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On non-forking spectra

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Abstract. Non-forking is one of the most important notions in modern model theory capturing the idea of a generic extension of a type (which is a far-reaching generalization of the concept of a generic point of a variety).

To a countable first-order theory we associate its *non-forking spectrum*—a function of two cardinals κ and λ giving the supremum of the possible number of types over a model of size λ that do not fork over a submodel of size κ. This is a natural generalization of the stability function of a theory.

We make progress towards classifying the non-forking spectra. On the one hand, we show that the possible values a non-forking spectrum may take are quite limited. On the other hand, we develop a general technique for constructing theories with a prescribed non-forking spectrum, thus giving a number of examples. In particular, we answer negatively a question of Adler whether NIP is equivalent to bounded non-forking.

In addition, we answer a question of Keisler regarding the number of cuts a linear order may have. Namely, we show that it is possible that $\operatorname{ded}\kappa < (\operatorname{ded}\kappa)^{\omega}$.

Keywords. Forking, dividing, NIP, NTP2, circularization, Dedekind cuts, cardinal arithmetic

1. Introduction

The notion of a non-forking extension of a type (see Definition 2.3) was introduced by Shelah for the purposes of his classification program to capture the idea of a "generic" extension of a type to a larger set of parameters which essentially does not add new constraints to the set of its solutions. In the context of stable theories non-forking gives rise to an independence relation enjoying a lot of natural properties (which in the special case of vector spaces amounts to linear independence and in the case of algebraically closed fields to algebraic independence) and is used extensively in the analysis of models.

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In a subsequent work of Shelah [She80] and Kim and Pillay [Kim98, KP97] the basic properties of forking were generalized to a larger class of simple theories. Recent work of the first and second authors shows that many properties of forking still hold in a larger class of theories without the tree property of the second kind [CK12].

Here we consider the following basic question: how many non-forking extensions can there be? More precisely, given a complete first-order theory T, we associate to it its non-forking spectrum, a function $f_T(\kappa, \lambda)$ from cardinals $\kappa \leq \lambda$ to cardinals defined as

$$f_T(\kappa, \lambda) = \sup\{S^{\mathrm{nf}}(N, M) \mid M \preceq N \models T,\ |M| \leq \kappa,\ |N| \leq \lambda\},$$

where $S^{\mathrm{nf}}(A, B) = \{p \in S_1(A) \mid p \text{ does not fork over } B\}$ (counting 1-types rather than n-types is essential, as the value may depend on the arity, see Section 5.8).

This is a generalization of the classical question "how many types can a theory have over a model?". Recall that the stability function of a theory is defined as $f_T(\kappa) = \sup\{S(M) \mid M \models T,\ |M| = \kappa\}$. It is easy to see that $f_T(\kappa, \kappa) = f_T(\kappa)$. This function has been studied extensively by Keisler [Kei76] and the third author [She71], where the following fundamental result was proved:

Fact 1.1. *For any complete countable first-order theory T, f_T is one of the following: κ, $\kappa + 2^{\aleph_0}$, $\kappa^{\aleph_0}$, $\operatorname{ded}\kappa$, $(\operatorname{ded}\kappa)^{\aleph_0}$, 2^κ.*

Here $\operatorname{ded}\kappa$ is the supremum of the number of cuts that a linear order of size κ may have (see Definition 6.1). While this result is unconditional, in some models of ZFC, some of these functions may coincide. Namely, if GCH holds, $\operatorname{ded}\kappa = (\operatorname{ded}\kappa)^{\aleph_0} = 2^\kappa$. By a result of Mitchell [Mit73], it was known that for any cardinal κ with $\operatorname{cof}\kappa > \aleph_0$, consistently $\operatorname{ded}\kappa < 2^\kappa$. In 1976, Keisler [Kei76, Problem 2] asked whether $\operatorname{ded}\kappa < (\operatorname{ded}\kappa)^{\aleph_0}$ is consistent with ZFC. We give a positive answer in Section 6.

The aim of this paper is to classify the possibilities of $f_T(\kappa, \lambda)$. The philosophy of "dividing lines" of the third author suggests that the possible non-forking spectra are quite far from being arbitrary, and that there should be finitely many possible functions, distinguished by the lack (or presence) of certain combinatorial configurations. We work towards justifying this philosophy and arrive at the following picture.

Main Theorem. *Let T be a countable complete first-order theory. Then for $\lambda \gg \kappa$, $f_T(\kappa, \lambda)$ can be one of the following, in increasing order (meaning that we have an example for each item in the list except for (11), and "???" means that we do not know if there is anything between the previous and the next item, while the lack of "???" means that there is nothing in between):*

(1) κ	(5) ???	(9) $\lambda^{\aleph_0}$	(13) ???
(2) $\kappa + 2^{\aleph_0}$	(6) $(\operatorname{ded}\kappa)^{\aleph_0}$	(10) ???	(14) $(\operatorname{ded}\lambda)^{\aleph_0}$
(3) $\kappa^{\aleph_0}$	(7) 2^{2^κ}	(11) $\lambda^{<\beth_{\aleph_1}(\kappa)}$	(15) ???
(4) $\operatorname{ded}\kappa$	(8) λ	(12) $\operatorname{ded}\lambda$	(16) 2^λ

In particular, note that the existence of an example of $f_T(\kappa, \lambda) = 2^{2^\kappa}$ answers negatively a question of Adler [Adl08, Section 6] whether NIP is equivalent to bounded non-forking.

The restriction $\lambda \gg \kappa$ is in order to make the statement clearer. It can be taken to be $\lambda \geq \beth_{\aleph_1}(\kappa)$. In fact we can say more about smaller λ in some cases. In the class of NTP_2 theories (see Section 4), we have a much nicer picture, meaning that there is a gap between (6) and (16).

In the first part of the paper, we prove that the non-forking spectra cannot take values which are not listed in the Main Theorem. The proofs here combine techniques from generalized stability theory (including results on stable and NIP theories, splitting and tree combinatorics) with a two-cardinal theorem for $L_{\omega_1,\omega}$.

The second part of the paper is devoted to examples.

We introduce a general construction which we call *circularization*. Roughly speaking, the idea is the following: modulo some technical assumptions, we start with an arbitrary theory T_0 in a finite relational language and an (essentially) arbitrary prescribed set F of formulas. We expand T by putting a circular order on the set of solutions of each formula in F, iterate the construction and take the limit. The point is that in the limit all the formulas in F are forced to fork, and we have gained some control on the set of non-forking types. This construction turns out to be quite flexible: by choosing the appropriate initial data, we can find a wide range of examples of non-forking spectra previously unknown.

2. Preliminaries

Our notation is standard: κ, λ, μ are cardinals; $\alpha, \beta, \ldots$ are ordinals; $M, N, \ldots$ are models; $\mathbb{M}$ is always a monster model of the theory in question; $B^{[\kappa]}$ is the set of subsets of B of size $\leq \kappa$; T is a complete countable first-order theory; for a sequence $\bar{a} = \langle a_i \mid i < \alpha \rangle$, $\mathrm{EM}(\bar{a}/A)$ denotes its Ehrenfeucht–Mostowski type over A. Given a formula $\phi(x)$ and a truth value t, $\phi^{\text{if } t}(x)$ denotes $\phi(x)$ if t is true, and $\neg\phi(x)$ if t is false.

2.1. *Basic properties of forking and dividing*

We recall the definition of forking and dividing (see e.g. [CK12, Section 2] for more details).

Definition 2.1 (Dividing). Let A be a set, and a a tuple. We say that the formula $\varphi(x, a)$ *divide*s over A if there is a number $k < \omega$ and tuples $\{a_i \mid i < \omega\}$ such that:

- $\mathrm{tp}(a_i/A) = \mathrm{tp}(a/A)$.
- The set $\{\varphi(x, a_i) \mid i < \omega\}$ is *k-inconsistent* (i.e. every subset of size k is not consistent).

In this case, we say that the formula *k-divides*.

Remark 2.2. From Ramsey and compactness it follows that $\varphi(x, a)$ divides over A if and only if there is an indiscernible sequence $\langle a_i \mid i < \omega \rangle$ over A such that $a_0 = a$ and $\{\varphi(x, a_i) \mid i < \omega\}$ is inconsistent.

Definition 2.3 (Forking). Let A be a set, and a a tuple.

- Say that the formula $\varphi(x, a)$ *forks* over A if there are formulas $\psi_i(x, a_i)$ for $i < n$ such that $\varphi(x, a) \vdash \bigvee_{i<n} \psi_i(x, a_i)$ and $\psi_i(x, a_i)$ divides over A for every $i < n$.

- Say that a type *p forks over A* if there is a finite conjunction of formulas from p which forks over A.

It follows immediately from the definition that if a partial type $p(x)$ does not fork over A then there is a global type $p'(x) \in S(\mathbb{M})$ extending $p(x)$ that does not fork over A.

Lemma 2.4. *Let $(A, \leq)$ be a κ^+-directed order and let $f : A \to \kappa$. Then there is a cofinal subset $A_0 \subseteq A$ such that f is constant on A_0.*

Proof. Assume not; then for every $\alpha < \kappa$ there is some $a_\alpha \in A$ such that $f(a) \neq \alpha$ for any $a \geq a_\alpha$. By κ^+-directedness there is some $a \geq a_\alpha$ for all $\alpha < \kappa$. But then whatever $f(a)$ is, we get a contradiction. □

Lemma 2.5. *Assume that $p(x) \in S(A)$ does not fork over B. Then there is some $B_0 \subseteq B$ such that $|B_0| \leq |A| + |T|$ and $p(x)$ does not fork over B_0.*

Proof. Let $\kappa = |A| + |T|$, and assume the conclusion fails. Then $p(x)$ forks over every $C \subseteq B$ with $|C| \leq \kappa$. That is, for every $C \in B^{[\kappa]}$ there are $p_C \subseteq p$ with $|p_C| < \omega$, $\psi_0^C(x, y_0), \ldots, \psi_{m_C-1}^C(x, y_{m_C}) \in L$ and $k_C < \omega$ such that for some $d_0^C, \ldots, d_{m_C-1}^C$, $p_C(x) \vdash \bigvee_{i<m_C} \psi_i^C(x, d_i^C)$ and each $\psi_i^C(x, d_i^C)$ is k_C-dividing over C. As $B^{[\kappa]}$ is κ^+-directed under inclusion and $|p(x)| \leq \kappa$, it follows by Lemma 2.4 that for some finite $p_0 \subseteq p$, $\{\psi_i \mid i < m\}$ and k, this holds for every $C \in B^{[\kappa]}$. But then by compactness $p_0(x)$ forks over B—a contradiction. □

2.2. *The non-forking spectra*

Definition 2.6. For a countable first-order T and infinite cardinals $\kappa \leq \lambda$, let

$$f_T(\kappa, \lambda) = \sup\{S^{\text{nf}}(N, M) \mid M \preceq N \models T,\ |M| \leq \kappa,\ |N| \leq \lambda\},$$

where $S^{\text{nf}}(A, B) = \{p \in S_1(A) \mid p \text{ does not fork over } B\}$. We call this function the *non-forking spectrum* of T.

For $n > 1$, we define $f_T^n(\kappa, \lambda)$ and S_n^{nf} similarly with 1-types replaced with n-types.

Note 2.7. All the proofs in Section 3 remain valid for f_T replaced by f_T^n.

Remark 2.8. A special case $f_T(\kappa, \kappa)$ is the well-known stability function $f_T(\kappa)$ because $S^{\text{nf}}(N, N) = S(N)$ (as no type over a model M forks over M).

Some easy observations:

Lemma 2.9. *For all $\kappa \leq \lambda$,*

(1) $f_T(\kappa) \leq f_T(\kappa, \lambda)$.
(2) $\kappa \leq f_T(\kappa, \lambda) \leq 2^\lambda$.
(3) *If $f_T(\kappa, \lambda) \geq \mu$ and $\kappa \leq \kappa'$ then $f_T(\kappa', \lambda) \geq \mu$.*
(4) $f_T^n(\kappa, \lambda) \leq f_T^{n+1}(\kappa, \lambda)$.

For set-theoretic preliminaries, see Section 6.

3. Gaps

In the following series of subsections, we exclude all the possibilities for f_T which are not in our list (except when "???" is indicated).

3.1. On (1)–(4)

Definition 3.1. Recall that a theory T is called *stable* if $f_T(\kappa) \leq \kappa^{\aleph_0}$ for all κ (see [She90, Theorem II.2.13] for equivalent definitions).

Remark 3.2. If T is stable then every type over a model M has a unique non-forking extension to any model containing M, so $f_T(\kappa) = f_T(\kappa, \lambda)$ for all $\lambda \geq \kappa \geq \aleph_0$.

If T is unstable, then $f_T(\kappa) \geq \operatorname{ded}\kappa$ for all κ (see [She90, Theorem II.2.49]), so $f_T(\kappa, \lambda) \geq \operatorname{ded}\kappa$ for all $\lambda \geq \kappa$.

Proposition 3.3. (1) *If $f_T(\kappa, \lambda) > \kappa$ for some $\lambda \geq \kappa$ then $f_T(\kappa, \lambda) \geq \kappa + 2^{\aleph_0}$ for all $\lambda \geq \kappa$.*
(2) *If $f_T(\kappa, \lambda) > \kappa + 2^{\aleph_0}$ for some $\lambda \geq \kappa$ then $f_T(\kappa, \lambda) \geq \kappa^{\aleph_0}$ for all $\lambda \geq \kappa$.*
(3) *If $f_T(\kappa, \lambda) > \kappa^{\aleph_0}$ for some $\lambda \geq \kappa$ then $f_T(\kappa, \lambda) \geq \operatorname{ded}\kappa$ for all $\lambda \geq \kappa$.*

Proof. (3): Suppose $f_T(\kappa, \lambda) > \kappa^{\aleph_0}$ for some $\lambda \geq \kappa$. Then T is unstable, so by Remark 3.2, $f_T(\kappa, \lambda) \geq \operatorname{ded}\kappa$ for all $\lambda \geq \kappa$.

(1): Suppose $f_T(\kappa, \lambda) > \kappa$ for some $\lambda \geq \kappa$. Without loss of generality T is stable. So $f_T(\kappa) = f_T(\kappa, \lambda) > \kappa$. By Fact 1.1, $f_T(\kappa) \geq \kappa + 2^{\aleph_0}$ for all κ, and we are done.

(2): Similar to (1). □

3.2. The gap between (6) and (7)

Definition 3.4. A formula $\varphi(x, y)$ has the *independence property* (IP) if there are sets $\{a_i \mid i < \omega\}$ and $\{b_s \mid s \subseteq \omega\}$ in $\mathbb{M}$ such that $\varphi(a_i, b_s)$ holds if and only if $i \in s$ for all $i < \omega$ and $s \subseteq \omega$.

A theory T is *NIP* (*dependent*) if no formula $\varphi(x, y)$ has IP.

See [Adl08] for more about NIP.

Fact 3.5. *If T is NIP and $M \models T$ then $|S(M)| \leq (\operatorname{ded}|M|)^{\aleph_0}$ [She71] and if $M \prec N$ and $p \in S(M)$ then p has at most $(\operatorname{ded}|M|)^{\aleph_0}$ non-forking extensions (e.g. follows from the proof of [Adl08, Theorem 42], noticing that $|S_\omega(M)| \leq (\operatorname{ded}|M|)^{\aleph_0}$). Consequently, $\left|S^{\mathrm{nf}}(N, M)\right| \leq (\operatorname{ded}|M|)^{\aleph_0}$.*

This is a generalization of a result due to Poizat [Poi81].

Proposition 3.6. *Assume that $f_T(\kappa, \lambda) > (\operatorname{ded}\kappa)^{\aleph_0}$ for some $\lambda \geq \kappa$. Then $f_T(\kappa, \lambda) \geq 2^{\min\{\lambda, 2^\kappa\}}$ for all $\lambda \geq \kappa$.*

Proof. By Fact 3.5, some formula $\varphi(x, y)$ in T has IP.

Recall that a set $S \subseteq \mathcal{P}(\kappa)$ is called *independent* if every finite intersection of elements of S or their complements is non-empty. By a theorem of Hausdorff there is such a family of size 2^κ. Fix some κ and $\mu \leq 2^\kappa$, and let S be a family of independent subsets of κ such that $|S| = \mu$.

Let $A = \{a_i \mid i < \kappa\}$ be such that $b_s \models \{\varphi(x, a_i)^{\text{if } i \in s} \mid i < \kappa\}$ for every $s \subseteq \kappa$. Let M be a model of size κ containing A, and N a model of size μ containing $M \cup \{b_s \mid s \in S\}$. Now for every $D \subseteq S$, there is an ultrafilter on κ containing D, and let $p_D \in S(N)$ be

$$\{\psi(x, c) \mid c \in N,\ \psi \in L,\ \{a \in M \mid \psi(a, c)\} \in D\},$$

so it is finitely satisfiable in A. Notice that if $D_1 \neq D_2$ then $p_{D_1} \neq p_{D_2}$, as $\varphi(x, b_s) \in p_{D_1} \wedge \neg\varphi(x, b_s) \in p_{D_2}$ for any $s \in D_1 \setminus D_2$. Thus $S^{\mathrm{nf}}(N, M) \geq 2^\mu$.

If $\lambda \leq 2^\kappa$, then let $\mu = \lambda$, and we have $f_T(\lambda, \kappa) \geq 2^\lambda$.

If $\lambda > 2^\kappa$, then let $\mu = 2^\kappa$, so $f_T(\kappa, \lambda) \geq 2^{2^\kappa}$, and we are done. □

Note that in the Main Theorem we have assumed that $\lambda \geq 2^{2^\kappa}$, so in this case we have $f_T(\kappa, \lambda) \geq 2^{2^\kappa}$.

3.3. *The gap between (7) and (8)*

We recall the basic properties of splitting.

Definition 3.7. Suppose $A \subseteq B$ are sets. A type $p(x) \in S(B)$ *splits* over A if there is some formula $\varphi(x, y)$ and $b, c \in B$ such that $\mathrm{tp}(b/A) = \mathrm{tp}(c/A)$ and $\varphi(x, b) \wedge \neg\varphi(x, c) \in p$.

Fact 3.8 (see e.g. [Adl08, Sections 5, 6]). *Let $M \prec N$ be models.*

(1) *The number of types in $S(N)$ that do not split over M is bounded by $2^{2^{|M|}}$.*
(2) *If N is $|M|^+$-saturated and $p \in S(N)$ splits over M, then there is an indiscernible sequence $\langle a_i \mid i < \omega\rangle$ in N over M such that $\varphi(x, a_0) \wedge \neg\varphi(x, a_1) \in p$ for some φ.*
(3) *If T is NIP, and $p \in S^{\mathrm{nf}}(N, M)$, then p does not split over M.*

Definition 3.9. A *non-forking pattern* of depth θ over A consists of an array $\{\bar{a}_\alpha \mid \alpha < \theta\}$ where $\bar{a}_\alpha = \langle a_{\alpha,i} \mid i < \omega\rangle$ and formulas $\{\varphi_\alpha(x, y) \mid \alpha < \theta\}$ such that

- $\bar{a}_{\alpha_0}$ is indiscernible over $\{\bar{a}_\alpha \mid \alpha < \alpha_0\} \cup A$.
- $\{\varphi_\alpha(x, a_{\alpha,0}) \wedge \neg\varphi_\alpha(x, a_{\alpha,1}) \mid \alpha < \theta\}$ does not fork over A.

Definition 3.10. A *pair non-forking pattern* of depth θ over a set A is defined similarly, but here we only demand that $\bar{a}_{\alpha_0}$ is indiscernible over $\{a_{\alpha,0}, a_{\alpha,1} \mid \alpha < \alpha_0\} \cup A$.

Lemma 3.11. *If there is a pair non-forking pattern of depth θ over A, then there is a non-forking pattern of depth θ over A.*

Proof. Suppose $\{\bar{a}_\alpha \mid \alpha < \theta\}$ is a pair non-forking pattern of depth θ. It is enough to find an array $\{\bar{b}_\alpha \mid \alpha < \theta\}$ as in the first point of Definition 3.9 such that $b_{\alpha,0}b_{\alpha,1} = a_{\alpha,0}a_{\alpha,1}$. By compactness we may assume that θ is finite. The proof is by induction on θ. For

$\theta = 0, 1$ there is nothing to do. Suppose $\theta = n + 1$. By induction, we may assume that the first n sequences satisfy the first point. By Ramsey and compactness (see e.g. [TZ12, Lemma 5.1.3]), there is an indiscernible sequence $\bar{b}'_n$ which is indiscernible over $A \cup \{\bar{a}_\alpha \mid \alpha < n\}$ and such that the type of any finite subtuple in $\bar{b}'_n$ is the same as a subtuple of the same length in $\bar{a}_n$ over $A \cup \{a_{\alpha,0}, a_{\alpha,1} \mid \alpha < n\}$. So there is an automorphism taking $\bar{b}'_n$ to $\bar{a}_n$ which fixes $A \cup \{a_{\alpha,0}, a_{\alpha,1} \mid \alpha < n\}$. Now let $\bar{b}_\alpha$ for $\alpha < n$ be the image of this automorphism, and $\bar{b}_n = \bar{a}_n$. □

Definition 3.12. For an infinite cardinal κ, let $g_T(\kappa)$ be the smallest cardinal θ such that there is no (pair) non-forking pattern of depth θ over some model of size κ.

Remark 3.13. It is clear that $g_T(\kappa') \geq g_T(\kappa)$ whenever $\kappa' \geq \kappa$. In addition, from Lemma 2.5 it follows that if $g_T(\kappa) > \theta$ then $g_T(\theta + \aleph_0) > \theta$.

Lemma 3.14. *If $g_T(\kappa) > \theta$ then there is M of size κ such that for any λ we can find a non-forking pattern $\{\bar{a}_\alpha, \varphi_\alpha \mid \alpha < \theta\}$ such that in addition:*

- $\bar{a}_\alpha = \langle a_{\alpha,i} \mid i < \lambda \rangle$.
- $\{\varphi_\alpha(x, a_{\alpha,0}) \mid \alpha < \theta\} \cup \{\neg\varphi_\alpha(x, a_{\alpha,i}) \mid \alpha < \theta,\ 0 < i < \lambda\}$ *does not fork over M.*

Proof. By assumption we have some non-forking pattern $\{\bar{a}_\alpha, \varphi_\alpha \mid \alpha < \theta\}$ over some M of size κ. By compactness, we may assume that $\bar{a}_\alpha$ is of length λ for all $\alpha < \theta$. Let $p(x) \in S(\mathbb{M})$ be a non-forking extension of $\{\varphi_\alpha(x, a_{\alpha,0}) \wedge \neg\varphi_\alpha(x, a_{\alpha,1}) \mid \alpha < \theta\}$. By omitting some elements from each sequence $\bar{a}_\alpha$ and maybe changing φ_α to $\neg\varphi_\alpha$ we may assume

$$\{\varphi_\alpha(x, a_{\alpha,0}) \mid \alpha < \theta\} \cup \{\neg\varphi_\alpha(x, a_{\alpha,i}) \mid \alpha < \theta,\ 0 < i < \lambda\} \subseteq p.$$ □

Proposition 3.15. *The following are equivalent:*

(1) *For some κ, $g_T(\kappa) > 1$.*
(2) *For every $\lambda \geq \kappa \geq \aleph_0$, $f_T(\kappa, \lambda) = 2^\lambda$ if $\lambda \leq 2^\kappa$ and $f_T(\kappa, \lambda) \geq \lambda$ otherwise.*
(3) *For some $\lambda \geq \kappa$, $f_T(\kappa, \lambda) > 2^{2^\kappa}$.*

Proof. (1) implies (2): By Remark 3.13, we may assume that $\kappa = \aleph_0$. By Lemma 3.14 there is some countable M such that for any λ there is some $\bar{b} = \langle b_i \mid i < \lambda \rangle$ such that $\{\varphi(x, b_0)\} \cup \{\neg\varphi(x, b_i) \mid i < \lambda\}$ does not fork over M. So, for every $i < \lambda$, $p_i(x) = \{\varphi(x, b_j)^{\text{if } j=i} \mid i \leq j < \lambda\}$ does not fork over M.

Taking some model $N \supseteq \bar{b}$ of size λ we can expand each p_i to some $q_i \in S^{\text{nf}}(N, M)$. Notice that for any $i < j < \lambda$, $q_i \neq q_j$ as $\neg\varphi(x, a_j) \in p_i$, but $\varphi(x, a_j) \in p_j$. So we conclude that $S^{\text{nf}}(N, M) \geq \lambda$. By Lemma 2.9, we see that $f_T(\kappa, \lambda) \geq \lambda$ for every $\lambda \geq \kappa$.

Note that by Fact 3.5, we know that T is not NIP, so if $\lambda \leq 2^\kappa$, then by Proposition 3.6, $f_T(\kappa, \lambda) = 2^\lambda$.

(2) implies (3) is clear.

(3) implies (1): Let $M \prec N$ witness that $f_T(\kappa, \lambda) > 2^{2^\kappa}$. By Fact 3.8(1), there is some $p \in S^{\text{nf}}(N, M)$ that splits over M.

Let $N' \succ N$ be $|M|^+$-saturated and $p' \in S^{\text{nf}}(N', M)$ a non-forking extension of p. By Fact 3.8(2) we find an indiscernible sequence $\bar{a} = \langle a_i \mid i < \omega \rangle$ in N' and a formula $\varphi(x, a_0) \wedge \neg\varphi(x, a_1) \in p$—and we get (1). □

3.4. The gap between (8) and (9)

Lemma 3.16. *For any cardinals λ and θ, if θ is regular or $\lambda \geq 2^{<\theta}$ then $(\lambda^{<\theta})^{<\theta} = \lambda^{<\theta}$.*

Proof. By [She86, Observation 2.11(4)], if $\lambda \geq 2^{<\theta}$, then $\lambda^{<\theta} = \lambda^{\nu}$ for some $\nu < \theta$. So $(\lambda^{<\theta})^{<\theta} = (\lambda^{\nu})^{<\theta} = \lambda^{<\theta}$. If θ is regular, then let $\lambda' = \lambda^{<\theta}$; since $\lambda' \geq 2^{<\theta}$, $(\lambda')^{<\theta} = (\lambda')^{\nu}$ for some $\nu < \theta$, so we have

$$(\lambda')^{<\theta} = (\lambda')^{\nu} = (\lambda^{<\theta})^{\nu} = \Big(\sum_{\mu<\theta} \lambda^{\mu}\Big)^{\nu} = \sum_{\mu<\theta} (\lambda^{\mu\cdot\nu}) = \lambda^{<\theta} = \lambda'. \qquad \square$$

Lemma 3.17. *Suppose $f_T(\kappa, \lambda) > \lambda^{<\theta}$ and $\lambda \geq \sum_{\mu<\theta} 2^{2^{\kappa+\mu}}$. Then $g_T(\kappa) > \theta$.*

Proof. Let $\lambda' = \lambda^{<\theta}$. By Lemma 3.16, $(\lambda')^{<\theta} = \lambda'$. Hence $f_T(\kappa, \lambda') \geq f_T(\kappa, \lambda) > \lambda^{<\theta} = (\lambda')^{<\theta}$, so we may replace λ with λ' and assume $\lambda^{<\theta} = \lambda$.

Let (N, M) be a witness to $f_T(\kappa, \lambda) > \lambda$. For every $A \subseteq N$ of size $< \theta$, let $M_A \subseteq \mathbb{M}$ be a $(\kappa + |A|)^+$-saturated model of size $\leq 2^{|A|+\kappa}$ containing $M \cup A$. Let $N_0 = \bigcup_{A \in N^{[<\theta]}} M_A$. So $N_0 \supseteq N$ and $|N_0| \leq \lambda \cdot 2^{<\theta+\kappa} = \lambda$. Repeating the construction with respect to (N_0, M), construct N_1, and more generally N_i for $i \leq \theta$, taking union at limit steps. So $|N_\theta| \leq \lambda \cdot \theta = \lambda$.

Fix $p(x) \in S^{\mathrm{nf}}(N_\theta, M)$.

We try to choose by induction on $\alpha < \theta$ formulas $\varphi^p_\alpha(x, y)$ and sequences $\bar{a}^p_\alpha = \langle a^p_{\alpha,i} \mid i < \omega\rangle$ in $N_{\alpha+1}$ such that $\bar{a}^p_\alpha$ is indiscernible over $\{a^p_{\beta,0}, a^p_{\beta,1} \mid \beta < \alpha\} \cup M$ and $\varphi^p_\alpha(x, a^p_{\alpha,0}) \wedge \neg\varphi^p_\alpha(x, a^p_{\alpha,1}) \in p$. If we succeed, then we get a pair non-forking pattern of depth θ over M as desired (by Lemma 3.11). Otherwise, we are stuck at some $\alpha_p < \theta$. Let $A_p = \bigcup\{a^p_{\beta,0}, a^p_{\beta,1} \mid \beta < \alpha_p\}$.

Let $F \subseteq S^{\mathrm{nf}}(N_\theta, M)$ be a set of size $> \lambda$ such that for $p \neq q \in F$, $p|_N \neq q|_N$. As the size of the set $\{A_p \mid p \in F\}$ is bounded by $\lambda^{<\theta} = \lambda$, there is some A of size $< \theta$ and α such that the set $S = \{p \in F \mid A_p = A \wedge \alpha_p = \alpha\}$ has $|S| > \lambda$. Let $M_0 \subseteq N_\alpha$ be some model containing $A \cup M$ of size $\kappa + |A|$. Suppose $p \in S$ and $p|_{N_\alpha}$ splits over M_0, so already $p|_{M_0 B}$ splits over M_0 for some finite B. Then there is some $(\kappa + |A|)^+$-saturated model $N' \subseteq N_{\alpha+1}$ containing $M \cup A \cup B$ and some $M'_0 \subseteq N'$ such that $M'_0 \equiv_{MAB} M_0$, so $p|_{N'}$ splits over M'_0. By Fact 3.8(2), we can find an M'_0-indiscernible sequence $\langle a^p_{\alpha,i} \mid i < \omega\rangle$ in $N' \subseteq N_{\alpha+1}$ such that $\varphi(x, a^p_{\alpha,0}) \wedge \neg\varphi(x, a^p_{\alpha,1}) \in p$—contradicting the choice of α. So, for every $p \in S$, $p|_{N_\alpha}$ does not split over M_0. But then by the choice of F and Fact 3.8(1), $|S| \leq 2^{2^{\kappa+|A|}}$—a contradiction. $\square$

Lemma 3.18. *If $g_T(\kappa) > \theta$ then $f_T(\kappa, \lambda) \geq \lambda^{\langle\theta\rangle_{\mathrm{tr}}}$ for all $\lambda \geq \kappa$* (see Definition 6.3).

Proof. Fix $\lambda \geq \kappa + \theta$ (if $\lambda < \theta$ then $\lambda^{\langle\theta\rangle_{\mathrm{tr}}}$ is 0). By Lemma 3.14, there is some non-forking pattern $\{\bar{a}_\alpha, \varphi_\alpha \mid \alpha < \theta\}$ over a model M of size κ such that $\bar{a}_\alpha = \langle a_{\alpha,i} \mid i < \lambda\rangle$ and $p(x) = \{\varphi_\alpha(x, a_{\alpha,0}) \mid \alpha < \theta\} \cup \{\neg\varphi_\alpha(x, a_{\alpha,i}) \mid \alpha < \theta, 0 < i < \lambda\}$ does not fork over M. By induction on $\beta \leq \theta$ we define elementary mappings F_η, $\eta \in \lambda^\beta$, with $\mathrm{dom}(F_\eta) = A_\beta = M \cup \{\bar{a}_\alpha \mid \alpha < \beta\}$:

- $F_\emptyset$ is the identity on M.
- If β is a limit ordinal, then let $F_\eta = \bigcup_{\alpha<\beta} F_{\eta\restriction\alpha}$.

- If $\beta = \alpha + 1$, let $F_{\eta 0}$ be an arbitrary extension of F_η to $A_{\alpha+1}$. For $i < \lambda$, let $F_{\eta i}$ be an arbitrary elementary mapping extending F_η such that $F_{\eta i}(a_{\alpha,j}) = F_{\eta 0}(a_{\alpha,i+j})$. This could be done by indiscernibility.

Let $p_\eta = F_\eta(p)$. Then:

- $p_\eta(x)$ does not fork over M—as F_η is an elementary map fixing M.
- If $\eta \neq \nu \in \lambda^\theta$, then $p_\eta \neq p_\nu$. To see this, let $\alpha = \min\{\beta < \theta \mid \eta \restriction \beta \neq \nu \restriction \beta\}$ and suppose $\alpha = \beta + 1$, $\rho = \eta \restriction \beta = \nu \restriction \beta$. Assume $\eta(\beta) = i < j = \nu(\beta)$ and $0 < k < \lambda$ is such that $i + k = j$. Then $\varphi(x, a_{\alpha,0}) \in p \Rightarrow \varphi(x, F_\nu(a_{\alpha,0})) \in p_\nu$. Similarly, $\neg\varphi(x, a_{\alpha,k}) \in p \Rightarrow \neg\varphi(x, F_\eta(a_{\alpha,k})) \in p_\eta$. But
$$F_\nu(a_{\alpha,0}) = F_{\rho j}(a_{\alpha,0}) = F_{\rho 0}(a_{\alpha,j}) = F_{\rho 0}(a_{i+k}) = F_{\rho i}(a_{\alpha,k}) = F_\eta(a_{\alpha,k}),$$
so $p_\eta \neq p_\nu$.

Let $T \subseteq \lambda^{<\theta}$ be a tree of size $\leq \lambda$ such that if $x \in T$ and $y < x$ then $y \in T$. Let $B = \bigcup\{F_\eta(\bar{a}_\alpha) \mid \alpha < \lg(\eta) \wedge \eta \in T\} \cup M$, so $|B| \leq \lambda + \kappa = \lambda$. Let N be some model containing B of size λ. Thus, $|S^{\mathrm{nf}}(N, M)|$ is at least the number of branches in T of length θ. By the definition of $\lambda^{\langle\theta\rangle_{\mathrm{tr}}}$ we are done. □

Proposition 3.19. *If $f_T(\kappa, \lambda) > \lambda$ for some $\lambda \geq 2^{2^\kappa}$, then $f_T(\kappa, \lambda) \geq \lambda^{\aleph_0}$ for all $\lambda \geq \kappa$.*

Proof. By Lemma 3.17 with $\theta = \aleph_0$, we have $g_T(\kappa) > \aleph_0$, and then by Remark 3.13, $g_T(\aleph_0) > \aleph_0$. By Lemma 3.18, $f_T(\aleph_0, \lambda) \geq \lambda^{\langle\aleph_0\rangle_{\mathrm{tr}}}$ for all λ, and $\lambda^{\langle\aleph_0\rangle_{\mathrm{tr}}} = \lambda^{\aleph_0}$ (see Remark 6.4). By Remark 2.9, $f_T(\kappa, \lambda) \geq f_T(\aleph_0, \lambda) \geq \lambda^{\aleph_0}$, so we are done. □

3.5. *On (10)*

Proposition 3.20. *If $f_T(\kappa, \lambda) > \lambda^\mu$ for some $\lambda \geq 2^{2^{\kappa+\mu}}$, then $f_T(\kappa, \lambda) \geq \lambda^{\langle\mu^+\rangle_{\mathrm{tr}}}$ for all $\lambda \geq \kappa \geq \mu^+$.*

Proof. By Lemma 3.17, $g_T(\kappa) > \mu^+$. By Lemma 2.5, $g_T(\mu^+) > \mu^+$. By Lemma 3.18, $f_T(\mu^+, \lambda) \geq \lambda^{\langle\mu^+\rangle_{\mathrm{tr}}}$ for all $\lambda \geq \mu^+$, and so by Lemma 2.9 , $f_T(\kappa, \lambda) \geq \lambda^{\langle\mu^+\rangle_{\mathrm{tr}}}$ for any $\lambda \geq \kappa \geq \mu^+$. □

Corollary 3.21. *If $f_T(\kappa, \lambda) > \lambda^{\aleph_n}$ for some $\lambda \geq 2^{2^{\kappa+\aleph_n}}$, then $f_T(\kappa, \lambda) \geq \lambda^{\langle\aleph_{n+1}\rangle_{\mathrm{tr}}}$ for all $\lambda \geq \kappa \geq \aleph_{n+1}$.*

This corollary says that morally there are gaps between λ and $\lambda^{\aleph_0}$, between $\lambda^{\aleph_0}$ and $\lambda^{\aleph_1}$ etc.

3.6. *On the gap between (11) and (12)*

The following fact follows from the proof of Morley's two-cardinal theorem. For details, see [Kei71, Theorem 23].

Fact 3.22. *Suppose $\psi \in L_{\omega_1,\omega}$, $<$ is a binary relation, P and Q are predicates in L and ψ implies that "$<$ is a linear order on Q". Suppose that for every countable ordinal ε there is a structure B such that:*

- *$B \models \psi$.*
- *There is an embedding of the order $\beth_\varepsilon(|P^B|)$ into $(Q^B, <^B)$.*

Then for every cardinal λ *there is some structure* B *such that:*

- $B \models \psi$.
- $|P^B| = \aleph_0$.
- *There is an embedding of* $(\lambda, <)$ *into* $(Q^B, <^B)$.

Lemma 3.23. *Let* $M \prec N$ *and* $a \in N$. *Then the following are equivalent:*

(1) $\varphi(x, a)$ *forks over* M.
(2) *The following holds in* N*:*

$$\bigvee_{\{\psi_0,\dots,\psi_{m-1}\}\subseteq L} \bigvee_{k_i<\omega, i<m} \bigwedge_{\Delta\subseteq L \text{ finite}} \bigwedge_{n<\omega} \forall c_0,\dots,c_{n-1} \in M\, \exists \bar{y}_0,\dots,\exists \bar{y}_{m-1}$$
$$\Big[\varphi(x,a) \vdash \bigvee_{i<n} \psi(x, y_{i,0}) \wedge \bigwedge_{i<m, j<n} (y_{i,j} \equiv^{\Delta}_{\bar{c}} y_{i,0}) \wedge \bigwedge_{i<m,\, s\in n^{[k_i]}} \forall x \left(\neg \bigwedge_{j\in s} \varphi(x, y_{i,j})\right)\Big]$$

where $\bar{y}_i = \langle y_{i,j} \mid j < n\rangle$ *for* $i < m$ *and* $\bar{c} = \langle c_i \mid i < n\rangle$.

Proof. By compactness. □

Lemma 3.24. *If* $g_T(\kappa) > \mu > \aleph_0$, *then there is a non-forking pattern* $\{\varphi_\alpha, \bar{a}_\alpha \mid \alpha < \mu\}$ *such that* $\varphi_\alpha = \varphi$ *for some formula* φ.

Proof. By the pigeon-hole principle. □

Proposition 3.25. *If for all* $\varepsilon < \aleph_1$, *there is some* κ *such that* $g_T(\kappa) > \beth_\varepsilon(\kappa)$, *then* $g_T(\aleph_0) = \infty$.

Proof. By Lemma 3.24, for every $\varepsilon < \aleph_1$ there is some formula φ_ε and a non-forking pattern $\{\varphi_\varepsilon, \bar{a}^\varepsilon_\alpha \mid \alpha < \beth_\varepsilon(\kappa)\}$ over a model M_ε of size κ. We may assume that $\varphi_\varepsilon = \varphi$ for all $\varepsilon < \aleph_1$.

Let ψ be the $L_{\omega_1,\omega}$ sentence in the language

$$\{P(x), S(x), Q(\alpha), <(\alpha,\beta), R(x,\alpha), <_R(x,y,\alpha)\} \cup L(T)$$

saying:

(1) $S \models T$.
(2) P is an L-elementary substructure of S.
(3) $S \cap Q = \emptyset$.
(4) The universe is $S \cup Q$.
(5) Q is infinite and $<$ is a linear order on Q.
(6) For each $\alpha \in Q$, $R(-,\alpha)$ is infinite and contained in S, and $<_R(-,-,\alpha)$ is a discrete linear order on $R(-,\alpha)$ with a first element.
(7) For each $\alpha \in Q$, $R(-,\alpha)$ is an L-indiscernible sequence over $P \cup \bigcup_{\beta<\alpha} R(-,\beta)$ ordered by $<_R(-,-,\alpha)$.
(8) The set $\{\varphi(x, y_{\alpha,0}) \wedge \neg\varphi(x, y_{\alpha,1}) \mid \alpha \in Q\}$ does not fork over P (in the sense of L), where $y_{\alpha,0}$ and $y_{\alpha,1}$ are the first elements in the sequence $R(-,\alpha)$.

Note that (6) can be expressed in $L_{\omega_1,\omega}$ by Lemma 3.23.

As the assumptions of Fact 3.22 are satisfied, for each λ we find a model B of ψ such that:

- $|P^B| = \aleph_0$.
- There is an embedding h of $(\lambda, <)$ into $(Q^B, <^B)$.

For all $\alpha < \lambda$ let $\bar{a}_\alpha$ be an infinite subsequence of $R(B, h(\alpha))$ and let $M = P(B)$. By (1)–(8), $\{\varphi, \bar{a}_\alpha \mid \alpha < \lambda\}$ is a non-forking pattern of depth λ over M—as desired. □

Corollary 3.26. (1) *If for all $\varepsilon < \aleph_1$ there is some κ such that $g_T(\kappa) > \beth_\varepsilon(\kappa)$, then $f_T(\lambda, \kappa) \geq \operatorname{ded}\lambda$ for all $\lambda \geq \kappa$.*

(2) *If for every $\varepsilon < \aleph_1$ there is some $\lambda \geq \beth_\varepsilon(\kappa)$ such that $f_T(\lambda, \kappa) > \lambda^{<\beth_\varepsilon(\kappa)}$, then $f_T(\lambda, \kappa) \geq \operatorname{ded}\lambda$ for all $\lambda \geq \kappa$.*

(3) *If $f_T(\lambda, \kappa) > \lambda^{<\beth_{\aleph_1}(\kappa)}$ for some $\lambda \geq \beth_{\aleph_1}(\kappa)$, then $f_T(\lambda, \kappa) \geq \operatorname{ded}\lambda$ for all $\lambda \geq \kappa$.*

Proof. (1) By Lemma 3.25, we know that $g_T(\aleph_0) = \infty$. For any $\lambda \geq \kappa$, by Lemma 3.18 we have $f_T(\kappa, \lambda) \geq \lambda^{\langle\theta\rangle_{\mathrm{tr}}}$ for all $\theta \leq \lambda$. As $\operatorname{ded}\lambda = \sup\{\lambda^{\langle\theta\rangle_{\mathrm{tr}}} \mid \theta \leq \lambda$, is regular$\}$ by Proposition 6.5(6) we get $f_T(\kappa, \lambda) \geq \operatorname{ded}\lambda$.

(2) Let $\varepsilon < \aleph_1$ be a limit ordinal and $\theta = \beth_\varepsilon(\kappa)$. Then

$$\sum_{\mu<\theta} 2^{2^{\kappa+\mu}} = \sum_{\alpha<\varepsilon} 2^{2^{\beth_\alpha(\kappa)}} = \sum_{\alpha<\varepsilon} \beth_{\alpha+2}(\kappa) = \beth_\varepsilon(\kappa).$$

By Lemma 3.17, $g_T(\kappa) > \beth_\varepsilon(\kappa)$. So we can apply (1) to conclude the proof.

(3) follows from (2). □

4. Inside NTP$_2$

NTP$_2$ is a large class of first-order theories containing both NIP and simple theories introduced by Shelah. For a general treatment, see [Che14]. In this section we show that for theories in this class, the non-forking spectrum is well behaved, i.e. it cannot take values between (6) and (16).

Fact 4.1 (see e.g. [HP11]). *Let $p(x)$ be a global type non-splitting over a set A. For any set $B \supseteq A$ and an ordinal α, let the sequence $\bar{c} = \langle c_i \mid i < \alpha\rangle$ be such that $c_i \models p|_{Bc_{<i}}$. Then $\bar{c}$ is indiscernible over B and its type over B does not depend on the choice of $\bar{c}$. Call this type $p^{(\alpha)}|_B$, and let $p^{(\alpha)} = \bigcup_{B\supseteq A} p^{(\alpha)}|_B$. Then $p^{(\alpha)}$ also does not split over A.*

Definition 4.2 (strict invariance). Let $p(x)$ be a global type. We say that p is *strictly invariant* over a set A if p does not split over A, and whenever $B \supseteq A$ and $c \models p|_B$ then $\operatorname{tp}(B/cA)$ does not fork over A.

Lemma 4.3. *Let p be a global type finitely satisfiable in A. Then there is some model $M \supseteq A$ with $|M| \leq |A| + \aleph_0$ such that $p^{(\omega)}$ is strictly invariant over M.*

Proof. Let M_0 be some model containing A of size $|A| + \aleph_0$. Construct by induction an increasing sequence of models M_i for $i < \omega$ such that $|M_i| = |M_0|$ and for every formula $\varphi(x, y)$ over M, if $\varphi(x, c) \in p^{(\omega)}$ for some c, then there is some $c' \in M_{i+1}$ such that $\varphi(x, c') \in p^{(\omega)}$. Let $M = \bigcup_{i<\omega} M_i$. □

In lieu of giving a definition of NTP$_2$, we only state the properties which we will be using.

Fact 4.4 ([CK12]). *Let T be* NTP$_2$ *and $M \models T$. Then:*

(1) *$\varphi(x, c)$ divides over M if and only if $\varphi(x, c)$ forks over M.*
(2) *Let $p(x)$ be a global type strictly invariant over M and $\langle c_i \mid i < \omega \rangle \models p^{(\omega)}|_M$. Then for any formula $\varphi(x, c_0)$ dividing over M, $\{\varphi(x, c_i) \mid i < \omega\}$ is inconsistent.*

Improving on [CK12, Theorem 4.3] we establish the following:

Theorem 4.5. *Let T be* NTP$_2$. *Then the following are equivalent:*

(1) *$f_T(\kappa, \lambda) > (\operatorname{ded} \kappa)^{\aleph_0}$ for some $\lambda \geq \kappa$.*
(2) *T has IP.*
(3) *$f_T(\kappa, \lambda) = 2^\lambda$ for every $\lambda \geq \kappa$.*

Proof. (1) implies (2) follows from Fact 3.5, and (3) implies (1) is clear.

(2) implies (3): Fix $\lambda \geq \kappa$. Let $\varphi(x, y)$ have IP, and $\bar{a} = \langle a_i \mid i < \omega \rangle$ be an indiscernible sequence such that $\forall U \subseteq \omega \, \exists b_U \, \varphi(a_i, b_U) \Leftrightarrow i \in U$. Let $p(x)$ be a global non-algebraic type finitely satisfiable in $\bar{a}$. By Lemma 4.3, there is a model $M \supseteq \bar{a}$ such that $|M| \leq \aleph_0$ and $p^{(\omega)}$ is strictly invariant over M.

Let $\bar{b} = \langle b_i \mid i < \lambda \rangle$ realize $p^{(\lambda)}|_M$. We show that $p_\eta(x) = \{\varphi(x, b_i)^{\text{if } \eta(i)=1} \mid i < \lambda\}$ does not divide over M for any $\eta \in 2^\lambda$.

First note that $p_\eta(x)$ is consistent for any η, as $\operatorname{tp}(\bar{b}/M)$ is finitely satisfiable in $\bar{a}$. But as for any $k < \omega$, $\langle (b_{k\cdot i}, b_{k\cdot i+1}, \ldots, b_{k\cdot(i+1)-1}) \mid i < \omega \rangle$ realizes $(p^{(k)})^{(\omega)}$, Fact 4.4(2) implies that $p_\eta(x)|_{b_0 \ldots b_{k-1}}$ does not divide over M for any $k < \omega$. Thus by indiscernibility of $\bar{b}$, $p_\eta(x)$ does not divide over M.

Take $N \supseteq \bar{b} \cup M$ of size λ. By Fact 4.4(1) every p_η extends to some $p'_\eta \in S^{\text{nf}}(N, M)$, thus $f_T(\kappa, \lambda) = 2^\lambda$. □

5. Examples

5.1. Examples of (1)–(6)

Proposition 5.1. (1) *If T is the theory of equality, then $f_T(\kappa, \lambda) = \kappa$ for all $\lambda \geq \kappa$.*

(2) *Let T be the model companion of the theory of countably many unary relations. Then $f_T(\kappa, \lambda) = \kappa + 2^{\aleph_0}$ for all $\lambda \geq \kappa$.*
(3) *Let T be the model companion of the theory of countably many equivalence relations. Then $f_T(\kappa, \lambda) = \kappa^{\aleph_0}$ for all $\lambda \geq \kappa$.*
(4) *Let $T = DLO$. Then $f_T(\kappa, \lambda) = \operatorname{ded} \kappa$ for all $\lambda \geq \kappa$.*
(5) *Let T be the model companion of infinitely many linear orders. Then $f_T(\kappa, \lambda) = (\operatorname{ded} \kappa)^{\aleph_0}$.*

Proof. (1)–(3): It is well known that these examples have the corresponding $f_T(\kappa)$'s, and that they are stable. It follows from Remark 3.2 that they have the corresponding $f_T(\kappa, \lambda)$.

(4): It is easy to check that every type has finitely many non-splitting global extensions, but DLO is NIP so by Fact 3.8 every non-forking extension is non-splitting. Since $f_T(\kappa) = \operatorname{ded} \kappa$ for this theory, we are done.

(5): This theory is NIP so $f_T(\kappa, \lambda) \leq (\operatorname{ded} \kappa)^{\aleph_0}$ by Fact 3.5, and clearly $f_T(\kappa) = (\operatorname{ded} \kappa)^{\aleph_0}$. □

5.2. *Circularization*

We shall first describe a general construction for examples of non-forking spectra functions.

For this section, a "formula" means an $\emptyset$-definable formula unless otherwise specified. Most formulas we work with are partitioned formulas, $\varphi(\bar{x}; \bar{y})$, where the variables are broken into two distinct sets. We write φ instead of $\varphi(\bar{x}; \bar{y})$ when the partition is clear from the context. We let $\varphi^1 = \varphi$ and $\varphi^0 = \neg\varphi$. We assume that our languages are relational in this section (so a subset is a substructure).

5.2.1. Circularization: Base step. The dense circular order was used as an example of a theory where forking is not the same as dividing (see e.g. [Kim96, Example 2.11]). The reason is that with circular ordering around, it is hard not to fork.

Definition 5.2. A *circular order* on a finite set is a ternary relation obtained by placing the points on a circle and taking all triples in clockwise order. For an infinite set, a circular order is a ternary relation such that the restriction to any finite set is a circular order. Equivalently, a circular order is a ternary relation C such that for every x, $C(x, -, -)$ is a linear order on $\{y \mid y \neq x\}$ and $C(x, y, z) \to C(y, z, x)$ for all x, y, z. Denote the theory of circular orders by T_C.

The following definitions are well-known.

Definition 5.3. Let K be a class of L-structures (where L is relational). We say that K has the *strong amalgamation property* (*SAP*) if for every $A, B, C \in K$ and embeddings $i_1 : A \to B$ and $i_2 : A \to C$ there exist a structure $D \in K$ and embeddings $j_1 : B \to D$ and $j_2 : C \to D$ such that

- $j_1 \circ i_1 = j_2 \circ i_2$ and
- $j_1(B) \cap j_2(C) = (j_1 \circ i_1)(A) = (j_2 \circ i_2)(A)$.

We say that K has the *disjoint embedding property* (*DEP*) if for any structures $A, B \in K$, there exists a structure $C \in K$ and embeddings $j_1 : B \to C$ and $j_2 : A \to C$ such that $j_1(A) \cap j_2(B) = \emptyset$.

We say that a first-order theory T has these properties if its class of (finite) models has them.

Remark 5.4. T_C is universal and it has DEP and SAP.

Fact 5.5. *Let T be a universal theory with DEP and SAP in a finite relational language L. Then:*

(1) ([Hod93, Theorem 7.4.1]) *T has a model completion T_0 which is ω-categorical and eliminates quantifiers.*
(2) ([Hod93, Theorem 7.1.8]) *If $A \subseteq M \models T_0$ then* $\mathrm{acl}(A) = A$.

Corollary 5.6. *Suppose that $\varphi(\bar{x}; \bar{y})$ is a formula in L, and $\bar{a} \in M \models T_0$. If $M \models \exists \bar{z}\, \varphi(\bar{z}; \bar{a}) \wedge \bar{z} \nsubseteq \bar{a}$ then $\{\bar{t} \in M \mid \varphi(\bar{t}; \bar{a})\}$ is infinite.*

Definition 5.7. For any formula $\varphi(\bar{x}; \bar{y})$ in L where $\bar{x}$ is not empty, let $C[\varphi(\bar{x}; \bar{y})]$ be a new $\lg(\bar{y}) + 3 \cdot \lg(\bar{x})$-place relation symbol. Denote $L[\varphi(\bar{x}; \bar{y})] = L \cup \{C[\varphi(\bar{x}; \bar{y})]\}$.

Definition 5.8. Suppose $\varphi(\bar{x}; \bar{y})$ is a quantifier free formula in L with $\bar{x}$ not empty. Let $T[\varphi(\bar{x}; \bar{y})]$ be the theory in $L[\varphi(\bar{x}; \bar{y})]$ containing T and the following axioms:

- For all $\bar{t}$ in the length of $\bar{y}$, the set

$$S[\varphi(\bar{x}; \bar{y})](\bar{t}) := \{\bar{s} \mid \bar{s} \cap \bar{t} = \emptyset \wedge \lg(\bar{s}) = \lg(\bar{x}) \wedge \varphi(\bar{s}; \bar{t})\}$$

  is circularly ordered by the relation

$$C[\varphi(\bar{x}; \bar{y})](\bar{t}) := \{(\bar{s}_1, \bar{s}_2, \bar{s}_3) \mid C[\varphi(\bar{x}, \bar{y})](\bar{t}, \bar{s}_1, \bar{s}_2, \bar{s}_3)\}$$

  (i.e. $C[\varphi(\bar{x}; \bar{y})]$ with index $\bar{t}$ orders this set in a circular order). Call $\bar{t}$ the *index variables*, and $\bar{s}$ the *main variables*.
- If $C[\varphi(\bar{x}; \bar{y})](\bar{t})(\bar{s}_1, \bar{s}_2, \bar{s}_3)$ then $\bar{s}_1, \bar{s}_2, \bar{s}_3 \in S[\varphi(\bar{x}; \bar{y})](\bar{t})$.

Claim 5.9. *If φ is as in the definition, then*

(1) *$T[\varphi]$ is universal.*
(2) *$T[\varphi]$ has DEP.*
(3) *$T[\varphi]$ has SAP.*

Proof. As T_C is universal, (1) is clear (note that this uses the fact that φ is quantifier free).

(3): Let M'_0, M'_1 and M'_2 be models of $T[\varphi]$ such that $M'_0 = M'_1 \cap M'_2$. Let $M_i = M'_i \restriction L$ for $i < 3$. By assumption, there is a model $M_3 \models T$ such that $M_1 \cup M_2 \subseteq M_3$. We define M'_3 as an expansion of M_3. Let $\bar{t} \in M_3$ be a tuple of length $\lg(\bar{y})$. Split into cases:

Case 1: $\bar{t} \in M'_0$. In this case, $(S^{M'_i}[\varphi](\bar{t}), C^{M'_i}[\varphi](\bar{t}))$ are circular orders for $i < 3$ and $S^{M'_1}[\varphi](\bar{t}) \cap S^{M'_2}[\varphi](\bar{t}) = S^{M'_0}[\varphi](\bar{t})$ so we can amalgamate them as circular orders and extend arbitrarily to $S^{M_3}[\varphi](\bar{t})$, and that will be $C^{M'_3}[\varphi](\bar{t})$.
Note that in the special case where $S^{M_0}[\varphi](\bar{t}) = \emptyset$, there are no restrictions on the place of $S^{M_i}[\varphi](\bar{t})$ for $i < 3$ in this order.
Case 2: $\bar{t} \in M_1 \backslash M_2$. Then $(S^{M'_1}[\varphi](\bar{t}), C^{M'_1}[\varphi](\bar{t}))$ is a circular order. Extend it so that its domain would be $S^{M_3}[\varphi](\bar{t})$ arbitrarily.
Case 3: $\bar{t} \in M_2 \backslash M_1$—the same.
Case 4: $\bar{t} \notin M_1$ and $\bar{t} \notin M_2$. Then $C^{M'_3}[\varphi](\bar{t})$ is any circular order on $S^{M_3}[\varphi](\bar{t})$.

(2): Similar to (3), but easier. □

Remark 5.10. It follows from the proof of amalgamation that if $M \models T$ contains models $M_0 \subseteq M_i \subseteq M$ for $i < n$ such that $M_0 = M_i \cap M_j$ for $i < j < n$, and for each M_i there is an expansion M'_i to a model of $T[\varphi]$ such that $M'_0 \subseteq M'_i$, then there is an expansion M' of M to a model of $T[\varphi]$ such that $M'_i \subseteq M'$.

Claim 5.11. (1) *If $M \models T$, then we can expand it to a model M' of $T[\varphi]$.*
(2) *Moreover, if $B \subseteq M$ and there is already an expansion B' of B to a model of $T[\varphi]$, then we can expand M in such a way that $B' \subseteq M'$.*

(3) *Moreover, suppose that:*

- $A \subseteq M$.
- $\langle \bar{c}_i \mid i < n \rangle$ *is a finite sequence of finite tuples from* M *such that* $\bar{c}_i \cap \bar{c}_j \subseteq A$ *and* $\mathrm{tp}_{\mathrm{qf}}(\bar{c}_i/A) = \mathrm{tp}_{\mathrm{qf}}(\bar{c}_j/A)$ *for all* $i < j < n$.
- M'_0 *is an expansion of* $A\bar{c}_0$ *to a model of* $T[\varphi]$.

Then we can find an expansion M' *such that the quantifier free types are still equal in the sense of* $L[\varphi]$ *and* $M'_0 \subseteq M'$.

Proof. (2): For any $\bar{t}$ in the length of $\bar{y}$, if $\bar{t} \in B$ then we choose a circular order $C^{M'}[\varphi](\bar{t})$ that extends $C^{B'}[\varphi](\bar{t})$ on $S^M[\varphi](\bar{t})$. If not, then define it arbitrarily.

(3): Let $M_i = A\bar{c}_i$. As $\bar{c}_0 \equiv^{\mathrm{qf}}_A \bar{c}_i$ for $i < n$, there are isomorphisms $f_i : M_0 \to M_i$ of L that fix A and take $\bar{c}_0$ to $\bar{c}_i$. So f_i induces expansions M'_i of M_i, isomorphic (via f_i) to M'_0. As the intersection of any two models M_i is exactly A, by Remark 5.10 there is an expansion M' of M to a model of $T[\varphi]$ that contains M'_i. In this expansion the quantifier free types will remain the same because the f_i are $L[\varphi]$-isomorphisms. □

Corollary 5.12. *Suppose that* $M' \models T[\varphi]$ *and* $M' \restriction L \subseteq N \models T$. *Then there is an expansion of* N *to a model* N' *of* $T[\varphi]$ *such that* $M' \subseteq N'$. *In particular, if* $M' \models T[\varphi]$ *is existentially closed, then* $M' \restriction L$ *is an existentially closed model of* T. *Denote by* $T_0[\varphi]$ *the model completion of* $T[\varphi]$. *We will call it the* <u>φ*-circularization*</u> *of* T_0. *It follows that* $T_0[\varphi] \restriction L = T_0$ (for more see [Hod93, Theorem 8.2.4]).

We turn to dividing:

Claim 5.13. *Assume that* $M \models T_0[\varphi]$, $A \subseteq M$, $\bar{a} \in M$, $S^M[\varphi](\bar{a}) \cap A^{\lg(\bar{x})} = \emptyset$, *and* $\bar{c} \neq \bar{d} \in S^M[\varphi](\bar{a})$. *Then the formula* $\psi(\bar{z}; \bar{a}, \bar{c}, \bar{d}) = C[\varphi](\bar{a}, \bar{c}, \bar{z}, \bar{d})$ 2*-divides over* $A\bar{a}$.

Proof. Let $M_0 = A\bar{a}$, $M_1 = M_0\bar{c}\bar{d}$ and $M_2 = M_0\bar{c}'\bar{d}'$ where $M_1 \cap M_2 = M_0$ and there is an isomorphism $f : M_1 \to M_2$ that fixes M_0 and takes $\bar{c}\bar{d}$ to $\bar{c}'\bar{d}'$.

By SAP, there is a model $M_3 \models T[\varphi]$ that contains $M_1 \cup M_2$. We wish to choose it carefully: in the proof of Claim 5.9, we saw that there are no constraints on the amalgamation of $C^{M_1}[\varphi](\bar{a})$ and $C^{M_2}[\varphi](\bar{a})$ (because $S^{M_0}[\varphi](\bar{a}) = \emptyset$, see the definition of $S[\varphi]$). In particular we can put $\bar{c}'$ and $\bar{d}'$ so that in the circular order we have $\bar{c} \to \bar{d} \to \bar{c}' \to \bar{d}' \to \bar{c}$, and in this case there is no $\bar{z}$ such that $C[\varphi](\bar{a})(\bar{c}, \bar{z}, \bar{d})$ and $C[\varphi](\bar{a})(\bar{c}', \bar{z}, \bar{d}')$.

Applying the same technique n times yields a model of $T[\varphi]$ with a sequence $\langle \bar{c}_i, \bar{d}_i \mid i < n \rangle$ that contains M_1 and satisfies $\mathrm{tp}_{\mathrm{qf}}(\bar{c}_i\bar{d}_i/A\bar{a}) = \mathrm{tp}_{\mathrm{qf}}(\bar{c}\bar{d}/A\bar{a})$, so that in the circular order $C[\varphi](\bar{a})$ the tuples will be ordered as follows: $\bar{c} \to \bar{d} \to \bar{c}_1 \to \bar{d}_1 \to \cdots \to \bar{c}_n \to \bar{d}_n \to \bar{c}$. Hence, there is a model of $T_0[\varphi]$ and an infinite such sequence, and this sequence witnesses the 2-dividing of $\psi(\bar{z}; a, \bar{c}, \bar{d})$.

Note that the tuples $\bar{c}_i\bar{d}_i$ were chosen so that the intersection of each pair $\bar{c}_i\bar{d}_i, \bar{c}_j\bar{d}_j$ is contained in A. □

The last sentence justifies the following auxiliary definition which will make life a bit easier:

Definition 5.14. Say that a formula $\varphi(\bar{x}, \bar{a})$ *k-divides disjointly* over A if there is an indiscernible sequence $\langle \bar{a}_i \mid i < \omega \rangle$ that witnesses k-dividing and moreover $\bar{a}_i \cap \bar{a}_j \subseteq A$.

Remark 5.15. Note that if $\varphi(\bar{x}, \bar{a})$ divides over A, then it divides disjointly over some $B \supseteq A$ (if I is an indiscernible sequence witnessing dividing, then $B = A \cup \bigcap I$).

We shall also need some kind of converse to the last claim. More precisely, we need to say when a formula does not divide.

Claim 5.16. *Suppose:*

(1) $A \subseteq M \models T_0[\varphi]$.
(2) $p(\bar{x}) = p_1(\bar{x}) \cup p_2(\bar{x})$ *is a complete quantifier free type over* M.
(3) $p_1(\bar{x})$ *is a complete* L *type over* M *and* $p_2(\bar{x})$ *is a complete* $\{C[\varphi]\}$ *type over* M.
(4) $p_1(\bar{x})$ *does not divide over* A (*as an* L-*type so also as an* $L[\varphi]$-*type*).
(5) *For all* $\bar{t} \in M^{\lg(\bar{y})}$, $p_2(\bar{x}){\restriction}\{C[\varphi](\bar{t}, -, -, -)\}$ *does not divide over* $A\bar{t}$ (*this means all formulas in* $p_2(\bar{x})$ *of the form* $C[\varphi](\bar{t}, \bar{z}_1, \bar{z}_2, \bar{z}_3)$ *where* $\bar{x}$ *substitutes the* $\bar{z}$*'s in some places and in the others there are parameters from* M).

Then $p(\bar{x})$ *does not divide over* A. *In particular, if neither* $p_1(\bar{x})$ *nor* $p_2(\bar{x})$ *divides over* A, *then* $p(\bar{x})$ *does not divide over* A.

Proof. Denote $\bar{x} = (x_0, \ldots, x_{m-1})$ and $p(\bar{x}, M) = p(\bar{x})$. We may assume that $p{\restriction}x_i$ is non-algebraic for all $i < m$ (otherwise, by Fact 5.5, $(x_i = c) \in p$ for some $c \in M$, so $c \in A$ as $x = c$ divides over A, and we can replace x_i by c). Suppose $\langle M_i \mid i < \omega\rangle$ is an $L[\varphi]$-indiscernible sequence over A in some model $N \supseteq M$ such that $M_0 = M$. We will show that $\bigcup\{p(\bar{x}, M_i) \mid i < \omega\}$ is consistent.

Let $\bar{c} \models \bigcup\{p_1(\bar{x}, M_i)\}$ (exists by (4)) and $B = \bigcup\{M_i \mid i < \omega\}$, and let $B' = B\bar{c}{\restriction}L$ (i.e. forget $C[\varphi]$). Also let $\bar{d} \models p(\bar{x})$ be in some other model $N' = M\bar{d}$ of $T[\varphi]$.

For $\bar{t} \in (B\bar{c})^{\lg(\bar{y})}$ we define a circular order on $S[\varphi](\bar{t})$ to make B' into a model U of $T[\varphi]$ extending B such that $\bar{c} \models \bigcup\{p(\bar{x}, M_i)\}$.

Case 1: $\bar{t} \nsubseteq M_i\bar{c}$ for any $i < \omega$. In this case, there is no information on $C[\varphi](\bar{t})$ in $\bigcup\{p_2(\bar{x}, M_i)\}$, so let $C[\varphi]^U(\bar{t})$ be any circular order on $S[\varphi](\bar{t})$ that extends the circular order $C[\varphi]^B(\bar{t})$ (in case $\bar{t} \subseteq B$).

Case 2: $\bar{t} \subseteq M_i\bar{c}$ for some $i < \omega$, but $\bar{t} \nsubseteq M_j\bar{c}$ for some other $j \neq i$. By indiscernibility, $\bar{t} \nsubseteq M_j\bar{c}$ for all $j \neq i$. Let $\sigma : M_i\bar{c} \to M\bar{d}$ be an L-isomorphism. There are two subcases:

- $\bar{t} \cap \bar{c} \neq \emptyset$. Let $C[\varphi]^U(\bar{t})$ be any extension of $\sigma^{-1}(C[\varphi]^{N'}(\sigma(\bar{t})))$ to $S^U[\varphi](\bar{t})$.
- $\bar{t} \cap \bar{c} = \emptyset$. Then $C[\varphi]^B(\bar{t})$ is already a circular order on $S^B[\varphi](\bar{t})$. On the other hand, $\sigma^{-1}(C[\varphi]^{N'}(\sigma(\bar{t})))$ defines some circular order on $S^{M_i\bar{c}}[\varphi](\bar{t})$. The intersection is $S^{M_i}[\varphi](\bar{t})$ on which they agree, so we can amalgamate the two circular orders.

Case 3: $\bar{t} \subseteq \bigcap M_i$. In this case, by (5), $p_2(\bar{x}){\restriction}\{C[\varphi](\bar{t}, -, -, -)\}$ does not divide over $A\bar{t}$, so let $\bar{c}' \models \bigcup\{p_2(\bar{x}, M_i){\restriction}C[\varphi](\bar{t}, -, -, -) \mid i < \omega\}$. Let U' be the $L[\varphi]$ structure $B\bar{c}'$. Let $f : B\bar{c} \to B\bar{c}'$ fix B and take $\bar{c}$ to $\bar{c}'$. Now, $C^{U'}[\varphi](f(\bar{t}))$ induces a circular order on

$$S = f^{-1}\big(S^{U'}[\varphi](f(\bar{t}))\big) \cap S^{B'}[\varphi](\bar{t}).$$

Extend it to some circular order on $S^U[\varphi](\bar{t})$ and let it be $C^U[\varphi](\bar{t})$.

Case 4: $\bar{t} \subseteq \bigcap M_i\bar{c}$ and $\bar{t} \cap \bar{c} \neq \emptyset$. Let $\sigma_i : M_i\bar{c} \to M\bar{d}$ be the L-isomorphism fixing $\bigcap M_i$ and taking $\bar{c}$ to $\bar{d}$. Then σ_i induces a circular order on $S^{M_i\bar{c}}[\varphi](\bar{t})$, and the intersection of any two $S^{M_i\bar{c}}[\varphi](\bar{t})$ and $S^{M_j\bar{c}}[\varphi](\bar{t})$ is $S^{\bigcap M_i\bar{c}}[\varphi](\bar{t})$, on which these circular orders agree. By amalgamation, we have a circular order on the union $\bigcup_i S^{M_i\bar{c}}[\varphi](\bar{t})$ that we can expand to a circular order on $S^U[\varphi](\bar{t})$. □

Claim 5.17. *Let $A \subseteq M \models T_0[\varphi]$ be $|A|^+$-saturated and $M' = M{\restriction}L$. Suppose that $\psi(\bar{z}, \bar{a})$, a quantifier free L-formula, k-divides disjointly over A in M'. Then the same is true in M.*

Proof. Suppose that $I = \langle \bar{a}_i \mid i < \omega\rangle \subseteq M$ witnesses k-dividing disjointly of $\psi(\bar{z}, \bar{a})$ over A in the sense of L. Assume that $\bar{a}_0 = \bar{a}$.

By Claim 5.11(3) and compactness, we can expand and extend M' to $M'' \models T_0[\varphi]$ that will keep the equality of types of the tuples in the sequence. In addition, the interpretation of the new relation $C[\varphi]$ on $A\bar{a}$ remains as it was in M. In particular, in M'', $\psi(\bar{z}, \bar{a})$ still k-divides over A. We may amalgamate a copy of M'' with M over $A\bar{a}$ to get a bigger model in which $\psi(\bar{z}, \bar{a})$ still k-divides disjointly, and by saturation this is still true in M.

□

5.2.2. Circularization: Iterations. Assume there are theories $\mathcal{T} = \langle T_i^\forall \mid i \leq \omega\rangle$ and formulas $\langle \varphi_i(\bar{x}_i; \bar{y}_i) \mid i < \omega\rangle$ in the finite relational languages $\langle L_i \mid i \leq \omega\rangle$ where:

- $T_0^\forall$ is a universal theory with SAP and DEP in L_0.
- $T_i^\forall$ is a theory in L_i for $i \leq \omega$.
- $\varphi_i(\bar{x}_i; \bar{y}_i)$ is a quantifier free formula in L_i.
- $L_i = L_i[\varphi_i(\bar{x}_i; \bar{y}_i)]$ and $T_{i+1}^\forall = T_i^\forall[\varphi_i(\bar{x}_i; \bar{y}_i)]$.
- $L_\omega = \bigcup\{L_i \mid i < \omega\}$ and $T_\omega^\forall = \bigcup\{T_i^\forall \mid i < \omega\}$.

Proposition 5.18. *In the situation above, $T_i^\forall$ has a model completion T_i, $T_i \subseteq T_{i+1}$ and $T_i \subseteq T_\omega$ which is the model completion of $T_\omega^\forall$ for all $i < \omega$.*

Proof. Follows from Claims 5.9 and 5.12. □

From now on, we work in $T := T_\omega$. Call T_ω the $\bar{\varphi}$*-circularization* of T_0 where $\bar{\varphi} = \langle \varphi_i \mid i < \omega\rangle$. Let $M \models T$ and $A \subseteq M$.

Claim 5.19. *Suppose $\varphi(\bar{x}; \bar{y}) = \varphi_i(\bar{x}_i; \bar{y}_i)$ for some $i < \omega$. Then for all $\bar{a} \in M^{\lg(\bar{y})}$, $\varphi(\bar{z}, \bar{a}) \wedge (\bar{z} \cap (\bar{a} \cap A) = \emptyset)$ forks over A if and only if it is not satisfied in A.*

Proof. Denote $\bar{a}' = \bar{a} \cap A$ and $\alpha(\bar{z}, \bar{a}) = \varphi(\bar{z}, \bar{a}) \wedge (\bar{z} \cap \bar{a}' = \emptyset)$. Obviously, if α is satisfied in A, it does not fork over A.

Suppose α is not satisfied in A. Consider the formula $\psi(\bar{z}, \bar{a}) = \varphi(\bar{z}, \bar{a}) \wedge (\bar{z} \cap \bar{a} = \emptyset)$. First we prove that ψ forks. It defines $S[\varphi]^M(\bar{a})$, and by assumption $S[\varphi]^M(\bar{a}) \cap A = \emptyset$. Note that for all $\bar{c} \neq \bar{d} \in S^M[\varphi](\bar{a})$, since $C^M[\varphi](\bar{a})$ orders this set in a circular order,

$$S[\varphi](\bar{a})(\bar{z}) \vdash C[\varphi](\bar{a})(\bar{c}, \bar{z}, \bar{d}) \vee C[\varphi](\bar{a})(\bar{d}, \bar{z}, \bar{c}) \vee \bar{z} = \bar{c} \vee \bar{z} = \bar{d}.$$

If $S[\varphi]^M(\bar{a}) = \emptyset$ we are done. If not, (by Corollary 5.6) this set is infinite and there are such $\bar{c}, \bar{d}$.

By Claims 5.13 and 5.17, $C[\varphi](\bar{a})(\bar{c},\bar{z},\bar{d})$ and $C[\varphi](\bar{a})(\bar{d},\bar{z},\bar{c})$ divide over $A\bar{a}$. By Corollary 5.6, both $\bar{z}=\bar{c}$ and $\bar{z}=\bar{d}$ divide over $A\bar{a}$. This means that $S[\varphi](\bar{a})(\bar{z})=\psi(\bar{z},\bar{a})$ forks over A.

Now, $\alpha(\bar{z},\bar{a})\vdash\psi(\bar{z},\bar{a})\vee\bigvee_{i,j}(z_i=a_j)$ (where z_i, a_j run over all the variables and parameters from $\bar{a}\backslash A$ in φ). But the formula $z_i=a_j$ divides over A when $a_j\notin A$ (by Corollary 5.6), so we are done. □

On the other hand, we have:

Claim 5.20. *Suppose that $p(\bar{x})$ is a (quantifier free) type over M such that:*

- *$p_0(\bar{x})=p\restriction L_0$ does not divide over A.*
- *$p_i(\bar{x})=p\restriction L_{i+1}\backslash L_i$ does not divide over A.*

Then p does not divide over A.

Proof. By induction on $i<\omega$ we show that $p_i'=p\restriction L_i$ does not divide over A. For $i=0$ this is given. For $i+1$ use Claim 5.16. □

The following definition is a bit vague:

Proposition 5.21. *Let $\mathcal{F}$ be a function defined on the class of all countable relational first-order languages such that $\mathcal{F}(L)$ is a set of quantifier free partitioned formulas in L. Let T_0 be a universal theory in the language L_0 satisfying SAP and DEP. We define:*

- *For $n<\omega$, let $L_{n+1}=\bigcup\{L_n[\varphi(\bar{x};\bar{y})]\mid\varphi(\bar{x};\bar{y})\in\mathcal{F}(L_n)\}$, and let $L_\omega=\bigcup\{L_n\mid n<\omega\}$.*
- *For $n<\omega$, let $T_n^\forall$ be a universal theory in L_n defined by induction on $n\leq\omega$:*
  - $T_0^\forall=T_0$.
  - $T_{n+1}^\forall=\bigcup\{T_n^\forall[\varphi(\bar{x};\bar{y})]\mid\varphi\in\mathcal{F}(L_n)\}$.
  - $T_\omega^\forall=\bigcup\{T_n^\forall\mid n<\omega\}$.

Then $T_\omega^\forall$ has a model completion which we denote by $\circlearrowleft_{T_0,L_0,\mathcal{F}}$. Moreover, it is a $\bar{\varphi}$-circularization for some choice of $\bar{\varphi}$.

Proof. By carefully choosing an enumeration of the formulas in L_ω, we can reconstruct $T_\omega^\forall, L_\omega$ in such a way that at each step we deal with one formula and it has a model completion by Proposition 5.18. □

5.3. Example of (7)

Definition 5.22. Let $L_0=\{=\}$ and T_0 be empty. Let $\mathcal{F}(L)$ be the set of all quantifier free partitioned formulas from L. Let $T=\circlearrowleft_{T_0,L_0,\mathcal{F}}$.

Remark 5.23. T has IP: Let $\varphi(x,y)=(x\neq y)$. Then $C[\varphi](y;x_1,x_2,x_3)$ has IP.

Corollary 5.24. *For any set A, a type $p(\bar{x})\in S(\mathbb{M})$ does not fork over A if and only if p is finitely satisfiable in A. In particular, by Fact 3.8, $f_T(\kappa,\lambda)\leq 2^{2^\kappa}$.*

Proof. Suppose $p(\bar{x})$ is a global type that is not finitely satisfiable in A. By quantifier elimination, there is a quantifier free formula $\varphi(\bar{x};\bar{y})$ and $\bar{a}\in\mathbb{M}$ such that $\varphi(\bar{x},\bar{a})\in p$, and this formula is not satisfiable in A. If $\bar{a}\cap A\neq\emptyset$, and $x_i=a\in p$ for some $a\in\bar{a}\cap A$, replace x_i by a in φ, and change the partition of the variables so that we get $\varphi(\bar{z},\bar{a})\wedge\bar{z}\cap(\bar{a}\cap A)=\emptyset\in p$. By Claim 5.19, this formula forks over A, and we are done. ◻

Proposition 5.25. *We have* $f_T(\kappa,\lambda)=2^{\min\{2^\kappa,\lambda\}}$.

Proof. By the proof of Proposition 3.6 and Remark 5.23. ◻

5.4. *Example of (8)*

In this section we are going to construct an example of a theory T with $f_T(\kappa,\lambda)=\lambda$. The idea is to start with the random graph and circularize it in order to ensure that any non-forking type $p\in S^{\mathrm{nf}}(N,M)$ can be R-connected to at most one point of N.

Definition 5.26. Suppose L is a relational language which includes a binary relation symbol R. For a quantifier free L-formula $\psi(\bar{x};\bar{y})$ and atomic formulas $\theta_0(\bar{x};\bar{y}_0)$, $\theta_1(\bar{x},\bar{y}_1)$, where $\lg(\bar{x})>0$, and both $\bar{x}$ and $\bar{y}_i$ occur in them, define the formula

$$\begin{aligned}\varphi_\psi^{\theta_0,\theta_1}(\bar{x};\bar{y}') &= \varphi_\psi^{\theta_0,\theta_1}(\bar{x};\bar{y},\bar{y}_0,\bar{y}_1,z_0,z_1,z_2)\\ &= \theta_0(\bar{x},\bar{y}_0)\wedge\theta_1(\bar{x},\bar{y}_1)\\ &\quad\wedge\psi(\bar{x},\bar{y})\\ &\quad\wedge\bigwedge_{i<j<3}R(z_i,z_j)\wedge\bigwedge_{\substack{i<3,\ y\in\bar{y}\bar{y}_0\bar{y}_1\\ \bar{y}_0\neq\bar{y}_1}}R(z_i,y).\end{aligned}$$

So z_0,z_1,z_2 form a triangle and are connected to all other parameters. The reason for this will be made clearer in the proof of Claim 5.28.

Definition 5.27. For a countable first-order relational language L containing a binary relation symbol R, let $\mathcal{F}(L)$ be the set of all formulas of the form $\varphi_\psi^{\theta_0,\theta_1}$ from L as above. Let $L_0=\{R\}$ where R is a binary relation symbol. Let T_0 say that R is a graph (symmetric and non-reflexive). Let $T=\circlearrowleft_{T_0,L_0,\mathcal{F}}$.

Claim 5.28. *Let $b\in M$. Let $p_b(z)$ be a non-algebraic type over M in one variable saying that $R(z,a)$ just when $a=b$. Then p_b isolates a complete type over M.*

Proof. We will show:

(1) $p_b\restriction L_0$ is complete.
(2) If $L\supseteq L_0$ is some subset of L_ω and for all atomic formulas $\theta(z)\in L\backslash L_0$ over M, $p_b(z)\models\neg\theta(z)$, then for all $\varphi\in L$ used in the circularization (as in Definition 5.26) and atomic formulas $\theta(z,\bar{y})\in L[\varphi]\backslash L$ and $\bar{c}\in M^{\lg(\bar{y})}$, $p_b(z)\models\neg\theta(z,\bar{c})$.

From (1) and (2) it follows by induction that p_b is complete.

(1) is immediate.

(2): Suppose $\theta(z, \bar{y})$ is an atomic formula in $L[\varphi]\backslash L$. Then it is of the form $C[\varphi](\dots)$ where $\varphi = \varphi_{\psi}^{\theta_0,\theta_1}(\bar{x}; \bar{y}')$ for some $\psi(\bar{x}; \bar{y})$ and $\theta_i(\bar{x}; \bar{y}_i)$ from L. Suppose z appears in $\theta(z, \bar{y})$ among the index variables. Then by the choice of φ, $\theta(z, \bar{c})$ implies that z is R-connected to at least two different elements from M, and this contradicts the choice of p_b (this is why we added the extra parameters forming an R-triangle in Definition 5.26). So assume that z appears only in the main variables.

Case 1: One of θ_0, θ_1 is not from L_0, say θ_0. Since $C[\varphi](\bar{y}', \bar{x}_1, \bar{x}_2, \bar{x}_3) \models \bigwedge \varphi(\bar{x}_i, \bar{y}')$, and $p_b(z) \models \neg\theta_0(\dots z \dots)$ by induction (this notation means: substituting some variables of θ_0 with z, and putting parameters from M elsewhere), $p_b(z) \models \neg\theta(z, \bar{c})$.

Case 2: Both $\theta_0, \theta_1 \in L_0$. Suppose $\bar{c} \in M^{\lg(\bar{y}')}$ and show that $p_b(z) \models \neg C[\varphi](\bar{c}; \dots z \dots)$. There are two possibilities for θ_i: $R(z, y)$ and $z = y$. If $C[\varphi](\bar{c}; \dots z \dots)$ holds, then we would infer that either $R(z, c_0) \wedge R(z, c_1)$ for some $c_0 \neq c_1 \in M$, or some equation $x = s'$ for $s' \in M$ is in p_b (here we use the fact that both x and $\bar{y}_i$ occur in θ_0, θ_1)—a contradiction. □

Claim 5.29. $f_T(\kappa, \lambda) \geq \lambda$.

Proof. Let $M \prec N \models T$, $|M| = \kappa$, $|N| = \lambda$. For each $b \in M$, let p_b be the type defined in the previous claim. Then p_b extends naturally to a global type q_b (i.e. the type over $\mathbb{M}$ that is R-connected only to b). This type does not divide over M (in fact, it does not divide over $\emptyset$), by Claim 5.20 and the proof of Claim 5.28 (all atomic formulas in L_n have exactly the same truth value for $n > 0$). □

Claim 5.30. $f_T^n(\kappa, \lambda) = \lambda$ *for all* n *and all* $\lambda \geq 2^{2^{\kappa}}$.

Proof. Suppose $f_T^n(\kappa, \lambda) > \lambda$. Let $M \prec N \models T$ where $|M| = \kappa$, $|N| = \lambda$ and $|S_n^{\mathrm{nf}}(N, M)| > \lambda$.

Let $\{p_i(\bar{x}) \mid i < \lambda^+\} \subseteq S_n^{\mathrm{nf}}(N, M)$ be pairwise distinct. By possibly replacing $\bar{x}$ with a subtuple and throwing away some i's, we may assume that for all $i < \lambda^+$, $p_i \models \bar{x} \cap M = \emptyset$. Since $\lambda \geq 2^{2^{\kappa}}$, we may assume that for all $i < \lambda^+$, p_i is not finitely satisfiable in M.

Then an easy computation shows that there must be some $i < \lambda^+$ such that p_i contains two positive occurrences of atomic formulas $\theta_0(\bar{x}, \bar{a}_0)$ and $\theta_1(\bar{x}, \bar{a}_1)$ for some $\bar{a}_0 \neq \bar{a}_1 \in N$. Let $p = p_i$. There is some quantifier free formula $\psi(\bar{x}, \bar{c}) \in p$ such that ψ is not realized in M. Let $\bar{a}$ be the tuple of parameters $\langle \bar{c}, \bar{a}_0, \bar{a}_1 \rangle$ and let $d_0, d_1, d_2 \in N$ be an R-triangle such that $R(d_i, a)$ for all $a \in \bar{a}$. Finally, let $\bar{a}' = \bar{a}d \cap M$. Then $\varphi_{\psi}^{\theta_0,\theta_1}(\bar{x}; \bar{c}, \bar{a}_0, \bar{a}_1, d) \wedge \bar{x} \cap \bar{a}' = \emptyset \in p$ forks over M by Claim 5.19. □

5.5. *Example of (9)*

In this subsection we prove the following:

Proposition 5.31. *For any theory* T*, there is a theory* T_* *such that* $f_{T_*}(\kappa, \lambda) = f_T(\kappa, \lambda)^{\aleph_0}$ *for all* $\lambda \geq \kappa$.

Let T be a theory in the language L and assume that T eliminates quantifiers. For each $n < \omega$, let L_n be a copy of L such that $L_n \cap L_m = \emptyset$ for $n < m$, and $L_n = \{R_n \mid R \in L\}$. Let $\langle M_n \mid n < \omega \rangle$ be a sequence of models of T. We define a structure M in the language $\{P_n(x), Q(x), f_n : Q \to P_n \mid n < \omega\} \cup \bigcup L_n$:

(1) $M = \bigsqcup_{n<\omega} M_n \sqcup \prod_{n<\omega} M_n$ ($\sqcup$ means disjoint union).
(2) $P_n^M = M_n$, $Q^M = \prod_{n<\omega} M_n$.
(3) If $R(\bar{x}) \in L(T)$ then for every $n < \omega$, $R_n^M \subseteq (P_n^M)^{\lg(\bar{x})}$ and P_n^M is the structure M_n.
(4) $f_n^M : Q^M \to P_n^M$, $f_n^M(\eta) = \eta(n)$—the projection onto the n-th coordinate.

Let $T_* = \mathrm{Th}(M)$.

Remark 5.32. The following properties are easy to check by back-and-forth:

(1) Doing the same construction with respect to any sequence $\langle M_n \mid n < \omega \rangle$ of models of T gives the same T_*.
(2) Moreover, if we have $M_n \preceq N_n$ for all n and do the construction, then $M \preceq N$.
(3) T_* eliminates quantifiers.

Now let $M \preceq N \models T$ with $|M| = \kappa$, $|N| = \lambda$.

Lemma 5.33. *Given $p(x) \in S_1(N)$ such that $Q(x) \in p$, for each $n < \omega$ let $p_n(y) = \{\varphi(y) \mid \varphi \in L_n, \varphi(f_n(x)) \in p\}$.*

(1) *$p(x)$ is equivalent to $\bigcup_{n<\omega} p_n(f_n(x))$.*
(2) *For each $n < \omega$, let $q_n(y)$ be a complete L_n-type over P_n^N. Then the type $(\bigcup_{n<\omega} q_n(f_n(x))) \cup \{Q(x)\}$ is consistent and complete.*
(3) *P_n is stably embedded and the induced structure on P_n is just the L_n-structure. Moreover, for any $n < \omega$ and L_*-formula $\varphi(\bar{x}, \bar{y}_1, \bar{y}_2, \bar{z})$ there is some L_n-formula $\psi(\bar{x}, \bar{y}_1, \bar{z}')$ such that for any $\bar{c}_1 \in P_n$, $\bar{c}_2 \in \bigcup_{m \neq n} P_m$ and $\bar{d} \in Q$, we have $\{\bar{a} \in P_n \mid \models \varphi(\bar{a}, \bar{c}_1, \bar{c}_2, \bar{d})\} = \bigcup\{\bar{a} \in P_n \mid \models \psi(\bar{a}, \bar{c}_1, f_n(\bar{d}))\}$.*
(4) *$p(x)$ forks over M if and only if for some $n < \omega$, $p_n(y) \restriction L_n$ forks over P_n^M (in the sense of T).*

Proof. (1), (2) and (3) follow by quantifier elimination, and (4) follows from (1)–(3). □

Proof of Proposition 5.31. We may assume that T eliminates quantifiers (by taking its Morleyzation). Consider T_* as above, and let us compute $f_{T_*}(\kappa, \lambda)$. Let $M \preceq N \models T_*$.

Let $S_n = \{p \in S^{\mathrm{nf}}(N, M) \mid P_n(x) \in p\}$. From Lemma 5.33, it follows that $|S_n| = |S^{\mathrm{nf}, L_n}(P_n^N, P_n^M)|$.

Let $S_Q = \{p \in S^{\mathrm{nf}}(N, M) \mid Q(x) \in p\}$. From Lemma 5.33, it follows that $|S_Q| = \prod_{n<\omega} |S^{\mathrm{nf}, L_n}(P_n^N, P_n^M)|$.

Let $S_\neg = \{p \in S^{\mathrm{nf}}(N, M) \mid \neg Q(x), \forall n < \omega(\neg P_n(x))\}$. Since there is no structure on elements outside of all the P_n and Q, we have $|S_\neg| \leq |M|$.

Note that $S^{\mathrm{nf}}(N, M) = \bigcup_{n<\omega} S_n \cup S_Q \cup S_\neg$. From this and Remark 5.32(2), it follows that $f_{T^*}(\kappa, \lambda) = f_T(\kappa, \lambda)^{\aleph_0}$. □

Remark 5.34. This analysis easily generalizes to show that $f^n_{T_*}(\kappa, \lambda) = f^n_T(\kappa, \lambda)^{\aleph_0}$.

5.6. *Examples of (12) and (14)*

Here we construct an example of a theory T with $f_T(\kappa, \lambda) = \operatorname{ded}\lambda$. The idea is that we start with an ordered random graph, and we circularize in order to ensure that for any $p \in S^{\mathrm{nf}}(N, M)$ there is some cut of N such that $R(x, a)$ is in p if any only if a is in the cut.

Notation 5.35. Here the language L contains an order relation $<$ which induces the natural lexicographic order on tuples, so abusing notation, we may write $\bar{y} < \bar{z}$.

In this section, we say that two atomic formulas $\theta_1(\bar{x}; \bar{y}_1)$ and $\theta_2(\bar{x}; \bar{y}_2)$ are different when the relation symbol is different (rather than just the variables are different).

Also, when we say "atomic formula" in the definition below, we mean that it does not use the order relation $<$.

Definition 5.36. Suppose L is a relational language which includes a binary relation symbol R, a unary predicate P and an order relation $<$.

For a quantifier free L-formula $\psi(\bar{x}; \bar{y})$ and two different atomic formulas $\theta_0(\bar{x}; \bar{y}_0)$, $\theta_1(\bar{x}, \bar{y}_1)$, where $\lg(\bar{x}) > 0$, and both $\bar{x}$ and $\bar{y}_i$ occur in them, define the formula

$$\begin{aligned}\varphi_\psi^{\theta_0,\theta_1}(\bar{x}; \bar{y}') &= \varphi_\psi^{\theta_0,\theta_1}(\bar{x}; \bar{y}, \bar{y}_0, \bar{y}_1, z_0, z_1)\\ &= \theta_0(\bar{x}, \bar{y}_0) \wedge \theta_1(\bar{x}, \bar{y}_1)\\ &\quad \wedge \psi(\bar{x}, \bar{y})\\ &\quad \wedge z_0 < z_1 \wedge P(z_0) \wedge P(z_1)\\ &\quad \wedge \bigwedge_{y \in \bar{y}\bar{y}_0\bar{y}_1,\, i<2} (y \neq z_i) \wedge R(y, z_1) \wedge \neg R(y, z_0).\end{aligned}$$

For an L-formula $\psi(\bar{x}; \bar{y})$ and an atomic formula $\theta(\bar{x}; \bar{y}_0)$ (in which $\bar{y}_0$ appears) , define the formula

$$\begin{aligned}\varphi_\psi^{\theta}(\bar{x}; \bar{y}') &= \varphi_\psi^{\theta}(\bar{x}; \bar{y}, \bar{y}_0, \bar{y}_1, z_0, z_1)\\ &= \neg\theta(\bar{x}, \bar{y}_0) \wedge \theta(\bar{x}, \bar{y}_1)\\ &\quad \wedge \psi(\bar{x}, \bar{y})\\ &\quad \wedge z_0 < z_1 \wedge P(z_0) \wedge P(z_1)\\ &\quad \wedge \bigwedge_{\substack{y \in \bar{y}\bar{y}_0\bar{y}_1,\, i<2\\ \bar{y}_0<\bar{y}_1}} (y \neq z_i) \wedge R(y, z_1) \wedge \neg R(y, z_0).\end{aligned}$$

Definition 5.37. For a countable first-order relational language L containing a binary relation symbol R, let $\mathcal{F}(L)$ be the set of all formulas from L of the form $\varphi_\psi^{\theta_0,\theta_1}$ or φ_ψ^{θ} as above. Let $L_0 = \{R, <\}$ where R and $<$ are binary relation symbols. Let T_0 say that R is a graph and that $<$ is a linear order. Let $T = \circlearrowleft_{T_0,L_0,\mathcal{F}}$.

Suppose $M \models T$.

Claim 5.38. *Let I be initial segments in M. Let $p_I(x)$ be a non-algebraic type over M saying that $x > M$, $\neg P(x)$ and $R(x, a)$ just when $a \in I$. Then p_I isolates a complete type over M.*

Proof. In fact, $p_I \restriction L_0$ is complete, and for all atomic formulas $\theta(x) \notin L_0$ over M, we have $p_I \models \neg\theta(x)$. The proof is very similar to the proof of Claim 5.28. □

Claim 5.39. $f_T(\kappa, \lambda) \geq \operatorname{ded} \lambda$.

Proof. Let $M \prec N \models T$, $|M| = \kappa$, $|N| = \lambda$. For each cut I in N, let p_I be the type defined in the previous claim. Then p_I extends naturally to a global type q_I (i.e. the type over $\mathbb{M}$ defined by $p_{I'}$ where $I' = \{c \in \mathbb{M} \mid \exists a \in I\ (c < a)\}$). This type does not divide over M (in fact, it does not divide over $\emptyset$) by Claim 5.20, and by the proof of the previous claim (all atomic formulas have exactly the same truth value in L_n for $n > 0$). □

Claim 5.40. $f_T^n(\kappa, \lambda) = \operatorname{ded} \lambda$ *for all* n *and all* $\lambda \geq 2^{2^\kappa}$.

Proof. Suppose $f_T^n(\kappa, \lambda) > \operatorname{ded} \lambda$. Let $M \prec N \models T$ where $|M| = \kappa$, $|N| = \lambda$.

Let $\{p_i(\bar{x}) \mid i < (\operatorname{ded} \lambda)^+\} \subseteq S^{\mathrm{nf}}(N, M)$ be a set of pairwise distinct types. As in the proof of Claim 5.30, we may assume that $p_i \models \bar{x} \cap M = \emptyset$ for all i, and p_i is not finitely satisfiable in N. Also we may assume that $p_i \restriction \{<\}$ is constant.

Then, by the choice of $\varphi_\psi^{\theta_0, \theta_1}$, for every $i < (\operatorname{ded} \lambda)^+$ there is at most one atomic formula of the form $\theta(\bar{x}; \bar{y})$ such that there is some positive instance $\theta(\bar{x}, \bar{a}) \in p_i$. [If not, suppose $\theta_0(\bar{x}, \bar{a}_0) \wedge \theta_1(\bar{x}, \bar{a}_1) \in p$. There is some quantifier free formula $\psi(\bar{x}, \bar{c}) \in p_i$ such that ψ is not realized in M. Let $\bar{a}$ be the tuple of parameters $\langle \bar{c}, \bar{a}_0, \bar{a}_1 \rangle$ and let $d_0, d_1, d_2 \in N$ be an R-triangle such that $R(d, b)$ for all $b \in \bar{a}$. Finally, let $\bar{a}' = \bar{a}d \cap M$. Then $\varphi_\psi^{\theta_0, \theta_1}(\bar{x}; \bar{c}, \bar{a}_0, \bar{a}_1, d) \wedge \bar{x} \cap \bar{a}' = \emptyset \in p$ forks over M by Claim 5.19.]

Similarly, by the choice of φ_ψ^θ, this formula induces a cut $I = \{\bar{a} \mid \theta(\bar{x}, \bar{a}) \in p_i\}$.

This formula and the cut it induces determine the type. But this is a contradiction to the definition of ded. □

Corollary 5.41. *There is a theory* T_* *such that* $f_{T_*}(\lambda, \kappa) = (\operatorname{ded} \lambda)^{\aleph_0}$.

Proof. By Proposition 5.31. □

5.7. *Example of (16)*

As a pleasant surprise to the reader who managed to get this far, the example is just the theory of the random graph (it is NTP_2 and has IP, see Proposition 4.5).

5.8. *Example of* $f_T^1(\kappa, \lambda) \leq 2^{2^\kappa}$ *but* $f_T^2(\kappa, \lambda) = 2^\lambda$

Again we use circularizations, but instead of considering all formulas, we consider only formulas with one variable.

Definition 5.42. Let $L_0 = \{=\}$ and T_0 be empty. Let $\mathcal{F}(L)$ be the set of all quantifier free partitioned formulas from L of the form $\varphi(x; \bar{y})$ where x is a singleton. Let $T = \circlearrowleft_{T_0, L_0, \mathcal{F}}$.

Let $A \subseteq M \models T$. By Claim 5.19 and as in the proof of Proposition 5.25, we get

Corollary 5.43. *If* $p(x) \in S_1(M)$ *then* p *does not fork over* A *if and only if it is finitely satisfiable in* A. *So* $f_T^1(\kappa, \lambda) \leq 2^{2^\kappa}$ *for all* $\kappa \leq \lambda$.

On the other hand, if we consider types in two variables, then there is no reason for them to fork.

Claim 5.44. $f_T^2(\kappa, \lambda) \geq 2^\lambda$.

Proof. Suppose $|M| = \lambda$, so $M = \{a_i \mid i < \lambda\}$, and $A \subseteq M$ of size κ. Let $q(z) \in S_1(M)$ be any 1-type which is finitely satisfiable in A but not algebraic over A. For $S \subseteq \lambda$, let $p_S(x, y)$ be a partial type over M such that:

(1) $p_S \restriction x = q(x)$, $p_S \restriction y = q(y)$.
(2) $R(x, y, a_i) \in p_S$ if and only if $i \in S$.

First, p_S is indeed a type. The proof is by induction, i.e. one proves that $p_S \restriction L_0$ is a type (which is clear), and that if L is some subset of L_ω such that $p_S \restriction L$ is a type, and $\varphi(x; \bar{y})$ is some partitioned L-formula with $\lg(x) = 1$, then also $p_S \restriction L[\varphi]$ is a type, which follows from Claim 5.11.

Let $N \supseteq M$ be an $|A|^+$-saturated model and $q' \supseteq q$ be a global type which is finitely satisfiable in A. Fix $c \models q'|_N$ and $d \models q'|_{Nc}$.

We want to construct a completion $r_S(x, y) \in S_2(N)$ containing p_S which does not divide over A. We start by $r_S \restriction x = q'|_N(x)$, $r_S \restriction y = q'_N(y)$ and $r_S \restriction L_0$ is any completion of $p_S \restriction L_0$. For each atomic formula $\theta(x, y, \bar{t})$ over N of the form $C[\varphi](\bar{t}, -, -, -)$ (so $\bar{t} \in N$) such that $\varphi(x, t) \in q'(x)$ define $\theta(x, y) \in r_S$ if and only if $\theta(c, d)$ holds. This is a type (by induction again, by Claim 5.11(3), but follow the proof a bit more carefully, and choose the amalgamation of the circular orders corresponding to $\bar{t}$ according to the choice of c, d). Let r_S by any completion.

Finally, r_S does not divide over A by Claim 5.16 (by induction and by the choice of c, d). ◻

6. On $\operatorname{ded} \kappa < (\operatorname{ded} \kappa)^{\aleph_0}$

6.1. *On* $\operatorname{ded} \lambda$

Definition 6.1. Let $\operatorname{ded} \lambda$ be the supremum of the set

$$\{|I| \mid I \text{ is a linear order with a dense subset of size } \leq \lambda\}.$$

Fact 6.2. *It is well known that* $\lambda < \operatorname{ded} \lambda \leq (\operatorname{ded} \lambda)^{\aleph_0} \leq 2^\lambda$. *If* $\operatorname{ded} \lambda = 2^\lambda$*, then* $\operatorname{ded} \lambda = (\operatorname{ded} \lambda)^{\aleph_0} = 2^\lambda$. *This is true for* $\lambda = \aleph_0$*, or more generally for any* λ *such that* $\lambda = \lambda^{<\lambda}$*. So in particular this holds for any* λ *under GCH.*

In addition, if $\operatorname{ded} \lambda$ *is not attained (i.e. it is a supremum rather than a maximum), then* $\operatorname{cof}(\operatorname{ded} \lambda) > \lambda$*. See also Corollary 6.12.*

Definition 6.3. Given a linear order I and two regular cardinals θ, μ, we say that S is a (θ, μ)*-cut* when it has cofinality θ from the left and cofinality μ from the right.

By a *tree* we mean a partial order $(T, <)$ such that for every $a \in T$, $T_{<a} = \{x \in T \mid x < a\}$ is well ordered. By a *branch* in T we mean a maximally linearly ordered subset of T. Its *length* is its order type.

For two cardinals λ and μ, let

$$\lambda^{\langle\mu\rangle_{\mathrm{tr}}} = \sup\{\kappa \mid \text{there is some tree } T \text{ with } \lambda \text{ nodes and } \kappa \text{ branches of length } \mu\}.$$

Remark 6.4. Note that $\lambda^{\langle\mu\rangle_{\mathrm{tr}}} \leq \lambda^{\mu}$ and if $\lambda = \lambda^{<\mu}$ then $\lambda^{\langle\mu\rangle_{\mathrm{tr}}} = \lambda^{\mu}$ (consider the tree $\lambda^{<\mu}$ ordered lexicographically).

Proposition 6.5. *The following cardinalities are the same:*

(1) $\operatorname{ded}\lambda$.
(2) $\sup\{\kappa \mid$ *there is a linear order I of size λ with κ cuts*$\}$.
(3) $\sup\{\kappa \mid \exists$ *a regular μ and a linear order I of size $\leq \lambda$ with κ (μ,μ)-cuts*$\}$.
(4) $\sup\{\kappa \mid \exists$ *a regular μ and a tree T with κ branches of length μ and $|T| \leq \lambda$*$\}$.
(5) $\sup\{\kappa \mid \exists$ *a limit ordinal δ and a tree T with κ branches of length δ and $|T| \leq \lambda$*$\}$.
(6) $\sup\{\lambda^{\langle\mu\rangle_{\mathrm{tr}}} \mid \mu \leq \lambda$ *is regular*$\}$.

Proof. (1)=(2), (4)=(6): obvious.

(2)=(3): By [KSTT05, Theorem 3.9], given a linear order I and two regular cardinals $\theta \neq \mu$, the number of (θ,μ)-cuts in I is at most $|I|$. Given I and a regular cardinal μ, let $D_{\mu}(I)$ be the set of (μ,μ)-cuts, and let $D(I)$ be the set of all cuts. Suppose $|I| = \lambda$; then $|D(I)| = \sup\{|D_{\mu}(I)| \mid \mu = \operatorname{cof}(\mu) \leq \lambda\}$ whenever $|D(I)| > \lambda$. By Fact 6.2, $\operatorname{ded}\lambda = \sup\{D_{\mu}(I) \mid \mu = \operatorname{cof}(\mu) \leq \lambda,\ |I| \leq \lambda\}$.

(2)=(4): Follows from [Bau76, Theorem 2.1(a)].

(4)=(5): Obviously (5) $\geq$ (4). Suppose T is a tree as in (5). Let $\mu = \operatorname{cof}(\delta)$ and let $U = \{\delta_i \mid i < \mu\}$ be increasing such that $\delta = \bigcup_{i<\mu}\delta_i$. Let $S = \{a \in T \mid \operatorname{lev}(a) \in U\}$. Then S is a subset of T, so a tree with the induced order. For a branch $B \subseteq T$ of length δ, let $B^S = B \cap S$; then B^S is a branch of S of length μ. If $B_1 \neq B_2$ are branches of length δ in T, then let $a \in B_1 \backslash B_2$, and let $a' > a$ in B_1 be such that $\operatorname{lev}(a') \in U$. Then $a' \in B_1^S \setminus B_2^S$. □

6.2. *Consistency of* $\operatorname{ded}\kappa < (\operatorname{ded}\kappa)^{\aleph_0}$

In [Kei76], the following fact is mentioned (without proof), attributed to Kunen:

Remark 6.6 (Kunen). If $\kappa^{\aleph_0} = \kappa$ then $(\operatorname{ded}\kappa)^{\aleph_0} = \operatorname{ded}\kappa$.

Proof. Suppose I is a linear order, and $J \subseteq I$ is dense, $|J| = \kappa$. Let $\mathcal{U}$ be a non-principal ultrafilter on ω. Then the linear order $I^{\omega}/\mathcal{U}$ has $J^{\omega}/\mathcal{U}$ as a dense subset. Now[1], $|J^{\omega}/\mathcal{U}| = \kappa^{\aleph_0} = \kappa$ and $|I^{\omega}/\mathcal{U}| = |I|^{\aleph_0}$. The remark follows from Fact 6.2. □

Answering a question of Keisler [Kei76, Problem 2], we show:

Theorem 6.7. *It is consistent with ZFC that* $\operatorname{ded}\kappa < (\operatorname{ded}\kappa)^{\aleph_0}$.

Our proof uses Easton forcing, so let us recall:

[1] If A is infinite then $A^{\omega}/\mathcal{U}$ has size $|A|^{\aleph_0}$: Let $g_n : A^n \to A$ be bijections. Then map $f \in A^{\omega}$ to $\bar{f} = \langle g_n(f(0),\ldots,f(n-1)) \mid n < \omega\rangle$, so that if $f \neq g$ then $\bar{f} \neq \bar{g}$ from some point onwards, and in particular modulo $\mathcal{U}$.

Theorem 6.8 (Easton). *Let M be a transitive model of ZFC and assume that the Generalized Continuum Hypothesis holds in M. Let F be a function (in M) whose arguments are regular cardinals and whose values are cardinals, such that for all regular κ and λ:*

(1) $F(\kappa) > \kappa$.
(2) $F(\kappa) \leq F(\lambda)$ *whenever* $\kappa \leq \lambda$.
(3) $\operatorname{cof}(F(\kappa)) > \kappa$.

Then there is a generic extension $M[G]$ of M such that M and $M[G]$ have the same cardinals and cofinalities, and for every regular κ, $M[G] \models 2^\kappa = F(\kappa)$.

See [Jec03, Theorem 15.18]. Easton forcing is a class forcing but we can just work with a set forcing, i.e. when F is a set. The following is the main claim:

Claim 6.9. *Suppose M is a transitive model of ZFC that satisfies GCH, and furthermore:*

- *κ is a regular cardinal.*
- *$\langle \theta_i \mid i < \kappa \rangle$, $\langle \mu_i \mid i < \kappa \rangle$ are strictly increasing sequences of cardinals and $\theta = \sup_{i<\kappa} \theta_i$, $\mu = \sup_{i<\kappa} \mu_i$.*
- *$\kappa < \theta_0$ and $\theta_i < \mu_0$ for all $i < \kappa$.*
- *θ_i is regular for all $i < \kappa$.*

*Then, letting P be Easton forcing with $F : \{\theta_i \mid i < \kappa\} \to$ **card**, $F(\theta_i) = \mu_i$ and G a generic for P, in $M[G]$ we have $\operatorname{ded}\theta = \mu$ and the supremum is attained.*

Remark 6.10. Note that in $M[G]$, since $2^{\theta_i} = \mu_i$ by Easton's Theorem 6.8, we also get $\operatorname{cof}(\theta) = \operatorname{cof}(\mu) = \kappa < \theta$ and $\mu^\kappa > \mu$.

Proof. First let us show that $\operatorname{ded}\theta \geq \mu$. Recall:

- $\operatorname{Add}(\kappa, \lambda)$ is the forcing notion that adjoins λ subsets to κ, i.e. it is the set of partial functions $p : \kappa \times \lambda \to 2$ such that $|\operatorname{dom}(p)| < \kappa$.
- The Easton forcing notion P is the set of all elements in $\prod_{i<\kappa} \operatorname{Add}(\theta_i, \mu_i)$ such that for every regular cardinal $\gamma \leq \kappa$, and for each $p \in P$, the support $s(p)$ satisfies $|s(p) \cap \gamma| < \gamma$.

If G is a generic of P, then the projection of G to i, G_i, is generic in $\operatorname{Add}(\theta_i, \mu_i)$.

For $i < \kappa$, consider the tree $T_i = (2^{<\theta_i})^M$. Since M satisfies GCH, it follows that $M[G] \models T_i| = \theta_i$. For all $\beta < \mu_i$, we can define a function $\eta_\beta : \theta_i \to 2$ by $\eta_\beta(\alpha) = p(\alpha, \beta)$ for some $p \in G_i$ such that $(\alpha, \beta) \in \operatorname{dom}(p)$. If $\alpha < \theta_i$, then $\eta_\beta \restriction \alpha \in M$ (consider the dense set $D = \{p \in \operatorname{Add}(\theta_i, \mu_i) \mid \alpha \times \{\beta\} \subseteq \operatorname{dom}(p)\}$), so for $\beta < \mu_i$, η_β defines a branch of T_i, and if $\beta_1 \neq \beta_2$ then $\eta_{\beta_1} \neq \eta_{\beta_2}$. By Proposition 6.5 we have $\operatorname{ded}\theta_i = \mu_i = 2^{\theta_i}$ in $M[G]$. Since $\operatorname{ded}\theta \geq \operatorname{ded}\theta_i$ for all $i < \kappa$, we are done.

Now let us show that $\operatorname{ded}\theta \leq \mu$. Let I be some linear order such that $|I| = \theta$. For any choice of cofinalities (κ_1, κ_2), we look at the set C_{κ_1,κ_2} of all (κ_1, κ_2)-cuts of I. Obviously for it to be non-empty, we must have $\kappa_1, \kappa_2 \leq \theta$, so let us assume that $\kappa_1, \kappa_2 \leq \theta_i$ for some i (note that θ is singular, so $\kappa_1, \kappa_2 \neq \theta$). We map each such cut to a pair of cofinal sequences (from the left and from the right). Hence we obtain $|C_{\kappa_1,\kappa_2}| \leq \theta^{\kappa_1+\kappa_2} \leq \theta^{\theta_i}$. Since $\theta \leq \mu_0$, we get $\theta^{\theta_i} \leq \mu_0^{\theta_i} \leq 2^{\theta_0+\theta_i} = \mu_i < \mu$. The number of regular cardinals below θ is $\leq \theta$, so we are done. ◻

Corollary 6.11. *Suppose GCH holds in M. Choose $\kappa = \aleph_0$, $\theta_i = \aleph_{i+1}$ and $\mu_i = \aleph_{\omega+i}$. Then in the generic extension, $\aleph_{\omega+\omega} = \operatorname{ded}\aleph_\omega < (\operatorname{ded}\aleph_\omega)^{\aleph_0}$. In fact, since the Singular Cardinal Hypothesis holds under Easton forcing (see* [Jec03, *Exercise* 15.12]*),* $(\operatorname{ded}\aleph_\omega)^{\aleph_0} = \aleph_{\omega+\omega+1}$.

Corollary 6.12. *It is consistent with ZFC that* $\operatorname{cof}(\operatorname{ded}\lambda) < \lambda$.

Problem 6.13. Is it consistent with ZFC that $\operatorname{ded}\kappa < (\operatorname{ded}\kappa)^{\aleph_0} < 2^\kappa$?

We remark that our construction is not sufficient for that: in the context of Claim 6.9, $(\operatorname{ded}\theta)^\kappa \le 2^\theta$, but $2^\theta = \prod_{i<\kappa} 2^{\theta_i} \le \prod_{i<\kappa}\mu_i \le \mu^\kappa = (\operatorname{ded}\theta)^\kappa$.

Some further properties relating the $\operatorname{ded}\kappa$ function and cardinal arithmetic are established in [CS16].

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