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## Segre classes as integrals over polytopes

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**Abstract.** We express the Segre class of a monomial scheme—or, more generally, a scheme monomially supported on a set of divisors cutting out complete intersections—in terms of an integral computed over an associated body in Euclidean space. The formula is in the spirit of the classical Bernstein–Kouchnirenko theorem computing intersection numbers of equivariant divisors in a torus in terms of mixed volumes, but deals with the more refined intersection-theoretic invariants given by Segre classes, and holds in the less restrictive context of ‘r.c. monomial schemes’.

**Keywords.** Segre classes, monomial ideals, Newton polyhedra

### 1. Introduction

Let  $V$  be a variety (or more generally a pure-dimensional scheme of finite type over an algebraically closed field) and let  $X_1, \dots, X_n$  be effective Cartier divisors in  $V$ . A choice of nonnegative multiplicities  $(a_1, \dots, a_n) \in \mathbb{Z}^n$  determines an effective divisor, obtained by taking  $X_i$  with multiplicity  $a_i$ . We call such a divisor a *monomial* in the  $X_i$ ’s, and we define a *monomial scheme* (with respect to the chosen divisors) to be an intersection of such monomials. In the literature (e.g., [Gow05]) this terminology is reserved for the case in which  $V$  is nonsingular and the divisors  $X_i$  are nonsingular and meet with normal crossings. We will refer to this as the ‘standard’ situation. In this paper we consider a substantially more general case, in which  $V$  may be any scheme as above and the divisors  $X_1, \dots, X_n$  are only required to intersect ‘completely’, in the sense that their local defining equations form regular sequences. The resulting monomial scheme is a ‘regular crossings (r.c.) monomial scheme’ in the terminology of [Har15]. (We should stress that an r.c. monomial scheme is *not* necessarily a complete intersection!)

A monomial scheme  $S$  is determined by an  $n$ -uple of divisors and the choice of finitely many lattice points in  $\mathbb{Z}^n$ . We call the complement of the convex hull of the positive orthants translated at these points the *Newton region* corresponding to (this description of)  $S$ . In [Alu13, Conjecture 1], we proposed a formula for the *Segre class* of a monomial subscheme  $S$  in the standard situation, and proved this formula in several representative

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cases. The purpose of this note is to prove a generalization of the full statement of [Alu13, Conjecture 1].

**Theorem 1.1.** *Let  $S \subseteq V$  be an r.c. monomial scheme with respect to a choice of  $n$  divisors  $X_1, \dots, X_n$ , and let  $N$  be the corresponding Newton region. Then*

$$s(S, V) = \int_N \frac{n! X_1 \cdots X_n da_1 \cdots da_n}{(1 + a_1 X_1 + \cdots + a_n X_n)^{n+1}}. \quad (1.1)$$

Note that different monomial schemes may have the same Newton region. According to (1.1), such schemes have the same Segre class. This phenomenon is due to the fact that the corresponding ideals have the same integral closure (cf. [Alu13, Remark 2.5]).

The right-hand side of (1.1) is interpreted by evaluating the integral formally with parameters  $X_1, \dots, X_n$ ; the result is a rational function in  $X_1, \dots, X_n$ , with a well-defined expansion as a power series in these variables, all of whose terms may be interpreted as intersection products of the corresponding divisors in  $V$ . These products are naturally supported on subschemes of  $S$  (cf. Lemma 2.10). The statement is that evaluating the terms of the series as these intersection products gives the Segre class  $s(S, V)$  of  $S$  in  $V$  as an element of the Chow group of  $S$ .

For a thorough treatment of Segre classes and their role in Fulton–MacPherson intersection theory, see [Ful84]. Segre classes are one of the basic ingredients in the definition of the intersection product [Ful84, Proposition 6.1(a)]. Residual intersection formulas are naturally written in terms of Segre classes [Ful84, Chapter 9]. Many problems in enumerative geometry can be phrased in terms of computations of Segre classes. Interesting invariants of singularities, such as Milnor classes or Donaldson–Thomas invariants, may also be expressed in terms of Segre classes. While the schemes considered in Theorem 1.1 may be too special for direct applications of this type, the approach to the computation of Segre classes presented in this result appears to be completely novel. We hope of course that it will be possible to extend the scope of (1.1) to yet more general schemes.

We refer the reader to [Alu13] for further contextual remarks concerning Segre classes and for examples illustrating (1.1) in the standard situation. In [Alu13], the formula is established in the case  $n = 2$ , and for several families of examples for arbitrary  $n$ . Also, the formula is stated in [Alu13] after push-forward to the ambient variety, and limited to standard monomial schemes. In this paper the formula is proved as an equality of classes in the Chow group of  $S$ , for any  $n$ , and for the larger class of r.c. monomial schemes. This is a substantially more general setting: the ambient scheme  $V$  and the divisors  $X_i$  need not be smooth, nor do the  $X_i$  need to meet transversally.

The proof of Theorem 1.1 is given in §2. It is based on the birational invariance of Segre classes and the fact that r.c. monomial schemes may be principalized by a sequence of blow-ups at r.c. monomial centers of codimension 2. This is proven by C. Harris [Har15], extending the analogous result for standard monomial schemes due to R. Goward [Gow05]. This fact and an explicit computation in the principal case reduce the proof to showing that the integral appearing in (1.1) is preserved under blow-ups at codimension-2 (r.c.) monomial centers. This in turn follows from an analysis of triangulations associated with these blow-ups.

In [Alu15b] we apply Theorem 1.1 to the computation of *multidegrees* of r.c. monomial rational maps. This is one point of contact of the result presented here with the existing literature: in the case of (standard) monomial rational maps between projective spaces, the multidegrees may be computed by *mixed volumes* of Minkowski sums of polytopes as an application of the classical Bernstein–Kouchnirenko theorem (see e.g. [GSP06, §4] or [Dol, §3.5]). In the application reviewed in [Alu15b], Theorem 1.1 leads to an alternative expression for the multidegrees, which reproduces the volume computation for the top multidegree, and generalizes it to the r.c. setting and to rational maps on any projective variety. Formulas for the other multidegrees lead to integral expressions for the mixed volumes appearing in the Bernstein–Kouchnirenko theorem.

It is natural to ask whether a result such as Theorem 1.1 may hold for nonmonomial schemes. One possibility is that an integral formula analogous to (1.1) may hold with  $N$  replaced by a suitable body; this would be in line with recent results for intersection numbers of divisors on open varieties, as in [KK12]. (In fact, these results were our first motivation to look for formulas for Segre classes in terms of ‘volumes’, which led to the formulation of Theorem 1.1.) For a different viewpoint, in [Alu15a] we propose a formulation of the result (in the standard monomial setting) in terms of the *log canonical thresholds* of suitable extensions of the ideal defining the subscheme. This expression makes sense for more general schemes, and may offer a candidate for a generalization of Theorem 1.1.

## 2. Proof

### 2.1. Preliminaries

We begin by recalling R. Goward’s theorem, and C. Harris’s generalization. First assume that we are in the standard monomial situation:  $V$  is a nonsingular variety,  $X_1, \dots, X_n$  are nonsingular divisors meeting with normal crossings, and  $S$  is the intersection of monomial hypersurfaces, i.e., effective divisors  $D_j$  supported on  $\bigcup_i X_i$ . According to [Gow05, Theorem 2], there exists a sequence of blow-ups at nonsingular centers producing a proper birational morphism  $\rho : V' \rightarrow V$  such that  $\rho^{-1}(S)$  is a divisor with normal crossings. Further, as explained in [Gow05, §4], the centers of the blow-ups may all be chosen to be codimension-2 intersections of (proper transforms of) the original components  $X_i$  and of the exceptional divisors produced in the process. In [Har15], C. Harris shows that both statements generalize verbatim to the *r.c. monomial case*. Here no nonsingularity restrictions are posed on  $V$  or on the divisors  $X_1, \dots, X_n$ . The divisors meet with *regular crossings* if for all  $A \subseteq \{1, \dots, n\}$  and all  $p \in \bigcap_{i \in A} X_i$ , the local equations for  $X_i$ ,  $i \in A$ , form a regular sequence at  $p$  [Har15]. An *r.c. monomial scheme* is a scheme defined by effective divisors supported on  $\bigcup_i X_i$ , where the  $X_i$  meet with regular crossings. According to [Har15, Theorem 1], every r.c. monomial scheme may be principalized by a sequence of blow-ups at centers of codimension 2; as in Goward’s result, these centers may be chosen to be intersections of proper transforms of the  $X_i$ ’s and of the exceptional divisors.

**Remark 2.1.** Let  $X_1, \dots, X_n$  be a set of divisors meeting with regular crossings, and let  $\pi : \tilde{V} \rightarrow V$  be the blow-up along the intersection of two of these hypersurfaces; without loss of generality this is  $X_1 \cap X_2$ , and (by definition of regular crossings) it is a regularly embedded subscheme of codimension 2. Then:

- The proper transforms  $\tilde{X}_i$  of the components  $X_i$ , together with the exceptional divisor  $E$ , form a divisor with regular crossings [Har15, Proposition 3].
- $\tilde{X}_i = \pi^{-1}(X_i)$  for  $i \geq 3$ .
- If  $D$  is a monomial in the  $X_i$ 's, then  $\pi^{-1}(D)$  is a monomial in the collection  $E, \tilde{X}_i$ .
- In fact, if  $D = \sum a_i X_i$ , then  $\pi^{-1}(D) = (a_1 + a_2)E + \sum_{i \geq 3} a_i \tilde{X}_i$ .

In view of the second point, we will write  $X_i$  for  $\tilde{X}_i = \pi^{-1}(X_i)$  for  $i \geq 3$ ; this abuse of notation is good mnemonic help when using the projection formula. For example, the projection formula gives  $\pi_*(\tilde{X}_1 \cdot \tilde{X}_3) = \pi_*(\tilde{X}_1 \cdot \pi^*(X_3)) = X_1 \cdot X_3$ . We find this easier to parse if we write  $\pi_*(\tilde{X}_1 \cdot X_3) = X_1 \cdot X_3$ , particularly as  $\pi_*(\tilde{X}_1 \cdot \tilde{X}_2) = 0$  since  $\tilde{X}_1 \cap \tilde{X}_2 = \emptyset$  to begin with. Also note that  $\pi_*(E \cdot \tilde{X}_2) = X_1 \cdot X_2$  and  $\pi_*(E \cdot X_i) = 0$  for  $i \geq 3$ .

As remarked here, at each step in the sequence considered by Harris the inverse image of  $S$  is a monomial scheme with respect to the collection of proper transforms of the  $X_i$ 's and of the previous exceptional divisors, and the next blow-up is performed along the intersection of two of these hypersurfaces. In order to prove Theorem 1.1, therefore, it suffices to prove the following two lemmas.

**Lemma 2.2.** *Let  $S$  be an r.c. monomial scheme, and assume  $S$  is a divisor. Then (1.1) holds for  $S$ .*

**Lemma 2.3.** *Let  $S$  be an r.c. monomial scheme, and let  $\pi : \tilde{V} \rightarrow V$  be the blow-up along  $X_1 \cap X_2$ . Then if (1.1) holds for  $\pi^{-1}(S)$ , then it also holds for  $S$ .*

The proofs of these two lemmas are given in the rest of this section, after some needed preliminaries. As pointed out above, these two lemmas imply Theorem 1.1.

## 2.2. Integrals and triangulations

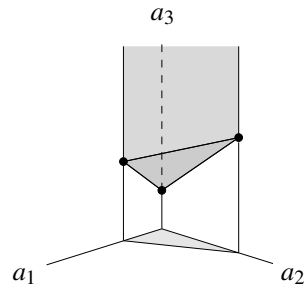
The integral appearing in (1.1) may be computed in terms of a triangulation of  $N$ . An  $n$ -dimensional simplex in  $\mathbb{R}^n$  is the convex hull of a set of  $n + 1$  points (its vertices) not contained in a hyperplane. Points are denoted by underlined letters:  $\underline{v} = (v_1, \dots, v_n)$ . The notation  $\underline{v} \cdot \underline{X}$  stands for  $v_1 X_1 + \dots + v_n X_n$ . We let  $\underline{e}_1 = (1, 0, \dots, 0), \dots, \underline{e}_n = (0, \dots, 0, 1)$ .

We also allow for the possibility that some of the vertices are at infinity, and we denote by  $\underline{a}_i$  the point at infinity in the direction of  $\underline{e}_i$ . Thus, the simplex  $T$  with 'finite' vertices  $\underline{v}_0, \dots, \underline{v}_r$  and 'infinite' vertices  $\underline{a}_{i_1}, \dots, \underline{a}_{i_{n-r}}$  is defined by

$$T = \left\{ \sum_{j=0}^r \lambda_j \underline{v}_j + \sum_{k=1}^{n-r} \mu_k \underline{e}_{i_k} \mid \forall j, k : \lambda_j \geq 0, \mu_k \geq 0, \text{ and } \sum_j \lambda_j = 1 \right\}.$$

Each simplex  $T$  has a normalized volume  $\widehat{\text{Vol}}(T)$ , defined as the normalized volume of the (finite) simplex obtained by projecting along its infinite directions. (The *normalized* volume is the ordinary Euclidean volume times the factorial of the dimension.)

**Example 2.4.** The simplex  $T$  with vertices  $v_0 = (0, 0, 1)$ ,  $v_1 = (1, 0, 2)$ ,  $v_2 = (0, 2, 3)$  and  $a_3$  (at infinity)



has normalized volume  $\widehat{\text{Vol}}(T) = 2$ .

We can associate with every simplex a contribution to the integral in (1.1):

**Lemma 2.5** ([Alu13, Proposition 3.1]). *If  $T$  has finite vertices  $v_0, \dots, v_r$  and infinite vertices  $a_1, \dots, a_{i_{n-r}}$ , then*

$$\int_T \frac{n! X_1 \cdots X_n da_1 \cdots da_n}{(1 + a_1 X_1 + \cdots + a_n X_n)^{n+1}} = \frac{\widehat{\text{Vol}}(T) X_1 \cdots X_n}{\prod_{\ell=0}^r (1 + v_\ell \cdot X) \prod_{j=1}^{n-r} X_j}.$$

Note that the numerator simplifies to give a multiple of the product of the parameters  $X_i$  corresponding to the ‘finite’ part of the simplex.

**Example 2.6.** For the simplex in Example 2.4,

$$\begin{aligned} \int_T \frac{3! X_1 X_2 X_3 da_1 da_2 da_3}{(1 + a_1 X_1 + \cdots + a_n X_n)^4} &= \frac{\widehat{\text{Vol}}(T) X_1 X_2 X_3}{(1 + X_3)(1 + X_1 + 2X_3)(1 + 2X_2 + 3X_3) X_3} \\ &= \frac{2 X_1 X_2}{(1 + X_3)(1 + X_1 + 2X_3)(1 + 2X_2 + 3X_3)}. \end{aligned}$$

It immediately follows from Lemma 2.5 that the integral over the whole positive orthant  $\mathbb{R}_{\geq 0}^n$  equals 1. Also:

**Corollary 2.7.** *For every monomial scheme  $S$ , the integral*

$$\int_N \frac{n! X_1 \cdots X_n da_1 \cdots da_n}{(1 + a_1 X_1 + \cdots + a_n X_n)^{n+1}}$$

*is a rational function in  $X_1, \dots, X_n$ . It may be expanded as a power series in  $X_1, \dots, X_n$  with integer coefficients.*

*Proof.* Triangulate  $N$ , then apply Lemma 2.5. □

2.3. Residual intersection

Let  $S$  be a monomial scheme, realized as the intersection of monomials  $D_1, \dots, D_r$ . Assume all  $D_i$ 's contain a fixed monomial divisor  $D = \sum_i d_i X_i$ ; we obtain a residual monomial scheme  $R$  by intersecting the residuals to  $D$  in  $D_i$ :

$$R := (D_1 - D) \cap \dots \cap (D_r - D).$$

The residual intersection formula in intersection theory (cf. [Ful84, Proposition 9.2]) gives a relation between the Segre classes of  $S$ ,  $D$ , and  $R$ . This formula should be expected to have a counterpart in terms of integrals.

**Lemma 2.8.** *With  $S, D, R$  as above:*

- The Newton region  $N_S$  for  $S$  is the intersection of the positive orthant with the translate by  $(d_1, \dots, d_n)$  of the Newton region  $N_R$  for  $R$ :

$$N_R = \{(v_1, \dots, v_n) \in \mathbb{R}_{\geq 0}^n \mid (v_1 + d_1, \dots, v_n + d_n) \in N_S\}.$$

- We have the equality

$$\int_{N_S} \frac{n! X_1 \cdots X_n da_1 \cdots da_n}{(1 + a_1 X_1 + \dots + a_n X_n)^{n+1}} = \frac{\sum_i d_i X_i}{1 + \sum_i d_i X_i} + \frac{1}{1 + \sum_i d_i X_i} \left( \int_{N_R} \frac{n! X_1 \cdots X_n da_1 \cdots da_n}{(1 + a_1 X_1 + \dots + a_n X_n)^{n+1}} \otimes \mathcal{O}\left(\sum_i d_i X_i\right) \right). \quad (2.1)$$

The notation introduced in [Alu94, §2], is used in this statement (and will be used in the following): for a line bundle  $\mathcal{L}$  and a class  $A = \sum_i a^{(i)}$  in the Chow group, where  $a^{(i)}$  has codimension  $i$  in the ambient scheme  $V$ ,  $A \otimes \mathcal{L}$  denotes the class  $\sum_i c(\mathcal{L})^{-i} \cap a^{(i)}$ . This notation determines an action of Pic on the Chow group, and is compatible with the effect of ordinary tensors on Chern classes (cf. [Alu94, Propositions 1 and 2]).

*Proof.* The first assertion is immediate. For the second, note that the complement  $N'_S$  of  $N_S$  in the positive orthant is precisely the translate of the complement  $N'_R$  by  $(d_1, \dots, d_n)$ . Since the integral over the positive orthant is 1, verifying the stated formula is equivalent to verifying that

$$\int_{N'_S} \frac{n! X_1 \cdots X_n da_1 \cdots da_n}{(1 + a_1 X_1 + \dots + a_n X_n)^{n+1}} = \frac{1}{1 + \sum_i d_i X_i} \left( \int_{N'_R} \frac{n! X_1 \cdots X_n da_1 \cdots da_n}{(1 + a_1 X_1 + \dots + a_n X_n)^{n+1}} \otimes \mathcal{O}\left(\sum_i d_i X_i\right) \right). \quad (2.2)$$

By the formal properties of the  $\otimes$  operation (cf. [Alu94, Proposition 1]),

$$\begin{aligned} \frac{X_1 \cdots X_n}{(1 + a_1 X_1 + \dots + a_n X_n)^{n+1}} \otimes \mathcal{O}\left(\sum_i d_i X_i\right) &= \frac{(1 + \sum_i d_i X_i) X_1 \cdots X_n}{(1 + (a_1 + d_1) X_1 + \dots + (a_n + d_n) X_n)^{n+1}}, \end{aligned}$$

showing that the right-hand side of (2.2) equals

$$\int_{N'_R} \frac{n!X_1 \cdots X_n da_1 \cdots da_n}{(1 + (a_1 + d_1)X_1 + \cdots + (a_n + d_n)X_n)^{n+1}}.$$

It is then clear that this equals the left-hand side, since  $N'_S$  is the translate of  $N'_R$  by  $(d_1, \dots, d_n)$ .  $\square$

**Corollary 2.9.** *With notation as above, formula (1.1) is true for  $S$  in  $A_*S$  if and only if it is true for  $R$ .*

*Proof.* Compare (2.1) with the formula for residual intersections of Segre classes, in the form given in [Alu94, Proposition 3].  $\square$

2.4. Proof of Lemma 2.2

Let  $D_1, \dots, D_r$  be monomials, and assume  $S = D_1 \cap \cdots \cap D_r$  is a divisor. Note that this may happen even if  $r > 1$ : for example, suppose  $X_1, X_2, X_3$  are divisors meeting with normal crossings, and  $X_1 \cap X_2 = \emptyset$ . If  $D_1 = X_1 + X_3$  and  $D_2 = X_2 + X_3$ , then  $S = D_1 \cap D_2$  is a divisor, in fact,  $X_3$ . However, the Newton region of this representation of  $S$  (depicted to the left) includes an infinite column that is not present in the representation as  $X_3$  (on the right).



We have to verify that if  $D_1 \cap \cdots \cap D_r = D$  is a monomial divisor, then these two representations lead to the same integral. By Corollary 2.9, we may in fact eliminate the common factor  $D$  in the monomials  $D_1, \dots, D_r$ , and we are reduced to showing that if  $D_1, \dots, D_r$  have empty intersection, then

$$\int_N \frac{n!X_1 \cdots X_n da_1 \cdots da_n}{(1 + a_1X_1 + \cdots + a_nX_n)^{n+1}} = 0.$$

This follows immediately from the following more general statement.

**Lemma 2.10.** *Let  $D_1, \dots, D_r$  be monomials in  $X_1, \dots, X_n$ , and let  $S = D_1 \cap \cdots \cap D_r$ , with Newton region  $N$ . Then the class computed by that integral in (1.1),*

$$\int_N \frac{n!X_1 \cdots X_n da_1 \cdots da_n}{(1 + a_1X_1 + \cdots + a_nX_n)^{n+1}},$$

*is supported on  $S$ .*

In particular, the class equals 0 if  $S = \emptyset$ . We will in fact prove that the class computed by that integral is a sum of classes obtained by applying Chow operators to classes of subschemes of  $S$ . As such, the integral defines an element in the Chow group  $A_*S$ .

*Proof of Lemma 2.10.* The integral may be computed by triangulating  $N$  and applying Lemma 2.5. Thus, it suffices to show that if  $T$  is a simplex contained in  $N$  with finite vertices  $\underline{v}_0, \dots, \underline{v}_r$  and infinite vertices  $\underline{a}_j, j \in J$ , then the support of the class

$$\frac{\widehat{\text{Vol}}(T)X_1 \cdots X_n}{\prod_{\ell=0}^r(1 + \underline{v}_\ell \cdot \underline{X}) \prod_{j \in J} X_j}$$

is contained in  $S$ . Hence, it suffices to show that the intersection product  $\prod_{j \notin J} X_j$  is supported on  $S$ , i.e.,  $\bigcap_{j \notin J} X_j$  is contained in  $S$ . The coordinates  $\underline{a}_j, j \notin J$ , span the subspace containing the (bounded) projection of  $T$  along its unbounded directions. Since  $T \subseteq N$ , the simplex spanned by  $\underline{e}_j, j \notin J$ , and  $\underline{a}_j, j \in J$ , is contained in  $N$ . This is the Newton region for the ideal generated by  $X_j, j \notin J$ . It follows that this ideal contains the ideal of  $S$ , concluding the proof.  $\square$

2.5. Proof of Lemma 2.3

With notation as at the beginning of this section (see Remark 2.1), we have to prove that the integral appearing in (1.1) is preserved by the push-forward by the blow-up morphism  $\pi : \tilde{V} \rightarrow V$ :

$$\pi_* \left( \int_{\widehat{N}} \frac{(n+1)!E\tilde{X}_1 \cdots \tilde{X}_n da_0 da_1 \cdots da_n}{(1+a_0E+a_1\tilde{X}_1+\cdots+a_n\tilde{X}_n)^{n+2}} \right) = \int_N \frac{n!X_1 \cdots X_n da_1 \cdots da_n}{(1+a_1X_1+\cdots+a_nX_n)^{n+1}}, \quad (2.3)$$

where  $N$  is the Newton region for the intersection of the monomials  $D_1, \dots, D_r$  in the  $X_i$ 's, and  $\widehat{N}$  is the Newton region for the intersection of the monomials  $\pi^{-1}(D_1), \dots, \pi^{-1}(D_r)$  in  $E$  and  $\tilde{X}_i$ . Here we are using coordinates  $a_1, \dots$  corresponding to  $X_1, \dots$ , and coordinates  $a_0, a_1, \dots$  corresponding to  $E, \tilde{X}_1, \dots$ .

Since the integral over the positive orthant is 1, we may equivalently (see Remark 2.11) show that

$$\pi_* \left( \int_{\widehat{N}'} \frac{(n+1)!E\tilde{X}_1 \cdots \tilde{X}_n da_0 da_1 \cdots da_n}{(1+a_0E+a_1\tilde{X}_1+\cdots+a_n\tilde{X}_n)^{n+2}} \right) = \int_{N'} \frac{n!X_1 \cdots X_n da_1 \cdots da_n}{(1+a_1X_1+\cdots+a_nX_n)^{n+1}}, \quad (2.4)$$

where  $N', \widehat{N}'$  are the complements of  $N, \widehat{N}$  in the corresponding positive orthants. We will construct compatible triangulations  $\widehat{\mathcal{U}}$  and  $\mathcal{U}$  of  $\widehat{N}', N'$  respectively, and use Lemma 2.5 to analyze the effect of  $\pi_*$  on the corresponding contributions to the integrals in (2.4).

**Remark 2.11.** A subtlety should be mentioned here. On the face of it, (2.4) is an equality in the Chow group of  $V$ , while our aim is to prove Theorem 1.1 as an equality in  $A_*S$ . This is only an apparent difficulty. The integral  $\int_{\widehat{N}'}$  on the left is shorthand for its expansion



as a power series in the parameters  $\tilde{X}_i$  (cf. Corollary 2.7), and this well-defined series has the form

$$1 + \hat{p}(E, \tilde{X}_1, \dots, \tilde{X}_n)$$

where  $\hat{p}(E, \tilde{X}_1, \dots, \tilde{X}_n)$  is supported on  $\pi^{-1}(S)$  after evaluation as a sum of intersection products in  $\tilde{V}$ . Indeed,  $\hat{p}(E, \tilde{X}_1, \dots, \tilde{X}_n)$  is simply the  $\int_{\tilde{N}}$  appearing in (2.3). Likewise, the right-hand side of (2.4) is a well-defined series  $1 + p(X_1, \dots, X_n)$ , where  $p(X_1, \dots, X_n)$  is a sum of terms supported on  $S$ , equaling the  $\int_N$  on the right-hand side of (2.3). The sense in which (2.4) should be interpreted, and in which it will be proven, is that the push-forward of each term in the series on the left contributes to a summand in the series on the right. With the exception of  $\pi_*(1) = 1$ , all these push-forwards map classes in  $A_*\pi^{-1}(S)$  to classes in  $A_*S$ . In particular,  $\pi_*(\hat{p}(E, \tilde{X}_1, \dots, \tilde{X}_n)) = p(X_1, \dots, X_n)$  in  $A_*S$ , which is precisely (2.3).

With the notation introduced in §2.2, we view  $N'$  as the convex hull of

$$v_1, \dots, v_r; \underline{a}_1, \dots, \underline{a}_n$$

where  $v_i, i = 1, \dots, r$ , are the lattice points corresponding to the monomials  $D_i$ . Likewise,  $\hat{N}'$  is the convex hull of

$$\hat{v}_1, \dots, \hat{v}_r; \underline{a}_0, \underline{a}_1, \dots, \underline{a}_n$$

where  $\hat{v}_i, i = 1, \dots, r$ , correspond to  $\pi^{-1}(D_i)$ . By Remark 2.1, the points  $\hat{v}_i$  are the lifts of the points  $v_i$  to the hyperplane  $H$  in  $\mathbb{R}^{n+1}$  with equation  $a_0 = a_1 + a_2$ . We let  $M = \{v_1, \dots, v_r\}, \hat{M} = \{\hat{v}_1, \dots, \hat{v}_r\}$ . Note that  $\underline{a}_3, \dots, \underline{a}_n$  belong to  $H$ , while  $\underline{a}_0, \underline{a}_1, \underline{a}_2$  do not. In fact,  $\underline{a}_0$  belongs to one of the two half-spaces determined by  $H$ , and  $\underline{a}_1, \underline{a}_2$  to the other.

To obtain the triangulations  $\hat{\mathcal{U}}, \mathcal{U}$ , we use the following procedure.

- Let  $\mathcal{T}$  be any triangulation of the convex hull of  $M \cup \{\underline{a}_3, \dots, \underline{a}_n\}$ .
- Let  $\hat{\mathcal{T}}$  be the lift of  $\mathcal{T}$  to the hyperplane  $H$ . This is a triangulation of the convex hull of  $\hat{M} \cup \{\underline{a}_3, \dots, \underline{a}_n\}$ .
- Let  $\hat{\mathcal{U}}_1$  be the triangulation of the convex hull of  $\hat{M} \cup \{\underline{a}_1, \underline{a}_2, \underline{a}_3, \dots, \underline{a}_n\}$  obtained by first taking pyramids over all simplices in  $\hat{\mathcal{T}}$  with apex  $\underline{a}_1$  (cf. [DLRS10, §4.2.1]), then placing  $\underline{a}_2$  on the resulting triangulation (cf. [DLRS10, §4.3.1]).
- Complete  $\hat{\mathcal{U}}_1$  to a triangulation  $\hat{\mathcal{U}}$  of  $\hat{N}'$ , by placing  $\underline{a}_0$ .

**Lemma 2.12.** *With notation as above:*

- (i) Each simplex  $\sigma$  of top dimension ( $= n$ ) in  $\hat{\mathcal{T}}$  determines two simplices  $\hat{\sigma}_0, \hat{\sigma}_1$  of top dimension ( $= n + 1$ ) in  $\hat{\mathcal{U}}$ , namely the pyramids over  $\sigma$  with apex  $\underline{a}_0$ , resp.,  $\underline{a}_1$ .
- (ii) Every top-dimensional simplex in  $\hat{\mathcal{U}}$  including  $\underline{a}_2$  also includes  $\underline{a}_0$  or  $\underline{a}_1$ .

*Proof.* Both points are direct consequences of the construction.

For (i), observe that all simplices of  $\hat{\mathcal{T}}$  (hence contained in  $H$ ) are visible from both  $\underline{a}_0$  and  $\underline{a}_1$ , since these points are on opposite sides of  $H$ , and  $\underline{a}_2$  is placed after  $\underline{a}_1$  in the construction.

For (ii), note that the top-dimensional simplices in  $\widehat{\mathcal{T}}$  are not visible from  $\underline{a}_2$  in the construction, since  $\underline{a}_2$  is on the same side of  $H$  as  $\underline{a}_1$ , and it is placed after  $\underline{a}_1$ .  $\square$

- By construction,  $N'$  is the convex hull of the projection in the  $a_0$  direction of  $\widehat{M}$  and  $\underline{a}_1, \dots, \underline{a}_n$ ; that is, it is the contraction of  $\widehat{N}'$  in the sense of [DLRS10, Definition 4.2.19]. By [DLRS10, Lemma 4.2.20], we obtain a triangulation  $\mathcal{U}$  of  $N'$  by taking the links of  $\underline{a}_0$  with respect to  $\widehat{\mathcal{U}}$  (cf. [DLRS10, Definition 2.1.6]).

The simplices in  $\mathcal{U}$  correspond precisely to the simplices in  $\widehat{\mathcal{U}}$  of which  $\underline{a}_0$  is a vertex.

We have constructed related triangulations  $\widehat{\mathcal{U}}$  of  $\widehat{N}'$  and  $\mathcal{U}$  of  $N'$ , and we have to study the effect of  $\pi_*$  on the contributions to  $\int_{\widehat{N}'}$  from the top-dimensional simplices in  $\widehat{\mathcal{U}}$  according to Lemma 2.5. We let  $U$ , resp.,  $\widehat{U}$  be the sets of top-dimensional simplices in  $\mathcal{U}$ , resp.,  $\widehat{\mathcal{U}}$ . The simplices in  $U$  are in one-to-one correspondence with the simplices in  $\widehat{U}$  containing  $\underline{a}_0$ .

Each top-dimensional simplex  $\hat{\sigma}$  in  $\widehat{\mathcal{U}}$  is the convex hull of an  $r$ -dimensional face in  $\widehat{\mathcal{T}}$  and  $n + 2 - r$  points at infinity, which may or may not include  $\underline{a}_0, \underline{a}_1, \underline{a}_2$ . We split the set  $\widehat{U}$  into a disjoint union  $\widehat{U}_0 \sqcup \widehat{U}_1 \sqcup \widehat{U}' \sqcup \widehat{U}''$ , according to these different possibilities:

- $\widehat{U}_0$  consists of the simplices  $\hat{\sigma}$  which include  $\underline{a}_0$  and none of  $\underline{a}_1, \underline{a}_2$ .
- $\widehat{U}_1$  likewise consists of the simplices  $\hat{\sigma}$  which include  $\underline{a}_1$  and none of  $\underline{a}_0, \underline{a}_2$ .
- $\widehat{U}'$  consists of the simplices  $\hat{\sigma}$  which contain  $\underline{a}_0$  and at least one of  $\underline{a}_1, \underline{a}_2$ .
- $\widehat{U}''$  consists of the simplices  $\hat{\sigma}$  which do not include  $\underline{a}_0$ , and include either both or none of  $\underline{a}_1$  and  $\underline{a}_2$ .

The remaining possibility, i.e., simplices which contain  $\underline{a}_2$  and none of  $\underline{a}_0, \underline{a}_1$ , is excluded by Lemma 2.12(ii). The simplices in  $\widehat{U}_0$ , resp.,  $\widehat{U}_1$  are pyramids over top-dimensional simplices in  $\widehat{\mathcal{T}}$  with apex  $\underline{a}_0$ , resp.,  $\underline{a}_1$  (Lemma 2.12(i)).

As noted above, by construction there is a one-to-one correspondence between  $\widehat{U}' \cup \widehat{U}_0$  and  $U$ , associating with each  $\hat{\sigma} \in \widehat{U}' \cup \widehat{U}_0$  the link of  $\underline{a}_0$  with respect to  $\hat{\sigma}$ . This natural bijection  $\widehat{U}' \cup \widehat{U}_0 \rightarrow U$  is not compatible with push-forward at the level of the contributions to the integral  $\int_{\widehat{N}'}$ . However, the decomposition found above allows us to define another realization of  $U$ . Define  $\alpha : \widehat{U}' \sqcup \widehat{U}_1 \rightarrow U$  as follows:

- If  $\hat{\sigma} \in \widehat{U}'$ , let  $\alpha(\hat{\sigma})$  be the link of  $\underline{a}_0$  with respect to  $\hat{\sigma}$ .
- If  $\hat{\sigma} \in \widehat{U}_1$ , then  $\hat{\sigma}$  is the pyramid over an  $n$ -dimensional simplex  $\sigma$  in  $\widehat{\mathcal{T}}$  with apex  $\underline{a}_1$ . Let  $\alpha(\hat{\sigma})$  be the link of  $\underline{a}_0$  with respect to the pyramid  $\hat{\sigma}$  over  $\sigma$  with apex  $\underline{a}_0$ .

In other words, in the second case  $\alpha(\hat{\sigma})$  is the simplex in  $\mathcal{T}$  (and hence in  $\mathcal{U}$ ) corresponding to the simplex  $\sigma$  of  $\widehat{\mathcal{T}}$ . The function  $\alpha$  is evidently a bijection.

**Lemma 2.13.** *Let  $\hat{\sigma} \in \widehat{U}$ .*

- *If  $\hat{\sigma} \in \widehat{U}_0 \sqcup \widehat{U}''$ , then the contribution of  $\hat{\sigma}$  to the integral over  $\widehat{N}'$  pushes forward to 0.*
- *If  $\hat{\sigma} \in \widehat{U}' \sqcup \widehat{U}_1$ , then the contribution of  $\hat{\sigma}$  to the integral over  $\widehat{N}'$  pushes forward to the contribution of  $\alpha(\hat{\sigma}) \in U$  to the integral over  $N'$ .*

Since  $\widehat{U} = \widehat{U}_0 \amalg \widehat{U}_1 \amalg \widehat{U}' \amalg \widehat{U}''$  and  $\alpha$  is a bijection onto  $U$ , this lemma verifies (2.4), proving Lemma 2.3 and hence concluding the proof of Theorem 1.1.

By construction, each simplex  $\hat{\sigma}$  in  $\widehat{U}$  has a set of finite vertices  $\hat{v}_0, \dots, \hat{v}_r$  in the hyperplane  $H$ , and a set of infinite vertices. According to Lemma 2.5, the corresponding contribution equals

$$\frac{\widehat{\text{Vol}}(\hat{\sigma})C}{\prod_{\ell=0}^r (1 + \hat{v}_\ell \cdot \widetilde{X})}$$

where  $\hat{v} \cdot \widetilde{X} = v_0E + v_1\widetilde{X}_1 + \dots$ , and  $C$  is a product of divisors from  $E, \widetilde{X}_1, \dots$ . A key observation here is that if  $\hat{v} = (v_0, \dots, v_n)$  is the lift to  $H$  of a corresponding vertex  $\underline{v} = (v_1, \dots, v_n)$ , then  $v_0 = v_1 + v_2$ , and it follows that

$$1 + \hat{v} \cdot \widetilde{X} = \pi^*(1 + \underline{v} \cdot X).$$

Thus, the ‘denominator’ in the contribution of  $\hat{\sigma}$  is a pull-back. Also, if  $\sigma$  is the contraction of a simplex  $\hat{\sigma}$  with respect to  $a_0$ , then the finite vertices of  $\sigma$  are precisely the projections  $\underline{v}_\ell$ , and  $\widehat{\text{Vol}}(\sigma) = \widehat{\text{Vol}}(\hat{\sigma})$ . By the projection formula, we see that

$$\pi_* \left( \frac{\widehat{\text{Vol}}(\hat{\sigma})C}{\prod_{\ell=0}^r (1 + \hat{v}_\ell \cdot \widetilde{X})} \right) = \frac{\widehat{\text{Vol}}(\sigma)\pi_*(C)}{\prod_{\ell=0}^r (1 + \underline{v}_\ell \cdot X)}.$$

This is the contribution of  $\sigma$  to  $\int_{N'}$ , provided that  $\pi_*(C)$  equals the correct product of divisors corresponding to the infinite vertices of  $\sigma$ . These are the infinite vertices of  $\hat{\sigma}$ , with  $\underline{a}_0$  removed. In the proof of Lemma 2.13,  $\pi_*(C)$  is either 0 (in the first point of the lemma) or equals the correct product (in the second).

*Proof of Lemma 2.13.* Assume first  $\hat{\sigma} \in \widehat{U}_0 \amalg \widehat{U}''$ .

If  $\hat{\sigma} \in \widehat{U}_0$ , then  $\hat{\sigma}$  includes  $\underline{a}_0$  and neither  $\underline{a}_1$  nor  $\underline{a}_2$ . According to Lemma 2.5 and the discussion preceding this proof, the contribution of  $\hat{\sigma}$  to  $\int_{N'}$  has the form

$$\frac{\widehat{\text{Vol}}(\hat{\sigma})\widetilde{X}_1 \cdot \widetilde{X}_2 \cdot X_{i_1} \cdots}{\pi^*(\cdots)}$$

with all the  $i_j \geq 3$ . (We are now using our convention of writing  $X_i$  for  $\widetilde{X}_i$  for  $i \geq 3$ , cf. Remark 2.1.) This term equals 0, since  $\widetilde{X}_1 \cap \widetilde{X}_2 = \emptyset$ . If  $\hat{\sigma} \in \widehat{U}''$ , then  $\hat{\sigma}$  does not contain  $\underline{a}_0$  and contains either both or neither of  $\underline{a}_1$  and  $\underline{a}_2$ . Its contribution has the form

$$\frac{\widehat{\text{Vol}}(\hat{\sigma})E \cdot X_{i_1} \cdots}{\pi^*(\cdots)} \quad \text{or} \quad \frac{\widehat{\text{Vol}}(\hat{\sigma})E \cdot \widetilde{X}_1 \cdot \widetilde{X}_2 \cdot X_{i_1} \cdots}{\pi^*(\cdots)}$$

with  $i_j \geq 3$ . In the first case it pushes forward to 0 by the projection formula, and in the second case it is 0, again because  $\widetilde{X}_1 \cap \widetilde{X}_2 = \emptyset$ .

This concludes the proof of the first part of the claim.

For the second part, assume  $\hat{\sigma} \in \widehat{U}' \amalg \widehat{U}_1$ . If  $\hat{\sigma} \in \widehat{U}'$ , then  $\hat{\sigma}$  contains  $\underline{a}_0$ , and  $\underline{a}_1$  or  $\underline{a}_2$  or both. The contribution of  $\hat{\sigma}$  to  $\int_{N'}$  has the form

$$\frac{\widehat{\text{Vol}}(\hat{\sigma})X_{i_1} \cdots}{\prod_{\ell=0}^r (1 + \hat{v}_\ell \cdot \widetilde{X})} \quad \text{or} \quad \frac{\widehat{\text{Vol}}(\hat{\sigma})\widetilde{X}_\ell \cdot X_{i_1} \cdots}{\prod_{\ell=0}^r (1 + \hat{v}_\ell \cdot \widetilde{X})}$$

with all the  $i_j \geq 3$  and  $\ell = 1$  or  $2$ . By the projection formula, these terms push forward to the corresponding contributions of  $\alpha(\hat{\sigma})$  to the integral over  $N'$ .

If  $\hat{\sigma} \in \hat{U}_1$ , then  $\hat{\sigma} = \hat{\sigma}_1$  for a top-dimensional simplex  $\sigma = \alpha(\hat{\sigma})$  in  $\hat{\mathcal{F}}$ ;  $\hat{\sigma}$  includes  $\underline{a}_1$ , but neither  $\underline{a}_0$  nor  $\underline{a}_2$ . The contribution of  $\hat{\sigma}$  to  $\int_{\hat{N}'}$  has the form

$$\frac{\widehat{\text{Vol}}(\hat{\sigma}) E \cdot \tilde{X}_2 \cdot X_{i_1} \cdots}{\prod_{\ell=0}^r (1 + \hat{v}_\ell \cdot \tilde{X})}$$

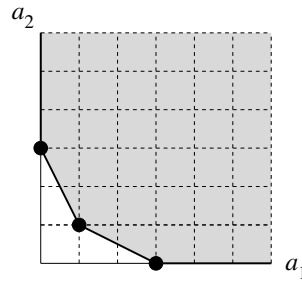
By the projection formula (cf. Remark 2.1), this term pushes forward to a contribution

$$\frac{\widehat{\text{Vol}}(\hat{\sigma}) X_1 \cdot X_2 \cdot X_{i_1} \cdots}{\prod_{\ell=0}^r (1 + v_\ell \cdot X)},$$

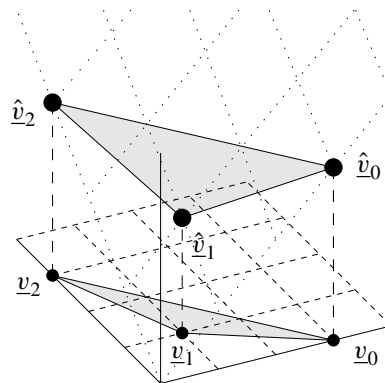
matching the contribution of  $\alpha(\sigma)$ , and concluding the proof. □

### 2.6. An example

A concrete example may clarify the argument presented in the previous section. Consider the ideal  $(x^3, xy, y^3)$ . The shaded area in the following picture depicts  $N'$  in the plane  $\mathbb{R}^2$  with coordinates  $(a_1, a_2)$ :

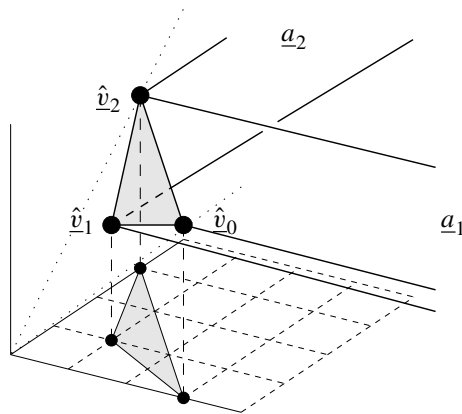


Lifting to  $\mathbb{R}^3$ , with ‘vertical’ coordinate  $a_0$ , gives



Here we have shaded the triangle determined by the three monomials in the  $(a_1, a_2)$  plane, as well as its lift to the hyperplane  $H$  with equation  $a_0 = a_1 + a_2$ .

With notation as in §2.5,  $\mathcal{T}$  and  $\widehat{\mathcal{T}}$  consist of the shaded triangles along with their faces. The one-dimensional faces of  $\widehat{\mathcal{U}}_1$  are included in the following picture.



The top-dimensional simplices in  $\widehat{\mathcal{U}}_1$  are

$$\widehat{v}_0\widehat{v}_1\widehat{v}_2a_1 \quad \text{and} \quad \widehat{v}_1\widehat{v}_2a_1a_2;$$

the triangle  $\widehat{v}_0\widehat{v}_1\widehat{v}_2$  is extended to a 3-simplex in the  $a_1$  direction; the triangle  $\widehat{v}_1\widehat{v}_2a_1$  is visible to  $a_2$ , so it produces a second 3-simplex. These two simplices and their faces form  $\widehat{\mathcal{U}}_1$ . The vertex  $v_0$  is not visible to  $a_2$ , since it is ‘behind’ the plane containing  $\widehat{v}_1, \widehat{v}_2, a_1$ . As remarked in Lemma 2.12(ii), a top-dimensional simplex including  $a_2$  must also include  $a_1$ .

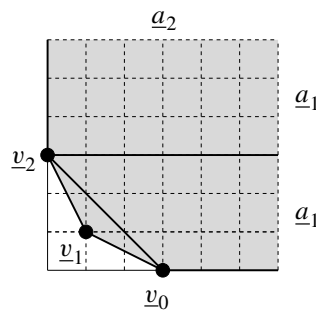
The 2-dimensional faces of  $\widehat{\mathcal{U}}_1$  visible to  $a_0$  are

$$\widehat{v}_0\widehat{v}_1\widehat{v}_2, \quad \widehat{v}_0\widehat{v}_2a_1, \quad \widehat{v}_2a_1a_2.$$

Therefore  $\widehat{\mathcal{U}}$  consists of the five 3-simplices

$$\widehat{v}_0\widehat{v}_1\widehat{v}_2a_1, \quad \widehat{v}_1\widehat{v}_2a_1a_2, \quad \widehat{v}_0\widehat{v}_1\widehat{v}_2a_0, \quad \widehat{v}_0\widehat{v}_2a_1a_0, \quad \widehat{v}_2a_1a_2a_0$$

and their faces. The corresponding triangulation  $\mathcal{U}$  is the projection of the faces of  $\widehat{\mathcal{U}}_1$  visible by  $a_0$ :



The five simplices listed for  $\widehat{\mathcal{U}}$  form the set  $\widehat{U}$  used in the proof. The decomposition  $\widehat{U}_0 \sqcup \widehat{U}_1 \sqcup \widehat{U}' \sqcup \widehat{U}''$  is as follows:

$$\widehat{U}_0 = \{\widehat{v}_0\widehat{v}_1\widehat{v}_2a_0\}, \quad \widehat{U}_1 = \{\widehat{v}_0\widehat{v}_1\widehat{v}_2a_1\}, \quad \widehat{U}' = \{\widehat{v}_0\widehat{v}_2a_1a_0, \widehat{v}_2a_1a_2a_0\}, \quad \widehat{U}'' = \{\widehat{v}_1\widehat{v}_2a_1a_2\}.$$

The bijection  $\alpha : \widehat{U}' \sqcup \widehat{U}_1 \rightarrow U$  maps

$$\widehat{v}_0\widehat{v}_2a_1a_0 \mapsto v_0v_2a_1, \quad \widehat{v}_2a_1a_2a_0 \mapsto v_2a_1a_2, \quad \widehat{v}_0\widehat{v}_1\widehat{v}_2a_1 \mapsto v_0v_1v_2.$$

Lemma 2.13 now states that the push-forward  $\pi_*$  will map the contributions from

$$\widehat{U}_0 \sqcup \widehat{U}'' = \{\widehat{v}_0\widehat{v}_1\widehat{v}_2a_0, \widehat{v}_1\widehat{v}_2a_1a_2\}$$

to 0, and those from

$$\widehat{U}' \sqcup \widehat{U}_1 = \{\widehat{v}_0\widehat{v}_2a_1a_1a_0, \widehat{v}_2a_1a_1a_2a_0, \widehat{v}_0\widehat{v}_1\widehat{v}_2a_1a_1\}$$

to the total contributions of the simplices in  $U$ . The contributions from the first set are

$$\frac{3\widetilde{X}_1\widetilde{X}_2}{(1+3\widetilde{X}_1+3E)(1+\widetilde{X}_1+\widetilde{X}_2+2E)(1+3\widetilde{X}_2+3E)} + \frac{E}{(1+\widetilde{X}_1+\widetilde{X}_2+2E)(1+3\widetilde{X}_2+3E)},$$

and vanish in the push-forward as prescribed by Lemma 2.13. Those from the second,

$$\frac{3\widetilde{X}_2}{(1+3\widetilde{X}_1+3E)(1+3\widetilde{X}_2+3E)} + \frac{1}{1+3\widetilde{X}_2+3E} + \frac{3\widetilde{X}_2E}{(1+3\widetilde{X}_1+3E)(1+\widetilde{X}_1+\widetilde{X}_2+2E)(1+3\widetilde{X}_2+3E)},$$

push forward to

$$\frac{3X_2}{(1+3X_1)(1+3X_2)} + \frac{1}{1+3X_2} + \frac{3X_1X_2}{(1+3X_1)(1+X_1+X_2)(1+3X_2)},$$

that is, to the sum of the contributions corresponding to the triangulation  $\mathcal{U}$  of  $N'$ .

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