathrm{OE}} \mathcal{S}$, if there exists an isomorphism of probability spaces $\theta : X \to Y$ such that $(x, y) \in \mathcal{R}$ if and only if $(\theta(x), \theta(y)) \in \mathcal{S}$ almost everywhere. Moreover, we say that $\mathcal{R}$ and $\mathcal{S}$ are *stably orbit equivalent* if $\mathcal{R}_{|X_0} \sim_{\mathrm{OE}} \mathcal{S}_{|Y_0}$ for some Borel subsets $X_0 \subset X$ and $Y_0 \subset Y$ of positive measure. Here, we denote by $\mathcal{R}_{|X_0} := \mathcal{R} \cap (X_0 \times X_0)$ the restriction of $\mathcal{R}$ to the Borel subset X_0 of X.

Two p.m.p. actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ are said to be [*stably*] *orbit equivalent* if the associated equivalence relations $\mathcal{R}(\Gamma \curvearrowright X)$ and $\mathcal{R}(\Lambda \curvearrowright Y)$ are [stably] orbit equivalent.

Let $\mathcal{R}$ be a countable ergodic p.m.p. equivalence relation on a standard probability space (X, μ). The *fundamental group* of $\mathcal{R}$, denoted $\mathcal{F}(\mathcal{R})$, is the multiplicative group of $t > 0$ for which there exist Borel sets $X_1, X_2 \subset X$ of positive measure such that $\mathcal{R}_{|X_1} \sim_{\mathrm{OE}} \mathcal{R}_{|X_2}$ and $\mu(X_1) = t\mu(X_2)$.

2.2. *Profinite and compact actions*

A p.m.p. action $\Gamma \curvearrowright (X, \mu)$ of a countable group Γ on a standard probability space (X, μ) is called *profinite* if it is the inverse limit of a sequence of p.m.p. actions of Γ on finite probability spaces (see [Io08, Definition 1.1]). This means that we can write

$(X,\mu) = \varprojlim (X_n,\mu_n)$, where X_n is a probability space of finite cardinality such that the subalgebra $\overleftarrow{L^\infty}(X_n) \subset L^\infty(X)$ is Γ-invariant for all n.

If such an action is ergodic, then it is isomorphic to an action of the form $\Gamma \curvearrowright \varprojlim (\Gamma/\Gamma_n,\mu_n)$, where $\{\Gamma_n\}_n$ is a descending chain of finite index subgroups of Γ, and $\overleftarrow{\mu_n}$ denotes the normalized counting measure on Γ/Γ_n (see [Io08, Remark 1.3]).

If Γ_n is a normal subgroup of Γ for all n, then the action $\Gamma \curvearrowright \varprojlim (\Gamma/\Gamma_n,\mu_n)$ can be identified with the left translation action $\Gamma \curvearrowright (G, m_G)$, where $G := \varprojlim \Gamma/\Gamma_n$ is the *profinite completion* of Γ with respect to $\{\Gamma_n\}$. Note that the action $\Gamma \curvearrowright (G, m_G)$ is free if and only if $\bigcap_n \Gamma_n = \{e\}$.

Now, recall that a p.m.p. action $\Gamma \curvearrowright (X,\mu)$ is *compact* if the image of Γ in the automorphism group of (X,μ) is compact. Any compact action is isomorphic to an action of the form $\Gamma \curvearrowright (G/K, m_{G/K})$, where G is some compact group in which Γ embeds densely and $K < G$ is a closed subgroup. Note that any profinite action is compact. More precisely, any ergodic profinite p.m.p. action $\Gamma \curvearrowright (X,\mu)$ is isomorphic to a compact action $\Gamma \curvearrowright (G/K, m_{G/K})$, where G is some profinite completion of Γ.

Convention. In order to distinguish the actions $\Gamma \curvearrowright G$ from the general profinite/compact actions $\Gamma \curvearrowright G/K$, we will refer to the former as *translation* profinite/compact actions.

Next, we give some examples of translation profinite actions that we will use later.

Example 2.1. Let S be a set of primes. We define the profinite groups

$$G_S = \prod_{p\in S} \mathrm{SL}_2(\mathbb{F}_p) \quad \text{and} \quad K_S = \prod_{p\in S} \mathrm{SL}_2(\mathbb{Z}_p),$$

where $\mathbb{F}_p$ is the field with p elements and $\mathbb{Z}_p$ is the ring of p-adic integers. The Strong Approximation Theorem states that the diagonal embeddings of $\mathrm{SL}_2(\mathbb{Z})$ into G_S and into K_S are dense. Therefore, the left translation actions $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright G_S$ and $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright K_S$ are profinite and ergodic.

For a subgroup $\Gamma < \mathrm{SL}_2(\mathbb{Z})$, we denote by $G_{\Gamma,S}$ and $K_{\Gamma,S}$ the closures of Γ in G_S and K_S. If Γ is non-amenable, then an extension of the Strong Approximation Theorem to linear groups established in the 1980s by Nori, Weisfeiler and others guarantees that $G_{\Gamma,S} < G_S$ and $K_{\Gamma,S} < K_S$ are open subgroups (see [LS03, Theorem 16.4.1]). Thus, the left translation actions $\Gamma \curvearrowright G_{\Gamma,S}$ and $\Gamma \curvearrowright K_{\Gamma,S}$ are also profinite and ergodic.

2.3. Spectral gap and strong ergodicity

An ergodic p.m.p. action $\Gamma \curvearrowright (X,\mu)$ is called *strongly ergodic* if for any sequence $\{A_n\}_n$ of Borel subsets of X satisfying $\mu(gA_n \triangle A_n) \to 0$ for all $g \in \Gamma$, we have $\mu(A_n)(1-\mu(A_n)) \to 0$. The action $\Gamma \curvearrowright (X,\mu)$ is said to have *spectral gap* if the associated unitary representation π of Γ on $L^2_0(X) = L^2(X) \ominus \mathbb{C}1$ has spectral gap, i.e. there is no sequence $\{\xi_n\}_n$ of unit vectors in $L^2_0(X)$ satisfying $\|\pi(g)(\xi_n) - \xi_n\|_2 \to 0$ for all $g \in \Gamma$.

If an action has spectral gap then it is strongly ergodic. The converse is false in general (see [HK05, Theorem A.3.2]), even within the class of profinite actions (see [AE10, Theorem 5]). Nevertheless, the converse is true for translation profinite actions [AE10]. Moreover, the following holds:

Proposition 2.2. *Let Γ be a residually finite group and $\{\Gamma_n\}_n$ a descending chain of finite index, normal subgroups with $\bigcap_n \Gamma_n = \{e\}$. Let $G = \varprojlim \Gamma/\Gamma_n$ be the profinite completion of Γ with respect to $\{\Gamma_n\}_n$. Consider the left translation action $\Gamma \curvearrowright (G, m_G)$, where m_G is the Haar measure of G. Then the following conditions are equivalent:*

(1) *$\Gamma \curvearrowright (G, m_G)$ is strongly ergodic.*
(2) *$\Gamma \curvearrowright (G, m_G)$ has spectral gap.*
(3) *Γ has* **property** (τ) *with respect to the family $\{\Gamma_n\}_n$ of subgroups, i.e. the representation of Γ on $\bigoplus_n \ell^2_0(\Gamma/\Gamma_n)$ has spectral gap, where $\ell^2_0(\Gamma/\Gamma_n)$ denotes the Hilbert space of functions $f \in \ell^2(\Gamma/\Gamma_n)$ with $\sum_{x\in\Gamma/\Gamma_n} f(x) = 0$.*

Moreover, if Γ is generated by a finite set S, then conditions (1)–(3) *are also equivalent to*

(4) *The Cayley graphs* $\mathrm{Cay}(\Gamma/\Gamma_n, S)$ *form a sequence of* **expanders**.

Proof. The implication (1)⇒(2) is clear, while (2)⇒(1) was proved in [AE10, Theorem 4]. The proof that (2)⇔(3)⇔(4) is well-known and straightforward (see e.g. [LZ03] or [Lu12]). □

Next, we collect from the literature several examples of profinite actions with spectral gap.

Example 2.3. If Γ has Kazhdan's property (T) (e.g. if $\Gamma = \mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$), then any ergodic action of Γ has spectral gap. In this paper, however, we are interested in profinite actions with spectral gap of groups such as $\mathrm{SL}_2(\mathbb{Z})$ and $\mathbb{F}_n$ ($n \geq 2$), which do not have property (T).

One main source of examples will be the famous theorem of Selberg asserting that $\mathrm{SL}_2(\mathbb{Z})$ has property (τ) with respect to the congruence subgroups $\Gamma(m) := \ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z}))$ (see[LZ03, Chapter 4]). This implies that the actions $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright G_S$ and $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright K_S$ defined in Example 2.1 have spectral gap. Moreover, it shows that whenever $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ is a finite index subgroup, then the actions $\Gamma \curvearrowright G_{\Gamma,S}$ and $\Gamma \curvearrowright K_{\Gamma,S}$ have spectral gap, where $G_{\Gamma,S}$ and $K_{\Gamma,S}$ denote the closures of Γ in G_S and K_S, respectively.

Recently, Selberg's theorem has been vastly generalized in a series of papers starting with J. Bourgain and A. Gamburd's breakthrough work [BG05]. In particular, J. Bourgain and P. Varjú have shown that any non-amenable subgroup $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ has property (τ) with respect to the family $\{\Gamma \cap \Gamma(m)\}_{m\in\mathbb{Z}}$ (see [BV10, Theorem 1]). Hence, the actions $\Gamma \curvearrowright G_{\Gamma,S}$ and $\Gamma \curvearrowright K_{\Gamma,S}$ have spectral gap whenever $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ is a non-amenable subgroup.

Remark 2.4. Note that if Γ is a *co-amenable* subgroup of $\mathrm{SL}_2(\mathbb{Z})$, then property (τ) of Γ relative to the family $\{\Gamma \cap \Gamma(m)\}_{m\in\mathbb{Z}}$ can be deduced from Selberg's theorem (see

[Sh99]). If we consider a finite index embedding of $\mathbb{F}_n$, $2 \leq n < \infty$, into $\mathrm{SL}_2(\mathbb{Z})$, then the commutator subgroup $\Gamma = [\mathbb{F}_n, \mathbb{F}_n] \cong \mathbb{F}_\infty$ is co-amenable inside $\mathrm{SL}_2(\mathbb{Z})$. We thus have an alternative way of seeing that the actions $\Gamma \curvearrowright G_{\Gamma,S}$ and $\Gamma \curvearrowright K_{\Gamma,S}$ have spectral gap, for this specific group Γ.

We continue this section with a lemma that will be a key ingredient in the proof of Theorem A.

Lemma 2.5. *Let $\Gamma \curvearrowright (X, \mu)$ be a strongly ergodic p.m.p. action and let $\varepsilon > 0$. Then we can find $\delta > 0$ and $F \subset \Gamma$ finite such that if a Borel map $\rho : X \to Y$ with values in a standard Borel space Y satisfies $\mu(\{x \in X \mid \rho(gx) = \rho(x)\}) \geq 1 - \delta$ for all $g \in F$, then there exists $y \in Y$ such that $\mu(\{x \in X \mid \rho(x) = y\}) \geq 1 - \varepsilon$.*

Proof. We may clearly assume that $\varepsilon \in (0, 1)$. Then we can find $\delta > 0$ and $F \subset \Gamma$ finite such that there is no Borel set $A \subset X$ satisfying $\mu(A) \in (\varepsilon/2, 1 - \varepsilon/2)$ and $\mu(g^{-1}A \triangle A) \leq \delta$ for all $g \in F$.

Let $\rho : X \to Y$ be as in the hypothesis and denote by $\nu = \rho_*\mu$ the push-forward of μ through ρ. Let $B \subset Y$ be a Borel set and set $A = \rho^{-1}(B)$. Then $g^{-1}A \triangle A \subset \{x \in X \mid \rho(gx) \neq \rho(x)\}$, and therefore $\mu(g^{-1}A\Delta A) \leq \delta$ for all $g \in F$. It follows that $\nu(B) = \mu(A) \notin (\varepsilon/2, 1 - \varepsilon/2)$.

Thus, ν is a Borel probability measure on Y such that $\nu(B) \notin (\varepsilon/2, 1-\varepsilon/2)$ for every Borel set $B \subset Y$. This implies that there is $y \in Y$ such that $\nu(\{y\}) \geq 1 - \varepsilon$. □

2.4. Smooth actions

Next, we recall the notion of smooth actions and prove an elementary result that we will need in Section 5.

Definition 2.6. A Borel space X is called *countably separated* if there exists a sequence of Borel sets which separate points. A Borel action $H \curvearrowright X$ of a topological group H on a standard Borel space X is called *smooth* if the quotient Borel structure on X/H is countably separated.

Lemma 2.7. *Let H be a locally compact Polish group, Ω a Polish space, and $H \curvearrowright \Omega$ a smooth continuous action. Denote by Ω_0 the set of $x \in \Omega$ such that $hx \neq x$ for all $h \in H \setminus \{e\}$. Then Ω_0 is a G_δ subset of Ω and there exists a Borel map $p : \Omega_0 \to H$ such that $p(hx) = hp(x)$ for all $h \in H$ and $x \in \Omega_0$.*

Proof. We denote by d_H and d_Ω the distance functions on H and Ω which give the respective Polish topologies. Since H is locally compact, the distance d_H can be chosen *proper*, i.e. such that the closed ball $\{h \in H \mid d_H(h, e) \leq M\}$ is compact for all $M > 0$.

Let $m, n \geq 1$. We denote by $U_{m,n}$ the set of $x \in \Omega$ such that $d_\Omega(hx, x) > 1/m$ for all $h \in H$ satisfying $1/n \leq d_H(h, e) \leq n$. We claim that $U_{m,n}$ is an open set. Indeed, let x_k be a sequence in $\Omega \setminus U_{m,n}$ which converges to a point $x \in \Omega$. Then we can find a sequence $h_k \in H$ such that $d_\Omega(h_k x_k, x_k) \leq 1/m$ and $1/n \leq d_H(h_k, e) \leq n$ for all $k \geq 1$. Since the set of $h \in H$ which satisfy $1/n \leq d_H(h, e) \leq n$ is compact, after passing

to a subsequence we may assume that h_k converges to some $h \in H$. But then $1/n \leq d_H(h, e) \leq n$ and $d_\Omega(hx, x) = \lim_{k\to\infty} d_\Omega(h_k x_k, x_k) \leq 1/m$. Thus, $x \in \Omega \setminus U_{m,n}$. This shows that $\Omega \setminus U_{m,n}$ is closed, and hence $U_{m,n}$ is open. Since $\Omega_0 = \bigcap_{n\geq 1} \bigcup_{m\geq 1} U_{m,n}$, we deduce that Ω_0 is a G_δ set.

Next, since the action $H \curvearrowright \Omega$ is smooth, it admits a *Borel selector*. More precisely, we can find a Borel map $s : \Omega \to \Omega$ such that $s(x) \in Hx$ and $s(x) = s(hx)$ for all $x \in \Omega$ and $h \in H$ (see [Ke95, Exercise 18.20 iii)]). Since the action $H \curvearrowright \Omega_0$ is free, for every $x \in \Omega_0$ there is a unique $p(x) \in H$ such that $x = p(x)s(x)$. The map $p : \Omega_0 \to H$ clearly satisfies $p(hx) = hp(x)$ for all $h \in H$ and $x \in \Omega_0$.

Let us show that p is Borel. To this end, let $F \subset H$ be a closed subset. Then the map $f : \Omega \to [0, \infty)$ given by $f(x) = \inf_{h\in F} d_\Omega(x, hs(x))$ is Borel. Note that if $x \in \Omega_0$ and $p(x) \in F$, then $f(x) = 0$. Conversely, let $x \in \Omega_0$ be such that $f(x) = 0$. We claim that $p(x) \in F$. Indeed, there is a sequence $\{h_n\}_{n\geq 1}$ in F such that $h_n s(x) \to x$ as $n \to \infty$. Since the action $H \curvearrowright \Omega$ is smooth and the stabilizer of x is trivial, the map $H \ni h \mapsto hx \in Hx$ is a homeomorphism [Zi84, Theorem 2.1.14]. Thus, we can find $h \in H$ such that $h_n \to h$ as $n \to \infty$. Since F is closed, we find that $h \in F$. This implies that $hs(x) = x$, and hence $p(x) = h \in F$.

Altogether, it follows that $\{x \in \Omega_0 \mid p(x) \in F\} = \Omega_0 \cap \{x \in \Omega \mid f(x) = 0\}$ is a Borel set. Since this holds for any closed set $F \subset H$, we conclude that p is Borel. □

2.5. *Extensions of homomorphisms*

We finish this section with an elementary result about extending homomorphisms from a dense subgroup of a Polish group to the whole group.

Lemma 2.8. *Let G be a locally compact Polish group, m_G a Haar measure of G, and H a Polish group. Let $\Gamma < G$ be a dense subgroup and $\delta : \Gamma \to H$ a homomorphism. Assume that $\theta : G \to H$ is a Borel map such that for all $g \in \Gamma$ we have $\theta(gx) = \delta(g)\theta(x)$ for m_G-almost every $x \in G$. Then δ extends to a continuous homomorphism $\delta : G \to H$ and we can find $h \in H$ such that $\theta(g) = \delta(g)h$ for m_G-almost every $g \in G$.*

Proof. Let $f : G \times G \to H$ be given by $f(x, y) = \theta(x)^{-1}\theta(y)$. Define S to be the set of $g \in G$ such that $f(gx, gy) = f(x, y)$ for $(m_G \times m_G)$-almost every $(x, y) \in G \times G$. Since H is Polish, S is a closed subgroup of G. Since $\Gamma < G$ is dense and $\Gamma \subset S$ by the hypothesis, we deduce that $S = G$.

This implies that given $g \in G$, we have $\theta(gx)\theta(x)^{-1} = \theta(gy)\theta(y)^{-1}$ for $(m_G \times m_G)$-almost every $(x, y) \in G \times G$. Thus, we can find $\delta(g) \in H$ such that $\theta(gx)\theta(x)^{-1} = \delta(g)$ for almost every $x \in G$. It is easy to see that $\delta : G \to H$ must be a continuous homomorphism.

Finally, since $\theta(gx)\theta(x)^{-1} = \delta(g)$ for $(m_G \times m_G)$-almost every $(g, x) \in G \times G$, we can find $x_0 \in G$ such that $\theta(gx_0)\theta(x_0)^{-1} = \delta(g)$ for m_G-almost every $g \in G$. Hence, for almost every $g \in G$ we have $\theta(g) = \delta(gx_0^{-1})\theta(x_0) = \delta(g)(\delta(x_0^{-1})\theta(x_0))$. Thus, the conclusion holds for $h = \delta(x_0^{-1})\theta(x_0)$. □

3. Cocycle rigidity

The purpose of this section is twofold. Firstly, we extract from [Io08] and [Fu09] two rigidity results for cocycles associated with profinite and compact actions. These results will be essential in the proofs of our main results. Secondly, we prove that for strongly ergodic actions, any cocycle that is approximately cohomologous to the trivial cocycle must be cohomologous to the trivial cocycle. We start by recalling some notions.

Let $\Gamma \curvearrowright (X, \mu)$ be a p.m.p. action of a countable group Γ on a standard probability space (X, μ). A measurable map $w : \Gamma \times X \to \Lambda$ into a group Λ is called a *measurable cocycle* (or just a *cocycle*) if it satisfies the identity $w(gh, x) = w(g, hx)w(h, x)$ for all $g, h \in \Gamma$ and almost every $x \in X$. Two cocycles $w_1, w_2 : \Gamma \times X \to \Lambda$ are said to be *cohomologous* if there exists a Borel map $\phi : X \to \Lambda$ such that $w_1(g, x) = \phi(gx)w_2(g, x)\phi(x)^{-1}$ for all $g \in \Gamma$ and almost every $x \in X$.

3.1. Cocycle rigidity for profinite actions

Now, assume that Γ has property (T) and let $\Gamma \curvearrowright (X, \mu) = \varprojlim (X_n, \mu_n)$ be a free ergodic profinite p.m.p. action. We proved that any cocycle $w : \Gamma \times X \to \Lambda$ with values in a countable group Λ is cohomologous to a cocycle which factors through the map $\Gamma \times X \to \Gamma \times X_n$ for some n (see [Io08, Theorem B]). In [Fu09], A. Furman extended this result from profinite to compact actions (see [Fu09, Theorem 5.21]).

A main ingredient in the proof of Theorem A is the following criterion for untwisting cocycles for profinite actions $\Gamma \curvearrowright (X, \mu)$. This criterion is an easy consequence of the proof of [Io08, Theorem B] and is implicitly proved in [Fu09, proof of Theorem 5.21]. A crucial aspect of this criterion is that it applies to arbitrary residually finite groups Γ, which are not assumed to have property (T).

Theorem 3.1 ([Io08], [Fu09]). *Let Γ be a residually finite group and $\{\Gamma_n\}_n$ a descending chain of finite index, normal subgroups with trivial intersection, $\bigcap_n \Gamma_n = \{e\}$. Let $G = \varprojlim \Gamma/\Gamma_n$ be the profinite completion of Γ with respect to $\{\Gamma_n\}_n$ and consider the left translation action $\Gamma \curvearrowright (G, m_G)$. Let Λ be a countable group and $w : \Gamma \times G \to \Lambda$ a cocycle. Assume that for some constant $C \in (31/32, 1)$ we can find a neighborhood V of the identity e in G such that*

$$m_G(\{x \in G \mid w(g, xt) = w(g, x)\}) \geq C \quad \text{for all } g \in \Gamma \text{ and every } t \in V. \tag{3.1}$$

Then we can find an open subgroup $G_0 < G$ such that the restriction of w to $(\Gamma \cap G_0) \times G_0$ is cohomologous to a homomorphism $\delta : \Gamma \cap G_0 \to \Lambda$.

For the reader's convenience, we give two proofs of Theorem 3.1, following [Io08] and [Fu09], respectively.

First proof of Theorem 3.1. Denote by c the counting measure on Λ. Following [Io08], consider the infinite measure preserving action of Γ on $(Z, \rho) := (G \times G \times \Lambda, m_G \times m_G \times c)$ given by

$$g \cdot (x, y, \lambda) = (gx, gy, w(g, x)\lambda w(g, y)^{-1}).$$

Denote by $\pi : \Gamma \to \mathcal{U}(L^2(Z, \rho))$ the associated Koopman representation. For $n \geq 0$, we let $r_n : G \to \Gamma/\Gamma_n$ be the quotient homomorphism and ζ_n the characteristic function of the subset $\{(x, y) \in G \times G \mid r_n(x) = r_n(y)\} \times \{e\}$ of Z.

Since V is a neighborhood of the identity, we can find $n \geq 0$ such that V contains $G_n := \ker(r_n)$. We claim that $\|\pi(g)(\zeta_n) - \zeta_n\|_2 \leq \sqrt{2 - 2C}\,\|\zeta_n\|_2$ for all $g \in \Gamma$. Towards this, let $g \in \Gamma$. Since $r_n(x) = r_n(y)$ if and only if $x^{-1}y \in G_n$, a simple computation shows that

$$\begin{aligned}\langle \pi(g)(\zeta_n), \zeta_n\rangle &= (m_G \times m_G)(\{(x, y) \in G \times G \mid r_n(x) = r_n(y) \text{ and } w(g, x) = w(g, y)\}\\ &= (m_G \times m_G)(\{x, t) \in G \times G \mid t \in G_n \text{ and } w(g, x) = w(g, xt)\}.\end{aligned}$$

Since $m_G(\{x \in G \mid w(g, xt) = w(g, x)\}) \geq C$ for all $t \in G_n$, and $m_G(G_n) = \|\zeta_n\|_2^2 = |\Gamma/\Gamma_n|^{-1}$, we deduce that $\langle \pi(g)\zeta_n, \zeta_n\rangle \geq C\mu(G_n) = C\|\zeta_n\|_2^2$. This implies the claim.

Now, define $\xi_n := \sqrt{|\Gamma/\Gamma_n|}\,\zeta_n \in L^2(Z, \rho)$. Then ξ_n is a unit vector and the above claim says that $\|\pi(g)(\xi_n) - \xi_n\|_2 \leq \sqrt{2 - 2C}$ for all $g \in \Gamma$. By using a standard averaging argument, we can find a $\pi(\Gamma)$-invariant vector $\eta \in L^2(Z, \rho)$ such that $\|\eta - \xi_n\|_2 \leq \sqrt{2 - 2C}$.

Since $\sqrt{2 - 2C} < 1/4$, continuing as in part 2 of the proof of [Io08, Theorem B] shows that w is cohomologous to a cocycle which factors through the map $\Gamma \times G \to \Gamma \times \Gamma/\Gamma_N$ for some $N \geq n$. Then $G_0 := \ker r_N$ clearly satisfies the conclusion. □

Second proof of Theorem 3.1. Let us now explain how the proof of [Fu09, Theorem 5.21] also implies Theorem 3.1. Following [Fu09], fix $t \in V$ and define a new cocycle $w_t : \Gamma \times G \to \Lambda$ by letting $w_t(g, x) = w(g, xt)$. Since $\Gamma \curvearrowright (G, m_G)$ is ergodic and $C > 7/8$, [Io08, Lemma 2.1] implies that w_t is cohomologous to w. Hence, we can find a Borel map $\phi_t : G \to \Lambda$ such that $w_t(g, x) = \phi_t(gx)w(g, x)\phi_t(x)^{-1}$ for all $g \in \Gamma$ and almost every $x \in G$.

Moreover, a close inspection of the proof of [Io08, Lemma 2.1] shows that ϕ_t satisfies the following: there exists $\eta_t \in L^2(G \times \Lambda, m_G \times c)$ such that $\phi_t(x)$ is the unique $\lambda \in \Lambda$ satisfying $|\eta_t(x, \lambda)| > 1/2$ for almost every $x \in G$, and $\|\eta_t - 1_{G\times\{e\}}\|_2 \leq \sqrt{2 - 2C} < 1/4$. Since

$$\begin{aligned}m_G(\{x \in G \mid |\eta_t(x, e)| \leq 1/2\}) &\leq 4\int_G |\eta_t(x, e) - 1|^2\, \mathrm{d}m_G(x)\\ &\leq 4\|\eta_t - 1_{G\times\{e\}}\|_2^2 < 1/4,\end{aligned}$$

we conclude that $m_G(\{x \in G \mid \phi_t(x) = e\}) > 3/4$. The proof of [Fu09, Theorem 5.21] now applies verbatim to give the conclusion. □

3.2. *Cocycle rigidity for compact actions*

Later on, we will also need the following variant of Theorem 3.1 for compact actions. This result is an immediate consequence of [Fu09].

Theorem 3.2 ([Fu09]). *Let Γ be a countable group together with a dense embedding $\tau : \Gamma \hookrightarrow G$ into a connected, compact group G. Consider the left translation action $\Gamma \curvearrowright (G, m_G)$, where m_G is the Haar measure of G. Assume that $\pi_1(G)$, the fundamental group of G, is finite. Let Λ be a countable group and $w : \Gamma \times G \to \Lambda$ a cocycle. Assume that for some constant $C \in (31/32, 1)$ we can find a neighborhood V of the identity e in G such that condition (3.1) holds.*

(1) *If any homomorphism $\pi_1(G) \to \Lambda$ is trivial (e.g. $\pi_1(G) = \{e\}$ or Λ is torsion free), then w is cohomologous to a homomorphism $\delta : \Gamma \to \Lambda$.*

(2) *In general, we can find a subgroup $\Lambda_0 < \Lambda$, a finite subgroup $\Lambda_1 < Z(\Lambda_0)$, a Borel map $\phi : G \to \Lambda$, and a homomorphism $\delta : \Gamma \to \Lambda_0/\Lambda_1$ such that if $p : \Lambda_0 \to \Lambda_0/\Lambda_1$ denotes the quotient homomorphism, then $w'(g, x) = \phi(gx)^{-1} w(g, x)\phi(x) \in \Lambda_0$ and $p(w'(g, x)) = \delta(g)$ for all $g \in \Gamma$ and almost every $x \in G$.*

Here and after we denote by $Z(\Lambda)$ the center of a group Λ.

Proof. (1) Assume that the only homomorphism $\pi_1(G) \to \Lambda$ is the trivial one. As in the second proof of Theorem 3.1, for every $t \in V$ we can find a Borel map $\phi_t : G \to \Lambda$ which satisfies $m_G(\{x \in G \mid \phi_t(x) = e\}) > 3/4$ and $w_t(g, x) = \phi_t(gx)w(g, x)\phi_t(x)^{-1}$ for all $g \in \Gamma$ and almost every $x \in G$. The conclusion then follows from the proof of [Fu09, Theorem 5.21].

(2) Let $\tilde{G}$ be the universal covering group of G, and denote by $\pi : \tilde{G} \to G$ the covering map. Let $\tilde{\Gamma} = \pi^{-1}(\Gamma)$ and define a cocycle $\tilde{w} : \tilde{\Gamma} \times \tilde{G} \to \Lambda$ by letting $\tilde{w}(g, x) = w(\pi(g), \pi(x))$. Since $\tilde{G}$ is connected, it follows that $\tilde{\Gamma} < \tilde{G}$ is dense. Since $\pi_1(\tilde{G}) = \{e\}$, the first part of the proof shows that $\tilde{w}$ is cohomologous to a homomorphism $\delta : \tilde{\Gamma} \to \Lambda$. Let $\psi : \tilde{G} \to \Lambda$ be a Borel map satisfying $\tilde{w}(g, x) = \psi(gx)\delta(g)\psi(x)^{-1}$ for all $g \in \tilde{\Gamma}$ and almost every $x \in \tilde{G}$.

Fix $k \in \ker \pi$ and define $\rho_k : \tilde{G} \to \Lambda$ by letting $\rho_k(x) = \psi(x)^{-1}\psi(xk)$. We claim that ρ_k is constant. Note that $\rho_k(gx) = \delta(g)\rho_k(x)\delta(g)^{-1}$ for all $g \in \tilde{\Gamma}$ and almost every $x \in \tilde{G}$. Let $h \in \Lambda$ be such that $A_h = \{x \in \tilde{G} \mid \rho_k(x) = h\}$ has positive measure. Then $gA_h = A_{\delta(g)h\delta(g)^{-1}}$ for all $g \in \tilde{\Gamma}$. This implies that for any $g_1, g_2 \in \tilde{\Gamma}$, the sets $g_1 A_h$ and $g_2 A_h$ are either disjoint or equal. Hence, A_h is invariant under some finite index subgroup $\Gamma_0 < \tilde{\Gamma}$. Since $\tilde{G}$ is connected and $\tilde{\Gamma} < \tilde{G}$ is dense, we see that $\Gamma_0 < \tilde{G}$ is also dense. As a consequence, $A_h = \tilde{G}$ almost everywhere. This proves the claim.

Thus, there is a homomorphism $\rho : \ker \pi \to \Lambda$ such that $\psi(x)^{-1}\psi(xk) = \rho(k)$ for all $k \in \ker \pi$ and almost every $x \in \tilde{G}$. Moreover, $\rho(k)$ commutes with $\delta(\tilde{\Gamma})$ for all $k \in \ker \pi$. Since $\tilde{w}(kg, x) = \tilde{w}(g, x)$, we also find that $\psi(x)^{-1}\psi(kx) = \delta(k)^{-1}$ for all $k \in \ker \pi$ and almost every $x \in G$. By combining these two facts, we deduce that $\delta(k)^{-1} = \rho(x^{-1}kx)$, which implies that $\delta(k)^{-1} = \rho(k)$ for all $k \in \ker \pi$. Thus, if $\Lambda_0 := \delta(\tilde{\Gamma})$ and $\Lambda_1 := \delta(\ker \pi) = \rho(\ker \pi)$, then Λ_1 is finite and $\Lambda_1 < Z(\Lambda_0)$. In particular, we have a homomorphism $\bar{\delta} : \Gamma = \tilde{\Gamma}/\ker \pi \to \Lambda_0/\Lambda_1$ given by $\bar{\delta}(x \ker \pi) = \delta(x)\Lambda_1$.

Finally, choose a Borel map $\phi : G \to \Lambda$ such that $\phi \circ \pi = \psi$. Note that if $y, y' \in \tilde{G}$ are such that $\pi(y) = \pi(y')$, then $y^{-1}y' \in \ker \pi$, hence $\psi(y') = \psi(y)\rho(y^{-1}y') \in \psi(y)\Lambda_1$. It follows that if $g \in \Gamma$ and $\tilde{g} \in \delta^{-1}(\{g\})$, then $w'(g, x) := \phi(gx)^{-1}w(g, x)\phi(x) \in \delta(\tilde{g})\Lambda_1 \subset \Lambda_0$ for almost every $x \in G$. This shows that $p(w'(g, x)) = \delta(\tilde{g})\Lambda_1 = \bar{\delta}(g)$ and finishes the proof. □

3.3. Approximately trivial cocycles

We end this section by proving Lemmas I and J.

Proof of Lemma I. Let $n \geq 1$. Since the action $\Gamma \curvearrowright (X, \mu)$ is strongly ergodic, by applying Lemma 2.5 we can find $\delta_n > 0$ and $F_n \subset \Gamma$ finite such that if a Borel map $\rho : X \to H$ satisfies $\mu(\{x \in X \mid \rho(gx) = \rho(x)\}) \geq 1 - \delta_n$ for every $g \in F_n$, then there exists $y \in H$ such that $\mu(\{x \in X \mid \rho(x) = y\}) \geq 1 - 1/2^n$. Moreover, we may clearly assume that $\delta_{n+1} \leq \delta_n \leq 1/2^n$ and $F_n \subset F_{n+1}$ for every $n \geq 1$.

Now, after replacing $\{\phi_n : X \to H\}_{n\geq 1}$ with a subsequence, we may assume that

$$\mu(\{x \in X \mid w(g,x) = \phi_n(gx)\phi_n(x)^{-1}\}) \geq 1 - \delta_n/2 \geq 1 - 1/2^{n+1} \quad \text{for all } g \in F_n \text{ and } n \geq 1. \tag{3.2}$$

Next, we claim that there exists a sequence $\{\psi_n : X \to H\}_{n\geq 1}$ of Borel maps such that

(1) $\mu(\{x \in X \mid w(g,x) = \psi_n(gx)\psi_n(x)^{-1}\}) \geq 1 - \delta_n/2$ for all $g \in F_n$ and every $n \geq 1$, and
(2) $\mu(\{x \in X \mid \psi_n(x) = \psi_{n+1}(x)\}) \geq 1 - 1/2^n$ for every $n \geq 1$.

We proceed by induction. Thus, let $\psi_1 := \phi_1$. Assume that we have constructed $\psi_1, \ldots, \psi_n$ and let us construct ψ_{n+1}. To this end, we define $\rho : X \to H$ by letting $\rho(x) = \psi_n(x)^{-1}\phi_{n+1}(x)$.

If $g \in F_n$, then by combining (3.2) with condition (1), we derive that the set of $x \in X$ such that $w(g,x) = \psi_n(gx)\psi_n(x)^{-1} = \phi_{n+1}(gx)\phi_{n+1}(x)^{-1}$ has measure $\geq 1 - \delta_n$. Hence $\mu(\{x \in X \mid \rho(gx) = \rho(x)\}) \geq 1 - \delta_n$ for all $g \in F_n$. This implies that we can find $h \in H$ such that $\mu(\{x \in X \mid \rho(x) = h\}) \geq 1 - 1/2^n$.

Therefore, if we define $\psi_{n+1}(x) := \phi_{n+1}(x)h^{-1}$, then $\mu(\{x \in X \mid \psi_n(x) = \psi_{n+1}(x)\}) \geq 1 - 1/2^n$ and also $\mu(\{x \in X \mid w(g,x) = \psi_{n+1}(gx)\psi_{n+1}(x)^{-1}\}) = \mu(\{x \in X \mid w(g,x) = \phi_{n+1}(gx)\phi_{n+1}(x)^{-1}\}) \geq 1 - \delta_n$ for all $g \in F_n$. This finishes the proof of the claim.

For $N \geq 1$, let $X_N := \{x \in X \mid \psi_n(x) = \psi_N(x) \text{ for all } n \geq N\}$. Then condition (2) implies that $\mu(X_N) \geq 1 - 1/2^{N-1}$. Thus, if $X' = \bigcup_{N\geq 1} X_N$, then $\mu(X') = 1$. We define $\psi : X \to H$ by letting

$$\psi(x) = \begin{cases} \psi_N(x) & \text{if } x \in X_N \text{ for some } N \geq 1, \\ e & \text{if } x \in X \setminus X'. \end{cases}$$

We claim that ψ satisfies the conclusion of the lemma. To see this, fix $g \in \Gamma$. For every $N \geq 1$, we define $Y_N := \{x \in X \mid w(g,x) = \psi_n(gx)\psi_n(x) \text{ for all } n \geq N\}$. Then (3.2) implies that $\mu(Y_N) \geq 1 - 1/2^{N+1}$. Hence, $Y' = \bigcup_{N\geq 1} Y_N$ satisfies $\mu(Y') = 1$. It is now clear that for every $x \in g^{-1}X' \cap X' \cap Y'$ we have $w(g,x) = \psi(gx)\psi(x)^{-1}$. Since $\mu(X') = \mu(Y') = 1$, we are done. □

Proof of Lemma J. Let $d : H \times H \to [0, \infty)$ be a left-right invariant metric on H. Define $\phi_{m,n} : X \to H$ by $\phi_{m,n}(x) = \phi_m^{-1}(x)\phi_n(x)$ for $m, n \geq 1$. Then $\lim_{m,n\to\infty} d(\phi_{m,n}(gx), \phi_{m,n}(x)) = 0$ for all $g \in \Gamma$ and almost every $x \in X$. By using the strong ergodicity of the action $\Gamma \curvearrowright (X, \mu)$ as in the proof of [Sc80, Proposition 2.3] we can find $h_{m,n} \in H$ such that $\lim_{m,n\to\infty} d(\phi_{m,n}(x), h_{m,n}) = 0$ for almost every $x \in X$. This further implies that we can find $h_n \in H$ such that the limit $\psi(x) = \lim_{n\to\infty} \phi_n(x)h_n$ exists for almost every $x \in X$. But then $\psi : X \to H$ is a Borel map such that $w(g, x) = \psi(gx)\psi(x)^{-1}$ for all $g \in \Gamma$ and almost every $x \in X$. □

4. Homomorphism rigidity and proof of Theorem A

4.1. Homomorphism rigidity for profinite actions

The main goal of this section is to prove Theorem A. The proof relies on the following rigidity result for homomorphisms between equivalence relations arising from translation profinite actions with spectral gap.

Theorem 4.1. *Let Γ be a residually finite group. Let $G = \varprojlim \Gamma/\Gamma_n$ be the profinite completion of Γ with respect to a chain of finite index normal subgroups with trivial intersection. Assume that the left translation action $\Gamma \curvearrowright (G, m_G)$ has spectral gap. Let Λ be a countable subgroup of a Polish group H and consider the left translation action $\Lambda \curvearrowright H$. Let $\theta : G \to H$ be a Borel map such that $\theta(\Gamma x) \subset \Lambda\theta(x)$ for almost every $x \in G$. Let $w : \Gamma \times G \to \Lambda$ be the cocycle defined by $\theta(gx) = w(g, x)\theta(x)$ for all $g \in \Gamma$ and almost every $x \in G$.*

Then we can find an open subgroup $G_0 < G$, a continuous homomorphism $\delta : G_0 \to H$, a Borel map $\phi : G_0 \to \Lambda$, and $h \in H$ such that

- $\delta(\Gamma \cap G_0) \subset \Lambda$,
- $w(g, x) = \phi(gx)\delta(g)\phi(x)^{-1}$ *for all $g \in \Gamma \cap G_0$ and almost every $x \in G_0$, and*
- $\theta(x) = \phi(x)\delta(x)h$ *for almost every $x \in G_0$.*

Remark 4.2. There are two useful ways of interpreting the conclusion of Theorem 4.1. Firstly, Theorem 4.1 describes all homomorphisms between $\mathcal{R}(\Gamma \curvearrowright G)$ and $\mathcal{R}(\Lambda \curvearrowright H)$. Thus, it shows that any Borel map $\theta : G \to H$ which satisfies $\theta(\Gamma x) \subset \Lambda\theta(x)$ for almost every $x \in G$ arises from a homomorphism $\delta : G_0 \to H$ for some open subgroup $G_0 < G$.

Secondly, Theorem 4.1 can be viewed as a rigidity result for cocycles $w : \Gamma \times G \to \Lambda$. More precisely, assume that if we view w as a cocycle with values in H, then w is cohomologous to the trivial cocycle. This means that there exists a Borel map $\theta : G \to H$ such that $w(g, x) = \theta(gx)\theta(x)^{-1}$ for all $g \in \Gamma$ and almost every $x \in G$. Theorem 4.1 then shows that there exists an open subgroup $G_0 < G$ such that the restriction of w to $(\Gamma \cap G_0) \times G_0$ is cohomologous to a homomorphism $\delta : \Gamma \cap G_0 \to \Lambda$.

Proof of Theorem 4.1. We claim that we can find an open subgroup $G_0 < G$ and a Borel map $\phi : G_0 \to \Lambda$ such that $w(g, x) = \phi(gx)\delta(g)\phi(x)^{-1}$ for all $g \in \Gamma \cap G_0$ and almost every $x \in G_0$. To this end, let $\varepsilon \in (0, 1/64)$.

Since the action $\Gamma \curvearrowright (G, m_G)$ has spectral gap, it is strongly ergodic. Lemma 2.5 yields $\delta > 0$ and a finite set $F \subset \Gamma$ such that if a Borel map $\rho : G \to H$ satisfies $m_G(\{x \in X \mid \rho(gx) = \rho(x)\}) \geq 1 - \delta$ for all $g \in F$, then $m_G(\{x \in X \mid \rho(x) = y\}) \geq 1 - \varepsilon$ for some $y \in H$.

Now, if $A \subset G$ is a Borel set, then $\lim_{t \to e} m_G(At \,\triangle\, A) = 0$. Since Λ is countable, it follows that $\lim_{t \to e} m_G(\{x \in G \mid w(g, xt) = w(g, x)\}) = 1$. Thus, we can find a neighborhood V of e in G such that

$$m_G(\{x \in G \mid w(g, xt) = w(g, x)\}) \geq 1 - \delta \quad \text{for all } \in F \text{ and } t \in V. \tag{4.1}$$

Fix $t \in V$ and define $\rho_t : G \to H$ by $\rho_t(x) = \theta(x)^{-1}\theta(xt)$. Then for almost every $x \in G$ we have $\rho_t(gx) = \theta(gx)^{-1}\theta(gxt) = \theta(x)^{-1}w(g, x)^{-1}w(g, xt)\theta(xt)$. Thus, $\rho_t(gx) = \rho_t(x)$ if and only if $w(g, xt) = w(g, x)$. Consequently, (4.1) implies that $m_G(\{x \in G \mid \rho_t(gx) = \rho_t(x)\}) \geq 1 - \delta$ for all $g \in F$.

We deduce that there is $y_t \in H$ such that $m_G(\{x \in G \mid \rho_t(x) = y_t\}) \geq 1 - \varepsilon$. This implies that $m_G(\{x \in G \mid \rho_t(gx) = \rho_t(x)\}) \geq 1 - 2\varepsilon$ for all $g \in \Gamma$. Equivalently,

$$m_G(\{x \in G \mid w(g, xt) = w(g, x)\}) \geq 1 - 2\varepsilon \quad \text{for all } g \in \Gamma.$$

Since $1 - 2\varepsilon > 31/32$ and $t \in V$ is arbitrary, Theorem 3.1 implies the claim.

Denoting $\tilde{\theta}(x) = \phi(x)^{-1}\theta(x)$ and using the claim we deduce that

$$\tilde{\theta}(gx) = \delta(g)\tilde{\theta}(x) \quad \text{for all } g \in \Gamma \cap G_0 \text{ and almost every } x \in G_0. \tag{4.2}$$

By applying Lemma 2.8 we conclude that δ extends to a continuous homomorphism $\delta : G_0 \to H$ and we can find $h \in H$ such that $\tilde{\theta}(g) = \delta(g)h$ for almost every $g \in G_0$. □

4.2. Proof of part (1) of Theorem A

In order to prove Theorem A, we handle parts (1) and (2) separately. First, we use Theorem 4.1 to describe the stable orbit equivalences between translation profinite actions with spectral gap. More generally, we have:

Corollary 4.3. *Let Γ, Λ be residually finite groups. Let $G = \varprojlim \Gamma/\Gamma_n$ and $H = \varprojlim \Lambda/\Lambda_n$ be the profinite completions of Γ and Λ with respect to chains of finite index normal subgroups with trivial intersection. Assume that the left translation action $\Gamma \curvearrowright (G, m_G)$ has spectral gap. Let $A \subset G$ and $B \subset H$ be Borel sets of positive measure endowed with the probability measures obtained by restricting and rescaling m_G and m_H. Let $\theta : A \to B$ be an isomorphism of probability spaces such that $\theta(\Gamma x \cap A) \subset \Lambda\theta(x) \cap B$ for almost every $x \in A$.*

Then we can find $\tau \in [\mathcal{R}(\Gamma \curvearrowright G)]$, $\rho \in [\mathcal{R}(\Lambda \curvearrowright H)]$, open subgroups $G_0 < G$ and $H_0 < H$, a continuous isomorphism $\delta : G_0 \to H_0$, and $h \in H$ such that $\tau(G_0) \subset A$, $\mu(A)/\nu(B) = [H : H_0]/[G : G_0]$,

$$\delta(\Gamma \cap G_0) \subset \Lambda \cap H_0 \quad \text{and} \quad (\rho \circ \theta \circ \tau)(x) = \delta(x)h \quad \text{for almost every } x \in G_0.$$

If moreover $\theta(\Gamma x \cap A) = \Lambda\theta(x) \cap B$ for almost every $x \in A$, then $\delta(\Gamma \cap G_0) = \Lambda \cap H_0$.

Corollary 4.3 clearly implies the "only if" assertion from part (1) of Theorem A. Note that the "if" assertion of (1) is obvious because $\mathcal{R}(\Gamma \curvearrowright G)_{|G_0} = \mathcal{R}((\Gamma \cap G_0) \curvearrowright G_0)$ whenever $G_0 < G$ is an open subgroup.

Proof of Corollary 4.3. After replacing θ with $\theta \circ \tau$ for some $\tau \in [\mathcal{R}(\Gamma \curvearrowright G)]$, we may assume that A contains an open subgroup $G_1 < G$. We will prove that the conclusion holds for $\tau = \mathrm{id}_G$.

Note that the action $\Gamma \cap G_1 \curvearrowright (G_1, m_{G_1})$ has spectral gap. Indeed, since the action $\Gamma \curvearrowright (G, m_G)$ has spectral gap, it follows that $\mathcal{R}(\Gamma \curvearrowright G)$ and hence $\mathcal{R}(\Gamma \curvearrowright G)_{|G_1} = \mathcal{R}((\Gamma \cap G_1) \curvearrowright G_1)$ are strongly ergodic. Thus, $\Gamma \cap G_1 \curvearrowright (G_1, m_{G_1})$ is strongly ergodic, and so it must have spectral gap by Proposition 2.2.

By applying Theorem 4.1 to $\theta_{|G_1}$ we can find an open subgroup $G_0 < G_1$, a continuous homomorphism $\delta : G_0 \to H$, a Borel map $\phi : G_0 \to \Lambda$, and $h \in H$ such that $\delta(\Gamma \cap G_0) \subset \Lambda \cap H_0$ and $\theta(x) = \phi(x)\delta(x)h$ for almost every $x \in G_0$.

Next, we claim that $K := \ker \delta$ is finite. Assume that is not the case. Let $\lambda \in \Lambda$ be such that $C := \{x \in G_0 \mid \phi(x) = \lambda\}$ satisfies $m_G(C) > 0$. Since K is assumed to be infinite, we can find $g \in K \setminus \{e\}$ such that $m_G(g^{-1}C \cap C) > 0$. Since $\theta(gx) = \theta(x)$ for almost every $x \in g^{-1}C \cap C$, this contradicts the fact that θ is 1-1.

Thus, after replacing G_0 with a smaller open subgroup of G, we may assume that δ is 1-1. Hence, if we let $H_0 := \delta(G_0)$, then $\delta : G_0 \to H_0$ is a continuous isomorphism. Note that H_0 is an open subgroup of H. Indeed, since $m_H(\theta(G_0)) > 0$ and $\theta(G_0) \subset \bigcup_{\lambda \in \Lambda} \lambda H_0 h$, we see that $m_H(H_0) > 0$. Therefore, $[H : H_0] < \infty$ and since $H_0 < H$ is a closed subgroup (being the continuous image of a compact group), it must be open.

Now, since the map $G_0 \ni x \mapsto \delta(x)h \in H_0 h$ is 1-1 and $\delta(x)h \in \Lambda\theta(x)$ for almost every $x \in G_0$, we can find $\rho \in [\mathcal{R}(\Lambda \curvearrowright H)]$ such that

$$\rho(\theta(x)) = \delta(x)h \quad \text{for almost every } x \in G_0. \tag{4.3}$$

In particular, $\nu(\theta(G_0)) = \nu(H_0 h) = \nu(H_0) = [H : H_0]^{-1}$. Since $\theta : A \to B$ is an isomorphism of probability spaces, we have $\nu(\theta(G_0)) = \frac{\nu(B)}{\mu(A)}\mu(G_0) = \frac{\nu(B)}{\mu(A)}[G : G_0]^{-1}$. By combining the last two identities we get $\mu(A)/\nu(B) = [H : H_0]/[G : G_0]$. This finishes the proof of the first assertion.

For the "moreover" assertion, assume that $\theta(\Gamma x \cap A) = \Lambda\theta(x) \cap B$ for almost every $x \in A$. This implies that $\theta(\Gamma x \cap G_0) = \Lambda\theta(x) \cap \theta(G_0)$ for almost every $x \in G_0$. By applying ρ to this identity and using (4.3) we get $\delta(\Gamma x \cap G_0) = \Lambda\delta(x) \cap H_0$ for almost every $x \in G_0$. In particular, if $\lambda \in \Lambda \cap H_0$, then $\lambda\delta(x) \in \delta(\Gamma x \cap G_0) = \delta((\Gamma \cap G_0)x)$ for some $x \in G_0$. Thus, $\lambda \in \delta(\Gamma \cap G_0)$, and therefore $\Lambda \cap H_0 \subset \delta(\Gamma \cap G_0)$. Since the other inclusion also holds, we are done. □

4.3. *Proof of part (2) of Theorem A*

Next, we use Theorem 4.1 to prove the second part of Theorem A. Moreover, we give an explicit characterization of when there exists a non-trivial homomorphism between two given equivalence relations arising from translation profinite actions with spectral gap. More generally, we have:

Corollary 4.4. *Let Γ be a residually finite group. Let $G = \varprojlim \Gamma/\Gamma_n$ be the profinite completion of Γ with respect to a chain of finite index normal subgroups with trivial intersection. Assume that the left translation action $\Gamma \curvearrowright (G, m_G)$ has spectral gap. Let Λ be a countable subgroup of a Polish group H and consider the left translation action $\Lambda \curvearrowright H$. Then:*

(1) *$\mathcal{R}(\Gamma \curvearrowright G) \leq_B \mathcal{R}(\Lambda \curvearrowright H)$ if and only if we can find an open subgroup $G_0 < G$, a closed subgroup $H_0 < H$, and a continuous isomorphism $\delta : G_0 \to H_0$ such that $\delta(\Gamma \cap G_0) = \Lambda \cap H_0$.*
(2) *There exists a non-trivial homomorphism from $\mathcal{R}(\Gamma \curvearrowright G)$ to $\mathcal{R}(\Lambda \curvearrowright H)$ if and only if there exist an open subgroup $G_0 < G$ and a continuous homomorphism $\delta : G_0 \to H$ such that $\delta(\Gamma \cap G_0) \subset \Lambda$ and $\delta(G_0) \not\subset \Lambda$.*

Remark 4.5. (1) If any homomorphism from $\mathcal{R}(\Gamma \curvearrowright G)$ to $\mathcal{R}(\Lambda \curvearrowright H)$ is trivial, then one says that $\mathcal{R}(\Gamma \curvearrowright G)$ is $\mathcal{R}(\Lambda \curvearrowright H)$-*ergodic* (see [HK05, Appendix A]).

(2) If a continuous homomorphism $\delta : G_0 \to H$ satisfies $\delta(G_0) \subset \Lambda$, then $\delta(G_0)$ must be finite. This implies that $\delta(G_0) = \delta(\Gamma \cap G_0)$ and the restriction of δ to some open subgroup $G_1 < G_0$ is trivial.

Proof of Corollary 4.4. (1) For the "if" assertion, assume that there exist an open subgroup $G_0 < G$, a closed subgroup $H_0 < H$, and a continuous isomorphism $\delta : G_0 \to H_0$ such that $\delta(\Gamma \cap G_0) = \Lambda \cap H_0$. Then δ witnesses the fact that $\mathcal{R}((\Gamma \cap G_0) \curvearrowright G_0) \leq_B \mathcal{R}(\Lambda \curvearrowright H)$.

Since $\Gamma < G$ is dense, we have $\Gamma G_0 = G$. This implies that we can find $F \subset \Gamma$ finite such that $G = \bigsqcup_{g \in F} gG_0$. Define a Borel map $\alpha : G \to G_0$ by $\alpha(x) = g^{-1}x$, where $g \in F$ is such that $x \in gG_0$. Then $y \in \Gamma x$ if and only if $\alpha(y) \in (\Gamma \cap G_0)\alpha(x)$, showing that $\mathcal{R}(\Gamma \curvearrowright G) \leq_B \mathcal{R}((\Gamma \cap G_0) \curvearrowright G_0)$. Altogether, we deduce that $\mathcal{R}(\Gamma \curvearrowright G) \leq_B \mathcal{R}(\Lambda \curvearrowright H)$.

For the "only if" assertion, let $\theta : G \to H$ be a Borel map such that $\Gamma x = \Gamma y$ if and only if $\Lambda\theta(x) = \Lambda\theta(y)$. By applying Theorem 4.1 we can find an open subgroup $G_0 < G$, a continuous homomorphism $\delta : G_0 \to H$, a Borel map $\phi : G_0 \to \Lambda$, and $h \in H$ such that $\delta(\Gamma \cap G_0) \subset \Lambda$ and

$$\theta(x) = \phi(x)\delta(x)h \quad \text{for almost every } x \in G_0. \tag{4.4}$$

We claim that $K := \ker \delta$ is finite. If $g \in K$, then (4.4) implies that $\Lambda\theta(gx) = \Lambda\theta(x)$ for almost every $x \in G_0$. From this we deduce that $\Gamma gx = \Gamma x$ for almost every (and thus for some) $x \in G_0$, and hence $g \in \Gamma$. This shows that $K \subset \Gamma$, so K is countable. It follows that K is a countable compact group. Since the Haar measure of K has finite mass, it must be the case that K is finite.

Next, since $\ker \delta$ is finite, after replacing G_0 with a smaller open subgroup of G we may assume that δ is 1-1. Define $H_0 := \delta(G_0)$; then H_0 is a closed subgroup of H and $\delta : G_0 \to H_0$ is a continuous isomorphism.

We know that $\delta(\Gamma \cap G_0) \subset \Lambda \cap H_0$, so it remains to show the reverse inclusion. Thus, let $\lambda \in \Lambda \cap H_0$ and let $g \in G_0$ be such that $\delta(g) = \lambda$. Then (4.4) implies that

$\Lambda\theta(gx) = \Lambda\delta(gx)h = \Lambda\delta(x)h = \Lambda\theta(x)$ for almost every $x \in G_0$. We deduce that $\Gamma gx = \Gamma x$ for almost every $x \in G_0$. Hence $g \in \Gamma$, and therefore $\lambda \in \delta(\Gamma \cap G_0)$.

(2) Assume that there is a homomorphism $\theta : G \to H$ from $\mathcal{R}(\Gamma \curvearrowright G)$ to $\mathcal{R}(\Lambda \curvearrowright H)$ which is non-trivial, in the sense that $\theta(x)$ does not belong to the same Λ-orbit for almost every $x \in G$. By Theorem 4.1 we can find an open subgroup $G_0 < G$, a continuous homomorphism $\delta : G_0 \to H$, a Borel map $\phi : G_0 \to \Lambda$, and $h \in H$ such that $\delta(\Gamma \cap G_0) \subset \Lambda$ and $\theta(x) = \phi(x)\delta(x)h$ for almost every $x \in G_0$. If $\delta(G_0) \subset \Lambda$, then $\theta(x) \in \Lambda h$ for almost every $x \in G_0$. Since $\Gamma G_0 = G$, we would get $\theta(x) \in \Lambda h$ for almost every $x \in G$, contradicting the assumption that θ is non-trivial.

Conversely, assume that we have a continuous homomorphism $\delta : G_0 \to H$ such that $\delta(\Gamma \cap G_0) \subset \Lambda$ and $\delta(G_0) \not\subset \Lambda$, where $G_0 < G$ is an open subgroup. Let $\alpha : G \to G_0$ be as in the proof of (1). Then $\beta := \delta \circ \alpha : G \to H$ is a homomorphism from $\mathcal{R}(\Gamma \curvearrowright G)$ to $\mathcal{R}(\Lambda \curvearrowright H)$. If β is trivial, then we can find $h \in H$ such that $\delta(x) \in \Lambda h$ for almost every $x \in G_0$. This implies that $\delta(G_0) \subset \Lambda$, a contradiction. □

4.4. Homomorphism rigidity for compact actions

We end this section by proving analogues of Theorem 4.1 and Corollary 4.4 for translation compact actions.

Theorem 4.6. *Let Γ be a countable group together with a dense embedding $\tau : \Gamma \hookrightarrow G$ into a connected compact Polish group G. Assume that $\pi_1(G)$ is finite and the left translation action $\Gamma \curvearrowright (G, m_G)$ has spectral gap. Let Λ be a countable subgroup of a Polish group H and consider the left translation action $\Lambda \curvearrowright H$. Let $\theta : G \to H$ be a Borel map such that $\theta(\Gamma x) \subset \Lambda\theta(x)$ for almost every $x \in G$. Let $w : \Gamma \times G \to \Lambda$ be the cocycle defined by $\theta(gx) = w(g, x)\theta(x)$ for all $g \in \Gamma$ and almost every $x \in G$.*

Then we can find a finite abelian subgroup $\Delta < \Lambda$ such that if H_1 denotes the centralizer of Δ in H and $\pi : H \to \Delta\backslash H$ the quotient, then we can find a continuous homomorphism $\delta : G \to H_1/\Delta$, a Borel map $\phi : G \to \Lambda$, and $h \in \Delta\backslash H$ satisfying

- $\delta(\Gamma) \subset (\Lambda \cap H_1)/\Delta$,
- $w'(g, x) = \phi(gx)^{-1}w(g, x)\phi(x) \in \Lambda \cap H_1$ *and* $\pi(w'(g, x)) = \delta(g)$ *for all* $g \in \Gamma$ *and almost every* $x \in G$, *and*
- $\pi(\phi(x)^{-1}\theta(x)) = \delta(x)h$ *for almost every* $x \in G$.

Proof. By repeating verbatim the first part of the proof of Theorem 4.1 we can find $C \in (31/32, 1)$ and a neighborhood V of e in G such that

$$m_G(\{x \in X \mid w(g, xt) = w(g, x)\}) \geq C \quad \text{for all } g \in \Gamma \text{ and } t \in V.$$

By applying Theorem 3.2 we find a subgroup $\Lambda_0 < \Lambda$, a finite subgroup $\Delta < Z(\Lambda_0)$, a Borel map $\phi : G \to \Lambda$, and a homomorphism $\delta : \Gamma \to \Lambda_0/\Delta$ such that $w'(g, x) = \phi(gx)^{-1}w(g, x)\phi(x) \in \Lambda_0$ and $p(w'(g, x)) = \delta(g)$ for all $g \in \Gamma$ and almost every $x \in G$. Here $p : \Lambda_0 \to \Lambda_0/\Delta$ is the quotient homomorphism. Let $\pi : H \to \Delta\backslash H$ denote the quotient. Since $\Delta < \Lambda_0$ is central, we can identify $\Lambda_0/\Delta = \Delta\backslash\Lambda_0$. Under

this identification we have $p = \pi_{|\Lambda_0}$, hence $\pi(w'(g,x)) = \delta(g)$ for all $g \in \Gamma$ and almost every $x \in G$.

Define $\tilde{\theta} : G \to \Delta\backslash H$ by letting $\tilde{\theta}(x) = \pi(\phi(x)^{-1}\theta(x))$. Let H_1 be the centralizer of Δ in H. Notice that H_1/Δ, and hence Λ_0/Δ, acts on $\Delta\backslash H$ by left multiplication. Then

$$\tilde{\theta}(gx) = \delta(g)\tilde{\theta}(x) \quad \text{for all } g \in \Gamma \text{ and almost every } x \in G. \tag{4.5}$$

Now, define the quotient topological space $\Sigma := (H_1/\Delta)\backslash(\Delta\backslash H)$. Since Σ is homeomorphic to $H_1\backslash H$ and $H_1 < H$ is a closed subgroup, [BK96, Proposition 1.2.3] implies that Σ is a Polish space. If $q : \Delta\backslash H \to \Sigma$ denotes the quotient map, then $q(\tilde{\theta}(gx)) = q(\tilde{\theta}(x))$ for all $g \in \Gamma$ and almost every $x \in G$. Since $\Gamma < G$ is dense, the map $G \ni x \mapsto q(\tilde{\theta}(x)) \in \Sigma$ is constant. Thus, we can find $h_0 \in \Delta\backslash H$ and a Borel map $\alpha : G \to H_1/\Delta$ such that $\tilde{\theta}(x) = \alpha(x)h_0$ for almost every $x \in G$.

In combination with (4.5), this implies that $\alpha(gx)h_0 = \delta(g)\alpha(x)h_0$ for all $g \in \Gamma$ and almost every $x \in G$. Since the left multiplication action $H_1/\Delta \curvearrowright \Delta\backslash H$ is free, we get

$$\alpha(gx) = \delta(g)\alpha(x) \quad \text{for all } g \in \Gamma \text{ and almost every } x \in G. \tag{4.6}$$

By Lemma 2.8, $\delta : \Gamma \to \Lambda_0/\Delta < H_1/\Delta$ extends to a continuous homomorphism $\delta : G \to H_1/\Delta$ and we can find $h_1 \in H_1/\Delta$ such that $\alpha(x) = \delta(x)h_1$ for almost every $x \in G$. Thus, if we let $h = h_1h_0 \in \Delta\backslash H$, then $\tilde{\theta}(x) = \delta(x)h$ for almost every $x \in G$. This finishes the proof of the theorem. □

As a consequence of Theorem 4.6 we obtain the following analogue of Corollary 4.4 for translation compact actions.

Corollary 4.7. *Assume the setting from Theorem 4.6. Then:*

(1) $\mathcal{R}(\Gamma \curvearrowright G) \leq_B \mathcal{R}(\Lambda \curvearrowright H)$ *if and only if there exist a finite subgroup* $\Sigma < \Gamma$ *such that* $\Sigma < Z(G)$*, a finite subgroup* $\Delta < \Lambda$ *and a closed subgroup* $H_0 < H$ *such that* $\Delta < Z(H_0)$*, and a continuous isomorphism* $\delta : G/\Sigma \to H_0/\Delta$ *such that* $\delta(\Gamma/\Sigma) = (\Lambda \cap H_0)/\Delta$.

(2) *There exists a non-trivial homomorphism from* $\mathcal{R}(\Gamma \curvearrowright G)$ *to* $\mathcal{R}(\Lambda \curvearrowright H)$ *if and only there exist a finite subgroup* $\Delta < \Lambda$ *and a closed subgroup* $H_0 < H$ *such that* $\Delta < Z(H_0)$*, and a non-trivial continuous homomorphism* $\delta : G \to H_0/\Delta$ *such that* $\delta(\Gamma) \subset (\Lambda \cap H_0)/\Delta$.

Proof. Since the "if" assertions of both (1) and (2) are immediate, let us prove the "only if" assertions. To this end, let $\theta : G \to H$ be a Borel map such that $\theta(x) \in \Lambda\theta(y)$ whenever $x \in \Gamma y$. By Theorem 4.6, we can find a finite subgroup $\Delta < \Gamma$ such that if H_1 is the centralizer of Δ in H and $\pi : H \to \Delta\backslash H$ the quotient map, then there exists a continuous homomorphism $\delta : G \to H_1/\Delta$, a Borel map $\phi : G \to \Lambda$, and $h \in \Delta\backslash H$ satisfying

$$\delta(\Gamma) \subset (\Lambda \cap H_1)/H_1, \qquad \pi(\phi(x)^{-1}\theta(x)) = \delta(x)h \quad \text{for } m_G\text{-almost every } x \in G.$$

Define $\bar{\theta} : G \to H$ by letting $\bar{\theta}(x) = \phi(x)^{-1}\theta(x)$, and $\tilde{\theta} : G \to \Delta\backslash H$ by $\tilde{\theta} = \pi \circ \bar{\theta}$.

Let H_0 be the unique subgroup of H_1 which contains Δ and satisfies $\delta(G) = H_0/\Delta$. Then $\delta(\Gamma) \subset (\Lambda \cap H_0)/\Delta$.

In the rest of the proof we derive the "only if" assertions by treating cases (1) and (2) separately.

(1) Assume that θ is a reduction, that is, $\theta(x) \in \Lambda\theta(y)$ if and only if $x \in \Gamma y$. By arguing as in the proof of Corollary 4.4(1), it follows that $\Sigma := \ker\delta$ is contained in Γ and is finite. Since G is connected, we find that $\Sigma \subset Z(G)$. We still denote by δ the resulting homomorphism $\delta : G/\Sigma \to H_0/\Delta$.

Then $\delta(\Gamma/\Sigma) \subset (\Lambda \cap H_0)/\Delta$. To show the reverse inclusion, let $\lambda \in \Lambda \cap H_0$. Then we can find $g \in G$ such that $\delta(g\Sigma) = \lambda\Delta$. Further, $\tilde{\theta}(gx) = \delta(g\Sigma)\tilde{\theta}(x) = (\lambda\Delta)\tilde{\theta}(x)$ for almost every $x \in G$. This implies that $\tilde{\theta}(gx) \in \Lambda\tilde{\theta}(x)$, and hence $\theta(gx) \in \Lambda\theta(x)$, for almost every $x \in G$. Thus, $gx \in \Gamma x$ for almost every $x \in G$. This implies that $g \in \Gamma$, and so $g\Sigma \in \Gamma/\Sigma$, as claimed. This finishes the proof of (1).

(2) Assume that θ is not trivial, i.e. $\theta(x)$ is not contained is a single Λ-orbit for almost every $x \in G$. We claim that δ is non-trivial. Suppose for contradiction that $\delta(g) = e$ for all $g \in G$. Then for every $g \in G$ we have $\tilde{\theta}(gx) = \tilde{\theta}(x)$ for almost every $x \in G$. By Fubini's theorem we can find $x_0 \in G$ such that $\tilde{\theta}(gx_0) = \tilde{\theta}(x_0)$ for almost every $g \in G$. This clearly implies that $\theta(y) \in \Lambda\theta(x_0)$ for almost every $y \in G$, leading to a contradiction. □

5. Homomorphism rigidity for general compact actions

In Section 4, we proved several rigidity results for translation compact actions with spectral gap. The purpose of this and the next section is to establish analogous rigidity results for more general compact actions. More precisely, in this section we will use Theorems 4.1 and 4.6 to prove the following result which generalizes both of these theorems.

Theorem 5.1. *Let Γ be a countable group together with a dense embedding $\tau : \Gamma \hookrightarrow G$ into a compact Polish group G. Assume that the left translation action $\Gamma \curvearrowright (G, m_G)$ has spectral gap. Let Λ be a countable group together with an embedding $\rho : \Lambda \hookrightarrow H$ into a locally compact Polish group H. Let $H \curvearrowright Y$ be a continuous action on a Polish space Y. Let $\theta : G \to Y$ be a Borel map such that $\theta(\Gamma x) \subset \Lambda\theta(x)$ for almost every $x \in G$. Assume that there exist $k \geq 1$ and an H-invariant G_δ subset $\Omega \subset Y^k$ such that*

- *the action $H \curvearrowright \Omega$ is smooth and free, and*
- *$(\theta(x_i))_{i=1}^k \in \Omega$ for $m_G^{\otimes k}$-almost every $x = (x_i)_{i=1}^k \in G^k$.*

Then:

(1) *Assume that $G = \varprojlim \Gamma/\Gamma_n$ is a profinite completion of Γ. Then we can find an open subgroup $G_0 < G$, a continuous homomorphism $\delta : G_0 \to H$, a Borel map $\phi : G_0 \to \Lambda$, and $y \in Y$ such that $\delta(\Gamma \cap G_0) \subset \Lambda$ and $\theta(x) = \phi(x)\delta(x)y$ for almost every $x \in G_0$.*

(2) *Assume that G is connected and $\pi_1(G)$ is finite. Then we can find a finite abelian subgroup $\Delta < \Lambda$ such that if H_1 denotes the centralizer of Δ in H and $\pi : Y \to \Delta\backslash Y$ the quotient, then we can find a continuous homomorphism $\delta : G \to H_1/\Delta$,*

a Borel map $\phi : G \to \Lambda$, and $y \in \Delta\backslash Y$ satisfying $\delta(\Gamma) \subset (\Lambda \cap H_1)/\Delta$ and $\pi(\phi(x)^{-1}\theta(x)) = \delta(x)y$ for almost every $x \in G$. (Here, we are using the action of H_1/Δ on $\Delta\backslash Y$ by left multiplication).

Since the action $H \curvearrowright \Omega = H$ is free and smooth, Theorem 5.1 generalizes Theorems 4.1 and 4.6. Before proceeding to the proof, let us mention that part (1) of Theorem 5.1 will be employed in Section 8 to prove Corollary D.

Proof of Theorem 5.1. Let $w : \Gamma \times G \to \Lambda$ be a Borel map such that $\theta(gx) = w(g, x)\theta(x)$ for all $g \in \Gamma$ and almost every $x \in G$. Since the action $\Lambda \curvearrowright Y$ is not assumed to be free, we cannot conclude that w is a cocycle at this point. However, as a consequence of the claim below, it will follow that w is indeed a cocycle.

Since $\Omega \subset Y^k$ is a G_δ set and Y^k is a Polish space, Ω is a Polish space (see [Ke95, Theorem 3.11]). Since the action $H \curvearrowright \Omega$ is continuous, free and smooth, Lemma 2.7 implies that there exists a Borel map $p : \Omega \to H$ such that $p(hx) = hp(x)$ for all $h \in H$ and $x \in \Omega$.

The rest of the proof relies on the following:

Claim. *There exists a sequence of Borel maps $\phi_n : G \to H$ such that for all $g \in \Gamma$ we have $m_G(\{x \in G \mid w(g, x) = \phi_n(gx)\phi_n(x)^{-1}\}) \to 1$ as $n \to \infty$.*

Proof of the claim. It suffices to show that for every $\varepsilon > 0$ and any finite subset $F \subset \Gamma$, we can find a Borel map $\phi : G \to H$ such that $m_G(\{x \in G \mid w(g, x) = \phi(gx)\phi(x)^{-1}\}) \geq 1 - \varepsilon$ for all $g \in F$.

Since Λ is countable, we can find a neighborhood V of the identity e in G such that

$$m_G(\{x \in G \mid w(g, xt) = w(g, x)\}) \geq 1 - \frac{\varepsilon}{k|F|} \quad \text{for all } g \in F \text{ and } t \in V. \tag{5.1}$$

Recall that $(\theta(x_i))_{i=1}^k \in \Omega$ for almost every $(x_i)_{i=1}^k \in G^k$. It follows that for all $x \in G$ we have $(\theta(xt_i))_{i=1}^k \in \Omega$ for almost every $(t_i)_{i=1}^k \in G^k$. Since $m_G^{\otimes k}(V^k) = m_G(V)^k > 0$, using Fubini's theorem we deduce that there exist $t_1, \ldots, t_k \in V$ such that $(\theta(xt_i))_{i=1}^k \in \Omega$ for almost every $x \in G$.

Further, we define $\psi : G \to Y^k$ by letting $\psi(x) := (\theta(xt_1), \ldots, \theta(xt_k))$. Then $\psi(x) \in \Omega$ for almost every $x \in G$. We denote by C the set of all $x \in G$ such that

- $\psi(x) \in \Omega$ and $\psi(gx) \in \Omega$ for all $g \in F$,
- $\theta(gxt_i) = w(g, xt_i)\theta(xt_i)$ for all $g \in F$ and $1 \leq i \leq k$, and
- $w(g, x) = w(g, xt_i)$ for all $g \in F$ and $1 \leq i \leq k$.

Since $\psi(x) \in \Omega$ and $\theta(gx) = w(g, x)\theta(x)$ for all $g \in \Gamma$ and almost every $x \in G$, (5.1) implies that $m_G(C) \geq 1 - \varepsilon$.

Finally, we define $\phi : G \to H$ by letting

$$\phi(x) = \begin{cases} \pi(\psi(x)) & \text{if } x \in \psi^{-1}(\Omega), \\ e & \text{if } x \notin \psi^{-1}(\Omega). \end{cases}$$

Then for every $x \in C$ and $g \in F$ we have

$$\begin{aligned}\phi(gx) &= \pi(\psi(gx)) = \pi(\theta(gxt_1), \dots, \theta(gxt_k)) \\ &= \pi(w(g, xt_1)\theta(xt_1), \dots, w(g, xt_k)\theta(xt_k)) \\ &= \pi(w(g, x)\theta(xt_1), \dots, w(g, x)\theta(xt_k)) = w(g, x)\pi(\theta(xt_1), \dots, \theta(xt_k)) \\ &= w(g, x)\phi(x).\end{aligned}$$

This finishes the proof of the claim. □

Now, since the action $\Gamma \curvearrowright G$ is strongly ergodic, the claim and Lemma I imply that there exists a Borel map $\psi : G \to H$ such that $w(g, x) = \psi(gx)\psi(x)^{-1}$ for all $g \in \Gamma$ and almost every $x \in G$. We are therefore in a position to apply Theorems 4.1 and 4.6, in cases (1) and (2), respectively.

Assume that we are in case (1). By Theorem 4.1 we can find an open subgroup $G_0 < G$, a continuous homomorphism $\delta : G_0 \to H$, and a Borel map $\phi : G_0 \to \Lambda$ such that $w(g, x) = \phi(gx)\delta(g)\phi(x)^{-1}$ for all $g \in \Gamma \cap G_0$ and m_G-almost every $x \in G_0$.

Thus, if we define $\tilde{\theta} : G_0 \to Y$ by $\tilde{\theta}(x) = \phi(x)^{-1}\theta(x)$, then $\tilde{\theta}(gx) = \delta(g)\tilde{\theta}(x)$ for all $g \in \Gamma \cap G_0$ and almost every $x \in G_0$. Since δ is continuous and $\Gamma \cap G_0 < G_0$ is dense, for every $g \in G_0$ we have $\tilde{\theta}(gx) = \delta(g)\tilde{\theta}(x)$ for almost every $x \in G_0$. By Fubini's theorem, we can find $x_0 \in G_0$ such that $\tilde{\theta}(gx_0) = \delta(g)\tilde{\theta}(x_0)$ for almost every $g \in G_0$. Denoting $y = \delta(x_0^{-1})\tilde{\theta}(x_0) \in Y$, we see that $\tilde{\theta}(g) = \delta(g)y$ for almost every $g \in G_0$. This clearly implies the conclusion.

Finalizing the proof in case (2) is similar; we leave the details to the reader. □

6. Orbit equivalence rigidity for compact actions

In this section, we study orbit equivalences between general compact actions with spectral gap. More precisely, assuming that G and H are connected, we provide conditions which ensure that any orbit equivalence between the actions $\Gamma \curvearrowright G/K$ and $\Lambda \curvearrowright H/L$ comes from a conjugacy between them. This result will be used in Section 10 to give examples of ergodic treeable equivalence relations without outer automorphisms.

Theorem 6.1. *Let Γ and Λ be countable groups together with dense embeddings $\tau : \Gamma \hookrightarrow G$ and $\sigma : \Lambda \hookrightarrow H$ into compact Polish groups G and H. Let $K < G$ and $L < H$ be closed subgroups. Assume that:*

(1) *The actions $\Gamma \curvearrowright (G/K, m_{G/K})$ and $\Lambda \curvearrowright (H/L, m_{H/L})$ are free.*
(2) *The left translation action $\Gamma \curvearrowright (G, m_G)$ has spectral gap.*
(3) *G and H are connected, $\pi_1(G)$ is finite, $\bigcap_{g\in G} gKg^{-1} = \{e\}$, and $\bigcap_{h\in H} hLh^{-1} = \{e\}$.*
(4) *Γ has no non-trivial finite normal subgroups, and Λ has infinite conjugacy classes (icc).*
(5) *For any $h \in \Lambda \setminus \{e\}$, we have*

$$(m_H \times m_H)(\{(x, y) \in H \times H \mid h \in xLx^{-1}yLy^{-1}\}) = 0.$$

Let $B \subset H/L$ be a Borel set with $m_{H/L}(B) > 0$. Endow B with the probability measure obtained by restricting and rescaling $m_{H/L}$. Let $\theta : G/K \to B$ be an isomorphism of probability spaces such that $\theta(\Gamma x) = \Lambda\theta(x) \cap B$ for $m_{G/K}$-almost every $x \in G/K$.

Then $B = H/L$ almost everywhere, and we can find a continuous isomorphism $\delta : G \to H$, $y \in H$, and $\alpha \in [\mathcal{R}(\Lambda \curvearrowright H/L)]$ such that

- *$\delta(\Gamma) = \Lambda$ and $\delta(K) = yLy^{-1}$, and*
- *$\tilde{\theta} := \alpha \circ \theta : G/K \to H/L$ is given by $\tilde{\theta}(gK) = \delta(g)yL$ for m_G-almost every $g \in G$.*

Remark 6.2. The technical condition (5) is imposed so that we can adapt the strategy of the proof of Theorem 4.1 to this new context. Condition (5) is trivially satisfied if $L = \{e\}$. More interestingly, it also holds if $L = \mathrm{SO}(n) < H = \mathrm{SO}(n+1)$ for $n \geq 2$ (see Claim 2 in the proof of Theorem 10.2).

Proof of Theorem 6.1. Let $\pi : G \to G/K$ denote the quotient map. Define the cocycle $w : \Gamma \times G/K \to \Lambda$ by $\theta(gx) = w(g,x)\theta(x)$. Denote $W = w \circ (\mathrm{id}_\Gamma \times \pi) : \Gamma \times G \to \Lambda$ and $\Theta = \theta \circ \pi : G \to H/L$. The proof of Theorem 6.1 relies on four claims.

Claim 1. *There exist a subgroup $\Lambda_0 < \Lambda$, a finite central subgroup $\Lambda_1 < Z(\Lambda_0)$, a Borel map $\phi : G \to \Lambda$ and a homomorphism $\delta : \Gamma \to \Lambda_0/\Lambda_1$ such that $W'(g,x) := \phi(gx)W(g,x)\phi(x)^{-1} \in \Lambda_0$ and $p(W'(g,x)) = \delta(g)$ for all $g \in \Gamma$ and almost every $x \in G$, where $p : \Lambda_0 \to \Lambda_0/\Lambda_1$ denotes the quotient homomorphism.*

Proof of Claim 1. We will apply Theorem 3.2. To this end, note that $W : \Gamma \times G \to \Lambda$ is a cocycle and we have

$$\Theta(gx) = W(g,x)\Theta(x) \quad \text{for all } g \in \Gamma \text{ and almost every } x \in G. \tag{6.1}$$

We start by adapting the beginning of the proof of Theorem 4.1. Let $\varepsilon \in (0, 1/64)$. Since the action $\Gamma \curvearrowright (G, m_G)$ has spectral gap, Lemma 2.5 provides $\delta > 0$ and $F \subset \Gamma$ finite such that whenever Y is a standard Borel space and $\rho : G \to Y$ is a Borel map satisfying $m_G(\{x \in G \mid \rho(gx) = \rho(x)\}) \geq 1 - \delta$ for all $g \in F$, we can find $y \in Y$ such that $m_G(\{x \in G \mid \rho(x) = y\}) \geq 1 - \varepsilon$. Thus, $\mu(\{x \in X \mid \rho(gx) = \rho(x)\}) \geq 1 - 2\varepsilon$ for all $g \in \Gamma$.

Since Λ is countable, we can find a neighborhood V of $e \in G$ such that

$$m_G(\{x \in G \mid W(g,xt) = W(g,x)\}) \geq 1 - \delta \quad \text{for all } g \in F \text{ and every } t \in V. \tag{6.2}$$

Let Y denote the double coset space $L\backslash H/L$. Fix $t \in V$ and define $\rho_t : G \to Y$ by letting $\rho_t(x) = L\Theta(x)^{-1}\Theta(xt)L$. By (6.1), for almost every $x \in G$ we have

$$\rho_t(gx) = L\Theta(gx)^{-1}\Theta(gxt)L = L\Theta(x)^{-1}W(g,x)^{-1}W(g,xt)\Theta(xt)L. \tag{6.3}$$

Combining (6.2) and (6.3) implies that $m_G(\{x \in G \mid \rho_t(gx) = \rho_t(x)\}) \geq 1 - \delta$ for all $g \in F$. Since Y is a standard Borel space, by using the above consequence of the spectral gap property we get

$$m_G(\{x \in G \mid \rho_t(gx) = \rho_t(x)\}) \geq 1 - 2\varepsilon \quad \text{for all } g \in \Gamma \text{ and every } t \in V. \tag{6.4}$$

Now, let Z_t be the set of $x \in G$ for which there is $h \in \Lambda \setminus \{e\}$ such that $h \in \Theta(x)L\Theta(x)^{-1}\Theta(xt)L\Theta(xt)^{-1}$. We claim that $m_G(Z_t) = 0$ for almost every $t \in G$. By Fubini's theorem, it suffices to show that the set Z of $(x, y) \in G \times G$ for which there is $h \in \Lambda \setminus \{e\}$ with $h \in \Theta(x)L\Theta(x)^{-1}\Theta(y)L\Theta(y)^{-1}$ has measure zero. To see this, note that assumption (5) implies that $(m_{H/L} \times m_{H/L})((\Theta \times \Theta)(Z)) = 0$. Since θ is measure preserving, it follows that $(m_G \times m_G)(Z) = 0$.

Next, by (6.3) we have $\{x \in G \mid \rho_t(gx) = \rho_t(x) \text{ and } W(g, x) \neq W(g, xt)\} \subset Z_t$. Since $m_G(Z_t) = 0$, by (6.4) we deduce that $m_G(\{x \in G \mid W(g, x) = W(g, xt)\}) \geq 1 - 2\varepsilon$ for all $g \in \Gamma$ and every $t \in V$. Since $1 - 2\varepsilon > 31/32$, G is connected and $\pi_1(G)$ is finite, part (2) of Theorem 3.2 gives the claim. □

Claim 2. *δ is injective.*

Proof of Claim 2. Since Γ has no non-trivial finite normal subgroups, in order to show that δ is injective it suffices to argue that $\ker\delta$ is finite. Assume that it is infinite and let $F \subset \Lambda$ be a finite set such that $m_G(\{x \in G \mid \phi(x) \in F\}) > 2/3$. Claim 1 implies that

$$m_{G/K}(\{x \in G/K \mid w(g, x) \in F^{-1}\Lambda_1 F\}) > 1/3 \quad \text{for all } g \in \ker\delta.$$

Since $\ker\delta$ is assumed to be infinite, we can find $h \in F^{-1}\Lambda_1 F$ and a sequence $\{g_n\}_n$ of distinct elements of $\ker\delta$ such that if $X_n = \{x \in G/K \mid w(g_n, x) = h\}$, then $m_{G/K}(X_n) > 1/(3|F^{-1}\Lambda_1 F|)$ for all n. Since the sets $\{X_n\}_n$ are mutually disjoint, this provides a contradiction. □

Claim 3. *$\Lambda_1 = \{e\}$, and $\delta(\Gamma) < \Lambda$ has finite index.*

Proof of Claim 3. Let $\Sigma := p^{-1}(\delta(\Gamma)) < \Lambda_0$. We claim that $\Sigma < \Lambda$ has finite index. Assume the index is infinite. We denote $\mathcal{R} = \mathcal{R}(\Lambda \curvearrowright H/L)$ and $\mathcal{S} = \mathcal{R}(\Sigma \curvearrowright H/L)$. We define $\varphi_{\mathcal{S}} : [[\mathcal{R}]] \to [0, 1]$ by letting

$$\varphi_{\mathcal{S}}(\beta) = m_{H/L}(\{x \in H/L \mid \beta(x) \text{ is defined and } \beta(x) \in [x]_{\mathcal{S}}\}) \quad \text{for every } \beta \in [[\mathcal{R}]].$$

Since $m_G(\{x \in G \mid \phi(x) \in F\}) > 2/3$, Claim 1 implies that

$$m_{G/K}(\{x \in G/K \mid \theta(gx) \in F^{-1}\Sigma F\theta(x)\}) > 1/3 \quad \text{for all } g \in \Gamma. \tag{6.5}$$

Now, for every $g \in \Gamma$, we define $\alpha_g = \theta g\theta^{-1} : B \to B$. Then $\alpha_g \in [[\mathcal{R}]]$, and inequality (6.5) gives

$$\sum_{h,k \in F} \varphi_{\mathcal{S}}(h\alpha_g k^{-1}) > 1/3 \quad \text{for all } g \in \Gamma. \tag{6.6}$$

Note that the equivalence relation associated to the action $(\alpha_g)_{g\in\Gamma}$ of Γ on B is equal to $\mathcal{R}_{|B}$. By using (6.6) and a straightforward modification of the proof of [IKT08, Theorem 2.5], it follows that we can find a Borel set $A \subset H/L$ of positive measure and $\kappa \geq 1$ such that every $\mathcal{R}_{|A}$-class contains at most κ $\mathcal{S}_{|A}$-classes. In other words, the inclusion $\mathcal{S}_{|A} \subset \mathcal{R}_{|A}$ has index at most κ.

We will show that this contradicts our assumption that $\Sigma < \Lambda$ has infinite index. Let us first show that if $g_1, \ldots, g_{\kappa+1} \in \Lambda$ and $g_i\Sigma \neq g_j\Sigma$ for all $i \neq j$, then

$m_{H/L}(\bigcap_{i=1}^{\kappa+1} g_i A) = 0$. Indeed, if $x \in \bigcap_{i=1}^{\kappa+1} g_i A$, let $y = g_1^{-1}x$. Then $y \in A$ and $(g_i^{-1}g_1)y \in [y]_{\mathcal{R}} \cap A = [y]_{\mathcal{R}_{|A}}$ for all $2 \leq i \leq \kappa+1$. Since $\Sigma \neq \Sigma(g_i^{-1}g_1) \neq \Sigma(g_j^{-1}g_1)$ for all $i, j \in \{2, \ldots, \kappa+1\}$ with $i \neq j$, we deduce that the $\mathcal{S}_{|A}$-classes of $y, (g_2^{-1}g_1)y, \ldots, (g_{\kappa+1}^{-1}g_1)y$ are disjoint. This would imply that $[y]_{\mathcal{R}_{|A}}$ contains more than κ $\mathcal{S}_{|A}$-classes, which is a contradiction.

Next, let p be the smallest natural number such that whenever $g_1, \ldots, g_p \in \Lambda$ are such that $g_i\Sigma \neq g_j\Sigma$ for all $1 \leq i < j \leq p$, then $m_{H/L}(\bigcap_{i=1}^{p} g_i A) = 0$. Then clearly $1 < p \leq \kappa+1$. Let q be a natural number such that $q < p$ and $2q \geq p$. Since $q < p$, we can find $g_1, \ldots, g_q \in \Lambda$ such that $g_i\Sigma \neq g_j\Sigma$ for all $i \neq j$, and $\tilde{A} = \bigcap_{i=1}^{q} g_i A$ satisfies $m_{H/L}(\tilde{A}) > 0$.

Since $\Sigma < \Lambda$ has infinite index, for any finite set $S \subset \Lambda$ we have $S\Sigma S \neq \Lambda$. Using this fact, we can find a sequence $\{h_a\}_{a=1}^{\infty}$ of elements of Λ such that $h_a(\bigcup_{i=1}^{q} g_i\Sigma) \cap h_b(\bigcup_{i=1}^{q} g_i\Sigma) = \emptyset$ for all $a \neq b$. Since $h_a\tilde{A} \cap h_b\tilde{A} = (\bigcap_{i=1}^{q} h_a g_i A) \cap (\bigcap_{i=1}^{q} h_b g_i A)$ and $2q \geq p$, we conclude that $m_{H/L}(h_a\tilde{A} \cap h_b\tilde{A}) = 0$ for all $a \neq b$. Since $m_{H/L}(\tilde{A}) > 0$, this gives the desired contradiction.

We have thus proved that $\Sigma < \Lambda$ has finite index. Since $\Sigma < \Lambda_0$ and $\Lambda_1 < Z(\Lambda_0)$, we see that Σ commutes with Λ_1. Since $\Sigma < \Lambda$ has finite index, the set $\{ghg^{-1} \mid g \in \Lambda\}$ is finite for every $h \in \Lambda_1$. As Λ is assumed icc, we get $\Lambda_1 = \{e\}$. Hence $\delta(\Gamma) = \Sigma$ has finite index in Λ. □

Claim 4. *$\phi : G \to \Lambda$ factors through $\pi : G \to G/K$.*

Proof of Claim 4. Let $k \in K$. Claim 1 implies that

$$\phi(gxk)\delta(g)\phi(xk)^{-1} = w(g, \pi(x)) = \phi(gx)\delta(g)\phi(x)^{-1}$$

for all $g \in \Gamma$ and almost every $x \in G$. Hence, if we denote $\lambda(x) = \phi(x)^{-1}\phi(xk)$, then $\lambda(gx) = \delta(g)\lambda(x)\delta(g)^{-1}$ for all $g \in \Gamma$ and almost every $x \in G$.

For $h \in \Lambda$, let $A_h = \{x \in G \mid \lambda(x) = h\}$. Then $gA_h = A_{\delta(g)h\delta(g)^{-1}}$ and $m_{H/L}(A_h) = m_{H/L}(A_{\delta(g)h\delta(g)^{-1}})$ for all $g \in \Gamma$. Since $\delta(\Gamma) < \Lambda$ has finite index by Claim 2 and Λ is icc, $\{\delta(g)h\delta(g)^{-1} \mid g \in \Gamma\}$ is infinite for any $h \in \Lambda \setminus \{e\}$. Since the sets $\{A_h\}_{h\in\Lambda}$ are mutually disjoint, we conclude that $m_{H/L}(A_h) = 0$ for all $h \in \Lambda \setminus \{e\}$. This implies that $\phi(x) = \phi(xk)$ for almost every $x \in G$. Therefore, ϕ factors through $\pi : G \to G/K$, as claimed. □

We are now ready to finish the proof of Theorem 6.1. By Claim 4, we can define $\tilde{\theta} : G/K \to H/L$ by letting $\tilde{\theta}(x) = \phi(x)^{-1}\theta(x)$. Also, let $\tilde{B} = \tilde{\theta}(G/K)$. Claims 1 and 3 show that $\tilde{\theta}(gx) = \delta(g)\tilde{\theta}(x)$ for all $g \in \Gamma$ and almost every $x \in G/K$.

Next, let $x \in G/K$ and $y \in \Gamma x$ be such that $\tilde{\theta}(x) = \tilde{\theta}(y)$. If $y = gx$ for some $g \in \Gamma$, then $\tilde{\theta}(y) = \delta(g)\tilde{\theta}(x) = \delta(g)\tilde{\theta}(y)$. By freeness, we infer that $\delta(g) = e$. Since δ is injective, we deduce that $g = e$ and hence $x = y$. This argument implies that there exists $\alpha \in [\mathcal{R}(\Lambda \curvearrowright H/L)]$ such that $\tilde{\theta} = \alpha \circ \theta$. In particular, $\tilde{\theta}(\Gamma x) = \Lambda\tilde{\theta}(x) \cap \tilde{B}$ for $m_{G/K}$-almost every $x \in G/K$.

Now, since $\delta(\Gamma) < \Lambda$ has finite index and $\Lambda < H$ is dense, it follows that $H_0 = \overline{\delta(\Gamma)} < H$ is a finite index closed subgroup. Since H is connected, we derive that

$H_0 = H$, hence $\delta(\Gamma) < H$ is dense. Therefore, the left translation actions of $\delta(\Gamma)$ on H and on H/L are ergodic. Since $\tilde{B} \subset H$ is a $\delta(\Gamma)$-invariant set of positive measure, we find that $\tilde{B} = K/L$, and hence $B = K/L$ almost everywhere.

By combining the last two paragraphs, we deduce that $\tilde{\theta}(\Gamma x) = \Lambda\tilde{\theta}(x)$ for almost every $x \in G/K$. Since $\tilde{\theta}(\Gamma x) = \delta(\Gamma)\tilde{\theta}(x)$ for almost every $x \in G/K$, we conclude that $\delta(\Gamma) = \Lambda$.

Altogether, $\tilde{\theta} : (G/K, m_{G/K}) \to (H/L, m_{H/L})$ is an isomorphism of probability spaces such that $\tilde{\theta}\Gamma\tilde{\theta}^{-1} = \Lambda$. Since $\bigcap_{g\in G} gKg^{-1} = \{e\}$ and $\bigcap_{h\in H} hLh^{-1} = \{e\}$, the closure of Γ in $\mathrm{Aut}(G/K, m_{G/K})$ is equal to G, and the closure of Λ in $\mathrm{Aut}(H/L, m_{H/L})$ is H. Hence $\tilde{\theta}G\tilde{\theta}^{-1} = H$. Thus, δ extends to a continuous isomorphism $\delta : G \to H$ such that for all $g \in G$ we have $\tilde{\theta}(gx) = \delta(g)\tilde{\theta}(x)$ for almost every $x \in G/K$.

By Fubini's theorem, there is $x_0K \in G/K$ such that $\tilde{\theta}(gx_0K) = \delta(g)\tilde{\theta}(x_0K)$ for almost every $g \in G$. Let $y \in H$ be such that $\delta(x_0)^{-1}\tilde{\theta}(x_0K) = yL$. Then $\tilde{\theta}(gK) = \delta(g)yL$ for almost every $g \in G$. By using this identity, it is easy to see that $\delta(K) = yLy^{-1}$. □

Specializing Theorem 6.1 to the case when $K = L = \{e\}$, we obtain the following analogue of the first part of Theorem A for certain translation compact actions.

Corollary 6.3. *Let Γ and Λ be countable icc groups together with dense embeddings $\tau : \Gamma \hookrightarrow G$ and $\sigma : \Lambda \hookrightarrow H$ into connected compact metrizable groups G and H. Assume that $\pi_1(G)$ and $\pi_1(H)$ are finite, and the left translation action $\Gamma \curvearrowright (G, m_G)$ has spectral gap. Then the actions $\Gamma \curvearrowright (G, m_G)$ and $\Lambda \curvearrowright (H, m_H)$ are stably orbit equivalent if and only if there exists a continuous isomorphism $\delta : G \to H$ such that $\delta(\Gamma) = \Lambda$.*

Proof. Assume that $\Gamma \curvearrowright (G, m_G)$ and $\Lambda \curvearrowright (H, m_H)$ are stably orbit equivalent. Since $\Gamma \curvearrowright (G, m_G)$ is strongly ergodic, $\Lambda \curvearrowright (H, m_H)$ is also strongly ergodic, and hence has spectral gap by Proposition 2.2. The conclusion then follows from Theorem 6.1. □

7. Proofs of Corollaries B and C

The aim of this section is to prove Corollaries B and C. We start by introducing some notation.

Notation 7.1. Let S be a set of primes.

- We define the profinite groups
$$G_S = \prod_{p\in S} \mathrm{SL}_2(\mathbb{F}_p), \qquad H_S = \prod_{p\in S} \mathrm{GL}_2(\mathbb{F}_p), \qquad K_S = \prod_{p\in S} \mathrm{SL}_2(\mathbb{Z}_p).$$
- For every p, we denote by $\pi_p : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{F}_p)$ reduction modulo p and by $\tau_p : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}_p)$ the natural embedding.
- Whenever S is infinite, we view $\mathrm{SL}_2(\mathbb{Z})$ as a subgroup of G_S via the diagonal embedding $\pi_S = (\pi_p)_{p\in S}$.
- We view $\mathrm{SL}_2(\mathbb{Z})$ as a subgroup of K_S via the diagonal embedding $\tau_S = (\tau_p)_{p\in S}$.
- For a subgroup $\Gamma < \mathrm{SL}_2(\mathbb{Z})$, we denote by $G_{\Gamma,S}$ and $K_{\Gamma,S}$ the closures of Γ in G_S and K_S, respectively.

Next, let us record the following fact that we will use several times in the next three sections:

Fact 7.2. If $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ is a non-amenable subgroup, then

- $G_{\Gamma,S} < G_S$ and $K_{\Gamma,S} < K_S$ are open subgroups (see Example 2.1), and
- the left translation actions $\Gamma \curvearrowright G_{\Gamma,S}$ and $\Gamma \curvearrowright K_{\Gamma,S}$ have spectral gap (see Example 2.3).

We continue with several elementary rigidity results about homomorphisms between the groups defined above. We start by collecting some properties of $\mathrm{SL}_2(\mathbb{F}_p)$ that we will use repeatedly.

Proposition 7.3. *Let p be a prime and denote by $I \in \mathrm{SL}_2(\mathbb{F}_p)$ the identity matrix. Then:*

(1) $|\mathrm{SL}_2(\mathbb{F}_p)| = p(p^2-1)$.
(2) *If $p \geq 5$, then the only proper normal subgroups of $\mathrm{SL}_2(\mathbb{F}_p)$ are $\{I\}$ and $\{\pm I\}$.*
(3) *Any automorphism of $\mathrm{SL}_2(\mathbb{F}_p)$ is given by conjugation with an element of $\mathrm{GL}_2(\mathbb{F}_p)$.*
(4) *Any proper subgroup $L < \mathrm{SL}_2(\mathbb{F}_p)$ with $|L| > 60$ is 2-step solvable. In particular, for every $a, b, c, d \in L$ we have $[[a, b], [c, d]] = I$.*

The first three facts are well-known. The last fact is a consequence of Dickson's classification of subgroups of $\mathrm{SL}_2(\mathbb{F}_p)$ (see [BG05, Theorem 7 and Proposition 3] and the references therein).

Below, whenever $S \subset T$, we consider the natural embeddings $G_S < G_T$ and $K_S < K_T$.

Lemma 7.4. *Let S and T be sets of primes ≥ 7. If $\delta : G_S \to G_T$ is an injective homomorphism, then $S \subset T$ and we can find $g \in H_S$ such that $\delta(x) = gxg^{-1}$ for all $x \in G_S$. In particular, $\delta(G_S) = G_S$.*

Proof. The lemma is an immediate consequence of the following:

Claim. *If $p \geq 7$ and $q \geq 5$ are primes and $\rho : \mathrm{SL}_2(\mathbb{F}_p) \to \mathrm{SL}_2(\mathbb{F}_q)$ is a non-trivial homomorphism, then $p = q$ and we can find $h \in \mathrm{GL}_2(\mathbb{F}_p)$ such that $\rho(x) = hxh^{-1}$ for all $x \in \mathrm{SL}_2(\mathbb{F}_p)$.*

Proof of the Claim. Since $\ker\rho$ is a proper normal subgroup of $\mathrm{SL}_2(\mathbb{F}_p)$, either $\ker\rho = \{I\}$ or $\ker\rho = \{\pm I\}$. Thus, $\rho(\mathrm{SL}_2(\mathbb{F}_p))$ is isomorphic to either $\mathrm{SL}_2(\mathbb{F}_p)$ or $\mathrm{PSL}_2(\mathbb{F}_p)$. As $p \geq 5$, neither of these groups is solvable (they are perfect groups, by Proposition 7.3(2)). Hence, $\rho(\mathrm{SL}_2(\mathbb{F}_p))$ is not solvable. Moreover, it has cardinality either $p(p^2-1)$ or $p(p^2-1)/2$.

Since $p(p^2-1)/2 > 60$, by using Proposition 7.3(4) we deduce that $\rho(\mathrm{SL}_2(\mathbb{F}_p)) = \mathrm{SL}_2(\mathbb{F}_q)$. Since the equation $q(q^2-1) = p(p^2-1)/2$ has no solutions $p \geq 7, q \geq 5$, we conclude that $p = q$ and that ρ is an automorphism of $\mathrm{SL}_2(\mathbb{F}_p)$. By Proposition 7.3(3), we are done. □

Corollary 7.5. *Let S and T be infinite sets of primes. Let $G < G_S$ be an open subgroup and $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ an infinite subgroup. If there exists an injective continuous homomorphism $\delta : G \to G_T$ such that $\delta(\Gamma \cap G) \subset \mathrm{SL}_2(\mathbb{Z})$, then $|S \vartriangle T| < \infty$.*

Proof. Let $T_0 = T \setminus \{2, 3, 5\}$. Since $G_{T_0} < G_T$ is an open subgroup and δ is continuous, we find that $\delta^{-1}(G_{T_0}) < G$ is an open subgroup. Thus, $\delta^{-1}(G_{T_0}) < G_S$ is an open subgroup, and therefore we can find a set $S_0 \subset S \setminus \{2, 3, 5\}$ such that $G_{S_0} \subset \delta^{-1}(G_{T_0})$ and $S \setminus S_0$ is finite. Altogether, $\delta_{|G_{S_0}} : G_{S_0} \to G_{T_0}$ is an injective homomorphism.

By Lemma 7.4 we know that $S_0 \subset T_0$ and $\delta(G_{S_0}) = G_{S_0}$. We claim that $T_0 \setminus S_0$ is finite. Assume it is infinite. Then $\mathrm{SL}_2(\mathbb{Z}) \cap G_{S_0} = \{e\}$ (where we view both $\mathrm{SL}_2(\mathbb{Z})$ and G_{S_0} as subgroups of G_{T_0}). On the other hand, since δ is injective, $\delta(\Gamma \cap G_{S_0})$ is an infinite subgroup of $\mathrm{SL}_2(\mathbb{Z}) \cap \delta(G_{S_0}) = \mathrm{SL}_2(\mathbb{Z}) \cap G_{S_0}$. This gives a contradiction.

Finally, the claim implies that $|S_0 \triangle T_0| < \infty$, and hence $|S \triangle T| < \infty$. □

For the rest of this section, for $m \geq 1$, we denote $\mathrm{SL}_2(m\mathbb{Z}) = \ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z}))$.

Lemma 7.6. *Let S be an infinite set of primes and $m, n \geq 1$. Assume that $p \nmid m$ for every $p \in S$. Assume that there exists $g \in H_S = \prod_{p \in S} \mathrm{GL}_2(\mathbb{F}_p)$ such that $g\pi_S(\mathrm{SL}_2(m\mathbb{Z}))g^{-1} = \pi_S(\mathrm{SL}_2(n\mathbb{Z}))$. Then $m = n$ and we can find $k \in \mathrm{GL}_2(\mathbb{Z})$ and l in the center of H_S such that $g = \pi_S(k)l$.*

Proof. Write $g = (g_p)_{p \in S}$. Denote by A the set of $x \in \mathbb{M}_2(\mathbb{Z})$ for which there exists $y \in \mathbb{M}_2(\mathbb{Z})$ such that $g_p \pi_p(x) g_p^{-1} = \pi_p(y)$ for all $p \in S$. Since S is infinite, such a y must be unique and we denote it by $\phi(x)$. Moreover, A is a subring of $\mathbb{M}_2(\mathbb{Z})$ and the map $\phi : A \to \mathbb{M}_2(\mathbb{Z})$ is an injective ring homomorphism.

By hypothesis, A contains the ring generated by $\mathrm{SL}_2(m\mathbb{Z})$ inside $\mathbb{M}_2(\mathbb{Z})$. Since the latter contains $m^2\mathbb{M}_2(\mathbb{Z})$, we deduce that $m^2\mathbb{M}_2(\mathbb{Z}) \subset A$.

Next, we find a matrix $k \in \mathbb{M}_2(\mathbb{Z})$ with non-zero determinant such that $\phi(x)k = kx$ for all $x \in A$. We view every $v \in \mathbb{Z}^2$ as a 2×1 matrix over $\mathbb{Z}$ and denote by v^t its transpose. Let $v \in \mathbb{Z}^2$ and $w \in (m^2\mathbb{Z})^2$ be non-zero vectors. Since $vw^t \in m^2\mathbb{M}_2(\mathbb{Z}) \setminus \{0\}$ and ϕ is injective, we see that $\phi(vw^t) \neq 0$. Let $z \in \mathbb{Z}^2$ be such that $\phi(vw^t)z \neq 0$.

We define $k : \mathbb{Z}^2 \to \mathbb{Z}^2$ by letting $k(\omega) = \phi(\omega w^t)z$. Fix $\omega \in \mathbb{Z}^2$ and $x \in A$. Since ϕ is multiplicative on A, we have $\phi(x\omega w^t) = \phi(x)\phi(\omega w^t)$. Thus, $k(x\omega) = \phi(x\omega w^t)z = \phi(x)\phi(\omega w^t)z = \phi(x)k(\omega)$. Hence, if we view k as an element of $\mathbb{M}_2(\mathbb{Z})$, then $kx = \phi(x)k$ for all $x \in A$. This implies that $\ker k$ is an A-invariant subgroup of $\mathbb{Z}^2$. Since $A \supset m^2\mathbb{M}_2(\mathbb{Z})$, $\ker k$ is either $\{0\}$ or $\mathbb{Z}^2$. Since $k(v) \neq 0$, it follows that k is injective, proving our claim.

Now, write $k = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where a, b, c, d are integers. After replacing k with $\frac{1}{N}k$ for some $N \geq 1$, we may assume that the greatest common divisor of a, b, c, d is 1. We claim that $k \in \mathrm{GL}_2(\mathbb{Z})$. Note that if $x \in \mathrm{SL}_2(m\mathbb{Z})$, then the hypothesis gives $\phi(x) \in \mathrm{SL}_2(n\mathbb{Z})$. Since also $x \in A$, we get $kxk^{-1} = \phi(x) \in \mathrm{SL}_2(n\mathbb{Z})$. By applying this relation to $x \in \left\{\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ m & 1 \end{smallmatrix}\right)\right\}$, it follows that $mk\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)k^{-1}, mk\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)k^{-1} \in n\mathbb{M}_2(\mathbb{Z})$.

These relations imply that $n \det k$ divides ma^2, mb^2, mc^2 and md^2. Thus, $n \det k \mid m$ and so $n \mid m$. By symmetry, we also get $m \mid n$. Hence $m = n$ and $\det k \in \{\pm 1\}$, and thus $k \in \mathrm{GL}_2(\mathbb{Z})$.

Finally, let $p \in S$. Then for every $x \in A$ we have

$$g_p \pi_p(x) g_p^{-1} \pi_p(k) = \pi_p(\phi(x)k) = \pi_p(kx) = \pi_p(k)\pi_p(x).$$

This shows that $g_p^{-1}\pi_p(k)$ commutes with $\pi_p(A)$. By using again the fact that $A \supset m^2\mathbb{M}_2(\mathbb{Z})$ and $p \nmid m$, we deduce that $g_p^{-1}\pi_p(k)$ is a multiple of the identity. Therefore, we can find $l_p \in \mathbb{Z}/p\mathbb{Z} \setminus \{0\}$ such that $\pi_p(k)l_p = g_p I$. Since $l = (l_p)_{p\in S}$ belongs to the center of G, the conclusion follows. □

Lemma 7.7. *Let p be a prime and S a set of primes such that $p \notin S$.*

(1) ([GG88]) *If $K < K_S$ is an open subgroup and $\delta : K \to \mathrm{SL}_2(\mathbb{Z}_p)$ is a continuous homomorphism, then $\delta(K)$ is finite.*
(2) *If $L < \mathrm{SL}_2(\mathbb{Z}_p)$ is an open subgroup, then there is no injective continuous homomorphism $\delta : L \to K_S$.*

Proof. We denote by $\rho_q : \mathrm{SL}_2(\mathbb{Z}_q) \to \mathrm{SL}_2(\mathbb{F}_q)$ for $q \geq 3$ and by $\rho_2 : \mathrm{SL}_2(\mathbb{Z}_2) \to \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$ the obvious surjective homomorphisms. Then $\Gamma_q := \ker \rho_q$ is a pro-q normal open subgroup of $\mathrm{SL}_2(\mathbb{Z}_q)$ for any prime q. (Recall that a profinite group G is *pro-q* if the index $[G : G_0]$ is a power of q for any open subgroup $G_0 < G$.)

Moreover, as is well-known, Γ_q is torsion free. Thus, if Δ is a finite subgroup of $\mathrm{SL}_2(\mathbb{Z}_q)$, then $\rho_q|\Delta$ is injective, and therefore $|\Delta|$ divides $[\mathrm{SL}_2(\mathbb{Z}_q) : \Gamma_q]$.

(1) This part follows from the proof of [GG88, Lemma A.6]. For completeness, we include the argument from [GG88]. We first treat the case when S has one element. Thus, let $\delta : K \to \mathrm{SL}_2(\mathbb{Z}_p)$ be a continuous homomorphism, where K is an open subgroup of $\mathrm{SL}_2(\mathbb{Z}_q)$ for some prime $q \neq p$. Since $[\delta(K) : \delta(K \cap \Gamma_q)] \leq [K : K \cap \Gamma_q] < \infty$, we may assume that $K < \Gamma_q$.

Fix $n \geq 2$ and define $\Gamma_{p,n} = \ker(\mathrm{SL}_2(\mathbb{Z}_p) \to \mathrm{SL}_2(\mathbb{Z}/p^n\mathbb{Z}))$. Then $\delta^{-1}(\Gamma_{p,n})$ is an open subgroup of $\delta^{-1}(\Gamma_p)$. Let $m = [\delta^{-1}(\Gamma_p) : \delta^{-1}(\Gamma_{p,n})]$. Since $\delta^{-1}(\Gamma_p)$ is a pro-q group (being an open subgroup of K), we can write $m = q^\alpha$ for some $\alpha \geq 0$. On the other hand, since $m \,|\, [\Gamma_p : \Gamma_{p,n}]$ and Γ_p is pro-p, we have $m = p^\beta$ for some $\beta \geq 0$. Since $p \neq q$, we must have $m = 1$.

In other words, $\delta^{-1}(\Gamma_p) = \delta^{-1}(\Gamma_{n,p})$ for all $n \geq 2$. Since $\bigcap_{n\geq 2} \Gamma_{p,n} = \{e\}$, we get $\delta^{-1}(\Gamma_p) = \{e\}$. This implies that $\delta(K)$ is finite.

Now, in general, let $K < K_S$ be an open subgroup and $\delta : K \to \mathrm{SL}_2(\mathbb{Z}_p)$ a continuous homomorphism. Then K contains an open subgroup $L < K_S$ of the form $L = \prod_{q\in S} L_q$, such that $L_q < \mathrm{SL}_2(\mathbb{Z}_q)$ is an open subgroup for all $q \in S$, and the set $\{q \in S \mid L_q \neq \mathrm{SL}_2(\mathbb{Z}_q)\}$ is finite. For every subset $F \subset S$, we denote $L_F = \prod_{q\in F} L_q < K_S$.

The first part of the proof shows that $\delta(L_q)$ is a finite subgroup of $\mathrm{SL}_2(\mathbb{Z}_p)$ for all $q \in S$. Since $\{\delta(L_q)\}_{q\in S}$ are mutually commuting subgroups of $\mathrm{SL}_2(\mathbb{Z}_p)$, we infer that $\delta(L_F)$ is a finite group whenever $F \subset S$ is finite. This implies that $|\delta(L_F)| \leq N := [\mathrm{SL}_2(\mathbb{Z}_p) : \Gamma_p]$. Thus, we can choose F such that $|\delta(L_F)| \geq |\delta(L_{F'})|$ for any other finite subset F' of S. In particular, if $F' \supset F$ then $\delta(L_{F'}) = \delta(L_F)$. Since δ is continuous, we deduce that $\delta(L) = \delta(L_F)$, hence $\delta(L)$ is finite. Since $L < K$ has finite index, we conclude that $\delta(K)$ is finite as well.

(2) Assume for contradiction that there is an injective continuous homomorphism $\delta : L \to K_S$, where $L < \mathrm{SL}_2(\mathbb{Z}_p)$ is an open subgroup and $p \notin S$. We may take $L < \Gamma_p$ so that L is pro-p.

We denote $\rho = (\rho_q)_{q\in S} : K_S \to \prod_{q\in S} \mathrm{SL}_2(\mathbb{Z}_q)/\Gamma_q$ and claim that $\rho\circ\delta$ is injective. To see this, for $q \in S$, we denote by $\sigma_q : K_S \to \mathrm{SL}_2(\mathbb{Z}_q)$ the quotient homomorphism. Then $\sigma_q\circ\delta : L \to \mathrm{SL}_2(\mathbb{Z}_q)$ is a continuous homomorphism. Since $p \notin S$, part (1) implies that $\sigma_q(\delta(L))$ is finite. Since Γ_q is torsion free, we derive that $\sigma_q(\delta(L)) \cap \Gamma_q = \{e\}$ for all $q \in S$.

Now, assume that $\rho(\delta(x)) = e$ for some $x \in L$. Then, given $q \in S$, we have $\rho_q(\sigma_q(\delta(x))) = e$, or equivalently $\sigma_q(\delta(x)) \in \Gamma_q$. Using the last paragraph, we find that $\sigma_q(\delta(x)) = e$, hence $\delta(x) = e$. Since δ is injective, we deduce that $x = e$. Hence, $\rho\circ\delta$ is also injective.

Next, we note that $\rho_q\circ\sigma_q\circ\delta : L \to \mathrm{SL}_2(\mathbb{Z}_q)/\Gamma_q$ is not onto, for any $q \in S$. This is because L is pro-p, while the cardinality of $\mathrm{SL}_2(\mathbb{Z}_q)/\Gamma_q$ is $q(q^2-1)$ if $q \geq 3$, and 40 if $q = 2$.

To finish the proof we use an idea from [Gam02]. Let $L_q = \rho_q(\sigma_q(\delta(L)))$. Then L_q is a proper subgroup of $\mathrm{SL}_2(\mathbb{Z}_q)/\Gamma_q \cong \mathrm{SL}_2(\mathbb{Z}_q)$ when $q \geq 3$, and of $\mathrm{SL}_2(\mathbb{Z}_2)/\Gamma_2 \cong \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$ when $q = 2$. By Proposition 7.3(4), either $|L_q| \leq 60$, or $[[a,b],[c,d]] = e$ for all $a,b,c,d \in L_q$. In either case we deduce that $[[a,b],[c,d]]^{60!} = e$ for all $a,b,c,d \in L_q$ and $q \in S$. Since $\rho(\delta(L)) \subset \prod_{q\in S} L_q$ and $\rho\circ\delta$ is injective, we conclude that $[[a,b],[c,d]]^{60!} = e$ for all $a,b,c,d \in L$. This is however impossible, since L contains a non-abelian free group (e.g. $L \cap \mathrm{SL}_2(\mathbb{Z})$). □

Corollary 7.8. *Let S and T be sets of primes. Let $K < K_S$ be an open subgroup and $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ an infinite subgroup. If there exists an injective continuous homomorphism $\delta : K \to K_T$ such that $\delta(\Gamma \cap K) \subset \mathrm{SL}_2(\mathbb{Z})$, then $S = T$.*

Proof. Firstly, Lemma 7.7(2) implies that $S \subset T$. Assume that we can find $p \in T \setminus S$, and let $\rho : K_T \to \mathrm{SL}_2(\mathbb{Z}_p)$ be the quotient homomorphism. By Lemma 7.7(1) we see that $\rho(\delta(K))$ is finite. On the other hand, by hypothesis, $\Lambda = \delta(\Gamma \cap K)$ is an infinite subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Since the restriction of ρ to $\mathrm{SL}_2(\mathbb{Z})$ is injective, we deduce that $\rho(\Lambda)$ is infinite, a contradiction. □

Proof of Corollary B. By Fact 7.2, $G_{\Gamma,S} < G_S$ is an open subgroup and the action $\Gamma \curvearrowright G_{\Gamma,S}$ has spectral gap for any infinite set of primes S and every non-amenable subgroup $\Gamma < \mathrm{SL}_2(\mathbb{Z})$.

Let S and T be infinite sets of primes. Below, we prove assertions (1) and (2) separately.

(1) Assume that either $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright G_S$ is stably orbit equivalent to $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright G_T$, or $\mathcal{R}(\Gamma \curvearrowright G_{\Gamma,S})$ is Borel reducible to $\mathcal{R}(\Lambda \curvearrowright G_{\Lambda,T})$. In either case, by Theorem A, we can find an open subgroup $G_0 < G_S$, a closed subgroup $G_1 < G_T$, and a continuous isomorphism $\delta : G_0 \to G_1$ such that $\delta(\pi_S(\mathrm{SL}_2(\mathbb{Z})) \cap G_0) = \pi_T(\mathrm{SL}_2(\mathbb{Z})) \cap G_1$.

Let $S_1 = S \setminus \{2,3,5\}$ and $T_0 = T \setminus \{2,3,5\}$. Then $G_{S_1} \cap \delta^{-1}(G_1 \cap G_{T_0})$ is an open subgroup of G_0, and thus of G_S. Hence, we can find a subset $S_0 \subset S_1$ such that $S \setminus S_0$ is finite, $G_{S_0} \subset G_0$, and $\delta(G_{S_0}) \subset G_{T_0}$. Lemma 7.4 implies that $S_0 \subset T_0$, $\delta(G_{S_0}) = G_{S_0}$, and there is $g \in H_{S_0}$ such that $\delta(x) = gxg^{-1}$ for all $x \in G_{S_0}$. Moreover, $\delta(\pi_S(\mathrm{SL}_2(\mathbb{Z})) \cap G_{S_0})) = \pi_T(\mathrm{SL}_2(\mathbb{Z})) \cap G_{S_0}$.

Next, we denote by m the product of the primes in $S \setminus S_0$. Then $\pi_S(\mathrm{SL}_2(\mathbb{Z})) \cap G_{S_0} = \pi_{S_0}(\mathrm{SL}_2(m\mathbb{Z}))$. Since $\pi_T(\mathrm{SL}_2(\mathbb{Z})) \cap G_{S_0}$ is infinite (hence, non-trivial), $T \setminus S_0$ is finite. Moreover, if n is the product of the primes in $T \setminus S_0$, then $\pi_T(\mathrm{SL}_2(\mathbb{Z})) \cap G_{S_0} = \pi_{S_0}(\mathrm{SL}_2(n\mathbb{Z}))$. Altogether, $g\pi_{S_0}(\mathrm{SL}_2(m\mathbb{Z}))g^{-1} = \pi_{S_0}(\mathrm{SL}_2(n\mathbb{Z}))$.

Lemma 7.6 implies that $m = n$, and thus $S \setminus S_0 = T \setminus S_0$. This gives $S = T$, as claimed.

(2) Let $\Gamma, \Lambda < \mathrm{SL}_2(\mathbb{Z})$ be non-amenable subgroups. Assume that either $\Gamma \curvearrowright G_{\Gamma,S}$ is stably orbit equivalent to $\Lambda \curvearrowright G_{\Lambda,T}$, or $\mathcal{R}(\Gamma \curvearrowright G_{\Gamma,S})$ is Borel reducible to $\mathcal{R}(\Lambda \curvearrowright G_{\Lambda,T})$. In either case, Theorem A implies that we can find an open subgroup $G_0 < G_{\Gamma,S}$, a closed subgroup $G_1 < G_{\Lambda,T}$, and a continuous isomorphism $\delta : G_0 \to G_1$ such that $\delta(\Gamma \cap G_0) = \Lambda \cap G_1$. Since $G_{\Gamma,S} < G_S$ is open, $G_0 < G_S$ is open. By Corollary 7.5, we conclude that $|S \vartriangle T| < \infty$. □

Proof of Corollary C. By Fact 7.2, $K_{\Gamma,S} < K_S$ is an open subgroup and the action $\Gamma \curvearrowright K_{\Gamma,S}$ has spectral gap for any non-empty set S of primes and every non-amenable subgroup $\Gamma < \mathrm{SL}_2(\mathbb{Z})$.

Let $\Gamma, \Lambda < \mathrm{SL}_2(\mathbb{Z})$ be non-amenable subgroups and S, T non-empty sets of primes. Assume that either $\Gamma \curvearrowright K_{\Gamma,S}$ is stably orbit equivalent to $\Lambda \curvearrowright K_{\Lambda,T}$, or $\mathcal{R}(\Gamma \curvearrowright K_{\Gamma,S})$ is Borel reducible to $\mathcal{R}(\Lambda \curvearrowright K_{\Lambda,T})$. Theorem A then implies that we can find an open subgroup $K_0 < K_{\Gamma,S}$, a closed subgroup $K_1 < K_{\Lambda,T}$, and a continuous isomorphism $\delta : K_0 \to K_1$ such that $\delta(\Gamma \cap K_0) = \Lambda \cap K_1$.

Since $K_{\Gamma,S} < K_S$ is open, so is $K_0 < K_S$, and Corollary 7.8 implies that $S = T$. □

We end this section by proving the following strengthening of part of Corollary C.

Corollary 7.9. *Let S, T be non-empty sets of primes, and $\Gamma, \Lambda < \mathrm{SL}_2(\mathbb{Z})$ non-amenable subgroups. If $T \not\subset S$, then any homomorphism from $\mathcal{R}(\Gamma \curvearrowright K_{\Gamma,S})$ to $\mathcal{R}(\Lambda \curvearrowright K_{\Lambda,T})$ is trivial.*

Proof. Assume that there is a non-trivial homomorphism from $\mathcal{R}(\Gamma \curvearrowright K_{\Gamma,S})$ to $\mathcal{R}(\Lambda \curvearrowright K_{\Lambda,T})$. By Corollary 4.4 we can find an open subgroup $K_0 < K_{\Gamma,S}$ and a continuous homomorphism $\delta : K_0 \to K_T$ such that $\delta(\Gamma \cap K_0) \subset \Lambda$ and $\delta(K_0) \not\subset \Lambda$. Also, Fact 7.2 implies that $K_0 < K_S$ is an open subgroup.

Let $p \in T \setminus S$ and let $\rho : K_T \to \mathrm{SL}_2(\mathbb{Z}_p)$ be the quotient homomorphism. By Lemma 7.7(1) we know that $\rho(\delta(K_0))$ is finite. Thus, we can find an open subgroup $K_1 < K_0$ such that $\rho(\delta(K_1)) = \{e\}$. Since the restriction of ρ to $\mathrm{SL}_2(\mathbb{Z})$ is injective and $\delta(\Gamma \cap K_1) \subset \mathrm{SL}_2(\mathbb{Z})$, we deduce that $\delta(\Gamma \cap K_1) = \{e\}$. As $\Gamma \cap K_1 < K_1$ is dense, it follows that $\delta(K_1) = \{e\}$. Since $\Gamma \cap K_0 < K_0$ is dense, we also have $(\Gamma \cap K_0)K_1 = K_0$. This yields $\delta(K_0) = \delta(\Gamma \cap K_0) \subset \Lambda$, a contradiction. □

8. Proof of Corollary D

This section is devoted to the proof of Corollary D. In fact, we will establish a stronger result from which Corollary D will follow easily. Before stating this result, let us recall some notation.

For a prime p, let $\mathrm{PG}(1,\mathbb{Q}_p)=\mathbb{Q}_p\cup\{\infty\}$ be the projective line over the field $\mathbb{Q}_p$ of p-adic numbers. Consider the Borel action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $\mathrm{PG}(1,\mathbb{Q}_p)$ by linear fractional transformations:

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}\cdot x=\frac{ax+b}{cx+d}.$$

Theorem 8.1. *Let $p\neq q$ be primes, and $\Gamma<\mathrm{SL}_2(\mathbb{Z})$, $\Lambda<\mathrm{GL}_2(\mathbb{Q}_q)$ countable subgroups. Assume that Γ is non-amenable and denote by G the closure of Γ in $\mathrm{SL}_2(\mathbb{Z}_p)$. Then any homomorphism from $\mathcal{R}(\Gamma\curvearrowright G)$ to $\mathcal{R}(\Lambda\curvearrowright\mathrm{PG}(1,\mathbb{Q}_q))$ is trivial with respect to m_G.*

Proof. Since $\Gamma<\mathrm{SL}_2(\mathbb{Z})$ is non-amenable, Fact 7.2 shows that $G<\mathrm{SL}_2(\mathbb{Z}_p)$ is an open subgroup and the action $\Gamma\curvearrowright(G,m_G)$ has spectral gap.

We denote $\mathrm{PGL}_2(\mathbb{Q}_q)=\mathrm{GL}_2(\mathbb{Q}_q)/Z_q$, where $Z_q=\{zI\mid z\in\mathbb{Q}_q^*\}$ is the center of $\mathrm{GL}_2(\mathbb{Q}_q)$. Let $\rho:\mathrm{GL}_2(\mathbb{Q}_q)\to\mathrm{PGL}_2(\mathbb{Q}_q)$ be the quotient homomorphism. Notice Z_q acts trivially on $\mathrm{PG}(1,\mathbb{Q}_q)$ and consider the resulting action $\mathrm{PGL}_2(\mathbb{Q}_q)\curvearrowright\mathrm{PG}(1,\mathbb{Q}_q)$.

Let $\Omega:=\{(x_1,x_2,x_3)\in\mathrm{PG}(1,\mathbb{Q}_q)^3\mid x_i\neq x_j \text{ for all } 1\leq i<j\leq 3\}$.

Claim 1. *The action $\mathrm{PGL}_2(\mathbb{Q}_q)\curvearrowright\Omega$ is transitive and free.*

Proof of Claim 1. This claim is well-known, but we include a proof for completeness.

Let $x_1,x_2,x_3\in\mathrm{PG}(1,\mathbb{Q}_q)$ be distinct. Consider the matrix

$$g=\begin{pmatrix} x_2-x_3 & -x_1(x_2-x_3)\\ x_2-x_1 & -x_3(x_2-x_1)\end{pmatrix}.$$

Then $g\in\mathrm{GL}_2(\mathbb{Q}_q)$ and the function $x\mapsto g\cdot x=\frac{(x-x_1)(x_2-x_3)}{(x-x_3)(x_2-x_1)}$ maps $x_1\mapsto 0$, $x_2\mapsto 1$, and $x_3\mapsto\infty$. This shows that the action $\mathrm{PGL}_2(\mathbb{Q}_q)\curvearrowright\Omega$ is transitive.

If $g=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\mathrm{GL}_2(\mathbb{Q}_q)$ and the equation $g\cdot x=\frac{ax+b}{cx+d}=x$ has at least three distinct solutions $x\in\mathrm{PG}(1,\mathbb{Q}_q)$, then $a=d$ and $b=c=0$, hence $g\in Z_q$. This shows that the action $\mathrm{PGL}_2(\mathbb{Q}_q)\curvearrowright\Omega$ is also free. □

Now, since Z_q acts trivially on $\mathrm{PG}(1,\mathbb{Q}_q)$, it follows that $\mathcal{R}(\Lambda\curvearrowright\mathrm{PG}(1,\mathbb{Q}_q))$ is Borel isomorphic to $\mathcal{R}(\rho(\Lambda)\curvearrowright\mathrm{PG}(1,\mathbb{Q}_q)$. Let $\theta:G\to\mathrm{PG}(1,\mathbb{Q}_q)$ be a homomorphism from $\mathcal{R}(\Gamma\curvearrowright G)$ to $\mathcal{R}(\rho(\Lambda)\curvearrowright\mathrm{PG}(1,\mathbb{Q}_q))$. Assume for contradiction that θ is not trivial.

Claim 2. $(\theta(x_1),\theta(x_2),\theta(x_3))\in\Omega$ *for almost every* $(x_1,x_2,x_3)\in G^3$.

Proof of Claim 2. If the claim is false, then $\{(x_1,x_2)\in G^2\mid\theta(x_1)=\theta(x_2)\}$ has positive measure. Then there exists $y\in\mathrm{PG}(1,\mathbb{Q}_q)$ such that $\{x\in G\mid\theta(x)=y\}$ has positive measure. Since $\Gamma<G$ is dense, we get $\theta(x)\in\Lambda y$ for almost every $x\in G$. This contradicts our assumption that θ is not trivial. □

Since $\Gamma\curvearrowright(G,m_G)$ has spectral gap and any transitive action is smooth, Claims 1 and 2 imply that the hypothesis of Theorem 5.1(1) is satisfied. Thus, we can find an open subgroup $G_0<G$, a continuous homomorphism $\delta:G_0\to\mathrm{PGL}_2(\mathbb{Q}_q)$, a Borel map $\phi:G_0\to\rho(\Lambda)$, and $y\in\mathrm{PG}(1,\mathbb{Q}_q)$ such that $\delta(\Gamma\cap G_0)\subset\rho(\Lambda)$ and $\theta(x)=\phi(x)\delta(x)y$ for almost every $x\in G_0$.

Claim 3. *We can find an open subgroup $G_1 < G_0$ such that $\delta(G_1) = \{e\}$.*

Proof of Claim 3. Let $K_q = I + q\mathbb{M}_2(\mathbb{Z}_q) = \left\{\left(\begin{smallmatrix} 1+a & b \\ c & 1+d \end{smallmatrix}\right) \mid a, b, c, d \in q\mathbb{Z}_q\right\}$. Then $K_q < \mathrm{GL}_2(\mathbb{Q}_q)$ is an open compact subgroup, hence $\rho(K_q) < \mathrm{PGL}_2(\mathbb{Q}_q)$ is an open subgroup. Since δ is continuous and $G_0 < \mathrm{SL}_2(\mathbb{Z}_p)$ is an open subgroup, we can find an open subgroup $G_1 < G_0$ such that $\delta(G_1) \subset \rho(K_q)$. Moreover, we may assume that G_1 is a pro-p group (see the proof of Lemma 7.7).

Now, let $K_{q,n} = I + q^n\mathbb{M}_2(\mathbb{Z}_q)$. Then $K_{q,n}$ is an open subgroup of K_q for every $n \geq 1$, and $\{K_{q,n}\}_{n\geq 1}$ is a basis of open neighborhoods of the identity in K_q. Since $K_q/K_{q,n} \cong I + q\mathbb{M}_2(\mathbb{Z}/q^n\mathbb{Z})$ and $|I + q\mathbb{M}_2(\mathbb{Z}/q^n\mathbb{Z})| = q^{4(n-1)}$, we deduce that K_q is a pro-q group. Thus so is $\rho(K_q)$, being a continuous image of K_q.

Since G_1 is pro-p and K_q is pro-q, and $p \neq q$, any continuous homomorphism from G_1 to K_q is trivial (see e.g. the proof of Lemma 7.7). This proves the claim. □

Finally, Claim 3 implies that $\theta(x) \in \Lambda y$ for almost every $x \in G_1$. Since $\Gamma < G$ is dense and $G_1 < G$ is open, we have $\Gamma G_1 = G$. From this we deduce that $\theta(x) \in \Lambda y$ for almost every $x \in G$, and hence θ is trivial. □

Proof of Corollary D. Let $\Gamma < \mathrm{GL}_2(\mathbb{Q}_p)$ and $\Lambda < \mathrm{GL}_2(\mathbb{Q}_q)$ be countable subgroups, for some primes $p \neq q$, such that $\Gamma_0 := \Gamma \cap \mathrm{SL}_2(\mathbb{Z})$ is non-amenable. Assume for contradiction that there exists a Borel reduction $\theta : \mathrm{PG}(1, \mathbb{Q}_p) \to \mathrm{PG}(1, \mathbb{Q}_q)$ from $\mathcal{R}(\Gamma \curvearrowright \mathrm{PG}(1, \mathbb{Q}_p))$ to $\mathcal{R}(\Lambda \curvearrowright \mathrm{PG}(1, \mathbb{Q}_q))$.

Next, note that $\mathrm{SL}_2(\mathbb{Z}_p) \curvearrowright \mathrm{PG}(1, \mathbb{Q}_p)$ is Borel isomorphic to the left multiplication action $\mathrm{SL}_2(\mathbb{Z}_p) \curvearrowright \mathrm{SL}_2(\mathbb{Z}_p)/K_p$, where $K_p < \mathrm{SL}_2(\mathbb{Z}_p)$ denotes the subgroup of lower triangular matrices. Indeed, the former action is transitive (see e.g. [Th01, Lemma 6.1]) and K_p is the stabilizer of $0 \in \mathrm{PG}(1, \mathbb{Q}_p)$. This implies that there exists a unique $\mathrm{SL}_2(\mathbb{Z}_p)$-invariant Borel probability measure on $\mathrm{PG}(1, \mathbb{Q}_p)$, which we denote by μ_p.

Let $\pi : \mathrm{SL}_2(\mathbb{Z}_p) \to \mathrm{SL}_2(\mathbb{Z}_p)/K_p$ be the quotient map, and define $\Theta := \theta \circ \pi : \mathrm{SL}_2(\mathbb{Z}_p) \to \mathrm{PG}(1, \mathbb{Q}_q)$. Denote also by G the closure of Γ_0 in $\mathrm{SL}_2(\mathbb{Z}_p)$. Since Γ_0 is non-amenable, Fact 7.2 implies that $G < \mathrm{SL}_2(\mathbb{Z}_p)$ is an open subgroup. Let $k = [\mathrm{SL}_2(\mathbb{Z}_p) : G]$ and let $g_1, \ldots, g_k \in \mathrm{SL}_2(\mathbb{Z}_p)$ be such that $\mathrm{SL}_2(\mathbb{Z}_p) = \bigsqcup_{i=1}^k Gg_i$. For $1 \leq i \leq k$, let $\Theta_i : G \to \mathrm{PG}(1, \mathbb{Q}_p)$ be given by $\Theta_i(x) = \Theta(xg_i)$.

Then Θ_i is a homomorphism from $\mathcal{R}(\Gamma_0 \curvearrowright G))$ to $\mathcal{R}(\Lambda \curvearrowright \mathrm{PG}(1, \mathbb{Q}_q))$. By Theorem 8.1 we conclude that Θ_i is trivial, i.e. there is $y_i \in \mathrm{PG}(1, \mathbb{Q}_q)$ such that $\Theta_i(x) \in \Lambda y_i$ for m_G-almost every $x \in G$. Hence $\theta(x) \in \bigcup_{i=1}^k \Lambda y_i$ for μ_p-almost every $x \in \mathrm{PG}(1, \mathbb{Q}_p)$. This contradicts the fact that θ is countable-to-one. □

9. Proofs of Corollaries E and F

In this section we will use Corollary 4.3 to deduce Corollaries E and F. We start by proving Corollary E. As at the beginning of Section 7, we denote $G_S = \prod_{p\in S} \mathrm{SL}_2(\mathbb{F}_p)$ and $H_S = \prod_{p\in S} \mathrm{GL}_2(\mathbb{F}_p)$ for any infinite set S of primes. Let $\pi_p : \mathbb{M}_2(\mathbb{Z}) \to \mathbb{M}_2(\mathbb{F}_p)$ be reduction modulo p.

Let $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$. We still denote by α the element $(\pi_p(\alpha))_{p\in S}$ of H_S. Then $\alpha^2 = I$ and $\alpha \in H_S$ normalizes G_S. Below we consider the semidirect product

$G_S \rtimes \mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Z}/2\mathbb{Z} = \{I, \alpha\}$ and α acts on G_S by conjugation. Note also that α commutes with the center $Z = \{\pm I\}$ of G_S.

We begin by proving the first part of Corollary E.

Corollary 9.1. *Let S be an infinite set of primes and denote $\mathcal{R} = \mathcal{R}(\mathrm{SL}_2(\mathbb{Z}) \curvearrowright G_S)$. Then $\mathrm{Out}(\mathcal{R}) \cong (G_S/Z) \rtimes \mathbb{Z}/2\mathbb{Z}$.*

Proof. We define $\sigma_h : G_S \to G_S$ for $h \in G_S$ and $\sigma_\beta : G_S \to G_S$ for $\beta \in \{I, \alpha\}$ by letting

$$\sigma_h(x) = xh \quad \text{and} \quad \sigma_\beta(x) = \beta x \beta^{-1} \quad \text{for all } x \in G.$$

As σ_h commutes with $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright G_S$ and β normalizes $\mathrm{SL}_2(\mathbb{Z})$, we have $\sigma_h, \sigma_\beta \in \mathrm{Aut}(\mathcal{R})$. Moreover, since $\sigma_\beta \sigma_h \sigma_\beta^{-1} = \sigma_{\beta h \beta^{-1}}$, we have a homomorphism $\sigma : G_S \rtimes \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\mathcal{R})$. We denote $\rho = \varepsilon \circ \sigma : G_S \rtimes \mathbb{Z}/2\mathbb{Z} \to \mathrm{Out}(\mathcal{R})$, where $\varepsilon : \mathrm{Aut}(\mathcal{R}) \to \mathrm{Out}(\mathcal{R})$ is the quotient homomorphism.

To prove the corollary, we only have to argue that $\ker \rho = Z$ and ρ is surjective. Firstly, let $h \in G_S$ and $\beta \in \{I, \alpha\}$ be such that $\sigma_{(h,\beta)} = \sigma_h \sigma_\beta$ belongs to $[\mathcal{R}]$. Thus, we can find $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and a Borel set $A \subset G_S$ of positive (Haar) measure such that

$$\beta x \beta^{-1} h = \gamma x \quad \text{for all } x \in A. \tag{9.1}$$

This implies that $\gamma^{-1}\beta$ commutes with xy^{-1} for all $x, y \in A$. Since AA^{-1} contains an open neighborhood of the identity in G_S, we deduce that $\gamma^{-1}\beta$ commutes with G_T for some subset $T \subset S$ with $S \setminus T$ finite. Hence, $\pi_p(\gamma^{-1}\beta) \in \{\pm I\}$ for all $p \in T$. Since T is infinite and $\gamma^{-1}\beta \in \mathrm{GL}_2(\mathbb{Z})$, it follows easily that $\gamma^{-1}\beta = \pm I$. Thus, we must have $\beta = I$ and $\gamma = \pm I$. By using (9.1), we conclude that $h = \pm I$, showing that $(h, \beta) \in Z$.

To show that ρ is surjective, let $\theta \in \mathrm{Aut}(\mathcal{R})$. Then by Corollary 4.3, after replacing θ with $\theta \circ \tau$ for some $\tau \in [\mathcal{R}]$, we can find open subgroups $G_0, G_1 < G_S$, an isomorphism $\delta : G_0 \to G_1$, and $h \in G_S$ such that $\delta(\mathrm{SL}_2(\mathbb{Z}) \cap G_0) = \mathrm{SL}_2(\mathbb{Z}) \cap G_1$ and $\theta(x) = \delta(x)h$ for all $x \in G_1$.

Let $S_0 \subset S$ be such that $S_0 \subset S \setminus \{2, 3, 7\}$, $S \setminus S_0$ is finite, $G_{S_0} \subset G_0$, and $\delta(G_{S_0}) \subset G_1 \cap G_{S_0}$. By Lemma 7.4 we infer that $\delta(G_{S_0}) = G_{S_0}$ and there is $g \in H_{S_0}$ such that $\delta(x) = gxg^{-1}$ for all $x \in G_{S_0}$. Since $\delta(\mathrm{SL}_2(\mathbb{Z}) \cap G_{S_0}) = \mathrm{SL}_2(\mathbb{Z}) \cap G_{S_0}$, from Lemma 7.6 we deduce that there is $k \in \mathrm{GL}_2(\mathbb{Z})$ such that $\delta(x) = kxk^{-1}$ for all $x \in G_{S_0}$.

Let $\beta \in \{I, \alpha\}$ be such that $k = \gamma\beta$ for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Then for almost every $x \in G_{S_0}$ we have $\theta(x) = \delta(x)h = \gamma\beta x \beta^{-1}\gamma^{-1}h = \gamma\sigma_{(\gamma^{-1}h,\beta)}(x)$. Thus, the set A of $x \in G_S$ such that $\theta(x) \in \Gamma\sigma_{\gamma^{-1}h,\beta}(x)$ contains G_{S_0}. Since A is invariant under $\mathcal{R}$, and $\mathcal{R}$ is ergodic, we deduce that $A = G_S$ almost everywhere.

This implies $\varepsilon(\theta) = \varepsilon(\sigma_{(\gamma^{-1}h,\beta)}) = \rho_{(\gamma^{-1}h,\beta)}$, which proves that ρ is surjective. □

We continue by establishing the second part of Corollary E.

Corollary 9.2. *Let S be an infinite set of primes and $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ a non-amenable subgroup. Let $G_{\Gamma,S}$ denote the closure of Γ in G_S. Then the equivalence relation $\mathcal{R} = \mathcal{R}(\Gamma \curvearrowright G_{\Gamma,S})$ and the II_1 factor $M = L^\infty(G_{\Gamma,S}) \rtimes \Gamma$ have trivial fundamental groups, i.e. $\mathcal{F}(\mathcal{R}) = \mathcal{F}(M) = \{1\}$.*

Proof. Fact 7.2 shows that $G_{\Gamma,S} < G_S$ is an open subgroup and the action $\Gamma \curvearrowright G_{\Gamma,S}$ has spectral gap. Let $t \in \mathcal{F}(\mathcal{R})$. Corollary 4.3 provides open subgroups $G_0, G_1 < G_{\Gamma,S}$ such that $t = [G_{\Gamma,S} : G_0]/[G_{\Gamma,S} : G_1]$ and a continuous isomorphism $\delta : G_0 \to G_1$.

Since $G_{\Gamma,S} < G_S$ is open, we see that $G_0 < G_S$ is open. Thus, we may find a subset S_0 of S such that $S_0 \subset S\setminus\{2,3,5\}$, $S\setminus S_0$ is finite, $G_{S_0} \subset G_0$, and $\delta(G_{S_0}) \subset G_1 \cap G_{S_0}$. But then applying Lemma 7.4 to the injective homomorphism $\delta_{|G_{S_0}} : G_{S_0} \to G_{S_0}$ gives $\delta(G_{S_0}) = G_{S_0}$. Hence,

$$t = \frac{[G_{\Gamma,S} : G_0]}{[G_{\Gamma,S} : G_1]} = \frac{[G_1 : G_{S_0}]}{[G_0 : G_{S_0}]} = \frac{[\delta(G_0) : \delta(G_{S_0})]}{[G_0 : G_{S_0}]} = 1.$$

Finally, by a result of N. Ozawa and S. Popa [OP07, Corollary 3], M has a unique Cartan subalgebra, up to unitary conjugacy. This implies that $\mathcal{F}(M) = \mathcal{F}(\mathcal{R}) = \{1\}$. □

Remark 9.3. Note that in the above proof one can use [Io11, Theorem 1.1 and Remark 4.1] instead of [OP07, Corollary 3] to conclude that $\mathcal{F}(M) = \mathcal{F}(\mathcal{R})$.

Proof of Corollary F. (1) Let p be a prime and denote $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, $G = \mathrm{SL}_2(\mathbb{Z}_p)$, $\mathcal{R} = \mathcal{R}(\Gamma \curvearrowright G)$. Our goal is to show that $\mathrm{Out}(\mathcal{R})$ is isomorphic to a $(\mathbb{Z}/2\mathbb{Z})^2$-extension of $\mathrm{PSL}_2(\mathbb{Q}_p)$. To this end, let $K = \mathrm{SL}_2(\mathbb{Q}_p)$. Then K is a unimodular locally compact group, i.e. it admits a Haar measure m_K which is invariant under both left and right multiplication by elements of K.

Claim 1. *m_K is invariant under the conjugation action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on K.*

Proof of Claim 1. Recall that if we parametrize (a co-null subset of) $\mathrm{SL}_2(\mathbb{Q}_p)$ by

$$\left\{ \begin{pmatrix} x & y \\ z & \frac{yz+1}{x} \end{pmatrix} \;\middle|\; x \in \mathbb{Q}_p \setminus \{0\},\ y, z \in \mathbb{Q}_p \right\},$$

then, up to a multiplicative factor, m_K is given by the differential form $\frac{1}{x} dx \wedge dy \wedge dz$.

Now, let $a \in \mathbb{Q}_p\setminus\{0\}$ and set $\zeta = \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Since $\zeta \left(\begin{smallmatrix} x & y \\ z & t \end{smallmatrix}\right) \zeta^{-1} = \left(\begin{smallmatrix} x & ay \\ z/a & t \end{smallmatrix}\right)$, we deduce that m_K is invariant under conjugation with ζ. Since m_K is also invariant under conjugation with elements from $\mathrm{SL}_2(\mathbb{Q}_p)$, and every $\eta \in \mathrm{GL}_2(\mathbb{Q}_p)$ can be written as $\eta = \left(\begin{smallmatrix} \det\eta & 0 \\ 0 & 1 \end{smallmatrix}\right)\eta'$ with $\eta' \in \mathrm{SL}_2(\mathbb{Q}_p)$, the claim follows. □

Next, we let

$$H = \{h \in \mathrm{GL}_2(\mathbb{Q}_p) \mid \det h = \pm p^n \text{ for some } n \in \mathbb{Z}\}, \qquad \Lambda = H \cap \mathrm{GL}_2(\mathbb{Z}[1/p]).$$

Following S. Gefter [Ge96, Remark 2.8] and A. Furman [Fu03, proof of Theorem 1.6], we will define a homomorphism $\rho : H \to \mathrm{Out}(\mathcal{R})$. Fix $h \in H$. We claim that there are $\lambda \in \Lambda$ and $g \in G$ such that $h = g^{-1}\lambda$. Note that $\lambda_0 = \left(\begin{smallmatrix} \det h & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \Lambda$ and $h\lambda_0^{-1} \in K = \mathrm{SL}_2(\mathbb{Q}_p)$. Since $\mathrm{SL}_2(\mathbb{Z}[1/p])$ and G are dense and, respectively, open in K, we can find $\lambda_1 \in \mathrm{SL}_2(\mathbb{Z}[1/p])$ and $g \in G$ such that $\lambda_0^{-1}h = g^{-1}\lambda_1$. Since $\lambda_1\lambda_0 \in \Lambda$, the claim is proven.

Let $G' = \lambda^{-1}G\lambda \cap G$ and define $\sigma_h : G' \to G$ by letting $\sigma_h(x) = \lambda x h^{-1} = (\lambda x \lambda^{-1})g$. Then $G' < G$ is an open subgroup. Moreover, Claim 1 implies that $m_G(\sigma_h(A)) = m_G(A)$ for any Borel subset $A \subset G'$. Furthermore, since $\Lambda \cap G = \Gamma$, if $x, y \in G$,

then $\Gamma x = \Gamma y$ if and only if $\Lambda x = \Lambda y$. Also, if $x, y \in G'$ then $\Lambda x = \Lambda y$ if and only if $\Lambda\sigma_h(x) = \Lambda\sigma_h(y)$. Altogether, if $x, y \in G'$, then

$$\Gamma x = \Gamma y \;\Leftrightarrow\; \Gamma\sigma_h(x) = \Gamma\sigma_h(y).$$

This implies that σ_h extends to an automorphism of $\mathcal{R}$, which we still denote by σ_h.

We define $\rho_h = \varepsilon(\sigma_h)$, where $\varepsilon : \mathrm{Aut}(\mathcal{R}) \to \mathrm{Out}(\mathcal{R})$ is the quotient homomorphism. Note that ρ_h only depends on h, and not on the choices made in its definition. Furthermore, it is easy to see that $\rho : H \to \mathrm{Out}(\mathcal{R})$ is a homomorphism.

The rest of the proof is divided between two claims.

Claim 2. $\ker\rho = Z := \{\pm p^n I \mid n \in \mathbb{Z}\} \subset H$.

Proof of Claim 2. Let $h \in \ker\rho$. Let $\lambda \in \Lambda$ and $g \in G$ be such that $h = g^{-1}\lambda$. Since $\sigma_h \in [\mathcal{R}]$, we can find $\gamma \in \Gamma$ such that $A = \{x \in G \mid \gamma x = \lambda x h^{-1}\}$ has positive measure. Notice that $\lambda^{-1}\gamma$ commutes with xy^{-1} for all $x, y \in A$. Since A has positive measure, AA^{-1} contains an open subgroup of G. Thus $\lambda^{-1}\gamma \in \Lambda$ commutes with $\left(\begin{smallmatrix}1 & p^n\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & 0\\ p^n & 1\end{smallmatrix}\right)$ for some $n \geq 1$. Hence $\lambda^{-1}\gamma = cI$ for some $c \in \mathbb{Z}[1/p]$. Since $\det(\lambda^{-1}\gamma) \in \{\pm p^n \mid n \in \mathbb{Z}\}$, we get $c = \pm p^m$ for some $m \in \mathbb{Z}$. Since there is $x \in G$ such that $\gamma x = \lambda x h^{-1}$, finally $h = c^{-1}I = \pm p^{-m} I \in Z$. □

Claim 3. *ρ is surjective.*

Proof of Claim 3. Let $\theta \in \mathrm{Aut}(\mathcal{R})$. We will prove that $\varepsilon(\theta) = \rho_h$ for some $h \in H$.

Since the action $\Gamma \curvearrowright (G, m_G)$ has spectral gap, Corollary 4.3 implies that after composing θ with an element from $[\mathcal{R}]$, we can find open subgroups $G_0, G_1 < G$, a continuous isomorphism $\delta : G_0 \to G_1$, and $g \in G$ such that $\delta(\Gamma \cap G_0) = \Gamma \cap G_1$ and $\theta(x) = \delta(x)g^{-1}$ for almost every $x \in G_0$.

Next, by a result of R. Pink, since $G_0, G_1 < H = \mathrm{SL}_2(\mathbb{Q}_p)$ are compact open subgroups, δ extends to a continuous automorphism of H (see [Pi98, Corollary 0.3]). Since the field $\mathbb{Q}_p$ has no non-trivial automorphisms, there exists $\lambda \in \mathrm{GL}_2(\mathbb{Q}_p)$ such that $\delta(x) = \lambda x \lambda^{-1}$ for all $x \in G_0$.

As $\delta(\Gamma \cap G_0) = \Gamma \cap G_1$, we get $\lambda(\Gamma \cap G_0)\lambda^{-1} \subset \Gamma$. Hence $\lambda\,\mathrm{SL}_2(p^n\mathbb{Z})\lambda^{-1} \subset \Gamma$ for some $n \geq 1$. As the subring of $\mathbb{M}_2(\mathbb{Z})$ generated by $\mathrm{SL}_2(m\mathbb{Z})$ contains $m^2 M_2(\mathbb{Z})$ for every $m \in \mathbb{Z}$, we deduce that $p^{2n}\lambda\mathbb{M}_2(\mathbb{Z})\lambda^{-1} \subset \mathbb{M}_2(\mathbb{Z})$. If we write $\lambda = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$, where $a, b, c, d \in \mathbb{Q}_p$, then $xy \in p^{-2n}(\det\lambda)\mathbb{Z}$ for all $x, y \in \{a, b, c, d\}$. Thus, after replacing λ with $k\lambda$ for some $k \in \mathbb{Q}_p \setminus \{0\}$, we may assume that $a, b, c, d \in \mathbb{Z}$, i.e. $\lambda \in \mathrm{GL}_2(\mathbb{Q}_p) \cap \mathbb{M}_2(\mathbb{Z})$.

Moreover, we may assume that the greatest common divisor of a, b, c, d is 1. Since $a^2, b^2, c^2, d^2 \in p^{-2n}(\det\lambda)\mathbb{Z}$, we see that $\det\lambda \mid p^{2n}$, hence $\det\lambda = \pm p^m$ for some $m \in \mathbb{N}$. This shows that $\lambda \in \Lambda$. Since $\theta(x) = \lambda x \lambda^{-1} g$ for almost every $x \in G_0$, it follows that $\varepsilon(\theta) = \rho_h$, where $h = g^{-1}\lambda \in H$. □

Claims 2 and 3 imply that $\mathrm{Out}(\mathcal{R}) \cong H/Z$. Let $\pi : H \to H/Z$ be the quotient homomorphism. Then $\pi(\mathrm{SL}_2(\mathbb{Q}_p)) \cong \mathrm{PSL}_2(\mathbb{Q}_p)$ and $(H/Z)/\mathrm{PSL}_2(\mathbb{Q}_p) = \left\{\left(\begin{smallmatrix}1 & 0\\ 0 & x\end{smallmatrix}\right)\mathrm{PSL}_2(\mathbb{Q}_p) \;\middle|\; x = \pm 1, \pm p\right\}$. It is now easy to see that $(H/Z)/\mathrm{PSL}_2(\mathbb{Q}_p) \cong (\mathbb{Z}/2\mathbb{Z})^2$.

(2) Let $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ be a non-amenable subgroup. Denote by L the closure of Γ in $\mathrm{SL}_2(\mathbb{Z}_p)$, and let $\mathcal{S} = \mathcal{R}(\Gamma \curvearrowright L)$ and $M = L^\infty(L) \rtimes \Gamma$. Our goal is to show that $\mathcal{F}(\mathcal{S}) = \mathcal{F}(M) = \{1\}$.

By [OP07, Corollary 3] , M has a unique Cartan subalgebra, up to unitary conjugacy. Thus, $\mathcal{F}(M) = \mathcal{F}(\mathcal{S})$, and hence it suffices to show that $\mathcal{F}(\mathcal{S}) = \{1\}$.

Towards this goal, let $t \in \mathcal{F}(\mathcal{S})$. By Fact 7.2, $L < \mathrm{SL}_2(\mathbb{Z}_p)$ is an open subgroup and the action $\Gamma \curvearrowright (L, m_L)$ has spectral gap. By applying Corollary 4.3 we can find open subgroups $L_0, L_1 < L$ such that $t = [L : L_0]/[L : L_1]$ and a continuous homomorphism $\delta : L_0 \to L_1$.

Since $L_0, L_1 < \mathrm{SL}_2(\mathbb{Q}_p)$ are compact open subgroups, [Pi98, Corollary 0.3] yields $\lambda \in \mathrm{GL}_2(\mathbb{Q}_p)$ such that $\delta(x) = \lambda x \lambda^{-1}$ for all $x \in L_0$. As in the proof of (1), we denote by m_K the Haar measure of $K = \mathrm{SL}_2(\mathbb{Q}_p)$. Since m_K is invariant under the conjugation action of $\mathrm{GL}_2(\mathbb{Q}_p)$ by Claim 1, we deduce that $m_K(L_0) = m_K(\lambda L_0 \lambda^{-1}) = m_K(L_1)$.

Since $L < K$ is an open subgroup, we have $m_K(L) > 0$. Since L is a compact group, the uniqueness of the Haar measure of L implies that we can find a constant $c > 0$ such that $m_K(A) = c m_L(A)$ for any Borel subset $A \subset L$. Thus, $m_L(L_0) = m_L(L_1)$, or equivalently $[L : L_0]^{-1} = [L : L_1]^{-1}$. This shows that $t = 1$, and thus $\mathcal{F}(\mathcal{S}) = \{1\}$. □

10. Proofs of Corollaries G and H

In this section we use Theorem 6.1 to calculate the outer automorphism groups of equivalence relations arising from the natural actions of rather general countable subgroups $\Gamma < \mathrm{SO}(n+1)$ on S^n and $P^n(\mathbb{R})$. In particular, we derive Corollaries G and H, leading to examples of treeable equivalence relations with trivial outer automorphism group.

We start by fixing some notation:

Notation 10.1. Let $n \geq 2$.

- We denote by λ_n the Lebesgue probability measure on the n-dimensional sphere S^n, and consider the p.m.p. action $\mathrm{SO}(n+1) \curvearrowright (S^n, \lambda_n)$.
- We denote by μ_n the probability measure on the n-dimensional real projective space $P^n(\mathbb{R})$ obtained by pushing forward λ_n through the quotient map $S^n \to P^n(\mathbb{R}) : \xi \mapsto [\xi]$.
- We consider the action $\mathbb{Z}/2\mathbb{Z} \curvearrowright (S^n, \lambda_n)$ given by the involution $T(x) = -x$. Then $(S^n, \lambda_n)/(\mathbb{Z}/2\mathbb{Z}) = (P^n(\mathbb{R}), \mu_n)$. Moreover, since the actions of $\mathbb{Z}/2\mathbb{Z}$ and $\mathrm{SO}(n+1)$ on S^n commute, we have a p.m.p. action $\mathrm{SO}(n+1) \curvearrowright (P^n(\mathbb{R}), \mu_n)$.

Theorem 10.2. *Let $\Gamma < G := \mathrm{SO}(n+1)$ be a countable icc dense subgroup for some $n \geq 2$. Assume that the left translation action $\Gamma \curvearrowright (G, m_G)$ has spectral gap.*

(1) *Let $\mathcal{R} = \mathcal{R}(\Gamma \curvearrowright (G, m_G))$. Then:*

  (a) $\mathcal{F}(\mathcal{R}) = \{1\}$.
  (b) *If n is even, then* $\mathrm{Out}(\mathcal{R}) \cong N_G(\Gamma)/\Gamma \times G$.
  (c) *If n is odd, then* $\mathrm{Out}(\mathcal{R}) \cong (N_G(\Gamma) \times G)/\tilde{\Gamma}$, *where* $\tilde{\Gamma} = \{(\alpha\gamma, \alpha I) \mid \gamma \in \Gamma, \alpha = \pm 1\}$ *and $I \in G$ is the identity matrix.*

(2) *Let* $\mathcal{S} = \mathcal{R}(\Gamma \curvearrowright (S^n, \lambda_n))$. *Then:*
   (a) $\mathcal{F}(\mathcal{S}) = \{1\}$.
   (b) *If* n *is even, then* $\mathrm{Out}(\mathcal{S}) \cong N_G(\Gamma)/\Gamma \times (\mathbb{Z}/2\mathbb{Z})$.
   (c) *If* n *is odd, then* $\mathrm{Out}(\mathcal{S}) \cong (N_G(\Gamma) \times \tilde{K})/K_0$, *where* $K = \left\{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right) \mid a \in \mathrm{SO}(n) \right\}$, $\tilde{K} = \left\{ \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha a \end{smallmatrix}\right) \mid a \in \mathrm{SO}(n),\ \alpha = \pm 1 \right\}$, *and* $K_0 = \{(\alpha\gamma, \alpha k) \mid \alpha = \pm 1,\ \gamma \in \Gamma,\ k \in K\}$.

(3) *Let* $\mathcal{T} = \mathcal{R}(\Gamma \curvearrowright (P^n(\mathbb{R}), \mu_n))$ *and assume that* $-I \notin \Gamma$. *Then:*
   (a) $\mathcal{F}(\mathcal{T}) = \{1\}$.
   (b) *If* n *is even, then* $\mathrm{Out}(\mathcal{T}) \cong N_G(\Gamma)/\Gamma$.
   (c) *If* n *is odd, then* $\mathrm{Out}(\mathcal{T}) \cong N_G(\Gamma)/\bar{\Gamma}$, *where* $\bar{\Gamma} = \{\alpha\gamma \mid \alpha = \pm 1, \gamma \in \Gamma\}$.

Remark 10.3. J. Bourgain and A. Gamburd proved that if $\Gamma < H = \mathrm{SU}(2)$ is a dense subgroup generated by finitely many matrices $\{g_1, \ldots, g_k\}$ having algebraic entries, then the representation $\pi : \Gamma \to \mathcal{U}(L^2(H, m_H))$ (equivalently, the action $\Gamma \curvearrowright (H, m_H)$) has spectral gap [BG06]. More recently, they proved that this result holds for $H = \mathrm{SU}(n)$ whenever $n \geq 2$ [BG11]. In particular, if $\Gamma < G = \mathrm{SO}(3)$ is any dense subgroup which is generated by finitely many matrices with algebraic entries, then the action $\Gamma \curvearrowright (G, m_G)$ has spectral gap. This provides a large family of actions to which the above theorem applies.

Proof of Theorem 10.2. Let us first record a fact that we will use repeatedly:

($\star$) $\Gamma \curvearrowright (G, m_G)$ has spectral gap, G is connected, $\pi_1(G) \cong \mathbb{Z}/2\mathbb{Z}$ is finite, and Γ is icc, hence has no non-trivial normal subgroups.

By using ($\star$) and applying Theorem 6.1 to $H = G$, $K = \{e\}$, $L = \{e\}$, we see that $\mathcal{F}(\mathcal{R}) = \{1\}$. Moreover, Theorem 6.1 shows that if $\theta \in \mathrm{Aut}(\mathcal{R})$, then after composing θ with an element of $[\mathcal{R}]$, we can find an automorphism $\delta : G \to G$ with $\delta(\Gamma) = \Gamma$, and $y \in G$, such that $\theta(g) = \delta(g)y$ for almost every $g \in G$. Since G has no outer automorphisms, $\delta(g) = zgz^{-1}$ for some $z \in N_G(\Gamma)$. Therefore, if $\theta : N_G(\Gamma) \times G \to \mathrm{Aut}(\mathcal{R})$ denotes the homomorphism defined by $\theta_{z,w}(g) = zgw$, and $\varepsilon : \mathrm{Aut}(\mathcal{R}) \to \mathrm{Out}(\mathcal{R})$ denotes the natural quotient homomorphism, then $\varepsilon \circ \theta$ is surjective.

Let $z \in N_G(\Gamma)$ and $w \in G$ be such that $\theta_{z,w} \in [\mathcal{R}]$. Let $\gamma \in \Gamma$ be such that $A = \{g \in G \mid zgw = \gamma g\}$ has positive measure. Then $zgh^{-1}z^{-1} = \gamma gh^{-1}\gamma^{-1}$ for all $g, h \in A$. Hence $\gamma^{-1}z$ commutes with AA^{-1} and further with $G_0 := \bigcup_{n \geq 1} (AA^{-1})^n$. Since A has positive measure, AA^{-1} contains a neighborhood of $I \in G$. Thus, G_0 is an open subgroup of G. Since G is connected, we deduce that $G_0 = G$, and therefore $\gamma^{-1}z$ must be in the center of G.

If n is even, then $Z(G) = \{I\}$, and hence $z = \gamma$. Since there is $g \in G$ such that $zgw = \gamma g$, we further get $w = 1$. This shows that the kernel of $\varepsilon \circ \theta$ is $\Gamma \times \{I\}$, hence $\mathrm{Out}(\mathcal{R}) \cong N_G(\Gamma)/\Gamma \times G$. If n is odd, then $Z(G) = \{\pm I\}$. It follows that $z = \alpha\gamma$ and $w = \alpha$ for $\alpha \in \{\pm I\}$. Thus $\ker(\varepsilon \circ \theta) = \tilde{\Gamma}$, hence $\mathrm{Out}(\mathcal{R}) \cong (N_G(\Gamma) \times G)/\tilde{\Gamma}$.

This finishes the proof of (1). We now turn to the proofs of (2) and (3).

We begin by giving an alternative description of the actions of $G = \mathrm{SO}(n+1)$ on S^n and $P^n(\mathbb{R})$. Let $\xi = (1, 0, \ldots, 0) \in S^n$. Since $G \curvearrowright S^n$ is transitive and $\mathrm{Stab}_G(\xi) = K$, it follows that $G \curvearrowright (S^n, \lambda_n)$ is isomorphic to $G \curvearrowright (G/K, m_{G/K})$.

Also, notice that $\tilde{K} = \{g \in G \mid g\xi = \pm\xi\}$ and $K < \tilde{K}$ is an open normal subgroup of index 2. Since the action $G \curvearrowright P^n(\mathbb{R})$ is transitive and $\mathrm{Stab}_G([\xi]) = \tilde{K}$, it follows that $G \curvearrowright (P^n(\mathbb{R}), \mu_n)$ is isomorphic to $G \curvearrowright (G/\tilde{K}, m_{G/\tilde{K}})$.

We are now ready to calculate $\mathcal{F}(\mathcal{S})$ and $\mathrm{Out}(\mathcal{S})$. Note that $(\star)$ ensures that conditions (1)–(3) of Theorem 6.1 are satisfied for $\Lambda = \Gamma$, $H = G$, and $L = K$. Since (4) is also satisfied by Claim 2, we can apply Theorem 6.1 to deduce that $\mathcal{F}(\mathcal{S}) = \{1\}$. Moreover, if $\theta \in \mathrm{Aut}(\mathcal{S})$, then after composing θ with an element from $[\mathcal{S}]$, we can find an automorphism $\delta : G \to G$ and $y \in G$ such that $\delta(\Gamma) = \Gamma$, $\delta(K) = yKy^{-1}$, and $\theta(gK) = \delta(g)yK$ for almost every $g \in G$.

Since G has no outer automorphisms, we can find $z \in G$ such that $\delta(g) = zgz^{-1}$ for all $g \in G$. Thus, if we set $w = z^{-1}y$, then $z \in N_G(\Gamma)$, $w \in N_G(K)$, and $\theta(gK) = zgwK$ for almost every $g \in G$. Since K stabilizes $\xi \in S^n$ and w normalizes K, it follows that K stabilizes $w\xi \in S^n$. This easily implies that $w\xi = \pm\xi$, i.e. $w \in \tilde{K}$.

Consider the well-defined homomorphism $\theta : N_G(\Gamma) \times \tilde{K} \to \mathrm{Aut}(\mathcal{S})$ given by $\theta_{z,w}(gK) = zgwK$. Then the above shows that $\varepsilon \circ \theta : N_G(\Gamma) \times \tilde{K} \to \mathrm{Out}(\mathcal{S})$ is surjective, where $\varepsilon : \mathrm{Aut}(\mathcal{S}) \to \mathrm{Out}(\mathcal{S})$ denotes the quotient homomorphism.

Let $z \in N_G(\Gamma)$ and $w \in \tilde{K}$ be such that $\theta_{z,w} \in [\mathcal{S}]$. Then we can find $\gamma \in \Gamma$ such that $A = \{gK \in G/K \mid zgwK = \gamma gK\}$ has positive measure. Let $\alpha \in \{\pm 1\}$ be such that $w\xi = \alpha\xi$. Then for every $gK \in A$ we have $\alpha z(g\xi) = \gamma(g\xi)$. Since $m_{G/K}(A) > 0$, Claim 1 implies that $\alpha z = \gamma$. Since there exists $gK \in G/K$ such that $zgwK = \gamma gK$, we deduce that $wK = \alpha K$, and hence $\alpha I \in G$.

If n is even, then $-I \notin G$, which forces $\alpha = 1$. Hence $w \in K$ and $z = \gamma \in \Gamma$. Therefore $\ker(\varepsilon \circ \theta) = \Gamma \times K$ and thus $\mathrm{Out}(\mathcal{S}) \cong (N_G(\Gamma) \times \tilde{K})/(\Gamma \times K) \cong N_G(\Gamma)/\Gamma \times (\mathbb{Z}/2\mathbb{Z})$. If n is odd, then $-I \in G$, and so $\ker(\varepsilon \circ \theta) = K_0$. This finishes the proof of (2).

Finally, we compute $\mathcal{F}(\mathcal{T})$ and $\mathrm{Out}(\mathcal{T})$. Recall that we may identify $\mathcal{T} = \mathcal{R}(\Gamma \curvearrowright G/\tilde{K})$. By $(\star)$, conditions (1)–(3) of Theorem 6.1 are satisfied for $H = G$, K and L both equal to $\tilde{K}$, and $\Lambda = \Gamma$. Since $-I \notin \Gamma$, Claim 2 shows that condition (4) of Theorem 6.1 also holds.

Hence $\mathcal{F}(\mathcal{T}) = \{1\}$. Moreover, if $\theta \in \mathrm{Aut}(\mathcal{S})$, then after composing θ with an element from $[\mathcal{S}]$, we can find $z \in N_G(\Gamma)$ and $w \in N_G(\tilde{K})$ such that $\theta(g\tilde{K}) = zgw\tilde{K}$ for almost every $g \in G$. Since $\tilde{K}$ stabilizes $[\xi] \in P^n(\mathbb{R})$ and w normalizes $\tilde{K}$, we see that $\tilde{K}$ stabilizes $[w\xi] \in P^n(\mathbb{R})$. Thus $[w\xi] = [\xi]$, hence $w \in \tilde{K}$. As a consequence, $\theta(g\tilde{K}) = zg\tilde{K}$ for almost every $g \in G$.

Define $\theta : N_G(\Gamma) \to \mathrm{Aut}(\mathcal{T})$ by letting $\theta_z(g\tilde{K}) = zg\tilde{K}$. If $\varepsilon : \mathrm{Aut}(\mathcal{T}) \to \mathrm{Out}(\mathcal{T})$ denotes the quotient homomorphism, then the above implies that $\varepsilon \circ \theta$ is surjective.

Let $z \in N_G(\Gamma)$ be such that $\theta_z \in [\mathcal{T}]$. Then $A = \{g\tilde{K} \in G/\tilde{K} \mid zg\tilde{K} = \gamma g\tilde{K}\}$ has positive measure for some $\gamma \in \Gamma$. If $g\tilde{K} \in A$, then $z(g\xi) = \pm\gamma(g\xi)$. By using Claim 1, we get $z = \alpha\gamma$ for $\alpha \in \{\pm 1\}$.

If n is even, then $-I \notin G$, so we must have $\alpha = 1$, hence $z = \gamma \in \Gamma$. This shows that $\ker\theta = \Gamma$, and therefore $\mathrm{Out}(\mathcal{T}) \cong N_G(\Gamma)/\Gamma$. If n is odd, then $-I \in G$ and we get $\ker(\varepsilon \circ \theta) \subset \bar{\Gamma}$. Since also $\bar{\Gamma} \subset \ker(\varepsilon \circ \theta)$, we deduce that $\mathrm{Out}(\mathcal{T}) \cong N_G(\Gamma)/\bar{\Gamma}$. □

10.1. Proof of Corollary G

Let $\Gamma < G$ be a subgroup which contains matrices $\{g_1, \dots, g_k\}$ with algebraic entries such that the subgroup Γ_0 of Γ generated by the $\{g_1, \dots, g_k\}$ is dense in G. Let us briefly explain how [BG06] implies that $\Gamma \curvearrowright (G, m_G)$ has spectral gap.

Recall that there is a surjective continuous homomorphism $\Phi : H = \mathrm{SU}(2) \to G = \mathrm{SO}(3)$ whose kernel is $\{\pm I\}$ (see e.g. [Ha03, Section 1.6.1]). More precisely, Φ is given by

$$\Phi\left(\begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix}\right) = \begin{pmatrix} \Re(x^2 - y^2) & \Im(x^2 + y^2) & -2\,\Re(xy) \\ -\Im(x^2 - y^2) & \Re(x^2 + y^2) & 2\,\Im(xy) \\ 2\,\Re(x\bar{y}) & 2\,\Im(x\bar{y}) & |x|^2 - |y|^2 \end{pmatrix} \quad \text{for all } x, y \in \mathbb{C} \text{ with } |x|^2 + |y|^2 = 1.$$

Let $h_1, \dots, h_k \in H$ be such that $\Phi(h_1) = g_1, \dots, \Phi(h_k) = g_k$, and denote by $\Lambda < H$ the subgroup generated by $\{h_1, \dots, h_k\}$. Since Γ is dense in G, the closure $H_0 = \bar{\Lambda} < H$ satisfies $\Phi(H_0) = G$. Since Φ is a 2-1 map, we get $[H : H_0] \leq 2$. Since H is connected, we must have $H_0 = H$, hence Λ is dense in H. Moreover, since the entries of $g_1, \dots, g_k$ are algebraic, the above formula for Φ implies that the entries of $h_1, \dots, h_k$ are also algebraic.

By a result of J. Bourgain and A. Gamburd [BG06, Theorem 1], since $\Lambda < H$ is a dense subgroup generated by the matrices $\{h_1, \dots, h_k\}$ with algebraic entries, the action $\Lambda \curvearrowright (H, m_H)$ has spectral gap. This readily implies that $\Gamma \curvearrowright (G, m_G)$ has spectral gap.

Corollary G now follows directly from Theorem 10.2. □

We continue by proving the following more general form of Corollary H.

Corollary 10.4. *Let $p, q \geq 3$ be natural numbers. Denote by α_p the rotation about the x-axis by angle $2\pi/p$, and by β_q the rotation about the z-axis by angle $2\pi/q$, i.e.*

$$\alpha_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{2\pi}{p} & -\sin\frac{2\pi}{p} \\ 0 & \sin\frac{2\pi}{p} & \cos\frac{2\pi}{p} \end{pmatrix} \quad \text{and} \quad \beta_q = \begin{pmatrix} \cos\frac{2\pi}{q} & -\sin\frac{2\pi}{q} & 0 \\ \sin\frac{2\pi}{q} & \cos\frac{2\pi}{q} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Denote by $\Gamma = \Gamma(p, q)$ the subgroup of $G = \mathrm{SO}(3)$ generated by α_p and β_q.

(1) *If p, q are odd and $p \neq q$, then $\Gamma \cong (\mathbb{Z}/p\mathbb{Z}) * (\mathbb{Z}/q\mathbb{Z})$, $\mathrm{Out}(\mathcal{R}(\Gamma \curvearrowright G)) \cong (\mathbb{Z}/2\mathbb{Z})^3 \times G$, $\mathrm{Out}(\mathcal{R}(\Gamma \curvearrowright S^2)) \cong (\mathbb{Z}/2\mathbb{Z})^3$, and $\mathrm{Out}(\mathcal{R}(\Gamma \curvearrowright P^2(\mathbb{R})) \cong (\mathbb{Z}/2\mathbb{Z})^2$.*

(2) *If $p \geq 4$ is even and $q \geq 3$ is odd, then $\Gamma \cong (\mathbb{Z}/p\mathbb{Z}) *_{\mathbb{Z}/2\mathbb{Z}} D_q$, $\mathrm{Out}(\mathcal{R}(\Gamma \curvearrowright G)) \cong (\mathbb{Z}/2\mathbb{Z})^2 \times G$, $\mathrm{Out}(\mathcal{R}(\Gamma \curvearrowright S^2)) \cong (\mathbb{Z}/2\mathbb{Z})^2$, and $\mathrm{Out}(\mathcal{R}(\Gamma \curvearrowright P^2(\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$.*

(3) *If $p \geq 4$ is even, $q = 2s$, $s \geq 3$ odd, and $p \neq q$, then $\Gamma \cong D_p *_{D_2} D_q$ and $\mathrm{Out}(\mathcal{R}(\Gamma \curvearrowright G)) \cong (\mathbb{Z}/2\mathbb{Z}) \times G$, $\mathrm{Out}(\mathcal{R}(\Gamma \curvearrowright S^2)) \cong \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{Out}(\mathcal{R}(\Gamma \curvearrowright P^2(\mathbb{R})) \cong \{e\}$.*

Here, D_n denotes the dihedral group with $2n$ elements. More precisely, D_n is the semi-direct product $D_n = (\mathbb{Z}/n\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z}$ associated with the order two automorphism $x \mapsto -x$ of $\mathbb{Z}/n\mathbb{Z}$.

The isomorphisms between Γ and the corresponding amalgamated free product groups are due to C. Radin and L. Sadun [RS98, Corollary 2]. Notice that they imply that Γ is icc.

Proof of Corollary 10.4. Note that α_p, β_q have algebraic entries, and $\Gamma < G = \mathrm{SO}(3)$ is dense (since it contains a copy of $\mathbb{F}_2$). Corollary G implies $\mathrm{Out}(\mathcal{R}(\Gamma \curvearrowright G)) \cong (N_G(\Gamma)/\Gamma) \times G$, $\mathrm{Out}(\mathcal{R}(\Gamma \curvearrowright S^2)) \cong N_G(\Gamma)/\Gamma \times (\mathbb{Z}/2\mathbb{Z})$, and $\mathrm{Out}(\mathcal{R}(\Gamma \curvearrowright P^2(\mathbb{R})) \cong N_G(\Gamma)/\Gamma$.

In the rest of the proof, we calculate $N_G(\Gamma)$ in each of the three cases. Towards this, we let $g \in N_G(\Gamma)$ and denote by ρ the automorphism of Γ given by $\rho(x) = gxg^{-1}$. Also, we define

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(1) Assume that p, q are odd and $p \neq q$. Denote $\Gamma_1 = \langle \alpha_p \rangle \cong \mathbb{Z}/p\mathbb{Z}$ and $\Gamma_2 = \langle \beta_q \rangle \cong \mathbb{Z}/q\mathbb{Z}$. Then $\Gamma = \Gamma_1 * \Gamma_2$ by [RS98, Corollary 2]. The Kurosh subgroup theorem implies that we can find $h_1, h_2 \in \Gamma$ and $i_1, i_2 \in \{1, 2\}$ such that $\rho(\Gamma_1) \subset h_1 \Gamma_{i_1} h_1^{-1}$ and $\rho(\Gamma_2) \subset h_2 \Gamma_{i_2} h_2^{-1}$. Since Γ_i and $h\Gamma_i h^{-1}$ cannot generate Γ for any $h \in \Gamma$ and $i \in \{1, 2\}$, we conclude that $i_1 \neq i_2$. Since $p \neq q$, we must have $i_1 = 1$ and $i_2 = 2$. It follows that $\rho(\Gamma_1) = h_1 \Gamma_1 h_1^{-1}$ and $\rho(\Gamma_2) = h_2 \Gamma_2 h_2^{-1}$.

Since $h_1 \Gamma_1 h_1^{-1}$ and $h_2 \Gamma_2 h_2^{-1}$ generate Γ, we can find $k_1 \in \Gamma_1$ and $k_2 \in \Gamma_2$ such that $h_2^{-1} h_1 = k_2^{-1} k_1$. Denote $l = h_1 k_1^{-1} = h_2 k_2^{-1} \in \Gamma$. Then $\rho(\Gamma_1) = l \Gamma_1 l^{-1}$ and $\rho(\Gamma_2) = l \Gamma_2 l^{-1}$.

Therefore, $l^{-1} g \in N_G(\Gamma_1) \cap N_G(\Gamma_2)$. It is easy to see that

$$N_G(\Gamma_1) = \left\{ \begin{pmatrix} \det A & 0 \\ 0 & A \end{pmatrix} \,\middle|\, A \in \mathrm{O}(2) \right\} \quad \text{and} \quad N_G(\Gamma_1) = \left\{ \begin{pmatrix} A & 0 \\ 0 & \det A \end{pmatrix} \,\middle|\, A \in \mathrm{O}(2)) \right\}. \tag{10.1}$$

If we let $D < G$ be the subgroup consisting of diagonal matrices, then (10.1) implies that $D = N_G(\Gamma_1) \cap N_G(\Gamma_2)$. Hence $l^{-1} g \in D$, showing that $g \in \langle \Gamma, D \rangle$. In conclusion, $N_G(\Gamma) = \langle \Gamma, D \rangle$.

Finally, notice that $D = \{I, a, b, ab\}$, $a = \alpha_{2p}^p$, $b = \beta_{2q}^q$, $\alpha_p = \alpha_{2p}^2$, $\beta_q = \beta_{2q}^2$. Therefore, $\langle \Gamma, D \rangle = \Gamma(2p, 2q)$. Moreover, [RS98, Corollary 2] shows that $\Gamma(2p, 2q) = \langle \alpha_p, b \rangle *_{\langle a,b \rangle} \langle \beta_q, a \rangle$. Since $\Gamma = \Gamma(p, q) = \langle \alpha_p \rangle * \langle \beta_q \rangle$, it follows that $N_G(\Gamma)/\Gamma = \Gamma(2p, 2q)/\Gamma(p, q) \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.

(2) Assume that $p \geq 4$ is even and $q \geq 3$ is odd. Note that $\alpha_p^{p/2} = a$ normalizes the cyclic group $\langle \beta_q \rangle$. Denote $\Gamma_1 = \langle \alpha_p \rangle \cong \mathbb{Z}/p\mathbb{Z}$, $\Gamma_2 = \langle \beta_q, a \rangle \cong D_q$, and $\Lambda = \langle a \rangle \cong \mathbb{Z}/2\mathbb{Z}$. Then by [RS98, Corollary 2] we have $\Gamma = \Gamma_1 *_\Lambda \Gamma_2$.

Since $\Gamma_1 \not\cong \Gamma_2$, by reasoning as in the proof of (1) we can find $l \in \Gamma$ such that $\rho(\Gamma_1) = l \Gamma_1 l^{-1}$ and $\rho(\Gamma_2) = l \Gamma_2 l^{-1}$. Hence $l^{-1} g \in N_G(\Gamma_1) \cap N_G(\Gamma_2)$. It is easy to see that $N_G(\Gamma_1) \cap N_G(\Gamma_2) = D$, which implies that $g \in \langle \Gamma, D \rangle$, and hence $N_G(\Gamma) = \langle \Gamma, D \rangle$.

Since $D = \{I, a, b, ab\}$, $a = \alpha_p^{p/2} \in \Gamma$, $b = \beta_{2q}^q$ and $\beta_q = \beta_{2q}^2$, we deduce that $\langle \Gamma, D \rangle = \Gamma(p, 2q)$. By [RS98, Corollary 2] we have $\Gamma(p, 2q) = \langle \alpha_p, b \rangle *_{\langle a,b \rangle} \langle \beta_{2q}, a \rangle$.

Since $\Gamma(p,q) = \langle \alpha_p \rangle *_{\langle a \rangle} \langle \beta_{2q}^2, a \rangle$, we conclude that $N_G(\Gamma)/\Gamma = \Gamma(p,2q)/\Gamma(p,q) \cong \mathbb{Z}/2\mathbb{Z}$.

(3) Assume that $p \geq 4$ is even, $q = 2s$, s odd, and $p \neq q$. Note that $\alpha_p^{p/2} = a$ normalizes the cyclic group $\langle \beta_q \rangle$, and $\beta_q^{q/2}$ normalizes the cyclic group $\langle \alpha_p \rangle$. If we denote $\Gamma_1 = \langle \alpha_p, b \rangle \cong D_p$, $\Gamma_2 = \langle \beta_q, a \rangle \cong D_q$, and $\Lambda = \langle a, b \rangle \cong (\mathbb{Z}/2\mathbb{Z})^2$, then [RS98, Corollary 2] yields $\Gamma = \Gamma_1 *_\Lambda \Gamma_2$.

Since $\Gamma_1 \ncong \Gamma_2$, by reasoning as in the proof of (1) we can find $l \in \Gamma$ such that $\rho(\Gamma_1) = l\Gamma_1 l^{-1}$ and $\rho(\Gamma_2) = l\Gamma_2 l^{-1}$. Hence $l^{-1}g \in N_G(\Gamma_1) \cap N_G(\Gamma_2)$. It is easy to see that $N_G(\Gamma_1) \cap N_G(\Gamma_2) = D$, which implies that $g \in \langle \Gamma, D \rangle$, and hence $N_G(\Gamma) = \langle \Gamma, D \rangle$. Since $D \subset \Gamma$, we get $N_G(\Gamma) = \Gamma$. □

Finally, let us note the the actions $\Gamma(p,q) \curvearrowright G$ are neither stably orbit equivalent nor Borel reducible to each other, for varying values of (p,q). More generally, we have:

Corollary 10.5. *Let Γ, Λ be countable icc dense subgroups of $G = \mathrm{SO}(3)$. Assume that Γ contains matrices $g_1, \ldots, g_k$ which have algebraic entries and generate a dense subgroup of G. Then:*

(1) *The actions $\Gamma \curvearrowright (G, m_G)$ and $\Lambda \curvearrowright (G, m_G)$ are stably orbit equivalent if and only if there exists $g \in G$ such that $g\Gamma g^{-1} = \Lambda$.*
(2) *$\mathcal{R}(\Gamma \curvearrowright G) \leq_B \mathcal{R}(\Lambda \curvearrowright G)$ if and only if there exists $g \in G$ such that $g\Gamma g^{-1} = \Lambda$.*
(3) *There exists a non-trivial homomorphism from $\mathcal{R}(\Gamma \curvearrowright G)$ to $\mathcal{R}(\Lambda \curvearrowright G)$ if and only if there exists $g \in G$ such that $g\Gamma g^{-1} \subset \Lambda$.*

Moreover, let $p, q, r, s \geq 3$ be integers such that $(p,q) \neq (r,s), (p,q) \neq (s,r)$, and $(p,q), (r,s) \notin (4\mathbb{Z})^2$. Then the actions $\Gamma(p,q) \curvearrowright (G, m_G)$ and $\Gamma(r,s) \curvearrowright (G, m_G)$ are not stably orbit equivalent, and the equivalence relations $\mathcal{R}(\Gamma(p,q) \curvearrowright G)$ and $\mathcal{R}(\Gamma(r,s) \curvearrowright G)$ are not comparable with respect to Borel reducibility.

Proof. As in the proof of Corollary G, the main result of [BG06] implies that the action $\Gamma \curvearrowright (G, m_G)$ has spectral gap. We claim that if $\delta : G \to H/\Delta$ is a non-trivial continuous homomorphism, where $H < G$ is a subgroup and $\Delta < Z(H)$ is a subgroup, then $H = G$, $\Delta = \{e\}$, and there exists $g \in G$ such that $\delta(x) = gxg^{-1}$ for all $x \in G$. This is because G is a simple group, any proper subgroup $H \lneq G$ has dimension strictly smaller than G, and G has no outer automorphisms.

By Corollaries 6.3 and 4.7, assertions (1)–(3) follow immediately.

To see the "moreover" assertion, note that the assumptions made and [RS98, Corollary 2] imply that $\Gamma(p,q), \Gamma(r,s)$ are non-isomorphic icc groups. By applying (1) and (2) we get the conclusion. □

Remark 10.6. In [Sw94], S. Świerczkowski exhibits an interesting family of embeddings of $\mathbb{F}_2$ into $G = \mathrm{SO}(3)$. Let a, b be integers such that $b > 0$ and $|a| \leq b$. Set $c = b^2 - a^2$ and define Γ to be the subgroup of G generated by the following rotation matrices:

$$A = \begin{pmatrix} a/b & -\sqrt{c}/b & 0 \\ \sqrt{c}/b & a/b & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a/b & -\sqrt{c}/b \\ 0 & \sqrt{c}/b & a/b \end{pmatrix}.$$

The main result of [Sw94] asserts that if $a/b \notin \{0, \pm 1/2, \pm 1\}$, then $\Gamma = \langle A \rangle * \langle B \rangle \cong \mathbb{F}_2$. Thus, if $a/b \notin \{0, \pm 1/2, \pm 1\}$, then since the entries of A and B are algebraic, Corollary G applies to Γ and, more generally, to any non-cyclic subgroup $\Gamma_0 < \Gamma$. Therefore, the calculation of the outer automorphism groups of the equivalence relations associated to the actions of Γ on G, S^2, $P^2(\mathbb{R})$ reduces to the calculation of $N_G(\Gamma)$. However, we have been unable to compute $N_G(\Gamma)$ or, more generally, $N_G(\Gamma_0)$.

Acknowledgments. I would like to thank Sorin Popa and Stefaan Vaes for many helpful comments, and the two referees for suggestions that helped improve the exposition.

The author was partially supported by NSF Grant DMS #1161047, NSF Career Grant DMS #1253402, and a Sloan Foundation Fellowship.

References

[AE10] Abért, M., Elek, G.: Dynamical properties of profinite actions. Ergodic Theory Dynam. Systems **32**, 1805–1835 (2012) Zbl 1297.37004 MR 2995875

[BK96] Becker, H., Kechris, A. S.: The Descriptive Set Theory of Polish Group Actions. London Math. Soc. Lecture Note Ser. 232, Cambridge Univ. Press, Cambridge (1996) Zbl 0949.54052 MR 1425877

[BG05] Bourgain, J., Gamburd, A.: Uniform expansion bounds for Cayley graphs of $\mathrm{SL}_2(\mathbb{F}_p)$. Ann. of Math. (2) **167**, 625–642 (2008) Zbl 1216.20042 MR 2415383

[BG06] Bourgain, J., Gamburd, A.: On the spectral gap for finitely-generated subgroups of SU(2). Invent. Math. **171**, 83–121 (2008) Zbl 1135.22010 MR 2358056

[BG11] Bourgain, J., Gamburd, A.: A spectral gap theorem in $\mathrm{SU}(d)$. J. Eur. Math. Soc. **14**, 1455–1511 (2012) Zbl 1254.43010 MR 2966656

[BV10] Bourgain, J., Varjú, P.: Expansion in $\mathrm{SL}_d(\mathbb{Z}/q\mathbb{Z})$, q arbitrary. Invent. Math. **188**, 151–173 (2012) Zbl 1247.20052 MR 2897695

[Bo09a] Bowen, L.: Orbit equivalence, coinduced actions and free products. Groups Geom. Dynam. **5**, 1–15 (2011) Zbl 1257.37004 MR 2763777

[Bo09b] Bowen, L.: Stable orbit equivalence of Bernoulli shifts over free groups. Groups Geom. Dynam. **5**, 17–38 (2011) Zbl 1259.37002 MR 2763778

[FM77] Feldman, J., Moore, C. C.: Ergodic equivalence relations, cohomology, and von Neumann algebras, I, II. Trans. Amer. Math. Soc. **234**, 289–324, 325–359 (1977) Zbl 0369.22010 MR 0578656(I) MR 0578730(II)

[Fu03] Furman, A.: Outer automorphism groups of some ergodic equivalence relations. Comment. Math. Helv. **80**, 157–196 (2005) Zbl 1067.37005 MR 2130572

[Fu09] Furman, A.: A survey of Measured Group Theory. In: Geometry, Rigidity, and Group Actions, Univ. of Chicago Press, Chicago, 296–374 (2011) Zbl 1267.37004 MR 2807836

[Ga99] Gaboriau, D.: Coût des relations d'équivalence et des groupes. Invent. Math. **139**, 41–98 (2000) Zbl 0939.28012 MR 1728876

[Ga01] Gaboriau, D.: Invariants ℓ^2 de relations d'éequivalence et de groupes. Publ. Math. Inst. Hautes Études Sci. **95**, 93–150 (2002) Zbl 1022.37002 MR 1953191

[Ga08] Gaboriau, D.: Relative property (T) actions and trivial outer automorphism groups. J. Funct. Anal. **260**, 414–427 (2011) Zbl 1211.46071 MR 2737406

[Ga10] Gaboriau, D.: Orbit equivalence and measured group theory. In: Proc. ICM (Hyderabad, 2010), Vol. III, Hindustan Book Agency, 1501–1527 (2010) Zbl 1259.37003 MR 2827853

[GP03] Gaboriau, D., Popa, S.: An uncountable family of non-orbit equivalent actions of $\mathbb{F}_n$. J. Amer. Math. Soc. **18**, 547–559 (2005) Zbl 1155.37302 MR 2138136

[Gam02] Gamburd, A: On the spectral gap for infinite index "congruence" subgroups of $\mathrm{SL}_2(\mathbb{Z})$. Israel J. Math. **127**, 157–200 (2002) Zbl 1009.22018 MR 1900698

[Ge96] Gefter, S. L.: Outer automorphism group of the ergodic equivalence relation generated by translations of dense subgroup of compact group on its homogeneous space. Publ. RIMS Kyoto Univ. **32**, 517–538 (1996) Zbl 0881.28014 MR 1409801

[GG88] Gefter, S. L., Golodets, V. Y.: Fundamental groups for ergodic actions and actions with unit fundamental groups. Publ. RIMS Kyoto Univ. **24**, 821–847 (1988) Zbl 0684.22003 MR 1000122

[Ha03] Hall, B. C.: Lie Groups, Lie Algebras, and Representations. An Elementary Introduction. Grad. Texts in Math. 222, Springer, New York (2003) Zbl 1316.22001 MR 3331229

[Hj03] Hjorth, G.: A converse to Dye's theorem. Trans. Amer. Math. Soc. **357**, 3083–3103 (2005) Zbl 1068.03035 MR 2135736

[Hj05] Hjorth, G.: A lemma for cost attained. Ann. Pure Appl. Logic **143**, 87–102 (2006) Zbl 1101.37004 MR 2258624

[Hj08] Hjorth, G.: Treeable equivalence relations. J. Math. Logic **12**, no. 1, 1250003, 21 pp. (2012) Zbl 1278.03081 MR 2950193

[HK05] Hjorth, G., Kechris, A.: Rigidity theorems for actions of product groups and countable Borel equivalence relations. Mem. Amer. Math. Soc. **177**, no. 833, viii + 109 pp. (2005) Zbl 1094.37002 MR 2155451

[HT04] Hjorth, G., Thomas, S.: The classification problem for p-local torsion-free abelian groups of rank two. J. Math. Logic **6**, 233–251 (2006) Zbl 1115.03062 MR 2317428

[Io06] Ioana, A.: Non-orbit equivalent actions of $\mathbb{F}_n$. Ann. Sci. École Norm. Sup. **42**, 675–696 (2009) Zbl 1185.37009 MR 2568879

[Io08] Ioana, A.: Cocycle superrigidity for profinite actions of property (T) groups. Duke Math. J. **157**, 337–367 (2011) Zbl 1235.37005 MR 2783933

[Io11] Ioana, A.: Compact actions and uniqueness of the group measure space decomposition of II_1 factors. J. Funct. Anal. **262**, 4525–4533 (2012) Zbl 1253.46067 MR 2900475

[IKT08] Ioana, A., Kechris, A. S., Tsankov, T.: Subequivalence relations and positive-definite functions. Groups Geom. Dynam. **4**, 579–625 (2009) Zbl 1186.37011 MR 2529949

[JKL01] Jackson, S., Kechris, A. S., Louveau, A.: Countable Borel equivalence relations. J. Math. Logic **2**, 1–80 (2002) Zbl 1008.03031 MR 1900547

[Ke95] Kechris, A. S.: Classical Descriptive Set Theory. Grad. Texts in Math. 156, Springer, New York (1995) Zbl 0819.04002 MR 1321597

[Lu12] Lubotzky, A.: Expander graphs in pure and applied mathematics. Bull. Amer. Math. Soc. **49**, 113–162 (2012) Zbl 1232.05194 MR 2869010

[LS03] Lubotzky, A., Segal, D.: Subgroup Growth. Progr. Math. 212, Birkhäuser, Basel (2003) Zbl 1071.20033 MR 1978431

[LZ03] Lubotzky, A., Żuk, A.: On property (τ). Preprint, http://www.ma.huji.ac.il/~alexlub/ (2003)

[MRV11] Meesschaert, N., Raum, S., Vaes, S.: Stable orbit equivalence of Bernoulli actions of free groups and isomorphism of some of their factor actions. Expo. Math. **31**, 274–294 (2013) Zbl 1321.37006 MR 3108102

[MS02] Monod, N., Shalom, Y.: Orbit equivalence rigidity and bounded cohomology. Ann. of Math. (2) **164**, 825–878 (2006) Zbl 1129.37003 MR 2259246

[OP07] Ozawa, N., Popa, S.: On a class of II_1 factors with at most one Cartan subalgebra. Ann. of Math. **172**, 713–749 (2010) Zbl 1201.46054 MR 2680430

[OP08] Ozawa, N., Popa, S.: On a class of II_1 factors with at most one Cartan subalgebra. II. Amer. J. Math. **132**, 841–866 (2010) Zbl 1213.46053 MR 2666909

[Pi98] Pink, R.: Compact subgroups of linear algebraic groups. J. Algebra **206**, 438–504 (1998) Zbl 0914.20044 MR 1637068

[Po01] Popa, S.: On a class of type II_1 factors with Betti numbers invariants. Ann. of Math. **163**, 809–899 (2006) Zbl 1120.46045 MR 2215135

[Po06a] Popa, S.: On the superrigidity of malleable actions with spectral gap. J. Amer. Math. Soc. **21**, 981–1000 (2008) Zbl 1222.46048 MR 2425177

[Po06b] Popa, S.: Deformation and rigidity for group actions and von Neumann algebras. In: Proc. ICM (Madrid, 2006), Vol. I, Eur. Math. Soc., 445–477 (2007) Zbl 1132.46038 MR 2334200

[Po09] Popa, S.: On the classification of inductive limits of II_1 factors with spectral gap. Trans. Amer. Math. Soc. **364**, 2987–3000 (2012) Zbl 1252.46063 MR 2888236

[PV08] Popa, S., Vaes, S.: Actions of $\mathbb{F}_\infty$ whose II_1 factors and orbit equivalence relations have prescribed fundamental group. J. Amer. Math. Soc. **23**, 383–403 (2010) Zbl 1202.46069 MR 2601038

[RS98] Radin, C., Sadun, L.: Subgroups of SO(3) associated with tilings. J. Algebra **202**, 611–633 (1998) Zbl 0911.20035 MR 1617675

[Sc80] Schmidt, K.: Asymptotically invariant sequences and an action of $\mathrm{SL}(2,\mathbb{Z})$ on the 2-sphere. Israel J. Math. **37**, 193–208 (1980) Zbl 0485.28018 MR 0599454

[Se65] Selberg, A.: On the estimation of Fourier coefficients of modular forms. In: Theory of Numbers, Proc. Sympos. Pure Math. 8, Amer. Math. Soc., 1–15 (1965) Zbl 0142.33903 MR 0182610

[Sh99] Shalom, Y.: Expander graphs and amenable quotients. In: Emerging Applications of Number Theory, IMA Vol. Math. Appl. 109, Springer, New York, 571–581 (1999) Zbl 0989.20028 MR 1691549

[Sw94] Świerczkowski, S.: A class of free rotation groups. Indag. Math. **5**, 221–226 (1994) Zbl 0809.20039 MR 1284564

[Th00] Thomas, S.: The classification problem for torsion-free abelian groups of finite rank. J. Amer. Math. Soc. **16**, 233–258 (2003) Zbl 1115.03062 MR 2317428

[Th01] Thomas, S.: Superrigidity and countable Borel equivalence relations. Ann. Pure Appl. Logic **120**, 237–262 (2003) Zbl 1016.03048 MR 1949709

[Th06] Thomas, S.: Borel superrigidity and the classification problem for the torsion-free abelian groups of finite rank. In: Proc. ICM (Madrid, 2006), Vol. II, Eur. Math. Soc., 93–116 (2007) Zbl 1104.03043 MR 2275590

[TS07] Thomas, S., Schneider, S.: Countable Borel equivalence relations. In: Appalachian Set Theory 2006–2012, to appear

[Zi84] Zimmer, R.: Ergodic Theory and Semisimple Groups. Monogr. Math. 81, Birkhäuser, Basel (1984) Zbl 0571.58015 MR 0776417