J. Eur. Math. Soc. 18, 2865–2924

DOI 10.4171/JEMS/656

Susanna Terracini · Gianmaria Verzini · Alessandro Zilio

Uniform Hölder bounds for strongly competing systems involving the square root of the laplacian

Received December 2, 2013 and in revised form February 18, 2015

Abstract. For a class of competition-diffusion nonlinear systems involving the square root of the laplacian, including the fractional Gross–Pitaevskii system

$$(-\Delta)^{1/2}u_i = \omega_i u_i^3 + \lambda_i u_i - \beta u_i \sum_{j\neq i} a_{ij} u_j^2, \quad i = 1, \dots, k,$$

we prove that L^∞ boundedness implies $\mathcal{C}^{0,\alpha}$ boundedness for every $\alpha \in [0, 1/2)$, uniformly as $\beta \to \infty$. Moreover we prove that the limiting profile is $\mathcal{C}^{0,1/2}$. This system arises, for instance, in the relativistic Hartree–Fock approximation theory for k-mixtures of Bose–Einstein condensates in different hyperfine states.

Keywords. Square root of the laplacian, spatial segregation, strongly competing systems, optimal regularity of limiting profiles, singular perturbations

Contents

1. Introduction 2866
   1.1. Notation 2869
2. Alt–Caffarelli–Friedman type monotonicity formulae 2869
   2.1. Segregated ACF formula 2870
   2.2. Perturbed ACF formula 2875
3. Almgren type monotonicity formulae 2878
   3.1. Almgren's formula for segregated entire profiles 2878
   3.2. Almgren's formula for coexisting entire profiles 2883
4. Liouville type theorems 2884
5. Some approximation results 2889

S. Terracini: Dipartimento di Matematica "Giuseppe Peano", Università degli Studi di Torino, Via Carlo Alberto 10, 10123 Torino, Italy; e-mail: susanna.terracini@unito.it

G. Verzini: Dipartimento di Matematica, Politecnico di Milano, p.za Leonardo da Vinci 32, 20133 Milano, Italy; e-mail: gianmaria.verzini@polimi.it

A. Zilio: Centre d'analyse et de mathématique sociales (CAMS), École des hautes études en sciences sociales (EHESS), 190-198, avenue de France, 75244 Paris Cedex 13, France; e-mail: azilio@ehess.fr, alessandro.zilio@polimi.it

Mathematics Subject Classification (2010): Primary 35J65; Secondary 35B40, 35B44, 35R11, 81Q05, 82B10

6. Local $\mathcal{C}^{0,\alpha}$ uniform bounds, α small . . . 2894
7. Liouville type theorems, reprise: the optimal growth . . . 2905
8. $\mathcal{C}^{0,\alpha}$ uniform bounds, $\alpha < 1/2$ . . . 2911
9. $\mathcal{C}^{0,1/2}$ regularity of the limiting profiles . . . 2914
References . . . 2922

1. Introduction

Regularity issues involving fractional laplacians are very challenging, because of the genuinely nonlocal nature of such operators, and for this reason they have recently become the object of intensive research, especially when associated with the asymptotic analysis and the study of free boundary problems (see for instance [9, 3, 21, 8, 11, 7, 20] and references therein). The present paper is concerned with this topic, when the creation of a free boundary is triggered by the interplay between fractional diffusion and *competitive interaction*.

Several physical phenomena can be described by a certain number of densities (of mass, population, probability, ...) distributed in a domain and subject to laws of diffusion, reaction, and competitive interaction. Whenever the competition is the prevailing feature, the densities tend to segregate, hence determining a partition of the domain. When anomalous diffusion is involved, one is led to consider the class of stationary systems of semilinear equations

$$\begin{cases} (-\Delta)^s u_i = f_i(x, u_i) - \beta u_i \sum_{j \neq i} g_{ij}(u_j), \\ u_i \in H^s(\mathbb{R}^N), \end{cases}$$

thus focusing on the singular limit problem obtained when the (positive) parameter β, accounting for the competitive interactions, diverges to ∞. Among others, the cases $f_i(s) = r_i s(1 - s/K_i)$, $g_{ij}(s) = a_{ij}s$ (logistic internal dynamics with Lotka–Volterra competition) and $f_i(s) = \omega_i s^3 + \lambda_i s$, $g_{ij}(s) = a_{ij}s^2$ (focusing-defocusing Gross–Pitaevskii system with competitive interactions, see for instance [13, 12]) are of most interest in the applications to population dynamics and theoretical physics, respectively.

For the standard Laplace diffusion operator (namely $s = 1$), the analysis of the qualitative properties of solutions to the corresponding systems has been undertaken, starting from [13, 14, 15], in a series of recent papers [16, 32, 5, 6, 22, 24], also in the parabolic case [31, 17, 18, 19]. In the singular limit one finds a vector $\mathbf{u} = (u_1, \ldots, u_k)$ of limiting profiles with mutually disjoint supports; indeed, the *segregated states* u_i satisfy $u_i \cdot u_j \equiv 0$ for $i \neq j$, and

$$-\Delta u_i = f_i(x, u_i) \quad \text{whenever } u_i \neq 0,\ i = 1, \ldots, k.$$

Natural questions concern the functional classes of convergence (a priori bounds), optimal regularity of the limiting profiles, equilibrium conditions at the interfaces, and regularity of the nodal set. In [16] (for Lotka–Volterra competition) and [22] (for the variational Gross–Pitaevskii system) it is proved that L^∞ boundedness implies $\mathcal{C}^{0,\alpha}$ boundedness, uniformly as $\beta \to \infty$, for every $\alpha \in (0, 1)$. Moreover, in the second case, it is shown that the limiting profiles are Lipschitz continuous. The proof relies upon elliptic estimates, the blow-up technique, and the monotonicity formulae of Almgren [1] and Alt–Caffarelli–Friedman [2], and it reveals a subtle interaction between the diffusion and competition aspects. This interaction mainly occurs at two levels: the validity and exactness of the Alt–Caffarelli–Friedman monotonicity formula and, consequently, the validity of Liouville type theorems for entire solutions to semilinear systems.

In this paper we address the problem of a priori bounds and optimal regularity of the limiting profiles in the simplest case of anomalous diffusion, driven by the square root of the laplacian. As is well known, anomalous diffusion arises when the Gaussian statistics of the classical Brownian motion is replaced by a different one, allowing for Lévy jumps (or flights). In the light of the already built theory for the regular laplacian, we focus on the joint effect of diffusion and competition as the (nonlocal) diffusion process acts on a longer range.

Our model problem will be the following:

$$\begin{cases} (-\Delta)^{1/2} u_i = f_{i,\beta}(u_i) - \beta u_i \sum_{j \neq i} u_j^2, \\ u_i \in H^{1/2}(\mathbb{R}^N). \end{cases} \tag{1.1}$$

This class of problems includes the already mentioned Gross–Pitaevskii systems with focusing or defocusing nonlinearities

$$\begin{cases} (-\Delta + m_i^2)^{1/2} u_i = \omega_i u_i^3 + \lambda_{i,\beta} u_i - \beta u_i \sum_{j \neq i} a_{ij} u_j^2, \\ u_i \in H^{1/2}(\mathbb{R}^N), \end{cases}$$

with $a_{ij} = a_{ji} > 0$, which is the relativistic version of the Hartree–Fock approximation theory for mixtures of Bose–Einstein condensates in different hyperfine states. Even though we will perform the proof in the case $m_i = 0$ (and $a_{ij} = 1$), the general case, allowing positive masses $m_i > 0$, follows with minor changes and it is actually a bit simpler.

As is well known (see e.g. [10]), the N-dimensional half-laplacian can be interpreted as a Dirichlet-to-Neumann operator and solutions to problem (1.1) as traces of harmonic functions on the $(N + 1)$-dimensional half-space having the right hand side of (1.1) as normal derivative. For this reason, it is worth stating our main results for harmonic functions with nonlinear Neumann boundary conditions involving strong competition terms. We use the following notation: for any dimension $N \geq 1$, we consider the half-ball $B_r^+(x_0, 0) := B_r(x_0, 0) \cap \{y > 0\}$, whose boundary contains the spherical part $\partial^+ B_r^+ := \partial B_r \cap \{y > 0\}$ and the flat part $\partial^0 B_r^+ := B_r \cap \{y = 0\}$ (here y denotes the $(N + 1)$-th coordinate).

Theorem 1.1 (Local uniform Hölder bounds). *Let the functions $f_{i,\beta}$ be continuous and uniformly bounded (with respect to β) on bounded sets, and let $\{\mathbf{v}_\beta = (v_{i,\beta})_{1 \leq i \leq k}\}_\beta$ be a family of $H^1(B_1^+)$ solutions to the problems*

$$\begin{cases} -\Delta v_i = 0 & \text{in } B_1^+, \\ \partial_\nu v_i = f_{i,\beta}(v_i) - \beta v_i \sum_{j \neq i} v_j^2 & \text{on } \partial^0 B_1^+. \end{cases} \tag{P)$_\beta$}$$

Assume that

$$\|\mathbf{v}_\beta\|_{L^\infty(B_1^+)} \leq M$$

for a constant M independent of β. Then for every $\alpha \in (0, 1/2)$ there exists a constant $C = C(M, \alpha)$, not depending on β, such that

$$\|\mathbf{v}_\beta\|_{\mathcal{C}^{0,\alpha}(\overline{B_{1/2}^+})} \leq C(M, \alpha).$$

Furthermore, $\{\mathbf{v}_\beta\}_\beta$ is relatively compact in $H^1(B_{1/2}^+) \cap \mathcal{C}^{0,\alpha}(\overline{B_{1/2}^+})$ for every $\alpha < 1/2$.

As a byproduct, up to a subsequence we have convergence of the above solutions to a limiting profile, whose components are segregated on the boundary $\partial^0 B^+$. If furthermore $f_{i,\beta} \to f_i$ uniformly on compact sets, we can prove that this limiting profile satisfies

$$\begin{cases} -\Delta v_i = 0 & \text{in } B_1^+, \\ v_i \partial_\nu v_i = f_i(v_i) v_i & \text{on } \partial^0 B_1^+. \end{cases}$$

One can see that, for solutions of this type of equation, the highest possible regularity corresponds to the Hölder exponent $\alpha = 1/2$. As a matter of fact, we can prove that the limiting profiles do enjoy such optimal regularity.

Theorem 1.2 (Optimal regularity of limiting profiles). *Under the assumptions above, assume moreover that the locally Lipschitz continuous functions f_i satisfy $f_i(s) = f_i'(0)s + O(|s|^{1+\varepsilon})$ as $s \to 0$ for some $\varepsilon > 0$. Then $\mathbf{v} \in \mathcal{C}^{0,1/2}(\overline{B_{1/2}^+})$.*

Once local regularity is established, we can move from $(P)_\beta$ and deal with global problems, adding suitable boundary conditions. An example of results that we can prove is the following.

Theorem 1.3 (Global uniform Hölder bounds). *Let the functions $f_{i,\beta}$ be continuous and uniformly bounded (with respect to β) on bounded sets, and let $\{\mathbf{u}_\beta\}_\beta$ be a family of $H^{1/2}(\mathbb{R}^N)$ solutions to the problems*

$$\begin{cases} (-\Delta)^{1/2} u_i = f_{i,\beta}(u_i) - \beta u_i \sum_{j \neq i} u_j^2 & \text{on } \Omega, \\ u_i \equiv 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$, with sufficiently smooth boundary. Assume that

$$\|\mathbf{u}_\beta\|_{L^\infty(\Omega)} \leq M$$

for a constant M independent of β. Then for every $\alpha \in (0, 1/2)$ there exists a constant $C = C(M, \alpha)$, not depending on β, such that

$$\|\mathbf{u}_\beta\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^N)} \leq C(M, \alpha).$$

Analogous results hold, for instance, when the square root of the laplacian is replaced with the spectral fractional laplacian with homogeneous Dirichlet boundary conditions on bounded domains (see [4]). Moreover, note that L^∞ bounds can be derived from $H^{1/2}$ ones, once suitable restrictions are imposed on the growth rate (subcritical) of the nonlinearities and/or on the dimension N, by means of a Brezis–Kato type argument.

In order to pursue the program just illustrated, compared with the case of the standard laplacian, a number of new difficulties has to be overcome. For instance, the polynomial decay of the fundamental solution of $(-\Delta)^{1/2}+1$ already affects the rate of segregation. Furthermore, since such segregation occurs only in the N-dimensional space, it is natural to expect free boundaries of codimension 2. But, perhaps, the most challenging issue is the lack of an exact Alt–Caffarelli–Friedman monotonicity formula. This reflects, at the spectral level, the lack of convexity of the eigenvalues with respect to domain variations (see Remark 2.4 below). To attack these problems, new tools are needed, involving different extremality conditions and new monotonicity formulas (associated with trace spectral problems).

Let us finally mention that general fractional laplacians arise in many models of enhanced anomalous diffusion; such operators are of real interest both in population dynamics and in relativistic quantum electrodynamics. This strongly motivates the extension of the theory in this direction, for any $s \in (0,1)$ [27], as well as for different kinds of competitive interaction [29].

1.1. Notation

Throughout the paper, we will write any $X \in \mathbb{R}^{N+1}$ as $X=(x,y)$ with $x \in \mathbb{R}^N$ and $y \in \mathbb{R}$. We set $\mathbb{R}^{N+1}_+ := \mathbb{R}^{N+1} \cap \{y>0\}$. For any $D \subset \mathbb{R}^{N+1}$ we write

$$D^+ := D \cap \{y>0\}, \qquad \partial^+ D^+ := \partial D \cap \{y>0\}, \qquad \partial^0 D^+ := D \cap \{y=0\}.$$

In most cases, we use this notation with $D = B_r(x_0,0)$ (the $(N+1)$-dimensional ball centered at a point of $\mathbb{R}^N$). In that case, we denote

$$S_r^{N-1}(x_0,0) := \{(x,0) : x \in \mathbb{R}^N,\ |x-x_0|=r\} = \partial B_r^+ \setminus (\partial^+ B_r^+ \cup \partial^0 B_r^+).$$

Beyond the usual function spaces, we will use

$$H^1_{\mathrm{loc}}(\overline{\mathbb{R}^{N+1}_+}) := \{v : \forall D \subset \mathbb{R}^{N+1} \text{ open and bounded}, v|_{D^+} \in H^1(D^+)\}.$$

Finally, we write B^+ for B_1^+, and we denote by C any constant we need not specify (possibly assuming different values even in the same expression).

2. Alt–Caffarelli–Friedman type monotonicity formulae

This section is devoted to the proof of some monotonicity formulae of Alt–Caffarelli–Friedman (ACF) type.

2.1. Segregated ACF formula

The validity of ACF type formulae depends on optimal partition problems involving spectral properties of the domain. In the present situation, the spectral problem we consider involves a pair of functions defined on $\mathbb{S}^N_+ := \partial^+ B^+$. As a peculiar fact, here such functions do not have disjoint supports on the whole $\mathbb{S}^N_+$, but only on its boundary $\mathbb{S}^{N-1}$. In this way we are led to consider the following optimal partition problem on $\mathbb{S}^{N-1}$.

Definition 2.1. For each open subset ω of $\mathbb{S}^{N-1} := \partial\mathbb{S}^N_+$ we define the first eigenvalue associated to ω as

$$\lambda_1(\omega) := \inf\left\{ \frac{\int_{\mathbb{S}^N_+} |\nabla_T u|^2 \, d\sigma}{\int_{\mathbb{S}^N_+} u^2 \, d\sigma} : u \in H^1(\mathbb{S}^N_+),\ u \equiv 0 \text{ on } \mathbb{S}^{N-1} \setminus \omega \right\}.$$

Here $\nabla_T u$ stands for the (tangential) gradient of u on $\mathbb{S}^N_+$.

Definition 2.2. On $\mathbb{S}^{N-1}$ we define the set $\mathcal{P}^2$ of 2-partitions by

$$\mathcal{P}^2 := \{(\omega_1, \omega_2) \colon \omega_i \subset \mathbb{S}^{N-1} \text{ open},\ \omega_1 \cap \omega_2 = \emptyset\},$$

and the number, only depending on N,

$$\begin{aligned} \nu^{\mathrm{ACF}} &:= \frac{1}{2} \inf_{(\omega_1,\omega_2)\in\mathcal{P}^2} \sum_{i=1}^{2} \left(\sqrt{\left(\frac{N-1}{2}\right)^2 + \lambda_1(\omega_i)} - \frac{N-1}{2} \right) \\ &= \frac{1}{2} \inf_{(\omega_1,\omega_2)\in\mathcal{P}^2} \sum_{i=1}^{2} \gamma(\lambda_1(\omega_i)). \end{aligned}$$

Remark 2.3. As is well known, u achieves $\lambda_1(\omega)$ if and only if it is of one sign, and its $\gamma(\lambda_1(\omega))$-homogeneous extension to $\mathbb{R}^{N+1}_+$ is harmonic.

Remark 2.4. By symmetrization arguments, one may try to restrict the study of the above optimal partition problem to the case when both ω_i are spherical caps. In such a situation, writing $\Gamma(\vartheta) := \gamma(\lambda_1(\omega_\vartheta))$ for the spherical cap ω_ϑ with opening ϑ, one is led to minimize the quantity

$$\varphi(\vartheta) := \tfrac{1}{2}[\Gamma(\vartheta) + \Gamma(\pi - \vartheta)], \qquad \vartheta \in [0, \pi].$$

It is worth noticing that the function φ is not convex: indeed, one can prove that

$$\varphi(0) = \varphi(\pi/2) = \varphi(\pi) = 1/2$$

(for details, see the proofs of Lemma 2.5 and Proposition 2.12 below). Thus, in particular, it is not clear whether the minimum of φ may be strictly less than $1/2$. As already mentioned, this brings a notable difference from the standard diffusion case.

Lemma 2.5. *For every dimension N, we have $0 < \nu^{\mathrm{ACF}} \le 1/2$.*

Proof. The bound from above easily follows by comparing with the value corresponding to the partition $(\mathbb{S}^{N-1}, \emptyset)$: indeed, we have $\lambda_1(\mathbb{S}^{N-1}) = 0$, achieved by $u(x, y) \equiv 1$, and $\lambda_1(\emptyset) = N$, achieved by $u(x, y) = y$. In order to prove the estimate from below, let us first observe that, for each $(\omega_1, \omega_2) \in \mathcal{P}^2$, there exist $u_1, u_2 \in H^1(\mathbb{S}^N_+)$ such that $u_i \equiv 0$ on $\mathbb{S}^{N-1} \setminus \omega_i$ and

$$\lambda_1(\omega_i) = \int_{\mathbb{S}^N_+} |\nabla_T u_i|^2 \, d\sigma, \qquad \int_{\mathbb{S}^N_+} u_i^2 \, d\sigma = 1.$$

This is a consequence of the compactness of both the embedding $H^1(\mathbb{S}^N_+) \hookrightarrow L^2(\mathbb{S}^N_+)$ and the trace operator from $H^1(\mathbb{S}^N_+)$ to $L^2(\mathbb{S}^{N-1})$ (recall that the constraint is continuous with respect to the $L^2(\mathbb{S}^{N-1})$ topology).

Towards a contradiction, suppose that there exists a sequence of 2-partitions $(\omega_1^n, \omega_2^n) \in \mathcal{P}^2$ such that

$$\gamma(\lambda_1(\omega_1^n)) + \gamma(\lambda_1(\omega_2^n)) \to 0.$$

Since γ is nonnegative and increasing, we must have $\lambda_1(\omega_i^n) \to 0$ for $i = 1, 2$, that is, there exist $u_1^n, u_2^n \in H^1(\mathbb{S}^N_+)$ such that $u_i^n \equiv 0$ on $\mathbb{S}^{N-1} \setminus \omega_i^n$ and

$$\int_{\mathbb{S}^N_+} |\nabla_T u_i^n|^2 \, d\sigma \to 0, \quad \text{while} \quad \int_{\mathbb{S}^N_+} |u_i^n|^2 \, d\sigma = 1.$$

Therefore, up to a subsequence,

$$u_1^n, u_2^n \rightharpoonup |\mathbb{S}^N_+|^{1/2} \quad \text{in } H^1(\mathbb{S}^N_+) \quad \text{and} \quad \int_{\mathbb{S}^{N-1}} u_1^n u_2^n \, d\sigma = 0,$$

which are incompatible. □

Under the previous notation, we can prove the following monotonicity formula.

Theorem 2.6. *Let $v_1, v_2 \in H^1(B_R^+(x_0, 0))$ be continuous functions such that*

- $v_1 v_2|_{\{y=0\}} = 0$, $v_i(x_0, 0) = 0$*;*
- *for every nonnegative $\phi \in \mathcal{C}_0^\infty(B_R(x_0, 0))$,*

$$\int_{\mathbb{R}^{N+1}_+} (-\Delta v_i) v_i \phi \, dx \, dy + \int_{\mathbb{R}^N} (\partial_\nu v_i) v_i \phi \, dx = \int_{\mathbb{R}^{N+1}_+} \nabla v_i \cdot \nabla(v_i \phi) \, dx \, dy \leq 0.$$

Then the function

$$\Phi(r) := \prod_{i=1}^{2} \frac{1}{r^{2\nu^{\mathrm{ACF}}}} \int_{B_r^+(x_0,0)} \frac{|\nabla v_i|^2}{|X - (x_0, 0)|^{N-1}} \, dx \, dy$$

is nondecreasing in $r \in (0, R)$.

Remark 2.7. Since

$$\int_{\mathbb{R}^{N+1}_+} \nabla v_i \cdot \nabla(v_i\phi)\,dx\,dy = \int_{\mathbb{R}^{N+1}_+} \left[|\nabla v_i|^2\phi + \tfrac{1}{2}\nabla(v_i)^2 \cdot \nabla\phi\right] dx\,dy, \tag{2.1}$$

if v_1, v_2 satisfy the assumptions of Theorem 2.6 then so do $|v_1|, |v_2|$.

By the above remark, we can assume without loss of generality that v_1 and v_2 are nonnegative. Since the theorem is trivial if either $v_1 \equiv 0$ or $v_2 \equiv 0$, we will prove it when both v_1 and v_2 are nonzero. Moreover, by translating and scaling, the theorem can be proved under the assumption that $x_0 = 0$ and $R = 1$. We will need some technical lemmas.

Definition 2.8. We define $\Gamma_1 \in \mathcal{C}^1(\mathbb{R}^{N+1}_+; \mathbb{R}^+)$ by

$$\Gamma_1(X) := \begin{cases} \frac{1}{|X|^{N-1}}, & |X| \geq 1, \\ \frac{N+1}{2} - \frac{N-1}{2}|X|^2, & |X| < 1. \end{cases}$$

We also let $\Gamma_\varepsilon(X) = \Gamma_1(X/\varepsilon)\varepsilon^{1-N}$, so that as $\varepsilon \to 0$, $\Gamma_\varepsilon \nearrow \Gamma = |X|^{1-N}$, a multiple of the fundamental solution of the half-laplacian.

Remark 2.9. Observe that each Γ_ε is radial and in particular $\partial_\nu \Gamma_\varepsilon = 0$ on $\mathbb{R}^N$. Moreover, Γ_ε is superharmonic on $\mathbb{R}^{N+1}_+$.

Lemma 2.10. *Let v_1, v_2 be as in Theorem 2.6. Then the function*

$$r \mapsto \int_{B_r^+} \frac{|\nabla v_i|^2}{|X|^{N-1}}\,dx\,dy \tag{2.2}$$

is well defined and bounded in any compact subset of $(0, 1)$.

Proof. Let $\varepsilon, \delta > 0$ and let $\eta_\delta \in \mathcal{C}_0^\infty(B_{r+\delta})$ be a smooth, radial cut-off function such that $0 \leq \eta_\delta \leq 1$ and $\eta_\delta = 1$ on B_r. Choosing $\phi = \eta_\delta\Gamma_\varepsilon$ in the second assumption of the theorem, and recalling (2.1), we obtain

$$\begin{aligned}\int_{\mathbb{R}^{N+1}_+} \left[|\nabla v_i|^2\Gamma_\varepsilon + \tfrac{1}{2}\nabla(v_i)^2 \cdot \nabla\Gamma_\varepsilon\right]\eta_\delta\,dx\,dy &\leq -\int_{\mathbb{R}^{N+1}_+} \tfrac{1}{2}\Gamma_\varepsilon\nabla(v_i)^2 \cdot \nabla\eta_\delta\,dx\,dy \\ &= \int_r^{r+\delta}\left[-\eta_\delta'(\rho)\int_{\partial^+B_\rho^+} \Gamma_\varepsilon v_i \nabla v_i \cdot \frac{X}{|X|}\,d\sigma\right]d\rho.\end{aligned}$$

Passing to the limit as $\delta \to 0$ we get, for almost every $r \in (0, 1)$,

$$\int_{B_r^+} \left[|\nabla v_i|^2\Gamma_\varepsilon + \tfrac{1}{2}\nabla(v_i)^2 \cdot \nabla\Gamma_\varepsilon\right] dx\,dy \leq \int_{\partial^+B_r^+} \Gamma_\varepsilon v_i \partial_\nu v_i\,d\sigma,$$

which, combined with the inequality $-\Delta\Gamma_\varepsilon \geq 0$ tested with $v_i^2/2$, leads to

$$\int_{B_r^+} |\nabla v_i|^2\Gamma_\varepsilon\,dx\,dy \leq \int_{\partial^+B_r^+} \left(\Gamma_\varepsilon v_i\partial_\nu v_i - \frac{v_i^2}{2}\partial_\nu\Gamma_\varepsilon\right) d\sigma.$$

Letting $\varepsilon \to 0^+$, by monotone convergence we infer

$$\int_{B_r^+} \frac{|\nabla v_i|^2}{|X|^{N-1}}\, dx\, dy \le \frac{1}{r^{N-1}} \int_{\partial^+ B_r^+} v_i \frac{\partial v_i}{\partial \nu}\, d\sigma + \frac{N-1}{2r^N} \int_{\partial^+ B_r^+} v_i^2\, d\sigma, \tag{2.3}$$

and this proves the lemma. □

Lemma 2.11. *Let v_1, v_2 be nontrivial functions satisfying the assumptions of Theorem 2.6. Then*

$$\sum_{i=1}^{2} \frac{\int_{\partial^+ B_r^+} \frac{|\nabla v_i|^2}{|X|^{N-1}}\, d\sigma}{\int_{B_r^+} \frac{|\nabla v_i|^2}{|X|^{N-1}}\, dx\, dy} \ge \frac{4}{r} \nu^{\mathrm{ACF}}. \tag{2.4}$$

Proof. First we use the estimate (2.3) to bound the left hand side of (2.4) from below:

$$\begin{aligned} \frac{\int_{\partial^+ B_r^+} \frac{|\nabla v_i|^2}{|X|^{N-1}}\, d\sigma}{\int_{B_r^+} \frac{|\nabla v_i|^2}{|X|^{N-1}}\, dx\, dy} &\ge \frac{\int_{\partial^+ B_r^+} |\nabla v_i|^2\, d\sigma}{\int_{\partial^+ B_r^+} v_i \partial_\nu v_i\, d\sigma + (N-1)\frac{r}{2} \int_{\partial^+ B_r^+} v_i^2\, d\sigma} \\ &= \frac{1}{r} \frac{\int_{\mathbb{S}^N_+} |\nabla v_i^{(r)}|^2\, d\sigma}{\int_{\mathbb{S}^N_+} v_i^{(r)} \partial_\nu v_i^{(r)}\, d\sigma + \frac{N-1}{2} \int_{\mathbb{S}^N_+} (v_i^{(r)})^2\, d\sigma}, \end{aligned}$$

where $v_i^{(r)} : \mathbb{S}^{N-1}_+ \to \mathbb{R}$ is defined as $v_i^{(r)}(\xi) = v_i(r\xi)$. We now estimate the right hand side above as follows. The numerator of the last fraction can be written as

$$\begin{aligned} \int_{\mathbb{S}^N_+} |\nabla v_i^{(r)}|^2\, d\sigma &= \int_{\mathbb{S}^N_+} |\partial_\nu v_i^{(r)}|^2\, d\sigma + \int_{\mathbb{S}^N_+} |\nabla_T v_i^{(r)}|^2\, d\sigma \\ &= \int_{\mathbb{S}^N_+} |v_i^{(r)}|^2\, d\sigma \left(\underbrace{\frac{\int_{\mathbb{S}^N_+} |\partial_\nu v_i^{(r)}|^2\, d\sigma}{\int_{\mathbb{S}^N_+} |v_i^{(r)}|^2\, d\sigma}}_{t^2} + \underbrace{\frac{\int_{\mathbb{S}^N_+} |\nabla_T v_i^{(r)}|^2\, d\sigma}{\int_{\mathbb{S}^N_+} |v_i^{(r)}|^2\, d\sigma}}_{\mathcal{R}} \right) \end{aligned}$$

where $\mathcal{R}$ stands for the Rayleigh quotient of $v_i^{(r)}$ on $\mathbb{S}^N_+$. On the other hand, by the Cauchy–Schwarz inequality, the denominator may be estimated from above by

$$\begin{aligned} &\int_{\mathbb{S}^N_+} v_i^{(r)} \partial_\nu v_i^{(r)}\, d\sigma + \frac{N-1}{2} \int_{\mathbb{S}^N_+} |v_i^{(r)}|^2\, d\sigma \\ &\quad \le \left(\int_{\mathbb{S}^N_+} |v_i^{(r)}|^2\, d\sigma \right)^{1/2} \left(\int_{\mathbb{S}^N_+} \partial_\nu v_i^{(r)}\, d\sigma \right)^{1/2} + \frac{N-1}{2} \int_{\mathbb{S}^N_+} |v_i^{(r)}|^2\, d\sigma \\ &\quad \le \int_{\mathbb{S}^N_+} |v_i^{(r)}|^2\, d\sigma \left[\underbrace{\left(\frac{\int_{\mathbb{S}^N_+} |\partial_\nu v_i^{(r)}|^2\, d\sigma}{\int_{\mathbb{S}^N_+} |v_i^{(r)}|^2\, d\sigma} \right)^{1/2}}_{t} + \frac{N-1}{2} \right]. \end{aligned}$$

As a consequence,

$$\frac{\int_{\partial^+ B_r^+} \frac{|\nabla v_i|^2}{|X|^{N-1}}\, d\sigma}{\int_{B_r^+} \frac{|\nabla v_i|^2}{|X|^{N-1}}\, dx\, dy} \geq \frac{1}{r} \min_{t\in\mathbb{R}^+} \frac{\mathcal{R}+t^2}{t+\frac{N-1}{2}}.$$

A simple computation shows that the minimum is achieved when

$$t=\gamma(\mathcal{R})=\sqrt{\left(\frac{N-1}{2}\right)^2+\mathcal{R}}-\frac{N-1}{2},$$

and it is equal to $2\gamma(\mathcal{R})$. Summing over $i=1,2$, we obtain

$$\sum_{i=1}^{2} \frac{\int_{\partial^+ B_r^+} \frac{|\nabla v_i|^2}{|X|^{N-1}}\, d\sigma}{\int_{B_r^+} \frac{|\nabla v_i|^2}{|X|^{N-1}}\, dx\, dy} \geq \frac{2}{r} \inf_{(\omega_1,\omega_2)\in\mathcal{P}^2} \sum_{i=1}^{2} \gamma(\lambda_1(\omega_i)) = \frac{4}{r}\nu^{\mathrm{ACF}},$$

where the inequality follows by replacing each $\mathcal{R}$ with its optimal value, that is, the eigenvalue $\lambda_1(\omega_i)$. □

Proof of Theorem 2.6. As already noticed, we may assume that $x_0=0$ and $R=1$ and that both v_1 and v_2 are nontrivial and nonnegative. We start by observing that the function $\Phi(r)$ is positive and absolutely continuous for $r\in(0,1)$, since it is the product of functions which are positive and absolutely continuous in $(0,1)$. Therefore, the theorem follows once we prove that $\Phi'(r)\geq 0$ for almost every $r\in(0,1)$. A direct computation of the logarithmic derivative of Φ shows that

$$\frac{\Phi'(r)}{\Phi(r)} = -\frac{4\nu^{\mathrm{ACF}}}{r} + \sum_{i=1}^{2} \frac{\int_{\partial^+ B_r^+}(|\nabla v_i|^2/|X|^{N-1})\, d\sigma}{\int_{B_r^+}(|\nabla v_i|^2/|X|^{N-1})\, dx\, dy} \geq 0,$$

where the last inequality follows by Lemma 2.11. □

As we mentioned, Theorem 2.6 will be crucial in proving interior regularity estimates. We now provide a related result, suitable for treating regularity up to the boundary. Unlike before, in this case we can show that the optimal exponent in the corresponding monotonicity formula is exactly $\gamma=1/2$.

Proposition 2.12. *Let $v\in H^1(B_R^+)$ be a continuous function such that*

- $v_1(x,0)=0$ *for* $x_1\leq 0$*;*
- *for every nonnegative* $\phi\in\mathcal{C}_0^\infty(B_R)$*,*

$$\int_{\mathbb{R}^{N+1}_+} (-\Delta v) v\phi\, dx\, dy + \int_{\mathbb{R}^N} (\partial_\nu v) v\phi\ dx = \int_{\mathbb{R}^{N+1}_+} \nabla v\cdot\nabla(v\phi)\, dx\, dy \leq 0.$$

Then the function

$$\Phi(r) := \frac{1}{r}\int_{B_r^+} \frac{|\nabla v|^2}{|X|^{N-1}}\, dx\, dy$$

is nondecreasing in $r\in(0,R)$.

Proof. Let $\bar\omega := \mathbb{S}^{N-1} \cap \{x_1 > 0\}$, and let v denote the 1/2-homogeneous, harmonic extension of $v(x,0) = \sqrt{x_1^+}$ to $\mathbb{R}^{N+1}_+$, that is,

$$v(x,y) = \sqrt{\frac{\sqrt{x_1^2 + y^2} + x_1}{2}}.$$

Since v is positive for $y > 0$, Remark 2.3 implies that $v|_{\mathbb{S}^N_+}$ is an eigenfunction associated to $\lambda_1(\bar\omega)$, so that

$$\gamma(\lambda_1(\bar\omega)) = 1/2.$$

But then, reasoning as in the proofs of Lemma 2.11 and Theorem 2.6, we readily obtain

$$\frac{\Phi'(r)}{\Phi(r)} \ge \frac{2}{r}\left[-\frac{1}{2} + \gamma(\lambda_1(\bar\omega))\right] = 0. \qquad \square$$

2.2. *Perturbed ACF formula*

We now move from Theorem 2.6 to a perturbed version of the monotonicity formula, suitable for functions which coexist on the boundary, rather than having disjoint supports.

Theorem 2.13. *Let ν^{ACF} be as in Definition 2.2, and let $v_1, v_2 \in H^1_{\mathrm{loc}}(\overline{\mathbb{R}^{N+1}_+})$ be continuous functions such that, for every nonnegative $\phi \in C^\infty_0(\overline{\mathbb{R}^{N+1}_+})$ and $j \neq i$,*

$$\begin{aligned}\int_{\mathbb{R}^{N+1}_+} (-\Delta v_i) v_i \phi \, dx\, dy + \int_{\mathbb{R}^N} (\partial_\nu v_i + v_i v_j^2) v_i \phi \, dx \\ = \int_{\mathbb{R}^{N+1}_+} \nabla v_i \cdot \nabla (v_i \phi) \, dx\, dy + \int_{\mathbb{R}^N} v_i^2 v_j^2 \phi \, dx \le 0.\end{aligned}$$

Then for any $\nu' \in (0, \nu^{\mathrm{ACF}})$ there exists $\bar r > 1$ such that the function

$$\Phi(r) := \prod_{i=1}^{2} \Phi_i(r)$$

is nondecreasing in r for $r \in (\bar r, \infty)$, where

$$\Phi_i(r) := \frac{1}{r^{2\nu'}} \left(\int_{B_r^+} |\nabla v_i|^2 \Gamma_1 \, dx\, dy + \int_{\partial^0 B_r^+} v_i^2 v_j^2 \Gamma_1 \, dx \right) \quad \textit{with } j \neq i.$$

Remark 2.14. We observe that, analogously to Remark 2.7, the main assumption of Theorem 2.13 can be equivalently rewritten as

$$\int_{\mathbb{R}^{N+1}_+} \left[|\nabla v_i|^2 \phi + \tfrac{1}{2} \nabla (v_i)^2 \cdot \nabla \phi \right] dx\, dy + \int_{\mathbb{R}^N} v_i^2 v_j^2 \phi \, dx \le 0$$

for every compactly supported $\phi \geq 0$. In particular, if v_1, v_2 satisfy that assumption, so do $|v_1|, |v_2|$. Moreover, reasoning as in the proof of Lemma 2.10, we find that, for every $\phi \geq 0$ and almost every r,

$$\int_{B_r^+} \left[|\nabla v_i|^2 \phi + \tfrac{1}{2} \nabla (v_i)^2 \cdot \nabla \phi \right] dx\, dy + \int_{\partial^0 B_r^+} v_i^2 v_j^2 \phi\, dx \leq \int_{\partial^+ B_r^+} (\partial_\nu v_i) v_i \phi\, d\sigma. \tag{2.5}$$

The proof of Theorem 2.13 follows the lines of the one of Theorem 2.6.

Lemma 2.15. *Let v_1, v_2 be nontrivial functions satisfying the assumptions of Theorem 2.13. Then, for any $r > 1$,*

$$\sum_{i=1}^{2} \frac{\int_{\partial B_r^+} |\nabla v_i|^2 \Gamma_1\, d\sigma + \int_{r\mathbb{S}^{N-1}} v_i^2 v_j^2 \Gamma_1\, d\sigma}{\int_{B_r^+} |\nabla v_i|^2 \Gamma_1\, dx\, dy + \int_{\partial^0 B_r^+} v_i^2 v_j^2 \Gamma_1\, dx} \geq \frac{2}{r} \sum_{i=1}^{2} \gamma(\Lambda_i(r)), \tag{2.6}$$

where $j \neq i$ and

$$\Lambda_i(r) = \frac{\int_{\mathbb{S}_+^N} |\nabla_T v_i^{(r)}|^2\, d\sigma + r \int_{\mathbb{S}^{N-1}} (v_i^{(r)} v_j^{(r)})^2\, d\sigma}{\int_{\mathbb{S}_+^N} |v_i^{(r)}|^2\, d\sigma}$$

(again, $v_i^{(r)} : \mathbb{S}_+^{N-1} \to \mathbb{R}$ is such that $v_i^{(r)}(\xi) = v_i(r\xi)$).

Proof. By choosing $\phi = \Gamma_1$ (Definition 2.8) in (2.5) we obtain, for a.e. $r > 0$,

$$\int_{B_r^+} \left[|\nabla v_i|^2 \Gamma_1 + \tfrac{1}{2} \nabla (v_i)^2 \cdot \nabla \Gamma_1 \right] dx\, dy + \int_{\partial^0 B_r^+} v_i^2 v_j^2 \Gamma_1\, dx \leq \int_{\partial^+ B_r^+} v_i \partial_\nu v_i \Gamma_1\, d\sigma.$$

The superharmonicity of Γ_1 then yields

$$\int_{B_r^+} |\nabla v_i|^2 \Gamma_1\, dx\, dy + \int_{\partial^0 B_r^+} v_i^2 v_j^2 \Gamma_1\, dx \leq \int_{\partial^+ B_r^+} \left(v_i \partial_\nu v_i \Gamma_1 - \frac{v_i^2}{2} \partial_\nu \Gamma_1 \right) d\sigma.$$

Recalling that $r > 1$ we can use this estimate to bound from below the left hand side of (2.6), obtaining

$$\frac{\int_{\partial B_r^+} |\nabla v_i|^2 \Gamma_1\, d\sigma + \int_{r\mathbb{S}^{N-1}} v_i^2 v_j^2 \Gamma_1\, d\sigma}{\int_{B_r^+} |\nabla v_i|^2 \Gamma_1\, dx\, dy + \int_{\partial^0 B_r^+} v_i^2 v_j^2 \Gamma_1\, dx} \geq \frac{1}{r} \frac{\int_{\mathbb{S}_+^N} |\nabla v_i^{(r)}|^2\, d\sigma + r \int_{\mathbb{S}^{N-1}} (v_i^{(r)} v_j^{(r)})^2\, d\sigma}{\int_{\mathbb{S}_+^N} v_i^{(r)} \partial_\nu v_i^{(r)}\, d\sigma + \frac{N-1}{2} \int_{\mathbb{S}_+^N} (v_i^{(r)})^2\, d\sigma}.$$

We now estimate the right hand side above. The numerator of the last fraction reads

$$\begin{aligned}
&\int_{\mathbb{S}_+^N} |\nabla v_i^{(r)}|^2\, d\sigma + r \int_{\mathbb{S}^{N-1}} (v_i^{(r)} v_j^{(r)})^2\, d\sigma \\
&\quad = \int_{\mathbb{S}_+^N} |\partial_\nu v_i^{(r)}|^2\, d\sigma + \int_{\mathbb{S}_+^N} |\nabla_T v_i^{(r)}|^2\, d\sigma + r \int_{\mathbb{S}^{N-1}} (v_i^{(r)} v_j^{(r)})^2\, d\sigma \\
&\quad = \int_{\mathbb{S}_+^N} |v_i^{(r)}|^2\, d\sigma \left(\underbrace{\frac{\int_{\mathbb{S}_+^N} |\partial_\nu v_i^{(r)}|^2\, d\sigma}{\int_{\mathbb{S}_+^N} |v_i^{(r)}|^2\, d\sigma}}_{t^2} + \underbrace{\frac{\int_{\mathbb{S}_+^N} |\nabla_T v_i^{(r)}|^2\, d\sigma + r \int_{\mathbb{S}^{N-1}} (v_i^{(r)} v_j^{(r)})^2\, d\sigma}{\int_{\mathbb{S}_+^N} |v_i^{(r)}|^2\, d\sigma}}_{\mathcal{R}} \right).
\end{aligned}$$

We can bound the denominator as in Lemma 2.11. As a consequence,

$$\frac{\int_{\partial B_r^+} |\nabla v_i|^2 \Gamma_1 \, d\sigma + \int_{r\mathbb{S}^{N-1}} v_i^2 v_j^2 \Gamma_1 \, d\sigma}{\int_{B_r^+} |\nabla v_i|^2 \Gamma_1 \, dx\, dy + \int_{\partial^0 B_r^+} v_i^2 v_j^2 \Gamma_1 \, dx} \geq \frac{1}{r} \min_{t \in \mathbb{R}^+} \frac{\mathcal{R} + t^2}{t + \frac{N-1}{2}}.$$

Minimizing with respect to t as in Lemma 2.11 and summing over $i = 1, 2$, we obtain (2.6). □

Proof of Theorem 2.13. Without loss of generality, we assume that both v_1 and v_2 are nontrivial. As in Theorem 2.6, we will prove that the logarithmic derivative of Φ is nonnegative for any $\nu' \in (0, \nu^{\mathrm{ACF}})$ and r sufficiently large. Again, a direct computation shows that

$$\begin{aligned} \frac{\Phi'(r)}{\Phi(r)} &= -\frac{4\nu'}{r} + \sum_{i=1}^{2} \frac{\int_{\partial B_r^+} |\nabla v_i|^2 \Gamma_1 \, d\sigma + \int_{r\mathbb{S}^{N-1}} v_i^2 v_j^2 \Gamma_1 \, d\sigma}{\int_{B_r^+} |\nabla v_i|^2 \Gamma_1 \, dx\, dy + \int_{\partial^0 B_r^+} v_i^2 v_j^2 \Gamma_1 \, dx} \\ &\geq \frac{4}{r}\left[-\nu' + \frac{1}{2} \sum_{i=1}^{2} \gamma(\Lambda(v_i^{(r)})) \right], \end{aligned}$$

and thus it is sufficient to prove that there exists $\bar{r} > 1$ such that, for every $r > \bar{r}$, the last term is nonnegative. Of course if $\Lambda_i(r) \to \infty$ for some i then there is nothing to prove; thus we can suppose that each $\Lambda_i(r)$ is uniformly bounded. To begin, we observe that, for r large,

$$H(r) := \|v_i^{(r)}\|^2_{L^2(\mathbb{S}^N_+)} = \int_{\mathbb{S}^N_+} (v_i^{(r)})^2 \, d\sigma \geq C > 0. \tag{2.7}$$

Indeed, the choice of $\phi \equiv 1$ in (2.5) yields

$$H'(r) = \int_{\mathbb{S}^N_+} r \partial_\nu (v_i^2)(r\xi) \, d\sigma \geq 0,$$

and since the functions are nontrivial, H cannot be identically 0.

Suppose for contradiction that there exists a sequence $r_n \to \infty$ such that

$$\frac{1}{2} \sum_{i=1}^{2} \gamma(\Lambda_i(r_n)) \leq \nu' < \nu^{\mathrm{ACF}}. \tag{2.8}$$

We introduce the renormalized sequence

$$w_{i,n} = \frac{v_i^{(r_n)}}{(\int_{\mathbb{S}^N_+} (v_i^{(r_n)})^2 \, d\sigma)^{1/2}}, \quad \text{so that} \quad \|w_{i,n}\|_{L^2(\mathbb{S}^N_+)} = 1.$$

Recall that $\Lambda_i(r_n)$ is uniformly bounded, that is,

$$K \geq \Lambda_i(r_n) = \int_{\mathbb{S}^N_+} |\nabla_T w_{i,n}|^2 \, d\sigma + \int_{\mathbb{S}^{N-1}} r_n w_{i,n}^2 w_{j,n}^2 \|v_j^{(r_n)}\|^2_{L^2(\mathbb{S}^N_+)} \, d\sigma.$$

Together with (2.7), this yields

$$\int_{\mathbb{S}^N_+} |\nabla_T w_{i,n}|^2 \, d\sigma \le K \quad \text{and} \quad \int_{\mathbb{S}^{N-1}} w_{i,n}^2 w_{j,n}^2 \, d\sigma \le \frac{1}{r_n} K'. \tag{2.9}$$

Hence there exist $\bar{w}_i \in H^1(\mathbb{S}^N_+)$ such that, up to a subsequence, $w_{i,n_k} \rightharpoonup \bar{w}_i$ weakly in $H^1(\mathbb{S}^N_+)$, with $\|\bar{w}_i\|_{L^2(\mathbb{S}^N_+)} = 1$. Moreover, from the weak lower semicontinuity of the norm,

$$\liminf_{k\to\infty} \Lambda_i(r_{n_k}) \ge \int_{\mathbb{S}^N_+} |\nabla_T \bar{w}_i|^2 \, d\sigma \ge \lambda_1(\{\bar{w}_i|_{y=0} > 0\}).$$

From (2.9) we see that $w_i^{(r)} w_j^{(r)} \to 0$ a.e. on $\mathbb{S}^{N-1}$ and $\bar{w}_i \bar{w}_j = 0$ on $\mathbb{S}^{N-1}$. This means that the limit configuration (w_1, w_2) induces a partition of $\mathbb{S}^N_+$ for which

$$\liminf_{k\to\infty} \frac{1}{2} \sum_{i=1}^{2} \gamma(\Lambda_i(r_{n_k})) \ge \nu^{\text{ACF}},$$

in contradiction with (2.8). □

3. Almgren type monotonicity formulae

In the following, we will be concerned with a number of *entire profiles*, that is, k-tuples of functions defined on the whole $\mathbb{R}^{N+1}_+$, which will be obtained from solutions to problem $(P)_\beta$ by suitable limiting procedures. This section is devoted to the proof of some monotonicity formulae of Almgren type, related to such profiles.

3.1. Almgren's formula for segregated entire profiles

To start with, we consider k-tuples $\mathbf{v}$ having components with segregated traces on $\mathbb{R}^N$. In such a situation, on the one hand each component of $\mathbf{v}$, when different from zero, satisfies a limiting version of $(P)_\beta$, where the internal dynamics is trivialized; on the other hand, the interaction between different components is now described by a Pohozaev type identity. We recall that, in order to prove the Almgren formula, it is sufficient to require the Pohozaev identity to hold only in spherical domains. Nonetheless, we prefer to assume its validity in the broader class of cylindrical domains, that is, domains which are products of spherical and cubic domains. This choice will be useful in classifying the possible limiting profiles in the process of dimensional reduction.

More precisely, let $C^+_{r,l}(x_0, 0) \subset \mathbb{R}^{N+1}_+$ be any set such that there exists $h \in \mathbb{N}$, $h \le N$, and a decomposition $\mathbb{R}^{N+1}_+ = \mathbb{R}^{h+1}_+ \oplus \mathbb{R}^{N-h}$ such that, writing

$$\mathbb{R}^{N+1}_+ \ni X = (x', x'', y) \quad \text{with } (x', y) \in \mathbb{R}^{h+1}_+,\ x'' \in \mathbb{R}^{N-h},$$

we have

$$C^+_{r,l}(x_0, 0) = B^+_r(x'_0, 0) \times Q_l(x''_0).$$

Here, $B^+_r \subset \mathbb{R}^{h+1}_+$ denotes a half-ball of radius r, and $Q_l \subset \mathbb{R}^{N-h}$ a cube of edge length $2l$.

Definition 3.1 (Segregated entire profiles). We denote by $\mathcal{G}_s$ the set of functions $\mathbf{v} \in H^1_{\mathrm{loc}}(\overline{\mathbb{R}^{N+1}_+}; \mathbb{R}^k)$, $\mathbf{v} = (v_1, \dots, v_k)$ continuous, which satisfy the following assumptions:

(1) $v_i v_j|_{y=0} = 0$ for $j \neq i$;

(2) for every i,

$$\begin{cases} -\Delta v_i = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ v_i \partial_\nu v_i = 0 & \text{on } \mathbb{R}^N \times \{0\}; \end{cases} \tag{3.1}$$

(3) for any $x_0 \in \mathbb{R}^N$ and a.e. $r, l > 0$,

$$\begin{aligned} &\int_{C^+_{r,l}} \Big(\sum_i 2|\nabla_{(x',y)} v_i|^2 - (h+1)|\nabla v_i|^2 \Big)\, dx\, dy + r \int_{\partial^+ B^+_r \times Q_l} \sum_i |\nabla v_i|^2 \, d\sigma \\ &= 2r \int_{\partial^+ B^+_r \times Q_l} \sum_i |\partial_\nu v_i|^2 \, d\sigma - 2 \int_{B^+_r \times \partial^+ Q_l} \sum_i \partial_\nu v_i \nabla_{(x',y)} v_i \cdot (x' - x'_0, y)\, d\sigma, \end{aligned} \tag{3.2}$$

where $\nabla_{(x',y)}$ is the gradient with respect to the directions in $\mathbb{R}^{h+1}_+$.

Remark 3.2. Let $\mathbf{v} \in \mathcal{G}_s$. By choosing $h = N$ in the above definition, we obtain the spherical Pohozaev identity

$$(1-N) \int_{B^+_r} \sum_i |\nabla v_i|^2 \, dx\, dy + r \int_{\partial^+ B^+_r} \sum_i |\nabla v_i|^2 \, d\sigma = 2r \int_{\partial^+ B^+_r} \sum_i |\partial_\nu v_i|^2 \, d\sigma \tag{3.3}$$

for a.e. $r > 0$.

For every $x_0 \in \mathbb{R}^N$ and $r > 0$, define

$$E(x_0, r) := \frac{1}{r^{N-1}} \int_{B^+_r(x_0,0)} \sum_i |\nabla v_i|^2 \, dx\, dy,$$

$$H(x_0, r) := \frac{1}{r^N} \int_{\partial^+ B^+_r(x_0,0)} \sum_i v_i^2 \, d\sigma.$$

Fix x_0. Since $\mathbf{v} \in H^1_{\mathrm{loc}}(\overline{\mathbb{R}^{N+1}_+}, \mathbb{R}^k)$, both E and H are locally absolutely continuous functions on $(0, \infty)$, that is, $E', H' \in L^1_{\mathrm{loc}}(0, \infty)$ (here, $' = d/dr$).

Theorem 3.3. *Let $\mathbf{v} \in \mathcal{G}_s$, $\mathbf{v} \not\equiv 0$. For every $x_0 \in \mathbb{R}^N$ the function (Almgren frequency function)*

$$N(x_0, r) := \frac{E(x_0, r)}{H(x_0, r)}$$

is well defined on $(0, \infty)$, absolutely continuous, nondecreasing, and satisfies

$$\frac{d}{dr} \log H(r) = \frac{2N(r)}{r}. \tag{3.4}$$

Moreover, if $N(r) \equiv \gamma$ on an open interval, then $N \equiv \gamma$ for every r, and $\mathbf{v}$ is a homogeneous function of degree γ.

Proof. Up to translation, we may suppose that $x_0 = 0$. Obviously $H \geq 0$, and $H > 0$ on a nonempty interval (r_1, r_2), otherwise $\mathbf{v} \equiv 0$. As a consequence, either $\mathbf{v}$ is a nontrivial constant, and the theorem easily follows; or, by harmonicity, $\mathbf{v}$ is not constant in the whole $B^+_{r_2}$, and also $E > 0$ for $r < r_2$. By passing to the logarithmic derivatives, the monotonicity of N will be a consequence of the claim

$$\frac{N'(r)}{N(r)} = \frac{E'(r)}{E(r)} - \frac{H'(r)}{H(r)} \geq 0 \quad \text{for } r \in (r_1, r_2).$$

Differentiating E and using the Pohozaev identity (3.3), we obtain

$$\begin{aligned} E'(r) &= \frac{1-N}{r^N} \int_{B_r^+} \sum_i |\nabla v_i|^2 \, dx\, dy + \frac{1}{r^{N-1}} \int_{\partial^+ B_r^+} \sum_i |\nabla v_i|^2 \, d\sigma \\ &= \frac{2}{r^{N-1}} \int_{\partial_+ B_r^+} \sum_i |\partial_\nu v_i|^2 \, d\sigma, \end{aligned}$$

while testing (3.1) with v_i in B_r^+ and summing over i yields

$$E(r) = \frac{1}{r^{N-1}} \int_{B_r^+} \sum_i |\nabla v_i|^2 \, dx\, dy = \frac{1}{r^{N-1}} \int_{\partial^+ B_r^+} \sum_i v_i \partial_\nu v_i \, d\sigma.$$

As far as H is concerned, we find

$$H'(r) = \frac{2}{r^N} \int_{\partial^+ B_r^+} \sum_i v_i \partial_\nu v_i \, d\sigma.$$

As a consequence, by the Cauchy–Schwarz inequality,

$$\frac{1}{2} \frac{N'(r)}{N(r)} = \frac{\int_{\partial^+ B_r^+} \sum_i |\partial_\nu v_i|^2 \, d\sigma}{\int_{\partial^+ B_r^+} \sum_i v_i \partial_\nu v_i \, d\sigma} - \frac{\int_{\partial^+ B_r^+} \sum_i v_i \partial_\nu v_i \, d\sigma}{\int_{\partial^+ B_r^+} \sum_i v_i^2 \, d\sigma} \geq 0 \quad \text{for } r \in (r_1, r_2). \tag{3.5}$$

Moreover, on the same interval,

$$\frac{d}{dr} \log H(r) = \frac{H'(r)}{H(r)} = \frac{2E(r)}{rH(r)} = \frac{2N(r)}{r}.$$

Let us show that we can choose $r_1 = 0$, $r_2 = \infty$. On the one hand, the above equation implies that if $\log H(\bar{r}) > -\infty$, then $\log H(r) > -\infty$ for every $r > \bar{r}$, so that $r_2 = \infty$. On the other hand, assume for contradiction that

$$r_1 := \inf\{r : H(r) > 0 \text{ on } (r, \infty)\} > 0.$$

By monotonicity, we have $N(r) < N(2r_1)$ for every $r_1 < r \leq 2r_1$. It follows that

$$\frac{d}{dr} \log H(r) \leq \frac{2N(2r_1)}{r}, \quad \text{so} \quad \frac{H(2r_1)}{H(r)} \leq \left(\frac{2r_1}{r}\right)^{2N(2r_1)},$$

and since H is continuous, $H(r_1) > 0$, a contradiction.

Now, assume $N(r) \equiv \gamma$ on some interval I. Recalling (3.5), we see that

$$\left(\int_{\partial^+ B_r^+} \sum_i v_i \partial_\nu v_i \, d\sigma\right)^2 = \int_{\partial^+ B_r^+} \sum_i v_i^2 \, d\sigma \int_{\partial^+ B_r^+} \sum_i |\partial_\nu v_i|^2 \, d\sigma,$$

which is true, by the Cauchy–Schwarz inequality, if and only if $\mathbf{v}$ and $\partial_\nu \mathbf{v}$ are parallel, that is,

$$v_i = \lambda(r)\partial_\nu v_i = \frac{\lambda(r)}{r} X \cdot \nabla v_i \qquad \text{for every } r \in I.$$

Using the definition of N, we obtain $\gamma = r/\lambda(r)$ for every $r \in I$, so that

$$\gamma v_i = X \cdot \nabla v_i \qquad \forall i = 1, \dots, k.$$

But this is the Euler equation for homogeneous functions, and it implies that $\mathbf{v}$ is homogeneous of degree γ. Since each v_i is also harmonic in $\mathbb{R}^{N+1}_+$, the homogeneity extends to the whole of $\mathbb{R}^{N+1}_+$, yielding $N(r) \equiv \gamma$ for every $r > 0$. □

In a standard way, from Theorem 3.3 we infer that the growth properties of the elements of $\mathcal{G}_s$ are related to their Almgren quotient.

Lemma 3.4. *Let $\mathbf{v} \in \mathcal{G}_s$, and let γ, $\bar{r}$ and C denote positive constants.*

(1) *If $|\mathbf{v}(X)| \leq C|X - (x_0, 0)|^\gamma$ for every $X \notin B^+_{\bar{r}}(x_0, 0)$, then $N(x_0, r) \leq \gamma$ for every $r > 0$.*
(2) *If $|\mathbf{v}(X)| \leq C|X - (x_0, 0)|^\gamma$ for every $X \in B^+_{\bar{r}}(x_0, 0)$, then $N(x_0, r) \geq \gamma$ for every $r > 0$.*

Proof. Let $\mathbf{v} \in \mathcal{G}_s$, and assume the growth condition holds for $r \geq \bar{r}$. Then, for r large, $H(r) \leq Cr^{2\gamma}$. Towards a contradiction, suppose that there exists $R > \bar{r}$ such that $N(x_0, R) \geq \gamma + \varepsilon$. By monotonicity of N we have

$$\frac{d}{dr} \log H(r) \geq \frac{2}{r}(\gamma + \varepsilon) \qquad \forall r \geq R,$$

and integrating over (R, r) we find

$$Cr^{2(\gamma+\varepsilon)} \leq H(r) \leq Cr^{2\gamma},$$

a contradiction for r large enough. On the other hand, if the growth condition holds for $r < \bar{r}$, we can argue in an analogous way, assuming that

$$\frac{d}{dr} \log H(r) \leq \frac{2(\gamma - \varepsilon)}{r}$$

for r small enough and again obtaining a contradiction. □

Corollary 3.5. *If* $\mathbf{v} \in \mathcal{G}_s$ *is globally Hölder continuous with exponent* γ *on* $\mathbb{R}^{N+1}_+$*, then it is homogeneous of degree* γ *with respect to any of its (possible) zeroes, and*

$$\mathcal{Z} := \{x \in \mathbb{R}^N : \mathbf{v}(x,0) = 0\} \quad \text{is an affine subspace of } \mathbb{R}^N.$$

Furthermore, if $\gamma < 1$*, then*

$$\mathcal{Z} = \emptyset \iff \mathbf{v} \text{ is a (nonzero) constant.}$$

Proof. On the one hand, if $(x_0, 0) \in \mathcal{Z}$, then Lemma 3.4 implies $N(x_0, r) = \gamma$ for every r, and the first part easily follows. On the other hand, suppose $\mathcal{Z} = \emptyset$. By continuity, up to relabeling, we have $v_1(x, 0) = \cdots = v_{k-1}(x, 0) = 0$ on $\mathbb{R}^N$, so that their odd extensions across $\{y = 0\}$ are harmonic and globally Hölder continuous with exponent $\gamma < 1$ on the whole of $\mathbb{R}^{N+1}$; but then the classical Liouville theorem implies that they are all trivial. Finally, by continuity, $v_k(x, 0)$ is always different from zero, so that $\partial_\nu v_k(x, 0) \equiv 0$ on $\mathbb{R}^N$. As a consequence, the Liouville theorem applies also to the even extension of v_k across $\{y = 0\}$, concluding the proof. □

Remark 3.6. We observe that $\mathbf{v} = (1, y, 0, \ldots, 0)$ belongs to $\mathcal{G}_s$ and it is globally Lipschitz continuous, but not homogeneous. This does not contradict Corollary 3.5, as the zero set is empty.

To conclude this section, we observe that the monotonicity of $N(x, r)$ implies that for both r small and r large the corresponding limits are well defined.

Lemma 3.7. *Let* $\mathbf{v} \in \mathcal{G}_s$*. Then*

(1) $N(x, 0^+)$ *is a nonnegative upper semicontinuous function on* $\mathbb{R}^N$*;*
(2) $N(x, \infty)$ *is constant (possibly* ∞*).*

Proof. The first assertion follows because $N(x, 0^+)$ is the infimum of continuous functions. On the other hand, let

$$\nu := \lim_{r\to\infty} N(0, r) > 0;$$

we prove the second assertion for $\nu < \infty$, the other case following with minor changes. Towards a contradiction, suppose that there exists $x_0 \in \mathbb{R}^N$ such that $\sup_{r>0} N(x_0, r) = \nu - 2\varepsilon$ for some $\varepsilon > 0$. Let $r_0 > 0$ be such that $N(0, r_0) \geq \nu - \varepsilon$. Reasoning as in the proof of Lemma 3.4 we see that when R_1, R_2 are sufficiently large, both $H(x_0, R_1) \leq CR_1^{2(\nu-2\varepsilon)}$ and $H(0, R_2) \geq CR_2^{2(\nu-\varepsilon)}$. By definition

$$\int_{B^+_{R_1}(x_0,0)\setminus B^+_{r_0}(x_0,0)} \sum_i v_i^2 \, dx\, dy = \int_{r_0}^{R_1} H(x_0, s) s^N \, ds \leq CR_1^{N+2(\nu-2\varepsilon)}$$

and

$$\int_{B^+_{R_2}(0,0)\setminus B^+_{r_0}(0,0)} \sum_i v_i^2 \, dx\, dy = \int_{r_0}^{R_2} H(0, s) s^N \, ds \geq CR_2^{N+2(\nu-\varepsilon)}.$$

Now, if we let $R_1 = R_2 + |x_0|$, we obtain

$$
\begin{aligned}
C R_2^{N+2(\nu-\varepsilon)} &\le \int_{B^+_{R_2}(0,0)\setminus B^+_{r_0}(0,0)} \sum_i v_i^2 \, dx\, dy \\
&\le \int_{B^+_{r_0}(x_0,0)} \sum_i v_i^2 \, dx\, dy - \int_{B^+_{r_0}(0,0)} \sum_i v_i^2 \, dx\, dy + \int_{B^+_{R_1}(x_0,0)\setminus B^+_{r_0}(x_0,0)} \sum_i v_i^2 \, dx\, dy \\
&\le C + C'(R_2 + |x_0|)^{N+2(\nu-2\varepsilon)},
\end{aligned}
$$

and we find a contradiction for R_2 sufficiently large. Exchanging the roles of 0 and x_0 we can conclude the proof. □

3.2. Almgren's formula for coexisting entire profiles

We now turn our attention to the case where $\mathbf{v}$ is a k-tuple of functions which a priori are not segregated, but satisfy a boundary equation on $\mathbb{R}^N$. In this setting, the validity of the Pohozaev identities is a consequence of the boundary equation.

Definition 3.8 (Coexisting entire profiles). We denote by $\mathcal{G}_c$ the set of functions $\mathbf{v} \in H^1_{\text{loc}}(\overline{\mathbb{R}^{N+1}_+})$ which are solutions to

$$
\begin{cases}
-\Delta v_i = 0 & \text{in } \mathbb{R}^{N+1}_+, \\
\partial_\nu v_i + v_i \sum_{j\neq i} v_j^2 = 0 & \text{on } \mathbb{R}^N \times \{0\},
\end{cases}
\tag{3.6}
$$

for every $i = 1, \dots, k$.

Remark 3.9. Of course, if $\mathbf{v} \in H^1_{\text{loc}}(\overline{\mathbb{R}^{N+1}_+})$ solves

$$
\begin{cases}
-\Delta v_i = 0 & \text{in } \mathbb{R}^{N+1}_+, \\
\partial_\nu v_i + \beta v_i \sum_{j\neq i} v_j^2 = 0 & \text{on } \mathbb{R}^N \times \{0\},
\end{cases}
$$

for some $\beta > 0$, then a suitable multiple of $\mathbf{v}$ belongs to $\mathcal{G}_s$.

Lemma 3.10. *Let* $\mathbf{v} \in \mathcal{G}_c$. *For any* $x_0 \in \mathbb{R}^N$ *and* $r > 0$,

$$
\begin{aligned}
(1-N)\int_{B_r^+} \sum_i |\nabla v_i|^2 \, dx\, dy + r \int_{\partial^+ B_r^+} \sum_i |\nabla v_i|^2 \, d\sigma - N \int_{\partial^0 B_r^+} \sum_{i,\, j<i} v_i^2 v_j^2 \, dx \\
+ r \int_{S_r^{N-1}} \sum_{i,\, j<i} v_i^2 v_j^2 \, d\sigma = 2r \int_{\partial^+ B_r^+} \sum_i |\partial_\nu v_i|^2 \, d\sigma.
\end{aligned}
$$

Proof. This follows by testing (3.6) with $\nabla v_i \cdot X$ in B_r^+ and exploiting some standard integral identities (see also Lemma 5.2 for a similar proof in a more general case). □

As before, we set

$$E(x_0, r) := \frac{1}{r^{N-1}} \int_{B_r^+(x_0,0)} \sum_i |\nabla v_i|^2 \,dx\,dy + \frac{1}{r^{N-1}} \int_{\partial^0 B_r^+(x_0,0)} \sum_{i,\, j<i} v_i^2 v_j^2 \,dx,$$

$$H(x_0, r) := \frac{1}{r^N} \int_{\partial^+ B_r^+(x_0,0)} \sum_i v_i^2 \,d\sigma.$$

Theorem 3.11. *Let* $\mathbf{v} \in \mathcal{G}_c$*. For every* $x_0 \in \mathbb{R}^N$ *the function*

$$N(x_0, r) := \frac{E(x_0, r)}{H(x_0, r)}$$

is nondecreasing, absolutely continuous and strictly positive for $r > 0$*. Moreover,*

$$\frac{d}{dr} \log H(r) \geq \frac{2N(r)}{r}.$$

Proof. The proof runs exactly as the one of Theorem 3.3, by using Lemma 3.10 instead of equation (3.3). □

As in the case of segregated profiles, we can state a first consequence of Theorem 3.11.

Lemma 3.12. *Let* $\mathbf{v} \in \mathcal{G}_c$*, and let* γ *and* C *denote positive constants. If* $|\mathbf{v}(X)| \leq C(1 + |X|^\gamma)$ *for every* X*, then* $N(x, \infty)$ *is constant and less than* γ*.*

Proof. Argue as in the proofs of Lemmas 3.4 and 3.7. □

4. Liouville type theorems

By combining the results obtained in Sections 2 and 3, we are in a position to prove that nontrivial entire profiles, both segregated and coexisting, exhibit a minimal rate of growth connected with the Alt–Caffarelli–Friedman exponent ν^{ACF}. To be precise, the result concerning coexisting profiles only relies on the arguments developed in Section 2.

Proposition 4.1. *Let* $\mathbf{v} \in \mathcal{G}_c$ *and* ν^{ACF} *be defined according to Definitions 3.8 and 2.2. If for some* $\gamma \in (0, \nu^{\mathrm{ACF}})$ *there exists* C *such that*

$$|\mathbf{v}(X)| \leq C(1 + |X|^\gamma)$$

for every X*, then some* $k - 1$ *components of* $\mathbf{v}$ *vanish and the remaining one is constant.*

Proof. We start by proving that only one component of $\mathbf{v}$ can be different from zero. Suppose that two components, say v_1 and v_2, are nontrivial. Then $|v_1|$, $|v_2|$ fit in the setting of Theorem 2.13 (recall Remark 2.14). Let r be suitably large, and let η be a nonnegative, smooth and radial cut-off function supported in B_{2r}^+ with $\eta = 1$ in B_r^+ and $|\nabla \eta| \leq Cr^{-1}$, $|\Delta \eta| \leq Cr^{-2}$. Moreover, let Γ_1 be as in Definition 2.8 (in particular, it is radial and superharmonic). Testing the equation for v_i with $\Gamma_1 v_i \eta$ we obtain

$$\int_{B_{2r}^+} |\nabla v_i|^2 \Gamma_1 \eta \,dx\,dy + \int_{\partial^0 B_{2r}^+} v_i^2 v_j^2 \Gamma_1 \eta \,dx \leq \int_{B_{2r}^+ \setminus B_r^+} \tfrac{1}{2} v_i^2 [\Gamma_1 \Delta \eta + 2\nabla \eta \cdot \nabla \Gamma_1] \,dx\,dy,$$

where in the last step we have used the fact that η is constant in B_r^+. Since $\Gamma_1(X) = |X|^{1-N}$ outside B_1, and $|v_i(X)| \leq Cr^\gamma$ outside a suitable $B_{\bar{r}}$, using the notation of Theorem 2.13 we infer

$$\Phi_i(r) = \frac{1}{r^{2\nu'}}\left(\int_{B_r^+} |\nabla v_i|^2 \Gamma_1 \, dx\, dy + \int_{\partial^0 B_r^+} v_i^2 v_j^2 \Gamma_1 \, dx\right) \leq \frac{1}{r^{2\nu'}} \cdot Cr^{2\gamma},$$

with C independent of $r > \bar{r}$. If we fix $\gamma < \nu' < \nu^{\mathrm{ACF}}$ and possibly take $\bar{r}$ larger, Theorem 2.13 states that

$$0 < \Phi(\bar{r}) \leq \Phi(r) = \prod_{i=1}^{2} \Phi_i(r) \leq Cr^{4(\gamma-\nu')},$$

a contradiction for r large enough. Finally, if v_1 is the unique nontrivial component of $\mathbf{v}$, an even extension of v_1 through $\mathbb{R}^N$ is harmonic in $\mathbb{R}^{N+1}$ and bounded everywhere by a function growing less than linearly, implying that v_1 is constant. □

For segregated entire profiles, the results of Section 3 become crucial.

Proposition 4.2. *Let $\mathbf{v} \in \mathcal{G}_s$ and ν^{ACF} be defined according to Definitions 3.1 and 2.2.*

(1) *If for some $\gamma \in (0, \nu^{\mathrm{ACF}})$ there exists C such that*

$$|\mathbf{v}(X)| \leq C(1 + |X|^\gamma)$$

for every X, then $k-1$ components of $\mathbf{v}$ vanish.

(2) *If furthermore $\mathbf{v} \in \mathcal{C}^{0,\gamma}(\overline{\mathbb{R}^{N+1}_+})$ then the only possibly nontrivial component is constant.*

Remark 4.3. We notice that the uniform Hölder continuity of exponent γ required in (2) readily implies the growth condition in (1), which we need not require explicitly. On the other hand, from the proof it will be clear that, once $k-1$ components vanish, (2) follows by assuming uniform Hölder continuity of *any* exponent $\gamma' \in (0, 1)$, not necessarily related to ν^{ACF}.

Proof of Proposition 4.2. To prove (1), we start as above by assuming that there exist two components, v_1 and v_2, which are nontrivial. We deduce that they must have a common zero on $\mathbb{R}^N$. As a consequence, we can reason as in the proof of Proposition 4.1, using Theorem 2.6 (and Remark 2.7) instead of Theorem 2.13, and obtain a contradiction.

Turning to (2), let v denote the only nontrivial component. By Corollary 3.5, the set

$$\mathcal{Z} = \{x \in \mathbb{R}^N : v(x, 0) = 0\}$$

is an affine subspace of $\mathbb{R}^N$. Now, if $\mathcal{Z} = \mathbb{R}^N$, then v satisfies

$$\begin{cases} -\Delta v = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ v = 0 & \text{on } \mathbb{R}^N, \end{cases}$$

so that the odd extension of v through $\{y = 0\}$ is harmonic in $\mathbb{R}^{N+1}$ and bounded everywhere by a function growing less than linearly, implying that v is constant. On the other hand, if $\dim \mathcal{Z} \le N-1$, then

$$\begin{cases} -\Delta v = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \partial_\nu v = 0 & \text{on } \mathbb{R}^N \setminus \mathcal{Z}, \end{cases}$$

and the even reflection of v through $\{y = 0\}$ is harmonic in $\mathbb{R}^{N+1} \setminus \mathcal{Z}$; since $\mathcal{Z}$ has null capacity with respect to $\mathbb{R}^{N+1}$, we infer that v is actually harmonic in $\mathbb{R}^{N+1}$, and the conclusion follows again since, by assumption, v is bounded everywhere by a function growing less than linearly. □

In the same spirit of the previous theorems, we now provide a result concerning single functions, rather than k-tuples.

Proposition 4.4. *Let $v \in H^1_{\mathrm{loc}}(\overline{\mathbb{R}^{N+1}_+})$ be continuous and satisfy*

$$\begin{cases} -\Delta v = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ v\partial_\nu v \le 0 & \text{on } \mathbb{R}^N, \\ v(x,0) = 0 & \text{on } \{x_1 \le 0\}, \end{cases}$$

and suppose that for some $\gamma \in [0, 1/2)$ and $C > 0$,

$$|v(X)| \le C(1 + |X|^\gamma)$$

for every X. Then v is constant.

Proof. It is trivial to check that v as above fulfills the assumptions of Proposition 2.12. Now, assuming that v is not constant, we can argue as in the proof of Proposition 4.1 to obtain a contradiction. □

To conclude the section, we provide another two theorems of Liouville type concerning single functions. The first one relies on the construction of a supersolution of a suitable problem, as in the following lemma.

Lemma 4.5. *Let $M, \delta > 0$ be fixed and let $h \in L^\infty(\partial^0 B_1^+)$ with $\|h\|_{L^\infty} \le \delta$. Any nonnegative solution $v \in H^1(B_1^+) \cap \mathcal{C}(\overline{B_1^+})$ to*

$$\begin{cases} -\Delta v \le 0 & \text{in } B_1^+, \\ \partial_\nu v \le -Mv + h & \text{on } \partial^0 B_1^+, \end{cases}$$

satisfies

$$\sup_{\partial^0 B_{1/2}^+} v \le \frac{1+\delta}{M} \sup_{\partial^+ B_1^+} v.$$

Proof. This follows by a simple comparison argument, once one notices that, for any $\delta > 0$, the function

$$w_\delta := \delta\frac{1}{M} + \frac{1}{N}\sum_{i=1}^N \frac{2}{\pi}\left[\frac{\pi}{2} - \arctan\left(\frac{x_i + 1}{y + 2/M}\right) + \frac{\pi}{2} - \arctan\left(\frac{1 - x_i}{y + 2/M}\right)\right]$$

satisfies

$$\begin{cases} -\Delta w_\delta = 0 & \text{in } B_1^+, \\ \partial_\nu w_\delta \geq -M w_\delta + \delta & \text{on } \partial^0 B_1^+, \\ w_\delta \geq 1 & \text{on } \partial^+ B_1^+, \\ w_\delta \leq (1+\delta)/M & \text{in } \partial^0 B_{1/2}^+. \end{cases}$$

For the reader's convenience, we sketch the argument in the case $N = 1$, $\delta = 0$.

For notational convenience, denote w_M by w. By a straightforward computation, w is positive and harmonic in $\mathbb{R}^2_+$. Using the elementary inequality $\frac{\pi}{2} - \arctan t \geq \frac{1}{1+t}$ for all $t \geq 0$, we can estimate

$$w(x,0) \geq \frac{2}{\pi}\left[\frac{1}{1+\frac{M}{2}(x+1)} + \frac{1}{1+\frac{M}{2}(1-x)}\right].$$

On the other hand, from $\frac{t}{1+t^2} \leq \frac{2}{1+t}$ for $t \geq 0$, we have

$$w_y(x,0) \leq \frac{2}{\pi} M\left[\frac{1}{1+\frac{M}{2}(x+1)} + \frac{1}{1+\frac{M}{2}(1-x)}\right].$$

Therefore, $\partial_\nu w(x,0) = -w_y(x,0) \geq -M w(x,0)$. For $(x,y) \in \overline{B_1^+}$ we have

$$\arctan\left(\frac{x+1}{y+2/M}\right) + \arctan\left(\frac{1-x}{y+2/M}\right) \leq \frac{\pi}{2},$$

that is, $w(x,y) \geq 1$ in B_1^+. Finally, we observe that $w(x,0)$, as a function of x, is strictly convex and even in $(-1,1)$. Consequently, if $|x| \leq 1/2$, using the elementary inequality $\pi/2 - \arctan t \leq 1/t$ for $t \geq 0$, we obtain $w(x,0) \leq 1/M$. □

Remark 4.6. One of the peculiar difficulties in dealing with fractional operators as compared with the standard local case is due to the slow decay of supersolutions. Indeed, in the pure laplacian case, it is well known that positive solutions of

$$-\Delta u \leq -Mu \quad \text{in } B \subset \mathbb{R}^N$$

exhibit exponential decay, that is, $u|_{B_{1/2}} \leq e^{-\frac{1}{2}\sqrt{M}} \sup_{\partial B} u$; see, for instance, [16, 22]. In contrast, in the previous lemma we proved that nonnegative solutions of

$$(-\Delta)^{1/2} u \leq -Mu \quad \text{in } B \subset \mathbb{R}^N$$

exhibit only polynomial decay, that is, $u|_{B_{1/2}} \leq \frac{1}{M} \sup_{\mathbb{R}^N \setminus B} u$. This estimate is sharp, since

$$\begin{cases} -\Delta v = 0 & \text{in } B^+, \\ v \geq 0 & \text{in } B^+, \\ \partial_\nu v = -Mv & \text{on } \partial^0 B^+ \end{cases}$$

implies

$$\inf_{\partial^0 B_{1/2}^+} v \geq \frac{1}{1+M} \inf_{\partial^+ B^+} v.$$

This follows by comparing v and the subsolution $w = \frac{1}{1+M}(1+My)\inf_{\partial^+ B^+} v$.

The previous estimate allows us to prove the following.

Proposition 4.7. *Let v satisfy*

$$\begin{cases} -\Delta v = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \partial_\nu v = -\lambda v & \text{on } \mathbb{R}^N, \end{cases}$$

for some $\lambda \geq 0$, and suppose that for some $\gamma \in [0, 1)$ and $C > 0$,

$$|v(X)| \leq C(1 + |X|^\gamma)$$

for every X. Then v is constant.

Proof. If $\lambda = 0$, then using even reflection through $\{y = 0\}$, we extend v to a harmonic function in all $\mathbb{R}^{N+1}$, and we conclude the proof as usual using the growth assumption. If $\lambda > 0$, let either $z = v^+$ or $z = v^-$. In both cases,

$$\begin{cases} -\Delta z \leq 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \partial_\nu z \leq -\lambda z & \text{on } \mathbb{R}^N. \end{cases}$$

By translating and scaling, Lemma 4.5 implies that

$$z(x_0, 0) \leq \sup_{\partial^0 B_{r/2}(x_0,0)} z \leq \frac{1}{\lambda r} \sup_{\partial^+ B_r(x_0,0)} z \leq C \frac{1 + r^\gamma}{r}.$$

Letting $r \to \infty$ yields the conclusion. □

Proposition 4.8. *Let v satisfy*

$$\begin{cases} -\Delta v = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \partial_\nu v = \lambda & \text{on } \mathbb{R}^N, \end{cases}$$

for some $\lambda \in \mathbb{R}$, and suppose that for some $\gamma \in [0, 1)$ and $C > 0$,

$$|v(X)| \leq C(1 + |X|^\gamma)$$

for every X. Then v is constant.

Proof. For $h \in \mathbb{R}^N$, let $w(x, y) := v(x + h, y) - v(x, y)$. Then w solves

$$\begin{cases} -\Delta w = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \partial_\nu w = 0 & \text{on } \mathbb{R}^N, \end{cases}$$

and, as usual, we can reflect and use the growth condition to infer that w has to be constant, that is, $v(x + h, y) = c_h + v(x, y)$. Differentiating this expression in x_i, we find that

$$v(x, y) = \sum_{i=1}^{k} c_i(y) x_i + c_0(y).$$

Using again the growth condition, we see that $c_i \equiv 0$ for $i = 1, \ldots, k$, while c_0 is constant. Consequently, $\lambda = 0$. □

5. Some approximation results

In the following, we want to apply the Liouville type theorems obtained in the previous section to suitable limiting profiles, obtained from solutions to the problem

$$\begin{cases} -\Delta v_i = 0 & \text{in } B^+, \\ \partial_\nu v_i = f_{i,\beta}(v_i) - \beta v_i \sum_{j\neq i} v_j^2 & \text{on } \partial^0 B^+, \end{cases} \qquad (P)_\beta$$

through some blow-up and blow-down procedures. From this point of view we have seen that, in the case of segregated entire profiles, the key property is the validity of some Pohozaev identities, which imply that the Almgren formula holds. In this section we prove that such identities can be obtained by passing to the limit in the corresponding identities for $(P)_\beta$, under suitable assumptions about the convergence. To be more precise, we will prove the following.

Proposition 5.1. *Let* $\mathbf{v}_n \in H^1(B^+_{r_n})$ *solve problem* $(P)_{\beta_n}$ *on* $B^+_{r_n}$, $n \in \mathbb{N}$, *and let* $\mathbf{v} \in H^1_{\text{loc}}(\overline{\mathbb{R}^{N+1}_+})$ *be such that, as* $n \to \infty$,

(1) $\beta_n \to \infty$;
(2) $r_n \to \infty$;
(3) *for every compact* $K \subset \mathbb{R}^{N+1}_+$, $\mathbf{v}_n \to \mathbf{v}$ *in* $H^1(K) \cap C(K)$;
(4) *the continuous functions* f_{i,β_n} *are such that, for every* $\bar{m} > 0$,

$$|f_{i,\beta_n}(s)| \leq C_n(\bar{m}) \quad \text{for } |s| < \bar{m},$$

where $C_n(\bar{m}) \to 0$.

Then $\mathbf{v} \in \mathcal{G}_s$.

We start by stating the basic identities for problem $(P)_\beta$. We recall that S_r^{N-1} denotes the $(N-1)$-dimensional boundary of $\partial^0 B_r^+$ in $\mathbb{R}^N$.

Lemma 5.2 (Pohozaev identity). *Let* $\mathbf{v} \in H^1(B^+)$ *solve problem* $(P)_\beta$ *on* B^+. *For every* $B_r^+ := B_r^+(x_0, 0) \subset B^+$ *the following Pohozaev identity holds:*

$$\begin{aligned}(1-N)\int_{B_r^+}\sum_i |\nabla v_i|^2\,dx\,dy + r\int_{\partial^+ B_r^+}\sum_i |\nabla v_i|^2\,d\sigma \\ + 2N\int_{\partial^0 B_r^+}\sum_i F_{i,\beta}(v_i)\,dx - N\beta\int_{\partial^0 B_r^+}\sum_{i,\,j<i} v_i^2 v_j^2\,dx - 2r\int_{S_r^{N-1}}\sum_i F_{i,\beta}(v_i)\,d\sigma \\ + r\beta\int_{S_r^{N-1}}\sum_{i,\,j<i} v_i^2 v_j^2\,d\sigma = 2r\int_{\partial^+ B_r^+}\sum_i |\partial_\nu v_i|^2\,d\sigma.\end{aligned}$$

Proof. Let the functions v_i solve problem $(P)_\beta$. Up to translation we assume that $x_0 = 0$. By multiplying the equation by $X \cdot \nabla v_i$ and integrating by parts over B_r^+, we obtain

$$\int_{B_r^+} \nabla v_i \cdot \nabla(X \cdot \nabla v_i)\,dx\,dy = r\int_{\partial^+ B_r^+} |\partial_\nu v_i|^2\,d\sigma + \int_{\partial^0 B_r^+} (\partial_\nu v_i)(x \cdot \nabla_x v_i)\,dx.$$

Using the identity

$$\nabla v_i \cdot \nabla(X \cdot \nabla v_i) = |\nabla v_i|^2 + X \cdot \nabla\left(\tfrac{1}{2}|\nabla v_i|^2\right)$$

and again integrating by parts, we can write the right hand side as

$$\int_{B_r^+} \nabla v_i \cdot \nabla(X \cdot \nabla v_i)\, dx\, dy = \frac{1-N}{2} \int_{B_r^+} |\nabla v_i|^2\, dx\, dy + \frac{r}{2} \int_{\partial^+ B_r^+} |\nabla v_i|^2\, d\sigma,$$

and this yields

$$\frac{1-N}{2} \int_{B_r^+} |\nabla v_i|^2\, dx\, dy + \frac{r}{2} \int_{\partial^+ B_r^+} |\nabla v_i|^2\, d\sigma - \int_{\partial^0 B_r^+} f_{i,\beta}(v_i)(x \cdot \nabla_x v_i)\, dx \\ + \frac{\beta}{2} \int_{\partial^0 B_r^+} (x \cdot \nabla_x v_i^2) \sum_{j \neq i} v_j^2\, dx = r \int_{\partial^+ B_r^+} |\partial_\nu v_i|^2\, d\sigma.$$

Summing the identities for $i = 1, \ldots, k$ we obtain

$$\frac{1-N}{2} \int_{B_r^+} \sum_i |\nabla v_i|^2\, dx\, dy + \frac{r}{2} \int_{\partial^+ B_r^+} \sum_i |\nabla v_i|^2\, d\sigma \\ - \int_{\partial^0 B_r^+} (x \cdot \nabla_x) \sum_i F_{i,\beta}(v_i)\, dx + \frac{\beta}{2} \int_{\partial^0 B_r^+} (x \cdot \nabla_x) \sum_{i,\, j<i} v_i^2 v_j^2\, dx = r \int_{\partial^+ B_r^+} \sum_i |\partial_\nu v_i|^2\, d\sigma. \tag{5.1}$$

The integrals over $\partial^0 B_r^+$ can be further simplified: by an application of the divergence theorem on $\mathbb{R}^N$ we have

$$\int_{\partial^0 B_r^+} (x \cdot \nabla_x) \sum_{i,\, j<i} v_i^2 v_j^2\, dx = \int_{\partial^0 B_r^+} \operatorname{div}\Big(x \sum_{i,\, j<i} v_i^2 v_j^2\Big)\, dx - \int_{\partial^0 B_r^+} \operatorname{div} x \sum_{i,\, j<i} v_i^2 v_j^2\, dx \\ = r \int_{S_r^{N-1}} \sum_{i,\, j<i} v_i^2 v_j^2\, d\sigma - N \int_{\partial^0 B_r^+} \sum_{i,\, j<i} v_i^2 v_j^2\, dx$$

and

$$\int_{\partial^0 B_r^+} (x \cdot \nabla_x) \sum_i F_{i,\beta}(v_i)\, dx \\ = \int_{\partial^0 B_r^+} \operatorname{div}\Big(x \sum_i F_{i,\beta}(v_i)\Big)\, dx - \int_{\partial^0 B_r^+} \operatorname{div} x \sum_i F_{i,\beta}(v_i)\, dx \\ = r \int_{r S^{N-1}} \sum_i F_{i,\beta}(v_i)\, d\sigma - N \int_{\partial^0 B_r^+} \sum_i F_{i,\beta}(v_i)\, dx;$$

the lemma follows by substituting this into (5.1). □

In a similar way, it is possible to prove the Pohozaev identities in cylinders (we use the notation introduced at the beginning of Section 3.1).

Lemma 5.3 (Pohozaev identity in cylinders). *Let* $\mathbf{v} \in H^1(B^+)$ *solve problem* $(P)_\beta$. *For every* $x \in \partial^0 B^+$ *and* $r, l > 0$ *such that* $C^+_{r,l} \subset B^+$ *the following Pohozaev identity holds:*

$$\begin{aligned}\int_{C^+_{r,l}} \Big(\sum_i 2|\nabla_{(x',y)} v_i|^2 - (h+1)|\nabla v_i|^2\Big)\,dx\,dy + r \int_{\partial^+ B^+_r \times Q_l} \sum_i |\nabla v_i|^2\,d\sigma \\ + 2h \int_{\partial^0 C^+_{r,l}} \sum_i F_{i,\beta}(v_i)\,dx - h\beta \int_{\partial^0 C^+_{r,l}} \sum_{i,\,j<i} v_i^2 v_j^2\,dx \\ - 2r \int_{S^{h-1}_r \times Q_l} \sum_i F_{i,\beta}(v_i)\,d\sigma + r\beta \int_{S^{h-1}_r \times Q_l} \sum_{i,\,j<i} v_i^2 v_j^2\,d\sigma \\ = 2r \int_{\partial^+ B^+_r \times Q_l} \sum_i |\partial_\nu v_i|^2\,d\sigma - 2 \int_{B^+_r \times \partial^+ Q_l} \sum_i \partial_\nu v_i \nabla_{(x',y)} v_i \cdot (x',y)\,d\sigma,\end{aligned}$$

where $\nabla_{(x',y)}$ *is the gradient with respect to the directions in* $\mathbb{R}^{h+1}_+$.

Remark 5.4. Even though the above Pohozaev identities are enough for our purposes, we point out that they are nothing but special cases of a more general class of identities, namely the domain variation formulas (see for instance [19]). They may be obtained by testing the equation of $(P)_\beta$ with $\nabla\mathbf{v} \cdot Y$ in a smooth domain $\omega \subset \mathbb{R}^{N+1}_+$, where $Y \in \mathcal{C}^1(\mathbb{R}^{N+1}_+; \mathbb{R}^{N+1}_+)$ is a smooth vector field such that $Y|_{y=0} \in \mathcal{C}^1(\mathbb{R}^N; \mathbb{R}^N)$.

To proceed, we need the following standard result.

Lemma 5.5. *Let* $f, \lambda \in L^\infty(\partial^0 B^+)$. *If* $w \in H^1(B^+)$ *is a solution to*

$$\begin{cases} -\Delta w = 0 & \text{in } B^+, \\ \partial_\nu w = f - \lambda w & \text{on } \partial^0 B^+, \end{cases}$$

then $|w| \in H^1(B^+)$ *and for any* $\phi \in H^1(B^+)$ *with* $\phi|_{\partial^+ B^+} = 0$ *and* $\phi \geq 0$,

$$\int_{B^+} \nabla|w| \cdot \nabla\phi\,dx\,dy - \int_{\partial^0 B^+} (|f| - \lambda|w|)\phi\,dx \leq 0.$$

Proof. Let $g_\varepsilon(s) = \sqrt{s^2 + \varepsilon} \in \mathcal{C}^\infty(\mathbb{R})$ be such that $g_\varepsilon(s) \to |s|$ and $g'_\varepsilon(s) \to \mathrm{sgn}(s)$. By the Stampacchia lemma,

$$g_\varepsilon(w) \to |w| \quad \text{in } H^1(B^+),$$

while, by Lebesgue's theorem,

$$g'_\varepsilon(w)w \to |w| \quad \text{in } L^2(\partial^0 B^+).$$

Thus, for any $\phi \in H^1(B^+)$ with $\phi|_{\partial^+ B^+} = 0$ and $\phi \geq 0$, we have

$$\begin{aligned}\int_{B^+} \nabla g_\varepsilon(w) \cdot \nabla\phi\,dx\,dy - \int_{\partial^0 B^+} g'_\varepsilon(w)(f - \lambda w)\phi\,dx \\ = \int_{B^+} g'_\varepsilon(w)\nabla w \cdot \nabla\phi\,dx\,dy - \int_{\partial^0 B^+} g'_\varepsilon(w)\partial_\nu w\phi\,dx \\ = \int_{B^+} -\mathrm{div}(g'_\varepsilon(w)\nabla w)\phi\,dx\,dy = \int_{B^+} \big(-g''_\varepsilon(w)|\nabla w|^2 - g'_\varepsilon(w)\Delta w\big)\phi\,dx\,dy \leq 0.\end{aligned}$$

Letting $\varepsilon \to 0$ we obtain the lemma. □

Going back to the notation of Proposition 5.1, we have the following lemma.

Lemma 5.6. *For every compact subset K of $\mathbb{R}^N$,*

$$\lim_{n\to\infty} \beta_n \int_K v_{i,n}^2 \sum_{j\neq i} v_{j,n}^2\, dx = 0.$$

Moreover, for every $x_0 \in \mathbb{R}^N$ and almost every $r > 0$,

$$\beta_n \int_{S_r^{N-1}} v_{i,n}^2 \sum_{j\neq i} v_{j,n}^2\, d\sigma \to 0.$$

Proof. Let $\eta \in \mathcal{C}_0^\infty(B_r)$ be a positive smooth cut-off function with $\eta \equiv 1$ on K. Taking into account Lemma 5.5, we obtain

$$0 \le \beta_n \int_K |v_{i,n}| \sum_{j\neq i} v_{j,n}^2\, dx \le \int_{\partial^0 B_r^+} (|f_{i,n}|\eta - |v_{i,n}|\partial_\nu \eta)\, dx + \int_{B_r^+} |v_{i,n}| \Delta\eta\, dx\, dy \le C.$$

In particular, on the one hand this implies that

$$\beta_n \int_K |v_{i,n}| \sum_{j\neq i} v_{j,n}^2\, dx \le C,$$

while on the other hand, by passing to the limit, we infer that $\{v_i = 0\} \cup \{v_j = 0\}$ contains K for any $i \neq j$. As a consequence, each term in the sum can be estimated as follows:

$$\begin{aligned}
\beta_n \int_K v_{i,n}^2 v_{j,n}^2\, dx &\le \beta_n \int_{K\cap\{v_i=0\}} v_{i,n}^2 v_{j,n}^2\, dx + \beta_n \int_{K\cap\{v_j=0\}} v_{j,n}^2 v_{i,n}^2\, dx \\
&\le \|v_{i,n}\|_{L^\infty(K\cap\{v_i=0\})} \beta_n \int_{K\cap\{v_i=0\}} |v_{i,n}| v_{j,n}^2\, dx \\
&\quad + \|v_{j,n}\|_{L^\infty(K\cap\{v_j=0\})} \beta_n \int_{K\cap\{v_j=0\}} |v_{j,n}| v_{i,n}^2\, dx \to 0,
\end{aligned}$$

and the first conclusion follows by summing over all $j \neq i$. The second conclusion follows by applying Fubini's theorem to the first when $K = \partial^0 B_R^+$. □

Proof of Proposition 5.1. First we notice that, by Lemma 5.6, $v_i v_j \equiv 0$ for all $i \neq j$. Moreover, since the uniform limit of harmonic functions is harmonic, $\Delta v_i = 0$ on $\mathbb{R}_+^{N+1}$. In order to obtain (3.1), we observe that, for any $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} v_{i,n} \partial_\nu v_{i,n} \phi\, dx = \int_{\mathbb{R}^N} \Big(v_{i,n} f_{i,\beta_n}(v_{i,n}) - \beta_n v_{i,n}^2 \sum_{j\neq i} v_{j,n}^2 \Big) \phi\, dx \to 0$$

by assumption (4) and Lemma 5.6. Finally, to prove that (3.2) holds, we are going to show that, for every $x_0 \in \mathbb{R}^N$ and almost every $r > 0$, the Pohozaev identity of Lemma 5.2

passes to the limit (the general case following by analogous arguments). Let us group the terms of the identity as

$$\underbrace{(1-N)\int_{B_r^+}\sum_i |\nabla v_{i,n}|^2\,dx\,dy}_{A_n} + \underbrace{r\int_{\partial^+ B_r^+}\sum_i |\nabla v_{i,n}|^2\,d\sigma}_{B_n^1}$$
$$+ \underbrace{2N\int_{\partial^0 B_r^+}\sum_i F_{i,n}(v_{i,n})\,dx - 2r\int_{S_r^{N-1}}\sum_i F_{i,n}(v_{i,n})\,d\sigma}_{I_n}$$
$$\underbrace{- N\beta_n\int_{\partial^0 B_r^+}\sum_{i,\,j<i} v_{i,n}^2 v_{i,n}^2\,dx + r\beta\int_{S_r^{N-1}}\sum_{i,\,j<i} v_{i,n}^2 v_{j,n}^2\,d\sigma}_{C_n} = \underbrace{2r\int_{\partial^+ B_r^+}\sum_i |\partial_\nu v_{i,n}|^2\,d\sigma.}_{B_n^2}$$

On the one hand, by strong H^1_{loc} convergence,

$$A_n \to (1-N)\int_{B_r^+}\sum_i |\nabla v_i|^2\,dx\,dy.$$

Moreover, both $I_n \to 0$ (by assumption (4)) and $C_n \to 0$ for a.e. r (by Lemma 5.6). We claim that

$$\lim_{n\to\infty} B_n^1 = r\int_{\partial^+ B_r^+}\sum_i |\nabla v_i|^2\,d\sigma \quad \text{and} \quad \lim_{n\to\infty} B_n^2 = 2r\int_{\partial^+ B_r^+}\sum_i |\partial_\nu v_i|^2\,d\sigma$$

in $L^1_{\text{loc}}[0,\infty)$; in particular, this will imply convergence for a.e. r. Let us prove the former limit, which implies the latter. The strong convergence $\mathbf{v}_n \to \mathbf{v}$ in $H^1_{\text{loc}}(\overline{\mathbb{R}^{N+1}_+})$ implies that

$$\int_0^R \int_{\partial^+ B_r^+}\sum_i |\nabla v_{i,n} - \nabla v_i|^2\,d\sigma\,dr \to 0,$$

so that $\int_{\partial^+ B_r^+} |\nabla v_{i,n}|^2\,d\sigma \to \int_{\partial^+ B_r^+} |\nabla v_{i,n}|^2\,d\sigma$ for a.e. r and there exists an integrable function $f \in L^1(0,R)$ such that, up to a subsequence,

$$\int_{\partial^+ B_r^+} |\partial_\nu v_{i,n_k}|^2\,d\sigma \le \int_{\partial^+ B_r^+} |\nabla v_{i,n_k}|^2\,d\sigma \le f(r) \quad \text{a.e. } r \in (0,R)$$

for every $i = 1,\ldots,k$. We can then use the Dominated Convergence Theorem. Since every subsequence of $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$ has a convergent subsubsequence, and the limit is the same, we deduce the convergence for the entire approximating sequence. □

6. Local $\mathcal{C}^{0,\alpha}$ uniform bounds, α small

In this section we begin our regularity analysis with a first partial result. We will obtain a localized version of uniform Hölder regularity for solutions to problem $(P)_\beta$ (introduced on page 2889), when the Hölder exponent is sufficiently small. We recall that, here and in the following, the functions $f_{i,\beta}$ are assumed to be continuous and uniformly bounded, with respect to β, on bounded sets.

Remark 6.1. By standard regularity results (see for instance [28]), we know that for all $r < 1$, $\alpha \in (0, 1)$ and $\bar{m}, \bar{\beta} > 0$, there exists a constant $C = C(r, \alpha, \bar{m}, \bar{\beta})$ such that

$$\|\mathbf{v}_\beta\|_{\mathcal{C}^{0,\alpha}(\overline{B_r^+})} \leq C$$

for every solution $\mathbf{v}_\beta$ of problem $(P)_\beta$ on B_1^+ satisfying

$$\beta \leq \bar{\beta} \quad \text{and} \quad \|\mathbf{v}_\beta\|_{L^\infty(B_1^+)} \leq \bar{m}.$$

The main result of this section is the following.

Theorem 6.2. *Let $\{\mathbf{v}_\beta\}_{\beta>0} \subset H^1(B_1^+)$ be a family of solutions to problem $(P)_\beta$ on B_1^+ such that*

$$\|\mathbf{v}_\beta\|_{L^\infty(B_1^+)} \leq \bar{m}$$

with $\bar{m}$ independent of β. Then for every $\alpha \in (0, \nu^{\mathrm{ACF}})$ there exists a constant $C = C(\bar{m}, \alpha)$, not depending on β, such that

$$\|\mathbf{v}_\beta\|_{\mathcal{C}^{0,\alpha}(\overline{B_{1/2}^+})} \leq C.$$

Furthermore, $\{\mathbf{v}_\beta\}_{\beta>0}$ is relatively compact in $H^1(B_{1/2}^+) \cap \mathcal{C}^{0,\alpha}(\overline{B_{1/2}^+})$ for each $\alpha < \nu^{\mathrm{ACF}}$.

Remark 6.3. Even though we prove it in $\overline{B_{1/2}^+}$, Theorem 6.2 also holds with $\overline{B_{1/2}^+}$ replaced by $K \cap B_1^+$ for every compact set $K \subset B_1$.

To ease notation, we write $B^+ = B_1^+$. Inspired by [22, 30], we develop a blow-up analysis. First, let η denote a smooth function such that

$$\begin{cases} \eta(X) = 1, & 0 \leq |X| \leq 1/2, \\ 0 < \eta(X) \leq 1, & 1/2 \leq |X| \leq 1, \\ \eta(X) = 0, & |X| = 1 \end{cases} \tag{6.1}$$

(in particular, η vanishes on $\partial^+ B^+$ but is strictly positive on $\partial^0 B^+$). We will prove that

$$\|\eta \mathbf{v}\|_{\mathcal{C}^{0,\alpha}(\overline{B^+})} \leq C,$$

and the theorem will follow by the regularity of η.

Assume for contradiction the existence of sequences $\{\beta_n\}_{n\in\mathbb{N}}$ and $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$ solving $(P)_{\beta_n}$ such that

$$L_n := \max_{i=1,\dots,k} \max_{X'\neq X''\in\overline{B^+}} \frac{|(\eta v_{i,n})(X') - (\eta v_{i,n})(X'')|}{|X'-X''|^\alpha} \to \infty$$

for some $\alpha \in (0, \nu^{\mathrm{ACF}})$, which we consider to be fixed from now on. By Remark 6.1 we readily infer that $\beta_n \to \infty$. Moreover, up to relabeling, we may assume that L_n is achieved for $i = 1$ and a sequence of points $(X'_n, X''_n) \in \overline{B^+} \times \overline{B^+}$. We start by proving some properties of such sequences.

Lemma 6.4. *Let $X'_n \neq X''_n$ and $r_n := |X'_n - X''_n|$ satisfy*

$$L_n = |(\eta v_{1,n})(X'_n) - (\eta v_{i,n})(X''_n)|/r_n^\alpha.$$

Then, as $n \to \infty$,

(1) $r_n \to 0$*;*
(2) $\mathrm{dist}(X'_n, \partial^+ B^+)/r_n \to \infty$ *and* $\mathrm{dist}(X''_n, \partial^+ B^+)/r_n \to \infty$.

Proof. By the uniform control on $\|\mathbf{v}_n\|_{L^\infty}$ we have

$$L_n \leq \frac{\bar{m}}{r_n^\alpha}(\eta(X'_n) + \eta(X''_n)),$$

which immediately implies $r_n \to 0$. Since η vanishes on $\partial^+ B^+$, we have

$$\eta(X) \leq \ell\, \mathrm{dist}(X, \partial^+ B^+)$$

for every $X \in \overline{B^+}$, where ℓ denotes the Lipschitz constant of η. As a consequence, the first inequality of the proof becomes

$$\frac{\mathrm{dist}(X'_n, \partial^+ B^+)}{r_n} + \frac{\mathrm{dist}(X''_n, \partial^+ B^+)}{r_n} \geq \frac{L_n r_n^{\alpha-1}}{\bar{m}\ell} \to \infty$$

(recall that $\alpha < 1$), and the lemma follows by recalling that $\mathrm{dist}(X'_n, X''_n) = r_n$. □

Our analysis is based on two different blow-up sequences, one having uniformly bounded Hölder quotients, the other satisfying a suitable problem. Let $\{P_n\}_{n\in\mathbb{N}} \subset \overline{B^+}$, $|P_n| < 1$, be a sequence of points, to be chosen later. We write

$$\tau_n B^+ := \frac{B^+ - P_n}{r_n}.$$

Note that $\tau_n B^+$ is a hemisphere, not necessarily centered on the hyperplane $\{y = 0\}$. We introduce the sequences

$$w_{i,n}(X) := \eta(P_n)\frac{v_{i,n}(P_n + r_n X)}{L_n r_n^\alpha} \quad \text{and} \quad \bar{w}_{i,n}(X) := \frac{(\eta v_{i,n})(P_n + r_n X)}{L_n r_n^\alpha},$$

where $X \in \tau_n B^+$. With this choice, on the one hand it follows immediately that, for every i,

$$\frac{|\bar{w}_{i,n}(X') - \bar{w}_{i,n}(X'')|}{|X'-X''|^\alpha} \leq \left| \bar{w}_{1,n}\left(\frac{X'_n - P_n}{r_n}\right) - \bar{w}_{1,n}\left(\frac{X''_n - P_n}{r_n}\right)\right| = 1,$$

in such a way that the functions $\{\bar{\mathbf{w}}_n\}_{n\in\mathbb{N}}$ share a uniform bound on the Hölder seminorm, and at least their first components are not constant. On the other hand, since $\eta(P_n) > 0$, each $\mathbf{w}_n$ solves

$$\begin{cases} -\Delta w_{i,n} = 0 & \text{in } \tau_n B^+, \\ \partial_\nu w_{i,n} = f_{i,n}(w_{i,n}) - M_n w_{i,n} \sum_{j\neq i} w_{j,n}^2 & \text{on } \tau_n \partial^0 B^+, \end{cases} \tag{6.2}$$

with $f_{i,n}(s) = \eta(P_n) r_n^{1-\alpha} L_n^{-1} f_{i,\beta_n}(L_n r_n^\alpha s/\eta(P_n))$ and $M_n = \beta_n L_n^2 r_n^{2\alpha+1}/\eta(P_n)^2$.

Remark 6.5. The uniform bound of $\|\mathbf{v}_\beta\|_{L^\infty}$ implies that

$$\sup_{\tau_n \partial^0 B^+} |f_{i,n}(w_{i,n})| = \eta(P_n) r_n^{1-\alpha} L_n^{-1} \sup_{\partial^0 B^+} |f_{i,\beta_n}(v_{i,n})| \leq C(\bar{m}) r_n^{1-\alpha} L_n^{-1} \to 0$$

as $n \to \infty$.

A crucial property is that the two blow-up sequences defined above have asymptotically equivalent behavior, as demonstrated in the following lemma.

Lemma 6.6. *Let $K \subset \mathbb{R}^{N+1}$ be compact. Then*

(1) $\max_{X\in K\cap \overline{\tau_n B^+}} |\mathbf{w}_n(X) - \bar{\mathbf{w}}_n(X)| \to 0$*;*
(2) *there exists C, only depending on K, such that $|\mathbf{w}_n(X) - \mathbf{w}_n(0)| \leq C$ for every $X \in K$.*

Proof. Again, this is a consequence of the Lipschitz continuity of η and of the uniform boundedness of $\{\mathbf{v}_\beta\}_\beta$. Indeed, for every $i = 1, \dots, k$,

$$|w_{i,n}(X) - \bar{w}_{i,n}(X)| \leq \bar{m} r_n^{-\alpha} L_n^{-1} |\eta(X_n + r_n X) - \eta(X_n)| \leq \ell \bar{m} r_n^{1-\alpha} L_n^{-1} |X|,$$

and the right hand side vanishes as $n \to \infty$, implying the first part. Moreover, by definition, $\mathbf{w}_n(0) = \bar{\mathbf{w}}_n(0)$, and $|\bar{\mathbf{w}}_n(X) - \bar{\mathbf{w}}_n(0)| \leq C|X|^\alpha$ for every $X \in \tau_n B^+$. But then we can conclude the proof by noticing that

$$|\mathbf{w}_n(X) - \mathbf{w}_n(0)| \leq |\mathbf{w}_n(X) - \bar{\mathbf{w}}_n(X)| + |\bar{\mathbf{w}}_n(X) - \bar{\mathbf{w}}_n(0)|$$

and applying the first part. □

Lemma 6.7. *Let, up to a subsequence, $\Omega_\infty := \lim \tau_n B^+$ and let*

$$\mathbf{W}_n(X) := \mathbf{w}_n(X) - \mathbf{w}_n(0) \quad \textit{and} \quad \bar{\mathbf{W}}_n(X) := \bar{\mathbf{w}}_n(X) - \bar{\mathbf{w}}_n(0).$$

Then there exists a function $\mathbf{W} \in \mathcal{C}^{0,\alpha}(\Omega_\infty)$ which is harmonic and such that $\mathbf{W}_n \to \mathbf{W}$ and $\bar{\mathbf{W}}_n \to \mathbf{W}$ uniformly on every compact set $K \subset \Omega_\infty$. Moreover, if we choose $\{P_n\}_{n\in\mathbb{N}}$ such that $|X_n' - P_n| < C r_n$ for some constant C and for every n, then $\mathbf{W}$ is nonconstant.

Proof. Let $K \subset \Omega_\infty$ be any fixed compact set. Then, by definition, K is contained in the half-sphere $\tau_n B^+$ for every n sufficiently large. By definition, $\{\bar{\mathbf{W}}_n\}_{n\in\mathbb{N}}$ is a sequence of functions which share the same $\mathcal{C}^{0,\alpha}$-seminorm and are uniformly bounded on K, since $\bar{\mathbf{W}}_n(0) = 0$. By the Ascoli–Arzelà theorem, there exists a function $\mathbf{W} \in C(K)$ which, up to a subsequence, is the uniform limit of $\{\bar{\mathbf{W}}_n\}_{n\in\mathbb{N}}$; taking a countable compact exhaustion of Ω_∞ we find that $\bar{\mathbf{W}}_n \to \mathbf{W}$ uniformly on every compact set. By Lemma 6.6, we also find that $\mathbf{W}_n \to \mathbf{W}$, and since the uniform limit of harmonic function is harmonic, we conclude that $\mathbf{W}$ is harmonic. Let $X, Y \in \Omega_\infty$. By definition, there exists $n_0 \in \mathbb{N}$ such that $X, Y \in \tau_n B^+$ for every $n \geq n_0$, and so

$$|\bar{\mathbf{W}}_n(X) - \bar{\mathbf{W}}_n(Y)| \leq \sqrt{k}\,|X - Y|^\alpha \qquad \text{for every } n \geq n_0.$$

Letting $n \to \infty$, we obtain $\mathbf{W} \in \mathcal{C}^{0,\alpha}(\Omega_\infty)$. Now fix $C > 0$, and choose $\{P_n\}_{n\in\mathbb{N}}$ such that $|X'_n - P_n| < Cr_n$. Then, up to a subsequence,

$$\frac{X'_n - P_n}{r_n} \to X' \quad \text{and} \quad \frac{X''_n - P_n}{r_n} \to X'',$$

where $X', X'' \in \overline{B_{C+1} \cap \Omega_\infty}$. Therefore, by equicontinuity and uniform convergence,

$$\left|\bar{W}_{1,n}\left(\frac{X'_n - P_n}{r_n}\right) - \bar{W}_{1,n}\left(\frac{X''_n - P_n}{r_n}\right)\right| = 1, \quad \text{so} \quad |W_1(X') - W_1(X'')| = 1,$$

and the lemma follows. □

In Lemma 6.4 we have shown that X'_n, X''_n cannot accumulate too fast towards $\partial^+ B^+$. Now we can prove that they converge to $\partial^0 B^+$.

Lemma 6.8. *There exists $C > 0$ such that, for every n sufficiently large,*

$$\frac{\operatorname{dist}(X'_n, \partial^0 B^+) + \operatorname{dist}(X''_n, \partial^0 B^+)}{r_n} \leq C.$$

Proof. Assuming otherwise and taking into account the second part of Lemma 6.4 yields

$$\frac{\operatorname{dist}(X'_n, \partial B^+) + \operatorname{dist}(X''_n, \partial B^+)}{r_n} \to \infty.$$

Choosing $P_n = X'_n$ in the definition of $\mathbf{w}_n$, $\bar{\mathbf{w}}_n$, we can apply Lemma 6.7. First of all, we notice that $\tau_n B^+ \to \Omega_\infty = \mathbb{R}^{N+1}$. But then $\mathbf{W}$ as in the aforementioned lemma is harmonic, globally Hölder continuous on $\mathbb{R}^{N+1}$ and nonconstant, in contradiction with the Liouville theorem. □

Now we can choose P_n in the definition of $\mathbf{w}_n$, $\bar{\mathbf{w}}_n$: from now on we set

$$P_n := (x'_n, 0),$$

where as usual $X'_n = (x'_n, y'_n)$. With this choice, it is immediate that $\tau_n B^+ \to \Omega_\infty = \mathbb{R}^{N+1}_+$, and that all the above results, and in particular Lemma 6.7, apply. This last fact follows from Lemma 6.8, since $Cr_n \geq \operatorname{dist}(X'_n, \partial^0 B^+) = |X'_n - P_n|$.

Our next aim is to prove that $\{\mathbf{w}_n\}_{n\in\mathbb{N}}$, $\{\bar{\mathbf{w}}_n\}_{n\in\mathbb{N}}$ are uniformly bounded. This will be done in a series of lemmas.

Lemma 6.9. *Under the previous blow-up configuration, if* $\bar{w}_{i,n}(0) \to \infty$ *for some* i*, then*

$$M_n w_{i,n}^2(0) = M_n \bar{w}_{i,n}^2(0) \le C$$

for a constant C *independent of* n*. In particular,* $M_n \to 0$.

Proof. Fix $r > 0$, and let B_{2r}^+ be the half-ball of radius $2r$; by Lemma 6.8, for n sufficiently large we have $B_{2r}^+ \subset \tau_n B^+$. Since the sequence $\{\bar{\mathbf{w}}_n\}_{n\in\mathbb{N}}$ consists of continuous functions with the same $\mathcal{C}^{0,\alpha}$-seminorm, we have $\inf_{B_{2r}^+} |\bar{w}_{i,n}| \to \infty$. Furthermore, by Lemma 6.6, $\inf_{B_{2r}^+} |w_{i,n}| \to \infty$ as well.

Towards a contradiction, assume that

$$I_n := \inf_{\partial^0 B_{2r}^+} M_n w_{i,n}^2 \to \infty.$$

We first show that for $j \neq i$, both $\{w_{j,n}\}_{n\in\mathbb{N}}$ and $\{\bar{w}_{j,n}\}_{n\in\mathbb{N}}$ are bounded in B_{2r}^+. We recall that $|w_{j,n}|$ is a subsolution of problem (6.2). More precisely, by Lemma 5.5,

$$\int_{B_{2r}^+} \nabla |w_{j,n}| \cdot \nabla \varphi \, dx\, dy - \int_{\partial^0 B_{2r}^+} (\|f_{j,n}\|_{L^\infty(B_{2r})} - I_n |w_{j,n}|)\varphi \, dx \le 0 \tag{6.3}$$

for every $\varphi \in H_0^1(B_{2r})$, $\varphi \ge 0$. Setting $\varphi = \eta^2 |w_{j,n}|$ with $\eta \in \mathcal{C}_0^\infty(B_{2r})$, we obtain

$$\int_{B_{2r}^+} \left(|\nabla(\eta |w_{j,n}|)|^2 - |\nabla \eta|^2 |w_{j,n}|^2 \right) dx\, dy + I_n \int_{\partial^0 B_{2r}^+} \eta^2 |w_{j,n}|^2 \, dx \\ \le \|f_{j,n}\|_{L^\infty} \int_{\partial^0 B_{2r}^+} \eta^2 |w_{j,n}| \, dx.$$

As a consequence,

$$\begin{aligned} I_n \int_{\partial^0 B_{2r}^+} \eta^2 |w_{j,n}|^2 \, dx &\le \int_{B_{2r}^+} |\nabla \eta|^2 |w_{j,n}|^2 \, dx\, dy + \|f_{j,n}\|_{L^\infty} \int_{\partial^0 B_{2r}^+} \eta^2 |w_{j,n}| \, dx \\ &\le \sup_{B_{2r}^+} |w_{j,n}|^2 \int_{B_{2r}^+} |\nabla \eta|^2 \, dx\, dy + \|f_{j,n}\|_{L^\infty} \int_{\partial^0 B_{2r}^+} \eta^2 \tfrac{1}{2}(1 + |w_{j,n}|^2) \, dx \\ &\le \sup_{B_{2r}^+} |w_{j,n}|^2 \left(\int_{B_{2r}^+} |\nabla \eta|^2 \, dx\, dy + C(r) \|f_{j,n}\|_{L^\infty} \right) + C(r) \|f_{j,n}\|_{L^\infty}, \end{aligned} \tag{6.4}$$

where, by Remark 6.5, $C(r)\|f_{j,n}\|_{L^\infty} \to 0$. On the other hand, using again the uniform Hölder bounds for $\{\bar{\mathbf{w}}_n\}_{n\in\mathbb{N}}$ and the uniform control given by Lemma 6.6, we infer

$$\begin{aligned} I_n \int_{\partial^0 B_{2r}^+} \eta^2 |w_{j,n}|^2 \, dx &\ge I_n \inf_{\partial^0 B_{2r}^+} |w_{j,n}|^2 \int_{\partial^0 B_{2r}^+} \eta^2 \, dx \\ &\ge C I_n \left(\sup_{B_{2r}^+} |w_{j,n}|^2 - (2r)^{2\alpha} \right) \int_{\partial^0 B_{2r}^+} \eta^2 \, dx. \end{aligned} \tag{6.5}$$

Combining (6.4) with (6.5) we deduce the uniform boundedness of $\sup_{\partial^+ B_{2r}^+} |w_{j,n}|$ for $j \neq i$. By (6.3), the variational counterpart of Lemma 4.5 implies

$$|w_{j,n}| \leq \frac{C}{I_n} \sup_{\partial^+ B_{2r}^+} |w_{j,n}| \quad \text{ on } \partial^0 B_r^+ \tag{6.6}$$

for a constant C independent of n. From the uniform bound it follows that $w_{j,n} \to 0$ uniformly in $\partial^0 B_r^+$ for every $r > 0$, and the same is true for $\bar{w}_{j,n}$, $j \neq i$; in particular, since $|\bar{w}_{1,n}(\tau_n X_n') - \bar{w}_{1,n}(\tau_n X_n'')| = 1$, we deduce that necessarily $i = 1$.

Now, $w_{1,n}$ satisfies

$$\int_{B_r^+} \nabla w_{1,n} \cdot \nabla \varphi \, dx\, dy = \int_{\partial^0 B_r^+} \Big(f_{1,n} - M_n w_{1,n} \sum_{j \neq 1} w_{j,n}^2 \Big) \varphi \, dx$$

for every $\varphi \in H_0^1(B_r)$. From the previous estimates and the definition of I_n we find

$$\begin{aligned} \Big| f_{1,n} - M_n w_{1,n} \sum_{j \neq 1} w_{j,n}^2 \Big| &\leq \| f_{1,n} \|_{L^\infty} + M_n (|w_{1,n}|^2 + 1) \sum_{j \neq 1} w_{j,n}^2 \\ &\leq \| f_{1,n} \|_{L^\infty} + C \frac{I_n + M_n (r^{2\alpha} + 1)}{I_n^2} \to 0 \end{aligned}$$

on $\partial^0 B_r^+$, and this holds for every r. As a consequence, we can define $\{\mathbf{W}_n\}_{n \in \mathbb{N}}$ as in Lemma 6.7, deducing that $W_{1,n}$ converges to W_1, which is a nonconstant, globally Hölder continuous function on $\mathbb{R}_+^{N+1}$ which satisfies

$$\begin{cases} -\Delta W_1 = 0 & \text{in } \mathbb{R}_+^{N+1}, \\ \partial_\nu W_1 = 0 & \text{on } \mathbb{R}^N. \end{cases}$$

But then the even extension of W_1 through $\{y = 0\}$ contradicts the Liouville theorem. □

Lemma 6.10. *In the previous blow-up setting, if there exists i such that $\bar{w}_{i,n}(0) \to \infty$, then for every r there exists a constant $C = C(r)$, independent of n, such that*

$$M_n |w_{i,n}(0)| \int_{\partial^0 B_r^+} \sum_{j \neq i} w_{j,n}^2 \, dx \leq C.$$

Proof. Fix $r > 1$. Multiplying (6.2) by $w_{i,n}$ and integrating on B_r^+ we obtain the identity

$$\int_{B_r^+} |\nabla w_{i,n}|^2 \, dx\, dy + \int_{\partial^0 B_r^+} \Big(-f_{i,n} w_{i,n} + M_n w_{i,n}^2 \sum_{j \neq i} w_{j,n}^2 \Big) dx = \int_{\partial^+ B_r^+} w_{i,n} \partial_\nu w_{i,n} \, d\sigma.$$

Define

$$\begin{aligned} E_i(r) &:= \frac{1}{r^{N-1}} \Big(\int_{B_r^+} |\nabla w_{i,n}|^2 \, dx\, dy + \int_{\partial^0 B_r^+} \Big(-f_{i,n} w_{i,n} + M_n w_{i,n}^2 \sum_{j \neq i} w_{j,n}^2 \Big) dx \Big), \\ H_i(r) &:= \frac{1}{r^N} \int_{\partial^+ B_r^+} w_{i,n}^2 \, d\sigma. \end{aligned}$$

A straightforward computation shows that $H_i \in AC(r/2, r)$ and

$$H_i'(r) = \frac{2}{r} E_i(r).$$

In particular, integrating from $r/2$ to r, we obtain the identity

$$H_i(r) - H_i\left(\frac{r}{2}\right) = \int_{r/2}^{r} \frac{2}{s} E_i(s)\, ds. \tag{6.7}$$

If r is suitably chosen, and n is large, after a scaling in the definition of H_i we see that the left hand side of (6.7) can be written as

$$\begin{aligned} H_i(r) - H_i\left(\frac{r}{2}\right) &= \int_{\partial^+ B^+} \left[w_{i,n}^2(rx) - w_{i,n}^2\left(\frac{r}{2}x\right)\right] d\sigma \\ &= \int_{\partial^+ B^+} \left[w_{i,n}(rx) - w_{i,n}\left(\frac{r}{2}x\right)\right]\left[w_{i,n}(rx) + w_{i,n}\left(\frac{r}{2}x\right)\right] d\sigma \\ &\le C(r)(|w_{i,n}(0)| + 1), \end{aligned}$$

where we have used the first part of Lemma 6.6 to estimate the difference in the integral above, and the second part of the same lemma for the sum. In a similar way, we obtain a lower bound of the right hand side of (6.7):

$$\begin{aligned} \frac{1}{r}\int_{r/2}^{r} \frac{2}{s} E_i(s) ds &\ge \min_{s \in [r/2, r]} \frac{1}{s} E_i(s) \\ &\ge M_n \min_{s\in[r/2,r]} \frac{1}{s^N} \int_{\partial^0 B_s^+} \sum_{j \neq i} w_{i,n}^2 w_{j,n}^2\, dx - \max_{s\in[r/2,r]} \frac{1}{s^N}\int_{\partial^0 B_s^+} |f_{i,n} w_{i,n}|\, dx \\ &\ge C(r)\cdot \left[M_n w_{i,n}^2(0) \int_{\partial^0 B_{r/2}^+} \frac{w_{i,n}^2}{w_{i,n}^2(0)} \sum_{j\neq i} w_{j,n}^2\, dx - \|f_{j,n}\|_{L^\infty}(|w_{i,n}(0)| + 1)\right], \end{aligned}$$

where $C(r)\|f_{j,n}\|_{L^\infty} \to 0$ as $n \to \infty$. Putting the two estimates together and recalling that M_n is bounded, while $w_{i,n}(x)/w_{i,n}(0) \to 1$ uniformly, we find

$$M_n w_{i,n}^2(0) \int_{\partial^0 B_{r/2}^+} \sum_{j \neq i} w_{j,n}^2\, dx \le C(r)(|w_{i,n}(0)| + 1).$$

The conclusion follows by dividing by $|w_{i,n}(0)|$ and using the uniform control of $\{w_{i,n}\}_{n\in\mathbb{N}}$ and $\{\bar{w}_{i,n}\}_{n\in\mathbb{N}}$, and the assumption that $|\bar{w}_{i,n}(0)| \to \infty$. □

Lemma 6.11. *Suppose $\{\mathbf{w}_n(0)\}_{n\in\mathbb{N}}$ is unbounded. If $\{w_{i,n}(0)\}_{n\in\mathbb{N}}$ is bounded, then*

$$w_{i,n} \to 0 \quad \text{uniformly on compact sets.}$$

In particular, $\{w_{1,n}(0)\}_{n\in\mathbb{N}}$ is unbounded.

Proof. By Lemma 6.9, $M_n \to 0$. Let i be such that $\{w_{i,n}(0)\}_{n\in\mathbb{N}}$ is bounded. Reasoning as in the proof of Lemma 6.7, we find that both $w_{i,n}$ and $\bar{w}_{i,n}$ converge to some w_i, uniformly on compact sets. Furthermore, w_i is harmonic, globally Hölder continuous, and nonconstant in the possible case $i = 1$. We claim that there exists a constant $\lambda \geq 0$ such that

$$\partial_\nu w_{i,n} = f_{i,n} - M_n w_{i,n} \sum_{j\neq i} w_{j,n}^2 \to -\lambda w_i$$

uniformly on compact sets. This, combined with Proposition 4.7, proves the lemma.

To prove the claim, let first $j \neq i$ be such that $\bar{w}_{j,n}(0)$ is unbounded. From Lemma 6.10 we see that $M_n \bar{w}_{j,n}^2(0)$ is bounded. Moreover, by uniform Hölder bounds,

$$M_n|\bar{w}_{j,n}^2(x,0) - \bar{w}_{j,n}^2(0,0)| \leq \underbrace{M_n \bar{w}_{j,n}^2(0,0)}_{\leq C \text{ (Lemma 6.9)}} \left| \frac{\bar{w}_{j,n}^2(x,0)}{\bar{w}_{j,n}^2(0,0)} - 1 \right| \to 0,$$

since $\bar{w}_{j,n}(x)/\bar{w}_{j,n}(0) \to 1$ uniformly, implying $M_n \bar{w}_{j,n}^2(x,0) \to \lambda_j \geq 0$.

Let now $j \neq i$ be such that $\bar{w}_{j,n}(0)$ is bounded. Then, again by uniform convergence,

$$M_n \bar{w}_{i,n} \bar{w}_{j,n}^2 \to 0$$

uniformly on every compact set. It follows that

$$f_{i,n} - M_n \bar{w}_{i,n} \sum_{j\neq i} \bar{w}_{j,n}^2 \to -\lambda w_i$$

uniformly on every compact set, and the same limit holds for $\{w_{i,n}\}_{n\in\mathbb{N}}$ by uniform convergence. □

Lemma 6.12. *The sequence* $\{\mathbf{w}_n(0)\}_{n\in\mathbb{N}}$ *is bounded.*

Proof. Assume that $\{\mathbf{w}_n(0)\}_{n\in\mathbb{N}}$ is unbounded. Then, by the above lemmas, $M_n \to 0$, $\{w_{1,n}(0)\}_{n\in\mathbb{N}}$ is unbounded, while $M_n w_{1,n}^2(0)$ is bounded. This implies $M_n|w_{1,n}| \to 0$ uniformly on compact sets.

Now, if $j \neq 1$ is such that $\{w_{j,n}(0)\}_{n\in\mathbb{N}}$ is bounded, then $M_n w_{1,n} w_{j,n}^2 \to 0$ uniformly on every compact set.

On the other hand, if $j \neq 1$ and $\{w_{j,n}(0)\}_{n\in\mathbb{N}}$ is unbounded, then Lemma 6.10 yields

$$C \geq M_n|w_{1,n}(0)| \int_{\partial^0 B_r^+} w_{j,n}^2\, dx = M_n|w_{1,n}(0)|w_{j,n}^2(0) \int_{\partial^0 B_r^+} \frac{w_{j,n}^2}{w_{j,n}^2(0)}\, dx,$$

so that $M_n|w_{1,n}(0)|w_{j,n}^2(0)$ is uniformly bounded. Of course, if $\{w_{j,n}(0)\}_{n\in\mathbb{N}}$ is unbounded then so is $\{w_{j,n}(x)\}_{n\in\mathbb{N}}$ for any fixed x, and the same argument shows that $M_n|w_{1,n}(x)|w_{j,n}^2(x)$ is bounded. Now,

$$\begin{aligned} &M_n\big||w_{1,n}(x)|w_{j,n}^2(x) - |w_{1,n}(0)|w_{j,n}^2(0)\big| \\ &\quad\leq M_n|w_{1,n}(x)|w_{j,n}^2(x)\left|1 - \frac{w_{j,n}^2(0)}{w_{j,n}^2(x)}\right| + M_n|w_{1,n}(0)|w_{j,n}^2(0)\left|\frac{w_{1,n}(x)}{w_{1,n}(0)} - 1\right| \to 0. \end{aligned}$$

This shows the existence of a constant $\lambda_j \in \mathbb{R}$ such that $M_n w_{1,n} w_{j,n}^2 \to \lambda_j$ uniformly on every compact set.

Summing up, at least up to a subsequence,

$$f_{1,n} - M_n w_{1,n} \sum_{h \neq 1} w_{h,n}^2 \to \lambda \in \mathbb{R},$$

uniformly on every compact subset of $\mathbb{R}^N$. Thus, as usual, $W_{1,n} = w_{1,n} - w_{1,n}(0)$ converges to W_1, a nonconstant, globally Hölder continuous solution to

$$\begin{cases} -\Delta W_1 = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \partial_\nu W_1 = \lambda & \text{on } \mathbb{R}^N. \end{cases}$$

Appealing to Proposition 4.8, we obtain a contradiction. □

The uniform bound on $\{\bar{\mathbf{w}}_n(0)\}_{n\in\mathbb{N}}$ allows us to prove the following convergence result.

Lemma 6.13. *Under the previous blow-up setting, there exists* $\mathbf{w} \in (H^1_{\text{loc}} \cap \mathcal{C}^{0,\alpha})(\overline{\mathbb{R}^{N+1}_+})$ *such that, up to a subsequence,*

$$\mathbf{w}_n \to \mathbf{w} \quad \textit{in } (H^1 \cap C)(K)$$

for every compact $K \subset \overline{\mathbb{R}^{N+1}_+}$.

Proof. Reasoning as in the proof of Lemma 6.7 we can easily see that, up to subsequences, both $\{\bar{\mathbf{w}}_n\}_{n\in\mathbb{N}}$ and $\{\mathbf{w}_n\}_{n\in\mathbb{N}}$ converge uniformly on compact sets to the same limit $\mathbf{w} \in \mathcal{C}^{0,\alpha}(\overline{\mathbb{R}^{N+1}_+})$, hence it remains to show the H^1_{loc} convergence of the latter sequence.

Let K be compact, let r be such that $K \subset \overline{B_r^+}$, and let $\eta \in \mathcal{C}_0^\infty(B_r^+)$ be any smooth cut-off function such that $0 \le \eta \le 1$ and $\eta \equiv 1$ on K. Testing the equation for $w_{i,n}$ with $w_{i,n}\eta^2$, we obtain

$$\begin{aligned} 0 &\le \int_K |\nabla w_{i,n}|^2 \, dx\, dy + M_n \int_{\partial^0 K} w_{i,n}^2 \sum_{j\neq i} w_{j,n}^2 \, dx \\ &\le \int_{B_r^+} |\nabla w_{i,n}|^2 \eta^2 \, dx\, dy + M_n \int_{\partial^0 B_r^+} w_{i,n}^2 \sum_{j \neq i} w_{j,n}^2 \eta^2 \, dx \\ &\le \frac{1}{2} \int_{B_r^+} w_{i,n}^2 |\Delta \eta^2| \, dx\, dy + \frac{1}{2} \int_{\partial^0 B_r^+} (w_{i,n}^2 |\partial_\nu \eta^2| + f_{i,n} w_{i,n} \eta^2) \, dx. \end{aligned}$$

Since the right hand side is bounded uniformly in n (recall Lemmas 6.12 and 6.6), we deduce that, up to a subsequence, $\{\mathbf{w}_n\}_{n\in\mathbb{N}}$ weakly converges in $H^1(K)$. Since this holds for every K, we infer that $\mathbf{w}_n \rightharpoonup \mathbf{w}$ in $H^1_{\text{loc}}(\overline{\mathbb{R}^{N+1}_+})$. To prove the strong convergence, let us now test the equation with $\eta^2(w_{i,n} - w_i)$. We obtain

$$\int_{B_r^+} \nabla w_{i,n} \cdot \nabla[\eta^2 (w_{i,n} - w_i)] \, dx\, dy = \int_{\partial^0 B_r^+} \eta^2 (w_{i,n} - w_i) \partial_\nu w_{i,n} \, dx. \tag{6.8}$$

We can estimate the right hand side as

$$\begin{aligned}&\int_{\partial^0 B_r^+} \eta^2 (w_{i,n} - w_i)\partial_\nu w_{i,n}\, dx\\ &\quad\leq \sup_{x\in B_r^+} |w_{i,n} - w_i| \int_{\partial^0 B_r^+} \eta^2 \Big[M_n |w_{i,n}| \sum_{j<i} w_{j,n}^2 + |f_{i,n}| \Big]\, dx \leq C(r) \sup_{x\in B_r^+} |w_{i,n} - w_i|,\end{aligned}$$

where the last step holds since the inequality for $|w_{i,n}|$ (Lemma 5.5) tested with η^2 yields

$$\begin{aligned}&\int_{\partial^0 B_r^+} \eta^2 M_n |w_{i,n}| \sum_{j<i} w_{j,n}^2\, dx\\ &\qquad\leq \int_{\partial^0 B_r^+} (|f_{i,n}|\eta^2 + |w_{i,n}\partial_\nu \eta^2|)\, dx + \int_{B_r^+} |w_{i,n}\Delta\eta^2|\, dx\, dy \leq C(r).\end{aligned}$$

Summing up, (6.8) implies

$$\begin{aligned}\int_{B_r^+} |\nabla(\eta w_{i,n})|^2\, dx\, dy \leq &\int_{B_r^+} (\eta^2 \nabla w_{i,n} \cdot \nabla w_i + 2\eta w_i \nabla w_{i,n} \cdot \nabla\eta + |\nabla\eta|^2 w_i^2)\, dx\, dy\\ &+ C(r) \sup_{x\in B_r^+} |w_{i,n} - w_i|.\end{aligned}$$

Using both weak H^1 and uniform convergence, we obtain

$$\limsup_{n\to\infty} \int_{B_r^+} |\nabla(\eta w_{i,n})|^2\, dx\, dy \leq \int_{B_r^+} |\nabla(\eta w_i)|^2\, dx\, dy$$

and we deduce the strong convergence in $H^1(B_r^+)$ of $\{\eta \mathbf{w}_n\}_{n\in\mathbb{N}}$ to $\eta\mathbf{w}$, that is, since η was arbitrary, the strong convergence of $\mathbf{w}_n$ to $\mathbf{w}$ in $H^1_{\mathrm{loc}}(\overline{\mathbb{R}^{N+1}_+})$. □

End of the proof of Theorem 6.2. Summing up, $\mathbf{w}_n \to \mathbf{w}$ in $(H^1 \cap C)_{\mathrm{loc}}$, and the limiting blow-up profile $\mathbf{w}$ is a nonconstant vector of harmonic, globally Hölder continuous functions. To reach the final contradiction, we distinguish, up to subsequences, between the following three cases.

Case 1: $M_n \to 0$. In this case also the equation on the boundary passes to the limit, and the nonconstant component w_1 satisfies $\partial_\nu w_1 \equiv 0$ on $\mathbb{R}^N$, so that its even extension through $\{y = 0\}$ contradicts the Liouville theorem.

Case 2: $M_n \to C > 0$. Even in this case the equation on the boundary passes to the limit, and $\mathbf{w}$ solves

$$\begin{cases} -\Delta w_i = 0 & \text{in } \mathbb{R}^{N+1}_+,\\ \partial_\nu w_i = -C w_i \sum_{j\neq i} w_j^2 & \text{on } \mathbb{R}^N \times \{0\}.\end{cases}$$

The contradiction is now reached by using Proposition 4.1, since $\mathbf{w} \in \mathcal{G}_c \cap \mathcal{C}^{0,\alpha}(\overline{\mathbb{R}^{N+1}_+})$ and $\alpha < \nu^{\mathrm{ACF}}$.

Case 3: $M_n \to \infty$. By Proposition 5.1, we infer $\mathbf{w} \in \mathcal{G}_s \cap \mathcal{C}^{0,\alpha}(\overline{\mathbb{R}^{N+1}_+})$ with $\alpha < \nu^{\mathrm{ACF}}$, in contradiction with Proposition 4.2.

The contradictions we have obtained imply that $\{\mathbf{v}_\beta\}_{\beta>0}$ is uniformly bounded in $\mathcal{C}^{0,\alpha}(\overline{B^+_{1/2}})$ for every $\alpha < \nu^{\mathrm{ACF}}$. But then the relative compactness in $\mathcal{C}^{0,\alpha}(\overline{B^+_{1/2}})$ follows by the Ascoli–Arzelà theorem, while the one in $H^1(B^+_{1/2})$ can be shown by reasoning as in the proof of Lemma 6.13. □

Remark 6.14. It is worth noticing that, in proving Theorem 6.2, the only part where we used the assumption $\alpha < \nu^{\mathrm{ACF}}$ is the concluding argument, while in the rest of the proof it is sufficient to suppose $\alpha < 1$.

As we mentioned, even though we are not able to show that $\nu^{\mathrm{ACF}} = 1/2$, nonetheless we will prove that the uniform Hölder bound holds for any $\alpha < 1/2$. In view of the previous remark, this can be done by means of some sharper Liouville type results, which will be obtained in the next section. To conclude the present discussion, we observe that a result analogous to Theorem 6.2 holds when segregated entire profiles are considered instead of solutions to $(P)_\beta$.

Proposition 6.15. *Let $\{\mathbf{v}_n\}_{n\in\mathbb{N}} \subset \mathcal{G}_s \cap \mathcal{C}^{0,\alpha}(\overline{B^+_1})$, for some $0 < \alpha \le \nu^{\mathrm{ACF}}$, be such that*

$$\|\mathbf{v}_n\|_{L^\infty(B^+_1)} \le \bar{m}$$

with $\bar{m}$ independent of n. Then for every $\alpha' \in (0, \alpha)$ there exists a constant $C = C(\bar{m}, \alpha')$, not depending on n, such that

$$\|\mathbf{v}_n\|_{\mathcal{C}^{0,\alpha'}(\overline{B^+_{1/2}})} \le C.$$

Furthermore, $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$ is relatively compact in $H^1(B^+_{1/2}) \cap \mathcal{C}^{0,\alpha'}(\overline{B^+_{1/2}})$ for every $\alpha' < \alpha$.

Proof. The proof follows the lines of the one of Theorem 6.2, being in fact easier, since we do not have to handle any competition term. Aiming at a contradiction, assume that, up to a subsequence,

$$L_n := \max_{i=1,\dots,k} \sup_{X',X''\in\overline{B^+}} \frac{|\eta(X')v_{i,n}(X') - \eta(X'')v_{i,n}(X'')|}{|X'-X''|^{\alpha'}} \to \infty,$$

where again η is a smooth cut-off function of the ball $B_{1/2}$ and $\alpha' < \alpha$. If L_n is achieved by (X'_n, X''_n), we introduce the sequences

$$w_{i,n}(X) := \eta(X_n)\frac{v_{i,n}(P_n + r_n X)}{L_n r_n^{\alpha'}} \quad \text{and} \quad \bar{w}_{i,n}(X) := \frac{(\eta v_{i,n})(P_n + r_n X)}{L_n r_n^{\alpha'}},$$

for $X \in \tau_n B^+$, where, as usual, on the one hand $\bar{\mathbf{w}}_n$ has Hölder seminorm (and oscillation) equal to 1, while on the other hand $\mathbf{w}_n$ belongs to $\mathcal{G}_s$. All the preliminary properties of (X'_n, X''_n), up to Lemma 6.8, are still valid, since they depend only on the harmonicity of $\{\mathbf{w}_n\}_{n\in\mathbb{N}}$. It follows that the choice $P_n = (x'_n, 0)$ for every $n \in \mathbb{N}$ guarantees the convergence of the rescaled domains $\tau_n B^+$ to $\mathbb{R}^{N+1}_+$, while on any compact set the sequences $\{\mathbf{w}_n\}_{n\in\mathbb{N}}$ and $\{\bar{\mathbf{w}}_n\}_{n\in\mathbb{N}}$ shadow each other. Up to relabeling and taking subsequences, we are left with two alternatives: either

- for any compact set $K \subset \mathbb{R}^N$ we have $\mathbf{w}_{1,n}(x,0) \neq 0$ for every $n \geq n_0(K)$ and $x \in K$; or
- there exists a bounded sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that $\mathbf{w}_n(x_n, 0) = 0$ for every n.

In the first case, if we define $\mathbf{W}_n = \mathbf{w}_n - \mathbf{w}_n(0)$ and $\bar{\mathbf{W}}_n = \bar{\mathbf{w}}_n - \bar{\mathbf{w}}_n(0)$, we find that the sequence $\{\bar{\mathbf{W}}_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $\mathcal{C}^{0,\alpha'}$, and hence $\{\mathbf{W}_n\}_{n\in\mathbb{N}}$ converges uniformly on compact sets to a nonconstant, globally Hölder continuous function $\mathbf{W}$, with $\partial_\nu W_1(x,0) \equiv 0$ and $W_i(x,0) \equiv 0$ for $i > 1$, on $\mathbb{R}^N$. Extending properly the vector $\mathbf{W}$ to the whole $\mathbb{R}^{N+1}$, we get a contradiction with the Liouville theorem.

For the second alternative, $\{\mathbf{w}_n\}_{n\in\mathbb{N}}$ itself converges, uniformly on compact sets, to a nonconstant, globally Hölder continuous function $\mathbf{w}$. We want to show that the convergence is also strong in H^1_{loc}; this will imply that also $\mathbf{w} \in \mathcal{G}_s$ (recall also the end of the proof of Proposition 5.1), in contradiction with Proposition 4.2. To prove the strong convergence, consider, for any i, the even extension of $|w_{i,n}|$ through $\{y = 0\}$, which we denote again with $|w_{i,n}|$. There exists a nonnegative Radon measure $\mu_{i,n}$ such that

$$-\Delta |w_{i,n}| = -\mu_{i,n} \quad \text{in } \mathcal{D}'(\tau_n B).$$

Indeed, on the one hand, for $X \in \{w_{i,n} \neq 0\}$, there exists a radius $r > 0$ such that the even extension of $w_{i,n}$ through $\{y = 0\}$ is harmonic in $B_r(X)$, yielding

$$|w_{i,n}|(X) \leq \frac{1}{|B_r|} \int_{B_r(X)} |w_{i,n}|(Y)\, dY;$$

on the other hand, $X \in \{w_{i,n} = 0\}$ immediately implies the same inequality, and the consequent subharmonicity of $|w_{i,n}|$. At this point, we can reason as in [26], showing that the L^∞ uniform bounds of $|w_{i,n}|$ on compact sets imply that the measures $\mu_{i,n}$ are bounded on compact sets [26, Lemma 3.7]; and that this, together with the uniform convergence of $\{|\mathbf{w}_n|\}_{n\in\mathbb{N}}$, implies its strong H^1_{loc} convergence [26, Lemma 3.11].

As a consequence of the previous contradiction argument, we deduce both the uniform bounds and the precompactness of $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$ in $\mathcal{C}^{0,\alpha'}(B_{1/2})$. Once we have (the uniform L^∞ bounds and) the uniform convergence of $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$, the strong H^1 precompactness can be obtained exactly as in the last part of the proof, upon replacing $|w_{i,n}|$ with $|v_{i,n}|$. □

7. Liouville type theorems, reprise: the optimal growth

In Section 6 we proved that nonexistence results of Liouville type imply uniform bounds in corresponding Hölder norms. This section is devoted to the study of the optimal Liouville exponents, which will allow us to enhance the regularity estimates. Our aim is to prove the following result.

Theorem 7.1. *Let $\nu \in (0, 1/2)$. If either*

(1) $\mathbf{v} \in \mathcal{G}_s \cap \mathcal{C}^{0,\nu}(\overline{\mathbb{R}^{N+1}_+})$, *or*
(2) $\mathbf{v} \in \mathcal{G}_c$ *and* $|\mathbf{v}(X)| \leq C(1 + |X|^\nu)$ *for every* X,

then $\mathbf{v}$ *is constant.*

The rest of the section is devoted to the proof of the above theorem. We already know from Propositions 4.1 and 4.2 that the conclusion holds whenever $\nu < \nu^{\mathrm{ACF}}$. In order to refine that result, we will prove that it holds for ν smaller than ν^{Liou}, according to the following definition.

Definition 7.2. For $\nu > 0$ and for every dimension N, we define the class

$$\mathcal{H}(\nu, N) := \left\{ \mathbf{v} \in \mathcal{G}_s : \begin{array}{l} \mathbf{v} \in \mathcal{C}^{0,\alpha}_{\mathrm{loc}}(\overline{\mathbb{R}^{N+1}_+}) \text{ for some } \alpha > 0, \\ \mathbf{v} \text{ is nontrivial and } \nu\text{-homogeneous} \end{array} \right\},$$

and the critical value

$$\nu^{\mathrm{Liou}}(N) = \inf\{\nu > 0 : \mathcal{H}(\nu, N) \text{ is nonempty}\}$$

Remark 7.3. Since $(y, 0, \dots, 0) \in \mathcal{H}(1, N)$, for every N, we have $\nu^{\mathrm{Liou}}(N) \le 1$.

Remark 7.4. By Corollary 3.5, if $\mathbf{v}$ is nonconstant and satisfies assumption (1) in Theorem 7.1 for some ν, then $\mathbf{v} \in \mathcal{H}(\nu, N)$.

To prove Theorem 7.1, we start by showing that, given any nonconstant $\mathbf{v}$ satisfying assumption (2) for some ν, we can construct a function $\bar{\mathbf{v}} \in \mathcal{H}(\nu', N)$, for a suitable $\nu' \le \nu$. This, together with the previous remark, will imply the equivalence between Theorem 7.1 and the inequality

$$\nu^{\mathrm{Liou}}(N) \ge 1/2.$$

To construct such a $\bar{\mathbf{v}}$, we will use the blow-down method. For any (nontrivial) $\mathbf{v} \in \mathcal{G}_c$ we denote by $N_{\mathbf{v}}(x_0, r)$, $H_{\mathbf{v}}(x_0, r)$ the related quantities involved in the Almgren frequency formula, defined in Section 3. For any $r > 0$, set

$$\mathbf{v}_r(X) := \frac{1}{\sqrt{H_{\mathbf{v}}(0, r)}} \mathbf{v}(rX).$$

Since H is a strictly positive increasing function in $\mathbb{R}_+$ (recall Theorem 3.11), $\mathbf{v}_r$ is well defined. We have the following.

Lemma 7.5 (Blow-down method). *Let $\mathbf{v}$ be a nonconstant function satisfying assumption (2) in Theorem 7.1 for some ν, and let*

$$0 < \nu' = \lim_{r \to \infty} N_{\mathbf{v}}(r) \le \nu.$$

Then there exists $\bar{\mathbf{v}} \in \mathcal{H}(\nu', N)$ such that, for a suitable sequence $r_n \to \infty$,

$$\mathbf{v}_{r_n} \to \bar{\mathbf{v}} \quad \textit{in } (H^1 \cap C)(K)$$

for every compact $K \subset \overline{\mathbb{R}^{N+1}_+}$.

Proof. First of all, by construction,

$$\|\mathbf{v}_r\|_{L^2(\partial^+ B^+)} = 1 \quad \text{so that} \quad \|v_{i,r}\|_{L^2(\partial^+ B^+)} \le 1 \quad \text{for } i = 1, \dots, k.$$

Each $\mathbf{v}_r$ is a solution to the system

$$\begin{cases} -\Delta v_{i,r} = 0 & \text{in } B^+, \\ \partial_\nu v_{i,r} + rH(r) v_{i,r} \sum_{j \neq i} v_{j,r}^2 = 0 & \text{on } \partial^0 B^+, \end{cases}$$

where $rH(r) \to \infty$ monotonically as $r \to \infty$. Reasoning as in the proof of Lemma 5.5, we find that the even reflection of $|v_{i,r}|$ through $\{y = 0\}$ satisfies

$$\begin{cases} -\Delta |v_{i,r}| \leq 0 & \text{in } B, \\ \|v_{i,r}\|_{L^2(\partial B)} \leq \sqrt{2}. \end{cases}$$

By the Poisson representation formula, there exists a constant C, not depending on r, such that

$$\|\mathbf{v}_r\|_{L^\infty(B^+_{3/4})} \leq C$$

for every r. Thus Theorem 6.2 shows that the family $\{\mathbf{v}_r\}_{r>1}$ is relatively compact in $(H^1 \cap \mathcal{C}^{0,\alpha})(B^+_{1/2})$ for all $\alpha < \nu^{\mathrm{ACF}}$. Furthermore, Proposition 5.1 implies that any limiting point of that family is an element of $\mathcal{G}_s$ on $B^+_{1/2}$.

In order to find a nontrivial limiting point, we claim that there exists a sequence $\{r_n\}_{n\in\mathbb{N}}$, $r_n \to \infty$, and a positive constant C such that

$$H(r_n) \leq CH(r_n/2) \qquad \forall r_n > 0.$$

Indeed, assume that there exists $r_0 > 0$ such that

$$H(r) \geq 3^{2\nu} H(r/2) \qquad \forall r \geq r_0.$$

Using the dyadic sequence $\{2^j r_0\}_{j\in\mathbb{N}}$ we see that

$$3^{2\nu j} H(r_0) \leq H(2^j r_0) \leq C(2^j)^{2\nu},$$

by assumption. This yields a contradiction for j sufficiently large, proving the claim. Denote by $\bar{\mathbf{v}}$ a limiting point of $\{\mathbf{v}_{r_n}\}_{n\in\mathbb{N}}$. Then

$$\|\mathbf{v}_{r_n}\|_{L^2(\partial^+ B^+_{1/2})} = \sqrt{\frac{H(r_n/2)}{H(r_n)}} \geq \sqrt{\frac{1}{C}},$$

implying in particular that $\bar{\mathbf{v}}$ is a nontrivial element of $\mathcal{G}_s$. Moreover, its Almgren quotient $N_{\bar{\mathbf{v}}}(0, r)$ is constant for all $r \in (0, 1/2)$: indeed,

$$N_{\bar{\mathbf{v}}}(0, r) = \lim_{r_n\to\infty} N_{\mathbf{v}_{r_n}}(0, r) = \lim_{r_n\to\infty} N_{\mathbf{v}}(0, r_n r) = \lim_{r\to\infty} N_{\mathbf{v}}(0, r) = \nu',$$

where the latter limit exists by the monotonicity of N (Theorem 3.11); moreover, since $\mathbf{v}$ is not constant, we have $\nu' > 0$, while $\nu' \leq \nu$ by Lemma 3.12. Since $N(0, r)$ is constant, we conclude by Theorem 3.3 that $\bar{\mathbf{v}}$ is homogeneous of degree ν', and so it can be extended on the whole $\mathbb{R}^{N+1}_+$ to a $\mathcal{C}^{0,\alpha}_{\mathrm{loc}}$ function for every $\alpha < \nu^{\mathrm{ACF}}$. □

By the previous lemma, if we show that $\nu^{\mathrm{Liou}}(N) \geq 1/2$ then Theorem 7.1 will follow. The next step in this direction consists in reducing the problem to estimating $\nu^{\mathrm{Liou}}(1)$.

Lemma 7.6 (Dimensional descent). *For any dimension $N \geq 2$,*

$$\nu^{\mathrm{Liou}}(N) \geq \nu^{\mathrm{Liou}}(N-1).$$

Proof. For every $\nu > 0$ such that there exists $\mathbf{v} \in \mathcal{H}(\nu, N)$, we will prove that $\nu^{\mathrm{Liou}}(N-1) \leq \nu$. Let ν, $\mathbf{v}$ be as above. By homogeneity, $\mathbf{v}(0,0) = 0$ and $N(0,r) = \nu$ for all $r > 0$. Since $\mathbf{v}$ is homogeneous, its boundary nodal set

$$\mathcal{Z} = \{x \in \mathbb{R}^N : \mathbf{v}(x,0) = 0\}$$

is a cone at $(0,0)$. We can easily rule out two degenerate situations:

- $\mathcal{Z} = \mathbb{R}^N$, in which case all the components of $\mathbf{v}$ have trivial trace on $\mathbb{R}^N$. As a consequence, the odd extension of $\mathbf{v}$ through $\{y = 0\}$ is a nontrivial vector of harmonic functions on $\mathbb{R}^{N+1}$, forcing $\nu \geq 1 \geq \nu^{\mathrm{Liou}}(N-1)$ by Remark 7.3.
- $\mathcal{Z} = \{(0,0)\}$, in which case all the components of $\mathbf{v}$ but one have trivial trace, and the last one has necessarily a vanishing normal derivative in $\{y = 0\}$. As before, extending the former functions oddly and the latter evenly through $\{y = 0\}$, we again obtain $\nu \geq 1 \geq \nu^{\mathrm{Liou}}(N-1)$.

It remains to analyze the third and most delicate case, when the boundary $\partial\mathcal{Z}$ is nontrivial. Let $x_0 \in \partial\mathcal{Z} \setminus \{(0,0)\}$, and let us introduce the following blow-up family (here $r \to 0$):

$$\mathbf{v}_r(X) = \frac{1}{\sqrt{H(x_0,r)}}\mathbf{v}((x_0,0) + rX).$$

We want to apply Proposition 6.15 to (a subsequence of) $\{\mathbf{v}_r\}_r$: the only assumption nontrivial to check is the uniform L^∞ bound. To prove it, we observe that the even extension of $|v_{i,r}|$ through $\{y = 0\}$ (denoted by the same symbol) is subharmonic: indeed, the inequality

$$|v_{i,r}|(X) \leq \frac{1}{|B_\rho|}\int_{B_\rho(X)} |v_{i,r}|(Y)\,dY$$

holds true if ρ is sufficiently small, both when $v_{i,r}(X) = 0$ and when $v_{i,r}(X) \neq 0$. Once we know that each $|v_{i,r}|$ is nonnegative and subharmonic, arguing as in the first part of the proof of Lemma 7.5 we can show that $w_{i,r}$ is uniformly bounded in $L^\infty(B_{3/4})$. Applying Proposition 6.15 we deduce that, up to a subsequence, $\mathbf{v}_r$ converges uniformly and strongly in H^1 to $\bar{\mathbf{v}}$, an element of $\mathcal{G}_s(N)$ on $B^+_{1/2}$. Reasoning as at the end of the proof of Lemma 7.5, we infer that $\bar{\mathbf{v}}$ is nontrivial, locally $\mathcal{C}^{0,\alpha}$, and

$$N_{\bar{\mathbf{v}}}(0,\rho) = \lim_{r\to 0} N_{\mathbf{v}_r}(0,\rho) = \lim_{r\to} N_{\mathbf{v}}(x_0, r\rho) = \lim_{r\to 0} N_{\mathbf{v}}(x_0,r) =: \nu',$$

where

$$\alpha \leq N_{\mathbf{v}}(x_0, 0^+) = \nu' \leq N_{\mathbf{v}}(x_0,\infty) = \nu$$

by Lemmas 3.4 and 3.7 and the monotonicity of N. In particular, $\bar{\mathbf{v}}$ is homogeneous of degree ν'.

To conclude the proof, we will show that $\bar{\mathbf{v}}$ is constant along the direction parallel to $(x_0, 0)$, and its restriction to the orthogonal half-plane belongs to $\mathcal{G}_s(N-1)$. Fix $(x, y) \in \mathbb{R}^{N+1}_+$ and $h \in \mathbb{R}$. By the homogeneity of $\mathbf{v}$ we have

$$\begin{aligned} &|\mathbf{v}_r(x + h(x_0 + rx), (1 + hr)y) - \mathbf{v}_r(x, y)| \\ &\qquad = \frac{|\mathbf{v}((1 + hr)(x_0 + rx, ry)) - \mathbf{v}(x_0 + rx, ry)|}{\sqrt{H(x_0, r)}} \\ &\qquad = |(1 + hr)^\nu - 1| \frac{|\mathbf{v}(x_0 + rx, ry)|}{\sqrt{H(x_0, r)}} = |(1 + hr)^\nu - 1| \, |\mathbf{v}_r(x, y)|. \end{aligned}$$

As $r \to 0$ (up to a subsequence) we infer, by uniform convergence,

$$|\bar{\mathbf{v}}(x + hx_0, y) - \bar{\mathbf{v}}(x, y)| = 0 \qquad \text{for every } h \in \mathbb{R}.$$

Denote by $\hat{\mathbf{v}}$ a section of $\bar{\mathbf{v}}$ with respect to the direction $\{h(x_0, 0)\}_{h\in\mathbb{R}}$. We claim that $\hat{\mathbf{v}} \in \mathcal{H}(\nu', N-1)$. It is a direct check to verify that $\hat{\mathbf{v}}$ is nontrivial, ν'-homogeneous, and $\mathcal{C}^{0,\alpha}_{\mathrm{loc}}$. In order to show that $\hat{\mathbf{v}} \in \mathcal{G}_s(N-1)$, we observe that the equations and the segregation conditions are trivially satisfied, therefore we only need to prove the Pohozaev identities on cylindrical domains (recall the discussion before Definition 3.1). To this end, let C' denote one of such domains in $\mathbb{R}^N_+$, and C'' the corresponding domain in $\mathbb{R}^{N+1}_+$ having C' as N-dimensional section, and the further axis parallel to $(x_0, 0)$. But then the Pohozaev identity for $\hat{\mathbf{v}}$ on C' immediately follows from the one for $\bar{\mathbf{v}}$ on C'', by using the Fubini theorem. □

We are ready to obtain the proof of Theorem 7.1 as a byproduct of the following classification result, which completely characterizes the elements of $\mathcal{H}(\nu, 1)$ and shows that $\nu^{\mathrm{Liou}}(1) = 1/2$.

Proposition 7.7. *Let $\nu > 0$. Then:*

(1) $\mathcal{H}(\nu, 1) = \emptyset \Leftrightarrow 2\nu \notin \mathbb{N}$*;*
(2) *if $\nu \in \mathbb{N}$ then any element of $\mathcal{H}(\nu, 1)$ consists of homogeneous polynomials, and only one of its components may have nontrivial trace on $\{y = 0\}$;*
(3) *if $\nu = k + 1/2$, $k \in \mathbb{N}$, then any element of $\mathcal{H}(\nu, 1)$ has exactly two nontrivial components, say v and w, and there exists $c \neq 0$ such that*

$$v(\rho, \theta) = c\rho^{1/2+k} \cos{(1/2 + k)\theta}, \qquad w(\rho, \theta) = \pm c\rho^{1/2+k} \sin{(1/2 + k)\theta}$$

(here (ρ, θ) denote polar coordinates in $\mathbb{R}^2_+$ around the homogeneity pole).

Proof. Let $\nu > 0$ be such that $\mathcal{H}(\nu, 1)$ is not empty, and $\mathbf{v} \in \mathcal{H}(\nu, 1)$. Since, by assumption, $\mathbf{v}$ is homogeneous, the Almgren quotient $N(0, r)$ is equal to ν for every $r > 0$. Moreover, for topological reasons, no more than two components of $\mathbf{v}$ can have nontrivial trace on $\{y = 0\}$. We will classify $\mathbf{v}$, and hence ν, according to the number of such components.

As a first case, suppose that two components of $\mathbf{v}$, say v and w, have nontrivial trace, in such a way that they solve

$$\begin{cases} -\Delta v = 0 & \text{in } \mathbb{R}^2_+, \\ v(x,0) = 0 & \text{on } x < 0, \\ \partial_\nu v(x,0) = 0 & \text{on } x > 0, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta w = 0 & \text{in } \mathbb{R}^2_+, \\ w(x,0) = 0 & \text{on } x > 0, \\ \partial_\nu w(x,0) = 0 & \text{on } x < 0. \end{cases}$$

By homogeneity, we can easily find v and w: indeed, for instance, v must be of the form $v(\rho,\theta) = \rho^\nu g(\theta)$ with ν and g solutions to

$$\begin{cases} \nu^2 g + g'' = 0 & \text{in } (0,\pi), \\ g(\pi) = 0, \ g'(0) = 0, \end{cases}$$

and an analogous argument holds for w. We conclude that

$$v(\rho,\theta) = c\rho^{1/2+k} \cos{(1/2+k)\theta}, \qquad w(\rho,\theta) = d\rho^{1/2+k} \sin{(1/2+k)\theta},$$

with $c, d \neq 0$ and $k \in \mathbb{N}$, forcing $\nu = k + 1/2$. All the other components of $\mathbf{v}$ must satisfy

$$\begin{cases} -\Delta v_i = 0 & \text{in } \mathbb{R}^2_+, \\ v_i = 0 & \text{on } \mathbb{R} \times \{0\}, \end{cases}$$

with homogeneity degree $k + 1/2$, which is impossible unless they are null. Let

$$\bar{v}(\rho,\theta) = \rho^{1/2+k} \cos{(1/2+k)\theta},$$

so that $v(x,y) = c\bar{v}(x,y)$, while $w(x,y) = d\bar{v}(-x,y)$. Since $\mathbf{v}$ must satisfy the Pohozaev identities for the elements of $\mathcal{G}_s$, we infer that

$$\int_{\partial^+ B_r^+(x_0,0)} (|\nabla v|^2 + |\nabla w|^2)\, d\sigma = 2 \int_{\partial^+ B_r^+(x_0,0)} (|\partial_\nu v|^2 + |\partial_\nu w|^2)\, d\sigma$$

for every $x_0 \in \mathbb{R}$ and $r > 0$. Considering the choices $x_0 = 1$ and $x_1 = -1$, and using the symmetries, we obtain

$$A_+ c^2 + A_- d^2 = 2B_+ c^2 + 2B_- d^2,$$
$$A_- c^2 + A_+ d^2 = 2B_- c^2 + 2B_+ d^2,$$

where

$$A_\pm = \int_{\partial^+ B_r^+(\pm 1,0)} |\nabla \bar{v}|^2\, d\sigma, \qquad B_\pm = \int_{\partial^+ B_r^+(\pm 1,0)} |\partial_\nu \bar{v}|^2\, d\sigma.$$

Since $A_\pm - 2B_\pm \neq 0$ at least for some r, the above equalities force $c^4 - d^4 = 0$, that is, $d = \pm c$. We want to show that this condition is also sufficient for $(v, w, 0, \dots, 0)$ to belong to $\mathcal{H}(\nu, 1)$. To this end, we only need to prove the actual validity of the Pohozaev identity for any x_0 and R. We begin by observing that v and w are conjugate harmonic functions, thus in particular

$$\nabla v \cdot \nabla w = 0 \quad \text{and} \quad |\nabla v| = |\nabla w| \quad \text{in } \mathbb{R}^2_+.$$

Hence, for any unit vector $\mathbf{n} \in \mathbb{R}^2$ we have

$$|\nabla v|^2 = |\nabla w|^2 = |\nabla v \cdot \mathbf{n}|^2 + |\nabla w \cdot \mathbf{n}|^2 = |\partial_{\mathbf{n}} v|^2 + |\partial_{\mathbf{n}} w|^2,$$

and the Pohozaev identity follows by integrating over half-circles, and choosing ν to be the outer normal. To sum up, the case when $\mathbf{v}$ has two components with nontrivial trace on $\{y = 0\}$ always falls under alternative (3) of the statement.

Secondly, assume that only one component, say v, has nontrivial trace on $\{y = 0\}$. Then $\{v(x, 0) > 0\}$ is either a half-line or the entire real line. The first case never happens, since v would solve

$$\begin{cases} -\Delta v = 0 & \text{in } \mathbb{R}^2_+, \\ v(x, 0) = 0 & \text{on } x < 0, \\ \partial_\nu v = 0 & \text{on } x > 0. \end{cases}$$

Reasoning as before, we would deduce that v is of the form

$$v(\rho, \theta) = c\rho^{1/2+k} \cos{(1/2 + k)\theta}$$

with $c \in \mathbb{R}$ and $k \in \mathbb{N}$, while all the (odd extensions of the) other components should be harmonic on $\mathbb{R}^2$ and homogeneous of degree $k + 1/2$, that is, null; the Pohozaev identity would force $c = 0$, and $\mathbf{v}$ would be trivial. In the second case, if $v(x, 0) \neq 0$ for every $x \neq 0$, then v is of the form

$$v(\rho, \theta) = c\rho^k \cos k\theta$$

with $c \in \mathbb{R} \setminus \{0\}$ and $k \in \mathbb{N}$, while all the other components of $\mathbf{v}$ are of the form

$$v_i(\rho, \theta) = c_i \rho^k \sin k\theta$$

for some $c_i \in \mathbb{R}$. Then the case of one nontrivial trace on $\{y = 0\}$ always falls under alternative (2) of the statement.

As the last case, suppose that $v_i(x, 0) \equiv 0$ for every i. Then each v_i is a ν-homogeneous solution to

$$\begin{cases} -\Delta v_i = 0 & \text{in } \mathbb{R}^2_+, \\ v_i = 0 & \text{on } \mathbb{R} \times \{0\}, \end{cases}$$

that is, for some $k \in \mathbb{N}$ and $c_i \in \mathbb{R}$, we have $\nu = 1 + k$ and

$$v_i(\rho, \theta) = c_i \rho^{1+k} \sin{(1 + k)\theta}.$$

Also this case always falls under alternative (2) of the statement, and the proposition follows. □

8. $\mathcal{C}^{0,\alpha}$ uniform bounds, $\alpha < 1/2$

This section is devoted to the proof of the uniform Hölder bounds, with any exponent less than $1/2$, for the problem with exterior boundary Dirichlet data. In this direction, let us consider the problem

$$\begin{cases} -\Delta v_i = 0 & \text{in } B^+, \\ \partial_\nu v_i = f_{i,\beta}(v_i) - \beta v_i \sum_{j\neq i} v_j^2 & \text{on } \partial^0 B^+ \cap \Omega, \\ v_i = 0 & \text{on } \partial^0 B^+ \setminus \Omega, \end{cases} \qquad (PD)_\beta$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ and the functions $f_{i,\beta}$ are continuous and uniformly bounded, with respect to β, on bounded sets.

Remark 8.1. For $(PD)_\beta$ it is known that, if Ω is of class $\mathcal{C}^3$, then any L^∞ solution is in fact $\mathcal{C}^{0,\alpha}$ for every $\alpha < 1/2$ (see [25]). Furthermore, a uniform bound holds when β is bounded, similarly to Remark 6.1. Actually, the assumption on the smoothness of Ω can be weakened, at least when considering global problems for $u(\cdot) = v(\cdot, 0)$, as done in the recent paper [23].

We prove the following.

Theorem 8.2. *Let $\{\mathbf{v}_\beta\}_{\beta>0} \subset H^1(B_1^+)$ be a family of solutions to problem $(PD)_\beta$ on B_1^+ such that*

$$\|\mathbf{v}_\beta\|_{L^\infty(B_1^+)} \leq \bar{m}$$

with $\bar{m}$ independent of β. Then for every $\alpha \in (0, 1/2)$ there exists a constant $C = C(\bar{m}, \alpha)$, not depending on β, such that

$$\|\mathbf{v}_\beta\|_{\mathcal{C}^{0,\alpha}(\overline{B_{1/2}^+})} \leq C.$$

Furthermore, $\{\mathbf{v}_\beta\}_{\beta>0}$ is relatively compact in $H^1(B_{1/2}^+) \cap \mathcal{C}^{0,\alpha}(\overline{B_{1/2}^+})$ for every $\alpha < 1/2$.

Actually, two particular cases of the above theorem can be obtained in a rather direct way.

Remark 8.3. If $\partial^0 B^+ \cap \Omega = \emptyset$ then the conclusion of Theorem 8.2 holds true. Indeed, the family of functions obtained from $\{\mathbf{v}_\beta\}_{\beta>0}$ by odd reflection across $\{y = 0\}$ consists of harmonic, L^∞ uniformly bounded functions on B_1.

Remark 8.4 (Proof of Theorem 1.1). If $\partial^0 B^+ \subset \Omega$ then the conclusion of Theorem 8.2 holds true. This is indeed the content of Theorem 1.1, that is, of Theorem 6.2 with ν^{ACF} replaced by $1/2$. In order to prove this result, one can reason as in the proof of the latter theorem, by using Theorem 7.1 instead of Propositions 4.1 and 4.2 (also recall Remark 6.14).

Proof of Theorem 8.2. The proof follows the lines of the proof of Theorem 6.2, to which we refer the reader for further details. To start with, let η be a smooth cut-off function as in (6.1), and fix $\alpha \in (0, 1/2)$. We assume for contradiction that

$$\begin{aligned} L_n &:= \max_{i=1,\dots,k} \max_{X' \neq X'' \in \overline{B^+}} \frac{|(\eta v_{i,n})(X') - (\eta v_{i,n})(X'')|}{|X' - X''|^\alpha} \\ &= \frac{|(\eta v_{1,n})(X_n') - (\eta v_{i,n})(X_n'')|}{r_n^\alpha} \to \infty, \end{aligned}$$

where, as usual, $\mathbf{v}_n$ solves $(PD)_{\beta_n}$, $\beta_n \to \infty$, and $r_n := |X'_n - X''_n| \to 0$. Furthermore, reasoning as in the proofs of Lemmas 6.4 and 6.8, one can prove that the sequences $\{X'_n\}_{n\in\mathbb{N}}$ and $\{X''_n\}_{n\in\mathbb{N}}$ accumulate near $\partial^0 B^+$ and far away from $\partial^+ B^+$, at least in the scale of r_n.

Under the previous notation, we define the blow-up sequences

$$w_{i,n}(X) := \eta(P_n)\frac{v_{i,n}(P_n + r_n X)}{L_n r_n^\alpha} \quad \text{and} \quad \bar{w}_{i,n}(X) := \frac{(\eta v_{i,n})(P_n + r_n X)}{L_n r_n^\alpha},$$

where

$$P_n := (x'_n, 0) \quad \text{and} \quad X \in \tau_n B^+ := \frac{B^+ - P_n}{r_n}.$$

They have the following properties:

- $\{\bar{\mathbf{w}}_n\}_{n\in\mathbb{N}}$ have uniformly bounded Hölder quotients on $\tau_n\overline{B^+}$, and $\operatorname{osc} w_{1,n} = 1$ for every n on a suitable compact set;
- each $\mathbf{w}_n$ solves

$$\begin{cases} -\Delta w_{i,n} = 0 & \text{in } \tau_n B^+,\\ \partial_\nu w_{i,n} = f_{i,n}(w_{i,n}) - M_n w_{i,n}\sum_{j\neq i} w_{j,n}^2 & \text{on } \tau_n(\partial^0 B^+ \cap \Omega),\\ w_{i,n} = 0 & \text{on } \tau_n(\partial^0 B^+ \setminus \Omega), \end{cases}$$

  where $\sup|f_{i,n}(w_{i,n})| \to 0$ as $n \to \infty$;
- $|\mathbf{w}_n - \bar{\mathbf{w}}_n| \to 0$ uniformly as $n \to \infty$ on every compact set.

By the regularity assumption on $\partial\Omega$ we infer that, up to translations, rotations and taking subsequences, one of the following three cases must hold.

Case 1: $\tau_n(\partial^0 B^+ \setminus \Omega) \to \mathbb{R}^N$. In particular, we have $\mathbf{w}_n(0) = \bar{\mathbf{w}}_n(0) = 0$ for n large. Reasoning as in Section 6 we find that both $\mathbf{w}_n$ and $\bar{\mathbf{w}}_n$ converge, uniformly on compact sets, to the same $\mathbf{w}$ which is harmonic and globally Hölder continuous on $\mathbb{R}^{N+1}_+$, vanishing on $\mathbb{R}^N$ and nonconstant. But then the odd extension of $\mathbf{w}$ across $\{y = 0\}$ contradicts the Liouville theorem.

Case 2: $\tau_n(\partial^0 B^+ \cap \Omega) \to \mathbb{R}^N$. In this case, for every compact set $K \subset \overline{\mathbb{R}^{N+1}_+}$, the sequences $\{\mathbf{w}_n|_K\}_{n\in\mathbb{N}}$ and $\{\bar{\mathbf{w}}_n|_K\}_{n\in\mathbb{N}}$, for n large, fit in the setting of Section 6. Consequently, we can argue exactly in the same way, recalling that the regularity for every $\alpha < 1/2$ is obtained by means of Theorem 7.1 (see also Remark 8.4).

Case 3: $\tau_n(\partial^0 B^+ \cap \Omega) \to \{x \in \mathbb{R}^N : x_1 > 0\}$. As in the first case, we have $\mathbf{w}_n(0) = \bar{\mathbf{w}}_n(0) = 0$ for n large, implying that $w_{1,n} \to w_1$ uniformly on compact sets of $\overline{\mathbb{R}^{N+1}_+}$, with w_1 nonconstant, harmonic, and such that $w_1(x, 0) = 0$ for $x_1 \le 0$. Finally, reasoning as in Lemma 6.13, we infer that $w_{1,n} \to w_1$ also strongly in H^1_{loc}, thus $w_1\partial_\nu w_1 \le 0$. We apply Proposition 4.4 to w_1 and reach a contradiction. □

Using the above result, we can prove the following global theorem.

Theorem 8.5. *Let* $\{\mathbf{v}_\beta\} \in H^1_{\mathrm{loc}}(\mathbb{R}^N \times (0,1))$ *solve*

$$\begin{cases} -\Delta v_{i,\beta} = 0 & \text{in } \mathbb{R}^N \times (0,1), \\ \partial_\nu v_{i,\beta} = f_{i,\beta}(v_{i,\beta}) - \beta v_{i,\beta} \sum_{j\neq i} v_{j,\beta}^2 & \text{on } \Omega, \\ v_{i,\beta} = 0 & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases}$$

If there exists a constant $\bar{m}$ *such that*

$$\|v_{i,\beta}\|_{L^\infty(\mathbb{R}^N \times (0,1))} \leq \bar{m},$$

then for any $\alpha \in (0, 1/2)$*,*

$$\|\mathbf{v}_\beta\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^N \times [0,1/3])} \leq C(\bar{m}, \alpha).$$

Furthermore, $\{\mathbf{v}_\beta\}_{\beta>0}$ *is relatively compact in* $(H^1 \cap \mathcal{C}^{0,\alpha})_{\mathrm{loc}}$ *for every* $\alpha < 1/2$*.*

Proof. The assertion easily follows by a covering argument. Indeed, we can cover $\mathbb{R}^N \times [0, 1/3]$ with a countable number of half-balls of radius $1/2$, centered on $\mathbb{R}^N$, and apply Theorem 8.2 to each of the corresponding half-balls of radius 1. □

Proof of Theorem 1.3. This is actually a corollary of Theorem 8.5: indeed, if $u \in (H^{1/2} \cap L^\infty)(\mathbb{R}^N)$, and $v \in H^1(\mathbb{R}^{N+1}_+)$ is its unique harmonic extension satisfying

$$(-\Delta)^{1/2} u(\cdot) = -\partial_y v(\cdot, 0),$$

then v is uniformly bounded in L^∞. □

Remark 8.6. Analogous results can be proved, with minor changes, when the fractional operator considered is the spectral square root of the laplacian, as studied in [4]. Indeed, in that situation, the corresponding extension problem is given by

$$\begin{cases} -\Delta v_{i,\beta} = 0 & \text{in } \Omega \times (0,\infty), \\ \partial_\nu v_{i,\beta} = f_{i,\beta}(v_{i,\beta}) - \beta v_{i,\beta} \sum_{j\neq i} v_{j,\beta}^2 & \text{on } \Omega \times \{0\}, \\ v_{i,\beta} = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases}$$

and the starting regularity for β bounded is even finer. As a consequence, one can consider the extension of v which is trivial outside $\Omega \times (0, \infty)$, and conclude by using a modified version of Proposition 4.4, suitable for subharmonic functions.

9. $\mathcal{C}^{0,1/2}$ regularity of the limiting profiles

In this section we consider the regularity of the limiting profiles, that is, the accumulation points of solutions to problem $(P)_\beta$ as $\beta \to \infty$. In Section 6 we proved that if $\{\mathbf{v}_\beta\}_{\beta>0}$ is a family of solutions to problem $(P)_\beta$, and $\|\mathbf{v}_\beta\|_{L^\infty(B^+)} \leq \bar{m}$ for a constant $\bar{m}$ independent of β, then there exists a sequence $\mathbf{v}_n := \mathbf{v}_{\beta_n}$ such that $\beta_n \to \infty$ and

$$\mathbf{v}_n \to \mathbf{v} \quad \text{in } (H^1 \cap \mathcal{C}^{0,\alpha})(K \cap B^+)$$

for every compact set $K \subset B$ and every $\alpha \in (0, 1/2)$. Now we turn to the proof of Theorem 1.2, that is, we show that $\mathbf{v} \in \mathcal{C}^{0,1/2}_{\mathrm{loc}}(B^+ \cup \partial^0 B^+)$. Actually, we will prove this under a more general assumption: from now on we will assume that the reaction terms in problem $(P)_\beta$ satisfy

$$\lim_{n\to\infty} f_{i,n} = f_i \quad \text{uniformly on every compact set,}$$

where $(f_1, \ldots, f_k)$ are locally Lipschitz and such that, for some $\varepsilon > 0$,

$$2F_i(s) - sf_i(s) \geq -C|s|^{2+\varepsilon} \qquad \text{for } s \text{ sufficiently small,} \tag{9.1}$$

for every i, where $F_i(s) = \int_0^s f_i(t)\,dt$ (in particular, $f_i(0) = 0$).

Remark 9.1. If $f_i \in \mathcal{C}^{1,\varepsilon}$ in a neighborhood of 0, and $f_i(0) = 0$, then assumption (9.1) holds true. Indeed, this implies that $2F_i(s) - sf_i(s) = O(s^{2+\varepsilon})$ as $s \to 0$.

We will obtain Theorem 1.2 as a byproduct of a stronger result:

Proposition 9.2. *Let $\mathbf{v} \in H^1(B^+)$ be such that*

(1) $\mathbf{v} \in (H^1 \cap \mathcal{C}^{0,\alpha})(K \cap B^+)$ *for every compact set $K \subset B$ and every $\alpha \in (0, 1/2)$;*
(2) $v_i v_j|_{\partial^0 B^+} = 0$ *for all $j \neq i$ and*

$$\begin{cases} -\Delta v_i = 0 & \text{in } B^+, \\ v_i \partial_\nu v_i = v_i f_i(v_i) & \text{on } \partial^0 B^+, \end{cases}$$

where f_i is locally Lipschitz continuous and satisfies (9.1) for every $i = 1, \ldots, k$;
(3) *for every $x_0 \in \partial^0 B^+$ and a.e. $r > 0$ such that $B_r^+(x_0, 0) \subset B^+$, the following Pohozaev identity holds:*

$$(1-N)\int_{B_r^+} \sum_i |\nabla v_i|^2\,dx\,dy + r\int_{\partial^+ B_r^+} \sum_i |\nabla v_i|^2\,d\sigma$$
$$+ 2N\int_{\partial^0 B_r^+} \sum_i F_i(v_i)\,dx - 2r\int_{S_r^{N-1}} \sum_i F_i(v_i)\,d\sigma = 2r\int_{\partial^+ B_r^+} \sum_i |\partial_\nu v_i|^2\,d\sigma.$$

Then $\mathbf{v} \in \mathcal{C}^{0,1/2}(K \cap B^+)$ for every compact $K \subset B$.

As we mentioned, Theorem 1.2 will follow from the above proposition by virtue of the following result.

Lemma 9.3. *Let $\beta_n \to \infty$, and let $\mathbf{v}_n \in H^1(B^+)$ solve problem $(P)_{\beta_n}$ for every n and be such that*

$$\mathbf{v}_n \to \mathbf{v} \quad \text{in } (H^1 \cap \mathcal{C}^{0,\alpha})(K \cap B^+)$$

for every compact set $K \subset B$ and every $\alpha \in (0, 1/2)$. Moreover, suppose the corresponding reaction terms $f_{i,n}$ converge, uniformly on compact sets, to locally Lipschitz functions f_i satisfying (9.1). Then $\mathbf{v}$ fulfills the assumptions of Proposition 9.2.

Proof. The proof follows the lines of the one of Proposition 5.1, with minor changes. □

In view of the previous lemma, with a slight abuse of terminology, we will use the name “limiting profiles” also for functions which simply satisfy the assumptions of Proposition 9.2. For the rest of this section we will denote by $\mathbf{v}$ a fixed limiting profile.

In the proof of Proposition 9.2 we shall use a further monotonicity formula of Almgren type. For every $x_0 \in \partial^0 B^+$ and $r > 0$ such that $B_r^+(x_0, 0) \subset B^+$, we set

$$E(x_0, r) := \frac{1}{r^{N-1}}\left(\int_{B_r^+(x_0,0)} \sum_i |\nabla v_i|^2\, dx\, dy - \int_{\partial^0 B_r^+(x_0,0)} \sum_i f_i(v_i)v_i\, dx\right),$$

$$H(x_0, r) := \frac{1}{r^N}\int_{\partial^+ B_r^+(x_0,0)} \sum_i v_i^2\, d\sigma.$$

As usual, $E(x_0, r)$ admits an equivalent expression: indeed, multiplying the equation in (2) by v_i, integrating over $B_r^+(x_0, 0)$ and summing over $i = 1, \dots, k$ we obtain

$$E(x_0, r) = \frac{1}{r^{N-1}}\int_{\partial^+ B_r^+(x_0,0)} \sum_i v_i \partial_\nu v_i\, d\sigma = \frac{2}{r} H'(x_0, r). \tag{9.2}$$

The presence of internal reaction terms in the definition of E has to be dealt with. To this end, the next two lemmas will provide a crucial estimate in order to bound the Almgren quotient. Before we state them, let us recall the following Poincaré inequality: for every $p \in [2, p^\#]$, where $p^\# = 2N/(N-1)$ denotes the critical Sobolev exponent for trace embedding (or simply $p \geq 2$ in dimension $N = 1$), there exists a constant $C_P = C_P(N, p)$ such that, for every $w \in H^1(B_r^+)$,

$$\left[\frac{1}{r^N}\int_{\partial^0 B_r^+} |w|^p\, dx\right]^{2/p} \leq C_P\left[\frac{1}{r^{N-1}}\int_{B_r^+} |\nabla w|^2\, dx\, dy + \frac{1}{r^N}\int_{\partial^+ B_r^+} w^2\, d\sigma\right] \tag{9.3}$$

(such an inequality follows from the one on B^+ by scaling arguments).

Lemma 9.4. *For every $p \in [2, p^\#]$ there exist constants $C, \bar r > 0$ such that*

$$\left[\frac{1}{r^N}\int_{\partial^0 B_r^+} \sum_i |v_i|^p\, dx\right]^{2/p} \leq C[E(r) + H(r)] \quad \textit{for every } r \in (0, \bar r).$$

Proof. Since $\mathbf{v} \in L^\infty(B^+)$, and each f_i is locally Lipschitz continuous with $f_i(0) = 0$, we have

$$\left|\frac{1}{r^{N-1}}\int_{\partial^0 B_r^+} \sum_i f_i(v_i)v_i\, dx\right| \leq C\frac{1}{r^{N-1}}\int_{\partial^0 B_r^+} \sum_i v_i^2\, dx$$
$$\leq C' r\left[\frac{1}{r^{N-1}}\int_{B_r^+} \sum_i |\nabla v_i|^2\, dx\, dy + \frac{1}{r^N}\int_{\partial^+ B_r^+} \sum_i v_i^2\, d\sigma\right],$$

where we have used inequality (9.3) with $p = 2$. As a consequence,

$$E(r) + H(r) \geq (1 - Cr)\left[\frac{1}{r^{N-1}}\int_{B_r^+} \sum_i |\nabla v_i|^2\, dx\, dy + \frac{1}{r^N}\int_{\partial^+ B_r^+} \sum_i v_i^2\, d\sigma\right], \tag{9.4}$$

and the lemma follows by taking into account (9.3) and choosing $\bar r$ sufficiently small. □

For the following lemma we introduce, for $p \in (2, p^{\#}]$, the auxiliary function

$$\psi(x_0, r) := \left(\frac{1}{r^N}\int_{\partial^0 B_r^+(x_0,0)} \sum_i |v_i|^p \, dx\right)^{1-2/p},$$

which is bounded for r small. We have the following.

Lemma 9.5. *For every $p \in (2, p^{\#}]$ there exist constants $C, \bar{r} > 0$ such that*

$$\frac{1}{r^{N-1}}\int_{S_r^{N-1}} \sum_i |v_i|^p \, d\sigma \le C[E(r) + H(r)] \cdot \frac{d}{dr}(r\psi(r)) \quad \textit{for every } r \in (0, \bar{r}).$$

Proof. A direct computation yields the identity

$$\begin{aligned}\frac{d}{dr}\psi(r) &= \left(1 - \frac{2}{p}\right)\psi^{-2/(p-2)}\left(\frac{1}{r^N}\int_{\partial^0 B_r^+(x_0,0)} \sum_i |v_i|^p \, dx\right)' \\ &= \left(1 - \frac{2}{p}\right)\psi(r)\frac{(r^{-N}\int_{\partial^0 B_r^+} \sum_i |v_i|^p \, dx)'}{r^{-N}\int_{\partial^0 B_r^+} \sum_i |v_i|^p \, dx}.\end{aligned}$$

As a consequence,

$$\frac{d}{dr}(r\psi(r)) = \psi(r)\left[r\left(1 - \frac{2}{p}\right)\frac{\int_{S_r^{N-1}} \sum_i |v_i|^p \, d\sigma}{\int_{\partial^0 B_r^+} \sum_i |v_i|^p \, d\sigma} + \left(1 - N\left(1 - \frac{2}{p}\right)\right)\right].$$

Now, $p \le p^{\#}$ implies $N(1 - 2/p) \le 1$, so that

$$\frac{d}{dr}(r\psi(r)) \ge r\psi(r)\left(1 - \frac{2}{p}\right)\frac{\int_{S_r^{N-1}} \sum_i |v_i|^p \, d\sigma}{\int_{\partial^0 B_r^+} \sum_i |v_i|^p \, d\sigma}.$$

Recalling the definition of ψ and using Lemma 9.4, we finally obtain

$$(E(r) + H(r))\frac{d}{dr}(r\psi(r)) \ge C\frac{1}{r^{N-1}}\int_{S_r^{N-1}} \sum_i |v_i|^p \, d\sigma,$$

where $C > 0$ since $p > 2$. □

As a matter of fact, we need to estimate the Almgren quotient only on the zero set of $\mathbf{v}$ (which is well defined since $\mathbf{v}$ is continuous).

Definition 9.6. We define the *boundary zero set* of the limiting profile $\mathbf{v}$ as

$$\mathcal{Z} = \{x \in \partial^0 B^+ : \mathbf{v}(x, 0) = 0\}.$$

Remark 9.7. A natural notion of free boundary, associated to a limiting profile $\mathbf{v}$, is the set in which the boundary condition of assumption (2) does not reduce to

$$\partial_\nu v_i = f_i(v_i), v_j \equiv 0 \quad \text{for some } j \ne i,$$

that is, *a posteriori*, the support of the singular part of the measure $\partial_\nu \mathbf{v}$. It is then clear that the free boundary is a subset of $\mathcal{Z} \subset \mathbb{R}^N$.

We are now in a position to state the Almgren type result which we use in this framework. As we mentioned, we prove it only at points of $\mathcal{Z}$; furthermore, it concerns boundedness of a (modified) Almgren quotient, rather than its monotonicity. More precisely, let

$$N(x_0, r) := \frac{E(x_0, r)}{H(x_0, r)} + 1.$$

Lemma 9.8. *There exist constants $C, \bar{r} > 0$ such that, for all $x_0 \in \mathcal{Z}$, $r \in (0, \bar{r})$ and $B_r^+(x_0, 0) \subset B^+$, we have:*

(1) $H(r), N(r) > 0$ *on* $(0, \bar{r})$*;*
(2) *the function* $r \mapsto e^{Cr(1+\psi(r))} N(x_0, r)$ *is nondecreasing;*
(3) $N(x_0, 0^+) \geq 1 + 1/2$.

Proof. The proof is similar to the one of Theorem 3.3, but in this case the internal reaction terms do not vanish. Let $x_0 \in \mathcal{Z}$ and let $\bar{r}$ be such that both the conclusions of Lemmas 9.4 and 9.5 hold. First, we ensure that the Almgren quotient, where defined, is nonnegative. Indeed, by Lemma 9.4,

$$E(r) + H(r) \geq 0, \quad \text{so} \quad N(r) = \frac{E(r)}{H(r)} + 1 \geq 0,$$

whenever $H(r) \neq 0$. By continuity of H we can consider, as in the proof of Theorem 3.3, a neighborhood of r where H does not vanish. We compute the derivative of E and we use the Pohozaev identity (assumption (3) of Proposition 9.2) to obtain

$$\begin{aligned}
E'(r) &= \frac{1-N}{r^N}\left(\int_{B_r^+} \sum_i |\nabla v_i|^2 \, dx\, dy - \int_{\partial^0 B_r^+} \sum_i v_i f_i(v_i)\, dx\right) \\
&\quad + \frac{1}{r^{N-1}}\left(\int_{\partial^+ B_r^+} \sum_i |\nabla v_i|^2 \, dx\, dy - \int_{S_r^{N-1}} \sum_i v_i f_i(v_i)\, dx\right) \\
&= \underbrace{\frac{2}{r^{N-1}} \int_{\partial^+ B_r^+} \sum_i |\partial_\nu v_i|^2 \, d\sigma}_{T} + \underbrace{\frac{1}{r^N} \int_{\partial^0 B_r^+} \Big[(N-1) \sum_i v_i f_i(v_i) - 2N \sum_i F_i(v_i)\Big] dx}_{I} \\
&\quad + \underbrace{\frac{1}{r^{N-1}} \int_{S_r^{N-1}} \Big[-\sum_i v_i f_i(v_i) + 2\sum_i F_i(v_i)\Big] d\sigma}_{Q}.
\end{aligned}$$

Since $\mathbf{v} \in L^\infty$, and since f_i are locally Lipschitz and $f_i(0) = 0$, there exists a positive constant C such that

$$|f(v_i)v_i| \leq C v_i^2 \quad \text{and} \quad |F(v_i)| \leq C v_i^2.$$

A direct application of Lemma 9.4 (with $p = 2$) yields

$$I \geq -C(E + H).$$

On the other hand, by assumption (9.1) and Lemma 9.5 (it is sufficient to choose $p = \min\{2+\varepsilon, p^\#\}$), we obtain

$$Q \geq -C(E+H)(r\psi)'.$$

The two estimates yield

$$E' \geq T - C[1+(r\psi)'](E+H).$$

Therefore, differentiating the Almgren quotient and using the Cauchy–Schwarz inequality, we obtain

$$\frac{N'}{N} = \frac{E'+H'}{E+H} - \frac{H'}{H} \geq \frac{TH - EH'}{H(E+H)} - C[1+(r\psi)'] \geq -C[1+(r\psi)'],$$

which implies that the function $e^{Cr(1+\psi(r))}N(r)$ is nondecreasing as far as $H(r) \neq 0$. Equation (9.2) directly implies

$$\frac{d}{dr}\log H(r) = \frac{H'(r)}{H(r)} = \frac{2E(r)}{rH(r)} = \frac{2(N(r)-1)}{r};$$

reasoning as in the proof of Theorem 3.3, we can use this formula, together with the bound

$$N(r) \leq e^{Cr^*(1+\psi(r^*))}N(r^*) \qquad \text{for every } r \leq r^*,$$

in order to obtain the strict positivity of H for $r \in (0, \bar{r})$ (for a possibly smaller $\bar{r}$). Finally, reasoning as in the proof of Lemma 3.4(2), assume for contradiction that, for some $r^* < \bar{r}$ and $\varepsilon > 0$, we have $e^{Cr^*(1+\psi(r^*))}N(r^*) \leq 3/2 - \varepsilon$. By the above bound we obtain

$$\frac{d}{dr}\log H(r) \leq \frac{2(e^{Cr^*(1+\psi(r^*))}N(r^*)-1)}{r} \leq \frac{1-2\varepsilon}{r}$$

for every $r \in (0, r^*)$. But this contradicts the fact that $\mathbf{v} \in \mathcal{C}^{0,\alpha}$ for $\alpha = (1-\varepsilon)/2$. □

The proof of Proposition 9.2 is based on a contradiction argument, involving the Morrey inequality. Indeed, let $K \subset B$ be compact, and define, for every $X \in K \cap \{y \geq 0\}$ and every $r < \operatorname{dist}(K, \partial B)$,

$$\Phi(X,r) := \frac{1}{r^N}\int_{B_r(X)\cap\{y>0\}} \sum_i |\nabla v_i|^2 \, dx\, dy.$$

It is well known that if Φ is bounded then $\mathbf{v} \in \mathcal{C}^{0,1/2}(K \cap B^+)$.

As a consequence of Lemma 9.8, we can prove a first estimate of Φ.

Lemma 9.9. *For every compact $K \subset B$ there exist constants $C, \bar{r} > 0$ such that for all $x_0 \in \mathcal{Z} \cap K$ and $r \in (0, \bar{r})$,*

$$\Phi(x_0, r) \leq C.$$

Proof. If $\bar{r}$ is sufficiently small, from Lemma 9.8 we know that

$$\tfrac{3}{2}e^{-Cr(1+\psi(r))} \le N(r) \le C$$

for every $r \in (0, \bar{r})$. Since $E + H = NH$, (9.4) implies that

$$\frac{1}{r^N}\int_{B_r^+(x_0,0)} \sum_i |\nabla v_i|^2\,dx\,dy \le C\frac{H(r)}{r}.$$

On the other hand, by the lower estimate on N,

$$\frac{d}{dr}\log\frac{H(r)}{r} \ge 3\frac{e^{-Cr(1+\psi(r))}-1}{r} \ge -3C(1+\psi(r)) \ge -C.$$

Integrating, we obtain

$$\frac{H(r)}{r} \le e^{C\bar{r}}\frac{H(\bar{r})}{\bar{r}},$$

and the lemma follows. □

The above result can be complemented with the following lemma.

Lemma 9.10. *For every compact $K \subset B$ there exist constants $C, \bar{r} > 0$ such that for all $x_0 \in (K \cap \{y = 0\}) \setminus \mathcal{Z}$ and*

$$0 < r < d := \min\{\mathrm{dist}(x_0, \mathcal{Z}), \bar{r}\},$$

we have

$$\Phi(x_0, d) \ge C\Phi(x_0, r).$$

Proof. Since $x_0 \notin \mathcal{Z}$ and $r \le \mathrm{dist}(x_0, \mathcal{Z})$, we can assume that $v_j \equiv 0$ on $\partial^0 B_r^+(x_0, 0)$ for, say, $j \ge 2$. As a consequence, the odd extension of v_j across $\{y = 0\}$ is harmonic on $B_r(x_0, 0)$, and the mean value property applied to the subharmonic function $|\nabla v_j|^2$ yields

$$\frac{1}{r^N}\int_{B_r^+(x_0,0)} |\nabla v_j|^2\,dx\,dy \le \frac{r}{d}\,\frac{1}{d^N}\int_{B_d^+(x_0,0)} |\nabla v_j|^2\,dx\,dy \quad \text{for every } j \ge 2. \tag{9.5}$$

We now show that a similar estimate also holds for v_1. Indeed, let $u := |\nabla v_1|^2$; by a straightforward computation,

$$\begin{cases} -\Delta u \le 0 & \text{in } B_d^+, \\ \partial_\nu u \le au & \text{in } \partial^0 B_d^+, \end{cases}$$

where $a := 2\|f_1'(v_1)\|_{L^\infty(B^+)}$ is bounded by assumption. Now, by scaling, one can show that if $\bar{r} = \bar{r}(a)$ is sufficiently small, then the equation

$$\begin{cases} -\Delta\varphi = 0 & \text{in } B_{\bar{r}}^+, \\ \partial_\nu\varphi = a\varphi & \text{on } \partial^0 B_{\bar{r}}^+, \end{cases}$$

admits a strictly positive (and smooth) solution. By the definition of d we deduce that

$$\begin{cases} -\operatorname{div}(\varphi^2\nabla(u/\varphi)) \leq 0 & \text{in } B_d^+, \\ \varphi^2\partial_\nu(u/\varphi) \leq 0 & \text{on } \partial^0 B_d^+, \end{cases}$$

so that the even extension of u is a solution to

$$-\operatorname{div}\left(\varphi^2\nabla\frac{u}{\varphi}\right) \leq 0 \quad \text{in } B_d.$$

Integrating this on any ball B_r, we obtain

$$\int_{\partial B_r} \varphi^2\partial_\nu\frac{u}{\varphi}\,d\sigma \geq 0.$$

If we set

$$H(r) = \frac{1}{r^N}\int_{\partial B_r} \varphi u\,d\sigma = \int_{\partial B} \varphi^2(rx)\frac{u(rx)}{\varphi(rx)}\,d\sigma,$$

a straightforward computation shows that

$$H'(r) = \frac{2}{r^N}\int_{\partial B_r} u\varphi\frac{\partial_\nu\varphi}{\varphi}\,d\sigma + \frac{1}{r^N}\int_{\partial B_r}\varphi^2\partial_\nu\frac{u}{\varphi}\,d\sigma \geq -2\left\|\frac{\partial_\nu\varphi}{\varphi}\right\|_{L^\infty(B)} H(r) \geq -CH(r),$$

that is, the function $r \mapsto e^{Cr}H(r)$ is nondecreasing in r. Hence, for all $0 < r_1 \leq r_2 \leq d$, we obtain $H(r_1) \leq CH(r_2)$. Multiplying by $r_1^N r_2^N$ and integrating over $(0,r)\times(r,d)$ with $r \leq d$, we obtain

$$\left(1 - \frac{r^{N+1}}{d^{N+1}}\right)\frac{1}{r^{N+1}}\int_{B_r}\varphi u\,dx\,dy \leq \frac{C}{d^{N+1}}\int_{B_d\setminus B_r}\varphi u\,dx\,dy.$$

Adding $Cd^{-N-1}\int_{B_r}\varphi u\,dx\,dy$, we infer

$$\frac{1}{r^{N+1}}\int_{B_r}\varphi u\,dx\,dy \leq C\frac{1}{d^{N+1}}\int_{B_d}\varphi u\,dx\,dy.$$

Recalling that φ is positive and bounded, and that $u = |\nabla v_1|^2$, we finally obtain

$$\frac{1}{r^N}\int_{B_r^+(x_0,0)}|\nabla v_1|^2\,dx\,dy \leq C\frac{r}{d}\frac{1}{d^N}\int_{B_d^+(x_0,0)}|\nabla v_1|^2\,dx\,dy.$$

The lemma now follows by adding this inequality to (9.5) for $j = 2,\dots,k$, and recalling that $d/r \geq 1$. □

End of the proof of Proposition 9.2. Assume for contradiction that there exists a sequence $\{(X_n,r_n)\}_{n\in\mathbb{N}}$ such that $X_n = (x_n,y_n) \in K\cap\{y \geq 0\}$, $r_n < \operatorname{dist}(K,\partial B)$, and

$$\Phi(X_n,r_n) \to \infty \quad \text{as } n \to \infty.$$

It is immediate to prove that $r_n \to 0$ and $y_n \to 0$: indeed, $\mathbf{v}$ is H^1 and harmonic for $\{y > 0\}$. In particular, the sequence $\{X_n\}_{n\in\mathbb{N}}$ accumulates on $\partial^0 K$. First we observe that, thanks to the subharmonicity of $\sum_i |\nabla v_i|^2$, if $r_n < y_n$ then

$$\Phi(X_n, y_n) \geq \frac{y_n}{r_n}\Phi(X_n, r_n) \geq \Phi(X_n, r_n);$$

as a consequence, we can assume without loss of generality that $r_n \geq y_n$. Analogously, once $r_n \geq y_n$, we have

$$\Phi((x_n, 0), 2r_n) \geq \frac{1}{2^N}\Phi(X_n, r_n),$$

and again, without loss of generality, we can assume that $y_n = 0$ for every n, and drop it from our notation.

Now, by the result of Lemma 9.10, the sequence (x_n, r_n) can be replaced by a sequence of points in $\mathcal{Z}$. Indeed, if $\operatorname{dist}(x_n, \mathcal{Z}) > \bar{r}$ for every $n \in \mathbb{N}$, then

$$\Phi(x_n, r_n) \leq C\Phi(x_n, \bar{r}),$$

and the right hand side is bounded since $\mathbf{v} \in H^1(B^+)$. As a consequence, $\operatorname{dist}(x_n, \mathcal{Z}) \leq \bar{r}$, and so

$$\Phi(x_n, r_n) \leq C\Phi(x_n, \operatorname{dist}(x_n, \mathcal{Z})).$$

Since the set $\mathcal{Z}$ is locally closed and $\operatorname{dist}(K, \partial^+ B) > 0$, for n sufficiently large, to each x_n we can associate $x'_n \in \mathcal{Z}$ such that $\operatorname{dist}(x_n, \mathcal{Z}) = |x_n - x'_n| \leq \frac{1}{2}\operatorname{dist}(x_n, \partial^+ B)$ and we can replace the pair $(x_n, \operatorname{dist}(x_n, \mathcal{Z}))$ with $(x'_n, 2\operatorname{dist}(x_n, \mathcal{Z}))$. Applying Lemma 9.9, we find a contradiction to the unboundedness of the Morrey quotient. □

Acknowledgments. This research was partly supported by PRIN2009 grant "Critical Point Theory and Perturbative Methods for Nonlinear Differential Equations".

References

[1] Almgren, F. J., Jr.: Dirichlet's problem for multiple valued functions and the regularity of mass minimizing integral currents. In: Minimal Submanifolds and Geodesics (Tokyo, 1977), North-Holland, Amsterdam, 1–6 (1979) Zbl 0439.49028 MR 0574247

[2] Alt, H. W., Caffarelli, L. A., Friedman, A.: Variational problems with two phases and their free boundaries. Trans. Amer. Math. Soc. **282**, 431–461 (1984) Zbl 0844.35137 MR 0732100

[3] Athanasopoulos, I., Caffarelli, L. A., Salsa, S.: The structure of the free boundary for lower dimensional obstacle problems. Amer. J. Math. **130**, 485–498 (2008) Zbl 1185.35339 MR 2405165

[4] Cabré, X., Tan, J.: Positive solutions of nonlinear problems involving the square root of the Laplacian. Adv. Math. **224**, 2052–2093 (2010) Zbl 1198.35286 MR 2646117

[5] Caffarelli, L. A., Karakhanyan, A. L., Lin, F.-H.: The geometry of solutions to a segregation problem for nondivergence systems. J. Fixed Point Theory Appl. **5**, 319–351 (2009) Zbl 1217.35015 MR 2529504

[6] Caffarelli, L. A., Lin, F.-H.: Singularly perturbed elliptic systems and multi-valued harmonic functions with free boundaries. J. Amer. Math. Soc. **21**, 847–862 (2008) Zbl 1194.35138 MR 2393430

[7] Caffarelli, L. A., Roquejoffre, J.-M., Savin, O.: Nonlocal minimal surfaces. Comm. Pure Appl. Math. **63**, 1111–1144 (2010) Zbl 1248.53009 MR 2675483

[8] Caffarelli, L. A., Roquejoffre, J.-M., Sire, Y.: Variational problems for free boundaries for the fractional Laplacian. J. Eur. Math. Soc. **12**, 1151–1179 (2010) Zbl 1221.35453 MR 2677613

[9] Caffarelli, L. A., Salsa, S., Silvestre, L.: Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian. Invent. Math. **171**, 425–461 (2008) Zbl 05248092 MR 2367025

[10] Caffarelli, L. A., Silvestre, L.: An extension problem related to the fractional Laplacian. Comm. Partial Differential Equations **32**, 1245–1260 (2007) Zbl 1143.26002 MR 2354493

[11] Caffarelli, L. A., Vasseur, A.: Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation. Ann. of Math. (2) **171**, 1903–1930 (2010) Zbl 1204.35063 MR 2680400

[12] Caliari, M., Squassina, M.: Location and phase segregation of ground and excited states for 2D Gross–Pitaevskii systems. Dynam. Partial Differential Equations **5**, 117–137 (2008) Zbl 1158.35322 MR 2435464

[13] Chang, S.-M., Lin, C.-S., Lin, T.-C., Lin, W.-W.: Segregated nodal domains of two-dimensional multispecies Bose–Einstein condensates. Phys. D **196**, 341–361 (2004) Zbl 1098.82602 MR 2090357

[14] Conti, M., Terracini, S., Verzini, G.: Nehari's problem and competing species systems. Ann. Inst. H. Poincaré Anal. Non Linéaire **19**, 871–888 (2002) Zbl 1090.35076 MR 1939088

[15] Conti, M., Terracini, S., Verzini, G.: An optimal partition problem related to nonlinear eigenvalues. J. Funct. Anal. **198**, 160–196 (2003) Zbl 1091.35051 MR 1962357

[16] Conti, M., Terracini, S., Verzini, G.: Asymptotic estimates for the spatial segregation of competitive systems. Adv. Math. **195**, 524–560 (2005) Zbl 1126.35016 MR 2146353

[17] Dancer, E. N., Wang, K., Zhang, Z.: Uniform Hölder estimate for singularly perturbed parabolic systems of Bose–Einstein condensates and competing species. J. Differential Equations **251**, 2737–2769 (2011) Zbl 1270.35043 MR 2831712

[18] Dancer, E. N., Wang, K., Zhang, Z.: Dynamics of strongly competing systems with many species. Trans. Amer. Math. Soc. **364**, 961–1005 (2012) Zbl 1252.35284 MR 2846360

[19] Dancer, E. N., Wang, K., Zhang, Z.: The limit equation for the Gross–Pitaevskii equations and S. Terracini's conjecture. J. Funct. Anal. **262**, 1087–1131 (2012) Zbl 1242.35119 MR 3004961

[20] De Silva, D., Roquejoffre, J. M.: Regularity in a one-phase free boundary problem for the fractional Laplacian. Ann. Inst. H. Poincaré Anal. Non Linéaire **29**, 335–367 (2012) Zbl 1251.35178 MR 2926238

[21] Garofalo, N., Petrosyan, A.: Some new monotonicity formulas and the singular set in the lower dimensional obstacle problem. Invent. Math. **177**, 415–461 (2009) Zbl 1175.35154 MR 2511747

[22] Noris, B., Tavares, H., Terracini, S., Verzini, G.: Uniform Hölder bounds for nonlinear Schrödinger systems with strong competition. Comm. Pure Appl. Math. **63**, 267–302 (2010) Zbl 1189.35314 MR 2599456

[23] Ros-Oton, X., Serra, J.: The Dirichlet problem for the fractional Laplacian: regularity up to the boundary. J. Math. Pures Appl. (9) **101**, 275–302 (2014) Zbl 1285.35020 MR 3168912

[24] Quitalo, V.: A free boundary problem arising from segregation of populations with high competition. Arch. Ration. Mech. Anal. **210**, 857–908 (2013) Zbl 1294.35177 MR 3116007

[25] Shamir, E.: Regularization of mixed second-order elliptic problems. Israel J. Math. **6**, 150–168 (1968) Zbl 0157.18202 MR 0239272

[26] Tavares, H., Terracini, S.: Regularity of the nodal set of segregated critical configurations under a weak reflection law. Calc. Var. Partial Differential Equations **45**, 273–317 (2012) Zbl 1263.35101 MR 2984134

[27] Terracini, S., Verzini, G., Zilio, A.: Uniform Hölder regularity with small exponent in competition-fractional diffusion systems. Discrete Contin. Dynam. Systems **34**, 2669–2691 (2014) Zbl 1285.35017 MR 3177654

[28] Troianiello, G. M.: Elliptic Differential Equations and Obstacle Problems. Plenum Press, New York (1987) Zbl 0655.35002 MR 1094820

[29] Verzini, G., Zilio, A.: Strong competition versus fractional diffusion: the case of Lotka–Volterra interaction. Comm. Partial Differential Equations **39**, 2284–2313 (2014) Zbl 1327.35418 MR 3259557

[30] Wang, K.: On the De Giorgi type conjecture for an elliptic system modeling phase separation. Comm. Partial Differential Equations **39**, 696–739 (2014) Zbl 1298.35062 MR 3178074

[31] Wang, K., Zhang, Z.: Some new results in competing systems with many species. Ann. Inst. H. Poincaré Anal. Non Linéaire **27**, 739–761 (2010) Zbl 1201.35113 MR 2595199

[32] Wei, J., Weth, T.: Asymptotic behaviour of solutions of planar elliptic systems with strong competition. Nonlinearity **21**, 305–317 (2008) Zbl 1132.35482 MR 2384550