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Legendrian knots and exact Lagrangian cobordisms

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Abstract. We introduce constructions of exact Lagrangian cobordisms with cylindrical Legendrian ends and study their invariants which arise from Symplectic Field Theory. A pair (X, L) consisting of an exact symplectic manifold X and an exact Lagrangian cobordism $L \subset X$ which agrees with cylinders over Legendrian links Λ_+ and Λ_- at the positive and negative ends induces a differential graded algebra (DGA) map from the Legendrian contact homology DGA of Λ_+ to that of Λ_- . We give a gradient flow tree description of the DGA maps for certain pairs (X, L) , which in turn yields a purely combinatorial description of the cobordism map for elementary cobordisms, i.e., cobordisms that correspond to certain local modifications of Legendrian knots. As an application, we find exact Lagrangian surfaces that fill a fixed Legendrian link and are not isotopic through exact Lagrangian surfaces.

Keywords. Contact structure, Legendrian knot, exact Lagrangian cobordism, contact homology

1. Introduction

The goal of this paper is to introduce constructions of exact Lagrangian cobordisms with cylindrical Legendrian ends and study their invariants which arise from the Symplectic Field Theory (SFT) of Eliashberg–Givental–Hofer [EGH].

1.1. Preliminaries

The *standard contact structure* on \mathbb{R}^3 is the 2-plane field $\xi_0 = \ker(\alpha_0)$, where $\alpha_0 = dz - ydx$ with respect to the standard coordinates (x, y, z) . A 1-dimensional submanifold $\Lambda \subset \mathbb{R}^3$ is *Legendrian* if it is everywhere tangent to ξ_0 . We assume that Legendrian knots and links are closed and oriented, unless stated otherwise.

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The *Reeb vector field* R_{α_0} of α_0 is given by $R_{\alpha_0} = \partial_z$. A *Reeb chord* of a Legendrian link $\Lambda \subset \mathbb{R}^3$ is an integral curve of R_{α_0} with initial and terminal points on Λ . We write $\mathcal{C}(\Lambda)$ for the set of Reeb chords of Λ .

An *exact symplectic manifold* is a triple (X, ω, β) consisting of a $2n$ -dimensional symplectic manifold (X, ω) and a 1-form β satisfying $d\beta = \omega$. We will often abbreviate an exact symplectic manifold as (X, β) , $(X, d\beta)$, or even X . An n -dimensional submanifold $L \subset (X, \omega, \beta)$ is *Lagrangian* if $\omega|_L = 0$ and is *exact Lagrangian* if $\beta|_L$ is exact in addition.

The *symplectization* of (\mathbb{R}^3, α_0) is the exact symplectic manifold $(\mathbb{R} \times \mathbb{R}^3, e^t \alpha_0)$, where t is the coordinate of the first \mathbb{R} -factor. A Legendrian link $\Lambda \subset \mathbb{R}^3$ gives rise to a cylindrical Lagrangian submanifold $\mathbb{R} \times \Lambda \subset \mathbb{R} \times \mathbb{R}^3$.

Definition 1.1. Let Λ_+ and Λ_- be Legendrian links in \mathbb{R}^3 , (X, β) an exact symplectic manifold whose positive and negative ends agree with the positive and negative ends of $(\mathbb{R} \times \mathbb{R}^3, e^t \alpha_0)$, and $L \subset (X, \beta)$ an oriented exact Lagrangian submanifold. Then the pair $((X, \beta), L)$ is an *exact Lagrangian cobordism from Λ_+ to Λ_- with cylindrical Legendrian ends $\mathcal{E}_{\pm}(L)$* if there exists $T > 0$ such that

$$\begin{aligned} \mathcal{E}_+(L) &= L \cap ((T, \infty) \times \mathbb{R}^3) = (T, \infty) \times \Lambda_+, \\ \mathcal{E}_-(L) &= L \cap ((-\infty, -T) \times \mathbb{R}^3) = (-\infty, -T) \times \Lambda_-, \end{aligned} \tag{1.1}$$

and

- (i) f is constant on each of $\mathcal{E}_+(L)$ and $\mathcal{E}_-(L)$ whenever $df = \beta|_L$ ¹ and
- (ii) L is compact with boundary $\Lambda_+ - \Lambda_-$ after removing the cylindrical ends $\mathcal{E}_{\pm}(L)$.

An exact Lagrangian cobordism $((X, \beta), L)$ from Λ to \emptyset is an *exact Lagrangian filling* of Λ .

Remark 1.2. The results of this paper carry over to the case where the exact Lagrangian L is unoriented, with the following exception: the grading of a cobordism map induced by L , as described in Section 1.2, is defined only mod 2.

The most general ambient exact symplectic manifold $(X, \omega, -\theta)$ that we consider in this paper is the *completion of the cotangent bundle T^*F of a surface F* , where $F = (\mathbb{R} \times [a_-, a_+]) \# \Sigma$ is the connected sum of a closed surface Σ and a strip $\mathbb{R} \times [a_-, a_+]$ and θ is the canonical 1-form on T^*F . Let $\partial_- F = \mathbb{R} \times \{a_-\}$ and $\partial_+ F = \mathbb{R} \times \{a_+\}$. Then X is the exact symplectic manifold obtained by attaching half-symplectizations $(-\infty, 0] \times \mathbb{R}^3$ and $[0, \infty) \times \mathbb{R}^3$ to T^*F along $T^*F|_{\partial_- F}$ and $T^*F|_{\partial_+ F}$; for more details, see Section 2.

Below we will also consider deformations of exact Lagrangian cobordisms as above. More precisely, we consider families (X, θ_s, L_s) , $s \in S$, of exact Lagrangian cobordisms, where both the 1-form θ_s and the Lagrangian L_s depend smoothly on the parameter $s \in S$

¹ It is automatic that f is constant on each component of $\mathcal{E}_{\pm}(L)$. If f is not the same constant on all the components of $\mathcal{E}_-(L)$ or on all the components of $\mathcal{E}_+(L)$, then the composition of exact Lagrangians is not necessarily exact Lagrangian and there are problems defining the DGA morphism $\Phi_{(X,L)}$ (see Section 1.2). These were communicated to the authors by Chantraine and Ghiggini; see [Cha2].

but are constant outside some compact subset. In what follows we often think of θ_s as part of the structure on X and denote the pair (X, θ_s) by X_s .

1.2. The Legendrian contact homology functor Φ

Let cob be the category whose objects are ‘‘chord generic’’ Legendrian links $\Lambda \subset \mathbb{R}^3$ (see Definition 3.1) and whose morphisms $\text{Hom}(\Lambda_+, \Lambda_-)$ are exact Lagrangian cobordisms from Λ_+ to Λ_- with cylindrical Legendrian ends. The pair $(\mathbb{R} \times \mathbb{R}^3, \mathbb{R} \times \Lambda)$ is the identity in $\text{Hom}(\Lambda, \Lambda)$. If $(X_{21}, L_{21}) \in \text{Hom}(\Lambda_2, \Lambda_1)$ and $(X_{10}, L_{10}) \in \text{Hom}(\Lambda_1, \Lambda_0)$, then their product $(X_{20}, L_{20}) = (X_{21}, L_{21}) \cdot (X_{10}, L_{10}) \in \text{Hom}(\Lambda_2, \Lambda_0)$ is obtained from (X_{21}, L_{21}) and (X_{10}, L_{10}) by concatenating the two along their common Λ_1 -end. We point out that the process of concatenation is noncanonical and some care is needed to make the concatenation associative. One way to do this is to remember the data of the cylindrical ends $\mathcal{E}_{\pm}(L)$. When we concatenate we truncate the ends so that we have a fixed width C that is left from each end, i.e., $(T, T + C) \times \Lambda_+$ and $(-T - C, -T) \times \Lambda_-$.

Let dgh be the homotopy category of unital (noncommutative) differential graded algebras (DGAs) over a field \mathbb{F} : the objects of dgh are unital DGAs and the morphisms are unital DGA morphisms up to chain homotopy (see Lemma 3.14).

We now describe a noncanonical functor $\Phi : \text{cob} \rightarrow \text{dgh}$, which we call the ‘‘Legendrian contact homology functor’’. The functor Φ associates a unital DGA $\mathcal{A}(\Lambda)$ to a generic Legendrian link $\Lambda \subset \mathbb{R}^3$ which is freely generated by $\mathcal{C}(\Lambda)$ over the group ring $\mathbb{F}[H_1(\Lambda; \mathbb{Z})]$ (see Section 3 for details). Here both the Reeb chords and the elements of $H_1(\Lambda; \mathbb{Z})$ are graded by a Maslov index. The differential of $\mathcal{A}(\Lambda)$ is defined by a count of punctured J -holomorphic disks in $\mathbb{R} \times \mathbb{R}^3$, where J is an ‘‘adjusted’’ almost complex structure; in particular, J is invariant with respect to the t -translation. (See Section 3.2 for more details.) The disks have boundary on $\mathbb{R} \times \Lambda$ and are asymptotic to strips over Reeb chords near the punctures (see the left panel of Figure 1). Furthermore, the disks that contribute to the differential are required to be rigid up to translation in the t -direction. When $\Lambda = \emptyset$, then $\mathcal{A}(\Lambda) = \mathbb{F}$ with the trivial differential $\partial = 0$ and the degree of elements in \mathbb{F} is zero.

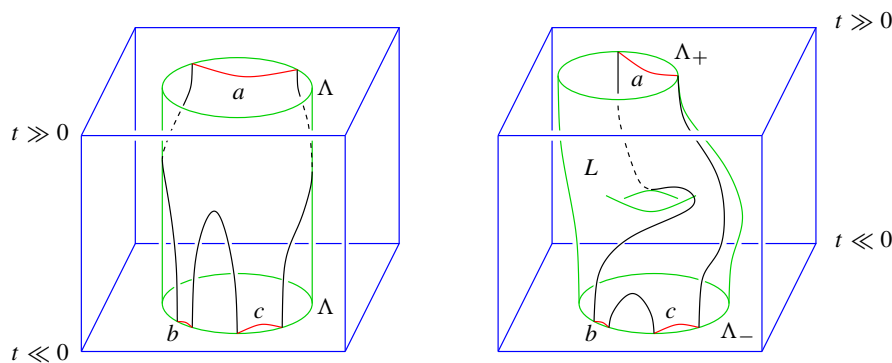


Fig. 1. Schematic picture of: \mathbb{R}^3 and $\mathbb{R} \times \mathbb{R}^3$ in blue (for colors, see the pdf file); the Legendrians $\Lambda, \Lambda_+, \Lambda_-$ and Lagrangians $\mathbb{R} \times \Lambda$ and L in green; the Reeb chords $a, b,$ and c in red; and the holomorphic disks in black, with a positive puncture at a and negative punctures at b and c .

The functor Φ associates to $(X, L) \in \text{Hom}(\Lambda_+, \Lambda_-)$ a chain homotopy class of DGA morphisms, one of whose representatives is

$$\Phi_{(X,L)}: \mathcal{A}(\Lambda_+) \rightarrow \mathcal{A}(\Lambda_-),$$

defined by a count of rigid punctured holomorphic disks in X with boundary on L , for a suitable almost complex structure. (See Section 3.2 for more details.) The right panel of Figure 1 depicts a holomorphic disk that is counted in $\Phi_{(X,L)}$, if we note that our statements hold not just in $\mathbb{R} \times \mathbb{R}^3$, but for more general exact Lagrangian cobordisms (X, L) as well.

The usual compactness and gluing results for holomorphic disks imply that the cobordism maps satisfy the following properties (see [E2] and Section 3.5):

Theorem 1.3. $\Phi: \text{cob} \rightarrow \text{dgh}$ is a functor, i.e.,

- (1) If $(X, L) = \text{id}$, then $\Phi_{(X,L)} = \text{id}$.
- (2) If $(X_{20}, L_{20}) = (X_{21}, L_{21}) \cdot (X_{10}, L_{10})$, then

$$\Phi_{(X_{20}, L_{20})} \simeq \Phi_{(X_{10}, L_{10})} \circ \Phi_{(X_{21}, L_{21})},$$

where \simeq indicates chain homotopy.

Theorem 1.4. If (X_t, L_t) , $0 \leq t \leq 1$, is an isotopy of exact Lagrangian cobordisms from Λ_+ to Λ_- with cylindrical ends, then $\Phi_{(X_0, L_0)} \simeq \Phi_{(X_1, L_1)}$.

For simplicity we use $\mathbb{F} = \mathbb{Z}/2$ -coefficients throughout, so that $\mathcal{A}(\Lambda_\pm)$ are algebras over \mathbb{F} and $\Phi_{(X,L)}$ is an \mathbb{F} -algebra morphism. In particular, all curve counts are mod 2 counts. The calculations of $\Phi_{(X,L)}$ are expected to carry over to the setting of more general coefficient rings using the orientation scheme developed in [EES3], for example.

An *augmentation* of Λ is a DGA morphism $\varepsilon: \mathcal{A}(\Lambda) \rightarrow \mathcal{A}(\emptyset) = \mathbb{F}$. Given an exact Lagrangian filling (X, L) of Λ , the DGA morphism $\Phi_{(X,L)}$ is an augmentation of Λ ; such augmentations will be called *geometric*.

1.3. Main results

The goal of this paper is to give Morse-theoretic and combinatorial descriptions of DGA maps induced by exact Lagrangian cobordisms $(X, L) \in \text{Hom}(\Lambda_+, \Lambda_-)$, where X is the completion of T^*F . This is done through an intermediary called a *Morse cobordism*, a compact immersed exact Lagrangian submanifold L^{Mo} in T^*F with double points only over ∂F and with a Legendrian lift $\tilde{L}^{\text{Mo}} \subset J^1F$, which is described in more detail in Section 2. Here $J^1F \simeq T^*F \times \mathbb{R}$ denotes the 1-jet space of F with its standard contact form $d\zeta - \theta$, where ζ is the \mathbb{R} -coordinate and θ is the canonical 1-form on T^*F . We assume that $\tilde{L}^{\text{Mo}} \cap J^1F|_{\partial F}$ agrees with the Legendrian links $\Lambda_\pm \subset J^1\partial_\pm F \subset J^1F$ and all the Reeb chords of \tilde{L}^{Mo} are contained in $J^1F|_{\partial F}$ so that they are in a natural one-to-one correspondence with the Reeb chords of Λ_\pm .

In Section 2 we associate to L^{Mo} a 2-parameter family of exact Lagrangian cobordisms $(X_\delta, L_{\delta;\sigma}) \in \text{Hom}(\Lambda_+, \Lambda_-)$, $\delta, \sigma > 0$, where X_δ is a completion of T^*F , and prove:

Lemma 1.5. *Given $(X, L) \in \text{Hom}(\Lambda_+, \Lambda_-)$, where X is the completion of T^*F , there exists a Morse cobordism L^{Mo} in T^*F such that the exact Lagrangian cobordisms $(X_\delta, L_{\delta;\sigma})$ associated to L^{Mo} are exact Lagrangian isotopic to (X, L) .*

As in [E1], we define Morse flow trees of a Morse cobordism $L^{\text{Mo}} \subset T^*F$ from Λ_+ to Λ_- . If a is a Reeb chord of Λ_+ and \mathbf{b} a monomial in $\mathcal{A}(\Lambda_-)$, then let $\mathcal{T}(a; \mathbf{b})$ denote the space of Morse flow trees of L^{Mo} with a positive puncture at a and negative punctures (as well as homotopy data) determined by \mathbf{b} (see Section 4 for details).

The following main technical result, proved in Section 5.6, gives a Morse-theoretic expression for the DGA morphism induced by $(X_\delta, L_{\delta;\sigma})$.

Theorem 1.6. *For all sufficiently small $\delta, \sigma > 0$, $\Phi_{(X_\delta, L_{\delta;\sigma})} : \mathcal{A}(\Lambda_+) \rightarrow \mathcal{A}(\Lambda_-)$ is given as follows: if $a \in \mathcal{C}(\Lambda_+)$ then*

$$\Phi_{(X_\delta, L_{\delta;\sigma})}(a) = \sum_{\dim(\mathcal{T}(a; \mathbf{b}))=0} |\mathcal{T}(a; \mathbf{b})| \mathbf{b},$$

where $|\mathcal{T}(a; \mathbf{b})|$ is the mod 2 count of points in the compact 0-dimensional moduli space $\mathcal{T}(a; \mathbf{b})$.

By combining Lemma 1.5 and Theorem 1.6, we obtain a Morse-theoretic description of any $\Phi_{(X, L)}$, where $(X, L) \in \text{Hom}(\Lambda_+, \Lambda_-)$ and X is the completion of T^*F . Although Theorem 1.6 is stated for general F , in applications we take $F = \mathbb{R} \times [a, b]$. In particular we use Theorem 1.6 to find explicit combinatorial formulas for $\Phi_{(\mathbb{R} \times \mathbb{R}^3, L)}$ in the following special cases:

- (a) L is induced by a Legendrian isotopy (see Sections 6.2 and 6.3);
- (b) L is a minimum cobordism, i.e., a disk bounding the standard unknot (see Definition 6.11); and
- (c) L is a saddle cobordism that is given by a 0-resolution of a contractible Reeb chord in the Lagrangian projection (see Definition 6.15).

Explicit formulas for (a) are given in Lemmas 6.6–6.8; the formula for (b) is given in Lemma 6.12; and the formula for (c) is given in Proposition 6.18.

Remark 1.7. The cobordisms in (b) and (c) above correspond to handle attachments of Morse index 0 and 1 when $\dim(L) = 2$ and admit generalizations to handle attachments of Morse index $0, \dots, n - 1$ when $\dim(L) = n$; see [R] for a similar study in higher dimensions.

As an application of these formulas we prove the following result:

Theorem 1.8. *Let Λ_n be the Legendrian $(2, n)$ -torus link in Figure 2, let $A_n = (2^{n+1} - (-1)^{n+1})/3$, and let $g_n = (n - 1)/2$ if n is odd and $g_n = (n - 2)/2$ if n is even. Then there are at least A_n (resp. $A_n - 1$) smoothly isotopic exact Lagrangian fillings of Λ_n of genus g_n that are pairwise nonisotopic through exact Lagrangian surfaces when n is odd (resp. even).*

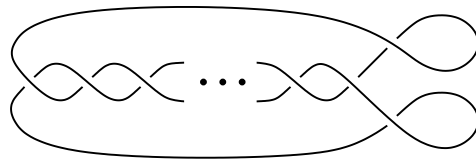


Fig. 2. Lagrangian diagram of a Legendrian $(2, n)$ -torus knot or link, oriented as the closure of a positive braid on two strands. The total number of crossings is $n + 2$.

Theorem 1.8 is in sharp contrast to the situation of the Legendrian unknot U with rotation number $r(U) = 0$ and Thurston–Bennequin invariant $tb(U) = -1$. By the isotopy uniqueness for Lagrangian planes in \mathbb{C}^2 that are standard at infinity [EP], there is a unique Lagrangian disk which fills U , up to exact Lagrangian isotopy. (See Figure 10 for such a disk.)

In Section 9 we discuss some relationships with Khovanov homology [Kh].

Outline of the paper. In Section 2 we construct and relate the various forms of exact Lagrangian cobordisms that are used in this paper. In particular we prove Lemma 1.5. We then review Legendrian contact homology in Section 3 and gradient flow trees from [E1] in Section 4. Section 5 is the main technical part of this paper where we prove Theorem 1.6. We then use Theorem 1.6 to compute the DGA maps of elementary exact Lagrangian cobordisms in Section 6. In Section 7 we collect some observations about exact Lagrangian fillings and augmentations. In Section 8 we give some applications including the proof of Theorem 1.8. The connections with Khovanov homology are given in Section 9.

2. Lagrangian cobordisms with Legendrian ends

In this section we introduce the various forms of exact Lagrangian cobordisms that we use in this paper.

2.1. Completion of T^*F

As in Section 1, let $F = (\mathbb{R} \times [a_-, a_+]) \# \Sigma$, where Σ is a compact surface and $a_+ > a_- > 0$. Let θ be the canonical 1-form on T^*F and let $\omega = -d\theta$. Let $U_+ = \mathbb{R} \times (a_+ - \epsilon, a_+]$ be a neighborhood $\partial_+ F$ and let $(\xi_1, \eta_1, \xi_2, \eta_2)$ be the coordinates on $T^*F|_{U_+}$ such that (ξ_1, ξ_2) are the coordinates on U_+ and (η_1, η_2) are the dual coordinates on \mathbb{R}^2 . Then $\theta = \eta_1 d\xi_1 + \eta_2 d\xi_2$ on $T^*F|_{U_+}$.

Consider the map

$$\Phi_+ : (\log(a_+ - \epsilon), \log a_+] \times \mathbb{R}^3 \rightarrow T^*F|_{U_+}, \quad (t, x, y, z) \mapsto (x, e^t y, e^t, z).$$

Then

$$\Phi_+^*(-\theta) = -e^t(ydx + zdt) = e^t(dz - ydx) - d(e^t z).$$

Hence Φ_+ is a symplectomorphism with respect to $d(e^t(dz - ydx))$ on the strip $(\log(a_+ - \epsilon), \log a_+] \times \mathbb{R}^3$ and the canonical symplectic form $\omega = -d\theta$ on $T^*F|_{U_+}$.

Next, on $(\log(a_+ - \epsilon), \infty) \times \mathbb{R}^3$, we consider the 1-form

$$\alpha = e^t(dz - ydx) - d(\beta(t)e^t z),$$

where $\beta(t) = 1$ for $t \in (a_+ - \epsilon, a_+]$ and $\beta(t) = 0$ for $t \gg a_+$. Then attaching $(\log(a_+ - \epsilon), \infty) \times \mathbb{R}^3, \alpha$ to T^*F using Φ_+ yields an exact symplectic manifold which extends $(T^*F, -\theta)$.

Similarly, let U_- be a neighborhood of $\partial_- F$ with coordinates $(\xi_1, \xi_2) \in \mathbb{R} \times [a_-, a_- + \epsilon)$. We attach $(-\infty, \log(a_- + \epsilon)) \times \mathbb{R}^3$ to $T^*F|_{U_-}$ using the map

$$\Phi_-(t, x, y, z) = (x, e^t y, e^t, z).$$

The result of these two attachments is an exact symplectic manifold X with cylindrical ends, called the *completion* of T^*F .

2.2. Conical cobordisms

We now explain how to translate the notion of a cylindrical Lagrangian cobordism (= Lagrangian cobordism with cylindrical ends) to the notion of a *conical Lagrangian cobordism* on T^*F . Conical Lagrangian cobordisms serve as the intermediary between cylindrical Lagrangian cobordisms and Morse cobordisms, where the latter is particularly suited for holomorphic disk counting.

Consider the ends

$$\begin{aligned} \mathcal{E}_+(L) &= (\log(a_+ - \epsilon), \infty) \times \Lambda_+ \subset [\log(a_+ - \epsilon), \infty) \times \mathbb{R}^3, \\ \mathcal{E}_-(L) &= (-\infty, \log(a_- + \epsilon)] \times \Lambda_- \subset (-\infty, \log(a_- + \epsilon)] \times \mathbb{R}^3, \end{aligned}$$

of a cylindrical Lagrangian cobordism L , where Λ_{\pm} are Legendrian links with parametrizations

$$(x_{\pm}(s), y_{\pm}(s), z_{\pm}(s)), \quad s \in S_{\pm}, \tag{2.1}$$

and S_{\pm} are closed 1-manifolds. Then $\Phi_{\pm}(\mathcal{E}_{\pm}(L)) \cap T^*F|_{U_{\pm}}$ are parametrized by

$$(x_{\pm}(s), \xi_2 y_{\pm}(s), \xi_2, z_{\pm}(s)),$$

where $(s, \xi_2) \in S_+ \times (a_+ - \epsilon, a_+]$ and $(s, \xi_2) \in S_- \times [a_-, a_- + \epsilon)$, respectively.

Definition 2.1. If Λ_+ and Λ_- are Legendrian links, then a Lagrangian submanifold $L^{\text{co}} \subset T^*F$ is a *conical Lagrangian cobordism* from Λ_+ to Λ_- if $L^{\text{co}} \cap T^*F|_{U_{\pm}}$ admits a parametrization

$$(x_{\pm}(s), (\xi_2 - a'_{\pm})y_{\pm}(s), \xi_2, z_{\pm}(s)), \tag{2.2}$$

where $a_- < a_+$ and $a'_{\pm} < a_{\pm}$. We do not require $a_- > 0$.

The reader can verify that a surface which admits a parametrization of the form

$$(x_{\pm}(s), f(\xi_2)y_{\pm}(s), \xi_2, f'(\xi_2)z_{\pm}(s))$$

is indeed exact Lagrangian.

Let \hat{F} be the surface obtained from F by gluing $E_- = \mathbb{R} \times (a'_-, a_-]$ to $\partial_- F$ and $E_+ = \mathbb{R} \times [a_+, \infty)$ to $\partial_+ F$ so that $X \simeq T^*\hat{F}$.

Definition 2.2. Given a conical Lagrangian cobordism $L^{\text{co}} \subset T^*F$, the *long conical Lagrangian cobordism* \hat{L}^{co} corresponding to L^{co} is a Lagrangian submanifold of $T^*\hat{F}$ such that $\hat{L}^{\text{co}} \cap T^*\hat{F}|_{E_- \cup E_+}$ admits a parametrization as in (2.2), where $(s, \xi_2) \in S_+ \times [a_+, \infty)$ and $(s, \xi_2) \in S_- \times (a'_-, a_-]$, respectively.

2.3. Morse cobordisms

We now define *Morse cobordisms*. Let $\Lambda_{\pm} \subset \mathbb{R}^3$ be Legendrian links with parametrizations as in (2.1).

Definition 2.3. An compact immersed Lagrangian submanifold L^{Mo} in T^*F with double points only over ∂F is a *Morse cobordism from Λ_+ to Λ_-* if $L^{\text{Mo}} \cap T^*F|_{U_{\pm}}$ admits parametrizations

$$(x_{\pm}(s), (A_{\pm} \mp (\xi_2 - a_{\pm})^2)y_{\pm}(s), \xi_2, \mp 2(\xi_2 - a_{\pm})z_{\pm}(s)), \tag{2.3}$$

where $(s, \xi_2) \in S_+ \times (a_+ - \epsilon, a_+]$ and $(s, \xi_2) \in S_- \times [a_-, a_- + \epsilon)$, respectively, and $A_{\pm} > 0$.

A Morse cobordism L^{Mo} has a Legendrian lift $\tilde{L}^{\text{Mo}} \subset J^1F$ which is unique up to \mathbb{R} -translation, and the Reeb chords of \tilde{L}^{Mo} over $\partial_+ F$ and $\partial_- F$ are canonically identified with the Reeb chords of Λ_+ and Λ_- , respectively.

2.4. From Morse to conical

We explain how to pass from a Morse cobordism to a conical cobordism.

Let $h_{\delta}: [0, \epsilon) \rightarrow \mathbb{R}$ be a family of increasing functions parametrized by $\delta \in (0, \epsilon/5)$ such that

$$h_{\delta}(u) = \begin{cases} \delta u - \frac{1}{2}\delta^2 & \text{for } u \in [0, \delta/2], \\ u^2 & \text{for } u \in [\delta, \epsilon]. \end{cases}$$

Let L^{Mo} be a Morse cobordism and let L'_{δ} be an exact Lagrangian on T^*F which agrees with L^{Mo} on $T^*F|_{F-(U_+ \cup U_-)}$ and is parametrized by

$$\begin{aligned} &(x_+(s), (A_+ - h_{\delta}(a_+ - \xi_2))y_+(s), \xi_2, h'_{\delta}(a_+ - \xi_2)z_+(s)), \\ &(x_-(s), (A_- + h_{\delta}(\xi_2 - a_-))y_-(s), \xi_2, h'_{\delta}(\xi_2 - a_-)z_-(s)) \end{aligned} \tag{2.4}$$

on $T^*F|_{U_{\pm}}$.

By adding ends $\mathbb{R} \times (a'_-(\delta), a_-]$ (for an appropriate $a'_-(\delta)$) and $\mathbb{R} \times [a_+, \infty)$ to $\partial_{\pm}F$ we obtain a surface F_{δ} and by adding conical ends to L'_{δ} we obtain a long conical Lagrangian $\hat{L}_{\delta}^{\text{co}} \subset X_{\delta} := T^*F_{\delta}$.

Given $\sigma > 0$, let s_{σ} be the fiber scaling map:

$$s_{\sigma} : T^*M \rightarrow T^*M, \quad s_{\sigma}(q, p) = (q, \sigma p),$$

where $q \in M$ and $p \in T_q^*M$. Taking $M = F_{\delta}$, we define $L_{\delta;\sigma} := s_{\sigma}(\hat{L}_{\delta}^{\text{co}})$. We refer to $(X_{\delta}, L_{\delta;\sigma})$ as the conical exact Lagrangian cobordisms associated to L^{Mo} .

Proof of Lemma 1.5. Given an exact Lagrangian cobordism L with cylindrical Legendrian ends, we apply Φ_{\pm} to transform it into a long conical Lagrangian cobordism \hat{L}^{co} . By truncating the ends of \hat{L}^{co} we obtain a conical Lagrangian cobordism L^{co} . Rounding L^{co} near its boundary yields a Morse cobordism L^{Mo} (details left to the reader). Now applying the above procedure gives conical exact Lagrangian cobordisms $L_{\delta;\sigma}$. One easily verifies that \hat{L}^{co} is exact Lagrangian isotopic to $L_{\delta;\sigma}$. \square

3. Legendrian contact homology

In this section we review the Legendrian contact homology of links in standard contact \mathbb{R}^3 , first from the SFT perspective of [EGH] and then from the combinatorial perspective of [Ch].

3.1. The graded algebra of Reeb chords

We use the notation from Section 1. Let Λ be a Legendrian link in (\mathbb{R}^3, ξ_0) , $\Phi^t = \Phi^t_{R_{\alpha_0}}$ the time- t flow of the Reeb vector field $R_{\alpha_0} = \partial_z$, and $\mathcal{C}(\Lambda)$ the set of Reeb chords of Λ . We remark that the discussion below easily extends to 1-jet spaces (and to \mathbb{R}^{2n+1} in particular) with the standard contact forms.

The α_0 -action of $c \in \mathcal{C}(\Lambda)$ is given by

$$\mathfrak{A}_{\alpha_0}(c) = \int_c \alpha_0.$$

If c^{\pm} are the endpoints of c , then

$$\mathfrak{A}_{\alpha_0}(c) = z(c^+) - z(c^-) > 0.$$

Hence we also refer to $\mathfrak{A}_{\alpha_0}(c)$ as the length of c .

Definition 3.1 (Chord genericity). A Legendrian link $\Lambda \subset (\mathbb{R}^3, \xi_0)$ is chord generic if, for any $c \in \mathcal{C}(\Lambda)$, $d\Phi^{\ell}(T_c-\Lambda)$ is transverse to $T_{c^+}\Lambda$ in the contact plane $(\xi_0)_{c^+}$ at c^+ , where $\ell = \mathfrak{A}_{\alpha_0}(c)$.

We assume that Λ is chord generic, since chord genericity can be attained by perturbing Λ in its isotopy class. In particular, $\mathcal{C}(\Lambda)$ is a finite set and is in one-to-one correspondence with the double points of the restriction of the Lagrangian projection

$$\Pi_{\mathbb{C}}: \mathbb{R}^3 \rightarrow \mathbb{R}^2, \quad (x, y, z) \mapsto (x, y),$$

to Λ .

Let $\Lambda = \Lambda_1 \cup \dots \cup \Lambda_k$, where $\Lambda_j, j = 1, \dots, k$, are the connected components of Λ . Fix a reference point $p_j \in \Lambda_j$ for each j . For each pair $i \neq j$, pick a path δ_{ij} in \mathbb{R}^3 that connects p_i to p_j and a path of lines in the contact planes along δ_{ij} that connects $T_{p_i}\Lambda_i$ to $T_{p_j}\Lambda_j$. When $i = j$, we let δ_{ii} be the constant path at p_i . For each Reeb chord $c \in \mathcal{C}(\Lambda)$ with $c^\pm \in \Lambda_{i^\pm}, i^\pm \in \{1, \dots, k\}$, pick capping paths γ_c^\pm in Λ_{i^\pm} that connect p_{i^\pm} to c^\pm . We use the complex trivialization \mathfrak{T} of ξ_0 that is induced from the Lagrangian projection $\Pi_{\mathbb{C}}: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ followed by the standard identification $\mathbb{R}^2 \simeq \mathbb{C}$. We also write homology classes in $H_1(\Lambda_j) \simeq \mathbb{Z}$ multiplicatively: the class which equals m times the generator is denoted by τ_j^m with $\tau_j^0 = 1_j$.

Let Γ_{Λ_j} be a loop of tangent lines of Λ_j induced by a loop which traverses Λ_j once in the direction determined by τ_j . Then the *degree* of τ_j is

$$|\tau_j| = \mu(\Gamma_{\Lambda_j}),$$

where the Maslov index μ is measured with respect to \mathfrak{T} .

Let $c \in \mathcal{C}(\Lambda)$ be a Reeb chord of length ℓ . Let $\hat{\Gamma}_c$ denote the loop of tangent lines in ξ_0 which is the concatenation of the following four paths:

- (i) the chosen path of tangent lines along $-\delta_{i^-i^+}$;
- (ii) the path of tangent lines of Λ along γ_c^- ;
- (iii) the smallest positive rotation in the contact plane ξ_{c^+} that takes $d\Phi^\ell(T_{c^-}\Lambda)$ to $T_{c^+}\Lambda$;
and
- (iv) the path of tangent lines of Λ along $-\gamma_c^+$.

Here $-\gamma_c^+$ is γ_c^+ with the opposite orientation. Then define the *degree* of c as

$$|c| = \mu(\hat{\Gamma}_c) - 1,$$

where the Maslov index μ is measured with respect to \mathfrak{T} .

The graded algebra $\mathcal{A}(\Lambda)$ underlying the Legendrian DGA of Λ is the unital algebra over \mathbb{F} which is freely generated by the Reeb chords in $\mathcal{C}(\Lambda)$ and the homology classes $\tau_j^{\pm 1} \in H_1(\Lambda_j), j = 1, \dots, k$, modulo the relations that the homology generators commute, with the grading given above.

Remark 3.2. It is often convenient to impose further commutation relations where the Reeb chords and the homology generators commute. See [EENS, Section 2.3.2] for a discussion of the various forms of Legendrian DGAs.

Remark 3.3. If L^{Mo} is a Morse cobordism in T^*F from Λ_+ to Λ_- , then the Reeb chords of $\tilde{L}^{\text{Mo}} \subset J^1F = T^*F \times \mathbb{R}$ over ∂_-F (resp. ∂_+F) are in natural one-to-one correspondence with the Reeb chords of Λ_- (resp. Λ_+). By abuse of notation, the Reeb chords of $\mathcal{C}(\tilde{L}^{\text{Mo}})$ corresponding to $c_{\pm} \in \mathcal{C}(\Lambda_{\pm})$ will be denoted by c_{\pm} . Then $|c_-|_{\tilde{L}^{\text{Mo}}} = |c_-|_{\Lambda_-}$ and $|c_+|_{\tilde{L}^{\text{Mo}}} = |c_+|_{\Lambda_+} + 1$, where $|\cdot|_{\tilde{L}^{\text{Mo}}}$ refers to the grading of $\mathcal{A}(\tilde{L}^{\text{Mo}})$ and $|\cdot|_{\Lambda_{\pm}}$ refers to the grading of $\mathcal{A}(\Lambda_{\pm})$.

3.2. Moduli spaces of holomorphic disks

Consider an exact Lagrangian cobordism (X, L) from Λ_+ to Λ_- with cylindrical Legendrian ends. For cobordisms (X, L) , we use the natural maps $H_1(\Lambda_{\pm}) \rightarrow H_1(L; \mathbb{Z})$ and consider $\mathcal{A}(\Lambda_{\pm})$ as algebras generated by Reeb chords and a basis of $H_1(L; \mathbb{Z})$ (modulo the commutation relations for the homology generators). When we want to make the coefficient ring R explicit we write $\mathcal{A}(\Lambda_{\pm}; R)$.

We describe $\mathcal{A}(\Lambda_+; \mathbb{F}[H_1(L; \mathbb{Z})])$ more precisely: Pick a reference point q_i in each component of L and for each $p_j \in \Lambda_{+,j}$ a path $\delta_{q_i p_j}$ in L from some q_i to p_j . Given $c \in \mathcal{C}(\Lambda_+)$, we define $\bar{\gamma}_c^{\pm}$ as the concatenation of γ_c^{\pm} with the appropriate $\delta_{p_i q_j}$. Then we repeat the construction of $\mathcal{A}(\Lambda_+)$ from Section 3.1 with γ_c^{\pm} replaced by $\bar{\gamma}_c^{\pm}$ and $H_1(\Lambda_+; \mathbb{Z})$ replaced by $H_1(L; \mathbb{Z})$. This gives us $\mathcal{A}(\Lambda_+; \mathbb{F}[H_1(L; \mathbb{Z})])$; the construction of $\mathcal{A}(\Lambda_-; \mathbb{F}[H_1(L; \mathbb{Z})])$ is similar.

Let $a \in \mathcal{C}(\Lambda_+)$ and $b_1, \dots, b_m \in \mathcal{C}(\Lambda_-)$ be Reeb chords, let τ_0, \dots, τ_m be elements of $H_1(L; \mathbb{Z})$, and let

$$\mathbf{b} = \tau_0 b_1 \tau_1 b_2 \tau_2 \dots \tau_{m-1} b_m \tau_m,$$

where b_s^- and b_{s+1}^+ both lie on Λ_{j_s} for $s \geq 1$; a^+ and b_1^+ both lie on Λ_{j_0} ; and b_m^- and a^- both lie on Λ_{j_m} . Fix an almost complex structure J on X which is *adjusted to the symplectic form on X* , i.e.,

- J is compatible with the symplectic form;
- J is \mathbb{R} -invariant on the symplectization ends; and
- $J(\xi) = \xi$ and $J(\partial_t) = R_{\alpha_0}$ on the symplectization ends.

If c is a Reeb chord of Λ_{\pm} , then the product $\mathbb{R}_{\pm} \times c \subset \mathbb{R}_{\pm} \times \mathbb{R}^3$ of c and a half-line \mathbb{R}_{\pm} in the \mathbb{R} -direction in the symplectization ends is J -holomorphic. We call $\mathbb{R}_{\pm} \times c$ a *strip over the Reeb chord c* .

Define $\mathcal{M}^{(X,L);J}(a; \mathbf{b})$ as the moduli space of J -holomorphic disks

$$u: (D_{m+1}, \partial D_{m+1}) \rightarrow (X, L)$$

with the following properties:

- (1) D_{m+1} is the closed unit disk with $m + 1$ boundary points ζ_0, \dots, ζ_m removed. Here ζ_0, \dots, ζ_m are arranged in counterclockwise order around the boundary of the disk.
- (2) u has a positive puncture at ζ_0 where it is asymptotic to the strip over the Reeb chord a at ∞ .
- (3) u has a negative puncture at ζ_j , $j > 0$, where it is asymptotic to the strip over the Reeb chord b_j at $-\infty$.

- (4) The loop in L obtained by concatenating the capping path $\overline{\gamma}_{b_s}^+$, the image of the boundary segment from ζ_s to ζ_{s-1} under u , and the capping path $-\overline{\gamma}_{b_{s-1}}^-$ represents the class τ_s . (Here we use the convention $b_0 = b_{m+1} = a$.)

The J in the notation of the moduli space will often be suppressed. We will also refer to $u \in \mathcal{M}^{(X,L);J}(a; \mathbf{b})$ as a J -holomorphic disk in (X, L) from a to \mathbf{b} .

Similarly we define $\mathcal{M}^{(X,L);J}(a_1, \dots, a_k; \mathbf{b}_1, \dots, \mathbf{b}_k)$ as the moduli space of J -holomorphic disks in (X, L) with positive punctures at a_1, \dots, a_k and negative punctures given by $\mathbf{b}_1, \dots, \mathbf{b}_k$ such that $a_1, \mathbf{b}_1, \dots, a_k, \mathbf{b}_k$ are in counterclockwise order on the boundary of the domain.

Definition 3.4. An $(a + b + 1)$ -level broken disk is a union $u_{-b} \cup \dots \cup u_a$, arranged from bottom to top, such that $u_j, j \neq 0$, maps to $X_0 = \mathbb{R} \times \mathbb{R}^3$, u_0 maps to X , each component of $u_j, j = -b, \dots, a$, has one positive puncture and zero or more negative punctures, and the positive punctures of $u_j, j = -b, \dots, a - 1$, are in one-to-one correspondence with the negative punctures of u_{j+1} .

An $(a + b + 1)$ -level broken disk is a special case of an $(a + b + 1)$ -level SFT building (see [BEHWZ]).

3.3. Energy

Let $((X, d\beta), L)$ be an exact Lagrangian cobordism from Λ_+ to Λ_- with cylindrical Legendrian ends. Let

$$X_+ = [T, \infty) \times \mathbb{R}^3, \quad X_- = (-\infty, -T] \times \mathbb{R}^3$$

be the positive and negative ends of X and let

$$X_0 = X - \text{int}(X_+ \cup X_-),$$

where $\text{int}(Y)$ denotes the interior of Y . We write $\beta_0 = \beta|_{X_0}$, $\beta_{\pm} = \beta|_{\partial X_{\pm}}$, and ω_0 for the exact 2-form on X which agrees with $d\beta_0$ on X_0 and with $d\beta_{\pm}$ on X_{\pm} .

Definition 3.5. If $u \in \mathcal{M}^{(X,L);J}(a; \mathbf{b})$, then we define its ω_0 -energy $E_{\omega_0}(u)$ and β_{\pm} -energy $E_{\beta_{\pm}}(u)$ as follows:

$$E_{\omega_0}(u) = \int_u \omega_0, \quad E_{\beta_{\pm}}(u) = \sup_{\phi_{\pm}} \int_{u|_{X_{\pm}}} d(\phi_{\pm}(t)) \wedge \beta_{\pm},$$

where $\phi_+ : [T, \infty) \rightarrow \mathbb{R}^{\geq 0}$ (resp. $\phi_- : (-\infty, -T] \rightarrow \mathbb{R}^{\geq 0}$) is a smooth nondecreasing function such that $\phi_+(T) = 0$ and $\lim_{t \rightarrow \infty} \phi_+(t) = 1$ (resp. $\lim_{t \rightarrow -\infty} \phi_-(t) = 0$ and $\phi_-(-T) = 0$).

The following lemma is immediate (cf. [E2, Lemma B.3] for details):

Lemma 3.6. If $u \in \mathcal{M}^{(X,L);J}(a; \mathbf{b})$, then $E_{\omega_0}(u) \leq \mathfrak{A}(a)$ and $E_{\beta_{\pm}}(u) \leq \mathfrak{A}(a)$.

3.4. Fredholm theory and compactness

The Fredholm index $\text{ind}(u)$ of $u \in \mathcal{M}^{(X,L);J}(a_1, \dots, a_k; \mathbf{b}_1, \dots, \mathbf{b}_k)$ is the full expected dimension count which includes conformal variations. If (X, L) is \mathbb{R} -invariant, then u is rigid if $\text{ind}(u) = 1$; otherwise u is rigid if $\text{ind}(u) = 0$.

Lemma 3.7. *If $u \in \mathcal{M}^{(X,L);J}(a_1, \dots, a_k; \mathbf{b}_1, \dots, \mathbf{b}_k)$, then*

$$\text{ind}(u) = \sum_{i=1}^k (|a_i| - |\mathbf{b}_i|) - 1 + k. \tag{3.1}$$

Proof. Follows from [CEL, Theorem A.1]. □

Lemma 3.8. *For generic J , the moduli space $\mathcal{M}^{(X,L);J}(a; \mathbf{b})$ is a transversely cut out manifold of dimension $\text{ind} = |a| - |\mathbf{b}|$ and admits a natural compactification by multiple-level broken disks.*

Proof. The transversality can be achieved by perturbing J in the contact planes near the positive puncture as in [EES4, Lemma 4.5(1)]. The compactness statement is a consequence of [BEHWZ, Section 11.3] (see also [E2, Lemma B.4]). □

In general, there are transversality problems when one counts holomorphic disks with more than one positive puncture, since such disks may be multiply-covered.

In the special case of a cylindrical cobordism $L = \mathbb{R} \times \Lambda$ in the symplectization $X_0 = \mathbb{R} \times \mathbb{R}^3$, we write $\mathcal{M}^J(a; \mathbf{b}) = \mathcal{M}^{(X_0, \mathbb{R} \times \Lambda);J}(a; \mathbf{b})$. Since J is \mathbb{R} -invariant, \mathbb{R} acts on $\mathcal{M}^J(a; \mathbf{b})$ and we write $\widehat{\mathcal{M}}^J(a; \mathbf{b}) = \mathcal{M}^J(a; \mathbf{b})/\mathbb{R}$ for the reduced moduli space.

Corollary 3.9. *For generic J , a reduced moduli space $\widehat{\mathcal{M}}^J(a; \mathbf{b})$ of dimension 0 is a compact 0-dimensional manifold and the boundary of a compactified reduced moduli space of dimension 1 is a possibly empty union of two-level broken disks $u_0 \cup u_1$ with $\text{ind}(u_0) = \text{ind}(u_1) = 1$.*

Proof. Follows from Lemma 3.8. □

When (X, L) is a nontrivial cobordism, we have the following description of the low-dimensional moduli spaces.

Corollary 3.10. *For generic J , the moduli spaces $\mathcal{M}^{(X,L);J}(a; \mathbf{b})$ of dimension 0 are compact 0-dimensional manifolds and the boundary of a compactified moduli space of dimension 1 is a possibly empty union of two-level broken disks of the following types:*

- (1) $u_0 \cup u_1$, where u_1 is an element of the form $\mathcal{M}^{J_+}(a, \mathbf{b}')$ with $\text{ind}(u_1) = 1$ and u_0 is the union of components of the form $\mathcal{M}^{(X,L);J}(a'; \mathbf{b}'')$ with $\text{ind}(u_0) = 0$; and
- (2) $u_{-1} \cup u_0$, where u_0 is an element of the form $\mathcal{M}^{(X,L);J}(a; \mathbf{b}')$ with $\text{ind}(u_0) = 0$ and u_{-1} consists of one component in some $\mathcal{M}^{J_-}(a', \mathbf{b}'')$ with $\text{ind} = 1$ and strips over Reeb chords.

Here J_+ and J_- are the almost complex structures on the positive and negative ends of X .

Proof. Follows from Lemma 3.8. □

If (X'', L'') and (X', L') are exact Lagrangian cobordisms with cylindrical Legendrian ends, where the positive end of (X', L') is the same as the negative end of (X'', L'') , then these cobordisms can be concatenated by truncating the ends $\{t \geq \rho\} \subset X'$ and $\{t \leq -\rho\} \subset X''$ for $\rho \gg 0$ and gluing the remaining pieces together. The resulting Lagrangian cobordism is denoted by (X_ρ, L_ρ) .

Corollary 3.11. *For all sufficiently large ρ , there is a natural one-to-one correspondence between rigid holomorphic disks in (X_ρ, L_ρ) and two-level broken disks $u_0 \cup u_1$, where u_0 is a rigid disk in (X', L') and u_1 is a rigid disk in (X'', L'') .*

Proof. Follows from Lemma 3.8 and a stretching argument (see [BEHWZ, Section 11.3] or [E2, Lemma 3.13]). □

3.5. Differentials and cobordism maps

The differential in the Legendrian DGA is defined by counting disks in 0-dimensional reduced moduli spaces in trivial cobordisms. More precisely, if $\Lambda \subset \mathbb{R}^3$ is a Legendrian link, then the differential of $\mathcal{A}(\Lambda)$ is the \mathbb{F} -linear map $\partial: \mathcal{A}(\Lambda) \rightarrow \mathcal{A}(\Lambda)$ which satisfies the Leibniz rule, maps homology generators (i.e., monomials of the group ring $\mathbb{F}[H_1(\Lambda; \mathbb{Z})]$) to 0, and is defined on the Reeb chord generators as follows:

$$\partial a = \sum_{\dim(\widehat{\mathcal{M}}(a; \mathbf{b}))=0} |\widehat{\mathcal{M}}(a; \mathbf{b})| \mathbf{b}.$$

Lemma 3.12. *The map $\partial: \mathcal{A}(\Lambda) \rightarrow \mathcal{A}(\Lambda)$ is a differential (i.e., $\partial^2 = 0$) of degree -1 .*

Proof. This is a direct consequence of Corollary 3.9. □

Similarly, if (X, L) is an exact Lagrangian cobordism from Λ_+ to Λ_- , then it induces a DGA map

$$\Phi_{(X,L)}: \mathcal{A}(\Lambda_+; \mathbb{F}[H_1(L; \mathbb{Z})]) \rightarrow \mathcal{A}(\Lambda_-; \mathbb{F}[H_1(L; \mathbb{Z})]),$$

defined as follows: on the homology generators $\Phi_{(X,L)} = \text{id}$ and on the Reeb chord generators a ,

$$\Phi_{(X,L)}(a) = \sum_{\dim(\mathcal{M}^{(X,L)}(a; \mathbf{b}))=0} |\mathcal{M}^{(X,L)}(a; \mathbf{b})| \mathbf{b}.$$

We write $\Phi_{(X,L)}^A$ when we want to emphasize the coefficient ring A .

Lemma 3.13. *The map $\Phi_{(X,L)}$ is a chain map (i.e., $\partial_- \circ \Phi_{(X,L)} = \Phi_{(X,L)} \circ \partial_+$, where ∂_\pm is the differential on $\mathcal{A}(\Lambda_\pm)$) of degree 0. Furthermore, if (X, L) is obtained from concatenating (X', L') and (X'', L'') , then*

$$\Phi_{(X,L)}^{\mathbb{F}[H_1(L; \mathbb{Z})]} \simeq \Phi_{(X'', L'')}^{\mathbb{F}[H_1(L; \mathbb{Z})]} \circ \Phi_{(X', L')}^{\mathbb{F}[H_1(L; \mathbb{Z})]}.$$

Proof. This is a direct consequence of Corollaries 3.10 and 3.11. □

Next consider a 1-parameter family of cobordisms (X_t, L_t) , $t \in [0, 1]$, together with almost complex structures J_t . Assume that the moduli spaces determined by (X_0, L_0, J_0) and (X_1, L_1, J_1) are transversely cut out so that the cobordism maps $\Phi_{(X_0, L_0)}$ and $\Phi_{(X_1, L_1)}$ are well-defined.

Lemma 3.14. *The DGA maps $\Phi_{(X_0, L_0)}$ and $\Phi_{(X_1, L_1)}$ are chain homotopic, i.e., there exists a degree +1 map $K : \mathcal{A}(\Lambda_+) \rightarrow \mathcal{A}(\Lambda_-)$ such that*

$$\Phi_{(X_1, L_1)} + \Phi_{(X_0, L_0)} = \Omega_K \circ \partial_+ + \partial_- \circ \Omega_K, \tag{3.2}$$

where Ω_K is \mathbb{F} -linear and is defined on monomials as follows:

$$\Omega_K(c_1 \dots c_m) = \sum_{j=1}^m \Phi_{(X_1, L_1)}(c_1 \dots c_{j-1}) K(c_j) \Phi_{(X_0, L_0)}(c_{j+1} \dots c_m).$$

Proof. The lemma follows from [E2, Lemma B.15]. Since the terminology of [E2, Lemma B.15] is slightly different from that of this paper, we sketch the argument, referring the reader to [E2, Section B.6] for details.

Consider the moduli space

$$\mathcal{M}(r; \mathbf{s}) = \coprod_{t \in [0, 1]} \mathcal{M}^{(X_t, L_t); J_t}(r; \mathbf{s}),$$

where $(X_t, L_t, J_t), t \in [0, 1]$, is generic. If $\text{ind}(r; \mathbf{s}) = -1$, then:

- (1) $\mathcal{M}^{(X_t, L_t); J_t}(r; \mathbf{s}) \neq \emptyset$ if and only if $t = t_j$ for some $t_j, j = 1, \dots, \ell$, where $0 < t_1 < \dots < t_\ell < 1$; and
- (2) $\#\mathcal{M}^{(X_{t_j}, L_{t_j}); J_{t_j}}(r; \mathbf{s}) = 1$, where $\#$ denotes cardinality.

For simplicity we assume that $\coprod_{\text{ind}(r; \mathbf{s})=-1} \mathcal{M}(r; \mathbf{s}) = \{v\}$ and v occurs at time t^* .

We then use v to construct the chain homotopy. A subtlety is that we need to treat the gluing of an $(a + b + 1)$ -level broken disk $u_{-b} \cup \dots \cup u_a$, where u_0 contains several copies of v . This is done using a *time-ordered, domain-dependent* abstract perturbation scheme, which we describe now.

Let $u : D_{m+1} \rightarrow X$ be a map that is close to breaking into $u_{-b} \cup \dots \cup u_a$. More precisely,

- let $c_{i_1}, \dots, c_{i_j}, i = -b, \dots, a$, be the Reeb chords of the negative ends of u_i , arranged in counterclockwise order around the boundary of the disk obtained by pre-gluing $u_i \cup \dots \cup u_a$;
- let $A_{ij} \subset D_{m+1}, i = -b + 1, \dots, a, j = 1, \dots, j_i$, be rectangles biholomorphic to $[0, 1] \times [\tau_{ij}, \tau'_{ij}]$ for some $\tau_{ij} < \tau'_{ij}$ such that $\{0, 1\} \times [\tau_{ij}, \tau'_{ij}] \subset \partial D_{m+1}$;
- let $A_{-bj} \subset D_{m+1}, j = 1, \dots, j_{-b}$, be half-infinite strips biholomorphic to $[0, 1] \times (-\infty, \tau'_{-bj}]$ for some τ'_{-bj} such that $\{0, 1\} \times (-\infty, \tau'_{-bj}] \subset \partial D_{m+1}$; similarly, let $A_{a+1, 1} \subset D_{m+1}$ be the half-infinite strip corresponding to the positive end of u_a ;
- the A_{ij} are disjoint and $u|_{A_{ij}}$ is close to a strip over c_{ij} ; and
- let $D_{m+1} - \bigcup_{i, j} A_{ij} = B_{-b} \sqcup \dots \sqcup B_a$ such that $u|_{B_i}, i = -b, \dots, a$, is close to u_i with ends truncated.

We will refer to the subscript ij in c_{ij} or A_{ij} as a *subscript at a negative end of u_i* .

Next choose $\epsilon > 0$ small and $N > 0$ large and let

$$\sigma : \mathcal{C}(\Lambda_+) \cup \mathcal{C}(\Lambda_-) \rightarrow (0, \epsilon)$$

be a map such that $\sigma(c)/\sigma(c') > N$ whenever $\mathfrak{A}(c) > \mathfrak{A}(c')$. We then inductively con-

struct the vector

$$(t_{ij}^*), \quad i = -b, \dots, a + 1, \quad j = 1, \dots, j_i$$

as follows: First set $t_{a+1,1}^* = 0$. Suppose we have constructed $t_{i_1}^* < \dots < t_{i_j}^*$. Then $t_{i-1,j}^*, j = 1, \dots, j_{i-1}$, is given by $t_{\tau(i-1,j)}^* + \sigma(c_{\tau(i-1,j)})(p_{i-1,j} - 1)$, where $\tau(i - 1, j)$ is the subscript at the positive end of the component \tilde{u} of u_{i-1} which has $(i - 1, j)$ as a subscript at the negative end, and $(i - 1, j)$ is the $p_{i-1,j}$ th negative end of \tilde{u} , arranged in counterclockwise order.

We use a *domain-dependent* almost complex structure \mathfrak{J}_t to define the $\bar{\partial}$ -operator, i.e., \mathfrak{J}_t depends smoothly on $x \in D_{m+1}$: we set $\mathfrak{J}_t = J_{t+t_{ij}^*}$ on A_{ij} and we extend \mathfrak{J}_t smoothly to B_i so that $\mathfrak{J}_t(x) = J_{t+t(x)}, x \in B_i$, and $t_{i+1,b}^* \leq t(x) \leq t_{ia}^*$, where $(i + 1, b)$ is the subscript at the positive end of the component \tilde{u} corresponding to $x \in B_i$ and (i, a) is the subscript of the last negative end of \tilde{u} .

Let $\mathcal{M}'(r; \mathbf{s})$ be the ‘‘perturbation of $\mathcal{M}(r; \mathbf{s})$ ’’, obtained using domain-dependent \mathfrak{J}_t for curves close to breaking near t^* . We first consider the case $\text{ind}(r; \mathbf{s}) = -1$. Observe that new $\text{ind} = -1$ perturbed disks might get created when the perturbation is turned on: for example, an $\text{ind} = -1$ disk might be created when gluing u_1 with $\text{ind}(u_1) = 1$ and two negative ends to u_0 consisting of two copies of v . On the other hand, all the $\text{ind} = -1$ perturbed broken disks are single-level disks. This is due to the spacing of (t_{ij}^*) and the fact that if $\text{ind}(\tilde{u}) = -1$ for \tilde{u} in some $\mathcal{M}'(r; \mathbf{s})$, then the domain-dependent \mathfrak{J}_t for \tilde{u} is close to J_{t^*} at the positive end of \tilde{u} ; the details are left to the reader.

Next we consider the case $\text{ind}(r; \mathbf{s}) = 0$. As in the case of $\text{ind}(r; \mathbf{s}) = -1$, the spacing for (t_{ij}^*) is chosen so that a broken disk of $\partial\mathcal{M}'(r; \mathbf{s})$ will use an $\text{ind} = -1$ curve of $\coprod_{r', \mathbf{s}'} \mathcal{M}'(r', \mathbf{s}')$ at most once (see Figure 3). Hence for $\text{ind}(r; \mathbf{s}) = 0$ we obtain

$$\partial\mathcal{M}'(r; \mathbf{s}) \simeq \mathcal{M}^{(X_0, L_0); J_0}(r; \mathbf{s}) \sqcup \mathcal{M}^{(X_1, L_1); J_1}(r; \mathbf{s}) \sqcup \mathcal{M}'', \tag{3.3}$$

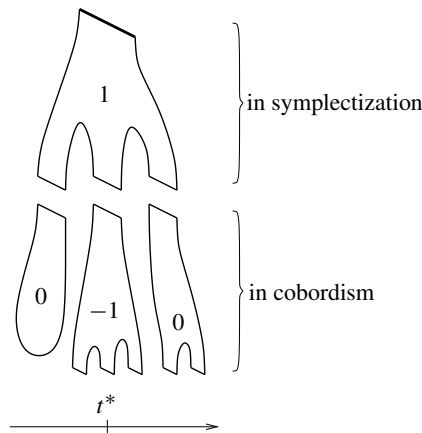


Fig. 3. The numbers indicate indices. Letting the complex structure around the negative punctures in the disks in the positive end depend on the parameter, we arrange that $\text{ind} = -1$ disks are glued at most once.

where \mathcal{M}'' is the set of perturbed broken disks of the form $u_0 \cup u_1$ or $u_{-1} \cup u_0$, where $\text{ind}(u_0) = -1$, $\text{ind}(u_{-1}) = \text{ind}(u_1) = 1$, and an $\text{ind} = -1$ disk appears only once in u_0 .

Finally, if we define $K(r)$, $r \in \Lambda_+$, as follows:

$$K(r) = \sum_{\text{ind}(r;s)=-1} |\mathcal{M}'(r; \mathbf{s})|_s,$$

then (3.2) holds in view of (3.3). □

3.6. Combinatorial description of the Legendrian DGA

Let $\Lambda \subset (\mathbb{R}^3, \xi_0)$ be a Legendrian link. Consider the Lagrangian projection $\Pi_{\mathbb{C}}: \mathbb{R}^3 \rightarrow \mathbb{R}^2$, $(x, y, z) \mapsto (x, y)$. Let J_0 be the adjusted almost complex structure on $\mathbb{R} \times \mathbb{R}^3$ which is induced from the complex structure on $\mathbb{R}^2 \simeq \mathbb{C}$ via $\Pi_{\mathbb{C}}$, i.e., J_0 maps

$$\partial_x \mapsto \partial_y + y\partial_t, \quad \partial_y \mapsto -\partial_x - y\partial_z, \quad \partial_t \mapsto \partial_z, \quad \partial_z \mapsto -\partial_t. \tag{3.4}$$

Then a J_0 -holomorphic disk in $(\mathbb{R} \times \mathbb{R}^3, \mathbb{R} \times \Lambda)$ from a to \mathbf{b} projects to a holomorphic disk in \mathbb{R}^2 whose boundary maps to the Legendrian knot diagram $\Pi_{\mathbb{C}}(\Lambda)$ and whose punctures “map to a and \mathbf{b} ”. On the other hand, by [EES2, Section 2.7], each such disk in \mathbb{R}^2 lifts to a unique \mathbb{R} -invariant family of disks in $\mathbb{R} \times \mathbb{R}^3$.

Thus we are led to the following combinatorial description of Legendrian contact homology, which is the version of the theory originally defined by Chekanov [Ch]: The set of Reeb chords $\mathcal{C}(\Lambda)$ is in one-to-one correspondence with the set of double points of $\Pi_{\mathbb{C}}(\Lambda)$. The Maslov index used to define the grading on $\mathcal{A}(\Lambda)$ is the Maslov index (i.e., twice the rotation number) of $\Pi_{\mathbb{C}}(\Lambda)$. The rigid holomorphic disks correspond to immersed polygons with convex corners at double points of $\Pi_{\mathbb{C}}(\Lambda)$. The sign of a puncture is positive (resp. negative) if the boundary orientation of the disk points towards (resp. away from) the double point along the lower strand and points away from (resp. towards) the double point along the upper strand. Consequently, the computation of the differential is reduced to the combinatorial problem of finding all the immersed polygons with convex corners, boundary on $\Pi_{\mathbb{C}}(\Lambda)$, and exactly one positive puncture.

4. Gradient flow trees of Morse cobordisms

4.1. Gradient flow trees

We briefly summarize the definitions and notation for *gradient flow trees* (or *flow trees* for short) from [E1, Sections 2.2 and 3.1]. We point out that a small part of the terminology used here is slightly different from the terminology used in that paper. We will mainly describe the case of a 2-dimensional Legendrian submanifold $\tilde{L} \subset J^1F$, leaving the simpler case of a 1-dimensional Legendrian link $\Lambda \subset \mathbb{R}^3 = J^1\mathbb{R}$ to Remark 4.8.

Let

$$\Pi_{J^0F} : J^1F \rightarrow J^0F = F \times \mathbb{R}, \quad \Pi_{T^*F} : J^1F \rightarrow T^*F, \quad \Pi_F : J^1F \rightarrow F$$

be the front projection, the Lagrangian projection, and the projection to the base. Let $\tilde{L} \subset J^1F$ be a 2-dimensional Legendrian submanifold. For generic \tilde{L} in its isotopy class, the singular set $\Sigma \subset \tilde{L}$ of Π_F consists of cusp edges and swallowtails (see [E1, equation (2-1) and Remark 2.5]). In particular, for generic \tilde{L} , $\Pi_F(\Sigma)$ admits a stratification:

$$\Pi_F(\Sigma) = \Sigma_1 \supset \Sigma_2^{\text{dbl}} \cup \Sigma_2^{\text{sw}},$$

where Σ_1 consists of all the critical values of Π_F and has codimension 1 in F , Σ_2^{dbl} is the set of transverse double points of Σ_1 , Σ_2^{sw} is the set of swallowtail points, and both Σ_2^{dbl} and Σ_2^{sw} have codimension 2 in F . We write

$$\Sigma_1^\circ = \Sigma_1 \setminus (\Sigma_2^{\text{dbl}} \cup \Sigma_2^{\text{sw}}), \quad \Sigma_0^\circ = \Pi_F(\tilde{L}) \setminus \Sigma_1.$$

On a small neighborhood U_q of a point $q \in \tilde{L} - \Sigma$, $\tilde{L}|_{U_q}$ is given as the 1-jet of a height function $f : \Pi_F(U_q) \rightarrow \mathbb{R}$.

Definition 4.1. Fix a Riemannian metric on F which agrees with the standard flat metric near the boundary and let ∇ denote the corresponding gradient operator. Let I be a compact interval or a half-line $[0, \infty)$.

(1) A flow line of \tilde{L} in F is a curve $\gamma : I \rightarrow F$, together with 1-jet lifts

$$\gamma_i : I \rightarrow \tilde{L} \subset J^1F, \quad \gamma = \Pi_F \circ \gamma_i, \quad i = 1, 2,$$

such that:

- for each $t_0 \in \text{int}(I)$ there is a neighborhood $N_i(t_0)$ of $\gamma_i(t_0) \subset \tilde{L}$ which is given by a height function $f_i : \Pi_F(N_i(t_0)) \rightarrow \mathbb{R}$; and
- on $\Pi_F(N_1(t_0) \cap N_2(t_0))$, γ satisfies the downward gradient equation

$$\dot{\gamma}(t) = -\nabla(f_1 - f_2)(\gamma(t)).$$

(2) The cotangent lifts of a flow line $(\gamma, \gamma_1, \gamma_2)$ are maps

$$\Pi_{T^*F} \circ \gamma_i : I \rightarrow L \subset T^*F, \quad i = 1, 2.$$

(3) The flow orientation is the choice of orientation on the 1-jet lifts γ_1, γ_2 such that locally $\Pi_F \circ \gamma_1$ is oriented by $-\nabla(f_1 - f_2)$ and $\Pi_F \circ \gamma_2$ by $-\nabla(f_2 - f_1)$.

Remark 4.2. A flow line is an immersion except when it is a constant map to a critical point of $f_1 - f_2$.

Definition 4.3. A source tree Γ is a metric space which is either a copy of \mathbb{R} or a tree with edges that are compact intervals or half-lines $[0, \infty)$. We will assume that Γ is connected unless stated otherwise. The endpoints of the edges are the vertices and the points at infinity of the half-lines or lines are the 1-valent punctures. We also designate certain vertices as interior punctures. Furthermore, at any vertex of the tree there is a cyclic ordering of adjacent edges. Let $E(\Gamma), V(\Gamma), P(\Gamma)$ be the sets of edges, vertices, and punctures (both 1-valent and interior) of Γ .

Definition 4.4 (Gradient flow tree). Let Γ be a source tree. A *gradient flow tree* of \tilde{L} is a map $\gamma : \Gamma \rightarrow F$ (called a *flow tree map*), together with 1-jet lifts γ_1^e, γ_2^e for each $e \in E(\Gamma)$, such that:

- (1) for $e \in E(\Gamma)$, $(\gamma|_e, \gamma_1^e, \gamma_2^e)$ is a flow line of \tilde{L} ;
- (2) a neighborhood of each 1-valent puncture maps to a flow line into or out of a critical point of some height function difference $f_i - f_j$, i.e., a Reeb chord;
- (3) the *cotangent lift* $C(\gamma)$ of γ , i.e., the union of the closures of the cotangent lifts $\Pi_{T^*F} \circ \gamma_i^e, i = 1, 2$, over all $e \in P(\Gamma)$, is a closed oriented curve; and
- (4) a vertex $v \in V(\Gamma)$ is an interior puncture if and only if there are adjacent cotangent lifts $\Pi_{T^*F} \circ \gamma_{i_1}^{e_1}$ and $\Pi_{T^*F} \circ \gamma_{i_2}^{e_2}$ of $C(\gamma)$ at v such that $\gamma_{i_1}^{e_1}(v) \neq \gamma_{i_2}^{e_2}(v)$ and are connected by a Reeb chord.

The restriction of a flow tree map to a half-infinite edge may be constant by Remark 4.2. See Figure 4 for examples of 1-valent punctures and interior punctures.

The cotangent and 1-jet lifts are oriented using the flow orientation. Near a puncture p , one of the 1-jet lifts of the edge e adjacent to p is incoming (i.e., oriented towards the critical point), and the other is outgoing (i.e., oriented away from the critical point).

Definition 4.5. A puncture of γ is *positive* if the height function of the incoming 1-jet lift is smaller than that of the outgoing 1-jet lift; otherwise it is *negative*.

The 1-jet lift $\tilde{C}(\gamma)$ of a flow tree γ is a union of paths which connect the endpoints of Reeb chords, just like the boundary of a holomorphic disk. In particular, $\tilde{C}(\gamma)$ determines $(a_1, \dots, a_k; \mathbf{b}_1, \dots, \mathbf{b}_k)$, where $a_1, \dots, a_k \in \mathcal{C}(\tilde{L})$ correspond to the positive punctures and $\mathbf{b}_1, \dots, \mathbf{b}_k \in \mathcal{A}(\tilde{L})$ correspond to the negative punctures; it also determines homology classes associated to the capped paths connecting them.

Definition 4.6. Two flow trees

$$(\gamma : \Gamma \rightarrow F; \gamma_i^e, i = 1, 2, e \in E(\Gamma)), (\gamma' : \Gamma' \rightarrow F; (\gamma')_i^e, i = 1, 2, e \in E(\Gamma'))$$

are *equivalent* if there is an isometry $\phi : \Gamma \xrightarrow{\sim} \Gamma'$ such that $\gamma = \gamma' \circ \phi, \gamma_i^e = (\gamma')_i^{\phi(e)} \circ \phi|_e$ for all $e \in E(\Gamma)$, and the cyclic orders around the vertices are preserved.

Definition 4.7. A *partial flow tree* $\gamma : \Gamma \rightarrow F$ satisfies the conditions of Definition 4.4 with the exception of (2). A 1-valent vertex of Γ where the cotangent lift of γ is nonclosed is called a *special vertex*. We write $V(\Gamma)$ for the set of vertices of Γ and $SV(\Gamma) \subset V(\Gamma)$ for the subset of special vertices. The sign of a special vertex is defined in the same way as in Definition 4.5.

For example, if we cut a flow tree in two along an edge, then we obtain two partial flow trees, each with one special puncture.

Remark 4.8. The definition of a flow tree of a 1-dimensional Legendrian link is exactly the same as above. Here the situation is simpler in that $\Sigma_1 = \Sigma_1^\circ$, i.e., generically the only singularities of the front projection are isolated cusps.

4.2. Formal dimension

Let $\mathcal{T}_{\tilde{L}}(a_1, \dots, a_k; \mathbf{b}_1, \dots, \mathbf{b}_k)$ be the space of flow trees from a_1, \dots, a_k to $\mathbf{b}_1, \dots, \mathbf{b}_k$ modulo equivalence. The formal dimension of $\gamma \in \mathcal{T}_{\tilde{L}} := \mathcal{T}_{\tilde{L}}(a_1, \dots, a_k; \mathbf{b}_1, \dots, \mathbf{b}_k)$ is given by

$$\dim(\gamma) = \dim(\mathcal{T}_{\tilde{L}}) = \sum_{i=1}^k (|a_i|_{\tilde{L}} - |\mathbf{b}_i|_{\tilde{L}}) + k - 2. \tag{4.1}$$

This formal dimension agrees with the Fredholm index of disks from a_1, \dots, a_k to $\mathbf{b}_1, \dots, \mathbf{b}_k$ in the cotangent bundle T^*F , which is one less than the Fredholm index in the symplectization $\mathbb{R} \times J^1F$, given by (3.1). This is due to the extra \mathbb{R} -translation.

We rewrite (4.1) in terms of Morse-theoretic data as in [E1, Definition 3.4]: Let n be the dimension of the Legendrian submanifold (which in our applications will be 1 or 2). Let $P_{\pm}(\gamma)$ be the set of positive/negative punctures of γ and let $R(\gamma)$ be the set of vertices of γ that are not punctures.

Definition 4.9 (Morse index $I(p)$). If $p \in P_{\pm}(\gamma)$, then let c be the Reeb chord corresponding to p and let f^+, f^- be the height functions for the two sheets containing c^{\pm} . Then the Morse index $I(p)$ is the Morse index of the height function difference $f^+ - f^-$ at p .

Definition 4.10 (Maslov content $\mu(r)$). Let $r \in R(\gamma)$. If $x \in \Sigma$ is a cusp point over r which lies in the 1-jet lift $\tilde{C}(\gamma)$, then let $\tilde{\mu}(x) = 1$ (resp. $\tilde{\mu}(x) = -1$) if the incoming arc of $\tilde{C}(\gamma)$ at x lies on the upper (resp. lower) sheet and the outgoing arc of $\tilde{C}(\gamma)$ lies on the lower (resp. upper) sheet. The Maslov content of r is

$$\mu(r) = \sum_x \tilde{\mu}(x),$$

where the sum is over all cusp points $x \in \Sigma$ over r .

Then we have

$$\dim(\gamma) = (n - 3) + \sum_{p \in P_+(\gamma)} (I(p) - (n - 1)) - \sum_{q \in P_-(\gamma)} (I(q) - 1) + \sum_{r \in R(\gamma)} \mu(r). \tag{4.2}$$

4.3. Generic flow trees on \tilde{L}

Suppose \tilde{L} is generic. Then by [E1, Theorem 1.1], $\mathcal{T}_{\tilde{L}} := \mathcal{T}_{\tilde{L}}(a; \mathbf{b})$ is a stratified space with strata that are manifolds, and its top-dimensional stratum has dimension $\dim(\mathcal{T}_{\tilde{L}})$. In particular, $\dim(\mathcal{T}_{\tilde{L}}) < 0$ implies $\mathcal{T}_{\tilde{L}} = \emptyset$, and $\dim(\mathcal{T}_{\tilde{L}}) = 0$ implies that $\mathcal{T}_{\tilde{L}}$ is a finite collection of flow trees that are transversely cut out. (A flow tree is transverse if it satisfies the preliminary transversality conditions from [E1, Section 3.1.1] and the conditions indicated in the proof of [E1, Proposition 3.14].)

A flow tree $\gamma : \Gamma \rightarrow F$ in $\mathcal{T}_{\tilde{L}}$ of dimension zero only has vertices of valency ≤ 3 whose neighborhoods are given as follows:

- (1) 1-valent punctures;
- (2) 2-valent interior punctures;
- (3) ends;

- (4) Y_0 -vertices;
- (5) Y_1 -vertices; and
- (6) switches.

See Figures 4, 5, and 6 for some examples. Observe that a flow line limits to a cusp edge corresponding to an end in finite time (cf. [E1, Lemma 2.8]).

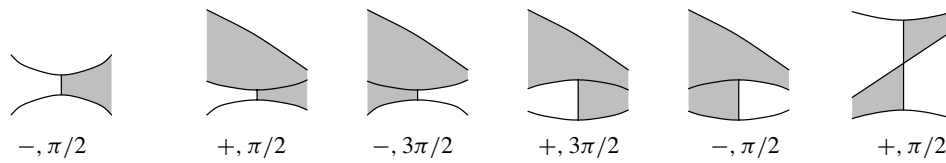


Fig. 4. A 1-valent puncture (on the far left) and several 2-valent interior punctures, drawn in the front projection. The sign of the puncture and the angle made in the Lagrangian projection are given.

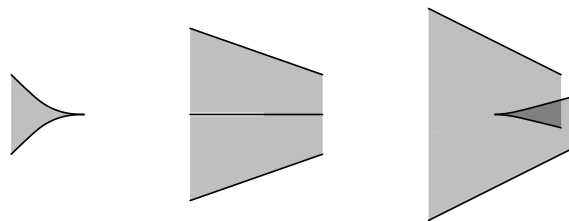


Fig. 5. From left to right, an end, a Y_0 -vertex, and a Y_1 -vertex, drawn in the front projection.

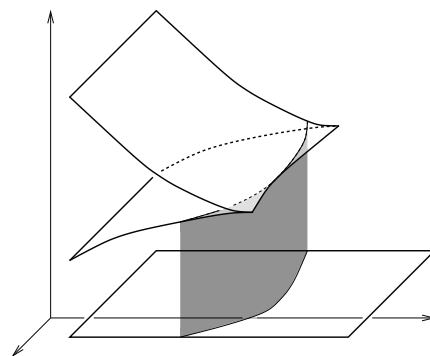


Fig. 6. A switch, drawn in the front projection.

4.4. Flow trees in Morse cobordisms

Let L^{Mo} be a Morse cobordism in T^*F from Λ_+ to Λ_- .

Lemma 4.11. *The 1-jet lift of a (connected) flow tree $\gamma : \Gamma \rightarrow F$ of $\tilde{L}^{\text{Mo}} \subset J^1F$ is contained in $J^1F|_{\partial F}$ if and only if $\gamma(x) \in \partial F$ for some $x \in \Gamma$. Furthermore, flow trees of \tilde{L}^{Mo} that are contained in $J^1F|_{\partial_+ F}$ and $J^1F|_{\partial_- F}$ are in natural one-to-one correspondence with flow trees of Λ_- and Λ_+ , respectively.*

Proof. This follows from Definition 2.3. \square

Regularity. If we view a flow tree in $\mathcal{T}_{\Lambda_+}(a; \mathbf{b})$ as a flow tree $\gamma : \Gamma \rightarrow \partial_+ F$ in $\mathcal{T}_{\tilde{L}^{\text{Mo}}}(a; \mathbf{b})$, then

$$\dim \mathcal{T}_{\tilde{L}^{\text{Mo}}}(a; \mathbf{b}) = \dim \mathcal{T}_{\Lambda_+}(a; \mathbf{b}) + 1 - \#\mathbf{b}, \quad (4.3)$$

where $\#\mathbf{b}$ is the number of ends of \mathbf{b} . If $\#\mathbf{b} > 1$ and $\dim \mathcal{T}_{\Lambda_+}(a; \mathbf{b}) = 0$, then there exist flow trees $\gamma : \Gamma \rightarrow \partial_+ F$ of $\tilde{L}^{\text{Mo}}|_{\partial_+ F}$ that are not generic as flow trees of \tilde{L}^{Mo} . This indicates that generically the Reeb chords of \tilde{L}^{Mo} in $J^1 F|_{\partial_+ F}$ would not be “aligned”, i.e., would not all lie above $\partial_+ F$.

However, for the purposes of this paper, there is no need to perturb out of this situation: A priori we may have a broken flow tree $\gamma_0 \cup \gamma_1$ (i.e., a flow tree with more than one level) from $a \in \mathcal{C}(\Lambda_+)$ to $\mathbf{b} \in \mathcal{A}(\Lambda_-)$, where γ_1 is a tree from a to \mathbf{b}' which is contained in $\tilde{L}^{\text{Mo}}|_{\partial_+ F}$ and γ_0 is the union of trees from \mathbf{b}' to \mathbf{b} . Since $\dim \mathcal{T}_{\Lambda_+}(a; \mathbf{b}') \geq 0$ and each of the $\#\mathbf{b}'$ components of γ_0 has $\dim \geq 0$,

$$\begin{aligned} \dim(\gamma_0 \cup \gamma_1) &\geq \dim \mathcal{T}_{\tilde{L}^{\text{Mo}}}(a; \mathbf{b}') + 0 + \#\mathbf{b}' \\ &= (\dim \mathcal{T}_{\Lambda_+}(a; \mathbf{b}') + 1 - \#\mathbf{b}') + \#\mathbf{b}' \geq 1, \end{aligned}$$

where the $\#\mathbf{b}'$ in the first line comes from the $\#\mathbf{b}'$ gluing conditions. Therefore, by perturbing outside a neighborhood of $J^1 F|_{\partial_+ F}$, the flow trees of $\dim(\mathcal{T}_{\tilde{L}}(a; \mathbf{b})) = 0$ that are not entirely contained in $J^1 F|_{\partial_+ F}$ are transversely cut out.

5. Holomorphic disks and flow trees

The goal of this section is to prove Theorem 1.6, i.e., the correspondence between rigid holomorphic disks and rigid Morse flow trees. The proof is an extension of the corresponding results in the compact case from [E1] to immersed exact Lagrangian submanifolds with cylindrical or conical ends.

This section is organized as follows: In Section 5.1 we show that certain rigid flow trees of a Morse cobordism L^{Mo} are in bijection with the rigid long conical flow trees (see Definition 5.1) of the associated long conical cobordisms $L_{\delta; \sigma}$. In Section 5.2 we make a small perturbation of $L_{\delta; \sigma}$ as in [E1, Section 4] that allows us to better control holomorphic disks under scaling and introduce an almost complex structure $J_{\delta; \sigma}$ on X_δ which agrees with the almost complex structure on T^*F induced by a Riemannian metric on F in the sense of [E1, Section 4.4] and is adjusted to the symplectization of (\mathbb{R}^3, α_0) at the ends. We then establish a key subharmonic estimate in Section 5.4 which allows us to gain control of holomorphic disks under rescaling and thereby prove the convergence of rigid holomorphic disks to rigid flow trees as $\sigma \rightarrow 0$ in Section 5.5. In Section 5.6 we construct rigid holomorphic disks near rigid flow trees and show that the construction captures all rigid disks.

5.1. Morse cobordisms, conical cobordisms, and flow trees

We use the notation from Section 2. Consider a Morse cobordism $L^{Mo} \subset T^*F$. In Section 2.4 we associated a family of long conical cobordisms $(X_\delta, L_{\delta;\sigma})$ to L^{Mo} , parametrized by $\delta, \sigma > 0$ small.

A Reeb chord flow line is a flow line $\{x\} \times [a_+, \infty) \subset F_\delta$ or $\{x\} \times (a'_-(\delta), a_-] \subset F_\delta$ which corresponds to a cylinder over a Reeb chord in $\mathcal{C}(\Lambda_\pm)$.

Definition 5.1. A long conical (LC) flow tree $\gamma : \Gamma \rightarrow F_\delta$ of $L_{\delta;\sigma}$ satisfies the conditions of Definition 4.4 with (2) and (3) replaced by:

- (2') a neighborhood of each puncture $p \in P(\Gamma)$ (i.e., a half-line) either limits to a critical point of some height function difference $f_i - f_j$ or is asymptotic to a Reeb chord flow line; and
- (3') if we partially compactify T^*F_δ by (i) attaching $\mathbb{R}^3 \times \{a'_-(\delta)\}$ and $\mathbb{R}^3 \times \{\infty\}$ and (ii) compactifying the cotangent fibers, and compactify the cotangent lift $C(\gamma)$ of γ to $\overline{C(\gamma)}$ by attaching suitable arcs corresponding to $c \times \{a'_-(\delta)\}$ or $c \times \{\infty\}$, where $c \in \mathcal{C}(\Lambda_\pm)$, then $\overline{C(\gamma)}$ is a closed oriented curve.

Lemma 5.2. There is a bijection between rigid flow trees of L^{Mo} that are not contained in $T^*F|_{\partial F}$ and rigid LC flow trees of $L_{\delta;\sigma}$.

Proof. The lemma follows from constructing a diffeomorphism $\phi_\delta : F - \partial F \xrightarrow{\sim} F_\delta$ which maps the gradient flow lines of L^{Mo} to the gradient flow lines of $L_{\delta;\sigma}$ after reparametrizations. (Note that fiber scaling does not alter gradient flow lines.)

The following model calculation can be generalized to give ϕ_δ : Let $I \subset \mathbb{R}$ be an interval and let $f = f_1 - f_2 : I \rightarrow \mathbb{R}$ be a height function difference. Consider the height function difference $(1 - t^2)f(s)$ on $I \times (-\varepsilon, 0)$ with coordinates (s, t) and the height function difference $\tau f(s)$ on $I \times (0, \infty)$ with coordinates (s, τ) . Their gradients are

$$((1 - t^2)\nabla f(s), -2tf(s)), \quad (\tau\nabla f(s), f(s)),$$

which are directed by

$$X_1 := \left(\nabla f(s), \frac{2t}{t^2 - 1} f(s) \right), \quad X_2 := \left(\nabla f(s), \frac{1}{\tau} f(s) \right).$$

We are looking for a function $\tau = g(t)$ such that

$$\phi : I \times (-\varepsilon, 0) \rightarrow I \times (0, \infty), \quad (s, t) \mapsto (s, g(t)),$$

is a diffeomorphism and satisfies $\phi_* X_1 = X_2$. Solving the differential equation $\frac{2t}{t^2 - 1} \frac{d\tau}{dt} = \frac{1}{\tau}$ gives $g(t) = \sqrt{t^2/2 - \log|t|} + c$, and for an appropriate c the desired condition holds. Finally, observe that ϕ is independent of the height function difference $f = f_1 - f_2$. \square

Given $\epsilon > 0$, let $N'_{\epsilon,\pm}$ denote the ϵ -neighborhood of the Reeb chord endpoints in Λ_\pm and let $N_\epsilon = (N'_{\epsilon,+} \times [a_+, \infty)) \cup (N'_{\epsilon,-} \times (a'_-(\delta), a_-])$.

Corollary 5.3. *If $\delta > 0$ is sufficiently small, then the restriction of any rigid LC flow tree of $L_{\delta;\sigma}$ to $X_\delta - T^*F$ is contained in $N_{C\delta}$ for some $C > 0$.*

Proof. As $\delta \rightarrow 0$, the diffeomorphism ϕ_δ^{-1} maps $F_\delta - F$ to smaller and smaller neighborhoods of ∂F .

We first prove the corollary with N_ϵ instead of $N_{C\delta}$ for any sufficiently small $\epsilon > 0$. If this weaker version of the corollary does not hold, then there is a sequence of rigid flow trees in L^{Mo} which converges to a broken flow tree with some level in $T^*F|_{\partial F}$, which necessarily has dimension at least one. This is a contradiction, which proves the corollary with N_ϵ instead of $N_{C\delta}$.

Now fix $\epsilon > 0$. If γ is a rigid flow tree of L^{Mo} and $\hat{\gamma}$ is the corresponding rigid LC flow tree of $L_{\delta;\sigma}$, then γ and $\hat{\gamma}$ agree on

$$\check{F}^\delta := F - \mathbb{R} \times ([a_-, a_- + \delta] \cup [a_+ - \delta, a_+]).$$

Moreover the endpoints of γ and $\hat{\gamma}$ on $\partial\check{F}^\delta$ are contained in $N'_{\epsilon,\pm}$ for $\delta > 0$ sufficiently small. The corollary then follows by looking at the gradient of a height function difference of L^{Mo} near a critical point and the continuation of $\gamma|_{\check{F}^\delta}$ as it approaches the critical point. \square

5.2. Almost complex structures and deformed Legendrian links

In this subsection we introduce small deformations of Λ_\pm and $L_{\delta;\sigma}$ and construct a family of almost complex structures on X_δ .

5.2.1. Deforming Legendrian links. By a preliminary small Legendrian isotopy, we may assume that the Lagrangian projection of Λ_\pm consists of straight lines near its double points. We then perturb Λ_\pm slightly in a neighborhood of all its Reeb chords by making the z -coordinates of Λ_\pm constant in a small neighborhood of the Reeb chord endpoints without changing the Lagrangian projection. The resulting link $\tilde{\Lambda}_\pm$ is not necessarily Legendrian. Similarly, we perturb the ends of $L_{\delta;\sigma}$ so that the resulting totally real submanifold $\tilde{L}_{\delta;\sigma}$ has conical ends over $\tilde{\Lambda}_\pm$.

We will consider holomorphic disks with boundary condition given by $\mathbb{R} \times \tilde{\Lambda}_\pm$ and $\tilde{L}_{\delta;\sigma}$, which are only totally real and not Lagrangian. It is well-known that in general the theory for holomorphic disks with boundary on totally real manifolds is very different from the Lagrangian case. As is clear from their definition, the totally real submanifolds considered here are very special: they have Lagrangian projections and lie very close to actual Lagrangian submanifolds. We do not intend to develop any general theory for holomorphic disks with totally real boundary conditions; rather we use these very specific totally real manifolds together with specific almost complex structures as a technical tool to obtain a combinatorial description of Legendrian contact homology chain maps. The necessary analytical estimates in this context are developed in Lemmas 5.4, 5.6, 5.8, 5.9, and 5.10.

Lemma 5.4. *Let J_0 be the almost complex structure on $\mathbb{R} \times \mathbb{R}^3$ given by (3.4). If Λ_\pm are generic, then for $\tilde{\Lambda}_\pm$ sufficiently close to Λ_\pm :*

(1) *There is a diffeomorphism*

$$\mathcal{M}^{(\mathbb{R} \times \mathbb{R}^3, \mathbb{R} \times \Lambda_{\pm}); J_0}(a; \mathbf{b}) \simeq \mathcal{M}^{(\mathbb{R} \times \mathbb{R}^3, \mathbb{R} \times \tilde{\Lambda}_{\pm}); J_0}(a; \mathbf{b}).$$

(2) *Consider DGAs associated to $\tilde{\Lambda}_{\pm}$ in direct analogy with the DGAs associated to Λ_{\pm} . Then (1) implies that the DGA of $\tilde{\Lambda}_{\pm}$ is canonically isomorphic to that of Λ_{\pm} . There are totally real cobordisms from $\tilde{\Lambda}_{\pm}$ to Λ_{\pm} (and vice versa) which induce the identity map on DGAs.*

Proof. (1) is immediate from the fact that in both cases the holomorphic disks are determined up to translation by their projections to \mathbb{C} . (2) is similar. Note that deforming only the z -coordinate in the cobordisms commutes with the projection to \mathbb{C} . Hence the only rigid disks of the cobordism are trivial strips. \square

5.2.2. *The almost complex structure J .* Let g be a Riemannian metric on F which is flat near ∂F and let g_{δ} be its extension to F_{δ} which restricts to the standard flat metric on $\mathbb{R} \times (a'_-(\delta), a_-]$ and $\mathbb{R} \times [a_+, \infty)$. Let J be the almost complex structure on X_{δ} which is compatible with g_{δ} in the sense of [E1, Section 4.4]. Although the precise definition of J is not important here, J is given by

$$\partial_{\xi_j} \mapsto \partial_{\eta_j}, \quad \partial_{\eta_j} \mapsto -\partial_{\xi_j}, \quad j = 1, 2,$$

on the flat parts $T^*(\mathbb{R} \times (a'_-(\delta), a_-])$ and $T^*(\mathbb{R} \times [a_+, \infty))$.

We now apply the coordinate change

$$\Phi_+ : [0, \infty) \times \mathbb{R}^3 \rightarrow T^*(\mathbb{R} \times [1, \infty)), \quad (t, x, y, z) \mapsto (x, e^t y, e^t, z),$$

from Section 2.1. (After composing with a ξ_2 -translation, we may assume that $a_+ = 1$.) At the positive symplectization end the almost complex structure J can be written as follows:

$$\partial_x \mapsto e^{-t} \partial_y, \quad \partial_y \mapsto -e^t \partial_x, \quad \partial_t \mapsto e^t \partial_z - e^t y \partial_x, \quad \partial_z \mapsto -e^{-t} \partial_t + e^{-t} y \partial_y.$$

The situation for the negative symplectic end is similar.

5.2.3. *The family of almost complex structures $J_{\delta; \sigma}$.* Next we define a family of almost complex structures $J_{\delta; \sigma}$ on X_{δ} that interpolates between J and the almost complex structure J_0 given by (3.4).

First consider the positive symplectization end which we take to be $[0, \infty) \times \mathbb{R}^3$ without loss of generality. Fix a sufficiently large constant $K \gg 1$. For $0 \leq t \leq \frac{1}{2} K \sigma$ and $|y| \leq 1$, $J_{\delta; \sigma}$ is defined as follows:

$$\begin{aligned} \partial_x &\mapsto k(t)^{-1} \partial_y, & \partial_y &\mapsto -k(t) \partial_x, & \partial_t &\mapsto k(t) \partial_z - k(t) b(t) y \partial_x, \\ \partial_z &\mapsto -k(t)^{-1} \partial_t + k(t)^{-1} b(t) y \partial_y, \end{aligned} \tag{5.1}$$

where $k(t) = e^t$ near $t = 0$ and $k(t) = 1$ near $t = \frac{1}{2} K \sigma$, and $b(t) = 1$ near $t = 0$ and $b(t) = 0$ near $t = \frac{1}{2} K \sigma$. Here $k(t)$ and $b(t)$ (as well as $a(t)$ below) depend on $\sigma > 0$.

Hence $J_{\delta;\sigma}$ is given as follows near $t = \frac{1}{2}K\sigma$:

$$\partial_x \mapsto \partial_y, \quad \partial_y \mapsto -\partial_x, \quad \partial_t \mapsto \partial_z, \quad \partial_z \mapsto -\partial_t. \quad (5.2)$$

For $\frac{1}{2}K\sigma \leq t \leq K\sigma$ and $|y| \leq 1$, $J_{\delta;\sigma}$ is defined as follows:

$$\partial_x \mapsto \partial_y + a(t)y\partial_t, \quad \partial_y \mapsto -\partial_x - a(t)y\partial_z, \quad \partial_t \mapsto \partial_z, \quad \partial_z \mapsto -\partial_t, \quad (5.3)$$

where $a(t) = 0$ near $t = \frac{1}{2}K\sigma$ and $a(t) = 1$ near $t = K\sigma$.

Lemma 5.5. *On the region $\{0 \leq t \leq K\sigma, |y| \leq 1\}$, $J_{\delta;\sigma}$ is tamed by the canonical symplectic form ω on T^*F_δ .*

Proof. For $\frac{1}{2}K\sigma \leq t \leq K\sigma, |y| \leq 1$ this is immediate from (5.3). We verify the assertion for $0 \leq t \leq \frac{1}{2}K\sigma, |y| \leq 1$. By the identification Φ_\pm from Section 2.1 we may assume that $\omega = e^t(dt \wedge (dz - ydx) + dx \wedge dy)$. We compute

$$\begin{aligned} & e^{-t}\omega(a_1\partial_x + a_2\partial_y + a_3\partial_t + a_4\partial_z, J_{\delta;\sigma}(a_1\partial_x + a_2\partial_y + a_3\partial_t + a_4\partial_z)) \\ &= (dx \wedge dy + dt \wedge (dz - ydx))(a_1\partial_x + a_2\partial_y + a_3\partial_t + a_4\partial_z, a_1k(t)^{-1}\partial_y - a_2k(t)\partial_x \\ &\quad + a_3(k(t)\partial_z - k(t)b(t)y\partial_x) + a_4(-k(t)^{-1}\partial_t + k(t)^{-1}b(t)y\partial_y)) \\ &= a_1^2k(t)^{-1} + a_1a_4k(t)^{-1}b(t)y + a_2^2k(t) + a_2a_3k(t)b(t)y \\ &\quad + a_3^2k(t) + a_2a_3k(t)y + a_3^2k(t)b(t)y^2 + a_4^2k(t)^{-1} - a_1a_4k(t)^{-1}y. \end{aligned}$$

Since $|y| \leq 1$ and we may take $|b(t) - 1| \leq 1$, the term $a_1^2 + (b(t) - 1)ya_1a_4 + a_4^2$ is positive. Similarly, since we may take $|b(t) + 1| \leq 2$, the term $a_2^2 + (b(t) + 1)ya_2a_3 + a_3^2$ is also positive. The positivity of the whole expression follows. \square

For $0 \leq t \leq K\sigma$ and $|y| \geq 1$, we arbitrarily extend $J_{\delta;\sigma}$ so that $J_{\delta;\sigma}$ is tamed by the canonical symplectic form ω on T^*F_δ . The precise form of $J_{\delta;\sigma}$ on this region is not important since the holomorphic curves that we consider do not enter this region by Sections 5.3 and 5.4.

Finally, $J_{\delta;\sigma}$ is t -translation invariant for $t \geq K\sigma$:

$$\partial_x \mapsto \partial_y + y\partial_t, \quad \partial_y \mapsto -\partial_x - y\partial_z, \quad \partial_t \mapsto \partial_z, \quad \partial_z \mapsto -\partial_t. \quad (5.4)$$

Similarly, we define $J_{\delta;\sigma}$ on the negative end using a completely analogous interpolation in a slice of width $K\sigma$.

5.3. A priori energy bounds and monotonicity

Lemma 5.6. *If $\tilde{L}_{\delta;\sigma}$ is sufficiently close to $L_{\delta;\sigma}$, then there exists $C > 1$ such that*

$$E_{\omega_0}(u) \leq \mathfrak{A}(a), \quad E_{\beta_\pm}(u) \leq C\mathfrak{A}(a) \quad \text{for any } u \in \mathcal{M}^{(X_\delta, \tilde{L}_{\delta;\sigma}); J_{\delta;\sigma}}(a; \mathbf{b}).$$

Proof. If $v \in \mathcal{M}^{(X_\delta, L_{\delta;\sigma}); J_{\delta;\sigma}}(a; \mathbf{b})$, then $E_{\omega_0}(v) \leq \mathfrak{A}(a)$ and $E_{\beta_\pm}(v) \leq \mathfrak{A}(a)$ by Lemma 3.6. Let $u \in \mathcal{M}^{(X_\delta, \tilde{L}_{\delta;\sigma}); J_{\delta;\sigma}}(a; \mathbf{b})$. The first inequality $E_{\omega_0}(u) \leq \mathfrak{A}(a)$ is immediate from the definition of $\tilde{L}_{\delta;\sigma}$ in Section 5.2.1. By the proof of Lemma 5.4, there is a diffeomorphism

$$\phi : \mathcal{M}^{(X_\delta, L_{\delta;\sigma}); J_{\delta;\sigma}}(a; \mathbf{b}) \xrightarrow{\sim} \mathcal{M}^{(X_\delta, \tilde{L}_{\delta;\sigma}); J_{\delta;\sigma}}(a; \mathbf{b})$$

such that $\phi(u)$ is C^1 -close to u . This implies that $E_{\beta_\pm}(u) \leq C\mathfrak{A}(a)$. □

Next we recall the monotonicity lemma for holomorphic curves (see for example [Si, Proposition 4.3.1]). Let h_δ be a metric which agrees with the standard Euclidean metric on the ends $[0, \infty) \times \mathbb{R}^3$ and $(-\infty, 0] \times \mathbb{R}^3$ and is commensurate with the metric on T^*F induced by g . All the distances will be measured with respect to h_δ . In particular, $B(p, r)$ is the ball of radius r around $p \in X_\delta$ with respect to h_δ .

Lemma 5.7 (Monotonicity lemma). *There exist $r_0, C > 0$ such that*

$$E_{\omega_0}(u) + E_{\beta_+}(u) + E_{\beta_-}(u) \geq C \cdot r^2 \tag{5.5}$$

for any $0 < r < r_0$ and any $J_{\delta;\sigma}$ -holomorphic map $u : M \rightarrow X_\delta$ with $u(0) = p$ and $u(\partial M) \subset \partial B(p, r)$.

We use the monotonicity lemma to obtain the following a priori $O(\sigma^{1/2})$ bounds on $|y|$ and $|z|$, where $|y|$ and $|z|$ are measured with respect to the standard Euclidean metric on \mathbb{R}^3 . In Section 5.4 we improve the $O(\sigma^{1/2})$ bound that comes from monotonicity to an $O(\sigma)$ bound that comes from the maximum principle for harmonic functions.

Lemma 5.8. *If $u \in \mathcal{M}^{(X_\delta, \tilde{L}_{\delta;\sigma}); J_{\delta;\sigma}}(a; \mathbf{b})$, then $|y \circ u| = O(\sigma^{1/2})$ and $|z \circ u| = O(\sigma^{1/2})$ on the region $\{0 \leq t \leq K\sigma\}$.*

Proof. Towards a contradiction, suppose there are sequences $\sigma_i \rightarrow 0$, $c_i \rightarrow \infty$, $u_i \in \mathcal{M}^{(X_\delta, \tilde{L}_{\delta;\sigma_i}); J_{\delta;\sigma_i}}(a; \mathbf{b})$, and p_i in the domain of u_i , such that $|y \circ u_i|(p_i) > c_i \sigma_i^{1/2}$ and $u_i(p_i) \in \{0 \leq t \leq K\sigma_i\}$. Then there is a ball $B(q_i, (c_i/3)\sigma_i^{1/2})$ such that:

- (i) u_i passes through q_i , where $q_i \in \{0 \leq t \leq K\sigma_i\}$; and
- (ii) $B(q_i, (c_i/3)\sigma_i^{1/2}) \cap \tilde{L}_{\delta;\sigma_i} = \emptyset$.

Here (ii) follows from the fact that the distance between $\tilde{L}_{\delta;\sigma}$ and the 0-section is $O(\sigma)$. By the Monotonicity Lemma 5.7,

$$E_{\omega_0}(u) + E_{\beta_+}(u) + E_{\beta_-}(u) \geq Cc_i^2\sigma/9.$$

On the other hand, by Lemma 5.6, $E_{\omega_0}(u) + E_{\beta_+}(u) + E_{\beta_-}(u)$ is bounded above by a constant times σ , a contradiction. This proves the lemma for $|y \circ u|$. The argument for $|z \circ u|$ is the same. □

In what follows we will work with $\tilde{\Lambda}_\pm$ and $\tilde{L}_{\delta;\sigma}$ and omit the tildes in the notation.

5.4. Subharmonicity

Define a function $p_{\delta;\sigma}^2 : X_\delta \rightarrow \mathbb{R}$ as follows:

$$p_{\delta;\sigma}^2(q) = \begin{cases} \eta_1^2 + \eta_2^2 & \text{for } q \in T^*F, \\ (k(t)y)^2 + z^2 & \text{for } q \notin T^*F, \end{cases}$$

where $k(t) = k_\sigma(t)$ is the interpolation function used in the definition of $J_{\delta;\sigma}$. Here $k(t)$ was defined on $[0, \frac{1}{2}K\sigma]$ and is extended by setting $k(t) = 1$ on $t \geq \frac{1}{2}K\sigma$.

Lemma 5.9. *There exist $K \gg 1$ and $\sigma_0, \delta_0 > 0$ such that, for any $0 < \sigma < \sigma_0, 0 < \delta < \delta_0$, and $u \in \mathcal{M}^{(X_\delta, L_{\delta;\sigma}); J_{\delta;\sigma}}(a; \mathbf{b})$, the function $p_{\delta;\sigma}^2 \circ u$ is subharmonic with respect to the metric g_δ . In particular, $p_{\delta;\sigma}^2 \circ u$ achieves its maximum on ∂D .*

Proof. On the region $t \geq K\sigma$, $k(t) = 1$ is constant. By a straightforward calculation the y - and z -coordinates are harmonic. This in turn implies that y^2 and z^2 are both subharmonic. On the other hand, $\eta_1^2 + \eta_2^2$ is subharmonic on T^*F by [E1, Section 5.1.2 and Lemma 5.5]. It remains to verify the subharmonicity on the interpolation region $0 \leq t \leq K\sigma$.

Consider the region $0 \leq t \leq \frac{1}{2}K\sigma$. The Cauchy–Riemann equations

$$\partial_{\tau_1} u + J_{\delta;\sigma} \partial_{\tau_2} u = 0$$

imply that

$$\begin{aligned} \partial_{\tau_1}(k(t)y) &= k(t)\partial_{\tau_1}y + yk'(t)\partial_{\tau_1}t \\ &= -k(t)(k(t)^{-1}(\partial_{\tau_2}x + b(t)y\partial_{\tau_2}z)) - yk'(t)(-k(t)^{-1}\partial_{\tau_2}z) \\ &= -\partial_{\tau_2}x - b(t)y\partial_{\tau_2}z + yk'(t)k(t)^{-1}\partial_{\tau_2}z, \end{aligned} \tag{5.6}$$

$$\partial_{\tau_2}(k(t)y) = \partial_{\tau_1}x + b(t)y\partial_{\tau_1}z - yk'(t)k(t)^{-1}\partial_{\tau_1}z. \tag{5.7}$$

Next we claim that

$$\partial_{\tau_1}^2(k(t)y) + \partial_{\tau_2}^2(k(t)y) = Q_1(t, y)(\partial_{\tau_1}y, \partial_{\tau_2}y, \partial_{\tau_1}z, \partial_{\tau_2}z), \tag{5.8}$$

where $Q_1(t, y)$ is a quadratic form whose coefficients satisfy an $O(K^{-1}\sigma^{-1/2})$ bound, where $K \gg 1$ is the constant in the definition of $J_{\delta;\sigma}$. By differentiating (5.6) and (5.7) and using the Cauchy–Riemann equations to express $\partial_{\tau_1}t$ and $\partial_{\tau_2}t$ in terms of $\partial_{\tau_1}z$ and $\partial_{\tau_2}z$, we obtain

$$\begin{aligned} \partial_{\tau_1}^2(k(t)y) + \partial_{\tau_2}^2(k(t)y) &= y(-b'(t) + (\log k(t))'')k(t)^{-1}((\partial_{\tau_1}z)^2 + (\partial_{\tau_2}z)^2) \\ &\quad + (-b(t) + (\log k(t))')(\partial_{\tau_1}y\partial_{\tau_2}z - \partial_{\tau_1}z\partial_{\tau_2}y). \end{aligned}$$

To see the bound on coefficients, note that we can choose $b(t)$ and $k(t)$ so that $b'(t) = O(K^{-1}\sigma^{-1})$, $c_0 < k(t) < c_1$ where $c_0, c_1 > 0$ are independent of σ , $(\log k(t))' = 1$ near $t = 0$ and $(\log k(t))' = 0$ near $t = \frac{1}{2}K\sigma$, and $(\log k(t))'' = O(K^{-1}\sigma^{-1})$. Since $|y| = O(\sigma^{1/2})$ by Lemma 5.8, the claim follows.

A similar but easier calculation gives

$$\partial_{\tau_1}^2 z + \partial_{\tau_2}^2 z = 0. \tag{5.9}$$

Using (5.8) and (5.9) we compute

$$\begin{aligned}
 &(\partial_{\tau_1}^2 + \partial_{\tau_2}^2)((k(t)y)^2 + z^2) \\
 &= 2(|\nabla(k(t)y)|^2 + |\nabla z|^2 + k(t)y \mathcal{Q}_1(t, y)(\partial_{\tau_1}y, \partial_{\tau_2}y, \partial_{\tau_1}z, \partial_{\tau_2}z)),
 \end{aligned}$$

where $\nabla = (\partial_{\tau_1}, \partial_{\tau_2})$. The bounds on the quadratic forms imply that there is a constant $C > 0$ such that

$$\mathcal{Q}_j(t, y)(\partial_{\tau_1}y, \partial_{\tau_2}y, \partial_{\tau_1}z, \partial_{\tau_2}z) \leq CK^{-1}\sigma^{-1/2}(|\nabla y|^2 + |\nabla z|^2)$$

for $j = 1, 2$. By Lemma 5.8, $|y| = O(\sigma^{1/2})$ and $|z| = O(\sigma^{1/2})$. Finally, since

$$\nabla(k(t)y) = k(t)\nabla y + (\log k(t))'(\partial_{\sigma}z, -\partial_{\tau}z)y,$$

the dominant term is $k(t)\nabla y$ and we conclude that

$$(\partial_{\tau_1}^2 + \partial_{\tau_2}^2)((k(t)y)^2 + z^2) \geq 0,$$

provided $K \gg 1$ is sufficiently large. Similarly, $p_{\delta;\sigma}^2 \circ u$ is subharmonic on the region $\frac{1}{2}K\sigma \leq t \leq K\sigma$. The lemma follows. \square

Lemma 5.10. *If $u \in \mathcal{M}^{(X_{\delta}, L_{\delta;\sigma}); J_{\delta;\sigma}}(a; \mathbf{b})$, then*

$$|p_{\delta;\sigma} \circ u| = \sqrt{p_{\delta;\sigma}^2 \circ u} = O(\sigma)$$

for all sufficiently small $\sigma > 0$.

Proof. This is immediate from the second assertion of Lemma 5.9 by observing that when restricted to $L_{\delta;\sigma} \cap (\{0 \leq t < \infty\} \cup \{-\infty < t \leq 0\})$, the function $p_{\delta;\sigma}^2 = (k_{\sigma}(t)y)^2 + z^2$ is bounded above by $Cy^2 + z^2$ for a positive constant C which is independent of σ and $|y|, |z| = O(\sigma)$. \square

5.5. From rigid disks to rigid trees

Consider a sequence

$$u_{\delta;\sigma} : (D_{m+1}, \partial D_{m+1}) \rightarrow (X_{\delta}, L_{\delta;\sigma}), \quad \delta, \sigma \rightarrow 0,$$

of rigid disks in $\mathcal{M}^{(X_{\delta}, L_{\delta;\sigma}); J_{\delta;\sigma}}(a; \mathbf{b})$.

Lemma 5.11. *After passing to a subsequence, there exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$ there exists $\sigma_0 = \sigma_0(\delta) > 0$ such that for any $\sigma \in (0, \sigma_0)$ there exists a rigid LC flow tree $\hat{\gamma}_{\delta}$ of $L_{\delta;\sigma}$ such that:*

- (1) $u_{\delta;\sigma}(D_{m+1}) \cap T^*F$ lies in an $O(\sigma \log(\sigma^{-1}))$ -neighborhood of the cotangent lift of $\hat{\gamma}_{\delta}$;
- (2) $\hat{\gamma}_{\delta}$ lies in a $\phi(\delta)$ -neighborhood of the Reeb chord flow lines over a and \mathbf{b} and $u_{\delta;\sigma}(D_{m+1}) \cap (X_{\delta} - T^*F)$ lies in a $\phi(\delta)$ -neighborhood of the strips over a and \mathbf{b} , where $\lim_{\delta \rightarrow 0} \phi(\delta) = 0$.

Proof. The lemma is almost a consequence of [E1, Theorem 1.2].

(1) First observe that $|p_{\delta;\sigma} \circ u_{\delta;\sigma}| = O(\sigma)$ by Lemma 5.10. Then we obtain local (= away from the points where the derivative blows up) flow tree convergence at a rate of $O(\sigma \log(\sigma^{-1}))$ as in [E1, Sections 5.2–5.3]. As in the usual proof of Gromov compactness, we add punctures at the points where the derivative blows up on the σ -scale, while noting that the number of such points is controlled by the average linking number (cf. [E1, Section 5.4]). This implies the flow tree convergence when restricted to T^*F .

(2) By Corollary 5.3, the restriction of $\hat{\gamma}_\delta$ to $X_\delta - T^*F$ is contained in $N_{\phi(\delta)}$ for some ϕ . The argument in [E1, Lemma 5.7] gives $O(\sigma)$ bounds on the derivative of $u_{\delta;\sigma}$ on $\{0 \leq t \leq K\sigma\}$. Hence, for $\sigma > 0$ sufficiently small, the restriction of $u_{\delta;\sigma}$ to $\{0 \leq t \leq K\sigma\}$ is close to cylinders over Reeb chords.

Finally, we claim that the restriction of $u_{\delta;\sigma}$ to $\{t \geq K\sigma\}$ is close to cylinders over Reeb chords for $\delta, \sigma > 0$ sufficiently small. Assume for contradiction that there is a sequence $u_{\delta_i;\sigma_i}$ such that some point of $u_{\delta_i;\sigma_i}|_{\{t \geq K\sigma_i\}}$ maps outside a fixed ε -neighborhood of cylinders over Reeb chords. Recalling that $\{t \geq K\sigma\}$ coincides with the symplectization of \mathbb{R}^3 , the flow tree convergence applied to the 1-dimensional Legendrian submanifold Λ_\pm yields a flow tree of Λ_\pm of dimension ≥ 0 . This implies that $\text{ind}(u_{\delta_i;\sigma_i}) \geq 1$, which is a contradiction. \square

5.6. From rigid trees to rigid disks

In this subsection we construct holomorphic disks near rigid LC flow trees as in [E1, Section 6].

The construction can be summarized as follows: As we take the limit $\sigma \rightarrow 0$, the Lagrangian boundary condition $L_{\delta;\sigma}$ degenerates to the 0-section and the domains S_σ of rigid $u_\sigma \in \mathcal{M}^{(X_\delta, L_{\delta;\sigma}); J_{\delta;\sigma}}(a; \mathbf{b})$ converge to a “broken domain” at the boundary of the space of conformal structures (see [E1, Remark 5.35]). After subdividing S_σ for $\sigma > 0$ small, we find explicit local solutions in a neighborhood of the rigid LC flow tree $\hat{\gamma}$, except in small shrinking regions where the neighboring local solutions are patched together. This yields a disk in $(X_\delta, L_{\delta;\sigma})$ which is close to being $J_{\delta;\sigma}$ -holomorphic as in [E1, Section 6.2]. Finally, we use Newton iteration in order to produce actual solutions as in [E1, Section 6.4].

In order to extend the argument to the case of a Lagrangian with cylindrical ends, we first construct explicit local solutions in the cylindrical ends near Reeb chords.

5.6.1. Construction of a truncated local solution. Let J_0 be the almost complex structure on $\mathbb{R} \times \mathbb{R}^3$ given by (3.4). In this subsection we will write down an explicit local J_0 -holomorphic map, i.e., a “truncated local solution”,

$$u = (t, x, y, z): [0, \infty) \times [0, 1] \rightarrow \mathbb{R} \times \mathbb{R}^3,$$

where $[0, \infty) \times [0, 1]$ is a half-strip with coordinates $\zeta = \tau_1 + i\tau_2$ and u is asymptotic to the Reeb chord $c \in \mathcal{C}(\Lambda_+)$ at the positive end. The situation at the negative end is analogous.

We may assume that there exist $-\pi/2 < \alpha_+ < \alpha_- < \pi/2$, $\alpha_{\pm} = \alpha_{\pm}(\sigma)$, such that the Lagrangian projections $\Pi_{\mathbb{C}}$ of the two branches of $\Lambda = \Lambda_+$ near c are contained in $e^{i\alpha_+}\mathbb{R}$ and $e^{i\alpha_-}\mathbb{R}$, and that $\Pi_{\mathbb{C}}$ of the positive end of u sweeps out the (small) sector from α_+ to α_- . As $\sigma \rightarrow 0$, we have $|\alpha_+|, |\alpha_-| = O(\sigma)$. Moreover, by the deformation of Λ from Section 5.2.1, the z -coordinates of the two branches are constant, i.e., $z = h_+$ and $z = h_-$, where $h_{\pm} = h_{\pm}(\sigma)$ and $|h_{\pm}| = O(\sigma)$ as $\sigma \rightarrow 0$. Here the subscript $+$ (resp. $-$) refers to the upper (resp. lower) strand.

The map u satisfies the Cauchy–Riemann equations if and only if $v = x + iy$ and $w = t + iz$ satisfy

$$\bar{\partial}v = \frac{1}{2}(\partial_{\tau_1} + i\partial_{\tau_2})v = 0, \tag{5.10}$$

$$\bar{\partial}w = \frac{1}{2}(\partial_{\tau_1} + i\partial_{\tau_2})w = -\frac{1}{2}y\partial_{\tau_2}\bar{v}. \tag{5.11}$$

If we choose the solution

$$v(\zeta) = e^{i\alpha_-}e^{(\alpha_+ - \alpha_-)\zeta} \tag{5.12}$$

for (5.10) and abbreviate $\theta = \alpha_+ - \alpha_-$, then (5.11) becomes

$$\bar{\partial}w = \frac{1}{4}\theta e^{2\theta\tau_1}(1 - e^{-2i(\alpha_- + \theta\tau_2)}) = \frac{1}{4}\theta e^{\theta(\zeta + \bar{\zeta})} - \frac{1}{4}\theta e^{-2i\alpha_+}e^{2\theta\bar{\zeta}}. \tag{5.13}$$

In order to solve for w , first let

$$w_0(\zeta) = (h_+ - h_-)\zeta + ih_- + c_0 \tag{5.14}$$

be a holomorphic map from $[0, \infty) \times [0, 1]$ to the strip over the Reeb chord c , postcomposed with the projection to the (t, z) -coordinates. Here c_0 is a constant. Next observe that

$$w_1(\zeta) = \frac{1}{4}e^{\theta(\zeta + \bar{\zeta})} - \frac{1}{8}e^{-2i\alpha_+}e^{2\theta\bar{\zeta}} + \frac{1}{8}e^{2i\alpha_+}e^{2\theta\zeta} \tag{5.15}$$

solves (5.13) and satisfies real boundary conditions, and $\lim_{\tau_1 \rightarrow \infty} w_1(\zeta) = 0$. Then we take $w = w_0 + w_1$.

The truncated local solution is then given by

$$u(\zeta) = (v(\zeta), w(\zeta)). \tag{5.16}$$

5.6.2. Gluing

Lemma 5.12. *If $\delta > 0$ is sufficiently small, then there exists $\sigma_0 > 0$ such that for any $0 < \sigma < \sigma_0$ and any rigid LC flow tree $\hat{\gamma}$ of $L_{\delta;\sigma}$, there is a unique rigid $J_{\delta;\sigma}$ -holomorphic disk with boundary on $L_{\delta;\sigma}$ in a neighborhood of $\hat{\gamma}$.*

Proof. This follows from the proof of [E1, Theorem 1.3], where the only new ingredient is the presence of the cylindrical end. The local solutions are constructed as in [E1, Section 6.1], where the solutions near the 1-valent punctures are replaced by the truncated local solutions u from Section 5.6.1. We then construct the approximate holomorphic disks by gluing as in [E1, Section 6.2]. The weight functions on the domains are still as in [E1, Section 6.3.1]. In particular, we impose small positive exponential weights near

the punctures and the weight function is equal to 1 on the strip regions of length $O(K\sigma)$ and width $O(\sigma)$ that map into the interpolation region $\{0 \leq t \leq K\sigma\}$.²

The necessary conditions for applying Newton iteration are still satisfied after these modifications are made. The uniform invertibility of the differential is shown as in [E1, Proposition 6.21], the quadratic estimate is established as in [E1, proof of Theorem 1.3, p. 1216]. The surjectivity of the construction, i.e., the fact that Newton iteration captures all rigid holomorphic disks, follows from an analog of the calculation in [E1, p. 1218], as follows. Let u be the truncated local solution from Section 5.6.1. It is not hard to check that the C^0 -norm near the ends controls the Fourier coefficients of u and the Fourier coefficients control the weighted Sobolev norm in the interpolation region $\{0 \leq t \leq K\sigma\}$ via the C^0 -norm. The C^0 -norm is controlled by Lemma 5.11. \square

Proof of Theorem 1.6. Let L^{Mo} be a Morse cobordism in T^*F and let $(X_\delta, L_{\delta;\sigma})$ be the associated conical exact Lagrangian cobordisms parametrized by $\delta, \sigma > 0$. For $\delta, \sigma > 0$ sufficiently small, any $u \in \mathcal{M}^{(X_\delta, L_{\delta;\sigma}); J_{\delta;\sigma}}(a; \mathbf{b})$ is “approximated” by a rigid LC flow tree $\hat{\gamma}_\delta$ in the sense of Lemma 5.11.

On the other hand, by Lemma 5.12, for $\sigma, \delta > 0$ sufficiently small, there is a unique rigid holomorphic disk in $\mathcal{M}^{(X_\delta, L_{\delta;\sigma}); J_{\delta;\sigma}}(a; \mathbf{b})$ which “approximates” the rigid LC flow tree $\hat{\gamma}_\delta$. The theorem now follows from Lemma 5.2, which gives a bijection between rigid flow trees of L^{Mo} and rigid LC flow trees of $L_{\delta;\sigma}$. \square

6. Elementary exact Lagrangian cobordisms and their DGA maps

In this section we introduce elementary exact Lagrangian cobordisms and compute the induced DGA maps. An *elementary exact Lagrangian cobordism* L is one of the following:

- (1) a cobordism induced by a $\Pi_{\mathbb{C}}$ -simple Legendrian isotopy;
- (2) a cobordism induced by a Legendrian Reidemeister move;
- (3) a minimum cobordism; or
- (4) a saddle cobordism.

The cobordisms will be discussed in Sections 6.2–6.5, respectively. An exact Lagrangian cobordism L is *decomposable* if it is exact Lagrangian isotopic to a concatenation of elementary exact Lagrangian cobordisms.

In this section the coefficient ring of the DGAs is \mathbb{F} , unless stated otherwise.

6.1. Lagrangian cobordisms from Legendrian isotopies

Let $\Lambda_\tau \subset \mathbb{R}^3$, $\tau \in [0, 1]$, be a 1-parameter family of Legendrian links from Λ_0 to Λ_1 . Choose a parametrization $\gamma_\tau: S \rightarrow \mathbb{R}^3$ of Λ_τ , where S is a (not necessarily connected) closed 1-manifold. We reparametrize the τ -parameter via a map $f: \mathbb{R} \rightarrow [0, 1]$ with

² The interpolation region interpolates between the local solution on the symplectization part and a local solution on T^*F which is “approximated” by a flow line.

small derivative so that $f(t) = 1$ for $t \geq t_0 \gg 0$ and $f(t) = 0$ for $t \leq -t_0$. Consider the trace

$$\Gamma: \mathbb{R} \times S \rightarrow \mathbb{R} \times \mathbb{R}^3, \quad (t, s) \mapsto (t, \gamma_{f(t)}(s)),$$

of the isotopy $\gamma_{f(t)}$. Then

$$\Gamma([t_0, \infty) \times S) = [t_0, \infty) \times \Lambda_1, \quad \Gamma((-\infty, -t_0] \times S) = (-\infty, -t_0] \times \Lambda_0.$$

The following lemma is a version of a standard result (cf. e.g. [E2, Lemma A.1]).

Lemma 6.1. *For any $\epsilon > 0$, there exists $\delta > 0$ such that if Γ satisfies $|\frac{\partial \Gamma}{\partial t}|_{C^0} < \delta$, then there is a cylindrical exact Lagrangian cobordism L from Λ_1 to Λ_0 which is ϵ -close (in the C^0 -metric) to the image of Γ .*

Proof. Let $H(t, s) = \alpha_0(\frac{\partial \Gamma}{\partial t}(t, s))$, where $\alpha_0 = dz - ydx$. We then write

$$\Gamma(t, s) = (t, x(t, s), y(t, s), z(t, s)),$$

and consider the deformed map

$$\Gamma'(t, s) = (t, x(t, s), y(t, s), z(t, s) + H(t, s)). \tag{6.1}$$

We calculate

$$\begin{aligned} (\Gamma')^*(e^t \alpha_0) &= (\Gamma')^*(e^t(dz - ydx)) \\ &= e^t \left(\frac{\partial z}{\partial t} dt + \frac{\partial z}{\partial s} ds - y \left(\frac{\partial x}{\partial t} dt + \frac{\partial x}{\partial s} ds \right) + \frac{\partial H}{\partial t} dt + \frac{\partial H}{\partial s} ds \right) \\ &= e^t \left(\left(\frac{\partial z}{\partial t} - y \frac{\partial x}{\partial t} \right) dt + \frac{\partial H}{\partial t} dt + \frac{\partial H}{\partial s} ds \right) \\ &= e^t \left(H(t, s) dt + \frac{\partial H}{\partial t} dt + \frac{\partial H}{\partial s} ds \right) \\ &= d(e^t H(t, s)). \end{aligned}$$

Hence Γ' is exact Lagrangian. Since $\Gamma: \mathbb{R} \times S \rightarrow \mathbb{R} \times M$ is an embedding, Γ' is also an embedding, provided the modification $H(t, s)$ is sufficiently small. Finally, condition (i) in Definition 1.1 is satisfied since $H(t, s) = 0$ at the ends of Γ' . The lemma follows. \square

6.2. Simple Legendrian isotopies

Let $\Lambda_\tau \subset \mathbb{R}^3$, $\tau \in [0, 1]$, be a 1-parameter family of Legendrian links from Λ_0 to Λ_1 . After a small perturbation, Λ_τ is chord generic except at isolated instances $0 < \tau_1 < \dots < \tau_k < 1$, where the Lagrangian projection $\Pi_{\mathbb{C}}(\Lambda_{\tau_i})$ has a self-tangency or a triple point and passing from $\Pi_{\mathbb{C}}(\Lambda_{\tau_i - \epsilon})$ to $\Pi_{\mathbb{C}}(\Lambda_{\tau_i + \epsilon})$, $\epsilon > 0$ small, corresponds to a Legendrian Reidemeister move (see [Kal, Figure 6]).

Definition 6.2. A Legendrian isotopy Λ_τ , $\tau \in [0, 1]$, is $\Pi_{\mathbb{C}}$ -simple if the Lagrangian projection of Λ_τ has only transverse double points for all $\tau \in [0, 1]$, i.e., there are no Reidemeister moves during the isotopy.

In particular, if $\Lambda_\tau, \tau \in [0, 1]$, is $\Pi_{\mathbb{C}}$ -simple, then there is a natural identification $\phi_{\tau, \tau^*} : \mathcal{C}(\Lambda_\tau) \xrightarrow{\sim} \mathcal{C}(\Lambda_{\tau^*})$ for all $\tau, \tau^* \in [0, 1]$.

We now use Lemma 6.1 to construct cobordisms of $\Pi_{\mathbb{C}}$ -simple isotopies and compute the corresponding cobordism maps.

Lemma 6.3. *Let $\Lambda_\tau, \tau \in [0, 1]$, be a $\Pi_{\mathbb{C}}$ -simple Legendrian isotopy. Then there exist $\epsilon > 0$ and a subdivision of $[0, 1]$ into intervals $[a, a + \epsilon]$ of length ϵ such that the Lagrangian cobordisms that correspond to $\Lambda_\tau, \tau \in [a, a + \epsilon]$, all induce the DGA isomorphisms*

$$(\mathcal{A}(\Lambda_a), \partial_a) \xrightarrow{\sim} (\mathcal{A}(\Lambda_{a+\epsilon}), \partial_{a+\epsilon})$$

which map $c \in \mathcal{C}(\Lambda_a)$ to the corresponding chord $\phi_{a, a+\epsilon}(c) \in \mathcal{C}(\Lambda_{a+\epsilon})$.

Proof. Arguing towards a contradiction, suppose there exist $\epsilon_i \rightarrow 0$ and intervals $[a_i, a_i + \epsilon_i]$ such that the concordance corresponding to $\Lambda_\tau, \tau \in [a_i, a_i + \epsilon_i]$, contains an $\text{ind} = 0$ holomorphic disk which is not close to a trivial strip. By passing to a subsequence we may assume that $a_i \rightarrow a$. By Gromov compactness, the trivial cylinder over Λ_a has an $\text{ind} = 0$ disk which is not a trivial strip. Such a disk projects to a nontrivial disk of Fredholm index -1 in \mathbb{C} with boundary on $\Pi_{\mathbb{C}}(\Lambda_a)$.

On the other hand, no such disk exists by the argument principle: If $u : D_m \rightarrow \mathbb{C}$ is a holomorphic disk whose boundary maps to $\Pi_{\mathbb{C}}(\Lambda_a)$, then an easy calculation shows that the Fredholm index $\text{ind}(u)$ of u is given by $\mu(\partial u) - 2$, where $\mu(\partial u)$ is the Maslov index along ∂u with positive $\pi/2$ -rotations at the corners. By the argument principle, $\mu(\partial u) \geq 2$. Hence $\text{ind}(u) \geq 0$ and the lemma follows. \square

6.3. Cobordisms corresponding to Reidemeister moves

Following [EK], we consider three Reidemeister moves:

- (L1) a triple point move;
- (L2) pair cancellation of Reeb chords; and
- (L3) pair creation of Reeb chords.

The (L1)-, (L2)-, and (L3)-moves are depicted in Figures 7, 8, and 9. The Morse cobordisms $L^{\text{Mo}} \subset T^*F, F = \mathbb{R} \times [0, 1]$, corresponding to the (L1)-, (L2)-, and (L3)-moves are referred to as the *triple point*, *death*, and *birth* cobordisms. Note that Morse cobordisms are called “Legendrian submanifolds with standard ends” in [EK, Section 3.2].

We remark that the exact Lagrangian cobordisms corresponding to the triple point, death, and birth Morse cobordisms, as well as the saddle and minimum cobordisms from Sections 6.4 and 6.5, satisfy condition (i) of Definition 1.1 since we may assume that the modifications occur inside a small region and outside this region the isotopy is trivial.

6.3.1. Abstract and geometric perturbations. We would like to apply Theorem 1.6 to compute the DGA morphisms corresponding the above Morse cobordisms L^{Mo} from Λ_+ to Λ_- . For simplicity we will be using \mathbb{F} -coefficients.

Let $L \subset T^*F$ be an immersed exact Lagrangian cobordism which satisfies the following:

- L is exact Lagrangian regular homotopic to L^{M_0} relative to $\Pi_{T^*F}(\Lambda_-) \times \{0\}$ and $\Pi_{T^*F}(\Lambda_+) \times \{1\}$;
- the exact Lagrangian regular homotopy is C^0 -small and supported on $\mathbb{R} \times [0, \epsilon]$ and $\mathbb{R} \times [1 - \epsilon, 1] \subset F$ for small $\epsilon > 0$; and
- L restricts to $\Pi_{T^*F}(\Lambda_-) \times [0, \epsilon]$ and $\Pi_{T^*F}(\Lambda_+) \times [1 - \epsilon, 1]$ over $\mathbb{R} \times [0, \epsilon]$ and $\mathbb{R} \times [1 - \epsilon, 1] \subset F$.

A *geometric perturbation* of L is a perturbation of the immersed exact Lagrangian L relative to ∂L together with a perturbation of the Riemannian metric on F which is used to define gradients. We assume additionally that the perturbation of L is a Morse cobordism. For a generic geometric perturbation, the formal dimension ≤ 1 moduli spaces of flow trees with one positive puncture on Λ_+ and arbitrarily many negative punctures on Λ_- are transversely cut out by [E1, Theorem 1.1]. A count of rigid flow trees from a to \mathbf{b} with respect to a generic geometric perturbation immediately gives $|\mathcal{T}(a; \mathbf{b})|$ in Theorem 1.6.

However, it is easier to compute the DGA morphisms using Morse–Bott type considerations since L is close to being Morse–Bott degenerate. In order to rigorously treat Morse–Bott theory for flow trees, we use an *abstract perturbation* scheme which counts *perturbed Morse–Bott flow tree cascades*. They are defined in [EK, Section 3.4] under the name “perturbed generalized flow trees”. At this point the reader is encouraged to review Sections 3.3 and 3.4 of [EK], including the notions of a *Morse–Bott flow tree cascade* (= “generalized flow tree”), a *slice tree*, a *connector*, and the *level* of a cascade.³ In this paper we will simply refer to a “Morse–Bott flow tree cascade” as a “cascade”. For a generic abstract perturbation, the formal dimension ≤ 1 moduli spaces of perturbed cascades with one positive puncture and arbitrarily many negative punctures are transversely cut out by [EK, Lemma 3.9]. In [EK, Lemmas 6.6–6.8], for each of the moves, all the rigid perturbed cascades were determined for a certain generic abstract perturbation of L .

Let

$$\Phi_a, \Phi_g : \mathcal{A}(\Lambda_+) \rightarrow \mathcal{A}(\Lambda_-)$$

be the maps defined by counting rigid perturbed cascades or rigid flow trees of \tilde{L} using an abstract perturbation and a geometric perturbation, respectively.

Lemma 6.4. *The maps Φ_a and Φ_g are DGA morphisms and are chain homotopic, i.e., there is a degree +1 map K which takes generators of $\mathcal{A}(\Lambda_+)$ to $\mathcal{A}(\Lambda_-)$ so that*

$$\Phi_a - \Phi_g = \Omega_K \circ \partial_+ + \partial_- \circ \Omega_K.$$

Here Ω_K is as in Lemma 3.14.

Proof. The fact that Φ_a and Φ_g are DGA morphisms follows from the compactness and transversality properties of perturbed cascades in J^1F . The proofs require only standard finite-dimensional arguments given in [EK, Lemma 3.9].

³ We need to slightly modify the definition of a *boundary adjusted Morse function* $\beta : N \rightarrow \mathbb{R}$ in [EK, Section 3.3, 2nd paragraph] so that on the collar neighborhood $C \times [0, \epsilon]$ of each component C of ∂N , $\beta(x, t) = \beta_\partial(x) + k_C t^2$ where k_C is a nonzero constant.

The proof of the chain homotopy is similar to that of [EK, Lemma 3.13] and is the usual chain homotopy argument in disguise. We make the following simplifying assumptions:

- (i) the abstract and geometric perturbations are close and are connected by a 1-parameter family of abstract perturbations $\mathfrak{P}_t, t \in [0, 1]$; and
- (ii) there is a single $\text{ind} = 0$ disk in the 1-parameter family of moduli spaces of rigid disks $\bigsqcup_{t \in [0,1]} \mathcal{M}_t$, where \mathcal{M}_t is with respect to \mathfrak{P}_t .⁴

Note that we can view a geometric perturbation as an instance of an abstract perturbation.

Let $F_I = F \times [0, 1]/\sim$, where $(x, t) \sim (x, t')$ for all $x \in \partial F$ and $t, t' \in [0, 1]$, and let $L_I = L \times [0, 1]/\sim$ be a Legendrian submanifold of $J^1(F_I)$. Also let $D = [0, 1] \times [0, 1]/\sim$, where $(s, t) \sim (s, t')$ for $s \in \{0, 1\}$ and $t, t' \in [0, 1]$, and let $\pi : T^*F_I \rightarrow D$ be the corresponding projection. We assume that $L_I \cap \pi^{-1}(D_{\pm})$ is a trivial cobordism over Λ_{\pm} , where

$$D_- = \{0 \leq s \leq 1/3\}, \quad D_+ = \{2/3 \leq s \leq 1\}$$

are subsets of D .

We think of D and F_I as smooth manifolds with boundary in the obvious way and choose a Morse function f on D such that:

- f has one saddle point $h = \{s = 1\}$, one maximum point e_+ in the interior of D_+ , one minimum point $e_- = \{s = 0\}$, and no other critical points;
- $-\nabla f = -\partial_s$ on $D - D_+ - D_-$; and
- $t = 0$ and $t = 1$ are gradient trajectories from h to e_- .

The set $\mathcal{C}(L_I)$ of Reeb chords of L_I is given by

$$\mathcal{C}(L_I) = \mathcal{C}(\Lambda_-) \cup \hat{\mathcal{C}}(\Lambda_+) \cup \tilde{\mathcal{C}}(\Lambda_+),$$

where $\mathcal{C}(\Lambda_-)$, $\hat{\mathcal{C}}(\Lambda_+)$, and $\tilde{\mathcal{C}}(\Lambda_+)$ are the sets of Reeb chords that lie above e_- , h , and e_+ , respectively. Each of $\hat{\mathcal{C}}(\Lambda_+)$ and $\tilde{\mathcal{C}}(\Lambda_+)$ is in one-to-one correspondence with $\mathcal{C}(\Lambda_+)$, except that $|\hat{c}| = |c| + 1$ and $|\tilde{c}| = |c| + 2$. Here $c \in \mathcal{C}(\Lambda_+)$ and \hat{c} and \tilde{c} are the corresponding elements in $\hat{\mathcal{C}}(\Lambda_+)$ and $\tilde{\mathcal{C}}(\Lambda_+)$; also let $d \in \mathcal{C}(\Lambda_-)$. Let $\hat{\mathcal{A}}(\Lambda_+)$ and $\tilde{\mathcal{A}}(\Lambda_+)$ be the algebras generated by $\hat{\mathcal{C}}(\Lambda_+)$ and $\tilde{\mathcal{C}}(\Lambda_+)$.

Assume that the cobordism L and the perturbations \mathfrak{P}_0 and \mathfrak{P}_1 correspond to the two flow lines connecting h to e_- . As in the proof of [EK, Lemma 3.13], we claim that the differential Δ of $\mathcal{A}(L_I)$ is given as follows:

$$\Delta d = \partial_- d, \tag{6.2}$$

$$\Delta \hat{c} = \Phi_a(c) + \Phi_g(c) + \hat{\mathbf{O}}(1), \tag{6.3}$$

$$\Delta \tilde{c} = \hat{c} + K(\hat{c}) + Q(\partial_+ c) + \mathbf{O}(2). \tag{6.4}$$

Here:

- $\hat{\mathbf{O}}(n)$ (resp. $\mathbf{O}(n)$) is a sum of words, each of which has at least n letters in $\hat{\mathcal{C}}(\Lambda_+)$ (resp. $\hat{\mathcal{C}}(\Lambda_+) \cup \tilde{\mathcal{C}}(\Lambda_+)$);

⁴ The reason we consider $\text{ind} = 0$ disks is that Φ_a and Φ_g count $\text{ind} = 1$ disks in $\mathbb{R} \times J^1 F$.

- $K : \hat{\mathcal{C}}(\Lambda_+) \rightarrow \mathcal{A}(\Lambda_-)$ is the chain homotopy corresponding to the $\text{ind} = 0$ disk; and
- $Q(b_1 \dots b_m) = \tilde{b}_1 \Phi_g(b_2 \dots b_m) + \Phi_a(b_1) \tilde{b}_2 \Phi_g(b_3 \dots b_m) + \dots + \Phi_a(b_1 \dots b_{m-1}) \tilde{b}_m$.

We briefly indicate how the terms of (6.4) are obtained. We first enumerate the rigid unperturbed cascades which contribute to $\mathbf{O}(0)$ and $\mathbf{O}(1)$. In the case of $\mathbf{O}(0)$, the chain homotopy term K is obtained by viewing f as a perturbation of a Morse function \tilde{f} such that:

- \tilde{f} has one maximum $e_+ = \{s = 1\}$ and one minimum $e_- = \{s = 0\}$ and no critical points in $\text{int}(D)$; and
- $t = 0$ and $t = 1$ are gradient trajectories from e_+ to e_- .

In the case of $\mathbf{O}(1)$, a rigid cascade Γ can have at most one level by an index computation. If the level of Γ is zero, then it is a connector from \tilde{c} to \hat{c} . If the level of Γ is one, then it consists of a slice tree Γ_{e_+} from \tilde{c} to $\tilde{b}_1, \dots, \tilde{b}_m$, together with $m - 1$ rigid trees $\gamma_i, i \neq i_0$, from \tilde{b}_i to some component $\mathbf{a}_i = a_{i_1} \dots a_{i_{j_i}}$ of $\Phi_a(b_i)$ or $\Phi_g(b_i)$. (In the case where L_I is not a product $\Lambda \times D$, we substitute rigid trees for connectors.)

An abstract perturbation analogous to the time-ordered, domain-dependent abstract perturbation from the proof of Lemma 3.14 then gives the term $Q(b_1 \dots b_m)$, as follows: Let $N(e_+) \subset D$ be a small neighborhood of e_+ , $q \in N(e_+)$, Γ_q a parallel copy of the slice tree Γ_{e_+} over q , and $c(q)$ the Reeb chord over q corresponding to $c \in \mathcal{C}(\Lambda_+)$. Suppose that

(*) the $\text{ind} = 0$ disk that contributes to K lies over the gradient trajectory $t = 1/2$ of f .

For $q \in N(e_+)$, we define the *perturbation function* (cf. [EK, Section 3.4])

$$v(\Gamma_q) : C(\Gamma_q) \rightarrow T_q D \simeq \mathbb{R}^2,$$

where $C(\Gamma_q)$ is the cotangent lift of Γ_q , so that the following holds for all rigid slice trees Γ_q :

- (1) $v(\Gamma_q)$ is independent of q ; and
- (2) $v(\Gamma_q)$ is zero near $c(q)$ and takes distinct constant values $v_i \in \mathbb{R}^2$ near $b_i(q)$.

We specify v_i further:

- (3) Let $\bar{v}_i = (s_i, -\varepsilon)$, where $\varepsilon > 0$ is small and $s_1 < \dots < s_m$. Then v_i is a generic point which is δ -close to \bar{v}_i , where $0 < \delta \ll \varepsilon$.

We will write $\Gamma_q + v(\Gamma_q)$ for Γ_q shifted in the D -direction via $v(\Gamma_q)$. A rigid perturbed cascade $\tilde{\Gamma}$ that corresponds to Γ consists of the following:

- (a) a connector from \tilde{c} to $c(q)$ for some point $q \in N(e_+)$;
- (b) a perturbed flow tree $\Gamma_q + v(\Gamma_q)$, where $b_i(q + v_i)$ is the perturbed negative puncture corresponding to $b_i(q)$; and
- (c) for each $i \neq i_0$, a flow tree starting at $b_i(q + v_i)$.

In order for such a cascade to be rigid and contribute to a term that is linear in the $\tilde{\mathcal{C}}(\Lambda_+)$ -variables, we must have $q + v_i = e_+$ when $i = i_0$; this uniquely determines q . By the

choice (3) of the v_i , together with $(*)$, there is exactly one perturbed cascade $\tilde{\Gamma}$ corresponding to Γ , and it contributes to $\Phi_a(b_1 \dots b_{i_0-1})\tilde{b}_{i_0}\Phi_g(b_{i_0+1} \dots b_m)$.

Applying Δ to (6.4) and using $\Delta^2 = 0$ we find that

$$\Phi_g(c) - \Phi_a(c) = \partial_-(K(\hat{c})) + \Omega_K(\partial_+c).$$

This follows from restricting to the terms without letters in $\hat{\mathcal{C}}(\Lambda_+)$ and $\tilde{\mathcal{C}}(\Lambda_+)$. □

Remark 6.5. In [EK], the full DGA differentials of the Morse cobordisms corresponding to (L1)–(L3) were computed using abstractly perturbed flow trees. The calculation needed here is simpler: we only need to consider trees with one positive puncture at the maximum and all other punctures at the minimum. In particular, the calculations for (L1) and (L2) can be carried out with only geometric perturbations.

6.3.2. *Triple point cobordisms.* Let L_{tr}^{Mo} be a triple point cobordism from Λ_+ to Λ_- and let (X_{tr}, L_{tr}) be the corresponding exact Lagrangian cobordism with cylindrical ends. Then there is a canonical identification $\mathcal{C}(\Lambda_+) \simeq \mathcal{C}(\Lambda_-)$. There are two types of (L1)-moves, denoted by (L1a) and (L1b) (see Figures 7 and 8).

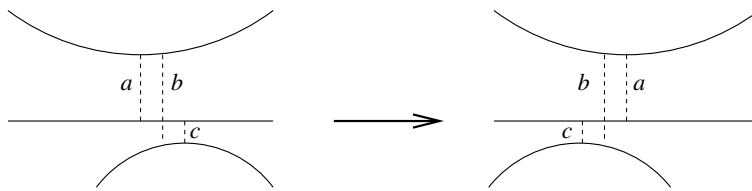


Fig. 7. An (L1a)-isotopy.

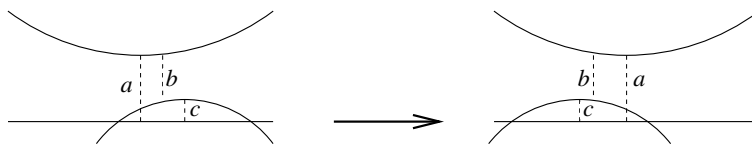


Fig. 8. An (L1b)-isotopy.

We define two DGA morphisms

$$\phi_{(L1a)}, \phi_{(L1b)} : \mathcal{A}(\Lambda_+) \rightarrow \mathcal{A}(\Lambda_-)$$

as follows: $\phi_{(L1a)}$ maps $x \in \mathcal{C}(\Lambda_+)$ to the corresponding $x \in \mathcal{C}(\Lambda_-)$ and $\phi_{(L1b)}$ maps $a \mapsto a + bc$ and all other $x \in \mathcal{C}(\Lambda_+)$ to the corresponding $x \in \mathcal{C}(\Lambda_-)$.

Lemma 6.6. *The cobordism map $\Phi_{(X_{tr}, L_{tr})} : \mathcal{A}(\Lambda_+) \rightarrow \mathcal{A}(\Lambda_-)$ is given by $\phi_{(L1a)}$ or $\phi_{(L1b)}$, as appropriate.*

Proof. Follows from [EK, Lemma 6.6] combined with Lemma 6.4 and Theorem 1.6. □

6.3.3. *Death cobordisms.* Let $L_{\text{de}}^{\text{Mo}}$ be a death cobordism from Λ_+ to Λ_- and let $(X_{\text{de}}, L_{\text{de}})$ be the corresponding exact Lagrangian cobordism with cylindrical ends. Then there is a canonical identification $\mathcal{C}(\Lambda_+) \simeq \mathcal{C}(\Lambda_-) \cup \{a, b\}$, where a and b are the canceling Reeb chords (see Figure 9).

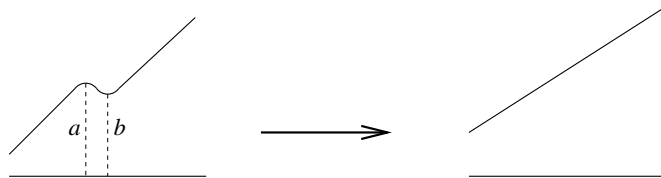


Fig. 9. An (L2)-isotopy from left to right. The (L3)-isotopy is from right to left.

We define the DGA morphism

$$\phi_{(L2)} : \mathcal{A}(\Lambda_+) \rightarrow \mathcal{A}(\Lambda_-)$$

as follows: Suppose $\partial_+ a = b + v$, where ∂_+ is the differential for $\mathcal{A}(\Lambda_+)$. Observe that v has no terms that contain a or b . Then $\phi_{(L2)}$ maps $a \mapsto 0, b \mapsto v$, and all other $x \in \mathcal{C}(\Lambda_+)$ to the corresponding $x \in \mathcal{C}(\Lambda_-)$.

Lemma 6.7. *The cobordism map $\Phi_{(X_{\text{de}}, L_{\text{de}})} : \mathcal{A}(\Lambda_+) \rightarrow \mathcal{A}(\Lambda_-)$ is given by $\phi_{(L2)}$.*

Proof. Follows from [EK, Lemma 6.7] combined with Lemma 6.4 and Theorem 1.6. \square

6.3.4. *Birth cobordisms.* Let $L_{\text{bi}}^{\text{Mo}}$ be a birth cobordism from Λ_+ to Λ_- and let $(X_{\text{bi}}, L_{\text{bi}})$ be the corresponding exact Lagrangian cobordism with cylindrical ends. Then there is a canonical identification $\mathcal{C}(\Lambda_+) \cup \{a, b\} \simeq \mathcal{C}(\Lambda_-)$, where a and b are the newly created Reeb chords.

We define the DGA morphism

$$\phi_{(L3)} : \mathcal{A}(\Lambda_+) \rightarrow \mathcal{A}(\Lambda_-)$$

inductively as follows (cf. [Kal, Remark 3.4]): Suppose $\partial_- a = b + v$, where ∂_- is the differential for $\mathcal{A}(\Lambda_-)$, and

$$\mathcal{C}(\Lambda_-) = \{b_1, \dots, b_m, b, a, a_1, \dots, a_l\},$$

arranged in action-nondecreasing order. We first set $\phi_{(L3)}(b_i) = b_i$. Suppose that

$$\partial_- a_1 = \sum B_1 b B_2 b \dots B_k b A,$$

where B_i is a monomial in b_1, \dots, b_m, A is a monomial in b_1, \dots, b_m, b, a , and every b in A is preceded by an a in A . Then

$$\begin{aligned} \phi_{(L3)}(a_1) = a_1 + \sum & (B_1 a B_2 b \dots B_k b A + B_1 v B_2 a B_3 b \dots B_k b A \\ & + B_1 v B_2 v B_3 a B_4 b \dots B_k b A + \dots + B_1 v B_2 v \dots B_k a A). \end{aligned}$$

Next suppose that

$$\partial_- a_i = \sum B_1 b B_2 b \dots B_k b A,$$

where B_i is a monomial in $b_1, \dots, b_m, a_1, \dots, a_{i-1}$, A is a monomial in $b_1, \dots, b_m, b, a, a_1, \dots, a_{i-1}$, and every b in A is preceded by an a in A . Then

$$\begin{aligned} \phi_{(L3)}(a_i) = a_i + \sum & (\overline{B_1} a B_2 b \dots B_k b A + \overline{B_1} v \overline{B_2} a B_3 b \dots B_k b A \\ & + \overline{B_1} v \overline{B_2} v \overline{B_3} a B_4 b \dots B_k b A + \dots + \overline{B_1} v \overline{B_2} v \dots \overline{B_k} a A), \end{aligned}$$

where the word $\overline{B_j}$ is obtained from B_j by replacing each occurrence of a_1, \dots, a_{i-1} by $\phi_{(L3)}(a_1), \dots, \phi_{(L3)}(a_{i-1})$.

Lemma 6.8. *The cobordism map $\Phi_{(X_{bi}, L_{bi})}: \mathcal{A}(\Lambda_+) \rightarrow \mathcal{A}(\Lambda_-)$ is given by $\phi_{(L3)}$.*

Proof. Follows from [EK, Lemma 6.8] combined with Lemma 6.4 and Theorem 1.6. \square

Remark 6.9. The chain maps $\phi_{(L1a)}$, $\phi_{(L1b)}$, $\phi_{(L2)}$, and $\phi_{(L3)}$ above are precisely the chain maps used by Chekanov [Ch] to prove the invariance of Legendrian contact homology under the Legendrian Reidemeister moves. The above exact Lagrangian cobordisms can be interpreted as providing a geometric context where these maps arise naturally.

6.4. Minimum cobordisms

In this subsection and the next we consider two types of exact Lagrangian cobordisms—minimum cobordisms and saddle cobordisms—that correspond to single Morse modifications of a Legendrian link.

Definition 6.10. A *trivial Morse cobordism* over a Legendrian $\Lambda \subset \mathbb{R}^3$ is a Morse cobordism in T^*F , $F = \mathbb{R} \times [0, 1]$, which is obtained from a conical Lagrangian in T^*F over Λ by a Morse modification near the boundary $\xi_2 = 0, 1$ as given in Definition 2.3. Here the coordinates of $\mathbb{R} \times [0, 1]$ are (ξ_1, ξ_2) .

Let $\Lambda_+ \subset \mathbb{R}^3$ be a Legendrian link, one of whose components is the standard Legendrian unknot U such that $\Pi_{\mathbb{C}}(U)$ is contained in a disk which is disjoint from $\Pi_{\mathbb{C}}(\Lambda_+ - U)$. Let $\Lambda_- = \Lambda_+ - U$.

Definition 6.11. A *minimum cobordism* from Λ_+ to Λ_- is a Morse cobordism in T^*F , $F = \mathbb{R} \times [0, 1]$, which is the union of a trivial Morse cobordism from $\Lambda_+ - U$ to Λ_- and a disk with boundary U in $J^1(\mathbb{R} \times [0, 1])$ with front in $J^0(\mathbb{R} \times [0, 1])$ as shown in Figure 10.

Let L_{mi}^{Mo} be a minimum cobordism from Λ_+ to Λ_- and let (X_{mi}, L_{mi}) be the corresponding exact Lagrangian cobordism with cylindrical ends. By assumption there are no Reeb chords of Λ_+ that connect U to any other component. Let a denote the unique Reeb chord from U to itself. Then $\mathcal{C}(\Lambda_+) = \mathcal{C}(\Lambda_-) \cup \{a\}$, where the identification is induced by the trivial cobordisms on $\Lambda_+ - U = \Lambda_-$.

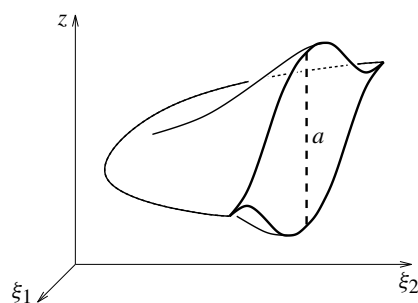


Fig. 10. A minimum cobordism.

Lemma 6.12. *The cobordism map $\Phi_{(X_{mi}, L_{mi})} : \mathcal{A}(\Lambda_+) \rightarrow \mathcal{A}(\Lambda_-)$ is given by*

$$\Phi_{(X_{mi}, L_{mi})}(c) = \begin{cases} 0 & \text{if } c = a, \\ c & \text{if } c \neq a, \end{cases}$$

where $c \in \mathcal{C}(\Lambda_+)$.

Proof. This is an immediate consequence of $|a| = 1$ and the fact that the rigid holomorphic disks in a trivial cobordism are strips over Reeb chords. \square

6.5. Saddle cobordisms

In this subsection we treat *saddle cobordisms*.

6.5.1. Contractibility

Definition 6.13 (Contractible Reeb chord). Let $\Lambda \subset \mathbb{R}^3$ be a Legendrian link. A Reeb chord $a \in \mathcal{C}(\Lambda)$ is *contractible* if there exists a homotopy $\Lambda_\tau, 0 \leq \tau \leq 1$, of Legendrian immersions such that:

- $\Lambda_0 = \Lambda$;
- $\Lambda_\tau, \tau \in [0, 1]$, is $\Pi_{\mathbb{C}}$ -simple, i.e., $\Pi_{\mathbb{C}}(\Lambda_\tau)$ has only transverse double points for all $\tau \in [0, 1]$; and
- Λ_1 has a transverse self-intersection which is obtained by sending $\mathfrak{A}(a_\tau) \rightarrow 0$ as $\tau \rightarrow 1$, where $a_\tau \in \mathcal{C}(\Lambda_\tau)$ is the Reeb chord corresponding to a .

Let $\Lambda'_+ \subset \mathbb{R}^3$ be a Legendrian link with a contractible Reeb chord a . Then, after Legendrian isotopy, we obtain $\Lambda_+ \subset \mathbb{R}^3$ with a contractible Reeb chord a , whose neighborhood is as shown on the left-hand side of Figure 11. Let Λ_- denote the Legendrian link obtained by modifying the front of Λ_+ as shown on the right-hand side of Figure 11.

Now we consider the Lagrangian projection $\Pi_{\mathbb{C}}$. The modification of Λ_+ given in Figure 11 corresponds to a 0-resolution of $\Pi_{\mathbb{C}}(\Lambda_+)$ at the crossing $\Pi_{\mathbb{C}}(a)$. Also the Legendrian regular homotopy can be continued so that a crossing of the opposite sign emerges where $\Pi_{\mathbb{C}}(a)$ used to be (see Figure 12).

We state a simple necessary and sufficient condition for contractibility with respect to the Lagrangian projection. We refer to a component of $\mathbb{C} - \Pi_{\mathbb{C}}(\Lambda_+)$ as a “region”.

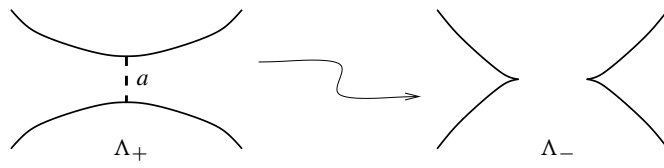


Fig. 11. Legendrian links at the ends of a saddle cobordism, viewed in the front projection.

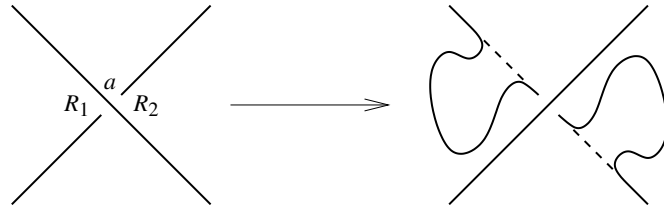


Fig. 12. Crossing change at a contractible Reeb chord in the Lagrangian projection.

Lemma 6.14 (Criterion for contractibility). *Let the quadrants with positive Reeb sign at $a \in \mathcal{C}(\Lambda_+)$ belong to the regions R_1 and R_2 . The Reeb chord a is contractible if and only if it is possible to apply a $\Pi_{\mathbb{C}}$ -simple isotopy to Λ_+ so that the areas of both R_1 and R_2 exceed the action $\mathfrak{A}(a)$. (If $R_1 = R_2$, the condition becomes $A(R_1) > 2 \cdot \mathfrak{A}(a)$.)*

Proof. If a contracts, then its action $\mathfrak{A}(a)$ approaches 0 and eventually becomes smaller than the area of any region in the diagram. If R_1 and R_2 have large enough areas, then it is possible to carry out the isotopy shown in Figure 12. □

6.5.2. Saddle cobordisms and simplicity

Definition 6.15. Let a be a contractible Reeb chord of a Legendrian link $\Lambda_+ \subset \mathbb{R}^3$ with $\mathfrak{A}(a) > 0$ small and let Λ_- be the Legendrian link obtained by the modification given in Figure 11. Then a *saddle cobordism* L_{sa}^{Mo} from Λ_+ to Λ_- corresponding to a is a Morse cobordism in T^*F , $F = \mathbb{R} \times [0, 1]$, for which there exists an open set $V \subset L_{sa}^{Mo}$ such that:

- (1) the front of V , viewed as a subset of $J^0(\mathbb{R} \times [0, 1])$, is obtained from Figure 13 by a Morse modification near the boundary $\xi_2 = 0, 1$ as given in Definition 2.3 and $\Pi_F(V)$ is a rectangle $(\xi_1^0, \xi_1^1) \times [0, 1] \subset \mathbb{R} \times [0, 1]$; and

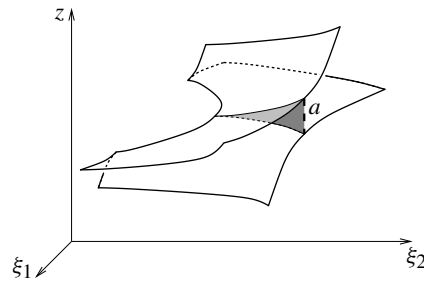


Fig. 13. A saddle cobordism and a gradient flow line ρ_0 from a to 1.

(2) $L_{sa}^{Mo} - V$ is a trivial Morse cobordism over $\Lambda_+ - N(a)$, where $N(a)$ is a neighborhood of a .

If L_{sa}^{Mo} is a saddle cobordism from Λ_+ to Λ_- corresponding to a , then we write (X_{sa}, L_{sa}) for the corresponding exact Lagrangian cobordism with cylindrical ends.

For $c \neq a$, let

$$\mathcal{M}(c, a^k; \mathbf{b}) = \mathcal{M}^{(\mathbb{R} \times \mathbb{R}^3, \mathbb{R} \times \Lambda_+)}(c, a^k; \mathbf{b})$$

be the moduli space of holomorphic disks in $\mathbb{R} \times \mathbb{R}^3$ with boundary on $\mathbb{R} \times \Lambda_+$, one positive puncture at c , k positive punctures at a , and negative punctures at \mathbf{b} .

We make the following simplifying definition:

Definition 6.16. A contractible Reeb chord $a \in \mathcal{C}(\Lambda_+)$ is *simple* if $\text{ind}(u) \geq k$ for all broken holomorphic disks u in $\mathbb{R} \times \mathbb{R}^3$ with boundary on $\mathbb{R} \times \Lambda_+$, one positive puncture at a chord $c \neq a$, and $k > 1$ positive punctures at a . A saddle cobordism for a simple contractible Reeb chord is a *simple saddle cobordism*.

6.5.3. *The DGA morphism for a simple saddle cobordism.* In view of the identification $\mathcal{C}(\Lambda_+) \simeq \mathcal{C}(\Lambda_-) \cup \{a\}$, we define an algebra map

$$\Psi_0 : \mathcal{A}(\Lambda_+) \rightarrow \mathcal{A}(\Lambda_-)$$

on the generators by setting $\Psi_0(a) = 1$, $\Psi_0(b) = b$ for $b \in \mathcal{C}(\Lambda_-)$, and $\Psi_0(A) = A$ for $A \in H_1(L_{sa})$. We also define

$$\Psi_1 : \mathcal{C}(\Lambda_+) \rightarrow \mathcal{A}(\Lambda_-)$$

by setting

$$\Psi_1(c) = \sum_{\dim(\mathcal{M}(c,a;\mathbf{b}))=1} |\mathcal{M}(c, a; \mathbf{b})/\mathbb{R}| \cdot \Psi_0(\mathbf{b}),$$

for $c \in \mathcal{C}(\Lambda_-)$ and $\Psi_1(a) = 0$.

Remark 6.17. We may assume that the moduli spaces $\mathcal{M}(c, a; \mathbf{b})$ are transversely cut out, since the disks in $\mathcal{M}(c, a; \mathbf{b})$ are not multiply-covered. Moreover, the bijection between flow trees with boundary on $\mathbb{R} \times \Lambda_+$ (resp. \tilde{L}_{sa}^{Mo}) and disks with boundary on $\mathbb{R} \times \Lambda_+$ (resp. \tilde{L}_{sa}^{Mo}) also holds for disks in $\mathcal{M}(c, a; \mathbf{b})$ of index 1.

Proposition 6.18. *If $a \in \mathcal{C}(\Lambda_+)$ is a simple contractible Reeb chord, then the corresponding cobordism map $\Phi_{(X_{sa}, L_{sa})} : \mathcal{A}(\Lambda_+) \rightarrow \mathcal{A}(\Lambda_-)$ is given by*

$$\Phi_{(X_{sa}, L_{sa})}(c) = \Psi_0(c) + \Psi_1(c).$$

We often refer to the map $\Phi_{(X_{sa}, L_{sa})}$ as the *0-resolution map at a* .

Proof. Let L_{sa}^{Mo} be a saddle cobordism from Λ_+ to Λ_- , a the simple contractible chord in Λ_+ , and Λ_1 the immersed Legendrian with a double point a' corresponding to the Reeb chord a of length 0. Given $c \in \mathcal{C}(\Lambda_-)$, we write \hat{c} for the corresponding Reeb chord in $\mathcal{C}(\Lambda_+)$.

First observe that there is a unique flow tree ρ_0 of L_{sa}^{Mo} from a to 1—it is the flow line given in Figure 13. To see that there is no other such tree, note that the positive function difference along the flow line emanating from a cannot be split further into two (or more) positive local function differences. The flow tree ρ_0 will be called the *basic tree* and the corresponding holomorphic disk the *basic disk*.

Step 1. We describe a degeneration $L_t, t \in [0, 1]$, of $L_0 = L_{sa}^{Mo}$ to the immersed exact Lagrangian $L_1 = \Lambda_1 \times [0, 1]$.

Let $\Lambda_t, t \in [0, 1]$, be a regular homotopy of Legendrian immersions with $\Lambda_0 = \Lambda_+$, which is guaranteed by Definition 6.13. We may assume that:

- $\Pi_{J^{0\mathbb{R}}}(\Lambda_t), t \in [0, 1]$, is independent of t outside a small rectangle $R \subset J^{0\mathbb{R}}$;
- $\Pi_{J^{0\mathbb{R}}}(\Lambda_t) \cap R, t < 1$, is as shown on the left-hand side of Figure 11; and
- $\Pi_{J^{0\mathbb{R}}}(\Lambda_t)$ limits to $\Pi_{J^{0\mathbb{R}}}(\Lambda_1)$ as $t \rightarrow 1$; in particular, $\mathfrak{A}(a_t) \rightarrow 0$ as $t \rightarrow 1$, where a_t is the contractible chord corresponding to a .

Next let $L_t, 0 \leq t < 1$, be a family of saddle cobordisms from Λ_t to Λ_- satisfying:

- $L_t \rightarrow L_1$ as $t \rightarrow 1$;
- if $V_t, t < 1$, is the open set V for L_t which appears in Definition 6.15 and C_t is the cusp edge corresponding to V_t , then $\Pi_F(C_t)$ is given by $\{\xi_2 = (t - 1)\xi_1^2 + 1/2, \xi_2 \in [0, 1]\}$.

Step 2. Let $\rho_0(t), 0 \leq t < 1$, be the basic tree for L_t .

Claim 6.19. *Let $\gamma_t, 0 \leq t < 1$, be a sequence of rigid flow trees of L_t with one positive end $\hat{c} \neq a_t$ and m negative ends b_1, \dots, b_m . If $\gamma_t \neq \rho_0(t)$ and γ_t is not a connector from \hat{c} to c , then γ_t has a subsequence which converges to a cascade Γ of L_1 as $t \rightarrow 1$, and Γ consists of:*

- a connector from \hat{c} to $c \times \{1/2\}$;
- a slice tree $\Gamma_1 \subset \Lambda_1 \times \{1/2\} \subset L_1$; and
- connectors from $b_i \times \{1/2\}$ to b_i for $i = 1, \dots, m$.

Proof. By the flow tree analog of Gromov compactness, there is a subsequence of γ_t which converges to a cascade Γ of L_1 . (Here the boundary condition L_t varies with t and Gromov compactness corresponds to local convergence of flow lines of the vector fields defined by L_t to the flow lines of the limiting vector field as $t \rightarrow 1$, provided an energy bound is satisfied.) If a level of Γ contains a slice tree of $\Lambda_1 \times \{\xi_2\}$ with $\xi_2 \neq 1/2$, then γ_t is not rigid. The claim follows. \square

Step 3. Consider the slice tree Γ_1 . Since the Reeb chords a_t limit to a_1 with $\mathfrak{A}(a_1) = 0$, we can view Γ_1 as a tree with punctures at a_1 , denoted by Γ_1° . Let k (resp. ℓ) be the number of positive (resp. negative) punctures at a_1 . For each negative (resp. positive) puncture at a_1 , there is a corresponding end (resp. switch) of $L_t, t < 1$ (see Figure 14).

We now apply (4.2) to compute $\dim(\Gamma_1^\circ)$ and $\dim(\gamma_t)$: Since $n = 1$ and $I(a_1) = 0$,

$$\dim(\Gamma_1^\circ) = -2 + I(c) - \sum_{i=1}^m (I(b_i) - 1) + \ell + \sum_r \mu(r).$$

On the other hand, since $n = 2, I(\hat{c}) = I(c) + 1$, each positive puncture at a_1 is converted into a switch which contributes -1 to the Maslov content, and each negative puncture at

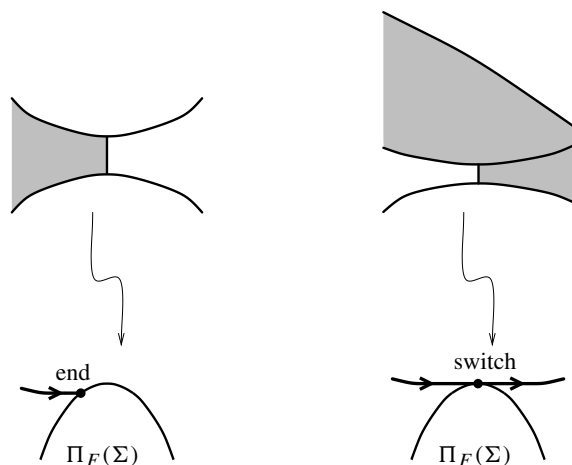


Fig. 14. Producing rigid trees in a saddle cobordism. Here Σ is the singular set of \tilde{L}_{sa}^{Mo} . The arrows in the bottom figures represent the negative gradients of the positive function differences.

a_1 is converted into an end which contributes $+1$ to the Maslov content, it follows that

$$\begin{aligned} \dim(\gamma_t) &= -1 + (I(\hat{c}) - 1) - \sum_{i=1}^m (I(b_i) - 1) + \sum_r \mu(r) - k + \ell \\ &= -1 + I(c) - \sum_{i=1}^m (I(b_i) - 1) + \sum_r \mu(r) - k + \ell \\ &= \dim(\Gamma_1^\circ) + 1 - k. \end{aligned}$$

Since $\dim(\gamma_t) = 0$, it follows that $\dim(\Gamma_1^\circ) = k - 1$. By the simplicity of the chord a_t , there is no such holomorphic disk when $k > 1$.

Note that for t close to 1, the rigid flow tree L_t must intersect the basic flow tree ρ_0 as well as $\Pi_F(\Sigma)$.

Step 4. It remains to construct a unique rigid tree γ_t in L_t , $1 - \varepsilon < t < 1$, corresponding to the cascade Γ . When we perturb Γ to γ_t , the unique positive puncture of the slice tree Γ_1° at a becomes a switch of γ_t as depicted on the right-hand side of Figure 14. The tree γ_t can be split into two partial flow trees γ_t^+ and γ_t^- at the switching point d , where γ_t^+ has a positive puncture at \hat{c} and γ_t^- has no positive punctures; similarly, Γ can be split into partial flow trees Γ^+ and Γ^- . At d , the incoming and outgoing gradient trajectories are uniquely determined. Hence there is a unique pair (γ_t^+, γ_t^-) , where γ_t^+ is close to Γ^+ and ends at some d , and γ_t^- is close to Γ^- and starts at the same d . This proves the lemma. \square

Remark 6.20. Consider a rigid tree γ in the cobordism L_{sa}^{Mo} with one positive end $c \neq a$. By applying “boundary gluing” to γ and the basic tree ρ_0 , we create a tree with two positive ends c and a . Here a “boundary gluing” of two flow trees is the flow tree analog

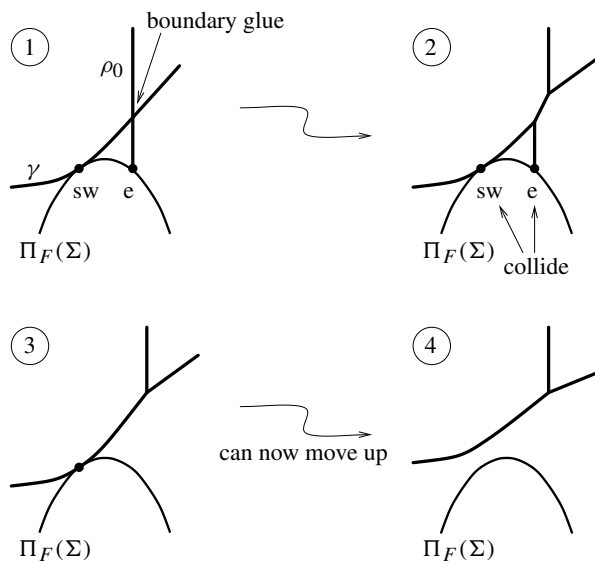


Fig. 15. Boundary gluing rigid trees in the cobordism. Here “sw” refers to a switch and “e” refers to an end.

of a gluing of two disks whose boundaries intersect at a point on the Lagrangian. The end and the switch of this tree cancel as in Figure 15, the tree moves upwards, and we eventually obtain a 1-dimensional family of trees in $\mathbb{R} \times \Lambda_+$ with two positive ends c and a .

Corollary 6.21. *If $a \in \mathcal{C}(\Lambda_+)$ is a simple contractible Reeb chord, then the corresponding cobordism map $\Phi_{(X_{sa}, L_{sa})} : \mathcal{A}(\Lambda_+) \rightarrow \mathcal{A}(\Lambda_-)$ is surjective.*

Proof. This follows from Proposition 6.18, together with the observation that

$$\mathfrak{A}(\Psi_0(c)) > \mathfrak{A}(\Psi_1(c)), \quad c \neq a.$$

Here $\mathfrak{A}(\Psi_1(c))$ is the supremum of $\mathfrak{A}(\mathbf{b})$ over all the nonzero monomial summands \mathbf{b} of $\Psi_1(c)$. □

6.5.4. Pushing forward augmentations. If $\Phi : \mathcal{A}(\Lambda_+) \rightarrow \mathcal{A}(\Lambda_-)$ is a DGA morphism and ε' is an augmentation of Λ_- , then the pullback $\Phi^*\varepsilon' := \varepsilon' \circ \Phi$ is naturally an augmentation of Λ_+ .

For saddle cobordisms corresponding to simple contractible Reeb chords, augmentations also behave well under *pushforwards*:

Lemma 6.22. *Let $a \in \mathcal{C}(\Lambda_+)$ be a simple contractible Reeb chord, let ε be an augmentation of Λ_+ with $\varepsilon(a) = 1$, and let $\Phi = \Phi_{(X_{sa}, L_{sa})}$ be the corresponding DGA morphism. Then there is a unique algebra map $\varepsilon' : \mathcal{A}(\Lambda_-) \rightarrow \mathbb{F}$ such that $\varepsilon = \varepsilon' \circ \Phi$. Furthermore, ε' is an augmentation of Λ_- .*

Proof. The existence and uniqueness of ε' uses the same key idea as that of Corollary 6.21. Since $\Phi(x) = \Psi_0(x) + \Psi_1(x)$ with $\mathfrak{A}(\Psi_0(x)) > \mathfrak{A}(\Psi_1(x))$, we can write $\varepsilon'(\Psi_0(x)) = \varepsilon(x) - \varepsilon'(\Psi_1(x))$ and use induction on the action.

To prove that $\varepsilon'\partial' = 0$, we compute

$$\begin{aligned} (\varepsilon'\partial')(\Psi_0(x)) &= (\varepsilon'\partial')(\Phi(x) - \Psi_1(x)) = (\varepsilon'\Phi\partial)(x) - (\varepsilon'\partial')(\Psi_1(x)) \\ &= (\varepsilon\partial)(x) - (\varepsilon'\partial')(\Psi_1(x)) = (\varepsilon'\partial')(-\Psi_1(x)), \end{aligned}$$

and use induction on the action. □

6.5.5. Dipped diagrams. In this subsection we explain how to modify a Legendrian link Λ with a contractible Reeb chord a so that a becomes simple. The modification is called *dipping*; it first appeared in [F] under the name “splashing” and was used extensively in [Sa2].

Lemma 6.23. *Let Λ be a Legendrian link with a contractible Reeb chord a . Then there exist a Legendrian isotopy Λ_τ , $\tau \in [0, 1]$, and a 1-parameter family a_τ , $\tau \in [0, 1]$, of contractible Reeb chords such that:*

- (1) $\Lambda_0 = \Lambda$ and $a_0 = a$;
- (2) $\Pi_{\mathbb{C}}(\Lambda_\tau)$ is a transverse intersection at $\Pi_{\mathbb{C}}(a_\tau)$ for all $\tau \in [0, 1]$; and
- (3) $a_1 \in \mathcal{C}(\Lambda_1)$ is simple.

Proof. We start with a $\Pi_{\mathbb{C}}$ -simple isotopy Λ_τ , $\tau \in [0, 1/3]$, with $\Lambda_0 = \Lambda$, such that the restriction of the front of $\Lambda_{1/3}$ to J^0I , where I is a small interval around the x -coordinate of $a_{1/3}$, satisfies the following: the two middle Legendrian arcs connected by $a_{1/3}$ are arranged as in Figure 16. The isotopy is obtained from the $\Pi_{\mathbb{C}}$ -simple isotopy given by Definition 6.13, by applying a C^0 -small perturbation near the Reeb chord $a_{1/3}$.

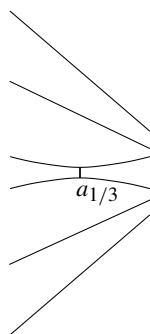


Fig. 16. Aligning all the local gradients in the front projection.

Next we shrink I to $I' \ni x(a_{1/3})$ and apply a C^0 -small isotopy to $\Pi_{J^0F}(\Lambda_{1/3}) \cap J^0I'$ so that the following holds:

- (*) in the front projection, the slopes of the Legendrian arcs above (resp. below) $\Pi_{J^0F}(a_{1/3})$ are negative (resp. positive) and decrease as the z -coordinate increases.

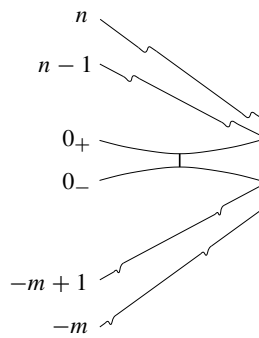


Fig. 17. Isolating the contractible chord by dipping.

In other words, $\Pi_{J^0 F}(\Lambda_{1/3}) \cap J^0 I'$ is as given in Figure 16. This yields Λ_τ , $\tau \in [1/3, 2/3]$.

Finally, we apply a C^0 -small isotopy to the front projection $\Pi_{J^0 F}(\Lambda_{2/3}) \cap J^0 I'$ so that the resulting front $\Pi_{J^0 F}(\Lambda_1) \cap J^0 I'$ is as in Figure 17. The strands are numbered

$$-m, \dots, -1, 0_-, 0_+, 1, \dots, n$$

from bottom to top. Let $I_{L,i} \subset I'$ (resp. $I_{R,i} \subset I'$) be the support of the perturbation of strand i to the left (resp. right) of $x(a_1)$. Then $I_{L,i} \cap I_{L,j} = \emptyset$ for $i \neq j$ and the $I_{L,i}$ move from left to right as i increases; the same holds for $I_{R,i}$. The resulting Legendrian isotopy will be denoted by Λ_τ , $\tau \in [2/3, 1]$. The reason this isotopy Λ_τ , $\tau \in [2/3, 1]$, is called a *dipping* is self-evident when the dipping is viewed in the Lagrangian projection (see Figure 18). Conditions (1) and (2) immediately hold by construction.

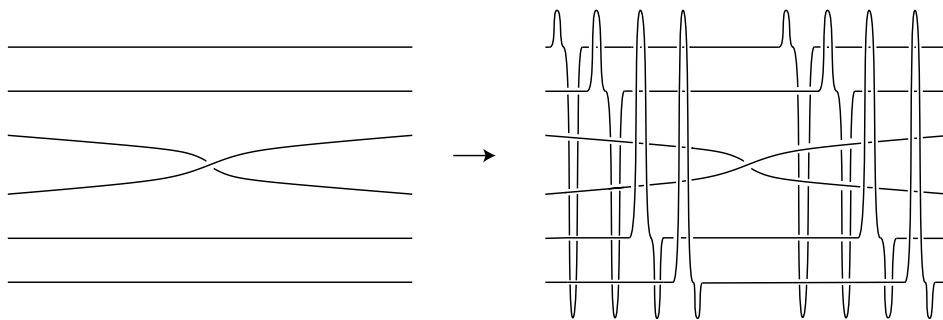


Fig. 18. A dipping in the Lagrangian projection.

Next we claim that if a tree γ of Λ_1 has one positive puncture at $c \neq a_1$ and k positive punctures at a_1 , then its formal dimension is at least $k - 1$. Recall that a tree has formal dimension $k - 1$ if and only if the Fredholm index of the corresponding disk is k . The reader might prefer to translate the proof below to the situation of a Lagrangian projection.

For each positive puncture of γ at a_1 with a $\pi/2$ -angle, there is a partial flow tree of one of the following types:

- γ' , which consists of a partial flow line $(\gamma|_e, \gamma_1^e, \gamma_2^e)$ where Π_{J^0F} maps γ_2^e to the upper sheet S_2 of a_1 and γ_1^e to a sheet S_3 above a_1 for $x \leq x(a_1)$ and a partial flow line $(\gamma|_{e'}, \gamma_1^{e'}, \gamma_2^{e'})$ where Π_{J^0F} maps $\gamma_2^{e'}$ to the lower sheet S_1 of a_1 and $\gamma_1^{e'}$ to S_3 for $x \geq x(a_1)$; γ' is given by the top right diagram of Figure 14; or
- γ'' , which is obtained from γ' by reflecting across the x -axis.

Note that there are no Y_1 -vertices, switches, or ends on J^0I' , since there are no cusps on J^0I' . A Y_0 -vertex and a 2-valent interior puncture with a $3\pi/2$ -angle both increase the formal dimension by 1.

Let $\gamma' \subset \gamma$ be a partial flow tree with a positive 2-valent interior puncture at a_1 . We consider the continuation of γ' to the left; the case of $\gamma'' \subset \gamma$ is similar. If there is no Y_0 -vertex and no 2-valent interior puncture with a $3\pi/2$ -angle, then by a case-by-case analysis there is always a positive puncture $c \neq a_1$ whose corresponding Reeb chord starts at 0_+ . Hence the continuation of γ' to the left has one of the following: (i) a positive puncture $c \neq a_1$; (ii) a Y_0 -vertex above 0_+ , when viewed in the front projection; or (iii) a 2-valent interior puncture with a $3\pi/2$ -angle, whose corresponding Reeb chord starts at or above 0_+ . Moreover, for each positive 2-valent puncture of γ at a_1 , the corresponding (i), (ii) or (iii) is distinct. Since γ has only one positive puncture $c \neq a_1$ and each Y_0 -vertex or $3\pi/2$ -angle increases the formal dimension by 1, it follows that $\dim(\gamma) \geq k - 1$. This proves the claim and the lemma. \square

6.5.6. *The DGA morphism for a general saddle cobordism.* Let L be a general saddle cobordism from Λ_+ to Λ_- with a contractible Reeb chord $a \in \mathcal{C}(\Lambda_+)$. We may assume that $\mathfrak{A}(a)$ is arbitrarily small. The DGA morphism Φ for L can then be computed as follows: Let Λ_τ and a_τ , $\tau \in [0, 1]$, be the isotopies from the proof of Lemma 6.23 such that $\Lambda_0 = \Lambda_+$, $a_0 = a$, and $a_1 \in \mathcal{C}(\Lambda_1)$ is simple and contractible. We then resolve a_1 to obtain a simple saddle cobordism from Λ_1 to Λ'_1 . Finally, we “undo” the isotopy Λ_τ , $\tau \in [0, 1]$, to obtain Λ'_τ , $\tau \in [0, 1]$, such that $\Lambda'_0 = \Lambda_-$. This gives a composition of three cobordisms with corresponding DGA morphism

$$\Phi_0 \circ \Phi_1 \circ \Phi_2: \mathcal{A}(\Lambda_+) \rightarrow \mathcal{A}(\Lambda_-),$$

where Φ_0 and Φ_2 correspond to Legendrian isotopies and can be computed using Lemmas 6.6–6.8, and Φ_1 corresponds to a simple saddle cobordism and can be computed using Proposition 6.18. Since the composition of the three cobordisms is isotopic to L , Φ is chain homotopic to $\Phi_0 \circ \Phi_1 \circ \Phi_2$ by Lemma 3.14.

7. Exact Lagrangian fillings and augmentations

In this section we collect some general results on exact Lagrangian fillings and augmentations.

7.1. tb , r , and slice genus

The following is due to Chantraine [Cha1, Theorem 1.3]:

Theorem 7.1 (Chantraine). *If the Legendrian knot Λ admits an orientable Lagrangian filling L , then:*

- (1) $g(L) = g_s(\Lambda)$; and
- (2) $tb(\Lambda) = 2g_s(\Lambda) - 1 = 2g(L) - 1$ and $r(\Lambda) = 0$.

Here $g(L)$ is the genus of L , $g_s(\Lambda)$ is the slice genus of Λ , and $tb(\Lambda)$ and $r(\Lambda)$ are the Thurston–Bennequin invariant and rotation number of Λ .

Sketch of proof. This is obtained by combining the fact that the tangent bundle of L is isomorphic to its normal bundle, the relation between linking on the boundary and intersections in the interior, and the slice Thurston–Bennequin inequality. \square

7.2. Restriction on linearized Legendrian contact homology

Suppose Λ admits an exact Lagrangian filling $(\mathbb{R} \times \mathbb{R}^3, L)$. Then the DGA morphism

$$\varepsilon = \Phi_{(\mathbb{R} \times \mathbb{R}^3, L)} : \mathcal{A}(\Lambda) \rightarrow \mathbb{F}$$

is an augmentation of Λ . The augmentation ε for $(\mathcal{A}(\Lambda), \partial)$ induces a “change of coordinates” $a_i \mapsto \bar{a}_i = a_i + \varepsilon(a_i)$ of $\mathcal{A}(\Lambda)$, where $a_i \in \mathcal{C}(\Lambda)$, such that $\partial \bar{a}_i$ does not have any constant terms when expressed in terms of sums of words in \bar{a}_j . In other words, with respect to the new generators \bar{a}_i , ∂ is nondecreasing with respect to the word length filtration, i.e., $\partial = \partial_1 + \partial_2 + \dots$, where ∂_j is the part of the boundary map that counts words of length j in the \bar{a}_i ’s. The ε -linearized Legendrian contact homology group $HC^\varepsilon(\Lambda)$ with respect to ε is the homology of $(\mathcal{A}_1, \partial_1)$, where \mathcal{A}_1 is the \mathbb{Q} -vector space generated by $\{\bar{a}_i, a_i \in \mathcal{C}(\Lambda)\}$; let $CC^\varepsilon(\Lambda)$ be the chain complex $(\mathcal{A}_1, \partial_1)$. For more details see [Ch].

The following theorem is essentially due to Seidel (see [E3]). Here we sketch a proof that uses wrapped Floer homology (cf. [AS, FSS]).

Theorem 7.2 (Seidel). $HC^\varepsilon(\Lambda) \simeq H^*(L)$.

Sketch of proof. Let $CF^{\text{wr}}(L)$ be the wrapped Floer chain complex of L , which is defined as follows: Let $H : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}$ be a Hamiltonian function which outside a finite height cylinder $[-T, T] \times \mathbb{R}^3$ is the composition of the projection $\pi : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}$ and the function $f : \mathbb{R} \rightarrow \mathbb{R}$, where:

- $f(t) = 0$ for $t \ll 0$;
- $f(t) = w_0 e^t$ for $t \gg 0$, where $w_0 > \mathfrak{A}_{\alpha_0}(a)$ for all $a \in \mathcal{C}(\Lambda)$;
- $f'(t) \geq 0$; and
- H is small in $\pi(L - \mathcal{E}_+(L))$, where $\mathcal{E}_+(L) \approx [T, \infty) \times \Lambda$ is the positive cylindrical end of L , and L is invariant under the gradient flow of H .

Let X_H be the Hamiltonian vector field of H , i.e., $i_{X_H}d(e^t\alpha_0) = dH$, and let ϕ_t be the time- t flow of X_H . Then $CF^{wr}(L)$ is defined as the Floer chain complex generated by time-1 Hamiltonian chords of L , i.e., flow segments $\phi_t(p)$, $t \in [0, 1]$, such that $\phi_0(p) = p \in L$ and $\phi_1(p) \in L$. For Hamiltonians H as above there are then two types of generators: chords corresponding to critical points of H on L , and Reeb chords of Λ , where a Reeb chord c of action $\mathfrak{A}_{\alpha_0}(c)$ appears in the t -slice where $e^{-t}f'(t) = \mathfrak{A}_{\alpha_0}(c)$. The differential on $CF^{wr}(L)$ then counts Floer holomorphic strips connecting Hamiltonian chords. More precisely, a Floer holomorphic strip is a map $u : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R} \times \mathbb{R}^3$ with boundary on L that solves the differential equation

$$(du - X_H \otimes d\tau_2) + J(du - X_H \otimes d\tau_2)i = 0,$$

and that is asymptotic to Hamiltonian chords at $\pm\infty$, where $(\tau_1, \tau_2) \in \mathbb{R} \times [0, 1]$ are standard coordinates, J is a compatible almost complex structure on $\mathbb{R} \times \mathbb{R}^3$ and i is the standard complex structure on $\mathbb{R} \times [0, 1]$.

Let $C' \subset CF^{wr}(L)$ be the “low energy” subcomplex generated by the critical point chords; its homology is isomorphic to $H^*(L)$ by the usual identification of Floer holomorphic strips and gradient trajectories. The “high energy part” $CF^{wr,+}(L)$ of $CF^{wr}(L)$ is the quotient $CF^{wr}(L)/C'$. We then obtain the exact triangle

$$\xrightarrow{\delta_*} H^*(L) \xrightarrow{i} HF^{wr}(L) \xrightarrow{j} HF^{wr,+}(L) \xrightarrow{\delta_*} . \tag{7.1}$$

Let $g : \mathbb{R} \rightarrow [0, 1]$ be a nonincreasing smooth function such that $g(\tau_1) = 0$ near ∞ and $g(\tau_1) = 1$ near $-\infty$. In analogy with the Morse–Bott description of symplectic homology from [BO], there is a chain map

$$\Psi : CC^\epsilon(\Lambda) \rightarrow CF^{wr}(L)$$

that counts interpolating Floer holomorphic strips $v : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R} \times \mathbb{R}^3$ with boundary on L that solves the differential equation

$$(dv - X_H \otimes g(\tau_1)d\tau_2) + J(du - X_H \otimes g(\tau_1)d\tau_2)i = 0,$$

and which are asymptotic to a Reeb chord in $\mathcal{C}(\Lambda)$ as $\tau_1 \rightarrow \infty$ and to a Hamiltonian chord as $\tau_1 \rightarrow -\infty$.

The chain map Ψ induces an isomorphism

$$\Psi_* : HC^\epsilon(\Lambda) \xrightarrow{\sim} HF^{wr,+}(L) \tag{7.2}$$

by an action filtration argument. In particular, the leading term of $\Psi(a)$, $a \in \mathcal{C}(\Lambda)$, in the action filtration is given by the reparametrized trivial strip over a .

Since L is displaceable, $HF^{wr}(L) = 0$ and the theorem follows from the exact triangle (7.1) and (7.2). □

Remark 7.3. The above theorem holds for displaceable Lagrangian fillings L in an exact symplectic manifold of dimension $2n$ for $n \geq 1$.

Remark 7.4 (Gradings). We are using the usual grading conventions for $CF^{\text{wr}}(L)$. In the case when the Maslov class of L is nonzero, $CF^{\text{wr}}(L)$ no longer has an integer grading. However, there is still a differential, and the resulting homology $HF^{\text{wr}}(L)$ is invariant under deformations.

Suppose $CF^{\text{wr}}(L)$ is \mathbb{Z} -graded. Then the map ι has degree 0, and in particular the intersection point of $CF(L, \phi_1(L))$ corresponding to the minimum of $H|_L$ has degree 0. The quotient map j has degree zero, the isomorphism Ψ_* has degree $-\dim(L) = -n$, and δ_* has degree $+1$.

7.3. Nontriviality of augmentations

We state a related result concerning the nontriviality of augmentations. Let $(\mathbb{R} \times \mathbb{R}^3, L)$ be an exact Lagrangian filling of Λ and let

$$\tilde{\varepsilon} : \mathcal{A}(\Lambda; \mathbb{F}[H_1(L; \mathbb{Z})]) \rightarrow \mathbb{F}[H_1(L; \mathbb{Z})]$$

be the induced augmentation map, i.e., the unital algebra map which satisfies the following:

- $\tilde{\varepsilon}$ is the identity map on the homology generators; and
- $\tilde{\varepsilon}$ takes any Reeb chord a of Λ to the count of rigid once-punctured disks with boundary on L which are asymptotic to a at the positive end.

Remark 7.5. If the Maslov class of L is nonzero, then $\tilde{\varepsilon}$ is an ungraded augmentation.

Proposition 7.6. *If $H_1(L; \mathbb{Z}) \neq 0$, then $\tilde{\varepsilon} \neq 0$, i.e., there exists a Reeb chord a of grading 0 such that $\tilde{\varepsilon}(a) \neq 0$.*

The proof below does not depend on whether $\tilde{\varepsilon}$ is graded.

Sketch of proof. Let $C^*(L; \mathbb{F})$ be the Morse cochain complex of L with respect to the function $H|_L$ from Theorem 7.2, which we assume is Morse. Let

$$\delta : CC^{\tilde{\varepsilon}}(\Lambda; \mathbb{F}) \rightarrow C^*(L; \mathbb{F})$$

be the chain map which induces the connecting homomorphism δ_* in the exact triangle (7.1). Since L is displaceable, δ_* is an isomorphism. The map δ admits the following geometric description (which follows from a generalization of the discussion in the proof of Theorem 7.2 that adds the Morse complex of L to $CC^{\tilde{\varepsilon}}$ and then includes also the low-energy generators in the isomorphism): Let p be a critical point of f and let $x \in \mathcal{C}(\Lambda)$. If we write $\delta(x) = \sum_p \langle \delta(x), p \rangle \cdot p$, then $\langle \delta(x), p \rangle$ is the count of rigid pairs (u, γ) , where u is a holomorphic disk with a positive end at x and boundary on L , and γ is a gradient flow line of L emanating from the boundary of u and ending at p . In particular, if $\text{ind}(p) = 1$, then $\langle \delta(x), p \rangle$ is the sum of the intersection numbers of γ' and the boundary of u , where we are ranging over all pairs (u, γ') such that

- γ' is a gradient trajectory that ends at p ; and
- u is a rigid disk with a positive end at x .

Assume for contradiction that $\tilde{\varepsilon} \equiv 0$. Then, for each $x \in \mathcal{C}(\Lambda)$ and $\eta \in H_1(L; \mathbb{Z})$, the number of rigid disks with a positive end at x and boundary on L that represent η is even. Let \bar{L} be the compactification of L obtained by adding $\{\infty\} \times \Lambda$. If $\text{ind}(p) = 1$, then the two gradient trajectories that end at p form a class in $H_1(\bar{L}, \partial\bar{L})$. In view of the above description of $\langle \delta(x), p \rangle$ for $\text{ind}(p) = 1$, it follows that $\delta(x) = 0$. This is a contradiction of $H_1(L; \mathbb{Z}) \neq 0$. \square

7.4. The fundamental class

Let $\varepsilon : \mathcal{A}(\Lambda) \rightarrow \mathbb{F}$ be an augmentation of $\Lambda \subset \mathbb{R}^3$.

Definition 7.7. An ε -augmented holomorphic disk in $(\mathbb{R} \times \mathbb{R}^3, \mathbb{R} \times \Lambda)$ from a to \mathbf{b} is a holomorphic disk

$$u : (D_{m+1}, \partial D_{m+1}) \rightarrow (\mathbb{R} \times \mathbb{R}^3, \mathbb{R} \times \Lambda)$$

from a to $\mathbf{b}' = \tau_0 b_1 \tau_1 \dots \tau_{m-1} b_m \tau_m$, together with a subset \mathfrak{c} of $\{1, \dots, m\}$ such that applying ε to all the $b_i, i \in \mathfrak{c}$, yields \mathbf{b} .

If (X, L) is an exact Lagrangian cobordism from Λ_+ to Λ_- and ε_- is an augmentation of Λ_- , then an ε_- -augmented holomorphic disk in (X, L) is defined similarly.

Given $y \in \mathbb{R} \times \Lambda$ and $a \in \mathcal{C}(\Lambda)$, let $\langle \partial_y a, \mathbf{b} \rangle$ be the count of ε -augmented $\text{ind} = 1$ holomorphic disks u from a to \mathbf{b} that pass through y . We then write $\partial_y a = \sum_{i=0}^\infty \partial_y^i a$, where ∂_y^i counts the ε -augmented disks from a to \mathbf{b} , and \mathbf{b} has i negative ends.

Definition 7.8. An ε -fundamental class is an element η_ε of $HC^\varepsilon(\Lambda)$ such that $\partial_y^0 x = 1$ for any generic $y \in \mathbb{R} \times \Lambda$ and representative x of η_ε .

The following is a theorem of Sabloff [Sa2] and Ekholm–Etnyre–Sabloff [EESa, Theorem 5.5].

Theorem 7.9. For any augmentation ε of $\Lambda \subset \mathbb{R}^3$, there exists an ε -fundamental class in $HC^\varepsilon(\Lambda)$.

Theorem 7.10. Let (X, L) be an exact Lagrangian cobordism from Λ_+ to Λ_- with corresponding chain map $\Phi_{(X,L)} : \mathcal{A}(\Lambda_+) \rightarrow \mathcal{A}(\Lambda_-)$, let ε_- be an augmentation of Λ_- , and let $\varepsilon_+ = \Phi_{(X,L)}^*(\varepsilon_-)$ be the induced augmentation of Λ_+ . Assume L, Λ_+ , and Λ_- are connected. Then the linearization

$$\Phi_{(X,L)} : HC^{\varepsilon_+}(\Lambda_+) \rightarrow HC^{\varepsilon_-}(\Lambda_-)$$

maps an η_{ε_+} -fundamental class to an η_{ε_-} -fundamental class.

Proof. The conclusion follows from observing that (X, L) induces a chain map from $\mathcal{A}(\Lambda_+)$ to $\mathcal{A}(\Lambda_-)$, where the differentials are “twisted by point conditions”. The proof is similar to the usual case (see [BEE, Section 4.2]).

More precisely, let $T > 0$ be sufficiently large and let $y_\pm \in \Lambda_\pm$ be generic. Take a generic path $y : \mathbb{R} \rightarrow L$ such that $y(t) = (t, y_+)$ for $t \geq T$ and (t, y_-) for $t \leq -T$.

Define $\mathcal{M}_{y(t)}(a)$, $a \in \mathcal{C}(\Lambda_+)$, as the moduli space of ε_- -augmented $\text{ind} = 1$ holomorphic disks in L from a to \emptyset that pass through $y(t)$. As $t \rightarrow \infty$, $\mathcal{M}_{y(t)}(a)$ limits to the moduli space of ε_+ -augmented $\text{ind} = 1$ disks in $\mathbb{R} \times \Lambda_+$ from a to \emptyset that pass through $(0, y_+)$. To see this, note that by SFT compactness a sequence in $\mathcal{M}_{y(t)}(a)$ limits to a multiple-level building, where the top level u_1 is a level in the symplectization (at the positive end) and satisfies $\text{ind}(u_1) \geq 1$ since it must pass through the generic point $(0, y_+)$. Hence $\text{ind}(u_1) = 1$ by transversality and the additivity of the Fredholm index. There is only one other level u_0 , which is a collection of rigid ε_- -augmented disks in the cobordism, i.e., a collection of disks contributing to the pullback augmentation ε_+ .

Similarly, as $t \rightarrow -\infty$, $\mathcal{M}_{y(t)}(a)$ limits to the moduli space of two-level buildings $u_{-1} \cup u_0$, where u_0 is an ε_- -augmented $\text{ind} = 0$ disk from a to some $b \in \mathcal{C}(\Lambda_-)$, and u_{-1} is an ε_- -augmented $\text{ind} = 1$ disk from b to \emptyset that passes through $(0, y_-)$. The resulting chain homotopy implies that if $x \in \eta_{\varepsilon_+}$ then the mod 2 count of ε_- -augmented disks from $\Phi_{(X,L)}(x)$ to \emptyset that pass through $(0, y_-)$ is equal to the mod 2 count of ε_+ -augmented disks from x to \emptyset that pass through $(0, y_+)$, which in turn is 1. This proves the theorem. \square

8. Applications

8.1. Lagrangian fillings of $(2, n)$ -torus links

In this subsection we consider the Legendrian $(2, n)$ -torus link Λ_n whose Lagrangian projection is given by Figure 2. The main result is Proposition 8.5, which immediately implies Theorem 1.8.

Each component of Λ_n has Maslov number 0. When n is even, Λ_n consists of two unknots with Thurston–Bennequin number $tb(\Lambda_n) = -1$. Let a_1 and a_2 be the two rightmost Reeb chords in Figure 2 with grading $|a_j| = 1$, $j = 1, 2$. The remaining Reeb chords, from left to right, are denoted by b_1, \dots, b_n and satisfy $|b_j| = 0$. (If n is even, we choose the reference path δ_{12} to be one of the Reeb chords connecting the two components of Λ_n . We then choose the path of lines in the contact planes along δ_{12} so it makes a $\pi/2$ -rotation with respect to the trivialization induced by \mathbb{C} .)

Claim 8.1. *The chords a_1 and a_2 are noncontractible and the chords b_j , $j = 1, \dots, n$, are all contractible. Moreover, all pairs b_j, b_k , $|k - l| > 1$, of nonadjacent degree 0 Reeb chords of Λ_n are simultaneously contractible.*

Proof. Follows from Lemma 6.14. \square

By Claim 8.1 there is a sequence of saddle cobordisms that resolves the crossings of Λ_n in each of the $n!$ possible orders. We denote a permutation $\sigma \in S_n = \text{Aut}(\{1, \dots, n\})$ by (i_1, \dots, i_n) if $\sigma(j) = i_j$. For each permutation $\sigma = (i_1, \dots, i_n)$, let L_σ be a Lagrangian cobordism in $\mathbb{R} \times \mathbb{R}^3$ which is deformation-equivalent to the composition of the saddle cobordisms that resolve the degree 0 crossings in the order b_{i_1}, \dots, b_{i_n} from top to bottom.

Let $\sigma = (i_1, \dots, i_n)$ be a permutation and i_j, i_{j+1} be adjacent entries such that there is some $i_p, p > j + 1$, with $i_j < i_p < i_{j+1}$ or $i_j > i_p > i_{j+1}$. If

$$\sigma' = (i_1, \dots, i_{j-1}, i_{j+1}, i_j, i_{j+2}, \dots, i_n),$$

then L_σ and $L_{\sigma'}$ are exact Lagrangian isotopic. Two permutations σ and σ' are *isotopy equivalent* if they are related by a sequence of transpositions of the above type.

Remark 8.2. If we drop the Lagrangian condition on the cobordism, then any two chords become simultaneously contractible. Thus all of our Lagrangian fillings Λ_σ are smoothly isotopic.

Lemma 8.3. *The number of isotopy equivalence classes of permutations is the Catalan number $C_n = \binom{2n}{n}/(n + 1)$.*

Proof. It is well-known that the Catalan numbers are determined by the initial value $C_0 = 1$ and the recurrence relation

$$C_{n+1} = \sum_{k=0}^n C_k C_{n-k}, \tag{8.1}$$

for $n \geq 0$. We prove that the number a_n of isotopy equivalence classes of permutations satisfies the same recurrence relation.

First note that $a_0 = 1$. If two permutations $(i_1, \dots, i_{n+1}), (i'_1, \dots, i'_{n+1}) \in S_{n+1}$ are equivalent, then $i_{n+1} = i'_{n+1}$ since the last entry can never be part of an isotopy move. Let S_n^{k+1} be the set of permutations such that $i_{n+1} = k + 1$ and let $\sigma = (i_1, \dots, i_n) \in S_n^{k+1}$. If $i_j > k + 1$ and $i_{j+1} < k + 1$, then i_j and i_{j+1} can be swapped by an isotopy equivalence. Hence any $\sigma \in S_n^{k+1}$ is equivalent to a permutation in normal form $\lambda = (\lambda_1, \lambda_2, k + 1)$, where $\lambda_1 \in \text{Aut}(\{1, \dots, k\})$ and $\lambda_2 \in \text{Aut}(\{k + 2, \dots, n + 1\})$. Now, two permutations in S_n^{k+1} in normal form are isotopy equivalent if and only if the corresponding permutations λ_1 and λ_2 are isotopy equivalent. Hence there are $a_k a_{n-k}$ isotopy equivalence classes in S_{n+1} with $i_{n+1} = k + 1$. Summing over k gives the recurrence relation (8.1). \square

Example 8.4. When $n = 3$, we have $C_3 = 5$ and the five sequences of resolutions produce the five different augmentations of the Legendrian right-handed trefoil. Hence the corresponding exact Lagrangian fillings are pairwise not exact Lagrangian isotopic. The only two isotopy equivalent permutations in this case are $(1, 3, 2)$ and $(3, 1, 2)$, and they both give rise to the augmentation $\varepsilon(b_1) = \varepsilon(b_2) = \varepsilon(b_3) = 1$. In Figure 19 we illustrate the sequence of resolutions and hence the Lagrangian cap corresponding to $(2, 1, 3)$. It yields the augmentation $\varepsilon(b_1) = 0, \varepsilon(b_2) = \varepsilon(b_3) = 1$.

For larger n , however, the pattern is less clear. The total number of augmentations is $A_n = (2^{n+1} - (-1)^{n+1})/3$ by [Ka2, Proposition 7.1], which is much smaller than C_n . When $n = 5$, we have $C_5 = 42$ and $A_5 = 21$. Computing the augmentations belonging to the 42 sequences of resolutions, we find that all 21 possibilities occur at least once.

We can prove by a relatively easy construction that this is the case for general Λ_n . More precisely, we have:

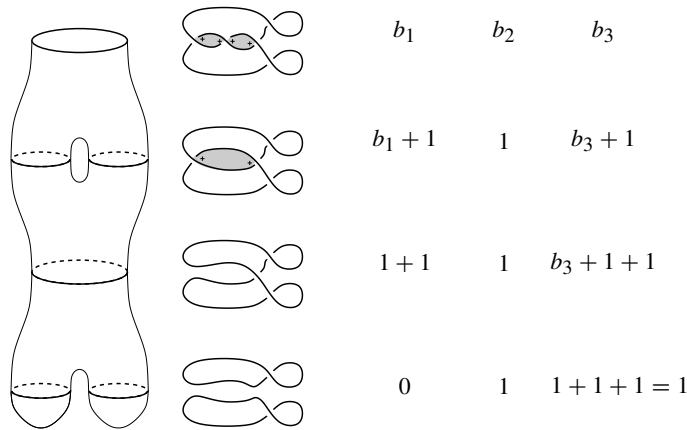


Fig. 19. Constructing a Lagrangian cap and computing its induced augmentation.

Proposition 8.5. Any augmentation of the $(2, 2k + 1)$ -torus knot is induced by a sequence of crossing resolutions of the standard diagram, and so are all nonzero augmentations of the $(2, 2k)$ -torus link.

An augmentation $\varepsilon : \mathcal{A}(\Lambda) \rightarrow \mathbb{F}$ is the *zero augmentation* (written as $\varepsilon \equiv 0$) if $\varepsilon|_{\mathcal{C}(\Lambda)} = 0$.

Proof. View $\Lambda_n, n = 2k$ or $2k + 1$, as the closure of a positive braid on two strands which is given by σ^n . Here σ is a positive generator of the braid group on two strands. Define the matrix

$$B_n = \begin{bmatrix} b_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b_2 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} b_n & 1 \\ 1 & 0 \end{bmatrix},$$

where the indeterminates b_1, \dots, b_n are the Reeb chord generators of $\mathcal{A}(\Lambda_n)$. Let $\varepsilon : \mathcal{A}(\Lambda_n) \rightarrow \mathbb{F}$ be an algebra map defined by sending each degree 0 generator b_i to 0 or 1 and each degree $\neq 0$ generator to 0. According to [Ka2, Theorem 5.3], the algebra map ε is an augmentation of Λ_n if and only if $\varepsilon(B_n)$ is of the form $\begin{bmatrix} 1 & * \\ * & * \end{bmatrix}$.

Given an augmentation ε for Λ_n , resolving a crossing q with $\varepsilon(q) = 1$ and changing the ε -value (from 1 to 0 or from 0 to 1) associated to the one or two adjacent degree zero crossings yields Λ_{n-1} with an algebra map $\varepsilon' : \mathcal{A}(\Lambda_{n-1}) \rightarrow \mathbb{F}$. We claim that ε' is an augmentation. Indeed, the Boolean algebra identity

$$\begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} y & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} x + 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} y + 1 & 1 \\ 1 & 0 \end{bmatrix}$$

implies that $\varepsilon'(B_{n-1}^j)$ is of the form $\begin{bmatrix} 1 & * \\ * & * \end{bmatrix}$, where

$$B_{n-1}^j = \begin{bmatrix} b_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} b_{j-1} + 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b_{j+1} + 1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} b_n & 1 \\ 1 & 0 \end{bmatrix}.$$

Next we claim that $\varepsilon' \not\equiv 0$ by a suitable choice of q : If there are index 0 crossings r with $\varepsilon(r) = 0$, then choose q with $\varepsilon(q) = 1$ so that it has a neighbor r with $\varepsilon(r) = 0$. If

$\varepsilon \equiv 1$ on the index 0 crossings, then let q be the leftmost or rightmost crossing. We have $\varepsilon' \neq 0$ as long as $n > 2$. Finally, observe that $\varepsilon(b_1) = \varepsilon(b_2) = 1$ is not an augmentation for the Hopf link Λ_2 .

The proof then proceeds by induction. □

Remark 8.6. Although $\varepsilon(b_1) = \varepsilon(b_2) = 0$ is an augmentation of the Hopf link Λ_2 , ε cannot come from an embedded Lagrangian by Theorem 7.2.

Not much is known about exact Lagrangian fillings of Legendrian links. We conclude this subsection with some open questions.

Question 8.7. *If $\sigma_1, \sigma_2 \in S_n$ are not isotopy equivalent but correspond to exact Lagrangian fillings $L_{\sigma_1}, L_{\sigma_2}$ of Λ_n which induce the same augmentation, are L_{σ_1} and L_{σ_2} isotopic through exact Lagrangians?*

Question 8.8. *Is any exact Lagrangian filling of Λ_n which induces a nonzero augmentation exact Lagrangian isotopic to one coming from a sequence of resolutions of the “standard model” of Λ_n given by Figure 2? This is not known even for $n = 3$.*

Question 8.9. *If Λ is a Legendrian knot and g_s is its slice genus, is every augmentation ε of Λ geometric, provided $HC^\varepsilon(\Lambda) \simeq H_*(\Sigma_{g_s})$, where Σ_{g_s} is the once-punctured oriented surface of genus g_s ? In other words, is Theorem 7.2 the only obstruction to the existence of an exact Lagrangian filling that induces ε ?*

Question 8.10. *Is every exact Lagrangian filling of a Legendrian knot Λ decomposable?*

In the case of a concave filling, there is an example of Sauvaget [Sau], which after some modification, yields an exact Lagrangian cobordism of genus 2 from \emptyset to a stabilized unknot. A concave filling cannot be decomposable.

8.2. Connected sums

8.2.1. Comultiplication. We now introduce the exact Lagrangian cobordism and the accompanying comultiplication map induced by (the inverse of) a connected sum. Let $\Lambda_1, \Lambda_2 \subset (\mathbb{R}^3, \xi_0)$ be Legendrian knots and let $\Lambda_1 \# \Lambda_2$ be their connected sum, schematically given in Figure 20.

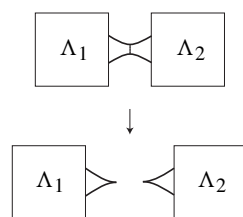


Fig. 20. The inverse of a connected sum of Λ_1 and Λ_2 in the front projection.

Suppose without loss of generality that $\Lambda_1 \sqcup \Lambda_2$ is obtained from $\Lambda_1 \# \Lambda_2$ by resolving a simple contractible Reeb chord a . Then

$$\mathcal{C}(\Lambda_1 \# \Lambda_2) \simeq \mathcal{C}(\Lambda_1) \cup \mathcal{C}(\Lambda_2) \cup \{a\}$$

and the corresponding simple saddle cobordism L induces the coproduct map

$$\Phi_L : \mathcal{A}(\Lambda_1 \# \Lambda_2) \rightarrow \mathcal{A}(\Lambda_1 \sqcup \Lambda_2) \tag{8.2}$$

which maps $x \mapsto x$ if $x \in \mathcal{C}(\Lambda_1)$ or $\mathcal{C}(\Lambda_2)$ and $a \mapsto 1$. This is due to the fact that there are no holomorphic disks in $\Lambda_1 \# \Lambda_2$ that have a at the positive end.

Example 8.11. Let Λ_1 and Λ_2 both be standard $tb = -1, r = 0$ Legendrian unknots U and let $x_i, i = 1, 2$, be the generators of $\mathcal{A}(\Lambda_i)$. Then $\Lambda_1 \# \Lambda_2$ is isotopic to U with generator x . Let $\mathbb{F}\{S\}$ be the free \mathbb{F} -algebra generated by the set S . On the level of homology the coproduct map is

$$\mathbb{F}\{x\} \rightarrow \mathbb{F}\{x_1, x_2\}, \quad x \mapsto x_1 + x_2.$$

If we abelianize the DGA morphism $\mathcal{A}(\Lambda_1 \# \Lambda_2) \rightarrow \mathcal{A}(\Lambda_1 \sqcup \Lambda_2)$, then on the level of homology the coproduct map is

$$\mathbb{F}[x] \rightarrow \mathbb{F}[x_1] \otimes \mathbb{F}[x_2], \quad x \mapsto x_1 \otimes 1 + 1 \otimes x_2,$$

which is the coproduct on the S^1 -equivariant cohomology of a point.

8.2.2. *Multiplication.* Next we introduce the exact symplectic cobordism induced by a stabilized connected sum $S_+S_-(\Lambda_1 \# \Lambda_2)$, where S_+ (resp. S_-) denotes a single positive (resp. negative) stabilization. This is given by Figure 21. Unfortunately, the multiplication map

$$\mathcal{A}(\Lambda_1) \otimes \mathcal{A}(\Lambda_2) \rightarrow \mathcal{A}(S_+S_-(\Lambda_1 \# \Lambda_2))$$

is the zero map and Legendrian contact homology gives no information.

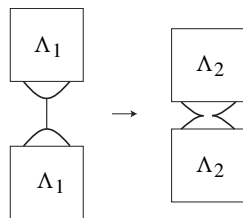


Fig. 21. The stabilized connected sum of Λ_1 and Λ_2 in the front projection.

The above operation nonetheless can be used to prove the following theorem:

Theorem 8.12. *Given any Legendrian link $\Lambda \subset \mathbb{R}^3$, there is an exact Lagrangian cobordism from Λ to some stabilized unknot.*

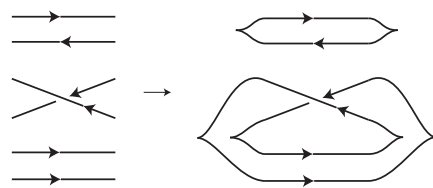


Fig. 22. Iterated pinching.

Proof. Consider the image of Λ under the front projection $\Pi_{J^0\mathbb{R}} : \mathbb{R}^3 \rightarrow \mathbb{R}^2, (x, y, z) \mapsto (x, z)$. We slice $\Pi_{J^0\mathbb{R}}(\Lambda)$ along $x = x_i$ for $x_1 < \dots < x_n$ such that there is at most one crossing on each $\{x_i < x < x_{i+1}\}$ and there are no cusps or crossings on each $x = x_i$. (Here we are assuming that Λ is sufficiently generic.) We then repeatedly apply the pinching procedure of Figure 11, making sure that at each step we pinch strands that are pointing in opposite directions. The result of applying the procedure to the slice $\{x_i < x < x_{i+1}\}$ is given in Figure 22, and is a disjoint union of possibly stabilized unknots. Finally, we apply the stabilized connected sum operation to obtain a stabilized unknot. \square

9. Khovanov homology

In this section we assume that the reader is familiar with basic facts about Khovanov homology [Kh]. We refer the reader to [Ja] for a discussion of the map on Khovanov homology induced by a cobordism of links.

The Khovanov homology of a link $\Lambda \subset \mathbb{R}^3$ will be denoted by $\text{Kh}(\Lambda) = \bigoplus_{i,j} K^{i,j}(\Lambda)$, where i is the homological grading and j is the q -grading. (Strictly speaking, when Λ is Legendrian, we consider the Lagrangian projection $\pi_{\mathbb{C}}(\Lambda)$.) The following is due to Jacobsson [Ja, Section 4.2].

Theorem 9.1 (Jacobsson). *If $\Lambda_+, \Lambda_- \subset \mathbb{R}^3$ are links and $L \subset [-1, 1] \times \mathbb{R}^3$ is a smooth cobordism from $\{1\} \times \Lambda_+$ to $\{-1\} \times \Lambda_-$, then it induces a map*

$$\Psi_L : \text{Kh}(\Lambda_-) \rightarrow \text{Kh}(\Lambda_+)$$

that is well-defined up to ± 1 and is invariant under smooth isotopies relative to ∂L .

The isotopy is required to fix Λ_+ and Λ_- pointwise to avoid monodromy issues. Note that any isotopy invariant derived from Ψ_L will have this ± 1 ambiguity as well.

Let \bar{K} denote the mirror of the link K . Given a Legendrian link $\Lambda \subset \mathbb{R}^3$ and an exact Lagrangian cobordism L from Λ to \emptyset , Theorem 9.1 gives a map

$$\Psi_{\bar{L}} : \text{Kh}(\emptyset) \simeq \mathbb{Z} \rightarrow \text{Kh}(\bar{\Lambda}).$$

Let $\Lambda \subset \mathbb{R}^3$ be a Legendrian link. Let \mathcal{S} (resp. \mathcal{S}^{dec}) be the set of isotopy classes of exact (resp. decomposable exact) Lagrangian cobordisms L from Λ to \emptyset . Then we define

$$\mathcal{L}_{\Lambda} = \bigsqcup_{L \in \mathcal{S}^{\text{dec}}} \{\Psi_{\bar{L}}(\pm 1)\} \subset \text{Kh}(\bar{\Lambda}), \quad \mathcal{L}'_{\Lambda} = \bigsqcup_{L \in \mathcal{S}} \{\Psi_{\bar{L}}(\pm 1)\} \subset \text{Kh}(\bar{\Lambda}).$$

By definition, \mathcal{L}_Λ and \mathcal{L}'_Λ are invariants of Λ , although not particularly computable.

Let L be a decomposable exact Lagrangian cobordism from Λ_+ to Λ_- which is the composition of elementary cobordisms L_i from Λ_i to Λ_{i+1} , $i = 1, \dots, n - 1$, where $\Lambda_+ = \Lambda_1$ and $\Lambda_- = \Lambda_n$. Then we define

$$\Psi'_{L_i} : \text{Kh}(\overline{\Lambda}_{i+1}) \rightarrow \text{Kh}(\overline{\Lambda}_i)$$

and Ψ'_L as the composition of the Ψ'_{L_i} as follows: If K_1 is the 1-resolution of K , then there exists a map $\text{Kh}(K_1) \rightarrow \text{Kh}(K)$. In our case, mirroring transforms a 0-resolution to a 1-resolution and we have a map $\text{Kh}(\overline{\Lambda}_{i+1}) \rightarrow \text{Kh}(\overline{\Lambda}_i)$ if Λ_{i+1} is a 0-resolution of Λ_i . If Λ_{i+1} is obtained from Λ_i by a Legendrian isotopy, then the isotopy induces the map $\text{Kh}(\overline{\Lambda}_{i+1}) \rightarrow \text{Kh}(\overline{\Lambda}_i)$.

Lemma 9.2. *The map $\Psi'_L : \text{Kh}(\overline{\Lambda}_n) \rightarrow \text{Kh}(\overline{\Lambda}_1)$ induced by the decomposable exact Lagrangian cobordism L agrees with $\Psi_{\overline{L}}$, where we view \overline{L} as a smooth surface from $\overline{\Lambda}_n$ to $\overline{\Lambda}_1$.*

Proof. In [Ja], the 1-resolution map $\text{Kh}(K_1) \rightarrow \text{Kh}(K)$ is not used in the definition of the cobordism map Ψ_L . Instead, the 1-resolution $K_1 \rightarrow K$ must be factored into the composition of a Reidemeister I move $K_1 \rightarrow K'$ followed by a Morse saddle move $K' \rightarrow K$. One easily verifies that the composition $\text{Kh}(K_1) \rightarrow \text{Kh}(K') \rightarrow \text{Kh}(K)$ agrees with the 1-resolution map. \square

Let L be an exact Lagrangian filling of Λ . We view L as a composition of a cobordism L_1 from Λ to the unknot U , followed by a minimal cobordism L_2 . Let ε_- and ε_+ be the augmentations induced by L_2 and L , respectively. The augmentation ε_+ gives rise to an automorphism of $\mathcal{A}(\Lambda)$ which takes $a \in \mathcal{C}(\Lambda)$ to $\bar{a} = a + \varepsilon_+(a)$, so that the differential respects the filtration by word length in the \bar{a} 's. Let $E^1(\mathcal{A}(\Lambda), \varepsilon_+)$ be the E^1 -term of the associated spectral sequence; one easily sees that $E^1(\mathcal{A}(\Lambda), \varepsilon_+)$ is the unital tensor algebra generated by $HC^{\varepsilon_+}(\Lambda)$. Then we can define a map

$$\Theta : E^1(\mathcal{A}(\Lambda), \varepsilon_+) \rightarrow \text{Kh}(\overline{\Lambda}),$$

obtained by composing the maps

$$\Phi_{L_1} : E^1(\mathcal{A}(\Lambda), \varepsilon_+) \rightarrow E^1(\mathcal{A}(U), \varepsilon_-), \quad \Psi_{L_1} : \text{Kh}(\overline{U}) \rightarrow \text{Kh}(\overline{\Lambda}_+),$$

in Legendrian contact homology and Khovanov homology, together with

$$E^1(\mathcal{A}(U), \varepsilon_-) \simeq \mathbb{F}[x] \rightarrow \text{Kh}(\overline{U}) \simeq V,$$

where $1 \mapsto \mathbf{v}_+$, $x \mapsto \mathbf{v}_-$, and V is a 2-dimensional \mathbb{F} -vector space generated by \mathbf{v}_+ of degree 1 and \mathbf{v}_- of degree -1 .

We close this subsection with some questions and a remark.

Question 9.3. *Is the map Θ good for anything?*

Question 9.4. *Can the set \mathcal{L}_Λ and \mathcal{L}'_Λ have more than two elements? Are \mathcal{L}_Λ and \mathcal{L}'_Λ finite?*

Observe that \mathcal{L}_Λ and \mathcal{L}'_Λ are empty if there are no exact Lagrangian cobordisms from Λ to \emptyset . By the ± 1 ambiguity, the cardinalities of \mathcal{L}_Λ and \mathcal{L}'_Λ are even.

Question 9.5. *Are there only finitely many exact Lagrangian isotopy classes of exact Lagrangians which bound a given Legendrian knot Λ ?*

Since there are no local Lagrangian 2-knots in 4-space [EP], one might expect that the answer is yes.

Example 9.6. Let Λ_3 be the Legendrian right-handed trefoil knot with $tb = 1$ from Section 8.1. For each of the five exact Lagrangian cobordisms L_1, \dots, L_5 constructed in Section 8.1, we can assign an element of $\text{Kh}(\overline{\Lambda})$. For each L_i , the element $\Psi_{L_i}^{-1}(1)$ coincides with Plamenevskaya's transverse knot invariant [PI], and also lies on Ng's [Ng] line $j - i = C$, where

$$C = \{\max(j - i) \mid \text{Kh}^{i,j}(\overline{L}) \neq 0\}.$$

By Remark 8.2, the exact Lagrangians L_i are all smoothly isotopic.

Question 9.7. *Clarify the relationship among \mathcal{L}_Λ and \mathcal{L}'_Λ , Plamenevskaya's transverse knot invariant, and Ng's line.*

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