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Bernstein inequalities with nondoubling weights

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Abstract. We answer Totik's question on weighted Bernstein inequalities by showing that

$$\|T_n'\|_{L_p(\omega)} \le C(p,\omega)n\|T_n\|_{L_p(\omega)}, \quad 0 < p \le \infty,$$

for all trigonometric polynomials T_n and certain nondoubling weights ω. Moreover, we find necessary conditions on ω for Bernstein's inequality to hold. We also prove weighted Markov, Remez, and Nikolskii inequalities for trigonometric and algebraic polynomials.

Keywords. Bernstein inequality, nondoubling weights, Remez inequality, Nikolskii inequality

Contents

1. Introduction

The famous Bernstein inequality for trigonometric polynomials T_n of degree at most n,

$$\|T_n'\|_{L_p(\mathbb{T})} \le Cn\|T_n\|_{L_p(\mathbb{T})}, \tag{1.1}$$

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plays an important role in modern analysis. Here, $\|\cdot\|_{L_p(\mathbb{T})}$ is the L_p-(quasi)norm,

$$\|f\|_{L_p(\mathbb{T})} = \left(\int_{\mathbb{T}} |f(t)|^p\, dt\right)^{1/p}, \qquad 0 < p < \infty,$$

with the usual modification for $p = \infty$. Bernstein proved (1.1) for $p = \infty$; the case $p < \infty$ was settled by Zygmund [Zy]. The best constant C is equal to 1 for any $p \in (0, \infty]$ (see [Ri, Zy, Ar]).

For algebraic polynomials P_n of degree at most n, the Bernstein inequality is given by

$$|P_n'(x)| \le \frac{n}{\sqrt{1-x^2}}\|P_n\|_{C[-1,1]}, \qquad x \in (-1, 1),$$

where $\|\cdot\|_{C[-1,1]}$ denotes the supremum norm on $[-1, 1]$. Its L_p-version is

$$\|\sqrt{1-x^2}\, P_n'(x)\|_{L_p[-1,1]} \le C(p)n\|P_n\|_{L_p[-1,1]}, \qquad 0 < p \le \infty. \tag{1.2}$$

Another important inequality for the derivative of algebraic polynomials is the following Markov inequality:

$$\|P_n'\|_{L_p[-1,1]} \le C(p)n^2\|P_n\|_{L_p[-1,1]}, \qquad 0 < p \le \infty. \tag{1.3}$$

Both Bernstein and Markov inequalities for trigonometric and algebraic polynomials respectively were extended to the case of smaller intervals (Privalov, Jackson, and Bary; see with [Ba]) and several intervals (see the recent paper by Totik [To1]).

In this paper we study weighted analogues of Bernstein's inequality,

$$\|T_n'\|_{L_p(\omega)} \le C(p, \omega)n\|T_n\|_{L_p(\omega)}, \tag{1.4}$$

where ω is a weight function, i.e., a nonnegative integrable function on $\mathbb{T}$. Here and in what follows, $\|T_n\|_{L_p(\omega)} = (\int_{\mathbb{T}} |T_n|^p \omega)^{1/p}$ if $p < \infty$, and $\|T_n\|_{L_\infty(\omega)} = \operatorname{ess\,sup}_{t\in\mathbb{T}} |T_n(t)\omega(t)|$.

First, we note that Muckenhoupt's A_p condition on weights ensures that (1.4) holds for $1 < p < \infty$. This follows from the fact that the Marcinkiewicz multiplier theorem and Littlewood–Paley decomposition hold in $L_p(\omega)$ with $\omega \in A_p$. In [MT], Mastroianni and Totik proved a much stronger result: (1.4) holds for any weight ω satisfying the doubling condition and for $1 \le p < \infty$. Later, a similar result was shown for $0 < p < 1$ (see [Er3]).

We recall that a periodic weight function ω satisfies the *doubling condition* if

$$W(2I) \le LW(I) \tag{1.5}$$

for all intervals I, where L is a constant independent of I, $2I$ is the interval twice the length of I and with the midpoint coinciding with that of I, and

$$W(I) = \int_I \omega(t)\, dt.$$

Also recall that a weight ω satisfies the A_∞ *condition* if for every $\alpha > 0$ there is $\beta > 0$ such that

$$W(E) \geq \beta W(I)$$

for any interval I and any measurable set $E \subset I$ with $|E| \geq \alpha|I|$. It is known [St, Ch. V] that any A_∞ weight satisfies the doubling condition. Here and in what follows, $|E|$ denotes the Lebesgue measure of the set E.

For the supremum norm, in addition to the natural assumption that ω is bounded, one needs the A^* *condition*: there exists a constant L such that for all intervals $I \subset [-\pi, \pi]$ and $t \in I$ we have

$$\omega(t) \leq \frac{L}{|I|} W(I).$$

This condition is stronger than the A_∞ condition and it is sufficient for (1.4) to hold when $p = \infty$.

In [To2], Totik posed the following question: under which condition on a general (not necessarily doubling) weight ω does the Bernstein inequality (1.4) hold for any trigonometric polynomial T_n of degree at most n? In this paper we aim to answer this question. We deal with the weight functions from the class Ω, defined below.

Definition. Let

$$\omega(t) = \exp(-F(g(t))), \quad t \in \mathbb{T},$$

where $g : \mathbb{T} \to [-A, A]$, $A > 0$, is an analytic function with

$$|g^{(n)}(t)| \leq D^n n!, \quad t \in \mathbb{T},\ n = 1, 2, \ldots, \tag{1.6}$$

such that each zero of g is of multiplicity one. Let also $F : [-A, A] \setminus \{0\} \to (0, \infty)$ be an even function, C^∞ on $(0, A]$, such that

(F1) $F(x) \to \infty$ as $x \to 0+$;
(F2) F is decreasing on $(0, A]$;
(F3) $|F^{(n)}(x)| \leq B^n n^n F(x)/x^n$ for all $x \in (0, A]$ and $n = 1, 2, \ldots$;
(F4) there exist $A_1, A_2 > 0$ such that

$$A_2 \leq \frac{|F'(x)|x}{F(x)} \leq A_1, \quad x \in (0, A].$$

Then we write $\omega \in \Omega$.

It is worth mentioning that all our results hold for weights $\omega(t) = \exp(-F(g(t)))$, where F satisfies (F1)–(F4) only for $x \in (0, \varepsilon)$ for some $0 < \varepsilon < A$ and

$$|F^{(n)}(x)| \leq B^n n^n F(x), \quad x \in [\varepsilon, A],\ n = 1, 2, \ldots.$$

A typical example of the function g is $\sin t$ or $\cos t$. Note that $\omega \in \Omega$ is nondoubling if and only if g has at least one zero on $\mathbb{T}$. In what follows this will be assumed to be the case. Below we give some examples of functions F satisfying properties (F1)–(F4). We define them for $x > 0$.

Examples. 1. Let

$$F(x) = x^{-\alpha},\ x^{-\alpha}|\log x|^{\xi_1},\ x^{-\alpha}|\log x|^{\xi_1}\cdots|\log_k x|^{\xi_k},\ x^{-\alpha}\exp|\log x|^{\xi},$$

where $\alpha > 0$, $\xi_j \in \mathbb{R}$, $\xi \in (0, 1)$, and $\log_j x = \log_{j-1}|\log x|$. Note that any such function F is of *regular variation* of index $-\alpha$, i.e., for all $r > 0$,

$$\lim_{x\to 0+} \frac{F(rx)}{F(x)} = r^{-\alpha}, \tag{1.7}$$

or equivalently

$$F(x) = \frac{1}{x^{\alpha}}\eta(x),$$

where η is a *slowly varying* function, i.e., $\lim_{x\to 0+}\eta(rx)/\eta(x) = 1$.

2. Note that there are functions satisfying (F1)–(F4) which are not regularly varying. For example, the function

$$F(x) = \exp\{-\log x(2 + \sin(\log_3 x))\}$$

is such that

$$\limsup_{x\to 0+} F(x)x^3 = 1 \quad \text{and} \quad \liminf_{x\to 0+} F(x)x = 1,$$

i.e., (1.7) does not hold. To show that F satisfies (F3) one can use Faà di Bruno's formula.

The main results of the paper are the following Theorems 1.1–1.3.

Theorem 1.1. *For $0 < p \le \infty$ and $\omega = \omega_1 \ldots \omega_s$ such that $\omega_i \in \Omega$, $i = 1, \ldots, s$, the Bernstein inequality*

$$\|T_n'\|_{L_p(\omega u)} \le Cn\|T_n\|_{L_p(\omega u)} \tag{1.8}$$

holds for any trigonometric polynomial T_n of degree at most n with $C = C(\omega, u, p)$ whenever u is doubling if $p < \infty$, and u satisfies the A^* condition if $p = \infty$.

For example, inequality (1.8) holds for the following weight:

$$\omega(t) = \exp(-1/\sin^2 t - 1/\cos^4 t).$$

To prove Bernstein's inequality (1.8) in the case when $\omega = \omega_1 \in \Omega$, i.e., $s = 1$, we use approximation properties of ω. To verify (1.8) with the product of weights each of which is from the class Ω, we need a new technique based on introduction of weighted classes for which Bernstein and Remez inequalities hold. In particular, $\omega_i \in \Omega$ and u as in Theorem 1.1 belong to these classes. This technique is developed in Sections 5 and 6.

A necessary condition for Bernstein's inequality to hold is given by the following result.

Theorem 1.2. *Let $\omega \in C(\mathbb{T})$ be a weight function with $\omega \searrow$ on $(-\epsilon, 0)$, $\omega(0) = 0$, $\omega \nearrow$ on $(0, \epsilon)$, and*

$$\limsup_{t \to 0} \frac{\log \omega(rt)}{\log \omega(t)} = \infty \quad \text{for some } r \in (0, 1). \tag{1.9}$$

Then for each $0 < p \le \infty$ there exist a sequence of positive integers $K_n \to \infty$ as $n \to \infty$ and a sequence of trigonometric polynomials Q_{K_n} of degree at most K_n such that

$$\lim_{n \to \infty} \frac{\|Q'_{K_n}\|_{L_p(\omega)}}{K_n \|Q_{K_n}\|_{L_p(\omega)}} = \infty.$$

Theorems 1.1 and 1.2 provide a sharp condition on the growth properties of a weight ω near the origin. Specifically, if a weight ω with $\omega \searrow$ on $(-\epsilon, 0)$, $\omega(0) = 0$, $\omega \nearrow$ on $(0, \epsilon)$ is such that Bernstein's inequality (1.4) holds, then necessarily, for all $r \in (0, 1)$,

$$\limsup_{t \to 0} \frac{\log \omega(rt)}{\log \omega(t)} = L < \infty. \tag{1.10}$$

On the other hand, any $\omega \in \Omega$ satisfies (1.10). Moreover, for each $r \in (0, 1)$ and $L > 1$, the weight $\omega(t) = \exp(-|\sin t|^{-\alpha})$ fulfills (1.10) with $\alpha = -\log_r L$. Thus $\omega \in \Omega$ and by Theorem 1.1 Bernstein's inequality (1.4) holds for this weight.

If in (1.9) the limit (not only the limit superior) exists, then a stronger result is true:

Theorem 1.3. *Let $\omega \in C(\mathbb{T})$ be a weight function with $w \searrow$ on $(-\epsilon, 0)$, $\omega(0) = 0$, $\omega \nearrow$ on $(0, \epsilon)$, and*

$$\lim_{t \to 0} \frac{\log \omega(rt)}{\log \omega(t)} = \infty \quad \text{for each } r \in (0, 1).$$

Then for each $0 < p \le \infty$ there exists a sequence of trigonometric polynomials Q_n of degree at most n such that

$$\lim_{n \to \infty} \frac{\|Q'_n\|_{L_p(\omega)}}{n \|Q_n\|_{L_p(\omega)}} = \infty.$$

The paper is organized as follows. In Section 2 we discuss growth properties of weights from the class Ω. Section 3 presents the order of trigonometric approximation of functions from Ω as well as of their derivatives. In Section 4 we give the proof of Bernstein's inequality with Ω-weights in L_1. We will use it as a model case to prove the general Bernstein inequality (1.8) in Section 6.

In Section 5 we establish weighted Remez inequalities for trigonometric and algebraic polynomials. Section 6 gives the proof of the general Bernstein inequality for $p \in (0, \infty]$. Theorem 1.1 is a corollary of that result. In Section 7 we study weighted Bernstein and Markov inequalities for algebraic polynomials on $[-1, 1]$. Section 8 provides weighted Nikolskii inequalities for trigonometric and algebraic polynomials.

Finally, in Section 9 we prove a necessary condition for Bernstein's inequality (1.4) to hold. Namely, we verify Theorems 1.2 and 1.3 as well as a result on the sharpness of Theorem 1.2.

Concerning algebraic polynomials on $[-1, 1]$, it is important to mention that for weights from the class $\mathcal{W}$ the Markov inequalities were obtained by Lubinsky and Saff (cf. [LS] and the book [LL]; see discussion in Section 7). A typical example of weights from the class $\mathcal{W}$ is $\omega_\alpha(x) = \exp(-(1-x^2)^\alpha)$, $\alpha > 0$. We note that using [LS] one can also derive the weighted Bernstein inequality

$$\|\sqrt{1-x^2}\, P_n'(x)\omega_\alpha(x)\|_{L_\infty[-1,1]} \le C(\alpha)n\|P_n(x)\omega_\alpha(x)\|_{L_\infty[-1,1]}$$

(see Remark 7.1). We also note that Bernstein's inequalities for algebraic polynomials were recently proved in [MN, No] for the weight $\omega = \omega_\alpha u$, where u is doubling. In Section 7 we deal with a more general class of weights. Our proof for the algebraic case is based on Bernstein's inequality for trigonometric polynomials from Section 6.

By C, C_i (c, c_i, respectively) we will denote positive large (small, respectively) constants that may be different on different occasions. Also, below we will write $C(\omega)$ for $C(A, A_1, A_2, B, D)$, where A, A_1, A_2, B, D are from the definition of the class Ω. Moreover, for positive sequences $\{a_n\}$ and $\{b_n\}$, $a_n \asymp b_n$ means that $c \le a_n/b_n \le C$.

2. Growth properties of Ω-functions

Let $F : [-A, A]\setminus\{0\} \to (0, \infty)$ be an even function, C^∞ on $(0, A]$, satisfying (F1)–(F4).

Definition. For each $a \ge F(A)$ we denote by $x_0(a)$ the unique positive solution of the equation

$$F(x) = a.$$

Definition. For each $a \ge F(A)/A$ we denote by $x_1(a)$ the unique positive solution of

$$F(x) = ax.$$

Note that both sequences $\{x_0(n)\}_{n\in\mathbb{N}}$ and $\{x_1(n)\}_{n\in\mathbb{N}}$ are decreasing.

Lemma 2.1. *There exist positive constants $C = C(A, A_1, A_2)$ and $c = c(A, A_1, A_2)$ such that*

$$cx^{-A_2} < F(x) < Cx^{-A_1}, \quad x \in (0, A].$$

Proof. By property (F4) we have

$$|F'(x)| = -F'(x) \ge A_2\frac{F(x)}{x}, \quad x \in (0, A].$$

Therefore,

$$\log F(x) - \log F(A) = \int_x^A -(\log F(t))'\,dt \ge \int_x^A \frac{A_2}{t}\,dt = A_2(\log A - \log x),$$

which yields

$$F(x) \ge F(A)A^{A_2}x^{-A_2}.$$

Similarly, the inequality

$$|F'(x)| \le A_1 \frac{F(x)}{x}, \quad x \in (0, A],$$

implies

$$F(x) \le F(A) A^{A_1} x^{-A_1}. \qquad \square$$

Lemma 2.2. *For each $R > 0$ there exist positive constants $C = C(R, A_1, A_2)$ and $c = c(R, A_1, A_2)$ such that*

$$c x_0(Rn) < x_0(n) < C x_0(Rn) \quad \textit{for all } n \textit{ large enough.}$$

Proof. Let us prove the lemma for $R \ge 1$; for $R < 1$ the proof is similar. Since F is decreasing on $(0, A]$, we can take $c = 1$. So, it is enough to show that $x_0(n) < C x_0(Rn)$. By definition of $x_0(n)$ and (F4) we have

$$(R-1)n = |F(x_0(n)) - F(x_0(Rn))| = \int_{x_0(Rn)}^{x_0(n)} -F'(t)\,dt$$
$$\ge A_2 \int_{x_0(Rn)}^{x_0(n)} F(t) \frac{dt}{t} \ge A_2 n (\log x_0(n) - \log x_0(Rn)).$$

Thus, one may choose $C = \exp((R-1)/A_2)$. $\square$

Lemma 2.3. *There exists a positive constant $\alpha = \alpha(A, A_1, A_2)$ such that*

$$n x_1(n) \ge n^{\alpha} \quad \textit{for all } n \textit{ large enough.}$$

Proof. Since $F(x_1(n)) = n x_1(n)$, the lemma follows immediately from Lemma 2.1. $\square$

By monotonicity of F we have $x_1(2n) < x_1(n)$, and hence $2n x_1(2n) = F(x_1(2n)) > F(x_1(n)) = n x_1(n)$. In other words, $x_1(2n) < x_1(n) < 2x_1(2n)$. However, the following stronger statement holds.

Lemma 2.4. *There exists a positive constant $\epsilon = \epsilon(A, A_1, A_2) < 1$ such that*

$$(1+\epsilon) x_1(2n) < x_1(n) < (2-\epsilon) x_1(2n) \quad \textit{for all } n \textit{ large enough.}$$

Proof. Note that both $x_0(n)$ and $x_1(n)$ are decreasing to zero. By definition of $x_1(n)$ and (F4) we have

$$2n x_1(2n) - n x_1(n) = F(x_1(2n)) - F(x_1(n)) = \int_{x_1(2n)}^{x_1(n)} -F'(t)\,dt$$
$$\le A_1 \int_{x_1(2n)}^{x_1(n)} \frac{F(t)}{t}\,dt \le A_1 (x_1(n) - x_1(2n)) \frac{F(x_1(2n))}{x_1(2n)} = 2n A_1 (x_1(n) - x_1(2n)).$$

Hence,

$$(2 + 2A_1) x_1(2n) \le x_1(n)(2A_1 + 1). \tag{2.1}$$

Similarly,

$$2nx_1(2n) - nx_1(n) \geq A_2 \int_{x_1(2n)}^{x_1(n)} \frac{F(t)}{t}\,dt$$
$$\geq A_2(x_1(n) - x_1(2n))\frac{F(x_1(n))}{x_1(n)} = nA_2(x_1(n) - x_1(2n)),$$

which gives

$$(1 + A_2)x_1(n) \leq (2 + A_2)x_1(2n). \tag{2.2}$$

Finally, by (2.1) and (2.2) we can take

$$\epsilon = \frac{1}{2}\min\left\{\frac{1}{1+2A_1}, \frac{A_2}{1+A_2}\right\}.$$ □

Corollary 2.1. *For each $C > 0$ there exists $K = K(C, A, A_1, A_2)$ such that*

$$Cx_1(n) < Kx_1(Kn) \quad \text{for all } n \text{ large enough.} \tag{2.3}$$

Proof. By Lemma 2.4 (the right-hand estimate) there exists a positive constant $\delta = \delta(A, A_1, A_2)$ such that $2x_1(2n) > (1+\delta)x_1(n)$. Take an integer m such that $(1+\delta)^m > C$. Then

$$2^m x_1(2^m n) > (1 + \delta)^m x_1(n) > Cx_1(n),$$

which is (2.3) with $K = 2^m$. □

Note that $Kx_1(Kn)$ increases with K, since

$$Knx_1(Kn) = F(x_1(Kn)) > F(x_1(K^*n)) = K^*nx_1(K^*n) \quad \text{for } K > K^*.$$

Similarly, using Lemma 2.4 (the left-hand estimate), we get

Corollary 2.2. *For each $L > 0$ there exists $Q = Q(L, A, A_1, A_2)$ such that*

$$x_1(Qn) < x_1(n)/L \quad \text{for all } n \text{ large enough.} \tag{2.4}$$

Corollary 2.3. *For each $K > 0$ there exists $L = L(K, A, A_1, A_2)$ such that*

$$F(x_1(n)/L) > Kx_1(n)n \quad \text{for all } n \text{ large enough.} \tag{2.5}$$

Proof. First, by (2.3) there exists $L = L(K, A, A_1, A_2)$ such that $Lnx_1(Ln) > Knx_1(n)$. Second, on account of monotonicity of F,

$$x_1(n)/L \leq x_1(Ln), \qquad L \geq 1.$$

Therefore,

$$F(x_1(n)/L) \geq F(x_1(Ln)) = Lnx_1(Ln) > Knx_1(n).$$ □

3. Approximation of Ω-functions

The aim of this section is to obtain the order of approximation of functions from the class Ω by trigonometric polynomials.

3.1. Estimates for the Fourier coefficients of $\omega \in \Omega$

We use the classical estimate for the n-th Fourier coefficient of ω :

$$|\hat{\omega}_n| = \left| \frac{1}{\pi} \int_{\mathbb{T}} \omega(t) \cos nt \, dt \right| \le 2 \inf_{k \ge 0} \frac{\|\omega^{(k)}\|_{C(\mathbb{T})}}{n^k}, \qquad n \ge 1. \tag{3.1}$$

Below we obtain a uniform upper bound of the n-th derivative of the function $\omega \in \Omega$, where $\omega(t) = H(g(t))$, $H(x) = \exp(-F(x))$. To this end, we use Faà di Bruno's formula

$$\big(u(v(x))\big)^{(k)} = \sum\nolimits^* \frac{k!}{m_1! \dots m_k!} u^{(m_1+\dots+m_k)}(v(x)) \left(\frac{v'(x)}{1!}\right)^{m_1} \dots \left(\frac{v^{(k)}(x)}{k!}\right)^{m_k}. \tag{3.2}$$

Here and below, $\sum^*$ indicates summation over all nonnegative integers $m_1, \dots, m_k$ such that $m_1 + 2m_2 + \dots + km_k = k$. We start with the following technical lemma.

Lemma 3.1. *For each $k \in \mathbb{N}$,*

$$\sum\nolimits^* \frac{k!}{m_1! \dots m_k! (k - m_1 - \dots - m_k)!} = \frac{1}{2} \binom{2k}{k}. \tag{3.3}$$

Proof. Denote the left-hand side of (3.3) by S_k. One can see that S_k is the coefficient of x^k in the polynomial

$$(1 + x + x^2 + \dots + x^k)^k,$$

and hence it is equal to the coefficient of x^k in the Taylor series expansion of the function

$$f(x) = \frac{1}{(1-x)^k}.$$

Therefore,

$$S_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{2} \binom{2k}{k}. \qquad \square$$

Now we are ready to estimate the maximum norm of the k-th derivative of the function H.

Lemma 3.2. *Let $H(x) = \exp(-F(x))$, where F satisfies* (F1)–(F4). *Then $H \in C^\infty[-A, A]$, and there exists $C = C(A, B, A_1, A_2) > 0$ such that for all $k > F(A)$,*

$$H^{(k)}(x) \le \left(\frac{Ck}{x_0(k)}\right)^k, \qquad x \in [-A, A].$$

Proof. Let $x \in (0, A]$. By (3.2),

$$H^{(k)}(x) = \sum\nolimits^* \frac{k!}{m_1! \dots m_k!} (-1)^{m_1+\dots+m_k} \exp(-F(x)) \left(\frac{F'(x)}{1!}\right)^{m_1} \dots \left(\frac{F^{(k)}(x)}{k!}\right)^{m_k}.$$

By (F3) we have

$$\frac{|F^{(s)}(x)|}{s!} \le C^s \frac{F(x)}{x^s}, \qquad 1 \le s \le k.$$

Hence,

$$\begin{aligned} |H^{(k)}(x)| &\le C^k \sum\nolimits^* \frac{k!}{m_1! \dots m_k!} \frac{H(x)(F(x))^{m_1+\dots+m_k}}{x^k} \\ &= C^k \sum\nolimits^* \frac{k!}{m_1! \dots m_k!} G_{m,k}(x), \end{aligned} \tag{3.4}$$

where $m = m_1 + \dots + m_k$ and

$$G_{m,k}(x) := \frac{H(x)(F(x))^m}{x^k}, \qquad x \in (0, A].$$

To estimate the maximum of $G_{m,k}(x)$ for $x \in (0, A)$, we write

$$G'_{m,k}(x) = \frac{H(x)(F(x))^{m-1}}{x^k}\Big(-F'(x)F(x) + mF'(x) - \frac{k}{x}F(x)\Big).$$

Therefore, if $F(x) < k/A_1$, then $G'_{m,k}(x) < 0$. Indeed, by (F4), we get

$$-F'(x)F(x) + mF'(x) - \frac{k}{x}F(x) < F(x)\Big(-F'(x) - \frac{k}{x}\Big) < F(x)\Big(\frac{A_1 F(x)}{x} - \frac{k}{x}\Big) < 0.$$

Similarly, if $F(x) > \max\{2, 2/A_2\}k$, then $G'_{m,k}(x) > 0$. In this case $F(x) > 2k \ge 2m$, and therefore

$$\begin{aligned} -F'(x)F(x) + mF'(x) - \frac{k}{x}F(x) &\ge -\frac{F'(x)F(x)}{2} - \frac{k}{x}F(x) = \frac{F(x)}{2}\Big(-F'(x) - \frac{2k}{x}\Big) \\ &\ge \frac{F(x)}{2}\Big(\frac{A_2 F(x)}{x} - \frac{2k}{x}\Big) > 0. \end{aligned}$$

Using the fact that each $G_{m,k}$ is a continuously differentiable function on $(0, A]$, we see that $\max_{0<x\le A} G_{m,k}(x)$ exists for all $1 \le m \le k$ and is attained at a point x^* such that

$$k/A_1 \le F(x^*) \le \max\{2, 2/A_2\}k. \tag{3.5}$$

Now, Lemma 2.1 implies that

$$G_{m,k}(x) \to 0 \quad \text{as } x \to 0+.$$

Thus, (3.4) yields $H^{(k)}(0) = 0$ for all $k \in \mathbb{N}$, and hence $H \in C^\infty[-A, A]$. Set $R = \max\{2, 2/A_2\}$. Since F is decreasing, by (3.5) we get

$$G_{m,k}(x) \le \exp(-F(A)) \frac{(Rk)^m}{x_0^k(Rk)} \le \frac{C^k k^m}{x_0^k(k)}, \qquad k > F(A).$$

Here the last inequality follows from Lemma 2.2. Combining the last display with (3.4) we obtain

$$|H^{(k)}(x)| \le \frac{C^k k!}{x_0^k(k)} \sum\nolimits^* \frac{k^{m_1+\cdots+m_k}}{m_1!\ldots m_k!}.$$

Finally, taking into account that $k^{k-m} \ge (k-m)!$ for $1 \le m \le k$, by (3.3) we get

$$\begin{aligned}|H^{(k)}(x)| &\le \frac{C^k k!}{x_0^k(k)} \sum\nolimits^* \frac{k!}{m_1!\ldots m_k!(k-m_1-\cdots-m_k)!} \le \frac{C^k k!}{x_0^k(k)} \\ &\le \left(\frac{Ck}{x_0(k)}\right)^k, \qquad x \in (0, A].\end{aligned}$$

For $x \in [-A, 0)$ the same inequality holds because F, and hence H, is even. □

We are now in a position to give a uniform estimate of $\omega^{(k)}$, where $\omega \in \Omega$.

Lemma 3.3. *Let $\omega \in \Omega$. Then there exists $C = C(\omega) > 0$ such that for all k large enough,*

$$\omega^{(k)}(t) \le \frac{C^k k^k}{x_0^k(k)}, \qquad t \in \mathbb{T}.$$

Proof. Take $k \ge F(A)$ so that $x_0(k)$ is well defined. Since g is an analytic function on $\mathbb{T}$, and $H \in C^\infty[-A, A]$, we have $\omega \in C^\infty(\mathbb{T})$. By Faà di Bruno's formula, for each $k \in \mathbb{N}$,

$$|\omega^{(k)}(t)| = |(H(g(t)))^{(k)}| = \sum\nolimits^* \frac{k!}{m_1!\ldots m_k!} H^{(m)}(g(t)) \left(\frac{g'(t)}{1!}\right)^{m_1} \ldots \left(\frac{g^{(k)}(t)}{k!}\right)^{m_k},$$

where $m = m_1 + \cdots + m_k$. We rewrite the last sum as $\sum_{m<F(A)} + \sum_{m \ge F(A)}$. Since $H^{(m)}(x) \le C(\omega)$ for any $m < F(A)$, we have

$$\sum_{m<F(A)} \le C(\omega) D^k \sum_{m<F(A)} \frac{k!}{m_1!\ldots m_k!} \le C^k k! \sum_{m<F(A)} \frac{1}{m_1!\ldots m_k!}. \tag{3.6}$$

To estimate $\sum_{m \ge F(A)}$, we can use

$$H^{(m)}(x) \le \left(\frac{Cm}{x_0(m)}\right)^m, \qquad m \ge F(A), \tag{3.7}$$

provided by Lemma 3.2, and (1.6) to get

$$\sum_{m \ge F(A)} \le D^k \sum\nolimits^* \frac{k!}{m_1!\ldots m_k!} \left(\frac{Cm}{x_0(m)}\right)^m.$$

Combining this with (3.6), we get

$$|\omega^{(k)}(t)| \le \frac{C^k k!}{x_0^k(k)} \sum\nolimits^* \frac{m^m}{m_1!\ldots m_k!}, \qquad t \in \mathbb{T}.$$

Noting that for any integers $1 \le m \le k$,

$$m^m \le \frac{k^k}{(k-m)!} \le \frac{C^k k!}{(k-m)!},$$

by (3.3) we have

$$|\omega^{(k)}(t)| \le \frac{C^k k!}{x_0^k(k)} \sum{}^* \frac{k!}{m_1! \dots m_k!(k - m_1 - \dots - m_k)!} \le \frac{C^k k!}{x_0^k(k)}, \qquad t \in \mathbb{T}. \qquad \square$$

The next result provides a nearly optimal k in estimate (3.1) for the n-th Fourier coefficient of $\omega \in \Omega$.

Lemma 3.4. *Let F be a function satisfying* (F1)–(F4). *Then for each $C > e$ and n large enough there exists an integer $k = k(C, n, F)$ such that*

$$\frac{C^k k^k}{n^k x_0^k(k)} \le \exp\left(-\frac{1}{C^2} n x_1(n) + 1\right).$$

Proof. Let k be the minimal integer such that

$$\frac{Ck}{n x_0(k)} > \frac{1}{e}. \tag{3.8}$$

Suppose $k < n x_1(n)/C^2$. Then

$$\frac{Ck}{n x_0(k)} < \frac{1}{C} \frac{n x_1(n)}{n x_0\big(\frac{1}{C^2} n x_1(n)\big)} < \frac{1}{C} \frac{x_1(n)}{x_0(n x_1(n))} = \frac{1}{C} < \frac{1}{e},$$

where we have used the definitions of $x_0(n)$ and $x_1(n)$. This contradicts (3.8). Thus, $k \ge n x_1(n)/C^2$. Therefore, since $n x_1(n) \to \infty$ as $n \to \infty$, we have $k \ge 2$ for n large enough. Finally, again applying (3.8) we get

$$\left(\frac{C(k-1)}{n x_0(k-1)}\right)^{k-1} \le \left(\frac{1}{e}\right)^{\frac{1}{C^2} n x_1(n) - 1},$$

and the claim easily follows. $\square$

We will also need the following technical result.

Lemma 3.5. *For each $\omega \in \Omega$ and $c > 0$ we have*

$$\sum_{v=n}^{\infty} \exp(-c v x_1(v)) \le \exp\left(-\frac{c}{2} n x_1(n)\right)$$

for all n large enough, i.e., for $n \ge n_0(\omega, c)$.

Proof. Indeed, since the sequence $n x_1(n)$ is increasing to infinity,

$$\sum_{v=n}^{\infty} \exp(-c v x_1(v)) = \sum_{s=0}^{\infty} \sum_{k=2^s n}^{2^{s+1} n - 1} \exp(-c k x_1(k)) \le \sum_{s=0}^{\infty} n 2^s \exp(-c 2^s n x_1(2^s n)).$$

By Lemma 2.4 there exists $\epsilon = \epsilon(A, A_1, A_2)$ such that $2x_1(2n) \geq (1+\epsilon)x_1(n)$ for all n large enough. Then

$$\sum_{v=n}^{\infty} \exp(-cvx_1(v)) \leq \sum_{s=0}^{\infty} n2^s \exp\bigl(-c(1+\epsilon)^s nx_1(n)\bigr) =: \sum_{s=0}^{\infty} h_s.$$

It is easy to check that, for $s \geq 0$ and $n \geq n_0(\omega, c)$,

$$h_{s+1}/h_s \leq 2\exp(-c\epsilon nx_1(n)) \leq 1/2.$$

Thus, Lemma 2.3 gives

$$\begin{aligned}\sum_{v=n}^{\infty} \exp(-cvx_1(v)) &\leq 2h_0 = 2n\exp(-cnx_1(n))\\ &\leq \exp\left(-\frac{c}{2}nx_1(n)\right), \qquad n \geq n_0(\omega, c).\end{aligned}$$ □

3.2. *Remez inequality for trigonometric polynomials*

We will need the following Remez inequality answering how large $\|T_n\|_{L_\infty(\mathbb{T})}$ can be if

$$|\{t \in \mathbb{T} : |T_n(t)| > 1\}| \leq \varepsilon < 1.$$

Lemma 3.6 ([Er1], [Er2]). *For any Lebesgue measurable set $B \subset \mathbb{T}$ such that $|B| < \pi/2$ we have*

$$\|T_n\|_{L_\infty(\mathbb{T})} \leq \exp(4n|B|)\|T_n\|_{L_\infty(\mathbb{T}\setminus B)}. \tag{3.9}$$

If $0 < p < \infty$ and $|B| < 1/4$ we have

$$\|T_n\|_{L_p(\mathbb{T})} \leq \bigl(1 + \exp(4n|B|p)\bigr)\|T_n\|_{L_p(\mathbb{T}\setminus B)}. \tag{3.10}$$

3.3. *Two approximation theorems for Ω-weights*

We are now ready to prove the following result on simultaneous trigonometric approximation of functions from the class Ω and of their derivatives.

Theorem 3.1. *For each $\omega \in \Omega$ there exists a positive constant $c = c(\omega)$ such that*

$$\|\omega - \omega_n\|_{C(\mathbb{T})} \leq \exp(-cnx_1(n)), \tag{3.11}$$

$$\|\omega' - \omega_n'\|_{C(\mathbb{T})} \leq \exp(-cnx_1(n)), \tag{3.12}$$

for n large enough, where ω_n is the n-th partial sum of the Fourier series of ω.

Proof. Integration by parts and Lemma 3.3 imply that, for some $C > 0$,

$$|\hat{\omega}_n| \leq 2\frac{\|\omega^{(k)}\|_{C(\mathbb{T})}}{n^k} \leq \frac{C^k k!}{x_0^k(k)n^k}.$$

Hence, by Lemma 3.4, there exists $c = c(\omega)$ such that, for $n \ge n_0(\omega)$,

$$|\hat{\omega}_n| \le \exp(-cnx_1(n)).$$

Let ω_n be the n-th partial sum of the Fourier series of ω, i.e.,

$$\omega_n(t) = \frac{\hat{\omega}_0}{2} + \sum_{k=1}^{n} \hat{\omega}_k \cos kt.$$

Since $\omega \in C^\infty(\mathbb{T})$, for each $t \in \mathbb{T}$ we have

$$\omega_n(t) \to \omega(t) \quad \text{and} \quad \omega_n'(t) \to \omega'(t) \quad \text{as } n \to \infty.$$

Therefore, taking into account Lemma 3.5, for each $t \in \mathbb{T}$ we have

$$|\omega(t) - \omega_n(t)| \le \sum_{v=n+1}^{\infty} |\hat{\omega}_v| \le \sum_{v=n+1}^{\infty} \exp(-cvx_1(v)) \le \exp\left(-\frac{c}{2} n x_1(n)\right),$$
$$|\omega'(t) - \omega_n'(t)| \le \sum_{v=n+1}^{\infty} v|\hat{\omega}_v| \le \sum_{v=n+1}^{\infty} \exp\left(-\frac{c}{2} v x_1(v)\right) \le \exp\left(-\frac{c}{4} n x_1(n)\right). \qquad \square$$

Let g be an analytic function as in the definition of Ω, i.e., satisfying (1.6) and such that each zero of g is of multiplicity one. Let $\{a_1, \dots, a_m\}$ be the set of all zeros of g on $\mathbb{T}$. For each $\epsilon > 0$ denote

$$B_\epsilon := \{t \in \mathbb{T} : |g(t)| < \epsilon\}.$$

Let us show that the measure of B_ϵ is at most linear in ϵ.

Lemma 3.7. *For every $\epsilon > 0$ we have*

$$|B_\epsilon| \le C(g)\epsilon.$$

Proof. Since all zeros of g have multiplicity one, we have

$$|g(t)| = |(t - a_1) \dots (t - a_m) h(t)|,$$

where $\min_{t \in \mathbb{T}} |h(t)| = b(g) =: b > 0$. Set

$$S := \left(\frac{3}{\min_{1 \le i < j \le m} |a_i - a_j|} \right)^{m-1}.$$

For given $\epsilon > 0$, let $t_0 \in \mathbb{T}$ be such that

$$|t_0 - a_i| > S\epsilon/b \quad \text{for all } i \in \overline{1, m}.$$

Since the inequality

$$|t_0 - a_j| \le \frac{\min_{1 \le i < j \le m} |a_i - a_j|}{3}$$

may hold at most for one $j \in \overline{1,m}$ we have

$$|g(t_0)| \geq \frac{S\epsilon}{b}\left(\frac{\min_{1\leq i<j\leq m}|a_i - a_j|}{3}\right)^{m-1} b = \epsilon.$$

Hence, $t_0 \notin B_\epsilon$. Therefore, for each $t \in B_\epsilon$, there exists $j \in \overline{1,m}$ such that

$$|t - a_j| \leq S\epsilon/b.$$

Thus, $|B_\epsilon| \leq (2mS/b)\epsilon$. □

Now we are in a position to prove the following approximation theorem.

Theorem 3.2. *For each $\omega \in \Omega$ there exists an integer $K = K(\omega)$ such that for each trigonometric polynomial T_n we have*

$$\frac{1}{2}\int_{\mathbb{T}} |T_n(t)\omega_{Kn}(t)|\,dt \leq \int_{\mathbb{T}} |T_n(t)|\omega(t)\,dt \leq 2\int_{\mathbb{T}} |T_n(t)\omega_{Kn}(t)|\,dt, \tag{3.13}$$

where ω_n is the n-th partial Fourier sum of ω.

Proof. It is enough to verify (3.13) for sufficiently large n. Using Theorem 3.1 we get

$$\int_{\mathbb{T}} |T_n(t)|\,|\omega(t) - \omega_{Kn}(t)|\,dt \leq \exp(-cKnx_1(Kn))\int_{\mathbb{T}} |T_n(t)|\,dt.$$

We define

$$B_{x_1(n)} = \{t \in \mathbb{T} : |g(t)| < x_1(n)\}.$$

Then Lemma 3.7 implies that $|B_{x_1(n)}| \leq Cx_1(n)$, where C depends only on ω. By the Remez inequality we get

$$\begin{aligned}\int_{\mathbb{T}} |T_n(t)||\omega(t) - \omega_{Kn}(t)|\,dt &\leq \exp(-cKnx_1(Kn))\exp(4n|B_{x_1(n)}|)\int_{\mathbb{T}\setminus B_{x_1(n)}} |T_n(t)|\,dt \\ &\leq \exp(-cKnx_1(Kn) + Cnx_1(n))\int_{\mathbb{T}\setminus B_{x_1(n)}} |T_n(t)|\,dt.\end{aligned}$$

Note that for each $t \in \mathbb{T} \setminus B_{x_1(n)}$,

$$\omega(t) = \exp(-F(g(t))) \geq \exp(-F(x_1(n))) = \exp(-nx_1(n)). \tag{3.14}$$

Therefore,

$$\begin{aligned}&\int_{\mathbb{T}} |T_n(t)|\,|\omega(t) - \omega_{Kn}(t)|\,dt \\ &\qquad \leq \exp\big(-cKnx_1(Kn) + Cnx_1(n) + nx_1(n)\big)\int_{\mathbb{T}\setminus B_{x_1(n)}} |T_n(t)|\omega(t)\,dt.\end{aligned}$$

Now, by Corollary 2.1 we can choose an integer K large enough such that

$$\int_{\mathbb{T}} |T_n(t)|\,|\omega(t) - \omega_{Kn}(t)|\,dt \leq \frac{1}{2}\int_{\mathbb{T}\setminus B_{x_1(n)}} |T_n(t)|\omega(t)\,dt \leq \frac{1}{2}\int_{\mathbb{T}} |T_n(t)|\omega(t)\,dt.$$

This immediately implies the statement of the theorem. □

4. Weighted Bernstein inequalities in L_1

In this section we prove the Bernstein inequality in $L_1(\omega)$, where $\omega \in \Omega$.

Theorem 4.1. *Let $\omega \in \Omega$. Then for each trigonometric polynomial T_n of degree at most n,*

$$\int_{\mathbb{T}} |T_n'(t)|\omega(t)\,dt \le C(\omega)n \int_{\mathbb{T}} |T_n(t)|\omega(t)\,dt. \tag{4.1}$$

Proof. Since

$$\int_{\mathbb{T}} |T_n'(t)|\omega(t)\,dt \le C(\omega,n) \int_{\mathbb{T}} |T_n(t)|\omega(t)\,dt \tag{4.2}$$

for any continuous weight ω, it is enough to prove (4.1) for n large enough. The proof is in three steps.

Step 1. By Theorem 3.2 there exists an integer $K = K(\omega)$ large enough such that the Kn-partial Fourier sum ω_{Kn} satisfies

$$\begin{aligned}\int_{\mathbb{T}} |T_n'(t)|\omega(t)\,dt &\le 2\int_{\mathbb{T}} |T_n'(t)\omega_{Kn}(t)|\,dt \\ &\le 2\int_{\mathbb{T}} |(T_n(t)\omega_{Kn}(t))'|\,dt + 2\int_{\mathbb{T}} |T_n(t)\omega_{Kn}'(t)|\,dt =: I_1 + I_2.\end{aligned} \tag{4.3}$$

Then by the classical Bernstein inequality and Theorem 3.2 we have

$$I_1 \le CKn \int_{\mathbb{T}} |T_n(t)\omega_{Kn}(t)|\,dt \le C(\omega)n \int_{\mathbb{T}} |T_n(t)|\omega(t)\,dt.$$

Further,

$$I_2 \le 2\int_{\mathbb{T}} |T_n(t)|\,|\omega'(t)|\,dt + 2\int_{\mathbb{T}} |T_n(t)|\,|\omega'(t) - \omega_{Kn}'(t)|\,dt =: I_{21} + I_{22}.$$

Step 2. To estimate I_{21}, define

$$B_{n,M} := \{t \in \mathbb{T} : g(t) \ne 0 \text{ and } |F'(g(t))g'(t)| \ge Mn\}.$$

Note that, for any $t \in B_{n,M}$, it follows from (F4) that

$$A_1 \frac{F(g(t))}{|g(t)|} \|g'\|_{C(\mathbb{T})} \ge Mn,$$

and therefore

$$\frac{F(g(t))}{|g(t)|} \ge M_2 n, \quad \text{where} \quad M_2 = \frac{M}{A_1 D}. \tag{4.4}$$

Using Corollary 2.2 we find that for each $L > 0$ there exists $Q = Q(L,\omega) > 1$ such that

$$F(x_1(n)/L) \le F(x_1(Qn)) = Qnx_1(Qn) < Qnx_1(n)$$

for n large enough. Hence, for all $x \in [x_1(n)/L, A]$,

$$F(x) \le F(x_1(n)/L) < Qnx_1(n) \le xQLn. \tag{4.5}$$

Therefore, if

$$M_2 = \frac{M}{A_1 D} > QL, \tag{4.6}$$

then (4.4) and (4.5) imply

$$|g(t)| < x_1(n)/L, \qquad t \in B_{n,M}.$$

Now, for each $K \in \mathbb{N}$, taking $L = L(K, \omega)$ as in Corollary 2.3 we get

$$F(g(t)) \geq F(x_1(n)/L) \geq K x_1(n) n. \tag{4.7}$$

Moreover, by Lemma 2.1,

$$F(g(t))/|g(t)| \leq C(\omega)(F(g(t)))^{1+1/A_2}, \qquad t \in B_{n,M}.$$

Let us estimate $|\omega'(t)|$ from above for $t \in B_{n,M}$. In view of (F1) and (F4),

$$\begin{aligned} |\omega'(t)| &= \omega(t)|F'(g(t))g'(t)| \leq A_1 D \omega(t) F(g(t))/|g(t)| \\ &\leq C(\omega)\exp(-F(g(t)))(F(g(t)))^{1+1/A_2} \\ &\leq C(\omega)\exp(-F(g(t))/2), \qquad t \in B_{n,M}, \end{aligned} \tag{4.8}$$

where in the last estimate we have used (4.7) and the fact that $nx_1(n) \to \infty$ as $n \to \infty$.

Hence, (4.7) and (4.8) imply

$$|\omega'(t)| \leq C(\omega)\exp(-K x_1(n) n/2), \qquad t \in B_{n,M}. \tag{4.9}$$

Step 3. Now we are ready to estimate I_{21}. We have

$$I_{21} = 2\int_{B_{n,M}} |T_n(t)|\,|\omega'(t)|\,dt + 2\int_{\mathbb{T}\setminus B_{n,M}} |T_n(t)|\,|\omega'(t)|\,dt =: I_{211} + I_{212}.$$

Let us estimate I_{211}. Thanks to (4.9), we obtain

$$\begin{aligned} I_{211} &= 2\int_{B_{n,M}} |T_n(t)|\,|\omega'(t)|\,dt \leq C(\omega)\exp(-K x_1(n) n/2)\int_{B_{n,M}} |T_n(t)|\,dt \\ &\leq C(\omega)\exp(-K x_1(n) n/2)\int_{\mathbb{T}} |T_n(t)|\,dt. \end{aligned}$$

Now, as in the proof of Theorem 3.2, we consider

$$B_{x_1(n)} = \{t \in \mathbb{T} : |g(t)| < x_1(n)\}.$$

By the Remez inequality and Lemma 3.7 we get

$$\begin{aligned} I_{211} &\leq C(\omega)\exp(-K x_1(n) n/2)\exp(4n|B_{x_1(n)}|)\int_{\mathbb{T}\setminus B_{x_1(n)}} |T_n(t)|\,dt \\ &\leq C(\omega)\exp\big(-K x_1(n) n/2 + C(\omega) n x_1(n)\big)\int_{\mathbb{T}\setminus B_{x_1(n)}} |T_n(t)|\omega(t)\,dt \\ &\leq C(\omega)\int_{\mathbb{T}} |T_n(t)|\omega(t)\,dt \end{aligned} \tag{4.10}$$

for $K \in \mathbb{N}$ large enough. On the other hand, it follows from the definition of $B_{n,M}$ that

$$I_{212} = 2\int_{\mathbb{T}\setminus B_{n,M}} |T_n(t)|\,|\omega'(t)|\,dt \le 2Mn\int_{\mathbb{T}} |T_n(t)|\omega(t)\,dt.$$

Thus,

$$I_{21} \le C(\omega)n\int_{\mathbb{T}} |T_n(t)|\omega(t)\,dt.$$

Regarding I_{22}, we first note that Theorem 3.1 yields

$$I_{22} \le \exp(-c(\omega)Knx_1(Kn))\int_{\mathbb{T}} |T_n(t)|\,dt.$$

Similarly to the case of I_{211}, we use Remez's inequality for the set $B_{x_1(n)}$ and Lemma 3.7 to deduce that

$$I_{22} \le C(\omega)\int_{\mathbb{T}} |T_n(t)|\omega(t)\,dt \tag{4.11}$$

for $K \in \mathbb{N}$ large enough.

Let us explain how we choose the constants K, L, Q, and M. First, $K \in \mathbb{N}$ is taken large enough such that (4.3), (4.10), and (4.11) hold. Further we choose $L = L(K,\omega)$ as in Corollary 2.3, $Q = Q(L,\omega)$ as in Corollary 2.2, and finally $M > QLA_1D$ so that (4.6) holds. □

5. Weighted Remez inequalities

For an arbitrary measurable set E, denote $\|T_n\|_{L_p(\omega,E)} = (\int_E |T_n|^p\omega)^{1/p}$ if $p < \infty$, and $\|T_n\|_{L_\infty(\omega,E)} = \operatorname{ess\,sup}_{t\in E} |T_n(t)\omega(t)|$. We write $\|T_n\|_{L_p(\omega)}$ for $\|T_n\|_{L_p(\omega,\mathbb{T})}$.

The following classes play an important role in our further study.

Definition. We say that a weight u satisfies the $\mathcal{R}(p)$ *condition*, $0 < p \le \infty$, and write $u \in \mathcal{R}(p)$, if for any trigonometric polynomial T_n the weighted Remez inequality holds, that is, there exists $C = C(p,u) > 0$ such that

$$\|T_n\|_{L_p(u,\mathbb{T})} \le \exp(Cn|E|)\|T_n\|_{L_p(u,\mathbb{T}\setminus E)} \tag{5.1}$$

for all measurable sets E with $|E| \le 1$.

Definition. We say that a weight u satisfies the $\mathcal{R}_{\rm int}(p)$ *condition*, $0 < p \le \infty$, and write $u \in \mathcal{R}_{\rm int}(p)$, if for any trigonometric polynomial T_n the restricted weighted Remez inequality holds, that is, there exists $C = C(p,u) > 0$ such that

$$\|T_n\|_{L_p(u,\mathbb{T})} \le \exp(Cn|E|)\|T_n\|_{L_p(u,\mathbb{T}\setminus E)} \tag{5.2}$$

for all sets E which are a finite union of intervals of length $\ge 1/n$ and such that $|E| \le 1$.

Remark 5.1. First, it is clear that $\mathcal{R}(p) \subset \mathcal{R}_{\rm int}(p)$. Note also that any doubling weight u satisfies the $\mathcal{R}_{\rm int}(p)$ condition if $0 < p < \infty$ (see [MT, Th. 5.3] and [Er1, Th. 7.2]), and any $u \in A^*$ satisfies the $\mathcal{R}_{\rm int}(\infty)$ condition (see [MT, (6.10)]).

Remark 5.2. One can similarly define the class $\mathcal{R}_{\text{int}}(p,d)$ of weights u such that for any T_n and every set E with $|E| \le 1$ that is a finite union of intervals of length $\ge d/n$ we have $\|T_n\|_{L_p(u,\mathbb{T})} \le \exp(Cn|E|)\|T_n\|_{L_p(u,\mathbb{T}\setminus E)}$ for some constant $C = C(p,u,d)$. However, it turns out that $\mathcal{R}_{\text{int}}(p,d) = \mathcal{R}_{\text{int}}(p)$, and therefore we can use $d = 1$.

We will need the following approximation inequalities for the weight $\omega^{1/p}$ that are similar to Theorems 3.1 and 3.2.

Lemma 5.1. *Let $\omega = \exp(-F(g(t))) \in \Omega$ and $v = \omega^{1/p}$ for $p \in (0,\infty)$. Let v_n be the n-th partial Fourier sum of v.*

(A) *We have*

$$\|v^p - |v_n|^p\|_{C(\mathbb{T})} \le \exp(-c(p,\omega)nx_1(n)), \tag{5.3}$$
$$\|v' - v_n'\|_{C(\mathbb{T})} \le \exp(-c(p,\omega)nx_1(n)), \tag{5.4}$$

for n large enough, where $x_1(n)$ is the unique positive solution of the equation $F(x_1(n)) = nx_1(n)$.

(B) *For any $u \in \mathcal{R}_{\text{int}}(p)$, there exists $K = K(\omega,u,p)$ such that*

$$\frac{1}{2}\int_{\mathbb{T}} |T_n(t)|^p |v_{Kn}(t)|^p u(t)\,dt \le \int_{\mathbb{T}} |T_n(t)|^p \omega(t) u(t)\,dt$$
$$\le 2\int_{\mathbb{T}} |T_n(t)|^p |v_{Kn}(t)|^p u(t)\,dt. \tag{5.5}$$

(C) *For any $u \in \mathcal{R}_{\text{int}}(\infty)$, there exists $K = K(\omega,u)$ such that*

$$\tfrac{1}{2}\|T_n\omega_{Kn}u\|_{L_\infty(\mathbb{T})} \le \|T_n\omega u\|_{L_\infty(\mathbb{T})} \le 2\|T_n\omega_{Kn}u\|_{L_\infty(\mathbb{T})}, \tag{5.6}$$

where ω_n is the n-th partial Fourier sum of ω.

Proof. We may assume that n is large enough. For any $\omega = \exp(-F(g(\cdot))) \in \Omega$ and any $p \in (0,\infty)$ we have, by definition of the class Ω,

$$v(t) = \omega^{1/p}(t) = \exp(-H(g(t))) \in \Omega, \qquad t \in \mathbb{T},$$

where $H(x) = F(x)/p$ satisfies (F1)–(F4). Moreover, by Corollary 2.3,

$$x_1^{\omega}(n) \asymp x_1^{v}(n),$$

where $x_1^{\omega}(n)$ is the unique positive solution of the equation $F(x_1^{\omega}(n)) = nx_1^{\omega}(n)$, and $x_1^{v}(n)$ is the unique positive solution of $H(x_1^{v}(n)) = nx_1^{v}(n)$.

To verify (5.3) and (5.4), we use Theorem 3.1 and the inequality

$$\left|v^p(t) - |v_{Kn}(t)|^p\right| \le C(p,\omega)|v(t) - v_{Kn}(t)|^{\min\{1,p\}}, \qquad 0 < p < \infty,\ t \in \mathbb{T}. \tag{5.7}$$

For $0 < p < 1$, the latter follows from the inequality $|a^p - b^p| \le C(p)|a-b|^p$, where $a, b \ge 0$. For $p > 1$, we get (5.7) using the fact that if $a > b > 0$ then $a^p - b^p = p\xi^{p-1}(a-b)$ for some $\xi \in (b,a)$. Thus, the proof of (A) is complete.

To show (B) and (C), we follow the proof of Theorem 3.2 using (5.3) and the following remark.

Remark 5.3. In the proofs of Theorems 3.1 and 3.2, we use the Remez inequalities only for the set $B_{x_1(n)} = \{t \in \mathbb{T} : |g(t)| < x_1(n)\}$. Analyzing the proof of Lemma 3.7, we note that there exists $\widehat{B}_{x_1(n)} \subset \mathbb{T}$ such that $B_{x_1(n)} \subseteq \widehat{B}_{x_1(n)}$, $|\widehat{B}_{x_1(n)}| \le Cx_1(n)$, and $\widehat{B}_{x_1(n)}$ is a union of m intervals of length $> 1/n$, where m is the number of zeros of g on $\mathbb{T}$. Therefore, in the proofs of Theorems 3.1 and 3.2 we can apply the Remez inequality for $\widehat{B}_{x_1(n)}$.

In this section we prove the following general Remez inequality in L_p.

Theorem 5.1. *Let $0 < p \le \infty$, $\omega \in \Omega$, and $u \in \mathcal{R}(p)$. Then for each trigonometric polynomial T_n we have*

$$\|T_n\|_{L_p(\omega u)} \le \exp(Cn|E|)\|T_n\|_{L_p(\omega u, \mathbb{T}\setminus E)}, \tag{5.8}$$

where $C = C(\omega, u, p)$ and E is a measurable set with $0 < |E| \le 1$.

Since any A_∞ weight u satisfies the $\mathcal{R}(p)$ condition for any $0 < p < \infty$ (see [MT, Th. 5.2] and [Er1, Th. 7.2]), and any A^* weight u satisfies the $\mathcal{R}(\infty)$ condition (see [MT, (6.10)]), Theorem 5.1 immediately implies the following result.

Corollary 5.1. *For $0 < p < \infty$, the Remez inequality (5.8) holds for any measurable set E with $|E| \le 1$ whenever $\omega \in \Omega$ and $u \in A_\infty$, and for $p = \infty$ whenever $\omega \in \Omega$ and $u \in A^*$. Moreover, applying Theorem 5.1 several times we obtain inequality (5.8) for the weight $\omega = \omega_1 \dots \omega_s$, where $\omega_i \in \Omega$, $i = 1, \dots, s$.*

Conditions on the weight u in Corollary 5.1 can be relaxed when E is a finite union of intervals. First, we give an analogue of Theorem 5.1 in this case.

Theorem 5.2. *Let $0 < p \le \infty$, $\omega \in \Omega$, and $u \in \mathcal{R}_{\text{int}}(p)$. Then for each trigonometric polynomial T_n we have*

$$\|T_n\|_{L_p(\omega u)} \le \exp(Cn|E|)\|T_n\|_{L_p(\omega u, \mathbb{T}\setminus E)}, \tag{5.9}$$

where $C = C(\omega, u, p)$ and E is a finite union of intervals of length $\ge 1/n$ each.

In particular, this and [MT, Th. 5.3] give a refinement of Corollary 5.1 for such sets E.

Corollary 5.2. *For $0 < p < \infty$ the Remez inequality (5.9) holds whenever $\omega \in \Omega$, u is doubling, and E is a union of intervals of length $\ge 1/n$ each. Moreover, applying Theorem 5.2 several times we obtain inequality (5.9) for the weight $\omega = \omega_1 \dots \omega_s$, where $\omega_i \in \Omega$, $i = 1, \dots, s$.*

Proof of Theorem 5.1. It is sufficient to show (5.8) for n large enough. Let first $p \in (0, \infty)$. It follows from Lemma 5.1 that for $v = \omega^{1/p} \in \Omega$ we have

$$\left\| v^p - |v_n|^p \right\|_{C(\mathbb{T})} \le \exp\bigl(-c(p, \omega)nx_1(n)\bigr), \tag{5.10}$$

where v_n is the n-th partial Fourier sum of the function v. Moreover, by (5.5),

$$\int_{\mathbb{T}} |T_n|^p v^p u \asymp \int_{\mathbb{T}} |T_n|^p |v_{Kn}|^p u \tag{5.11}$$

for $K = K(\omega, u, p)$ large enough. Let us also recall that

$$B = B_{x_1(n)} = \{t \in \mathbb{T} : |g(t)| \le x_1(n)\}.$$

Case 1: $|B| \le |E|$. Using (5.11) and (5.1) for $u \in \mathcal{R}(p)$, we obtain

$$\begin{aligned}\int_{\mathbb{T}} |T_n|^p \omega u &\le \exp\bigl(C(p,u)Kn(|E| + |B|)\bigr) \int_{\mathbb{T}\setminus(E\cup B)} |T_n|^p |v_{Kn}|^p u \\ &\le \exp(C(p,u)Kn|E|) \int_{\mathbb{T}\setminus(E\cup B)} |T_n|^p |v_{Kn}|^p u.\end{aligned}$$

The latter can be estimated by $I_1 + I_2$, where

$$\begin{aligned}I_1 &:= \exp(C(p,u)Kn|E|) \int_{\mathbb{T}\setminus(E\cup B)} |T_n|^p v^p u, \\ I_2 &:= \exp(C(p,u)Kn|E|) \int_{\mathbb{T}\setminus(E\cup B)} |T_n|^p \bigl|v^p - |v_{Kn}|^p\bigr| u.\end{aligned}$$

Corollary 2.1 implies that, for any $c > 0$, there exists $K = K(c, \omega)$ such that $x_1(n) < cKx_1(Kn)$, and therefore $\exp(-cKnx_1(Kn)) \le \exp(-nx_1(n))$. Then, by (5.10) for $c = c(p, \omega)$,

$$\bigl|v^p - |v_{Kn}|^p\bigr| \le \exp(-cKnx_1(Kn)) \le \exp(-nx_1(n)) \le \omega(t), \qquad t \in \mathbb{T} \setminus B,$$

where the last inequality follows from (3.14). Thus,

$$I_1 + I_2 \le 2I_1 \le 2\exp(C(p,\omega,u)n|E|) \int_{\mathbb{T}\setminus E} |T_n|^p \omega u.$$

Case 2: $|B| > |E|$. Similarly to Case 1, using (5.1), we get

$$\int_{\mathbb{T}} |T_n|^p \omega u \le I_1 + I_2,$$

where

$$\begin{aligned}I_1 &:= \exp(C(p,u)Kn|E|) \int_{\mathbb{T}\setminus E} |T_n|^p v^p u, \\ I_2 &:= \exp(C(p,u)Kn|E|) \int_{\mathbb{T}\setminus E} |T_n|^p \bigl|v^p - |v_{Kn}|^p\bigr| u.\end{aligned}$$

By (5.10),

$$I_2 \le \exp(C(p,u)Kn|E|) \exp\bigl(-c(p,\omega)Knx_1(Kn)\bigr) \int_{\mathbb{T}} |T_n|^p u.$$

Applying again the Remez inequality (5.1), we obtain

$$\begin{aligned}I_2 \le{}& \exp(C(p,u)Kn|E|) \exp\bigl(-c(p,\omega)Knx_1(Kn)\bigr) \exp\bigl(C(p,u)n(|B| + |E|)\bigr) \\ &\times \int_{\mathbb{T}\setminus(E\cup B)} |T_n|^p u.\end{aligned}$$

Since $\omega(t) \geq \exp(-nx_1(n))$, $t \in \mathbb{T} \setminus B$, we get

$$I_2 \leq \exp(C(p,u)Kn|E|)\exp\big(-c(p,\omega)Knx_1(Kn)\big)\exp(C(p,u)n|B|)\exp(nx_1(n)) \times \int_{\mathbb{T}\setminus(E\cup B)} |T_n|^p \omega u.$$

Taking into account that $|B| \leq C(\omega)x_1(n)$, we deduce that

$$\exp\big(C(p,u)Kn|E|-c(p,\omega)Knx_1(Kn)+C(p,u)n|B|+nx_1(n)\big) \leq \exp(C(p,u)Kn|E|)$$

for $K = K(\omega, u, p)$ large enough. Thus,

$$I_2 \leq \exp(C(p,u)Kn|E|)\int_{\mathbb{T}\setminus E} |T_n|^p \omega u.$$

Collecting the estimates for I_1 and I_2, we arrive at

$$\int_{\mathbb{T}} |T_n|^p \omega u \leq \exp(C(p,\omega,u)n|E|)\int_{\mathbb{T}\setminus E} |T_n|^p \omega u, \qquad p \in (0,\infty),$$

which is the required inequality.

The proof in the case $p = \infty$ follows the same lines and is left to the reader. □

The proof of Theorem 5.2 is similar to the proof of Theorem 5.1 thanks to Remark 5.3.

We now give the following important corollary of the Remez inequalities for product weights.

Corollary 5.3. *Let* $\omega = \omega_1 \dots \omega_s$, *where* $\omega_i \in \Omega$, $i = 1, \dots, s$. *Let also* $0 < p \leq \infty$ *and* $u \in \mathcal{R}_{\text{int}}(p)$. *Then*

$$\int_{\mathbb{T}} |T_n|^p \omega u \asymp \int_{\mathbb{T}} |T_n|^p |v_{Kn}^{(1)}|^p \cdots |v_{Kn}^{(s)}|^p u, \qquad 0 < p < \infty,$$

where $v_n^{(i)}$ *is the* n*-th partial Fourier sum of* $\omega_i^{1/p}$, $i = 1, \dots, s$, *and* $K = K(\omega, u, p)$ *is large enough.*

Moreover,

$$\|T_n \omega u\|_{L_\infty(\mathbb{T})} \asymp \|T_n v_{Kn}^{(1)} \cdots v_{Kn}^{(s)} u\|_{L_\infty(\mathbb{T})}$$

where $v_n^{(i)}$ *is the* n*-th partial Fourier sum of* ω_i, $i = 1, \dots, s$, *and* $K = K(\omega, u)$ *is large enough.*

To prove this, we use induction, Lemma 5.1(B), and the following result provided by Theorem 5.2: if $\omega_i \in \Omega$ and $u \in \mathcal{R}_{\text{int}}(p)$, then $\omega_1 \dots \omega_l u \in \mathcal{R}_{\text{int}}(p)$ for any integer $1 \leq l \leq s-1$ (for $p < \infty$ see also Corollary 5.2).

We finish this section by proving the following Remez inequality for algebraic polynomials P_n.

Corollary 5.4. *Let $0 < p < \infty$ and $\omega = \omega_1 \dots \omega_s$, where $\omega_i(\cos t) \in \Omega$, $i = 1, \dots, s$. Then*

$$\|P_n\|_{L_p(\omega u,[-1,1])} \le \exp(C(p,\omega,u)n\sqrt{|E|})\|P_n\|_{L_p(\omega u,[-1,1]\setminus E)} \tag{5.12}$$

for all measurable sets E with $|E| \le 1/4$ and any weight $u \in A_\infty$. For $p = \infty$, (5.12) holds for $u \in A^*$.

Proof. To prove (5.12), we use the change of variables $x = \cos t$, Corollary 5.1, and the following two facts:

$$u \in A_\infty \text{ on } [-1,1] \quad \text{if and only if} \quad u(\cos t)|\sin t| \in A_\infty \text{ on } \mathbb{T} \tag{5.13}$$

(see [MT, p. 63]), and

$$u \in A^* \text{ on } [-1,1] \quad \text{if and only if} \quad u(\cos t) \in A^* \text{ on } \mathbb{T} \tag{5.14}$$

(see [MT, p. 68]).

To conclude the proof, we remark that for the map $\Phi(t) = \cos t$ and any measurable set $E \subset [-1,1]$ with $|E| \le 1/4$, we have $|\Phi^{-1}(E)| \le 2\sqrt{|E|} \le 1$. □

An analogue of Theorem 5.2 for algebraic polynomials can be established similarly.

6. Weighted Bernstein inequalities in L_p

The goal of this section is to establish the weighted Bernstein inequality in L_p for product weights generalizing Theorem 4.1. The proof combines the approximation technique that was used in Theorem 4.1 and the Remez inequalities from Section 5.

Definition. We say that a weight u satisfies the *$\mathcal{B}(p)$ condition*, $0 < p \le \infty$, and write $u \in \mathcal{B}(p)$, if for any trigonometric polynomial T_n of degree at most n the weighted Bernstein inequality holds, that is,

$$\|T_n'\|_{L_p(u)} \le C(p,u)n\|T_n\|_{L_p(u)}. \tag{6.1}$$

Theorem 6.1. *Let $0 < p \le \infty$, $\omega \in \Omega$, and $u \in \mathcal{B}(p) \cap \mathcal{R}_{\mathrm{int}}(p)$. Then for any trigonometric polynomial T_n of degree at most n we have*

$$\|T_n'\|_{L_p(\omega u)} \le Cn\|T_n\|_{L_p(\omega u)}, \tag{6.2}$$

where $C = C(\omega, u, p)$.

Proof. It is enough to prove (6.2) for n large enough. We start with the case $0 < p < \infty$.

First, by (5.5) we have, for some $K = K(\omega, u, p)$,

$$\int_{\mathbb{T}} |T_n'|^p \omega u \le 2 \int_{\mathbb{T}} |T_n'|^p |v_{Kn}|^p u \le 2^{1+p}\Big(\int_{\mathbb{T}} |(T_n v_{Kn})'|^p u + \int_{\mathbb{T}} |T_n v_{Kn}'|^p u\Big).$$

Since $u \in \mathcal{B}(p)$, we get

$$\int_{\mathbb{T}} |(T_n v_{Kn})'|^p u \le C(\omega, u, p) n^p \int_{\mathbb{T}} |T_n v_{Kn}|^p u \le C(\omega, u, p) n^p \int_{\mathbb{T}} |T_n|^p \omega u.$$

Also,

$$\int_{\mathbb{T}} |T_n v_{Kn}'|^p u \le 2^p \Big(\int_{\mathbb{T}} |T_n v'|^p u + \int_{\mathbb{T}} |T_n|^p |v' - v_{Kn}'|^p u\Big).$$

To conclude the proof, we follow the estimation of I_{21} and I_{22} in the proof of Theorem 4.1 taking into account (5.4). Note that in view of Remark 5.3 it suffices to assume that $u \in \mathcal{R}_{\text{int}}(p)$.

Finally, we arrive at

$$\int_{\mathbb{T}} |T_n'|^p \omega u \le C(\omega, u, p) n^p \int_{\mathbb{T}} |T_n|^p \omega u.$$

The proof for $p = \infty$ is similar, using Theorem 3.1 and the inequality

$$\tfrac{1}{2}\|T_n \omega_{Kn} u\|_{L_\infty(\mathbb{T})} \le \|T_n \omega u\|_{L_\infty(\mathbb{T})} \le 2\|T_n \omega_{Kn} u\|_{L_\infty(\mathbb{T})}, \tag{6.3}$$

for K large enough provided by Lemma 5.1(C). First,

$$\begin{aligned}\|T_n' \omega u\|_{L_\infty(\mathbb{T})} &\le C\big(\|(T_n \omega_{Kn})' u\|_{L_\infty(\mathbb{T})} + \|T_n \omega_{Kn}' u\|_{L_\infty(\mathbb{T})}\big)\\ &\le C\big(n\|T_n \omega_{Kn} u\|_{L_\infty(\mathbb{T})} + \|T_n \omega_{Kn}' u\|_{L_\infty(\mathbb{T})}\big),\end{aligned}$$

where $C = C(\omega, u, p)$. In view of (6.3), $n\|T_n \omega_{Kn} u\|_{L_\infty(\mathbb{T})} \le 2n\|T_n \omega u\|_{L_\infty(\mathbb{T})}$. To estimate the second term, we write

$$\|T_n \omega_{Kn}' u\|_{L_\infty(\mathbb{T})} \le \Big(\operatorname*{ess\,sup}_{t \in B_{n,M}} + \operatorname*{ess\,sup}_{t \in \mathbb{T}\setminus B_{n,M}}\Big)|T_n(t)\omega_{Kn}'(t)u(t)|$$

and use Remez's inequality with $u \in \mathcal{R}_{\text{int}}(p)$ and Theorem 3.1 to get $\|T_n \omega_{Kn}' u\|_{L_\infty(\mathbb{T})} \le Cn\|T_n \omega u\|_{L_\infty(\mathbb{T})}$. □

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. First, any doubling weight u satisfies Bernstein's inequality (6.1) for $0 < p < \infty$ (see [MT, Th. 4.1] and [Er1, Th. 3.1]). Concerning the restricted Remez inequality, (5.2) holds for any doubling weight u (see [Er1, Th. 7.2]), and therefore $u \in \mathcal{B}(p) \cap \mathcal{R}_{\text{int}}(p)$, $0 < p < \infty$. Then, by Corollary 5.2, $\omega_1 \dots \omega_{s-1} u \in \mathcal{R}_{\text{int}}(p)$. Thus, if $0 < p < \infty$, the statement of Theorem 1.1 follows from Theorem 6.1 by induction.

Let now $p = \infty$ and $u \in A^*$. Bernstein's inequality (6.1) is proved in [MT, (6.7)], and Remez's inequality in [MT, (6.10)]. Therefore, $u \in A^*$ implies $u \in \mathcal{B}(\infty) \cap \mathcal{R}_{\text{int}}(\infty)$. Similarly to the case $p < \infty$, Theorem 1.1 immediately follows from Corollary 5.1 and Theorem 6.1. □

7. Weighted Bernstein and Markov inequalities for algebraic polynomials

In this section, we deal with weights ω and $u : [-1, 1] \to [0, \infty)$. The weight u is either doubling or satisfies the A^* condition on $[-1, 1]$; both notions are defined similarly to those on $\mathbb{T}$ (see, e.g., [MT, p. 62]). First, we obtain a weighted Bernstein inequality for algebraic polynomials on $[-1, 1]$.

Theorem 7.1. *Let $0 < p < \infty$ and $\omega = \omega_1 \dots \omega_s$, where $\omega_i(\cos t) \in \Omega$, $i = 1, \dots, s$, and let u be a doubling weight. Then*

$$\int_{-1}^{1} \varphi^p(x)|P_n'(x)|^p \omega(x)u(x)\,dx \le C(p, \omega, u)n^p \int_{-1}^{1} |P_n(x)|^p \omega(x)u(x)\,dx, \qquad \varphi(x) = \sqrt{1 - x^2}. \tag{7.1}$$

Proof. This follows immediately from Theorem 1.1, the change of variables $x = \cos t$, and the fact that u is doubling on $[-1, 1]$ if and only if $u(\cos t)|\sin t|$ is doubling on $\mathbb{T}$ (see [MT, p. 63]). □

A counterpart for $p = \infty$ reads as follows.

Theorem 7.2. *Let $\omega = \omega_1 \dots \omega_s$, where $\omega_i(\cos t) \in \Omega$, $i = 1, \dots, s$, and let $u \in A^*$. Then*

$$\|\varphi P_n' \omega u\|_{L_\infty[-1,1]} \le C(\omega, u)n\|P_n \omega u\|_{L_\infty[-1,1]}. \tag{7.2}$$

The proof is similar to the proof of Theorem 7.1, using (5.14).

Let us now discuss Markov's inequality for algebraic polynomials.

Theorem 7.3. *Let $0 < p < \infty$ and $\omega = \omega_1 \dots \omega_s$, where $\omega_i(\cos t) \in \Omega$, $i = 1, \dots, s$, and let u be a doubling weight. Then*

$$\int_{-1}^{1} |P_n'(x)|^p \omega(x)u(x)\,dx \le C(p, \omega, u)n^{2p} \int_{-1}^{1} |P_n(x)|^p \omega(x)u(x)\,dx. \tag{7.3}$$

Proof. First, the Bernstein inequality (7.1) yields

$$Cn^{2p} \int_{-1}^{1} |P_n(x)|^p \omega(x)u(x)\,dx \ge n^p \int_{-1}^{1} \varphi^p(x)|P_n'(x)|^p \omega(x)u(x)\,dx.$$

Therefore, it is enough to show that

$$Cn^p \int_{-1}^{1} \varphi^p(x)|P_n'(x)|^p \omega(x)u(x)\,dx \ge \int_{-1}^{1} |P_n'(x)|^p \omega(x)u(x)\,dx,$$

or, taking an even trigonometric polynomial $T_n(t) = P_n'(\cos t)$,

$$Cn^p \int_{\mathbb{T}} |T_n(t) \sin t|^p \omega(\cos t)u(\cos t)|\sin t|\,dt \ge \int_{\mathbb{T}} |T_n(t)|^p \omega(\cos t)u(\cos t)|\sin t|\,dt,$$

or equivalently

$$Cn^p \int_{\mathbb{T}} |T_n(t)\sin t|^p \bar{\omega}(t)\bar{u}(t)\,dt \ge \int_{\mathbb{T}} |T_n(t)|^p \bar{\omega}(t)\bar{u}(t)\,dt, \tag{7.4}$$

where $\bar{\omega}(t) = \bar{\omega}_1(t)\dots\bar{\omega}_s(t)$, $\bar{\omega}_i = \omega_i(\cos t) \in \Omega$, $i = 1,\dots,s$, and $\bar{u}(t) = u(\cos t)|\sin t|$ is doubling on $\mathbb{T}$.

Now, (7.4) follows from Corollary 5.2 for $E = [-1/n, 1/n] \cup [\pi - 1/n, \pi + 1/n]$:

$$\int_{\mathbb{T}} |T_n(t)|^p \bar{\omega}(t)\bar{u}(t)\,dt \le C \int_{\mathbb{T}\setminus E} |T_n(t)|^p \bar{\omega}(t)\bar{u}(t)\,dt$$
$$\le Cn^p \int_{\mathbb{T}\setminus E} |T_n(t)\sin t|^p \bar{\omega}(t)\bar{u}(t)\,dt. \qquad \square$$

Markov's inequality for $p = \infty$ is written as follows.

Theorem 7.4. *Let $\omega = \omega_1 \dots \omega_s$, where $\omega_i(\cos t) \in \Omega$, $i = 1,\dots,s$, and let u be an A^* weight on $[-1,1]$. Then*

$$\|P_n' \omega u\|_{L_\infty[-1,1]} \le C(\omega, u) n^2 \|P_n \omega u\|_{L_\infty[-1,1]}.$$

The proof repeats the argument of the proof of Theorem 7.3, using the inequality

$$\|T_n(t)\bar{\omega}(t)\tilde{u}(t)\|_{L_\infty(\mathbb{T})} \le Cn \|T_n(t)\bar{\omega}(t)\tilde{u}(t)\sin t\|_{L_\infty(\mathbb{T})}, \tag{7.5}$$

where $\bar{\omega}(t) = \bar{\omega}_1(t)\dots\bar{\omega}_s(t)$, $\bar{\omega}_i = \omega_i(\cos t) \in \Omega$, $i = 1,\dots,s$, and $\tilde{u}(t) = u(\cos t)$ is an A^* weight on $\mathbb{T}$. Inequality (7.5) follows from Corollary 5.1.

Let us remark that the non-weighted version of (7.4)–(7.5),

$$\|T_n(t)\|_{L_p(\mathbb{T})} \le C(p) n \|T_n(t)\sin t\|_{L_p(\mathbb{T})}, \qquad 0 < p \le \infty,$$

was proved in [Be2, Ba].

Remark 7.1. Note that for some weights the Bernstein inequality (7.2) for algebraic polynomials can be derived from known results. First, let us recall the definition of the Mhaskar–Rakhmanov–Saff number, which is a crucial concept to analyze weighted inequalities. Suppose that $\omega(x) = \exp(Q(x))$, where $Q : (-1,1) \to \mathbb{R}$ is even and differentiable on $(0,1)$. Also suppose that $xQ'(x)$ is positive and increasing in $(0,1)$ with limits zero and infinity at 0 and 1, respectively, and

$$\int_0^1 \frac{xQ'(x)}{\sqrt{1-x^2}}\,dx = \infty.$$

Then the n-th *Mhaskar–Rakhmanov–Saff number*, $a_n = a_n(Q)$, is defined to be the root of

$$n = \frac{2}{\pi} \int_0^1 \frac{a_n x Q'(a_n x)}{\sqrt{1-x^2}}\,dx, \qquad n \ge 1.$$

The importance of this number lies in the Mhaskar–Saff identity

$$\|P_n\omega\|_{C[-1,1]} = \|P_n\omega\|_{C[-a_n,a_n]}, \qquad n \geq 1,$$

and asymptotically as $n \to \infty$, a_n is the smallest such number (see [MS1, MS2]). In particular, for the weight

$$\omega_\alpha(x) = \exp(-(1-x^2)^\alpha), \qquad \alpha > 0, \tag{7.6}$$

we have

$$1 - a_n \asymp n^{-1/(\alpha+1/2)}, \qquad n \to \infty. \tag{7.7}$$

For this weight, Lubinsky and Saff proved the following inequalities [LS, p. 531]:

$$\left|P_n'(x)\omega_\alpha(x)\sqrt{1-|x|/a_n}\right| \leq C(\alpha)n\|P_n\omega_\alpha\|_{C[-1,1]}, \qquad |x| < a_n, \tag{7.8}$$

$$\|P_n'\omega_\alpha\|_{C[-1,1]} \leq C(\alpha)n^{\frac{2\alpha+2}{2\alpha+1}}\|P_n\omega_\alpha\|_{C[-1,1]}. \tag{7.9}$$

In fact, similar results hold for a wide class of functions, which we denote by $\mathcal{W}$. By definition, $\omega = \exp(-Q) \in \mathcal{W}$ if

(i) Q is even and continuously differentiable in $(-1, 1)$, while Q'' is continuous in $(0, 1)$;
(ii) $Q' \geq 0$ and $Q'' \geq 0$ in $(0, 1)$;
(iii) $\int_0^1 (xQ'(x)/\sqrt{1-x^2})\,dx = \infty$;
(iv) for $T(x) = 1 + xQ''(x)/Q'(x)$, $x \in (0, 1)$, one has: T is increasing in $(0, 1)$, $T(0+) > 1$, and $T(x) = O(Q'(x))$ as $x \in 1-$.

Let us show that both (7.8) and (7.9) imply (7.2) for ω_α given by (7.6) and $u(x) \equiv 1$. Indeed, let $x \in (0, 1)$. If $1 - C^2n^{-1/(\alpha+1/2)} \leq x$ for some positive $C = C(\alpha)$, then $n^{(2\alpha+2)/(2\alpha+1)} \leq 2Cn/\sqrt{1-x^2}$, and (7.9) implies

$$|P_n'(x)\varphi(x)\omega_\alpha(x)| \leq Cn\|P_n\omega_\alpha\|_{C[-1,1]} \tag{7.10}$$

for such x. If $x \leq (C^2 - 1)/(C^2/a_n - 1)$, then $\sqrt{1-x^2} \leq 2C\sqrt{1-|x|/a_n}$ and (7.8) implies (7.10) for such x. Further, (7.7) shows that $a_n > 1 - Bn^{-1/(\alpha+1/2)}$ for some $B = B(\alpha) > 0$. Then, taking $C^2 = 2B + 2$, we obtain

$$1 - C^2n^{-1/(\alpha+1/2)} < (C^2 - 1)/(C^2/a_n - 1)$$

for sufficiently large n. Finally,

$$\|P_n'\varphi\omega_\alpha\|_{C[-1,1]} \leq Cn\|P_n\omega_\alpha\|_{C[-1,1]}.$$

We also mention that in the recent papers [MN, No] the authors obtained weighted Bernstein, Nikolskii, and Remez inequalities for algebraic polynomials for the weights $\omega(x) = \exp(-(1-x^2)^\alpha)u(x)$, $\alpha > 0$, where u is doubling on $[-1, 1]$.

8. Weighted Nikolskii inequalities

Nikolskii's inequality for trigonometric polynomials, that is,

$$\|T_n\|_{L_q(\mathbb{T})} \le Cn^{1/p-1/q}\|T_n\|_{L_p(\mathbb{T})}, \qquad p < q,$$

plays an important role in approximation theory and functional analysis, in particular, in the proofs of embedding theorems for function spaces (see, e.g., [DW]). It is known that if u is an A_∞ weight, then for any $0 < p \le q < \infty$ there is a constant $C = C(u, p, q)$ such that

$$\left(\int_{\mathbb{T}} |T_n|^q u\right)^{1/q} \le Cn^{1/p-1/q}\left(\int_{\mathbb{T}} |T_n|^p u^{p/q}\right)^{1/p} \tag{8.1}$$

(see [MT, Th. 5.5] and [Er3, Th. 8.1]). Moreover, if $u \in A^*$, then for any $1 \le p < \infty$ there is a constant $C = C(u, p)$ such that

$$\|T_n u\|_{L_\infty(\mathbb{T})} \le Cn^{1/p}\left(\int_{\mathbb{T}} |T_n|^p u^p\right)^{1/p} \tag{8.2}$$

(see [MT, (6.9)]). Note that (8.2) holds for $0 < p < 1$ as well, provided $u \in A^*$. Indeed, we first apply (8.2) with $p = 1$ to get

$$\|T_n\|_{L_\infty(u)} \le Cn\|T_n\|_{L_1(u)}. \tag{8.3}$$

Then, since $u \in A^*$ yields $u \in A_\infty$, we use (8.1) with $0 < p < 1$ and $q = 1$:

$$\|T_n\|_{L_1(u)} \le Cn^{1/p-1}\|T_n\|_{L_p(u^p)}. \tag{8.4}$$

We prove the following weighted Nikolskii inequalities for trigonometric polynomials.

Theorem 8.1. *Let $0 < p \le q \le \infty$ and $\omega = \omega_1 \dots \omega_s$, where $\omega_i \in \Omega$, $i = 1, \dots, s$, and let $u \in \mathcal{R}_{\text{int}}(q)$.*

(A) *Suppose $q < \infty$, $u^{p/q} \in \mathcal{R}_{\text{int}}(p)$, and (8.1) holds for each trigonometric polynomial T_n. Then*

$$\|T_n\|_{L_q(\omega u)} \le Cn^{1/p-1/q}\|T_n\|_{L_p((\omega u)^{p/q})}, \tag{8.5}$$

where $C = C(\omega, u, p, q)$.

(B) *Suppose $p < q = \infty$, $u^p \in \mathcal{R}_{\text{int}}(p)$, and (8.2) holds for each trigonometric polynomial T_n. Then*

$$\|T_n\|_{L_\infty(\omega u)} \le Cn^{1/p}\|T_n\|_{L_p((\omega u)^p)}, \tag{8.6}$$

where $C = C(\omega, u, p)$.

In particular, this implies

Corollary 8.1. *Let $\omega = \omega_1 \dots \omega_s$, where $\omega_i \in \Omega$, $i = 1, \dots, s$. Then inequality (8.5) holds provided $u \in A_\infty$ and $0 < p \le q < \infty$, and (8.6) holds provided $u \in A^*$ and $0 < p < \infty$.*

Proof of Theorem 8.1. First, by definition of Ω, any weight $\omega_i \in \Omega$, $1 \le i \le s-1$, is such that $\omega_i^{p/q} \in \Omega$ for any $0 < p, q < \infty$. Then, by Corollary 5.2, $\omega_1 \dots \omega_{s-1} u \in \mathcal{R}_{\text{int}}(q)$ and $(\omega_1 \dots \omega_{s-1} u)^{p/q} \in \mathcal{R}_{\text{int}}(p)$. Thus, it is enough to prove (8.5) and (8.6) for $\omega = \omega_s \in \Omega$.

(A) By (5.5) we have

$$\int_{\mathbb{T}} |T_n|^q \omega u \asymp \int_{\mathbb{T}} |T_n|^q |v_{Kn}|^q u, \qquad u \in \mathcal{R}_{\text{int}}(q), \tag{8.7}$$

where v_n is the n-th partial Fourier sum of $\omega^{1/q}$ and $K = K(\omega, u)$ is large enough. Moreover, applying again (5.5) for the weight $\omega^{p/q}$, where $0 < p \le q < \infty$, we have

$$\int_{\mathbb{T}} |T_n|^p \omega^{p/q} u^{p/q} \asymp \int_{\mathbb{T}} |T_n|^p |v_{Kn}|^p u^{p/q} \tag{8.8}$$

for $K = K(\omega, u, p, q)$ large enough, provided that $u^{p/q} \in \mathcal{R}_{\text{int}}(p)$. Now we apply (8.1) to get (8.5).

(B) The case $q = \infty$ is similar since $\omega_1 \dots \omega_{s-1} u \in \mathcal{R}_{\text{int}}(\infty)$ and $(\omega_1 \dots \omega_{s-1} u)^p \in \mathcal{R}_{\text{int}}(p)$. □

Proof of Corollary 8.1. To show (8.5) for $0 < p < q < \infty$ and (8.6) for $1 \le p < \infty$, we use results from [MT], [Er3], and the following two facts:

(i) $u^{p/q} \in A_\infty$ whenever $u \in A_\infty$ and $0 < p < q < \infty$ (see [St, Ch. V]), and
(ii) $u^p \in A^* \subset A_\infty$ whenever $u \in A^*$ and $p > 1$; this follows from Jensen's inequality.

To prove (8.6) for $0 < p < 1$, we first apply (8.6) with $p = 1$ and then (8.5) with $0 < p < 1$ and $q = 1$ as in (8.3) and (8.4). □

We finish this section by proving Nikolskii inequalities for algebraic polynomials.

Corollary 8.2. *Let $0 < p \le q \le \infty$ and $\omega = \omega_1 \dots \omega_s$, where $\omega_i(\cos t) \in \Omega$, $i = 1, \dots, s$. Then for each algebraic polynomial P_n we have*

$$\|P_n\|_{L_q([-1,1],\omega u)} \le C(p,q,\omega,u) n^{2/p - 2/q} \|P_n\|_{L_p([-1,1],(\omega u)^{p/q})}, \qquad 0 < p \le q < \infty, \tag{8.9}$$

provided $u \in A_\infty$, and

$$\|P_n\|_{L_\infty([-1,1],\omega u)} \le C(p,q,\omega,u) n^{2/p} \|P_n\|_{L_p([-1,1],(\omega u)^p)}, \qquad 0 < p < \infty, \tag{8.10}$$

provided $u \in A^*$.

Proof. First, let $0 < p \le q < \infty$. We give a straightforward proof applying the Remez inequalities for algebraic polynomials given by Corollary 5.4. Define

$$E := \left\{ x \in [-1,1] : n^2 \int_{-1}^{1} |P_n|^q \omega u \le |P_n(x)|^q \omega(x) u(x) \right\}.$$

Then, since $|E| \le n^{-2}$, inequality (5.12) yields

$$\begin{aligned}\|P_n\|^q_{L_q([-1,1],\omega u)} &\le C(q,\omega,u)\int_{[-1,1]\setminus E}|P_n|^q\omega u\\ &\le C(q,\omega,u)\big\||P_n|^q\omega u\big\|^{(q-p)/q}_{L_\infty([-1,1]\setminus E)}\int_{[-1,1]\setminus E}|P_n|^p(\omega u)^{p/q}\\ &\le C(q,\omega,u)n^{2(q-p)/q}\left(\int_{-1}^1|P_n|^q\omega u\right)^{(q-p)/q}\int_{-1}^1|P_n|^p(\omega u)^{p/q},\end{aligned}$$

which gives (8.9).

Let now $0<p<\infty$ and $u\in A^*$. Let $v_n^{(i)}(\cos t)$ be the n-th partial Fourier sum of $\omega_i(\cos t)\in\Omega$, $i=1,\dots,s$. Then, by Corollary 5.3, changing variables gives

$$\|P_n\omega u\|_{L_\infty[-1,1]}\asymp\|P_n v^{(1)}_{Kn}\cdots v^{(s)}_{Kn}u\|_{L_\infty[-1,1]}$$

provided that $u(\cos t)|\sin t|$ is an A^* weight on $\mathbb{T}$. The latter holds by (5.14).

Moreover, since $u^p\in A^*\subset A_\infty$, $p>1$, Corollary 5.3 implies that

$$\int_{-1}^1|P_n|^p(\omega u)^p\asymp\int_{-1}^1|P_n|^p|v^{(1)}_{Kn}|^p\cdots|v^{(s)}_{Kn}|^pu^p.$$

Then (8.10) for $1\le p<\infty$ follows from

$$\|P_n\|_{L_\infty(u)}\le C(p,u)n^{2/p}\|P_n\|_{L_p(u^p)},\qquad u\in A^*,\ 1\le p<\infty$$

(see [MT, (7.31)]). The case $0<p<1$ can be treated as in the proof of Corollary 8.1. □

9. Necessary conditions for a weighted Bernstein inequality

We will use the following properties of the Chebyshev polynomials $\mathcal{T}_n$ defined by $\mathcal{T}_n(\cos t)=\cos nt$:

$$|\mathcal{T}_n(x)|\le 1,\qquad |x|\le 1; \tag{9.1}$$

$$\mathcal{T}_n(x)\text{ is increasing on }(1,\infty); \tag{9.2}$$

$$\mathcal{T}_n(x)=\tfrac12\big((x+\sqrt{x^2-1})^n+(x-\sqrt{x^2-1})^n\big)\qquad\text{for all }x\in\mathbb{R}\setminus(-1,1). \tag{9.3}$$

The last identity readily implies that

$$\mathcal{T}_n(1+1/n^2)\le C_1,\qquad n\in\mathbb{N}, \tag{9.4}$$

$$\frac{\mathcal{T}_n'(x)}{\mathcal{T}_n(x)}\ge\frac14\frac{n}{\sqrt{x^2-1}},\qquad x>1+1/n^2,\ n\in\mathbb{N}. \tag{9.5}$$

To prove the main theorems of this section we need two auxiliary results.

Lemma 9.1. *Let* ξ *be a negative increasing continuous function on* $(0, \epsilon)$*, for some* $\epsilon > 0$*, such that* $\xi(0+) = -\infty$ *and, for each* $r \in (0, 1)$*,*

$$\xi(rx)/\xi(x) \to \infty \quad \text{as } x \to 0+. \tag{9.6}$$

Then for each positive sequence h_n *such that* $h_n \to 0$ *as* $n \to \infty$ *there exists a positive sequence* $\beta_n \to 0$ *as* $n \to \infty$ *such that, for each* $r \in (0, 1)$*,*

$$\inf_{x\in(0,h_n)} \frac{\xi(rx)}{\xi(x)}\beta_n \to \infty \quad \text{as } n \to \infty.$$

Proof. Fix a positive sequence $h_n \to 0$. By (9.6), there exists an increasing sequence of positive integers $n(k)$ such that for each $n > n(k)$,

$$\inf_{x\in(0,h_n)} \frac{\xi((1-1/k)x)}{\xi(x)} > k^2. \tag{9.7}$$

Set $\beta_n = 1/k$ for $n \in [n(k) + 1, n(k + 1)]$. Fix $r \in (0, 1)$. Consider a positive integer K such that $1 - 1/K > r$ and $h_n < \epsilon$ for $n > n(K)$. Applying monotonicity of ξ and (9.7), we get

$$\inf_{x\in(0,h_n)} \frac{\xi(rx)}{\xi(x)}\beta_n > \inf_{x\in(0,h_n)} \frac{\xi((1-1/K)x)}{\xi(x)} \frac{1}{K} > K, \qquad n \in [n(K) + 1, n(K + 1)].$$

This establishes the statement of the lemma. □

The proof of the next lemma is a trivial corollary of the mean value theorem.

Lemma 9.2. *Let* ξ *be an increasing continuous function on* $(0, \epsilon)$*, for some* $\epsilon > 0$*, such that* $\xi(0+) = -\infty$*. Then, for each* M *large enough, the equation*

$$\xi(x) = -Mx$$

has a unique solution $y(M) \in (0, \epsilon)$*, which is continuous in* M *and decreasing to* 0 *as* $M \to \infty$*.*

Now we give the following extension of Theorem 1.3.

Theorem 9.1. *Let* $\omega \in C(\mathbb{T})$ *be a weight function satisfying the following conditions:*

$$\omega(t_0) = 0 \quad \text{for some } t_0 \in \mathbb{T}, \tag{9.8}$$

$$\omega \text{ is increasing on } (t_0, t_0 + \epsilon) \text{ and decreasing on } (t_0 - \epsilon, t_0) \text{ for some } \epsilon > 0, \tag{9.9}$$

$$\lim_{t\to t_0} \frac{\log\omega(t_0 + r(t - t_0))}{\log\omega(t)} = \infty \quad \text{for each } r \in (0, 1). \tag{9.10}$$

Then for each $0 < p \le \infty$ *there exists a sequence of trigonometric polynomials* Q_n *of degree at most* n *such that*

$$\lim_{n\to\infty} \frac{\|Q_n'\|_{L_p(\omega)}}{n\|Q_n\|_{L_p(\omega)}} = \infty.$$

Remark 9.1. (i) Note that if ω is a continuous nondoubling weight then $\omega(t_0) = 0$ for some $t_0 \in \mathbb{T}$, i.e., condition (9.8) holds. Without loss of generality we assume below that $t_0 = 0$ and $\|\omega\|_{C(\mathbb{T})} \le 1$.

(ii) Condition (9.9) has been assumed to simplify the proof. The principal condition is (9.10), which implies that ω goes to 0 fast enough as $t \to 0$. Condition (9.10) can be equivalently written as follows: for each $r \in (0, 1)$,

$$\lim_{t\to 0} \frac{\log \omega(rt)}{\log \omega(t)} \text{ exists, possibly } \infty,$$

and, for some $r^* \in (0, 1)$,

$$\lim_{t\to 0} \frac{\log \omega(r^* t)}{\log \omega(t)} = \infty.$$

Example 9.1. A typical example of a weight satisfying the conditions of Theorem 9.1 is

$$\omega^*_\alpha(t) = \exp(-F(g(t))), \quad \text{where} \quad F(x) = \exp(|x|^{-\alpha}),\ \alpha > 0,$$

and $g : \mathbb{T} \to [-1, 1]$ is an analytic function, $g(0) = 0$. Although $\omega^*_\alpha \in C^\infty(\mathbb{T})$, the result of Theorem 1.1 is not true for this kind of function.

Proof of Theorem 9.1. Our proof is in five steps. First, we will prove the theorem for $p = \infty$ (Steps 1–4).

Step 1. Recall that $t_0 = 0$ and $\|\omega\|_{C(\mathbb{T})} \le 1$. We choose Q_n as follows:

$$Q_n(t) := \mathcal{T}_n(1 + a_n^2 - \sin^2 t),$$

where $a_n \to 0$ is a positive sequence depending on ω to be chosen later. For each $n \in \mathbb{N}$, we denote by b_n any point on $\mathbb{T}$ such that

$$\|Q_n\omega\|_{C(\mathbb{T})} = |Q_n(b_n)\omega(b_n)|.$$

Without loss of generality we may assume that $b_n \in (0, \pi)$. Suppose that the sequence $\{a_n\}$ is such that

$$\lim_{n\to\infty} Q_n(b_n)\omega(b_n) = \infty, \tag{9.11}$$

$$b_n = a_n(1 + o(1)) \quad \text{as } n \to \infty. \tag{9.12}$$

Then (9.4) and (9.11) imply

$$1 + a_n^2 - \sin^2 b_n > 1 + 1/n^2 \quad \text{for } n \text{ large enough.} \tag{9.13}$$

Hence,

$$\begin{aligned}\frac{\|Q_n'\omega\|_{C(\mathbb{T})}}{n\|Q_n\omega\|_{C(\mathbb{T})}} &\ge \frac{|Q_n'(b_n)\omega(b_n)|}{nQ_n(b_n)\omega(b_n)} = \frac{\mathcal{T}_n'(1 + a_n^2 - \sin^2 b_n)|\sin 2b_n|}{n\mathcal{T}_n(1 + a_n^2 - \sin^2 b_n)} \\ &\ge \frac{|\sin 2b_n|}{4\sqrt{(1 + a_n^2 - \sin^2 b_n)^2 - 1}},\end{aligned}$$

where in the last inequality we have used (9.5). Finally, taking into account (9.12), we obtain

$$\lim_{n\to\infty} \frac{\|Q_n'\omega\|_{C(\mathbb{T})}}{n\|Q_n\omega\|_{C(\mathbb{T})}} = \infty,$$

which is the statement of the theorem in the case $p = \infty$.

Step 2. Let us now focus on the search of the sequence a_n which satisfies (9.11) and (9.12). Note that if we take sequences $a_n \to 0$ and $\lambda_n \to 1$ such that

$$\mathcal{T}_n(1 + a_n^2 - \sin^2(\lambda_n a_n))\omega(\lambda_n a_n) \to \infty \quad \text{as } n \to \infty, \tag{9.14}$$

and, for each $r \in (0, 1)$,

$$\mathcal{T}_n(1 + a_n^2)\omega(ra_n) \to 0 \quad \text{as } n \to \infty, \tag{9.15}$$

then a_n satisfies (9.11) and (9.12). Indeed, condition (9.14) immediately implies (9.11), so (9.13) holds as well, and hence

$$\limsup_{n\to\infty} b_n/a_n \le 1.$$

If

$$\liminf_{n\to\infty} b_n/a_n < r < 1,$$

then, applying (9.2) and (9.9), we have

$$Q_n(b_n)\omega(b_n) \le \mathcal{T}_n(1 + a_n^2)\omega(ra_n)$$

for infinitely many $n \in \mathbb{N}$. This inequality together with (9.15) contradicts (9.11). So,

$$\liminf_{n\to\infty} b_n/a_n \ge 1, \qquad \text{and therefore} \qquad \lim_{n\to\infty} b_n/a_n = 1,$$

which is (9.12).

Step 3. Set $\xi := \log\omega$. Taking the logarithm on both sides of (9.14) and (9.15), and applying (9.3), we find that if $\{a_n\}$ and $\{\lambda_n\}$ satisfy, as $n \to \infty$,

$$n \log\Big(1 + a_n^2 - \sin^2(\lambda_n a_n) + \sqrt{(1 + a_n^2 - \sin^2(\lambda_n a_n))^2 - 1}\Big) + \xi(\lambda_n a_n) \to \infty,$$

and, for each $r \in (0, 1)$,

$$n \log\Big(1 + a_n^2 + \sqrt{(1 + a_n^2)^2 - 1}\Big) + \xi(ra_n) \to -\infty,$$

then $\{a_n\}$ and $\{\lambda_n\}$ satisfy (9.14) and (9.15) as well. Finally, since $\log(1 + t + \sqrt{(1+t)^2 - 1}) \sim \sqrt{2t}$ as $t \to 0$, it is enough to choose $a_n \to 0$ and $\lambda_n \to 1-$ such that

$$n\lambda_n a_n\sqrt{1 - \lambda_n^2} + \xi(\lambda_n a_n) \to \infty, \tag{9.16}$$

and, for each $r \in (0, 1)$,

$$2na_n + \xi(ra_n) \to -\infty. \tag{9.17}$$

Step 4. Now we are in a position to choose $\{a_n\}$ and $\{\lambda_n\}$. For n large enough, let h_n be the unique solution of the equation

$$\xi(x) = -n^{1/2}x,$$

provided by Lemma 9.2. It follows from Lemma 9.1 that there exists a sequence $\{\lambda_n\}$ which goes to 1 slowly enough such that

$$\sqrt{1-\lambda_n^2} > n^{-1/3}$$

and, for each $r \in (0, 1)$,

$$\inf_{t\in(0,2h_n)} \frac{\xi(rt)}{\xi(t)}\sqrt{1-\lambda_n^2} \to \infty \quad \text{as } n \to \infty.$$

Moreover, for each $r \in (0, 1)$ and $r_1 \in (r, 1)$, we have

$$\inf_{t\in(0,h_n/\lambda_n)} \frac{\xi(rt)}{\xi(\lambda_n t)}\sqrt{1-\lambda_n^2} = \inf_{t\in(0,h_n)} \frac{\xi(rt/\lambda_n)}{\xi(t)}\sqrt{1-\lambda_n^2} \geq \inf_{t\in(0,h_n/\lambda_n)} \frac{\xi(rt/\lambda_n)}{\xi(t)}\sqrt{1-\lambda_n^2}$$
$$\geq \inf_{t\in(0,h_n/\lambda_n)} \frac{\xi(r_1 t)}{\xi(t)}\sqrt{1-\lambda_n^2} \to \infty \quad \text{as } n \to \infty. \tag{9.18}$$

Set $a_n := z_n/\lambda_n$, where z_n is the unique solution of the equation

$$\xi(z) = -\tfrac{1}{2}nz\sqrt{1-\lambda_n^2}, \tag{9.19}$$

provided by Lemma 9.2. Then Lemma 9.2 implies that $z_n \to 0$, and hence $a_n \to 0$. Therefore,

$$n\lambda_n a_n\sqrt{1-\lambda_n^2} + \xi(\lambda_n a_n) = -\xi(\lambda_n a_n) \to \infty,$$

i.e., (9.16) holds.

On the other hand, Lemma 9.2 together with the condition $\frac{1}{2}n\sqrt{1-\lambda_n^2} > n^{1/2}$ for n large enough implies that $z_n = a_n\lambda_n < h_n$. Thus, (9.18) yields

$$\lim_{n\to\infty} \frac{\xi(ra_n)}{\xi(\lambda_n a_n)}\sqrt{1-\lambda_n^2} = \infty.$$

Moreover, (9.19) implies $\xi(\lambda_n a_n) = -\frac{1}{2}n\lambda_n a_n\sqrt{1-\lambda_n^2}$. Hence,

$$\lim_{n\to\infty} \frac{\xi(ra_n)}{n\lambda_n a_n/2} = -\infty,$$

which gives (9.17).

Thus, the sequence $\{a_n\}$ satisfies (9.16) and (9.17), and therefore (9.11) and (9.12), which concludes the proof of Theorem 9.1 in the case $p = \infty$.

Step 5. The proof for $0 < p < \infty$ follows the same lines. We again choose

$$Q_n(t) := \mathcal{T}_n(1 + a_n^2 - \sin^2 t),$$

where $a_n = a_n(\omega, p) \to 0$ is a positive sequence to be chosen later. Similarly to Steps 1 and 2, it is enough to find a sequence $\{a_n\}$ such that $a_n \to 0$ and, for each $r \in (0, 1)$,

$$|\mathcal{T}_n(1 + a_n^2)|^p \omega(ra_n) \to 0, \qquad \int_{\mathbb{T}} |\mathcal{T}_n(1 + a_n^2 - \sin^2 t)|^p \omega(t)\, dt \to \infty.$$

The latter holds if for some sequence $\{\lambda_n\}$ with $\lambda_n \to 1-$ one has

$$\int_{(2\lambda_n - 1)a_n}^{\lambda_n a_n} |\mathcal{T}_n(1 + a_n^2 - \sin^2 t)|^p \omega(t)\, dt \\ \geq |\mathcal{T}_n(1 + a_n^2 - \sin^2(\lambda_n a_n))|^p \omega((2\lambda_n - 1)a_n)(1 - \lambda_n)a_n \to \infty.$$

Similarly to Step 3 (cf. (9.16) and (9.17)) it is enough to choose sequences $\{\lambda_n\}$ and $\{a_n\}$ such that

$$pn\lambda_n a_n \sqrt{1 - \lambda_n^2} + \log(1 - \lambda_n) + \log a_n + \xi((2\lambda_n - 1)a_n) \to \infty \tag{9.20}$$

and, for each $r \in (0, 1)$,

$$2pna_n + \xi(ra_n) \to -\infty. \tag{9.21}$$

Similarly to Step 4 one can choose sequences $\{\lambda_n\}$ and $\{a_n\}$ satisfying

$$\xi((2\lambda_n - 1)a_n) = -pn\lambda_n a_n(1 - \lambda_n^2) \tag{9.22}$$

and (9.21). Finally, (9.22) together with $\lim_{n\to\infty} \omega(a_n)/a_n = 0$ implies (9.20). □

The next theorem (cf. Theorem 1.2) provides a necessary condition for the weighted Bernstein inequality to hold.

Theorem 9.2. *Let $\omega \in C(\mathbb{T})$ be a weight function satisfying* (9.8), (9.9), *and*

$$\limsup_{t\to t_0} \frac{\log \omega(t_0 + r(t - t_0))}{\log \omega(t)} = \infty \quad \text{for each } r \in (0, 1). \tag{9.23}$$

Then for each $0 < p \leq \infty$ there exists a sequence of positive integers $K_n \to \infty$ as $n \to \infty$ and a sequence of trigonometric polynomials Q_{K_n} of degree at most K_n such that

$$\lim_{n\to\infty} \frac{\|Q'_{K_n}\|_{L_p(\omega)}}{K_n \|Q_{K_n}\|_{L_p(\omega)}} = \infty.$$

Remark 9.2. If (9.23) holds for some $r \in (0, 1)$, then it holds for any $r \in (0, 1)$.

Proof of Theorem 9.2. Without loss of generality we assume $t_0 = 0$ and $\|\omega\|_{C(\mathbb{T})} \le 1$. We will prove the theorem only for $p = \infty$; the case $0 < p < \infty$ is similar (see the proof of Theorem 9.1, Step 5). Define

$$Q_{K_n}(t) := \mathcal{T}_{K_n}(1 + a_n^2 - \sin^2 t),$$

with K_n and $a_n \to 0$ to be chosen later. Set $\xi := \log \omega$. Now following step by step the proof of Theorem 9.1 up to (9.16) and (9.17), one can see that it is enough to choose $a_n \to 0$, an increasing sequence of integers K_n, and $\lambda_n \to 1-$ such that

$$K_n \lambda_n a_n \sqrt{1 - \lambda_n^2} + \xi(\lambda_n a_n) \to \infty \tag{9.24}$$

and, for each $r \in (0, 1)$,

$$2K_n a_n + \xi(r a_n) \to -\infty. \tag{9.25}$$

Since

$$\limsup_{t \to 0} \frac{\xi(rt)}{\xi(t)} = \infty \quad \text{for each } r \in (0, 1),$$

there exists a decreasing positive sequence c_n such that $c_n \to 0$ and

$$\frac{\xi((1 - 1/n)c_n)}{\xi(c_n)} > n^2. \tag{9.26}$$

Set

$$\lambda_n := 1 - 1/n, \quad a_n := c_n/\lambda_n, \quad K_n := 2\left[\frac{-\xi(c_n)}{\lambda_n a_n \sqrt{1 - \lambda_n^2}}\right].$$

Since $\lim_{t \to 0} \xi(t) = -\infty$, we have $K_n \to \infty$, and hence (9.24) holds.

To complete the proof, take an arbitrary $r \in (0, 1)$. Since $r < \lambda_n^2$ for n large enough, by monotonicity of ξ we have

$$2K_n a_n + \xi(r a_n) < 2K_n a_n + \xi((1 - 1/n)c_n).$$

Thus, by (9.26),

$$2K_n a_n + \xi(r a_n) < 2K_n a_n + n^2 \xi(c_n) \le \frac{-4\xi(c_n)}{\lambda_n \sqrt{1 - \lambda_n^2}} + n^2 \xi(c_n) \to -\infty.$$

This proves (9.25). □

The next theorem shows an essential difference between Theorems 9.1 and 9.2 in the case when the weight satisfies (9.23) but not (9.10). In this case Bernstein's inequality may hold for some subsequence of integers K_n but not for all $n \in \mathbb{N}$. For simplicity we consider only the case $p = \infty$ and $t_0 = 0$.

Theorem 9.3. *There exists an even weight function* $\omega \in C^\infty(\mathbb{T})$ *satisfying* (9.8) *and* (9.9) *and*

$$\limsup_{t\to 0} \frac{\log \omega(rt)}{\log \omega(t)} = \infty \quad \text{for each } r \in (0,1), \tag{9.27}$$

such that for some increasing sequence of positive integers K_n *the Bernstein inequality*

$$\|T'_{K_n}\omega\|_{C(\mathbb{T})} \le C K_n \|T_{K_n}\omega\|_{C(\mathbb{T})}$$

holds for any trigonometric polynomial T_{K_n} *of degree at most* K_n.

Proof. Let

$$W(x) = \frac{\int_0^{\pi x} \exp(-1/\sin^2 t)\,dt}{\int_0^{\pi} \exp(-1/\sin^2 t)\,dt}, \qquad x \in [0,1].$$

Define an even weight ω as follows:

$$\omega(t) := \begin{cases} 1 & \text{if } t \in [\alpha_1, \pi], \\ d_n & \text{if } t \in [\alpha_n, \alpha_{n-1}/2],\ n \ge 2, \\ d_{n+1} + (d_n - d_{n+1}) W(2t/\alpha_n - 1) & \text{if } t \in [\alpha_n/2, \alpha_n],\ n \ge 1, \\ 0 & \text{if } t = 0, \end{cases}$$

where $d_n := \exp(-\exp(n^2))$ and $\alpha_n := d_n^2$. By construction, $\omega \in C^\infty(\mathbb{T})$. Since

$$\lim_{n\to\infty} \frac{\log \omega(\alpha_n/2)}{\log \omega(\alpha_n)} = \infty,$$

ω satisfies (9.27).

For each $n \in \mathbb{N}$, we also define an even weight ω_n by

$$\omega_n(t) := \begin{cases} \omega(t) & \text{if } t \in [\alpha_n, \pi], \\ d_n & \text{if } t \in [0, \alpha_n]. \end{cases}$$

Set

$$K_n := \left\lceil \frac{1}{100\alpha_n} \right\rceil. \tag{9.28}$$

Take a polynomial T_{K_n} of degree at most K_n. Since $\omega_n(t) \ge \omega(t)$, $t \in \mathbb{T}$, we have $\|T_{K_n}\omega\|_{C(\mathbb{T})} \le \|T_{K_n}\omega_n\|_{C(\mathbb{T})}$.

On the other hand,

$$\|T_{K_n}\omega_n\|_{C(\mathbb{T})} \le 2\|T_{K_n}\omega\|_{C(\mathbb{T})}. \tag{9.29}$$

Indeed, let $t_0 \in \mathbb{T}$ be a point where $|T_{K_n}\omega_n|$ attains its maximum. If $|t_0| \ge \alpha_n$, then (9.29) is obvious. If $|t_0| < \alpha_n$, then using Remez's inequality and (9.28) we get

$$\begin{aligned} \|T_{K_n}\omega_n\|_{C(\mathbb{T})} &= d_n \|T_{K_n}\|_{C(\mathbb{T})} \le d_n \exp(8\alpha_n K_n) \max_{t\in\mathbb{T}\setminus[-\alpha_n,\alpha_n]} |T_{K_n}(t)| \\ &< 2 \max_{t\in\mathbb{T}\setminus[-\alpha_n,\alpha_n]} |T_{K_n}(t)\omega(t)| \le 2\|T_{K_n}\omega\|_{C(\mathbb{T})}. \end{aligned} \tag{9.30}$$

Note that by definition of ω_n we have

$$|\omega_n'(t)| \le C \max_{1\le k\le n-1} \frac{d_k}{\alpha_k} \le C\frac{d_{n-1}}{\alpha_{n-1}}, \qquad t\in\mathbb{T}. \tag{9.31}$$

Therefore, since $d_n \le |\omega_n(t)|$,

$$|\omega_n'(t)| \le C\frac{d_{n-1}}{d_n\alpha_{n-1}}|\omega_n(t)|, \qquad t\in\mathbb{T}. \tag{9.32}$$

Moreover,

$$|\omega_n''(t)| \le C \max_{1\le k\le n-1} \frac{d_k}{\alpha_k^2} \le C\frac{d_{n-1}}{\alpha_{n-1}^2}, \qquad t\in\mathbb{T}. \tag{9.33}$$

Since $\omega_n \in C^\infty(\mathbb{T})$, by Jackson's theorem there exists a trigonometric polynomial Q_{K_n} of degree at most K_n such that

$$\|\omega_n - Q_{K_n}\|_{C(\mathbb{T})} \le C\frac{\|\omega_n'\|_{C(\mathbb{T})}}{K_n} \quad \text{and} \quad \|\omega_n' - Q_{K_n}'\|_{C(\mathbb{T})} \le C\frac{\|\omega_n''\|_{C(\mathbb{T})}}{K_n}.$$

By [DL, Theorem 2.7, p. 207], Q_{K_n} can be taken as the best approximant of ω_n in $C(\mathbb{T})$.

Thus, (9.31) and (9.33) yield

$$\|\omega_n - Q_{K_n}\|_{C(\mathbb{T})} \le C\frac{d_{n-1}\alpha_n}{\alpha_{n-1}} \le \frac{d_n}{2}, \tag{9.34}$$

$$\|\omega_n' - Q_{K_n}'\|_{C(\mathbb{T})} \le C\frac{d_{n-1}\alpha_n}{\alpha_{n-1}^2} \le K_n d_n, \tag{9.35}$$

for n large enough. Now by (9.34) we get

$$\begin{aligned}\|T_{K_n}'\omega\|_{C(\mathbb{T})} &\le \|T_{K_n}'\omega_n\|_{C(\mathbb{T})} \le \|T_{K_n}'Q_{K_n}\|_{C(\mathbb{T})} + \|T_{K_n}'\|_{C(\mathbb{T})}\|\omega_n - Q_{K_n}\|_{C(\mathbb{T})}\\ &\le \|T_{K_n}'Q_{K_n}\|_{C(\mathbb{T})} + \tfrac12 d_n\|T_{K_n}'\|_{C(\mathbb{T})} \le \|T_{K_n}'Q_{K_n}\|_{C(\mathbb{T})} + \tfrac12\|T_{K_n}'\omega_n\|_{C(\mathbb{T})}.\end{aligned}$$

Therefore,

$$\|T_{K_n}'\omega\|_{C(\mathbb{T})} \le \|T_{K_n}'\omega_n\|_{C(\mathbb{T})} \le 2\|T_{K_n}'Q_n\|_{C(\mathbb{T})}.$$

Similarly applying the inequality

$$\|T_{K_n}'\omega_n\|_{C(\mathbb{T})} \ge \|T_{K_n}'Q_{K_n}\|_{C(\mathbb{T})} - \|T_{K_n}'\|_{C(\mathbb{T})}\|\omega_n - Q_{K_n}\|_{C(\mathbb{T})},$$

we get

$$\|T_{K_n}Q_{K_n}\|_{C(\mathbb{T})} \le 2\|T_{K_n}\omega_n\|_{C(\mathbb{T})}. \tag{9.36}$$

Thus,

$$\|T_{K_n}'\omega\|_{C(\mathbb{T})} \le 2\|T_{K_n}'Q_{K_n}\|_{C(\mathbb{T})} \le 2\|(T_{K_n}Q_{K_n})'\|_{C(\mathbb{T})} + 2\|T_{K_n}Q_{K_n}'\|_{C(\mathbb{T})} =: I_1 + I_2.$$

By Bernstein's inequality for polynomials and (9.34) we have

$$I_1 \le CK_n\|T_{K_n}Q_{K_n}\|_{C(\mathbb{T})} \le 4CK_n\|T_{K_n}\omega\|_{C(\mathbb{T})}.$$

Regarding I_2, we first note that

$$I_2 \le 2\|T_{K_n}\omega_n'\|_{C(\mathbb{T})} + 2\|T_{K_n}\|_{C(\mathbb{T})}\|\omega_n' - Q_{K_n}'\|_{C(\mathbb{T})} =: I_{21} + I_{22}.$$

By (9.32) and (9.30) we get

$$I_{21} \le C\frac{d_{n-1}}{d_n\alpha_{n-1}}\|T_{K_n}\omega_n\|_{C(\mathbb{T})} < CK_n\|T_{K_n}\omega\|_{C(\mathbb{T})}.$$

Moreover, (9.35) and (9.30) imply

$$I_{22} \le 2K_n d_n\|T_{K_n}\|_{C(\mathbb{T})} \le 2K_n\|T_{K_n}\omega_n\|_{C(\mathbb{T})} \le 4K_n\|T_{K_n}\omega\|_{C(\mathbb{T})}$$

for n large enough. Hence, for any $n \in \mathbb{N}$,

$$\|T_{K_n}'\omega\|_{C(\mathbb{T})} \le CK_n\|T_{K_n}\omega\|_{C(\mathbb{T})}. \qquad \square$$

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