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Strictly convergent analytic structures

Abstract. We give conclusive answers to some questions about definability in analytic languages that arose shortly after the work by Denef and van den Dries [DD] on *p*-adic subanalytic sets, and we continue the study of non-archimedean fields with analytic structure of [LR3], [CLR1] and [CL]. We show that the language $\mathcal{L}_K$ consisting of the language of valued fields together with all strictly convergent power series over a complete, rank one valued field K can be expanded, in a definitional way, to a larger language corresponding to an analytic structure (with separated power series) from [CL], hence inheriting all properties from loc. cit., including geometric properties for the definable sets like certain forms of quantifier elimination. Our expansion comes from adding specific, existentially definable functions, which are solutions of certain henselian systems of equations. Moreover, we show that, even when K is algebraically closed, one does *not* have quantifier elimination in $\mathcal{L}_K$ itself, and hence passing to expansions is unavoidable in general. We pursue this study in the wider generality of extending non-separated power series rings to separated ones, and give new examples, in particular of the analytic structure over $\mathbb{Z}[[t]]$ that can be interpreted and understood now in all complete valued fields. In a separate direction, we show in rather large generality that Weierstrass Preparation implies Weierstrass Division.

Keywords. Henselian valued fields, Tate algebras, strictly convergent power series, subanalytic sets, quantifier elimination, analytic structure, separated power series, non-archimedean geometry, Weierstrass Preparation and Division, Artin approximation, Weierstrass systems

1. Introduction

We study a new notion of strictly convergent analytic structure (consisting of strictly convergent power series), linked to the notion of separated analytic structure of [CL] (consisting of power series with two kinds of variables, cf. Remark 1.0.1). We use this study to answer some natural questions that arose shortly after the work by Denef and

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van den Dries on p-adic subanalytic sets [DD]. In a separate direction, we show in a rather large generality that Weierstrass Preparation implies Weierstrass Division, which is the converse of the usual implication.

To explain these questions and their context, we first fix some notation. Let K be a complete, rank one valued field (not necessarily algebraically closed). Let K° be the valuation ring of K with maximal ideal $K^{\circ\circ}$. For $m \geq 0$, set $A_m := K^\circ\langle\xi\rangle = T_m(K)^\circ$, the ring of strictly convergent power series over K° in the variables $\xi = (\xi_1, \ldots, \xi_m)$, where a power series $\sum_{i\in\mathbb{N}^m} a_i \xi^i$ is called *strictly convergent* if its coefficients a_i go to zero as $|i| \to \infty$, where we write ξ^i for $\prod_{j=1}^m \xi_j^{i_j}$. In any complete, rank one valued field $F \supset K$, any power series $f = \sum_i a_i \xi^i \in A_m$ gives rise to a function from $(F^\circ)^m$ to F by evaluating the series. More generally, for L any valued field (not necessarily complete, nor necessarily of rank one), a system $\sigma = (\sigma_m)_{m\geq 0}$ of ring homomorphisms

$$\sigma_m : A_m \to \mathcal{F}((L^\circ)^m, L^\circ),$$

with $\mathcal{F}((L^\circ)^m, L^\circ)$ the ring of L°-valued functions on $(L^\circ)^m$, satisfying

(1) $\sigma_0(K^{\circ\circ}) \subset L^{\circ\circ}$,
(2) $\sigma_m(\xi_i) =$ the i-th coordinate function on $(K^\circ)^m$ for $i = 1, \ldots, m$, and
(3) σ_{m+1} extends σ_m with the natural inclusions $A_m \hookrightarrow A_{m+1}$ and $(L^\circ)^m \hookrightarrow (L^\circ)^{m+1}$: $\xi \mapsto (\xi, 0)$ inducing $\mathcal{F}((L^\circ)^m, L^\circ) \hookrightarrow \mathcal{F}((L^\circ)^{m+1}, L^\circ)$

is called an *analytic $\{A_m\}_m$-structure* on L.

Let $\mathcal{L}_K$ be the valued field language (including the field inverse extended by zero on zero and the division symbol | satisfied by a pair (x, y) if and only if y/x lies in the valuation ring), together with a function symbol for each series in $\bigcup_m A_m$. For any valued field L that is equipped with an analytic $\{A_m\}_{m\in\mathbb{N}}$-structure $(\sigma_m)_m$, the field L can be turned into an $\mathcal{L}_K$-model with the valued field structure for the valued field language and, for each function symbol f corresponding to a series f_0 in A_m, the interpretation of f is given by $\sigma_m(f_0)$ on $(L^\circ)^m$ extended by zero to a function on L^m. An important part of this paper is the study of $\mathcal{L}_K$-definable sets in henselian valued fields with analytic $\{A_m\}_{m\in\mathbb{N}}$-structure, although this is in fact just one of the examples that we give for our more general concept of strictly convergent analytic structure, introduced in Section 3.6. We have chosen in much of this paper to put the full field inverse in the language, instead of restricted division, as in some previous papers. The two approaches are equivalent (see Remark 3.2.12).

In several contexts (for example on p-adic fields), the definable sets are already well understood, but the general case has remained rather mysterious for a long time. In [DD], Denef and van den Dries obtained a natural quantifier elimination result for $\mathcal{L}_{\mathbb{Q}_p}$-definable subsets of $\mathbb{Q}_p^n$, based on the semialgebraic quantifier elimination result for semialgebraic sets by A. Macintyre. Precisely, joining the language of Macintyre with the language $\mathcal{L}_{\mathbb{Q}_p}$ one obtains quantifier elimination for $\mathbb{Q}_p$. The existence of quantifier elimination in the language of valued fields for algebraically closed valued fields (of any characteristic) led to the question of whether a similar transition could be made from the semialgebraic sets to the subanalytic ones, that is, for $\mathcal{L}_K$-definable subsets of K^n when the complete, rank

one valued field K is algebraically closed. The problem is subtle. An elaborate attempt to prove quantifier elimination in the language $\mathcal{L}_K$ via a "flattening theorem" ultimately failed—see [LR5] for an account of this history and for a counterexample to that strategy. In Section 4 of this paper we show, among other things, that such a direct analogue to the p-adic case is false: a complete, rank one, algebraically closed valued field K does *not* have quantifier elimination in the language $\mathcal{L}_K$. Some additional existentially definable separated functions are needed for quantifier elimination.

Recall that, by [LR2], there is a definitional expansion $\mathcal{L}_K^{[\text{LR2}]}$ of $\mathcal{L}_K$ such that K_{alg} has quantifier elimination in $\mathcal{L}_K^{[\text{LR2}]}$. The expansion $\mathcal{L}_K^{[\text{LR2}]}$ is obtained from $\mathcal{L}_K$ by adding function symbols for certain functions, $\mathcal{L}_K$-existentially definable on K_{alg} that give the "Weierstrass data" for regular power series. In this paper we extend this positive result of [LR2] in several directions.

Generally, for not necessarily algebraically closed K and a henselian valued field L with analytic $\{A_m\}_m$-structure as above, we can still control the geometry of the $\mathcal{L}_K$-definable sets, in the sense that we give a definitional expansion $\mathcal{L}'_K$ of $\mathcal{L}_K$ that corresponds to a separated analytic structure as defined in [CL]. In particular, if L is assumed to have characteristic zero (and arbitrary residual characteristic), then in a Denef–Pas style expansion $\mathcal{L}''_K$ of $\mathcal{L}'_K$ (or just an expansion with RV_N-sorts, see below), as in [CL], one still gets elimination of valued field quantifiers in the language $\mathcal{L}''_K$. The expansion $\mathcal{L}'_K$ is obtained from $\mathcal{L}_K$ by adding existentially definable functions, which are solutions of certain henselian systems of equations. The proof that Weierstrass Preparation and Division hold is reminiscent of the proof of Artin Approximation.

The simpler nature of the p-adic situation is not unique. Indeed, the case that K and L are discretely valued, complete and of characteristic zero, and when σ_0 maps a uniformizer of K° to a uniformizer of L° is already well understood and is treated by [CLR1, Theorem 4.2] and [CL, Example 4.4(1)]. The general study of $\mathcal{L}_K$-definable sets is more subtle and is captured by Theorem 3.4.1.

Another direction in which we extend the results from [LR2] is by starting from rings other than the A_m above, for example, instead of A_m we can take the ring B_m of strictly convergent power series over $\mathbb{Z}[[t]]$ in the variables $\xi = (\xi_1, \ldots, \xi_m)$, where a power series is called strictly convergent if its coefficients t-adically go to zero; an analytic $\{B_m\}$-structure $(\sigma_m)_m$ on a valued field L is axiomatized as above, where (1) now reads $\sigma_0(t\mathbb{Z}[[t]]) \subset L^{\circ\circ}$. Also such analytic $\{B_m\}$-structures are related to the ones of [CL] by definitional expansions (see Section 3.5).

More generally, we define a concept of *strictly convergent analytic structure* in Section 3.6, which is a general concept for systems of (non-separated) power series that can be extended, by adding certain power series, to a separated analytic Weierstrass system as in [CL], and the above systems $\{A_m\}$ and $\{B_m\}$ turn out to be examples by Theorems 3.4.1 and 3.5.1. The (separated) power series that one adds to the non-separated ones are solutions to certain systems of henselian analytic equations, and correspond naturally to functions whose graph is existentially definable (see Section 3.2).

Remark 1.0.1 (Separated versus non-separated power series). In the present context of rings of power series and the analytic interpretations they give on henselian valued fields,

often there are one or two different kinds of variables, which play different roles. When two different kinds of variables occur, we speak of separated power series, usually denoting one kind of variables by ξ_i and the other kind by ρ_j with indices i, j. When only one kind of variables occurs, one may speak of non-separated power series, or of strictly convergent power series. For a separated power series f in ξ and ρ, in an interpretation in a henselian valued field L, the variables ξ will run over L°, while the variables ρ will run over $L^{\circ\circ}$ (namely, the ξ_i over the valuation ring and the ρ_j over the maximal ideal). In this paper, the starting point will be power series in one kind of variables, ξ-variables, and, gradually, separated power series will appear, in ξ and ρ. Indeed, the solutions of certain systems of equations may have power series solutions whose natural domain of convergence will be products of the valuation rings with the maximal ideals of the valued fields under consideration (see Section 3.2). Finally, note that "strictly convergent" in this paper often means non-separated, and does not designate convergence in the classical sense.

Previous attempts to understand $\mathcal{L}_K$-definable sets include a quantifier elimination for algebraically closed valued fields in a (much) smaller language including only overconvergent power series by Schoutens [S1]–[S3], with a correction by F. Martin [Mar1], [Mar2] and in a much larger language by the second author. This larger language of [LL1] consisted of rings of separated power series. The study of definable sets was continued in e.g. [LL2], [LR1], [LR2], [Ce1]–[Ce5], [BMS], [Ri]. Dimension theory for sets definable in the larger (separated) language was developed in [LR6] and a more complete dimension theory has been developed in Martin's thesis [Mar1]. A desire for uniformity and for analytic Ax–Kochen principles as in [vdD] (to change the residue field characteristic) led to the notion of a henselian field with analytic $\mathcal{A}$-structure, developed in [vdD], [DHM], [LR3], [CLR1], [CL]. In [CL] we gave a general theory of henselian valued fields with separated analytic structure, including a cell decomposition for definable sets and elimination of valued field quantifiers for such fields of characteristic zero. Those results in particular apply to the separated analytic structures discussed in this paper.

The development in [CL] was carried out in a quite general context—the ring A over which the Weierstrass system was defined was allowed to be quite general. The most subtle part of the development was in connection with the *strong noetherian property* (SNP). We showed in [CL] that an SNP strong enough for our purposes followed from a very weak form of the SNP by Weierstrass Division. Indeed, the weak form is easily true in the basic examples. This development put Weierstrass Division (or Weierstrass Preparation—see Section 2 below) at the center of the theory. In the Appendix to this paper we further elucidate the SNP and discuss the relationship among various forms of the SNP.

In [CL] we also gave a cursory treatment of fields with *strictly convergent* analytic structure. We considered only two cases: (1) the field is discretely valued and there is a designated uniformizing parameter, in which case no "separated" power series are needed; and (2) the strictly convergent analytic structure is the strictly convergent part of a separated analytic structure. (See Remark A.1.13 for more details.) The natural question of when a strictly convergent analytic structure on a henselian field could be extended to a separated analytic structure in a canonical way by adjoining only existentially definable

functions was left unaddressed. This is the question we deal with in Section 3. As mentioned above, the results of Section 4 show that such an expansion is necessary in order to obtain quantifier elimination results.

We begin our paper by showing that Weierstrass Preparation implies Weierstrass Division in a rather large generality. This is the converse of the generally known implication.

2. Weierstrass Preparation implies Weierstrass Division

In many contexts, it is well-known that the Weierstrass Division Theorem implies the Weierstrass Preparation Theorem. In this section, we show that the converse implication holds too, under very mild conditions. Perhaps this observation has been made elsewhere, but we have not seen it before.

2.1. The strictly convergent case

Let A be a commutative ring and let $I \subset A$ be an ideal with $I \neq A$.

We consider a family $\{A_m\}$ of rings for $m \geq 0$ with $A_0 = A$, and for $1 \leq m \leq m'$,

$$A[\xi_1, \ldots, \xi_m] \subset A_m \subset A_{m'} \subset A[[\xi_1, \ldots, \xi_{m'}]].$$

For $f = \sum a_\mu \xi^\mu \in A_m$, let $\widetilde{f} := \sum \widetilde{a}_\mu \xi^\mu \in \widetilde{A}_m$, where $\sim : A \to A/I$ is the residue map, and $\widetilde{A}_m := \{\widetilde{f} : f \in A_m\} \subset \widetilde{A}[[\xi]]$. We assume the rings A_m are closed under permutation of the variables and satisfy

- (∗) For all m and m' with $0 \leq m \leq m'$ and with $\xi = (\xi_1, \ldots, \xi_m)$ and $\xi'' = (\xi_{m+1}, \ldots, \xi_{m'})$, and for all f in $A_{m'}$, say $f = \sum_{\mu \in \mathbb{N}^m} \overline{f}_\mu(\xi)(\xi'')^\mu$, the $\overline{f}_\mu$ are in A_m.
- (∗∗) $\widetilde{A}_m = \widetilde{A}[\xi]$.

The notion of "regularity" in a given degree $d \geq 0$ is the following.

Definition 2.1.1 (Strictly convergent regular). Let $d \geq 0$ and $m \geq 0$ be integers. A power series $f \in A_m$, say $f = \sum_{i \in \mathbb{N}^m} a_i \xi^i$, is called *regular in ξ_m of degree d* when the power series $\widetilde{f} \in \widetilde{A}[\xi]$ is a monic polynomial in ξ_m of degree d in $\widetilde{A}_{m-1}[\xi_m]$.

Proposition 2.1.2. *If the family of rings $\{A_m\}$ satisfies the Weierstrass Preparation Theorem, then the family also satisfies the corresponding Weierstrass Division Theorem. More precisely, suppose that:*

(WP) *For all $d, m \geq 0$ and all $f \in A_m$ regular in ξ_m of degree d (Definition 2.1.1), there exist unique unit $u \in A_m$ and $r_i \in A_{m-1}$ such that $f = u \cdot (\xi_m^d + \sum_{i=0}^{d-1} r_i \xi_m^i)$.*

Then:

(WD) *For all $d, m \geq 0$, all $f \in A_m$ regular in ξ_m of degree d, and all $g \in A_m$, there exist unique $q \in A_m$ and $r_1, \ldots, r_{d-1} \in A_{m-1}$ such that*

$$g = q \cdot f + r_{d-1}\xi_m^{d-1} + \cdots + r_0.$$

Proof. Below. □

Note that since the rings A_m are closed under permutation of the variables, from Weierstrass Preparation for f regular in ξ_m, Weierstrass Preparation for f regular in other ξ_i follows.

2.2. The separated case

Let B be a commutative ring and let $I \subset B$ be an ideal with $I \neq B$. We consider a family $\{B_{m,n}\}$ of rings for $m, n \geq 0$ with $B_{0,0} = B$, and for $m \leq m'$ and $n \leq n'$,

$$B[\xi_1, \ldots, \xi_m, \rho_1, \ldots, \rho_n] \subset B_{m,n} \subset B_{m',n'} \subset B[[\xi_1, \ldots, \xi_{m'}, \rho_1, \ldots, \rho_{n'}]].$$

We define the residue map $\sim$ modulo I as above. In particular, $\widetilde{B} = B/I$. We assume that the rings $B_{m,n}$ are closed under permutation of the ξ_i and under permutation of the ρ_j, and satisfy the following very mild conditions:

(∗) For all m and m' with $0 \leq m \leq m'$ and with $\xi = (\xi_1, \ldots, \xi_m)$ and $\xi'' = (\xi_{m+1}, \ldots, \xi_{m'})$, and all n and n' with $0 \leq n \leq n'$ and with $\rho = (\rho_1, \ldots, \rho_n)$ and $\rho'' = (\rho_{n+1}, \ldots, \rho_{n'})$, and for all f in $B_{m',n'}$, say

$$f = \sum_{\mu, \nu \in \mathbb{N}^{m'-m+n'-n}} \overline{f}_{\mu\nu}(\xi, \rho)(\xi'')^\mu (\rho'')^\nu,$$

the $\overline{f}_{\mu,\nu}$ are in $B_{m,n}$.

(∗∗) $\widetilde{B}[\xi, \rho] \subset \widetilde{B}_{m,n} \subset \widetilde{B}[\xi][[\rho]]$.

Definition 2.2.1 (Separated regular). (i) f is called *regular in ξ_m of degree d* when in $\widetilde{B}[\xi]$, $\widetilde{f} \bmod (\rho_1, \ldots, \rho_n)$ is a monic polynomial in ξ_m of degree d.

(ii) f is called *regular in ρ_n of degree d* when, in $\widetilde{B}[\xi][[\rho]]$,

$$\widetilde{f} \equiv \rho_n^d \bmod (\rho_1, \ldots, \rho_{n-1}, \rho_n^{d+1}).$$

Proposition 2.2.2. *If the family $\{B_{m,n}\}$ of rings satisfies the Weierstrass Preparation Theorems in both kinds of variables, then the family also satisfies the corresponding Weierstrass Division Theorems in both kinds of variables. More precisely, suppose that:*

(WP) *For all non-negative integers m, n, d, and all $f \in B_{m,n}$ which are regular of degree d in ξ_m, resp. in ρ_n, there exist unique unit $u \in B_{m,n}$ and $r_i \in B_{m-1,n}$ such that $f = u \cdot (\xi_m^d + \sum_{i=0}^{d-1} r_i \xi_m^i)$, resp. there exist unique unit $u \in B_{m,n}$ and $r_i \in B_{m,n-1}$ such that $f = u \cdot (\rho_n^d + \sum_{i=0}^{d-1} r_i \rho_n^i)$.*

Then:

(WD) *For all non-negative integers m, n, d, all $f \in B_{m,n}$ which are regular of degree d in ξ_m, resp. in ρ_n, and for all $g \in B_{m,n}$, there are unique $q \in B_{m,n}$ and $r_1, \ldots, r_{d-1} \in B_{m-1,n}$ such that*

$$g = q \cdot f + r_{d-1}\xi_m^{d-1} + \cdots + r_1\xi_m + r_0,$$

resp. there are unique $q \in B_{m,n}$ and $r_1, \ldots, r_{d-1} \in B_{m,n-1}$ such that

$$g = q \cdot f + r_{d-1}\rho_n^{d-1} + \cdots + r_1\rho_n + r_0.$$

Note that since the rings $A_{m,n}$ are closed under permutation of the ξ variables and the ρ variables, from Weierstrass Preparation for f regular in ξ_m or ρ_n, Weierstrass Preparation for f regular in the other ξ_i or ρ_j follows.

The polynomial case of A a field (and $I = \{0\}$) is included in Definition 2.1.1. Also included in Definition 2.1.1 and Proposition 2.1.2 are the rings of strictly convergent and overconvergent power series. Included in Definition 2.2.1 and Proposition 2.2.2 are the rings of separated power series. Also included in this case (with $m = 0$) are the classical cases of formal power series over a field or complete valuation ring, algebraic power series over a field or a complete valuation ring, or germs of convergent power series over for example $\mathbb{R}$ or $\mathbb{C}$. (Note that the proof of Proposition 2.1.2 in the case $m = 0$ stays valid—in that case—i.e. Weierstrass Division in $B_{0,n}$ follows from Weierstrass Preparation in $B_{0,n+1}$). In each of these examples it is well-known that Weierstrass Preparation is an easy consequence of Weierstrass Division, i.e. if the rings satisfy Weierstrass Division they also satisfy Weierstrass Preparation. The above propositions treat the converse implication.

Proof of Proposition 2.2.2. Let $f(\xi, \rho) \in B_{m,n}$ be regular in ξ_m of degree s. By Weierstrass Preparation we may write

$$f(\xi, \rho) = U(\xi, \rho) \cdot [\xi_m^s + a_{s-1}(\xi', \rho)\xi_m^{s-1} + \cdots + a_0(\xi', \rho)]$$
$$= U(\xi, \rho) \cdot P(\xi, \rho),$$

say, where $\xi' = (\xi_1, \ldots, \xi_{m-1})$ and $a_i \in B_{m-1,n}$ for all i. Let $g \in B_{m,n}$. Then

$$P(\xi, \rho) + \rho_{n+1}g(\xi, \rho) \in B_{m,n+1}$$

is also regular in ξ_m of degree s. Hence, by Weierstrass Preparation (in $B_{m,n+1}$),

$$P(\xi, \rho) + \rho_{n+1}g(\xi, \rho) = \mathcal{U}(\xi, \rho, \rho_{n+1}) \cdot \mathcal{P}(\xi, \rho, \rho_{n+1})$$

where $\mathcal{P}(\xi, \rho, \rho_{n+1})$ is a (Weierstrass) polynomial of degree s in ξ_m. Let

$$\mathcal{P}(\xi, \rho, \rho_{n+1}) = R_0(\xi, \rho) + \rho_{n+1}R_1(\xi, \rho) + \rho_{n+1}^2 R_2(\xi, \rho) + \cdots.$$

Then $R_0(\xi, \rho) = P(\xi, \rho)$, $\mathcal{U}(\xi, \rho, 0) = 1$ and each $R_i(\xi, \rho)$ is a polynomial in ξ_m of degree $\leq s$. We can rewrite this equation as

$$\mathcal{U}(\xi, \rho, \rho_{n+1})\mathcal{P}(\xi, \rho, \rho_{n+1})$$
$$= P(\xi, \rho) + \rho_{n+1}\left[\left.\frac{\partial \mathcal{U}}{\partial \rho_{n+1}}\right|_{\rho_{n+1}=0} P(\xi, \rho) + R_1(\xi, \rho)\right] + \rho_{n+1}^2[\cdots] + \cdots.$$

Hence, equating the coefficients of ρ_{n+1}, we see that

$$g(\xi, \rho) = Q(\xi, \rho) \cdot P(\xi, \rho) + R_1(\xi, \rho),$$

where $Q(\xi, \rho) = \left.\frac{\partial \mathcal{U}}{\partial \rho_{n+1}}\right|_{\rho_{n+1}=0}$. Hence we have obtained the Weierstrass data for the division of g by P from that corresponding to Weierstrass Preparation for $P + \rho_{n+1}g$.

Then $g(\xi, \rho) = Q(\xi, \rho)U(\xi, \rho)^{-1} \cdot U(\xi, \rho)P(\xi, \rho) + R_1(\xi, \rho)$, and since $f = UP$, this gives the Weierstrass data for the division of g by f.

The case that f is regular in ρ_n is similar.   □

Proof of Proposition 2.1.2. In the case of Proposition 2.1.2 where we only have one sort of variable we can proceed as follows. In the special case that A is the valuation ring of a non-trivially valued field instead of $P(\xi, \rho) + \rho_{n+1}g(\xi, \rho)$ we consider $P(\xi) + \varepsilon\xi_{m+1}g(\xi)$ for some $\varepsilon \in A$ with $0 < |\varepsilon| < 1$. This series is regular in the appropriate sense in ξ_m, and we can proceed exactly as in the proof of Proposition 2.2.2. In the general case we can do some *Euclidean* division of g by f to write $g = q' \cdot f + g'$ for q' a polynomial in ξ_m and g' such that $\widetilde{g'}$ is a polynomial in ξ of degree $< s$. Then we need only find the Weierstrass data of the division of g' by f, and we can do that as in the proof of Proposition 2.2.2 considering $f + \xi_{m+1}g'$ which is regular in ξ_m of degree s.   □

3. Strictly convergent Weierstrass systems and their properties

In this section we show how to extend a strictly convergent pre-Weierstrass system to a separated pre-Weierstrass system in a uniformly, existentially definable way, by adjoining certain "henselian" power series. Under mild additional conditions this extension leads to a separated Weierstrass system. In particular, that is the case for the basic examples (Theorems 3.4.1 and 3.5.1). In Subsection 3.6 we discuss the general case. The results generalize and extend the partial results about strictly convergent Weierstrass systems given in [CL] (see Remark A.1.13).

3.1. Definitions

By a valued field we mean a field together with a valuation map to a (possibly trivial) ordered abelian group (not necessarily of rank one) satisfying the usual non-archimedean properties. We denote the valuation ring of a valued field F by F°, the maximal ideal of F° by $F^{\circ\circ}$ and the residue field $F^\circ/F^{\circ\circ}$ by $\widetilde{F}$. By a *henselian field* we mean a valued field for which Hensel's Lemma holds on F°. For a henselian field F we let F_{alg} denote the algebraic closure of F; the valuation on F has a unique extension to a valuation on F_{alg}. Unless otherwise stated we write $\xi = (\xi_1, \ldots, \xi_m)$ when m is clear from the context.

Let A be a commutative ring with unit, and let I be an ideal of A with $I \neq A$. Let $\widetilde{A} := A/I$ and write $\sim: A \to \widetilde{A}$ for the projection map. We also write $\sim$ for the residue map $A[[\xi]] \to \widetilde{A}[[\xi]]$ which sends $\sum_\mu a_\mu \xi^\mu$ to $\sum_\mu \widetilde{a}_\mu \xi^\mu$, and $(A_m)^\sim$ for the image of A_m under this map when $A_m \subset A[[\xi]]$.

Definition 3.1.1 (Strictly convergent regular). A power series $f \in A[[\xi]]$ is called *regular in ξ_m of degree d* when $\widetilde{f} \in \widetilde{A}[[\xi]]$ is a monic polynomial in ξ_m of degree d.

Definition 3.1.2 (System). A collection $\mathcal{A} = \{A_m\}_{m \in \mathbb{N}}$ of A-algebras A_m satisfying, for all $m \geq 0$:

(i) $A_0 = A$,

(ii) A_m is a subalgebra of $A[[\xi_1, \ldots, \xi_m]]$ which is closed under permutation of the variables,

(iii) $A_m[\xi_{m+1}] \subset A_{m+1}$,

(iv) $(A_m)^\sim$ is the polynomial ring $\widetilde{A}[\xi_1, \ldots, \xi_m]$,

is called a *system*, or an (A, I)-*system*, if we want to emphasize the dependence on A and I.

Definition 3.1.3 (Analytic structure). Let $\mathcal{A} = \{A_m\}$ be a system as in Definition 3.1.2, and let K be a valued field. A *strictly convergent analytic $\mathcal{A}$-structure on K* is a collection $\sigma = \{\sigma_m\}_{m \in \mathbb{N}}$ such that, for each $m \geq 0$, σ_m is a homomorphism from A_m to the ring of K°-valued functions on $(K^\circ)^m$ satisfying:

(1) $I \subset \sigma_0^{-1}(K^{\circ\circ})$,

(2) $\sigma_m(\xi_i) = $ the i-th coordinate function on $(K^\circ)^m$, $i = 1, \ldots, m$, and

(3) σ_{m+1} extends σ_m where we identify in the obvious way functions on $(K^\circ)^m$ with functions on $(K^\circ)^{m+1}$ that do not depend on the last coordinate.

Definition 3.1.4 (Pre-Weierstrass system). A system $\mathcal{A} = \{A_m\}_{m \in \mathbb{N}}$ of A-algebras A_m satisfying, for all $m \leq m'$:

(v) if $f \in A_{m'}$, say $f = \sum_\mu \overline{f}_\mu(\xi)(\xi'')^\mu$, then the $\overline{f}_\mu$ are in A_m, where $\xi'' = (\xi_{m+1}, \ldots, \xi_{m'})$ and $\xi = (\xi_1, \ldots, \xi_m)$,

(vi) (Weierstrass Preparation) if $f \in A_m$ is regular in ξ_m of degree d, then there exist unique unit $u \in A_m$ and $r_i \in A_{m-1}$ such that $f = u \cdot (\xi_m^d + \sum_{i=0}^{d-1} r_i \xi_m^i)$,

(viiWNP) (Weak noetherian property) if $f = \sum_\mu a_\mu \xi^\mu \in A_m$ there is a finite set $J \subset \mathbb{N}^m$, and for each $\mu \in J$ a power series $g_\mu \in A_m$ with $\widetilde{g}_\mu = 0$ such that

$$f = \sum_{\mu \in J} a_\mu \xi^\mu (1 + g_\mu),$$

(viii) the ideal $\{a \in A : \sigma_0(a) = 0 \text{ for all analytic structures } \sigma = (\sigma_m)\}$ equals the zero ideal,

is called a *strictly convergent pre-Weierstrass system*, or an (A, I)-strictly convergent pre-Weierstrass system, if we want to emphasize the dependence on A and I.

Let us fix, until Subsection 3.4, a strictly convergent pre-Weierstrass system $\{A_m\}_m$.

Later on, a stronger form of (viiWNP) will play a role.

Remark 3.1.5. (a) By the results of Section 2, from (vi) we also see that Weierstrass Division holds in the A_m:

If $f \in A_m$ is regular in ξ_m of degree d, and $g \in A_m$, there are unique $q \in A_m$ and $r_1, \ldots, r_{d-1} \in A_{m-1}$ such that $g = q \cdot f + r_{d-1}\xi_m^{d-1} + \cdots + r_0$. Furthermore, if $\widetilde{g} = 0$ then $\widetilde{q} = \widetilde{r}_0 = \cdots = \widetilde{r}_{d-1} = 0$.

(b) It follows from Weierstrass Preparation that if $f \in A_m$ satisfies $\widetilde{f} = 1$ (i.e. f is regular of degree 0) then f is a unit.

(c) From (viiWNP), it follows for $f \in A_m$ with $\widetilde{f} = 0$ that $f \in I \cdot A_m$. The above conditions (i)–(vi) do not impose much structure on "small" power series (e.g. those in $I \cdot A_m$),

as they do not provide a mechanism for dividing by a biggest coefficient and bringing such a power series to the "top" level where conditions (iv) and (vi) apply. Condition (viiWNP) (and even more so, Definition A.1.6 below) imposes structure on "small" power series in the A_m and provides such a mechanism.

Lemma-Definition 3.1.6. *Let $\mathcal{A}$ be a pre-Weierstrass system and let K be a henselian field with analytic $\mathcal{A}$-structure σ. Let $m \geq 0$ be an integer. We associate with $f \in A_m$ the power series*

$$\mathcal{S}^\sigma(f) := \sum_\mu \sigma(a_\mu)\xi^\mu \quad in \ K^\circ[[\xi]].$$

Then the map $\mathcal{S}$ is an A-algebra homomorphism. Moreover, if $\mathcal{S}^\sigma(f) = 0$ then $f^\sigma = 0$. Hence, the homomorphism σ_m factors through A_m^σ, the image of $\mathcal{S}$. On A_m^σ we can define the Gauss norm as follows, with $f = \sum_\mu a_\mu \xi^\mu$ and where J is as in (viiWNP)*:*

$$\|\mathcal{S}^\sigma(f)\| := \max\{|\sigma(a_\mu)| : \mu \in J\}.$$

Proof. That $\mathcal{S}^\sigma(f) = 0$ implies $f^\sigma = 0$ follows immediately from condition (viiWNP). The rest is clear. □

Remark 3.1.7. (a) If the valuation on K is non-trivial, then one also has the converse implication that $f^\sigma = 0$ implies $\mathcal{S}^\sigma(f) = 0$. This also follows from condition (viiWNP).

(b) Weierstrass Division for the rings A_m implies that composition of tuples of power series in A_m corresponds again to power series in A_m, as explained in [CL, Remark 4.5.2].

3.2. Extension by henselian functions

We sometimes wish to consider some of the variables to be of the first kind (namely to indicate that their natural domain is the valuation ring), and some of the second kind (namely to indicate that their intended domain is the maximal ideal)—cf. Remark 1.0.1. We will use $\rho = (\rho_1, \ldots, \rho_n)$ to denote variables of the second kind, and keep $\xi = (\xi_1, \ldots, \xi_m)$ for variables of the first kind. Variables of the second kind typically arise in making a power series regular, or guaranteeing that the existence of a solution to an equation, or system of equations, is implied by Hensel's Lemma.

We will sometimes write $A_{m,n}$ for A_{m+n} to indicate that the last n variables will be considered to be of the second kind and run over the maximal ideal; this change in notation will be convenient to gradually introduce genuinely new rings of separated power series.

This is crucial in our use of the henselian functions h_Σ defined below. These h_Σ are solutions of henselian systems (also defined below), and correspond to unique power series over A, which may be new, that is, not lie in any of the A_m.

For example, the power series $1 + y - \rho_1 \cdot y^2$ is regular of degree 1 in y (Definition 2.2.1(i) or Definition 3.3.2(i)), and the equation $1 + y - \rho_1 \cdot y^2 = 0$ defines $y = y(\rho)$ as a henselian function of (indeed, as a henselian power series in) ρ_1 since ρ_1 is considered to run over the maximal ideal, but the equation $1 + y - \xi_1 \cdot y^2$ only defines y as a henselian function of ξ_1 when $|\xi_1| < 1$, and does not define y as a power series in ξ_1; indeed not

even as a function of ξ_1 when $|\xi_1| = 1$. We will later enrich the A_m by adding power series like $y(\rho)$, and get a richer system, genuinely in the two kinds of variables.

Basic henselian functions are defined in Definition 3.2.10. A more general kind of henselian witnessing functions than the basic one will be defined using systems of equations of the following form.

Definition 3.2.1 (Systems of henselian equations). Let $\eta = (\eta_1, \ldots, \eta_M)$ and $\lambda = (\lambda_1, \ldots, \lambda_N)$. A *system, Σ, of henselian equations for (η, λ) over (ξ, ρ)* is a system of the form

$$\begin{aligned}
0 &= \eta_i - b_{0,i}(\xi, \rho) - b_{1,i}(\xi, \rho, \eta, \lambda), \quad i = 1, \ldots, M, \\
0 &= \lambda_j - c_{0,j}(\xi, \rho) - c_{1,j}(\xi, \rho, \eta, \lambda), \quad j = 1, \ldots, N,
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
b_{0,i} &\in A[\xi, \rho], & b_{1,i} &\in (I, \rho, \lambda)A_{m+M,n+N}, \\
c_{0,j} &\in (\rho)A[\xi, \rho], & c_{1,j} &\in (I, \rho, \lambda^2)A_{m+M,n+N}.
\end{aligned} \tag{3.2}$$

Here λ^2 is shorthand for $\{\lambda_i \lambda_j : i, j = 1, \ldots, N\}$. By the ideal I_Σ of the system Σ is meant the ideal of $A_{m+M,n+N}$ generated by the right hand sides of the equations in Σ. When Σ is a set of *polynomial* equations in $(\xi, \rho, \eta, \lambda)$ we call Σ a *polynomial henselian system*. We will often write a henselian system in the equivalent form:

$$\begin{aligned}
\eta_i &= b_{0,i}(\xi, \rho) + b_{1,i}(\xi, \rho, \eta, \lambda), \quad i = 1, \ldots, M, \\
\lambda_j &= c_{0,j}(\xi, \rho) + c_{1,j}(\xi, \rho, \eta, \lambda), \quad j = 1, \ldots, N.
\end{aligned} \tag{3.3}$$

Definition 3.2.2. Let g be in A_{m+n+s} for some integer s and consider power series h_i in $A_m[[\rho_1, \ldots, \rho_n]]$ for $i = 1, \ldots, s$. Then we call $h = (h_i)_{i=1}^s$ a *power series solution* of the equation $g = 0$ when for each integer $k > 0$ there exists a tuple p_k of polynomials in the variables ρ of degree $< k$ with coefficients in A_m such that $h \equiv p_k \bmod (\rho)^k$ and $g(\xi, \rho, p_k(\xi, \rho))$ vanishes modulo $(\rho)^k$, where the composition is as in Remark 3.1.7(b). We write $g(\xi, \rho, h(\xi, \rho)) = 0$ to denote that h is a power series solution for g.

The next lemma shows that any system Σ of henselian equations has a unique power series solution in $A_m[[\rho]]$.

Lemma 3.2.3. *Let Σ be a system of henselian equations as in Definition 3.2.1. Then Σ has a unique power series solution, namely, there exist unique power series h_i in $A_m[[\rho_1, \ldots, \rho_n]]$ for $i = 1, \ldots, M + N$ such that $g(\xi, \rho, h(\xi, \rho)) = 0$ for each g in I_Σ with $h = (h_i)_i$. We denote the tuple $(h_i)_i$ by h_Σ.*

Proof. We prove this by induction on n, with notation from 3.2.1. For the induction step, assume $n > 0$. We proceed exactly as in the proof of Lemma 3.3.3 below. Let $k > 0$ and write $\eta_i = \sum_{\ell=0}^k \eta_{i\ell} \rho_n^\ell$ and $\lambda_j = \sum_{\ell=0}^k \lambda_{j\ell} \rho_n^\ell$ (where the $\eta_{i\ell}, \lambda_{j\ell}$ are new variables). Substitute this into the system Σ and do Weierstrass Division by ρ_n^k to obtain a henselian system that, by induction, determines the $\eta_{i\ell}$ and $\lambda_{j\ell}$ as power series in ξ and $\rho_1, \ldots, \rho_{n-1}$. See the proof of Lemma 3.3.3 for details. Since k is arbitrary, this determines η and λ as power series. $\blacksquare$

We must still treat the case $n = 0$, m arbitrary. In this case the η_i, λ_j are defined by a system Σ of the form

$$\begin{aligned}
\eta_i &= b_{0,i} + b_{1,i}(\eta, \lambda), \quad i = 1, \ldots, m + M, \\
\lambda_j &= c_{0,j} + c_{1,j}(\eta, \lambda), \quad j = 1, \ldots, N,
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
b_{0,i} &\in A_m, \quad b_{1,i} \in (I, \lambda) A_{m+M,N}, \\
c_{0,j} &\in I A_m, \quad c_{1,j} \in (I, \lambda^2) A_{m+M,N}.
\end{aligned} \tag{3.5}$$

If we allow the $c_{0,j}$ to be in A_{m+M} with $\widetilde{c}_{0,j} = 0$, may assume that the $c_{1,j}$ all have 0 as constant (i.e. λ-free) term, and, after an A_{m+M}-linear change of variables among the λ_j with Jacobian a unit in A_{m+M}, we may assume that the $c_{1,j}$ have no linear terms in λ. Below we will see that the equations

$$\lambda_j = c_{0,j} + c_{1,j}(\eta, \lambda), \quad j = 1, \ldots, N, \tag{3.6}$$

with $c_{0,j} \in A_{m+M}, \widetilde{c}_{0,j} = 0$ and $c_{1,j} \in (\lambda^2) A_{m+M,N}$, have a unique solution with $\lambda_j = \lambda_j(\eta) \in A_{m+M}$ and $\widetilde{\lambda}_j(\eta) = 0$. Substituting into the equations

$$\eta_i = b_{0,i} + b_{1,i}(\eta, \lambda(\eta)) \quad i = 1, \ldots, m + M,$$

yields a system of equations for the η_i, all regular of degree 1, which we can solve successively by (strictly convergent) Weierstrass Preparation and Division, to get the solution with the η_i in A_m. Substituting into the solution for the λ_j determines these as elements of $I \cdot A_m$.

Finally, it remains to show that the equations (3.6) with $c_{0,j} \in A_{m+M}$ satisfying $\widetilde{c}_{0,j} = 0$ and $c_{1,j} \in (\lambda^2) A_{m+M,N}$ have a unique solution in A_{m+M} with $\widetilde{\lambda} = 0$. In (3.6) we make the substitution

$$\lambda_j = \sum_\ell c_{0,\ell} U_{j,\ell}$$

where the $U_{j,\ell}$ are new variables. This yields the equations

$$\sum_\ell c_{0,\ell} U_{j,\ell} = c_{0,j} + \sum_{\alpha,\beta} c_{0,\alpha} c_{0,\beta} U_{0,\alpha} U_{0,\beta} G_{j\alpha\beta}(\eta, U)$$

for $j = 1, \ldots, N$. In place of the first equation (i.e. $j = 1$) consider the system of equations

$$\begin{aligned}
c_{0,1} U_{1,1} &= c_{0,j} + c_{0,j} H_{1,1}(U), \\
c_{0,2} U_{1,2} &= c_{0,2} H_{1,2}(U), \\
&\cdots \\
c_{0,N} U_{1,N} &= c_{0,N} H_{1,N}(U),
\end{aligned} \tag{3.7}$$

where in $c_{0,1}H_{1,1}$ we have collected all the terms of $\sum_{\alpha,\beta} c_{0,\alpha}c_{0,\beta}U_{0,\alpha}U_{0,\beta}G_{j\alpha\beta}(\eta, U)$ that have a factor $c_{0,1}U_{0,1}$, in $c_{0,2}H_{1,2}$ all the remaining terms that have a factor $c_{0,2}U_{0,2}$, and so on. Write down the analogous systems for $j = 2, \ldots, N$. Now consider the system

$$
\begin{aligned}
U_{1,1} &= 1 + H_{1,1}(U), \\
U_{1,2} &= H_{1,2}(U), \\
&\cdots \\
U_{1,N} &= H_{1,N}(U),
\end{aligned}
\tag{3.8}
$$

and the analogous equations for $j = 2, \ldots, N$. In this system all the $H_{\alpha,\beta} \in A_{m+M+N^2}$ satisfy $\widetilde{H}_{\alpha,\beta} = 0$, and hence the system can be solved for the $U_{\alpha,\beta}$ (as elements of A_{m+M}) by successive (strictly convergent) Weierstrass Preparation and Division. This proves the existence of a solution $\overline{\lambda}(\eta) \in A_{m+M}^N$ to (3.6). For uniqueness substitute $\lambda_j = \overline{\lambda}_j + \mu_j, j = 1, \ldots, N$, in (3.6), where the μ_j are new variables. We must show that the resulting equations (in the μ_j), which are of the form

$$
\mu_j = \sum_\alpha d_\alpha \mu_\alpha + \sum_{\alpha,\beta} e_{\alpha,\beta}\mu_\alpha\mu_\beta
\tag{3.9}
$$

where the d_α are in $(\lambda)A_{m+M,N}$ and the $e_{\alpha,\beta}$ are in $A_{m+M,N}$, have only the zero solution. By (viiWNP) and (viii) (this is the only place where we use (viii)) it is sufficient to prove this in $A_{m+M,N}^\sigma$ for each analytic structure σ. But in A_{m+M}^σ we have a Gauss norm by 3.1.6 and it is easy to see that a non-zero solution to these equations is impossible. Indeed, suppose that we had a non-zero solution $\mu(\eta)$. Look at the jth equation where $\mu_j(\eta)$ has biggest Gauss norm. The Gauss norm of the right hand side would be smaller than that of the left hand side. This completes the proof of the lemma. □

Definition 3.2.4. Let h be the power series solution of a system Σ of henselian equations as in Lemma 3.2.3 and let g be any power series in $A_{m+M,n+N}$. Write g as $g_0 + g_1$ where g_0 is in $A[\xi, \rho]$ and g_1 in $(I, \rho, \lambda)A_{m+M,n+N}$. Consider the system Σ' for $(\eta, \eta_{M+1}, \lambda)$ over (ξ, ρ) consisting of Σ and of the additional equation

$$
0 = \eta_{M+1} - g_0 - g_1.
$$

Then the $M + 1$-th entry h_{M+1} of the unique power series solution $h_{\Sigma'} = (h_i)_{i=1}^{M+1+N}$ of Σ' is called the *composition* of g with the power series tuple (ξ, ρ, h_Σ) and is denoted by $g(\xi, \rho, h_\Sigma(\xi, \rho))$.

The proof of Lemma 3.2.3 actually establishes the following version of the implicit function theorem in the power series rings $A_{m,n} = A_{m+n}$.

Lemma 3.2.5. *Let $\mathcal{A}$ be a strictly convergent pre-Weierstrass system.*

(i) *Suppose that $F_1(\xi, \eta), \ldots, F_N(\xi, \eta) \in A_{m+M}$ and that $\overline{y}_1, \ldots, \overline{y}_N \in A_m$ satisfy $F_i(\xi, \overline{y})^\sim = 0$ and $J(\xi, \overline{y})$ is a unit in A_m, where $J = \left|\left(\frac{\partial F_i}{\partial \eta_j}\right)_{i,j=1,\ldots,N}\right|$. Then there is a unique $y \in A_m^N$ such that $F(\xi, y) = 0$ and $\widetilde{y} = \widetilde{\overline{y}}$.*

(ii) *Suppose that $F_1(\xi, \rho, \eta, \lambda), \ldots, F_N(\xi, \rho, \eta, \lambda) \in A_{m+M,n+N}$ and that polynomials $\overline{y}_1, \ldots, \overline{y}_{M+N} \in A_{m,n}$ satisfy $F_i(\xi, \rho, \overline{y})^\sim \equiv 0 \mod (\rho)$ and $J(\xi, \rho, \overline{y})^\sim \equiv 1 \mod (\rho)$, where*

$$J = \left| \left(\frac{\partial F_i}{\partial \eta_j \, \partial \lambda_k} \right)_{\substack{j=1,\ldots,M, \, k=1,\ldots,N \\ i=1,\ldots,M+N}} \right|.$$

Then there is a unique $y \in A_m[[\rho]]^{M+N}$ such that $\widetilde{y} \equiv \widetilde{\overline{y}} \mod (\rho)$ and $F(\xi, y) = 0$, where the latter means that y is a solution in the sense of Lemma 3.2.3. In fact (by Proposition 3.2.7 below) there is a polynomial henselian system Σ' and a power series $G \in A_{m+n+M+N}^{M+N}$ such that $y(\xi, \rho) = G(\xi, \rho, h_{\Sigma'})$.

Definition 3.2.6. By Lemma 3.2.3 and Definition 3.2.4 we can define the following sub-algebras of $A[[\xi, \rho]]$:

$$A_{m,n}^H := \{g(\xi, \rho, h_\Sigma(\xi, \rho)) : g \in A_{m+M,n+N}, \ \Sigma \text{ a henselian system}\},$$

$$A_{m,n}^{H'} := \{g(\xi, \rho, h_\Sigma(\xi, \rho)) : g \in A_{m+M,n+N}, \ \Sigma \text{ a polynomial henselian system}\}.$$

Set $\mathcal{A}^H := \{A_{m,n}^H\}_{m,n}$.

Proposition 3.2.7. *Let $\mathcal{A}$ be a strictly convergent pre-Weierstrass system. Then $A_{m,n}^H = A_{m,n}^{H'}$. Indeed, every element f of $A_{m,n}^H$ is of the form $f(\xi, \rho) = g(\xi, \rho, h_\Sigma(\xi, \rho))$ where $g \in A_{m',n'}$ for some $m', n' \in \mathbb{N}$, and h_Σ is the solution to a **polynomial** henselian system.*

Proof. Let

$$\eta_i = b_{0,i}(\xi, \rho) + b_{1,i}(\xi, \rho, \eta, \lambda), \quad i = 1, \ldots, M,$$

$$\lambda_j = c_{0,j}(\xi, \rho) + c_{1,j}(\xi, \rho, \eta, \lambda), \quad j = 1, \ldots, N,$$

be a henselian system. We will obtain the solution as the composition of functions in $\mathcal{A}$ and the solution to a *polynomial* henselian system. We can find polynomials $\overline{b}_{1,i}, \overline{c}_{1,j}$ such that all the coefficients of $b_{1,i} - \overline{b}_{1,i}, c_{1,j} - \overline{c}_{1,j}$ are in $I \cdot A$, i.e. $(b_{1,i} - \overline{b}_{1,i})^\sim = (c_{1,j} - \overline{c}_{1,j})^\sim = 0$, or equivalently, using condition (viiWNP), $b_{1,i} - \overline{b}_{1,i}, c_{1,j} - \overline{c}_{1,j} \in I \cdot A_{m+M+n+N}$. Let $\overline{y} = (\overline{y}_1, \ldots, \overline{y}_M, \overline{z}_1, \ldots, \overline{z}_N)$ be the solution to the polynomial henselian system

$$\eta_i = b_{0,i}(\xi, \rho) + \overline{b}_{1,i}(\xi, \rho, \eta, \lambda), \quad i = 1, \ldots, M,$$

$$\lambda_j = c_{0,j}(\xi, \rho) + \overline{c}_{1,j}(\xi, \rho, \eta, \lambda), \quad j = 1, \ldots, N,$$

so the $\overline{y}_i, \overline{z}_j$ are terms of $\mathcal{L}_{\mathcal{A}}^{D,H'}$ (see Definition 3.2.11 below). Make the change of variables $\eta = \overline{y} + \zeta$ and $\lambda = \overline{z} + \zeta'$. Proceed exactly as in the last stage of the proof of Lemma 3.2.3 (i.e. use Lemma 3.2.5), writing $\zeta_i = \sum_\alpha \varepsilon_\alpha U_{i,\alpha}$ and $\zeta'_j = \sum_\alpha \varepsilon_\alpha U'_{j,\alpha}$ to obtain a system of equations for the $U_{i,\alpha}, U'_{j,\alpha}$ that can be solved by (strictly convergent) Weierstrass Division. $\qquad \square$

Lemma-Definition 3.2.8 (Henselian functions). *Let σ be an analytic $\mathcal{A}$-structure on a henselian valued field K. For any henselian system Σ and any $\xi \in (K^\circ)^m$ and $\rho \in (K^{\circ\circ})^n$, there exist unique values $\eta \in (K^\circ)^M$ and $\lambda \in (K^{\circ\circ})^N$ such that $f^\sigma(\xi, \rho, \eta, \lambda) = 0$ for all*

$f \in I_\Sigma$; we denote this tuple (η, λ) by $k_\Sigma^\sigma(\xi, \rho)$. Moreover, the graph of the function k_Σ^σ on $(K^\circ)^m \times (K^{\circ\circ})^N$ is quantifier-free definable in $\mathcal{L}_{\mathcal{A}}$ by a formula which is independent of σ.

Proof. For a polynomial system Σ this is clear by the usual form of Hensel's Lemma for systems of polynomial equations with Jacobian of norm 1—see Lemma 3.2.13(i) below. For a general Σ the lemma now follows from the observation that the argument of Proposition 3.2.7 applies also in any field with analytic $\mathcal{A}$-structure, and hence applies also to the k_Σ^σ. □

Remark 3.2.9. For a henselian system Σ we have defined h_Σ as a power series and k_Σ as a (new) function symbol (or term) whose interpretation k_Σ^σ in a field with analytic $\mathcal{A}$-structure is defined in Lemma-Definition 3.2.8. Later we will extend strictly convergent analytic structures σ to separated analytic structures by defining $h_\Sigma^\sigma = k_\Sigma^\sigma$. This will require an additional assumption ("goodness") on $\mathcal{A}$ (Definition 3.3.5).

The following henselian witnessing functions are much more basic than the ones associated to a system of equations in 3.2.8. Nevertheless, as we will see in Lemma 3.2.13(ii), they can in fact replace h_Σ.

Definition 3.2.10 (Basic henselian functions). For K a henselian field, $h_n : K^{n+1} \to K$ is the function that associates to $(a_0, \ldots, a_n, b) \in K^\circ$ the unique zero, c, of the polynomial $p(x) := a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ that satisfies $|c - b| < 1$ if $|p(b)| < 1$ and $|p'(b)| = 1$. (Let h_n output 0 in all other cases.) The graphs of these functions are obviously quantifier-free definable in $\mathcal{L}$.

Definition 3.2.11 (Languages). For L a language containing the language of valued fields $\mathcal{L} := (+, \cdot, ^{-1}, 0, 1, |)$, let L^D be L with the field inverse replaced by the binary function symbol D for restricted division (or two such symbols in the separated case). Further, let L_A be L with constants for the elements of A adjoined, let $L_{\mathcal{A}}$ be L with function symbols for the elements of $\bigcup_m A_m$ adjoined, and let L^h be L with function symbols for the basic henselian functions h_n adjoined. Recall that $|$ is a binary symbol on a valued ring (or valued field) such that $x \mid y$ holds if and only if x is non-zero and the valuation of x is less than or equal to the valuation of y. When we use a language including the field inverse, the intended interpretations are valued fields F, and when we replace the field inverse by restricted division, the intended interpretations are valuation rings F°.

In the next remark we comment on the different kinds of division, and how the results relate.

Remark 3.2.12. In some papers (e.g. [LL2], [LR2], [LR3], and also in Theorems 4.1 and 4.2 below) we have worked in the languages $\mathcal{L}_{\mathcal{A}(K)}^D$ and $\mathcal{L}_{\mathcal{A}_{\text{sep}}(K)}^D$, in which we omit the function for field inverse and include symbols for restricted division, and in others (e.g. [CL]) we have worked in the language $\mathcal{L}_{\mathcal{A}(K)}$ of the Introduction, in which we have a symbol for field inverse. We remark that K_{alg}° has quantifier elimination in the language $\mathcal{L}_{\mathcal{A}(K)}^D$ (or $\mathcal{L}_{\mathcal{A}^H(K)}^D$ or $\mathcal{L}_{\mathcal{A}_{\text{sep}}(K)}^D$) if and only if K_{alg} has quantifier elimination in

the language $\mathcal{L}_{\mathcal{A}(K)}$ (or $\mathcal{L}_{\mathcal{A}^H(K)}$ or $\mathcal{L}_{\mathcal{A}_{\text{sep}}(K)}$). We can cover $(K_{\text{alg}})^n$ by polydiscs of the form $\{x : \bigwedge_i (|x_i| \square_i 1)\}$ where the $\square_i$ are in $\{\leq, \geq\}$, and we can map such a polydisc to $(K^\circ_{\text{alg}})^n$ by $x_i \mapsto x_i$ if $\square_i$ is $\leq$ and $x_i \mapsto (x_i)^{-1}$ if $\square_i$ is $\geq$. For $\mathcal{F}$ any of $\mathcal{A}(K)$, $\mathcal{A}^H(K)$ or $\mathcal{A}_{\text{sep}}(K)$, it is clear that any $\mathcal{L}^D_{\mathcal{F}}$-quantifier-free definable subset of $(K^\circ_{\text{alg}})^n$ is $\mathcal{L}_{\mathcal{F}}$-quantifier-free definable. For the converse we can easily show by induction on the number, k, of occurrences of symbols from $\bigcup A_{m,n}$ in the formula, that an $\mathcal{L}_{\mathcal{F}}$-quantifier-free definable subset of $(K^\circ_{\text{alg}})^n$ is indeed $\mathcal{L}^D_{\mathcal{F}}$-quantifier-free definable. (By the same induction one can show that if an $\mathcal{L}_{\mathcal{F}}$-term on a quantifier-free $\mathcal{L}^D_{\mathcal{F}}$-definable subset X of $(K^\circ_{\text{alg}})^n$ is always ≤ 1 in size, then it is equivalent to an $\mathcal{L}^D_{\mathcal{F}}$-term. Actually it is sufficient to just establish this piecewise on a quantifier-free $\mathcal{L}^D_{\mathcal{F}}$-definable cover of X, but see the proof of Lemma 3.2.13 for combining terms defined piecewise into one term.)

Statement (ii) of the following proposition may seem surprising but comes in part from the fact that piecewise definitions of terms on certain pieces can be combined into a single term.

Lemma 3.2.13. (i) *If K is a henselian field, then K° satisfies the Implicit Function Theorem: Given a system $P_i(y_1, \ldots, y_n) = 0$, $i = 1, \ldots, n$, of n polynomial equations in n unknowns over K°, and $\overline{y} = (\overline{y}_1, \ldots, \overline{y}_n) \in (K^\circ)^n$ such that the $P_i(\overline{y})$ are in $K^{\circ\circ}$ and the Jacobian $|(\frac{\partial P_i}{\partial y_j})_{i,j=1,\ldots,n}|(\overline{y})$ at the point $\overline{y}$ is a unit, there is a unique $\widetilde{y} \in (K^\circ)^n$ with $\overline{y} - \widetilde{y} \in (K^{\circ\circ})^n$ satisfying $P(\widetilde{y}) = 0$.*
(ii) *Let τ be a term of $\mathcal{L}^D_{\mathcal{A}^H}$. There is a term τ' of $\mathcal{L}^{D,h}_{\mathcal{A}}$ such that for every field K with analytic $\mathcal{A}$-structure, say via σ, we have $\tau^\sigma = \tau'^\sigma$.*
(iii) *Let τ be a term of $\mathcal{L}_{\mathcal{A}^H}$. There is a term τ' of $\mathcal{L}^h_{\mathcal{A}}$ such that for every field K with analytic $\mathcal{A}$-structure, say via σ, we have $\tau^\sigma = \tau'^\sigma$.*

Proof. (i) See [Kuh1, Theorem 24] or [Kuh2, Theorem 9.13].

(ii) By Lemma 3.2.7 and Lemma-Definition 3.2.8, we need only prove this for terms k_Σ defined by polynomial henselian systems Σ. Let Σ be given. It follows from 3.2.8 by a standard compactness argument that there are finitely many terms $\tau_i(\xi, \rho)$ of $\mathcal{L}^{D,h}_{\mathcal{A}}$ such that in each henselian field K with analytic $\mathcal{A}$-structure and for any $(\xi, \rho) \in (K^\circ)^m \times (K^{\circ\circ})^n$ at least one of the τ_i satisfies Σ at (ξ, ρ), that is, $\tau_i(\xi, \rho) = k^\sigma_\Sigma(\xi, \rho)$. Let $\varphi_i(\xi, \rho)$ be the quantifier-free formula that says that $\tau_i(\xi, \rho)$ satisfies Σ and for $j < i$, $\tau_j(\xi, \rho)$ does not satisfy Σ. Observe, by induction on formulas, that for every quantifier-free $\mathcal{L}^{D,h}_{\mathcal{A}}$-formula φ there is an $\mathcal{L}^{D,h}_{\mathcal{A}}$-term χ_φ that gives the characteristic function of φ. (The characteristic function of $x \neq 0$ is $D(x, x)$ and the characteristic function of $0 \neq |x| \leq |y|$ is $D(D(x, y), D(x, y))$.) Now take $\tau' := \sum \chi_{\varphi_i} \cdot \tau_i$.

(iii) The proof is similar to (ii).   □

3.3. Weierstrass Preparation for $\{A^H_{m,n}\}$

In this section we will show that the $A^H_{m,n}$ satisfy *separated* Weierstrass Preparation (and thus Division).

Our first main result of this paper can be stated as follows with terminology from [CL] which is recalled in the Appendix to this paper. The proof of the theorem boils down to the Weierstrass Preparation statement provided by Proposition 3.3.3 below.

Theorem 3.3.1. *Let $\mathcal{A}$ be a strictly convergent (A, I)-pre-Weierstrass system (Definition 3.1.4). Then $\mathcal{A}^H := \{A_{m,n}^H\}$ is an (A, I)-separated pre-Weierstrass system.*

Proof. We need to verify the properties (1) up to (6) from Definition A.1.1. Each of properties (1)–(4) is either immediate or easy. To prove (6) it is sufficient, by the results of Section 2, to prove Weierstrass Preparation, which we do in Proposition 3.3.3. Property (5) is an immediate consequence of Weierstrass Division in the form of (6). □

Let us recall the definition of regularity in the case of separated power series.

Definition 3.3.2 (Separated regular). With the above notation where A is a ring and $I \subset A$ is a proper ideal, let $f \in A[[\xi_1, \ldots, \xi_m, \rho_1, \ldots, \rho_n]]$, and let J be the ideal

$$J := \left\{ \sum_{\mu, \nu} a_{\mu, \nu} \xi^\mu \rho^\nu \in A[[\xi, \rho]] : a_{\mu, \nu} \in I \right\}$$

of $A[[\xi, \rho]]$.

 (i) f is called *regular in ξ_m of degree d* when f is congruent in $A[[\xi, \rho]]$ to a monic polynomial in ξ_m of degree d modulo the ideal $B_1 \subset A[[\xi, \rho]]$ where

$$B_1 := J + (\rho)A[[\xi, \rho]].$$

 (ii) f is called *regular in ρ_n of degree d* when f is congruent in $A[[\xi, \rho]]$ to ρ_n^d modulo the ideal $B_2 \subset A[[\xi, \rho]]$ where

$$B_2 := J + (\rho_1, \ldots, \rho_{n-1}, \rho_n^{d+1})A[[\xi, \rho]].$$

Proposition 3.3.3. (i) *Let $f(\xi, \rho) \in A_{m,n}^H$ be regular in ξ_m of degree s. Let $\xi' = (\xi_1, \ldots, \xi_{m-1})$. There are unique $f_0, \ldots, f_{s-1} \in A_{m-1,n}^H$ and unit $U \in A_{m,n}^H$ such that*
$$f = U \cdot \left(\xi_m^s + f_{s-1}(\xi', \rho)\xi_m^s + \cdots + f_1(\xi', \rho)\xi_m + f_0(\xi', \rho) \right).$$

(ii) *Let $f(\xi, \rho) \in A_{m,n}^H$ be regular in ρ_n of degree s. Let $\rho' = (\rho_1, \ldots, \rho_{n-1})$. There are unique $f_0, \ldots, f_{s-1} \in (I, \rho_1, \ldots, \rho_{n-1})A_{m,n-1}^H$ and unit $U \in A_{m,n}^H$ such that*

$$f = U \cdot \left(\rho_n^s + f_{s-1}(\xi, \rho')\rho_n^s + \cdots + f_1(\xi, \rho')\rho_n + f_0(\xi, \rho') \right).$$

Proof. We prove (i); the proof of (ii) is similar. Let $f(\xi, \rho) \in A_{m,n}^H$ be regular in ξ_m of degree s. By Proposition 3.2.7, we may write

$$f = F_0(\xi, \rho) + F_1(\xi, \rho, g_1, \ldots, g_M, h_1, \ldots, h_N) \tag{3.10}$$

where $F_0 \in A[\xi, \rho]$, $F_1(\xi, \rho, \eta, \lambda) \in (I, \rho, \lambda)A_{m+M,n+N}$, and the g_i, h_j are defined by

a henselian system $\Sigma(\eta, \lambda)$:

$$\begin{aligned}
\eta_i &= b_{0,i}(\xi, \rho) + b_{1,i}(\xi, \rho, \eta, \lambda), \quad i = 1, \dots, M, \\
\lambda_j &= c_{0,j}(\xi, \rho) + c_{1,j}(\xi, \rho, \eta, \lambda), \quad j = 1, \dots, N,
\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}
b_{0,i} &\in A[\xi, \rho], & b_{1,i} &\in (I, \rho, \lambda) A_{m+M, n+N}, \\
c_{0,j} &\in (\rho) A[\xi, \rho], & c_{1,j} &\in (I, \rho, \lambda^2) A_{m+M, n+N}.
\end{aligned}$$

Let $\xi' = (\xi_1, \dots, \xi_{m-1})$.

We follow the outline of the proof of [Ar]. Let $a_0, \dots, a_{s-1}$ be new variables and let

$$a := \xi_m^s + a_{s-1}\xi_m^{s-1} + \cdots + a_0.$$

Let $g_{i,k}^*, h_{j,k}^*$ be new variables, $i = 1, \dots, M$, $j = 1, \dots, N$, $k = 0, \dots, s-1$, and let

$$g_i^* := \sum_{k=0}^{s-1} g_{i,k}^* \xi_m^j \quad \text{and} \quad h_j^* := \sum_{k=0}^{s-1} h_{j,k}^* \xi_m^j.$$

We seek a henselian system that will determine $a_0, \dots, a_{s-1}$ as the Weierstrass data of f, and the $g_{i,k}^*$ (respectively $h_{j,k}^*$) as the Weierstrass data of the remainder on dividing g_i (respectively h_j) by f.

In (3.10), setting $f = 0$ and replacing the g_i by g_i^* and the h_j by h_j^* (i.e. working modulo a) we get

$$0 = F_0 + F_1(\xi', \rho, g_1^*, \dots, g_M^*, h_1^*, \dots, h_N^*). \tag{3.12}$$

By strictly convergent Weierstrass Division by a, which is (strictly convergent) regular in ξ_m of degree s, we get equations

$$0 = \sum_{k=0}^{s-1} H_k(\xi', \rho, \{a_k\}, \{g_{ik}^*\}, \{h_{jk}^*\})\xi_m^j, \tag{3.13}$$

where we have written $\{a_k\}$ to denote the variables $a_0, \dots, a_{s-1}$, and so on. From the fact that f (and hence also $F_0 = F_0(\xi, \rho)$) is regular of degree s in ξ_m, and that modulo a we have $\xi_m^s = -a_{s-1}\xi_m^{s-1} - \cdots - a_0$, we see directly, equating the coefficients of ξ_m^i for $i = 0, \dots, s-1$ to 0, that (3.13) can be rewritten as a henselian system determining $a_0, \dots, a_{s-1}$ in terms of the other variables $\xi', \rho, \{g_{ik}^*\}, \{h_{jk}^*\}$.

Let $\Sigma(\eta, \lambda)$ be the henselian system determining the g_i and h_j. Divide the equations $\Sigma(\xi, \rho, g_1^*, \dots, g_M^*, h_1^*, \dots, h_N^*)$ (i.e. the system obtained from $\Sigma(\eta, \lambda)$ by replacing η_i by $\sum_k g_{ik}^*\xi_m^k$ and λ_j by $\sum_k h_{jk}^*\xi_m^k$) by a to get a system of the form

$$\begin{aligned}
g_i^* &= \sum_{k=0}^{s-1} G_{ik}(\xi', \rho, \{a_k\}, \{g_{ik}^*\}, \{h_{jk}^*\})\xi_m^k, \quad i = 1, \dots, M, \\
h_j^* &= \sum_{k=0}^{s-1} H_{jk}(\xi', \rho, \{a_k\}, \{g_{ik}^*\}, \{h_{jk}^*\})\xi_m^k, \quad j = 1, \dots, N.
\end{aligned} \tag{3.14}$$

Equating the coefficients of like powers of ξ_m yields a henselian system Σ^* of the form

$$
\begin{aligned}
g_{ik}^* &= G_{ik}(\xi', \rho, \{a_k\}, \{g_{ik}^*\}, \{h_{jk}^*\}), & i &= 1, \ldots, M, \\
h_{jk}^* &= H_{jk}(\xi', \rho, \{a_k\}, \{g_{ik}^*\}, \{h_{jk}^*\}), & j &= 1, \ldots, N.
\end{aligned}
\tag{3.15}
$$

Equations (3.15) and the above described henselian system for $a_0, \ldots, a_{s-1}$ together form the required henselian system for the g_{ik} and h_{jk}, the Weierstrass data of f and the remainders on dividing the g_i and h_j by f. Lemma 3.2.5(ii) shows that these two henselian systems can be combined in one system determining all the variables $a_0, \ldots, a_{s-1}, g_{ik}^*$ and h_{jk}^*.

Finally, we must show that $U \in A_{m,n}^H$. Let Θ_i, Θ_j' be new variables, and write

$$
\begin{aligned}
\eta_0 &:= \Theta_0 \cdot a, \\
\eta_i &:= \Theta_i \cdot a + \sum_k g_{ik}^* \xi_m^k, \\
\lambda_j &:= \Theta_j' \cdot a + \sum_k h_{jk}^* \xi_m^k.
\end{aligned}
$$

Making this substitution in (3.10) and (3.11), doing Weierstrass Division by a and using (3.15) and (3.16) yields equations of the form

$$
\begin{aligned}
a \cdot \Theta_0 &= a \cdot G_0(\xi, \rho, \{\Theta^*\}, \{a_k\}, \{g_{ik}^*\}, \{h_{jk}^*\}), \\
a \cdot \Theta_i &= a \cdot G_i(\xi, \rho, \{\Theta^*\}, \{a_k\}, \{g_{ik}^*\}, \{h_{jk}^*\}), \\
a \cdot \Theta_j' &= a \cdot G_j'(\xi, \rho, \{\Theta^*\}, \{a_k\}, \{g_{ik}^*\}, \{h_{jk}^*\}),
\end{aligned}
$$

where $\Theta^* = (\Theta_i, \Theta_j')$. The required system of henselian equations for Θ_0 and Θ^* over the other variables is then

$$
\begin{aligned}
\Theta_0 &= G_0(\xi, \rho, \{\Theta^*\}, \{a_k\}, \{g_{ik}^*\}, \{h_{jk}^*\}), \\
\Theta_i &= G_i(\xi, \rho, \{\Theta^*\}, \{a_k\}, \{g_{ik}^*\}, \{h_{jk}^*\}), \\
\Theta_j' &= G_j'(\xi, \rho, \{\Theta^*\}, \{a_k\}, \{g_{ik}^*\}, \{h_{jk}^*\}),
\end{aligned}
$$

and we can again use Lemma 3.2.5(ii) to combine the systems. $\square$

Remark 3.3.4. It follows from Weierstrass Division that if $f \in A_{m,n}^H$ and $f(\xi', 0, \rho) = 0$ then there is a $g \in A_{m,n}^H$ such that $f = \xi_m \cdot g$ (and similarly with respect to the other variables).

With a suitable definition of "regular", the analogue of Proposition 3.3.3 holds for the terms k_Σ. (Of course uniqueness holds only at the level of k_Σ^σ.)

In order to be able to extend any $\mathcal{A}$-structure on any field K to an $\mathcal{A}^H$-structure on K, we introduce the following condition of goodness.

Definition 3.3.5. Let $\mathcal{A}$ be a strictly convergent pre-Weierstrass system. We call $\mathcal{A}$ a *good pre-Weierstrass system* if for every system Σ of henselian equations, if $F \in A_{m+M,n+N}$ and $F \circ h_\Sigma = 0$ (as a power series) then $F \in I_\Sigma \cdot (A_{m+M,n+N})_{1+(I,\rho,\lambda)}$.

Lemma 3.3.6. *Let $\mathcal{A}$ be a good strictly convergent pre-Weierstrass system, and let σ be an analytic $\mathcal{A}$-structure on a henselian valued field K. Then there exists a unique separated $\mathcal{A}^H$-structure σ^H on K which extends σ (and which will often be also denoted by σ). More precisely, if a series $f \in A_{m,n}^H$ is a composition $f = g(\xi, \rho, h_\Sigma(\xi, \rho))$ as in Definition 3.2.4, then $\sigma^H(f)$ is the function from $(K^\circ)^{m+M} \times (K^{\circ\circ})^{n+N}$ to K given by*

$$F^\sigma \circ k_\Sigma^\sigma,$$

with notation from Lemma-Definition 3.2.8.

Proof. Lemma-Definition 3.2.8 and Definition 3.3.5 guarantee that the separated $\mathcal{A}^H$-structure σ^H on K is well defined, that is, if $g \circ h_\Sigma = 0$ then there is a unit $u \in A_{m+M,n+N 1+(I,\rho,\lambda)}$ such that $u \cdot g \in I_\Sigma$ and hence $g^\sigma \circ k_\Sigma^\sigma = 0$. $\square$

The following result will be useful in Sections 3.4 and 3.5.

Proposition 3.3.7 (Criterion). *Assume that a strictly convergent pre-Weierstrass system $\mathcal{A} = \{A_m\}$ has the property that there is a separated Weierstrass system $\mathcal{B} = \{B_{m,n}\}$ such that for all (m, n) we have $A_{m,n} \subset B_{m,n}$ and $B_{m,n}$ is faithfully flat over $(A_{m,n})_{1+(\rho)A_{m+n}}$. Then*

(i) *$\mathcal{A}$ is a good pre-Weierstrass system, and*
(ii) *$\mathcal{A}^H$ is a separated Weierstrass system.*

Proof. (i) From [Mat, Theorem 7.5] it follows that $I_\Sigma \cdot (A_{m,n})_{1+(\rho)A_{m,n}} = (I_\Sigma \cdot B_{m+M,n+N}) \cap (A_{m,n})_{1+(\rho)A_{m,n}}$. Hence if $F \circ h_\Sigma = 0$ in $B_{m,n}$ then $F \in I_\Sigma \cdot (A_{m,n})_{1+(\rho)A_{m,n}}$.

For (ii) observe that property (vii)′ of Definition A.2.1 (from which property (vii) follows) can be expressed as a linear equation over $A_{m,n}^H$ which has a solution in $B_{m,n}$. We can capture the condition $g_{\mu\nu} \in B_{m,n}^\circ$ by writing $g_{\mu\nu} = \sum_i c_i g_{\mu\nu i} + \sum_j \rho_j g'_{\mu\nu j}$ for suitable $c_i \in I$ and $g_{\mu\nu i}, g'_{\mu\nu j} \in B_{m,n}$. Let $f = F \circ h_\Sigma$ and $\overline{f}_{\mu\nu} = F_{\mu\nu} \circ h_{\Sigma_{\mu\nu}}$ with $F, F_{\mu\nu} \in A_{p,q}$ for suitable p, q. Let Σ' be the conjunction of the henselian systems Σ and the $\Sigma_{\mu\nu}$. Then the linear equation

$$F = \sum_{(\mu,\nu) \in J} F_{\mu\nu}(\xi'')^\mu (\rho'')^\nu \left(1 + \sum_i c_i g_{\mu\nu i} + \sum_j \rho_j g'_{\mu\nu j}\right) \bmod (I_{\Sigma'}) \qquad (3.16)$$

has coefficients from $A_{p,q}$ and has a solution in $B_{p,q}$, for suitable (p, q), and hence in $(A_{p,q})_{1+(\rho*)A_{p,q}}$. Substituting $h_{\Sigma'}$ for the appropriate variables gives us (vii)′. $\square$

Lemma 3.2.8 yields the following corollary.

Corollary 3.3.8. *Let $\mathcal{A}$ be a good pre-Weierstrass system and let $f \in \bigcup_{m,n} A_{m,n}^H$. Let σ be an analytic $\mathcal{A}$-structure on a henselian valued field K. Write σ for the $\mathcal{A}^H$-structure on K given by Lemma 3.3.6. The graph of f^σ is existentially $\mathcal{L}_\mathcal{A}$-definable. Moreover, the existential formula defining the graph of f^σ can be taken to be independent of the analytic structure σ.*

Now we come to two other main results in the following two subsections, which cover and extend a number of examples considered in previous papers [vdD], [LR2], [LR3], [CLR1], [CL]. In Subsection 3.6, we introduce a new general concept of analytic structures and provide some further examples, complementing Theorems 3.4.1 and 3.5.1. In Section 4 we show that the extension of $\mathcal{A}$ to $\mathcal{A}^H$ is necessary for quantifier elimination.

3.4. The case of K a complete field and $A_m = T_m(K)^\circ$

In this subsection we discuss the "classical" example. Let K be a complete, rank one valued field (not necessarily algebraically closed) and let $A_m := K^\circ\langle\xi\rangle = T_m(K)^\circ$, the ring of strictly convergent power series over K° and let $\mathcal{A} := \{A_m\}$. This is certainly a pre-Weierstrass system. Any complete, rank one field $F \supset K$ has analytic $\mathcal{A}$-structure. In this case, $A = K^\circ$ and $I = K^{\circ\circ}$.

Theorem 3.4.1. *With the above notation, $\mathcal{A}$ is a good pre-Weierstrass system and $\mathcal{A}^H :=\{A_{m,n}^H\}$ is a separated Weierstrass system.*

Proof. The proof is immediate from Proposition 3.3.7, by taking $B_{m,n} = S_{m,n}(K)^\circ$, and using [R, Lemma 3.1] which establishes that $S_{m,n}(K)^\circ$ is faithfully flat over $(T_{m+n}^\circ)_{1+(\rho)T_{m+n}^\circ}$. $\qquad\square$

In [LR2] we showed that for K rank one and complete there is a separated Weierstrass system $\mathcal{E}$ extending $\mathcal{A}$ all of whose additional functions are uniformly existentially definable in every *algebraically closed* field with analytic $\mathcal{A}$-structure. We did not make explicit what the additional functions are, nor did the uniform definitions work in non-algebraically closed fields with analytic $\mathcal{A}$-structure.

In this section we do not assume that K is algebraically closed, nor do we restrict consideration to algebraically closed fields with analytic $\mathcal{A}$-structure. In this larger generality we obtain quantifier elimination in the forms of Theorems 3.4.2 and 3.4.4, and we make explicit what the additional existentially definable functions are, namely they are defined by terms of $\mathcal{L}_{\mathcal{A}^H}$ (or even $\mathcal{L}_{\mathcal{A}}^h$), on *all* fields with analytic $\mathcal{A}$-structure, and the graphs of the functions h_Σ are indeed quantifier-free definable in all such fields. We also give a result that links definable functions to terms in Theorem 3.4.5. (The corresponding results in a more abstract framework are bundled in Theorem 3.6.2 below.) Similar results hold with $\mathcal{L}^D$ in place of $\mathcal{L}$ (see Remark 3.2.12).

In the following result, $\mathcal{L}_{\mathcal{A}^H}$ is the valued field language $\mathcal{L}$ with function symbols for the elements of the $A_{m,n}^H$ adjoined for all m and n.

Theorem 3.4.2. *Let L be any algebraically closed valued field with strictly convergent analytic $\mathcal{A}$-structure. Then L allows quantifier elimination in both $\mathcal{L}_{\mathcal{A}^H}$ and $\mathcal{L}_{\mathcal{A}}^h$ (and thus quantifier simplification, i.e. every formula is equivalent to an existential formula, in $\mathcal{L}_{\mathcal{A}}$). Moreover, the theory of algebraically closed valued fields with strictly convergent analytic $\mathcal{A}$-structure eliminates quantifiers in both the languages $\mathcal{L}_{\mathcal{A}^H}$ and $\mathcal{L}_{\mathcal{A}}^h$ (both considered as the intended definitional expansions of $\mathcal{L}_{\mathcal{A}}$).*

Proof. Follows from Theorem 3.4.1, Lemma 3.2.13(iii), and [CL, Theorem 4.5.15] for $\mathcal{A}^H$, amended by Remark A.1.8. $\qquad\square$

Theorem 3.4.3. (i) (characteristic 0) *Let $x = (x_1, \ldots, x_n)$ be several variables and y one variable. Suppose that a formula $\varphi(x, y) \in \mathcal{L}_\mathcal{A}$ defines y as a function of x in all algebraically closed fields of characteristic 0 with analytic $\mathcal{A}$-structure. Then there is a term τ of $\mathcal{L}_\mathcal{A}^h$ such that $L \models \varphi(x, y) \leftrightarrow y = \tau(x)$ for all algebraically closed valued fields L of characteristic 0 with analytic $\mathcal{A}$-structure.*

(ii) (characteristic $p \neq 0$) *In characteristic $p > 0$ let $\mathcal{L}'_\mathcal{A}$ be the language $\mathcal{L}_\mathcal{A}$ with the p-th root function $(\cdot)^{1/p}$ adjoined. In characteristic p the analogous result to* (i) *holds with $\mathcal{L}'^h_\mathcal{A}$ in place of $\mathcal{L}^h_\mathcal{A}$. Namely, for any $\mathcal{L}_\mathcal{A}$-definable function f for the theory T of algebraically closed valued fields of characteristic p with analytic $\mathcal{A}$-structure, defined on a Cartesian power of the valued field, there exists an $\mathcal{L}'^h_\mathcal{A}$-term t such that T proves that t and f are the same function.*

Proof. Follows from Proposition A.1.11 and Lemma 3.2.13(iii). $\qquad\qquad\square$

Similarly, but invoking [CL, Theorem 6.3.7] instead of [CL, Theorem 4.5.15], we find a more general but more involved quantifier elimination result, using leading terms sorts. Let $\mathcal{L}_{\mathrm{Hen},\mathcal{A}}$ (resp. $\mathcal{L}_{\mathrm{Hen},\mathcal{A}^H}$) be as in [CL, Section 6.2], namely the multi-sorted language which has the field language and function symbols for all f in A_m and all m (resp. in $A_{m,n}^H$ for all m, n) on the valued field sort, the maps rv_N for each integer $N > 0$ and the full induced structure on the sorts RV_N. Recall that on a valued field L, $RV_N(L)$ is the quotient of multiplicative semigroups $L/(1 + nL^{\circ\circ})$, and $rv_N : L \to RV_N(L)$ the projection.

Theorem 3.4.4. *The theory of characteristic zero henselian valued fields with strictly convergent analytic $\mathcal{A}$-structure eliminates the valued field quantifiers in both the languages $\mathcal{L}_{\mathrm{Hen},\mathcal{A}^H}$ and $\mathcal{L}^h_{\mathrm{Hen},\mathcal{A}}$ (both considered as the intended definitional expansions of $\mathcal{L}_{\mathrm{Hen},\mathcal{A}}$).*

We can also formulate the following result on the term structure of definable functions. To this end, let $\mathcal{L}^*_{\mathrm{Hen},\mathcal{A}}$ (resp. $\mathcal{L}^*_{\mathrm{Hen},\mathcal{A}^H}$) be the language $\mathcal{L}_{\mathrm{Hen},\mathcal{A}}$ (resp. $\mathcal{L}_{\mathrm{Hen},\mathcal{A}^H}$) together with all the functions $h_{m,n}$ as in [CL, Definition 6.1.7]. Note that these functions $h_{m,n}$ are multi-sorted variants of the basic henselian functions h_n of Definition 3.2.10.

Theorem 3.4.5 (Term structure). *The theory of characteristic zero henselian valued fields with strictly convergent analytic $\mathcal{A}$-structure has the following property, uniformly in its models K. Let $X \subset K^n$ be definable and let $f : X \to K$ be an $\mathcal{L}_{\mathrm{Hen},\mathcal{A}}$-definable function. Then there exist an $\mathcal{L}_{\mathrm{Hen},\mathcal{A}}$-definable function $g : X \to RV_N(K)^N$ for some $N > 0$ and an $\mathcal{L}^*_{\mathrm{Hen},\mathcal{A}}$-term t such that*

$$f(x) = t(x, g(x)) \quad \text{for all } x \in X. \tag{3.17}$$

Proof. By Theorem [CL, 6.3.8], we can find g and a term t in $\mathcal{L}^*_{\mathrm{Hen},\mathcal{A}^H}$ such that (3.17) holds. By compactness and using characteristic functions as at the end of the proof of Lemma 3.2.13, we see that such g and t can be taken independently of the choice of the model K. By Lemma 3.2.13(iii), we can take t to be in $(\mathcal{L}^*_{\mathrm{Hen},\mathcal{A}})^h$, using the same g. Now, if we change g we can rephrase the h_n in terms of the $h_{m,n}$ to conclude the proof. $\qquad\square$

Remark 3.4.6. Note that the elimination of valued field quantifiers of Theorem 3.4.4 also holds for $\mathcal{L}^*_{\mathrm{Hen},\mathcal{A}}$ instead of $\mathcal{L}^h_{\mathrm{Hen},\mathcal{A}}$, by the argument at the end of the proof of Theorem 3.4.5.

Because henselian functions, reciprocals and D-functions are the unique solutions of their defining equations, with the above notation, we have the following corollary. The result analogous to (i) implicit in [LR2] gave only a finite-to-one projection. In [LR6] we proved the result analogous to (ii) for $\mathcal{L}_{\mathcal{A}_{\mathrm{sep}}}$-subanalytic sets.

Corollary 3.4.7. (i) *Every $\mathcal{L}_{\mathcal{A}}$-subanalytic set (i.e. a subset of K^n_{alg} defined by an $\mathcal{L}_{\mathcal{A}}$-formula) is the one-to-one projection of an $\mathcal{L}_{\mathcal{A}}$-semianalytic set (i.e. a subset of K^{n+N}_{alg} defined by a quantifier-free $\mathcal{L}_{\mathcal{A}}$-formula which does not involve $(\cdot)^{-1}$).*
(ii) *Every $\mathcal{L}_{\mathcal{A}}$-subanalytic set is the finite disjoint union of $\mathcal{L}_{\mathcal{A}}$-subanalytic manifolds.*

Remark 3.4.8. There are several different definitions of (rigid) semianalytic sets (cf. [LL2], [LR1], [Mar1]). The above definition is perhaps the most restrictive, and best called "globally affinoid (or strictly convergent) semianalytic".

3.5. The case when A is a noetherian ring, complete in its I-adic topology

We now treat the case that A is a noetherian ring which is complete in its I-adic topology (i.e. $A = A^\wedge = \varprojlim_n A/I^n$, so A is also I-adically separated). This case complements the study of [CLR1].

For example, one could take $A = \mathbb{Z}[[t]]$ and

$$A_m = \mathbb{Z}[[t]]\langle \xi_1, \ldots, \xi_m \rangle = \mathbb{Z}[[t]][\xi_1, \ldots, \xi_m]^\wedge,$$

the t-adic completion of $\mathbb{Z}[[t]][\xi_1, \ldots, \xi_m]$. Then $\mathcal{A} := \{A_m\}$ is a strictly convergent pre-Weierstrass system. The fields $\mathbb{Q}_p, \mathbb{C}_p$ for p prime, and all ultraproducts of these fields, have analytic $\mathcal{A}$-structure via the natural maps $t \mapsto p$. The fields $\mathbb{F}_p((t))$ for p prime, and all ultraproducts of these fields, have analytic $\mathcal{A}$-structure via the natural maps $p \mapsto 0$. We could consider all of these to be fields with analytic structure satisfying the side conditions that for all p either $p = 0$ or $|p| = 1$ or p is prime, and that t is prime. This situation is discussed in Appendix A.4.

There are many more ways that a henselian field can have analytic $\mathcal{A}$-structure. For example, the field $\mathbb{C}((t^{1/\infty}))$ of Puiseux series has analytic $\mathcal{A}$-structure via $t \mapsto t$. As another example, let $\mathbb{C}^*_p$ be a non-principal ultrapower of $\mathbb{C}_p$ and consider the mapping σ that sends t to $[p, p^{1/2}, p^{1/3}, \ldots]$, so in $\mathbb{C}^*_p$ the element $\sigma(t)$ is an "infinitesimal" power of p. This σ gives an analytic $\mathcal{A}$-structure on $\mathbb{C}^*_p$. Alternatively, one could map t to a finite but irrational power of p.

Theorem 3.5.1 below explains, via [CL], aspects of the geometry and model theory of definable sets in all these analytic structures uniformly.

Note that Theorem 3.4.1 above is not covered by Theorem 3.5.1, except when K is discretely valued.

Theorem 3.5.1. *Let A be a noetherian ring, I an ideal of A with $I \neq A$, and assume that A is complete in its I-adic topology. Let $A_m := A[\xi_1, \ldots, \xi_m]\widehat{\ }$ be the I-adic completion of $A[\xi_1, \ldots, \xi_m]$. Then $\mathcal{A}$ is a good pre-Weierstrass System and $\mathcal{A}^H := \{A_{m,n}^H\}$ is a separated Weierstrass system.*

Proof. We showed in [CLR1, Lemma 2.9] that if we define

$$B_{m,n} := A\langle\xi\rangle[[\rho]] := A[\xi_1, \ldots, \xi_m]\widehat{\ }[[\rho_1, \ldots, \rho_n]]$$

and $\mathcal{B} := \{B_{m,n}\}$, then $\mathcal{B}$ is a separated Weierstrass system. Indeed, we showed that these rings satisfy condition $(7)'$ of Definition A.2.1. Hence the result follows from Proposition 3.3.7 and the fact that $B_{m,n}$ is faithfully flat over $(A_{m,n})_{1+(I,\rho)A_{m,n}}$. Since the (ρ)-adic completion of $(A_{m,n})_{1+(I,\rho)A_{m,n}}$ is $B_{m,n}$, faithful flatness follows from [Mat, Theorem 8.14].   □

We have the following quantifier elimination result, as in Section 3.4.

Theorem 3.5.2. *Theorems 3.4.2–3.4.5 go through in literally the same way with $\mathcal{A}$ as in this section instead of that in Section 3.4.*

3.6. A general concept and examples

Motivated by Theorems 3.4.1 and 3.5.1, we introduce the following terminology, of which these theorems in fact provide examples. See the appendix for a summary and elaboration of some of the definitions of [CL].

Definition 3.6.1. We call a strictly convergent pre-Weierstrass system $\mathcal{A} = \{A_m\}$ (Definitions 3.1.4 and 3.3.5) a *strictly convergent Weierstrass system* if $\mathcal{A}$ is good and $\mathcal{A}^H$ is a separated (A, I)-Weierstrass system (as defined in [CL] and also in Definitions A.1.1 and A.1.4 in the Appendix below).

By the very nature of Theorems 3.4.1 and 3.5.1, any analytic structure $\mathcal{A}$ as in these theorems is a strictly convergent Weierstrass system (as opposed to just a pre-Weierstrass system).

Before giving further examples we outline an alternative proof for Theorem 3.4.1. Let $A_m = T_m(K)^\circ$ and $f = F \circ h_\Sigma \in A_{m,n}^H \subset S_{m,n}(K)$. Let $|c| = \|f\|$ be the Gauss norm of f in $S_{m,n}(K)$. We allow the possibility that $c = 0$. Let F and h_Σ be defined over the B-ring E. If $|c| < 1$ then $\widetilde{F} \in \widetilde{I}_\Sigma \cdot \widetilde{K}[\xi][[\rho]]$, and hence by Krull's Theorem [Mat, Theorem 8.10], also $\widetilde{F} \in \widetilde{I}_\Sigma \cdot \widetilde{K}[\xi][\rho]$. By the usual induction on the levels of E we eventually find $c \in K^\circ$ and G such that $f = c \cdot G \circ h_\Sigma$ and $F - c \cdot G \in I_\Sigma$. The case $c = 0$ gives "goodness", and the case $c \neq 0$ reduces proving (vii) or (vii)$'$ to the special case of $\|f\| = 1$. That case can be dealt with by induction on $m + n$, by breaking up f into pieces exactly as in [CL, Remark 4.1.11(ii)].

Theorem 3.6.2. *Theorems 3.4.2–3.4.5 go through in literally the same way with $\mathcal{A}$ any strictly convergent Weierstrass system as in Definition 3.6.1 instead of the more concrete structure of Section 3.4.*

Using Theorem 3.5.1, it is easy to write down many examples of strictly convergent Weierstrass systems other than those discussed at the beginning of the previous subsections. Here are some further examples of a different kind.

(a) (cf. [CL, Section 4.4(10)]) Let K be complete, rank one and let $\mathcal{A} = \{A_m\}$ where $A_m = K^\circ\langle\langle\xi_1, \ldots, \xi_m\rangle\rangle$, the ring of *overconvergent* power series in $(\xi_1, \ldots, \xi_m)$. Then $\mathcal{A} = \{A_m\}$ is a good pre-Weierstrass system and $\mathcal{A}^H := \{A_{m,n}^H\}$ is a separated Weierstrass system. The proofs of these statements are similar to the proof of Theorem 3.4.1.

(b) (cf. [CL, Section 4.4(4)]) Let F be a *maximally complete field* and let $A_m = T_m(\mathcal{B})$ be the full ring of strictly convergent power series as defined in loc. cit. Then $\mathcal{A} = \{A_m\}$ is a good pre-Weierstrass system and $\mathcal{A}^H := \{A_{m,n}^H\}$ is a separated Weierstrass system. The proof is similar to the alternative proof of Theorem 3.4.1 outlined above, except that transfinite induction on the support of E is used.

(c) (cf. [CL, Section 4.4(7)]) Let L be a field and let

$$A_m = \bigcup_i L[[x_1, \ldots, x_i]]\langle\xi_1, \ldots, \xi_m\rangle$$

where $L[[x_1, \ldots, x_i]]\langle\xi_1, \ldots, \xi_m\rangle$ is the $(x_1, \ldots, x_i)$-adic completion of the ring $L[[x_1, \ldots, x_i]][\xi_1, \ldots, \xi_m]$. Then $\mathcal{A} = \{A_m\}$ is a good pre-Weierstrass system and $\mathcal{A}^H := \{A_{m,n}^H\}$ is a separated Weierstrass system. This is immediate from Theorem 3.5.1, since A_m is the direct limit of rings covered by that theorem.

Several results for general strictly convergent Weierstrass systems now follow from [CL] such as quantifier elimination results similar to Theorem 3.5.2, and the cell decomposition and Jacobian property of [CL, Theorem 6.3.7].

Remark 3.6.3. We also observe that the above methods allow one to give a sharpening of the results of [CL, Section 4.6]. Let K be a rank one henselian field, and let $A(K)_{\mathrm{alg},m,n}$ be the K-algebra of all power series in $(\xi_1, \ldots, \xi_m, \rho_1, \ldots, \rho_n)$ that are defined by polynomial henselian systems over $(K^\circ, K^{\circ\circ})$. Let $\mathcal{A}_{\mathrm{alg,str}} := \{A(K)_{\mathrm{alg},m,0}\}_m$ and $\mathcal{A}_{\mathrm{alg,sep}} := \{A(K)_{\mathrm{alg},m,n}\}_{m,n}$. Then

Proposition 3.6.4. $\mathcal{A}_{\mathrm{alg,str}}$ *is the smallest* $(K^\circ, K^{\circ\circ})$-*strictly convergent Weierstrass system and* $\mathcal{A}_{\mathrm{alg,sep}}$ *is the smallest* $(K^\circ, K^{\circ\circ})$-*separated Weierstrass system.*

4. The counterexample

The quantifier elimination of Denef and van den Dries [DD] for $K = \mathbb{Q}_p$ in the strictly convergent analytic language $\mathcal{L}_{\mathcal{A}}^D$ (see definitions below) raised the obvious question of whether a similar result held for $\mathbb{C}_p$, the completion of the algebraic closure of $\mathbb{Q}_p$ (or in general for complete, rank one, algebraically closed valued fields) since there is quantifier elimination for algebraically closed valued fields in the *algebraic* language. In [S1], [S2]

Schoutens gave such a quantifier elimination in the more restrictive language with function symbols for the functions defined by *overconvergent* power series, and developed the corresponding theory of *strongly subanalytic sets*. In [LL2] a quantifier elimination was given in a language ($\mathcal{L}^D_{\mathcal{A}_{\text{sep}}}$) with function symbols for a larger class of functions—those defined by the so called separated power series. This led to the theory of rigid subanalytic sets which was developed further in [LR1]–[LR4], [LR6], [CLR1], [CL], and [Ce1]–[Ce5]. An elaborate proof of quantifier elimination in the strictly convergent language, based on a "Flattening Theorem", was published, but the proof was not correct. (See [LR5] for the history and a counterexample to the "Flattening Theorem".) Theorem 4.3 shows that there is no quantifier elimination for complete, rank one, algebraically closed fields of characteristic zero in the strictly convergent language $\mathcal{L}^D_{\mathcal{A}}$ (see definition below) with $\mathcal{A} = \{T_m(K)^\circ\}$. Fortunately, this negative result does not lead to wild behavior, as there are Theorem 3.4.1 and the quantifier elimination results of Theorem 4.2.

The results of [LR2] and, in a more explicit form, Theorem 3.4.1 of this paper show that for K a henselian field, K_{alg} has quantifier *simplification* in the strictly convergent language $\mathcal{L}_{\mathcal{A}}$. We show in this section that there is no quantifier *elimination* in the strictly convergent language $\mathcal{L}^D_{\mathcal{A}}$, and hence neither in the language denoted by $\mathcal{L}_K$ in the introduction. To be precise: Let K be a complete, rank one, non-trivially valued field of characteristic 0, and let K_{alg} be the algebraic closure of K (we do not exclude the case that K is itself algebraically closed). The assumption that K has characteristic 0 is not essential, but it simplifies the arguments in a number of places. For a valued field F let $F^\circ := \{x \in F : |x| \leq 1\}$ and $F^{\circ\circ} := \{x \in F : |x| < 1\}$. Let $T_m(K)^\circ$ be the ring of strictly convergent power series in m variables with coefficients from K°, and let $S_{m,n}(K)^\circ$ be the ring of separated power series in m variables of the first kind and n of the second kind (cf. [LL2], [LR1]) with coefficients from K°. Moreover

$$A_m := T_m(K)^\circ, \qquad A_{m,n} := S_{m,n}(K)^\circ,$$
$$\mathcal{A} := \{A_m\}_{m\in\mathbb{N}}, \qquad \mathcal{A}_{\text{sep}} := \{A_{m,n}\}_{m,n\in\mathbb{N}}.$$

For $\mathcal{F}$ a collection of functions we let $\mathcal{L}_{\mathcal{F}}$ denote the language of valued fields augmented with symbols for the functions in $\mathcal{F}$, and let $\mathcal{L}^D_{\mathcal{F}}$ denote $\mathcal{L}_{\mathcal{F}}$ with the field inverse replaced by a symbol D for restricted division adjoined (or two such symbols in the separated case).

Before proceeding we recall Remark 3.2.12 about the equivalence of the languages with full field inverse and languages with only restricted division.

We know

Theorem 4.1 ([LL2]). *K°_{alg} has quantifier elimination in $\mathcal{L}^D_{\mathcal{A}_{\text{sep}}}$.*

And, as a special instance of Theorem 3.4.2, we have

Theorem 4.2. *K°_{alg} has quantifier simplification in $\mathcal{L}_{\mathcal{A}}$ and quantifier elimination in both $\mathcal{L}^D_{\mathcal{A}^H}$ and $\mathcal{L}^{D,h}_{\mathcal{A}}$.*

The counterexample to the question described in the introduction is provided by

Theorem 4.3. K_{alg}° *does not have quantifier elimination in $\mathcal{L}_{\mathcal{A}}^{D}$. Indeed, there is a strictly convergent subanalytic set $X \subset (K_{\text{alg}}^{\circ})^3$ (i.e. a set described by an existential $\mathcal{L}_{\mathcal{A}}^{D}$-formula) that is not described by any quantifier-free $\mathcal{L}_{\mathcal{A}}^{D}$-formula (i.e. is not $\mathcal{L}_{\mathcal{A}}^{D}$-semianalytic).*

The rest of this section is devoted to the proof of Theorem 4.3.

Let K_{alg}^{*} be a non-principal ultrapower of K_{alg}, and let $\mathfrak{p}^{\infty} = \{a \in (K_{\text{alg}}^{*})^{\circ} : |a| < 1/n$ for all $n \in \mathbb{N}\}$ be the ideal of infinitesimals in $(K_{\text{alg}}^{*})^{\circ}$. Let $K_1 = Q((K_{\text{alg}}^{*})^{\circ}/\mathfrak{p}^{\infty})$ be the field of fractions of $(K_{\text{alg}}^{*})^{\circ}/\mathfrak{p}^{\infty}$. By the results of [LR3] and [CL] we know that the fields K_{alg}, K_{alg}^{*} and K_1 all are algebraically closed valued fields with analytic $\mathcal{A}$-structure, and also analytic $\mathcal{A}_{\text{sep}}$-structure. Hence these fields are elementarily equivalent in each of the languages $\mathcal{L}_{\mathcal{A}}$, $\mathcal{L}_{\mathcal{A}}^{D}$, $\mathcal{L}_{\mathcal{A}}^{D,h}$, and $\mathcal{L}_{\mathcal{A}_{\text{sep}}}^{D}$. Fix $p \in K$ with $0 < |p| < 1$, and let

$$f(\zeta) := \sum_{n=1}^{\infty} p^n \zeta^{n!},$$

so f is strictly convergent, but definitely not overconvergent. Let $X \subset (K_1^{\circ})^3$ be defined by the formula

$$(\exists z)[|\rho| < 1 \,\wedge\, \xi_1 + \xi_2 z = f(z) \,\wedge\, \rho z^2 - z + 1 = 0 \wedge |z| \le 1] \tag{4.1}$$

and for $\overline{\rho} \in K_1^{\circ}$ let $X(\overline{\rho}) := \{(\overline{\rho}, \xi_1, \xi_2) : (\overline{\rho}, \xi_1, \xi_2) \in X\}$.

For $|\rho| < 1$ let $z_1 = z_1(\rho)$ be the zero of $\rho \zeta^2 - \zeta + 1$ with $|z| \le 1$, and let $z_2 = z_2(\rho)$ be the other zero (which is of size $|1/\rho|$). We shall show that there are $\overline{\rho}, \overline{\xi}_1, \overline{\xi}_2 \in K_1^{\circ}$ such that X is not described by a quantifier-free $\mathcal{L}_{\mathcal{A}}^{D}$-formula in any K-rational domain $\mathcal{U} \subset (K_1^{\circ})^3$ with $(\overline{\rho}, \overline{\xi}_1, \overline{\xi}_2) \in \mathcal{U}$. This will show that X is not $\mathcal{L}_{\mathcal{A}}^{D}$-semianalytic. Observe that X is an $\mathcal{L}_{\text{sep}}$- (indeed $\mathcal{L}_{\mathcal{A}}^{h}$-) analytic variety in the open disc $|\rho| < 1$.

Let $\overline{\rho} \in K_1$ satisfy $1 - 1/n < |\overline{\rho}| < 1$ for all $n \in \mathbb{N}$, and define $\overline{\xi}_1, \overline{\xi}_2 \in K_1^{\circ}$ by the equations

$$\overline{\xi}_1 + \overline{\xi}_2 z_1(\overline{\rho}) = f(z_1(\overline{\rho})), \quad \overline{\xi}_1 + \overline{\xi}_2 z_2(\overline{\rho}) = f(z_2(\overline{\rho})). \tag{4.2}$$

Remark 4.4. Note that $f(z_2(\overline{\rho}))$ as an element of K_1 or K_{alg}^{*} is *not* defined by the analytic $\mathcal{A}$-structure since $|z_2| > 1$. However, $f(z_2(\overline{\rho}))$ is well defined as an element of K_1° by the completeness of K_1. Furthermore, for each $n \in \mathbb{N}$ there are $\xi_1^{(n)} = \xi_1^{(n)}(\rho)$ and $\xi_2^{(n)} = \xi_2^{(n)}(\rho) \in (1/\rho^{n!})K[\rho]$ such that if $\overline{\xi}_i^{(n)} := \xi_i^{(n)}(\overline{\rho})$ then $|\overline{\xi}_i - \overline{\xi}_i^{(n)}| < |p^n|$. Solve equations (4.2) for $\overline{\xi}_1, \overline{\xi}_2$ using $z_1 + z_2 = z_1 z_2 = 1/\rho$. Indeed, $\overline{\xi}_1, \overline{\xi}_2$ are just the "Weierstrass data" evaluated at $\overline{\rho}$ on formally dividing $f(\zeta)$ by $\zeta^2 - (1/\rho)\zeta + 1/\rho$. Let $\xi_i(\rho) := \lim_{n \to \infty} \xi_i^{(n)}(\rho) \in K^{\circ}[[1/\rho]]$. Direct calculation shows that $\xi_1(\rho)$ and $\xi_2(\rho)$ are strictly convergent power series in $1/\rho$ that do not converge on any annulus of the form $1 - 1/n \le \rho \le 1$. In other words, they converge for $|1/\rho| \le 1$ but not on any bigger disc $|1/\rho| \le 1 + \delta, 0 < \delta \in \mathbb{R}$.

We will need the following definitions and lemmas.

Definition 4.5. (i) Let $\overline{\alpha} = (\alpha_1, \ldots, \alpha_\ell) \in K_{\mathrm{alg}}^\circ \setminus K_{\mathrm{alg}}^{\circ\circ}$ and let $\mathcal{W}_n(\overline{\alpha})$ be the *K-rational domain* (i.e. rational domain defined over K, see Remark 4.6(ii) below)

$$\Big\{ (\rho, \xi_1, \xi_2) : |p| \le |\rho^{n!}| \le 1 \wedge \bigwedge_{i=1}^{\ell} |\rho - \alpha_i| = 1 \wedge |\xi_1 - \xi_1^{(n)}(\rho)| \le |p^n|$$
$$\wedge |\xi_2 - \xi_2^{(n)}(\rho)| \le |p^n| \Big\}.$$

Clearly $(\overline{\rho}, \overline{\xi}_1, \overline{\xi}_2) \in \mathcal{W}_n(\overline{\alpha})$.

(ii) $\mathcal{W}_n^1(\overline{\alpha})$ is the *K*-rational domain

$$\Big\{ (\rho, \xi_1) : |p| \le |\rho^{n!}| \le 1 \wedge \bigwedge_{i=1}^{\ell} |\rho - \alpha_i| = 1 \wedge |\xi_1 - \xi_1^{(n)}(\rho)| \le |p^n| \Big\},$$

and $\mathcal{W}_n^2(\overline{\alpha})$ is the *K*-rational domain

$$\Big\{ (\rho, \xi_2) : |p| \le |\rho^{n!}| \le 1 \wedge \bigwedge_{i=1}^{\ell} |\rho - \alpha_i| = 1 \wedge |\xi_2 - \xi_2^{(n)}(\rho)| \le |p^n| \Big\}.$$

(iii) Let $n = \max(n_1, n_2)$. Then $\mathcal{W}_{n_1, n_2}(\overline{\alpha})$ is the *K*-rational domain

$$\Big\{ (\rho, \xi_1, \xi_2) : |p| \le |\rho^{n!}| \le 1 \wedge \bigwedge_{i=1}^{\ell} |\rho - \alpha_i| = 1$$
$$\wedge |\xi_1 - \xi_1^{(n_1)}(\rho)| \le |p^{n_1}| \wedge |\xi_2 - \xi_2^{(n_2)}(\rho)| \le |p^{n_2}| \Big\}.$$

(iv) For $\mathcal{U}$ a *K*-rational domain we denote by $\mathcal{O}_{\mathcal{U}}$ the ring of (strictly convergent) analytic functions on $\mathcal{U}$ (see [BGR, 7.3.2]).

Remark 4.6. (i) $\mathcal{O}_{\mathcal{U}}$ depends only on the set $\mathcal{U}$ and not on a particular definition of $\mathcal{U}$. Observe that if $n_1 \le n_2$ then

$$\mathcal{W}_{n_1, n_2}(\overline{\alpha}) = \Big\{ (\rho, \xi_1, \xi_2) : |p| \le |\rho^{n!}| \le 1 \wedge \bigwedge_{i=1}^{\ell} |\rho - \alpha_i| = 1$$
$$\wedge |\xi_1 - \xi_1^{(n_2)}(\rho)| \le |p^{n_1}| \wedge |\xi_2 - \xi_2^{(n_2)}(\rho)| \le |p^{n_2}| \Big\}$$

(here we have replaced $|\xi_1 - \xi_1^{(n_1)}(\rho)| \le |p^{n_1}|$ by $|\xi_1 - \xi_1^{(n_2)}(\rho)| \le |p^{n_1}|$ in the definition; recall that $n = \max(n_1, n_2)$).

(ii) We are being a bit sloppy with the definitions of the $\mathcal{W}_*(\overline{\alpha})$ since we are allowing $\alpha_i \in K_{\mathrm{alg}}$. However, if the set $\{\alpha_1, \ldots, \alpha_\ell\}$ is taken to be closed under conjugation (which it can always be), then $\mathcal{W}_*(\overline{\alpha})$ can always be defined over K (see [CLR1, Section 3] or [CL, Section 5]).

Lemma 4.7. *Let $\mathcal{U}$ be a K-rational domain with $(\overline{\rho}, \overline{\xi}_1, \overline{\xi}_2) \in \mathcal{U}$. Then there is an $n \in \mathbb{N}$ and $\overline{\alpha} = (\alpha_1, \ldots, \alpha_\ell) \in (K_{\mathrm{alg}}^\circ \setminus K_{\mathrm{alg}}^{\circ\circ})^\ell$ such that $(\overline{\rho}, \overline{\xi}_1, \overline{\xi}_2) \in \mathcal{W}_n(\overline{\alpha}) \subset \mathcal{U}$.*

Proof. $\mathcal{U}$ can be defined by a finite set of inequalities of the form

$$\varepsilon \leq |g_i(\rho, \xi_1, \xi_2)| \leq h_i(\rho, \xi_1, \xi_2)|. \tag{4.3}$$

Here $0 < \varepsilon \in \mathbb{R}$ and $g_i, h_i \in K^{\circ}[\rho, \xi_1, \xi_2]$. Restricted to $\mathcal{U} \cap \mathcal{W}_n(\emptyset)$ for n such that $|p^n| < \varepsilon^2$, these inequalities reduce to

$$\varepsilon \leq |g_i(\rho, \xi_1^{(n)}(\rho), \xi_2^{(n)}(\rho))| \leq |h_i(\rho, \xi_1^{(n)}(\rho), \xi_2^{(n)}(\rho))|. \tag{4.4}$$

Since these inequalities are satisfied by $\overline{\rho}$, we have

$$\|(\rho^{n!})^k g_i(\rho, \xi_1^{(n)}(\rho), \xi_2^{(n)}(\rho))\| \leq \|(\rho^{n!})^k h_i(\rho, \xi_1^{(n)}(\rho), \xi_2^{(n)}(\rho))\|$$

where $k \in \mathbb{N}$ is $\geq$ the degrees of the g_i and h_i, and $\| \cdot \|$ denotes the Gauss norm. But then this inequality is also satisfied for all ρ with $|\rho| = 1$ except for ρ in a finite union of open discs of radius 1, all defined over K_{alg} (or even K). Since the inequalities (4.4) are satisfied by $\overline{\rho}$, they are satisfied for all ρ with $1 - \delta \leq |\rho| < 1$ for some $0 < \delta \in \mathbb{R}$. Increasing n so that also $|p^n| < \delta$ and using $\overline{\alpha}$ to exclude the open discs gives us the required $\mathcal{W}_n(\overline{\alpha})$. $\qquad\square$

Lemma 4.8. *Let $F(\rho, \xi_1, \xi_2) \in \mathcal{O}_{\mathcal{W}_n(\overline{\alpha})}$ (the ring of analytic functions on the rational domain $\mathcal{W}_n(\overline{\alpha})$), and assume that*

$$X(\overline{\rho})|_{\mathcal{W}_n} \subset \{(\rho, \xi_1, \xi_2) \in \mathcal{W}_n : F(\rho, \xi_1, \xi_2) = 0\} =: V_F.$$

Define

$$\overline{X}(\overline{\rho}) := \{(\overline{\rho}, \xi_1, \xi_2) : \xi_1 + \xi_2 z_2(\overline{\rho}) = f(z_2(\overline{\rho}))\} \subset (K_1^{\circ})^3.$$

Then also $\overline{X}(\overline{\rho})|_{\mathcal{W}_n} \subset V_F$.

Proof. Since $F \in \mathcal{O}_{\mathcal{W}_n(\overline{\alpha})}$ and $X(\overline{\rho})|_{\mathcal{W}_n((\overline{\alpha})} \subset V_F$, for $(\overline{\rho}, \xi_1, \xi_2) \in X(\overline{\rho})|_{\mathcal{W}_n(\overline{\alpha})}$ we have $F(\overline{\rho}, \xi_1, \xi_2) = 0$. Now for $(\rho, \xi_1, \xi_2) \in X$ we have $(\xi_1 + \zeta\xi_2 - f(\zeta))|_{\zeta = z_1(\rho)} = 0$, and on $\mathcal{W}_n(\overline{\alpha})$ we have $\xi_i = \xi_i^{(n)}(\rho) + p^n \eta_i$ for $i = 1, 2$ where η_1, η_2 are new variables that vary over K_{alg}° (or K_1°).

Let $\mathcal{W}_n^+(\overline{\alpha}) := \{(\xi_1, \xi_2, \rho, \zeta) : (\xi_1, \xi_2, \rho) \in \mathcal{W}_n(\overline{\alpha}) \wedge |\rho\zeta^2 - \zeta + 1| \leq |p^n \rho^{n!}|\}$. On $\mathcal{W}_n^+(\overline{\alpha})$ we have $\xi_i = \xi_i^n(\rho) + p^n \eta_i$ for $i = 1, 2$, so we may consider the variables to be $\eta_1, \eta_2, \zeta, \rho, \frac{p}{\rho^{n!}}, \frac{1}{\rho - \alpha_i}$ and $\frac{\rho\zeta^2 - \zeta + 1}{p^n \rho^{n!}}$. By direct computation, using the fact that the $\xi_i^{(n)}$ are approximations to the Weierstrass data of $f(\zeta)$ on division by $\zeta^2 - (1/\rho)\zeta + 1$, we see that $\xi_1 + \zeta\xi_2 - f(\zeta)$ is regular in η_1 of degree 1. Hence, on $\mathcal{W}_n^+(\overline{\alpha})$ we have

$$F(\rho, \xi_1, \xi_2) = G(\rho, \xi_2, \zeta) + (\xi_1 + \zeta\xi_2 - f(\zeta))Q$$

where $G, Q \in \mathcal{O}_{\mathcal{W}_n^+(\overline{\alpha})}$.

Now, by (formal) Weierstrass Division of G by $\zeta^2 - (1/\rho)\zeta + 1/\rho$ we have

$$F(\rho, \xi_1, \xi_2) = H_1(\rho, \eta_2) + H_2(\rho, \eta_2)\zeta + \left(\zeta^2 - \tfrac{1}{\rho}\zeta + \tfrac{1}{\rho}\right)Q_1 + (\xi_1 + \zeta\xi_2 - f(\zeta))Q. \tag{4.5}$$

Here H_1, H_2 and Q_1 are *not* strictly convergent on the domain $\mathcal{W}_n^+(\overline{\alpha})$. However H_1, H_2 and Q_1 are strictly convergent in η_2 with coefficients functions in

$$K^\circ\left[\rho, \frac{1}{\rho}, \frac{1}{\rho - \alpha_1}, \ldots, \frac{1}{\rho - \alpha_\ell}, \zeta, \frac{\rho\zeta^2 - \zeta + 1}{p^n \rho^{n!}}\right]^{\widehat{}} =: A^+$$

where $\widehat{}$ denotes completion in the norm on K°, i.e. we can write

$$H_1(\rho, \eta_2) = \sum_{j=0}^{\infty} h_{1j}(\rho)\eta_2^j, \quad H_2(\rho, \eta_2) = \sum_{j=0}^{\infty} h_{2j}(\rho)\eta_2^j$$

where $h_{ij} \in K^\circ\left[\rho, \frac{1}{\rho}, \frac{1}{\rho-\alpha_1}, \ldots, \frac{1}{\rho-\alpha_\ell}\right]^{\widehat{}}$ and $\|h_{ij}\| \to 0$ as $j \to \infty$. (Here $\|\cdot\|$ is the Gauss norm on $K^\circ\left[\rho, \frac{1}{\rho}, \frac{1}{\rho-\alpha_1}, \ldots, \frac{1}{\rho-\alpha_\ell}\right]^{\widehat{}}$). Because K_1 is complete, and contains no infinitesimals, all the power series in A^+ define functions on

$$\mathcal{W}_n^+(\overline{\alpha}) \cap \{(\xi_1, \xi_2, \rho, \zeta) : 1 - 1/k \le |\rho| < 1 \,\forall k \in \mathbb{N} \wedge |\rho\zeta^2 - \zeta + 1| \le |p^n \rho^{n!}|\}.$$

Indeed, they define functions on $\mathcal{W}_n^{++}(\overline{\alpha})$ defined like $\mathcal{W}_n^+(\overline{\alpha})$ but allowing ζ to vary over the slightly bigger disc $\{\zeta \in K_1 : |\zeta| \le 1 + 1/k \,\forall k \in \mathbb{N}\}$, i.e. on

$$\mathcal{W}_n^{++}(\overline{\alpha}) := \{(\xi_1, \xi_2, \rho, \zeta) \in (K_1)^3 \times K_1 : (\xi_1, \xi_2, \rho) \in \mathcal{W}_n(\overline{\alpha})$$
$$\wedge |\rho\zeta^2 - \zeta + 1| \le |p^n \rho^{n!}| \wedge (1 - 1/k \le |\rho| < 1 \wedge |\zeta| \le 1 + 1/k \,\forall k \in \mathbb{N})\}.$$

Note that also $h_{ij}(\overline{\rho}) \in K_1$ and $h_{ij}(\overline{\rho}) \to 0$ as $j \to \infty$. Hence substituting $z_1(\overline{\rho})$ for ζ and $\overline{\rho}$ for ρ in (4.5) we obtain

$$H_1(\overline{\rho}, \eta_2) + H_2(\overline{\rho}, \eta_2)z_1(\overline{\rho}) = 0 \quad \text{for all } \eta_2 \in K_1^\circ,$$

i.e.

$$\sum_{j=0}^{\infty}[h_{1j}(\overline{\rho}) + h_{2j}(\overline{\rho})z_1(\overline{\rho})]\eta_2^j = 0 \quad \text{for all } \eta_2 \in K_1^\circ.$$

Hence

$$h_{1j}(\overline{\rho}) + h_{2j}(\overline{\rho})z_1(\overline{\rho}) = 0 \quad \text{for all } j.$$

Next we will show that $h_{ij}(\overline{\rho}) = 0$ for $i = 1, 2$ and $j = 0, 1, 2, \ldots$. Suppose not. Then for some j we have $h_{1j}(\overline{\rho}) + h_{2j}(\overline{\rho})z_1(\overline{\rho}) = 0$ but either $h_{1j}(\overline{\rho}) \ne 0$ or $h_{2j}(\overline{\rho}) \ne 0$. Since $|z_1(\overline{\rho})| = 1$ we must have both $h_{1j}(\overline{\rho}) \ne 0$ and $h_{2j}(\overline{\rho}) \ne 0$. Hence $|h_{1j}(\overline{\rho})| = |h_{2j}(\overline{\rho})|$. From the definition of $\overline{\rho}$ we see that $\|h_{ij}(\rho)\| - |h_{ij}(\overline{\rho})|$ is infinitesimal. Hence, multiplying by a suitable element of $K \setminus \{0\}$ we may assume that $\|h_{1j}(\rho)\| = (\|h_{2j}(\rho)\| = 1$. Let $\widetilde{}$ denote on K° the residue modulo $K^{\circ\circ}$ and on K_1° the residue modulo $K_1^\circ \cdot K^{\circ\circ} = \{x \in K_1^\circ : |x| \le |y| \text{ for some } y \in K^{\circ\circ}\}$. Then

$$\widetilde{h}_{1j}(\widetilde{\overline{\rho}}) + \widetilde{h}_{2j}(\widetilde{\overline{\rho}})\widetilde{z_1(\overline{\rho})} = 0.$$

Since $\widetilde{h}_{1j}(\rho)$ and $\widetilde{h}_{2j}(\rho)$ are rational functions of ρ, we find that $\widetilde{h}_{1j}(\widetilde{\overline{\rho}})/\widetilde{h}_{2j}(\widetilde{\overline{\rho}})$ is a solution of the equation $\widetilde{\overline{\rho}}\zeta^2 - \zeta + 1 = 0$ in $\widetilde{K}(\widetilde{\overline{\rho}})$. But this equation has no rational

solutions ($\widetilde{\overline{\rho}}$ is transcendental over $\widetilde{K}$; look at the zeros and poles of the rational function). Since $\overline{\rho}$ is transcendental over K, also $h_{ij}(\rho) = 0$ for $i = 1, 2$ and $j = 0, 1, 2, \ldots$. Thus

$$F(\rho, \xi_1, \xi_2) = \left(\zeta^2 - \tfrac{1}{\rho}\zeta + \tfrac{1}{\rho}\right)Q_1 + (\xi_1 + \zeta\xi_2 - f(\zeta))Q. \tag{4.6}$$

This is an equation among elements of A^+, and hence an equation among the functions they define on $\mathcal{W}_n^{++}(\overline{\alpha})$. Hence from (4.6), for any $(\overline{\rho}, \xi_1, \xi_2) \in \overline{X}(\overline{\rho})|_{\mathcal{W}_n(\overline{\alpha})}$ we have

$$\begin{aligned} F(\overline{\rho}, \xi_1, \xi_2) = {} & \left(z_2^2(\overline{\rho}) - \tfrac{1}{\overline{\rho}}z_2(\overline{\rho}) + \tfrac{1}{\overline{\rho}}\right)Q_1(\overline{\rho}, \eta_2, z_2(\overline{\rho})) \\ & + ((\xi_1 + z_2(\overline{\rho})\xi_2 - f(z_2(\overline{\rho}))Q(\overline{\rho}, \xi_1, \xi_2, z_2(\overline{\rho})), \end{aligned} \tag{4.7}$$

i.e. $F(\overline{\rho}, \xi_1, \xi_2) = 0$. Again, this equation is to be understood as an equation among functions on $\mathcal{W}_n^{++}(\overline{\alpha})$. [Since $|z_2(\overline{\rho})| > 1$, $f(z_2(\overline{\rho}), Q(\overline{\rho}, \xi_1, \xi_2, z_2(\overline{\rho}))$ and $Q_1(\overline{\rho}, \eta_2, z_2(\overline{\rho}))$ are not evaluated using the analytic structure, but by using the completeness of K_1. As we observed above, power series in A^+ define functions on $\mathcal{W}_n^{++}(\overline{\alpha})$. Indeed, for every m, modulo p^{m+1} all the functions F, f, Q, Q_1 are congruent to polynomials in the various variables with coefficients rational functions of $\overline{\rho}$. Hence $|F(\overline{\rho}, \xi_1, \xi_2)| < |p^m|$ for every $m \in \mathbb{N}$. Since there are no infinitesimals in K_1 we have $F(\overline{\rho}, \xi_1, \xi_2) = 0$.] This completes the proof of the lemma. $\qquad\square$

Remark 4.9. Here is an alternative proof of Lemma 4.8 that may be of some interest. Let $\mathbb{R}^* \supset \mathbb{R}$ be the non-principal ultraproduct of $\mathbb{R}$. Let $\mathbb{R}_f^* := \{x \in \mathbb{R}^* : \exists y \in \mathbb{R} \ (|x| \le |y|)\}$ be the finite part of $\mathbb{R}^*$, and $I_{\inf} := \{x \in \mathbb{R}^* : \forall n \in \mathbb{N} \ (|y| < 1/n)\}$ be the ideal of infinitesimals in $\mathbb{R}_f^*$. Let $\mathbb{R}_1 = \mathbb{R}_f^*/I_{\inf}$. Then K_1 is a normed field with norm $|\cdot|_1 \to \mathbb{R}_1$ and K_1 is complete in $|\cdot|_1$, i.e. if $|a_i|_1 \to 0$ then $\sum_i a_i \in K_1$. Consider the norm $|\cdot|_2 : K_1 \to \mathbb{R}$ defined by $|x|_2 :=$ the standard part of $|x|_1$. Denote K_1 with norm $|\cdot|_2$ by K_2. Observe that for $a_i \in K_1$ we have $|a_i|_1 \to 0$ if and only if $|a_i|_2 \to 0$, so K_2 is also complete. Let $K_2^\circ := \{x \in K_2 : |x|_2 \le 1\} = \{x \in K_1 : \forall n \in \mathbb{N} \ (|x|_1 < 1 + 1/n)\}$. Then $K_2^\circ \supsetneq K_1^\circ = \{x \in K_1 : |x|_1 \le 1\}$. Both K_1 and K_2 are algebraically closed valued fields with analytic $\mathcal{A}$-structure, say via σ_1 and σ_2 respectively. If $g \in A_m$ and $x_1, \ldots, x_m \in K_1^\circ$ then $\sigma_1(g)(x_1, \ldots, x_m) = \sigma_2(g)(x_1, \ldots, x_m)$ and $|\sigma_2(g)(x_1, \ldots, x_m)|_2 =$ the standard part of $|\sigma_1(g)(x_1, \ldots, x_m)|_1$. However, if at least one of the x_i is in $K_2^\circ \setminus K_1^\circ$ then $\sigma_1(g)(x_1, \ldots, x_m) = 0$. Consider the lines

$$\begin{aligned} X(\overline{\rho}) &:= \{(\overline{\rho}, \xi_1, \xi_2) : \xi_1 + z_1(\overline{\rho})\xi_2 = f(z_1(\overline{\rho}))\}, \\ \overline{X}(\overline{\rho}) &:= \{(\overline{\rho}, \xi_1, \xi_2) : \xi_1 + z_2(\overline{\rho})\xi_2 = f(z_2(\overline{\rho}))\}. \end{aligned}$$

There is a $|\cdot|_2$-preserving K-automorphism π of K_2 that leaves $\overline{\rho}$ fixed but interchanges $z_1(\overline{\rho})$ and $z_2(\overline{\rho})$. Since $|\pi(x)|_2 = |x|_2$ we have $\|\pi(x)|_1 - |x|_1| \in I_{\inf}$. Suppose $F(\rho, \xi_1, \xi_2) \in \mathcal{O}_{\mathcal{W}_n(\overline{\alpha})}$ is such that $\sigma_1(F)(\overline{\rho}, \xi_1, \xi_2)$ vanishes on $X(\overline{\rho})|_{\mathcal{W}_n(\overline{\alpha})}$. Then also $\sigma_2(F)(\overline{\rho}, \xi_1, \xi_2)$ vanishes on $X(\overline{\rho})|_{\mathcal{W}_n(\overline{\alpha})}$, and hence $\sigma_2(F)(\overline{\rho}, \xi_1, \xi_2)$ vanishes on $\pi(X(\overline{\rho})|_{\mathcal{W}_n(\overline{\alpha})}) = \overline{X}(\overline{\rho})|_{\mathcal{W}_n(\overline{\alpha})}$. Now for $(\overline{\rho}, \xi_1, \xi_2) \in \mathcal{W}_n(\overline{\alpha})$ we have $|\xi_i^{(n)}(\overline{\rho})|_1 = |p/\overline{\rho}| \le |p|^{1/2}$ and $|\xi_i - \xi_i^{(n)}(\overline{\rho})|_1 \le 1$. Hence also $|\pi(\xi_i)|_1 \le 1$, so we conclude that $\sigma_1(F)(\overline{\rho}, \xi_1, \xi_2)$ vanishes on $\overline{X}(\overline{\rho})|_{\mathcal{W}_n(\overline{\alpha})}$.

Lemma 4.10. *Let $F(\rho, \xi_1, \xi_2) \in \mathcal{O}_{\mathcal{W}_n(\overline{\alpha})}$ and suppose that $F(\overline{\rho}, \overline{\xi}_1, \overline{\xi}_2) = 0$ and $\frac{\partial F}{\partial \xi_1}(\overline{\rho}, \overline{\xi}_1, \overline{\xi}_2) \neq 0$. Then there are $n', n'' \in \mathbb{N}$ and an $\overline{\alpha}'$ such that $F(\rho, \xi_1, \xi_2)$ as an element of $\mathcal{O}_{\mathcal{W}_{n,n''}(\overline{\alpha}')}$ is regular of degree 1 in ξ_1, after division by a suitable unit. Hence there is a $\widehat{\xi}_1 = \widehat{\xi}_1(\rho, \xi_2) \in \mathcal{O}_{\mathcal{W}_{n''}^2(\overline{\alpha}')}$ such that $F(\rho, \widehat{\xi}_1(\rho, \xi_2), \xi_2) = 0$ as an element of $\mathcal{O}_{\mathcal{W}_{n''}(\overline{\alpha}')}$. The same holds with the roles of ξ_1 and ξ_2 interchanged.*

Proof. For strictly convergent g, let $g^{(n')}$ be a polynomial that is congruent to g modulo $p^{n'}$, i.e. the truncation of g at size $p^{n'}$. Since $\frac{\partial F}{\partial \xi_1}(\overline{\rho}, \overline{\xi}_1, \overline{\xi}_2) \neq 0$, we have $\left| \frac{\partial F}{\partial \xi_1}(\overline{\rho}, \overline{\xi}_1, \overline{\xi}_2) \right|^2 > |p^{n'}|$ for some n'. Then $\left| \frac{\partial F^{(n')}}{\partial \xi_1}(\overline{\rho}, \overline{\xi}_1^{(n')}, \overline{\xi}_2^{(n')}) \right|^2 > |p^{n'}|$. Observe that

$$\frac{\partial F^{(n')}}{\partial \xi_1}(\overline{\rho}, \overline{\xi}_1^{(n')}, \overline{\xi}_2^{(n')}) \in \frac{1}{\overline{\rho}^{n''!}} K^{\circ}[\overline{\rho}]$$

for some n''. Hence for suitable $n'' \geq n'$ and $\overline{\alpha}'$,

$$\left| \frac{\partial F^{(n')}}{\partial \xi_1}(\rho, \xi_1, \xi_2) \right|^2 \geq |p^{n'}|$$

for all $(\rho, \xi_1, \xi_2) \in \mathcal{W}_{n''}(\overline{\alpha}')$. On $\mathcal{W}_{n',n''}(\overline{\alpha}')$ we can write $\xi_1 = \xi_1^{(n'')}(\rho) + p^{n'}\eta_1$ and $\xi_2 = \xi_2^{(n'')}(\rho) + p^{n''}\eta_2$, where the η_i are new variables. Indeed, on the set $\{\rho : |p| \leq |\rho^{n''!}| \leq 1\}$ we have $|\xi_1^{(n')}(\rho) - \xi_1^{(n'')}(\rho)| < |p^{n'}|$, so taking $\xi_1^{(n'')}(\rho)$ as the "center" of the disc $|\xi_1 - \xi_1^{(n')}| \leq |p^{n'}|$ instead of $\xi_1^{n'}(\rho)$ does not change the rational domain or its ring of functions (Remark 4.6). Writing $F(\rho, \xi_1(\rho, \eta_1), \xi_2(\rho, \eta_2))$ as a power series in η_1 and dividing by the unit

$$p^{n'} \frac{\partial F}{\partial \xi_1}(\rho, \xi_1^{(n'')}, \xi_2^{(n'')} + p^{n''}\eta_2),$$

which is the coefficient of η_1, makes F regular of degree 1 in η_1. The existence of $\widehat{\xi}_1(\rho, \xi_2)$ follows by Weierstrass Preparation. $\qquad\square$

We now proceed with the proof that X is not $\mathcal{L}_{\mathcal{A}}^D$-semianalytic on any $\mathcal{W}_n(\overline{\alpha})$.

Claim 1. *X is not $\mathcal{L}_{\mathcal{A}}$-semianalytic on any $\mathcal{W}_n(\overline{\alpha})$.*

Suppose it were. Then there would be an $F \in \mathcal{O}_{\mathcal{W}_n(\overline{\alpha})}$ such that $X(\overline{\rho})|_{\mathcal{W}_n(\overline{\alpha})} \subset V_F$. Since $X(\overline{\rho})|_{\mathcal{W}_n(\overline{\alpha})}$ is a line, we may assume that F is irreducible in $\mathcal{O}_{\mathcal{W}_n(\overline{\alpha})}$ and in $\mathcal{O}_{\mathcal{W}_{n'}(\overline{\alpha}')}$ for all $n' \geq n$ and all $\overline{\alpha}'$. By Lemma 4.8, also $\overline{X}(\overline{\rho})|_{\mathcal{W}_n(\overline{\alpha})} \subset V_F$. Hence the curve $V_F(\overline{\rho})$ in $\mathcal{W}_n$ defined by $F(\overline{\rho}, \xi_1, \xi_2) = 0$ would be singular at $(\overline{\rho}, \overline{\xi}_1, \overline{\xi}_2)$. Increasing n if necessary, we may assume that $(\overline{\rho}, \overline{\xi}_1, \overline{\xi}_2)$ is the only singular point on the curve $F(\overline{\rho}, \xi_1, \xi_2) = 0$ in $\mathcal{W}_n(\overline{\alpha})$. But then, since a point is a smooth 0-dimensional variety, by [LR6] or [Ce1], for some $n' \geq n$ and some $\overline{\alpha}'$ we would deduce that $\overline{\xi}_1$ and $\overline{\xi}_2$ are $\mathcal{O}_{\mathcal{W}_n(\overline{\alpha}')}$-analytic functions of $\overline{\rho}$. But this is not the case by Remark 4.4.

Alternatively we can see this as follows: For some suitable derivative G of F the curve $V_G(\overline{\rho}, \xi_1, \xi_2)$ contains the point $(\overline{\rho}, \overline{\xi}_1, \overline{\xi}_2)$ and is smooth at that point. If $V_G \not\subset V_F$, by

Lemma 4.10 we can solve $G = 0$ for ξ_1, say, as an analytic function $\widehat{\xi}_1(\rho, \xi_2)$ of ρ and ξ_2, and substitute this function into F to get $F(\rho, \widehat{\xi}_1(\rho, \xi_2), \xi_2) \not\equiv 0$. Use the equation $F(\rho, \widehat{\xi}_1(\rho, \xi_2), \xi_2) = 0$ and Lemma 4.10 to solve for $\widehat{\xi}_2$ as an analytic function of ρ, contradicting Remark 4.4. If $V_G \subset V_F$, since $V_G(\overline{\rho})$ is smooth, we know that $X(\overline{\rho})|_{\mathcal{W}_n(\overline{\alpha})} \not\subset V_G$, and hence $V_F \neq V_G$. Then, since G is (strictly convergent) regular in ξ_1 of degree 1, by (strictly convergent) Weierstrass Division, G would divide F, contradicting the fact that F is irreducible.

Claim 2. *X is not $\mathcal{L}_{\mathcal{A}}^D$-semianalytic.*

Suppose it were. Choose an $\mathcal{L}_{\mathcal{A}}^D$-semianalytic representation on a $\mathcal{W}_n(\overline{\alpha})$ of lowest possible complexity—i.e. using the smallest number of non-analytic D-functions (necessarily ≥ 1 by Claim 1) and with innermost D-functions reduced (i.e. with numerator and denominator having no common factor). Let $D(g, h)$ be an innermost D-function that is not analytic. Then $g, h \in \mathcal{O}_{\mathcal{W}_n(\overline{\alpha})}$ and necessarily $g(\overline{\rho}, \overline{\xi}_1, \overline{\xi}_2) = h(\overline{\rho}, \overline{\xi}_1, \overline{\xi}_2) = 0$, and g and h have no common factor in any $\mathcal{O}_{\mathcal{W}_n'(\overline{\alpha}')}$. Considering the two curves $\mathcal{C}_g := \{(\overline{\rho}, \xi_1, \xi_2) : g(\overline{\rho}, \xi_1, \xi_2) = 0\}$ and $\mathcal{C}_h := \{(\overline{\rho}, \xi_1, \xi_2) : h(\overline{\rho}, \xi_1, \xi_2) = 0\}$, we now proceed as in the proof of Claim 1 to show that $\overline{\xi}_1, \overline{\xi}_2$ are analytic functions of $\overline{\rho}$, again contradicting Remark 4.4. This completes the proof of Theorem 4.3.

Remark 4.11. The set X is clearly both $\mathcal{L}_{\text{sep}}$-semianalytic and $\mathcal{L}_{\mathcal{A}}^h$-semianalytic. It is also $\mathcal{L}_{\mathcal{A}}$-semianalytic in a neighborhood of every point of K_{alg}. Indeed, it is $\mathcal{L}_{\mathcal{A}}$-semi-analytic on the sets described by $|\rho| = 1$ and $|\rho| \leq 1 - \delta$, for every $\delta < 1, \delta \in \mathbb{R}$.

Appendix A. Some comments on analytic structures from [CL]

In this appendix we recall some definitions from [CL], sometimes in a slightly modified form, we give some complements and small amendments to [CL], notably in Definition A.1.3, Lemma A.1.5, and Remarks A.1.8, A.1.9, A.1.12. We also give some supplementary discussion of various topics—other strong noetherian properties, extension of parameters, and Weierstrass systems with side conditions.

A.1.

Let us first recall the notion of separated pre-Weierstrass system of [CL] ("separated" means that the power series involve two kinds of variables, see Remark 1.0.1).

Definition A.1.1 (Separated pre-Weierstrass system, cf. [CL, Definitions 4.1.2, 4.1.3 and 4.1.5]). Let B be a ring and $I \subset B$ a proper ideal. Let $B_{m,n} \subset B[[\xi, \rho]]$ be B-algebras. We call $\mathcal{B} := \{B_{m,n}\}_{m,n \in \mathbb{N}}$ a (B, I)-*separated pre-Weierstrass system* if it satisfies the following conditions (1)–(6) for all m, n, m', n'. (Let $m \leq m'$ and $n \leq n'$ be natural numbers, and $\xi = (\xi_1, \ldots, \xi_m)$, $\xi' = (\xi_1, \ldots, \xi_{m'})$, $\xi'' = (\xi_{m+1}, \ldots, \xi_{m'})$, $\rho = (\rho_1, \ldots, \rho_n)$, $\rho' = (\rho_1, \ldots, \rho_{n'})$, and $\rho'' = (\rho_{n+1}, \ldots, \rho_{n'})$ be variables.)

(1) $B_{0,0} = B$,
(2) $B_{m,n} \subset B[[\xi, \rho]]$ and is closed under any permutation of the ξ_i or of the ρ_j,
(3) $B_{m,n}[\xi'', \rho''] \subset B_{m',n'}$,

(4) the image $(B_{m,n})^{\sim}$ of $B_{m,n}$ under the residue map $\sim : B[[\xi, \rho]] \rightarrow \widetilde{B}[[\xi, \rho]]$ is a subring of $\widetilde{B}[\xi][[\rho]]$, and

(5) if $f \in B_{m',n'}$, say $f = \sum_{\mu\nu} \overline{f}_{\mu\nu}(\xi, \rho)(\xi'')^{\mu}(\rho'')^{\nu}$, then the $\overline{f}_{\mu\nu}$ are in $B_{m,n}$,

(6) The two usual Weierstrass Division Theorems hold in the $B_{m,n}$, namely, for $f, g \in B_{m,n}$:

    (a) If f is regular in ξ_m of degree d (Definition 3.3.2(i)), then there exist unique $q \in B_{m,n}$ and $r \in B_{m-1,n}[\xi_m]$ of degree at most $d - 1$ such that $g = qf + r$.

    (b) If f is regular in ρ_n of degree d (Definition 3.3.2(ii)), then there exist unique $q \in B_{m,n}$ and $r \in B_{m,n-1}[\rho_n]$ of degree at most $d - 1$ such that $g = qf + r$.

Since (A, I) is quite general, there may be pairs (a, b) in A whose relative size is not determined by (A, I). In each field with analytic $\mathcal{A}$-structure, say via σ, the relative size of $\sigma(a)$ and $\sigma(b)$ is determined. We handle this situation by means of Definition A.1.2 (and Definition A.1.4). In the case that $A = K^{\circ}$ and $I = K^{\circ\circ}$ with K a valued field this complication does not arise. In the case of Theorem 3.5.1 we can also avoid this complication by using the stronger property (7$'$) of Definition A.2.1 in place of Definition A.1.4.

    We recall the following definitions from [CL].

Definition A.1.2 (Rings of fractions). Let $\mathcal{A} = \{A_{m,n}\}_{m,n\in\mathbb{N}}$ be a separated (A, I)-pre-Weierstrass system. We inductively define the concept of an A-algebra C being a *ring of $\mathcal{A}$-fractions with proper ideal C°* and with *rings $C_{m,n}$ of separated power series "over"* C:

 (i) The ring A is a ring of $\mathcal{A}$-fractions with ideal $A^{\circ} = I$ and with rings of separated power series the $A_{m,n}$ from the system $\mathcal{A}$.

 (ii) If B is a ring of $\mathcal{A}$-fractions and d in B satisfies $C^{\circ} \neq C$, with

$$C := B/dB, \quad C^{\circ} := B^{\circ}/dB,$$

    then C is a ring of $\mathcal{A}$-fractions with proper ideal C° and $C_{m,n} := B_{m,n}/dB_{m,n}$.

(iii) If B is a ring of $\mathcal{A}$-fractions and c, d in B satisfy $C^{\circ} \neq C$ with

$$C = B\langle c/d \rangle := B_{1,0}/(d\xi_1 - c), \quad C^{\circ} := (B^{\circ})B\langle c/d \rangle$$

    then C is a ring of $\mathcal{A}$-fractions with proper ideal C° and $C_{m,n} := B_{m+1,n}/(d\xi_1 - c)$.

(iv) If B is a ring of $\mathcal{A}$-fractions and c, d in B satisfy $C^{\circ} \neq C$, with

$$C = B[[c/d]]_{\mathrm{s}} := B_{0,1}/(d\rho_1 - c), \quad C^{\circ} := (B^{\circ}, \rho_1)B_{0,1}/(d\rho_1 - c),$$

    and (B°, ρ_1) the ideal generated by B° and ρ_1, then C is a ring of $\mathcal{A}$-fractions with proper ideal C° and $C_{m,n} := B_{m,n+1}/(d\rho_1 - c)$.

Definition A.1.3. In Definition A.1.2 we put the word "over" in quotes because $C_{m,n}$ is not in general a ring of power series with coefficients from C. However, if $f \in C_{m,n}$, the Taylor coefficients of f are well defined as elements of C by repeated Weierstrass Division. In other words, for each (μ, ν) there is a unique $\overline{c}_{\mu,\nu} \in C$ such that for $\mu < \mu'$ and $\nu < \nu'$, $\overline{c}_{\mu,\nu}$ is the coefficient of $\xi^{\mu}\rho^{\nu}$ in f mod $(\xi^{\mu'}, \rho^{\nu'})$. We can define the *series of f* by $\mathcal{S}(f) := \sum \overline{c}_{\mu,\nu}\xi^{\mu}\rho^{\nu} \in C[[\xi, \rho]]$.

    In all cases of Definition A.1.2 define $C_{m,n}^{\circ} := \{f \in C_{m,n} : \overline{c}_{\mu,0} \in C^{\circ} \text{ for all } \mu\}$.

Definition A.1.4. We call a separated pre-Weierstrass system $\mathcal{B}$ (Definition A.1.1) a *separated Weierstrass system* if

(7) For every ring C of $\mathcal{B}$-fractions, if $f \in C_{m,n}$ and $\mathcal{S}(f) = \sum_{\mu,\nu} \overline{c}_{\mu\nu}\xi^\mu\rho^\nu$, then there is a finite set $J \subset \mathbb{N}^{m+n}$ and for each $(\mu, \nu) \in J$ there is a $g_{\mu\nu} \in C^\circ_{m,n}$ such that

$$f = \sum_{(\mu,\nu)\in J} \overline{c}_{\mu\nu}\xi^\mu\rho^\nu(1 + g_{\mu\nu}).$$

Lemma A.1.5. *If $\mathcal{B} = B_{m,n}$ is a separated Weierstrass system, then for every ring C of $\mathcal{B}$-fractions, $C_{m,n}$ is indeed a ring of power series over C.*

Proof. Property (7) above guarantees that the homomorphism $f \mapsto \mathcal{S}(f) = \sum \overline{c}_{\mu,\nu}\xi^\mu\rho^\nu$ is an isomorphism. $\qquad\square$

In the light of this lemma we will henceforth identify $C_{m,n}$ with $\mathcal{S}(C_{m,n})$.

The treatment in [CL] implicitly assumed that $C_{m,n}$ is a power series ring over C. The treatment here is better in that it omits that assumption, and the statement of the strong noetherian property uses $\mathcal{S}(f)$ to define the $\overline{c}_{\mu,\nu}$. (Definition 4.1.5(c) of [CL] should be replaced by Definition A.1.4 above.) Once we have assumed property (7), the distinction between $C_{m,n}$ and $\mathcal{S}(C_{m,n})$ disappears and $C_{m,n}$ is indeed a power series ring, as observed above.

Restating (7) in our context, we have:

Lemma A.1.6 (Criterion). *A good strictly convergent pre-Weierstrass system $\mathcal{A} = \{A_m\}$ is a strictly convergent Weierstrass system if (and only if) $\mathcal{A}^H$ satisfies condition (7) of Definition A.1.4, i.e. if*

(vii) *For every ring C of $\mathcal{A}^H$-fractions, if $f \in C_{m,n}$ and $\mathcal{S}(f) = \sum_{\mu,\nu} \overline{c}_{\mu\nu}\xi^\mu\rho^\nu$, then there is a finite set $J \subset \mathbb{N}^{m+n}$ and for each $(\mu, \nu) \in J$ there is a $g_{\mu\nu} \in C^\circ_{m,n}$ such that*

$$f = \sum_{(\mu,\nu)\in J} \overline{c}_{\mu\nu}\xi^\mu\rho^\nu(1 + g_{\mu\nu}).$$

Remark A.1.7. It does not follow that a field F with strictly convergent analytic $\mathcal{A}$-structure is necessarily henselian. See the example in [CL, Remark 4.5.13]. This is why we must require that the fields with analytic $\mathcal{A}$-structure be henselian.

Remark A.1.8. We amend Definition 4.5.14 and Theorem 4.5.15 of [CL]. In Definition 4.5.14 of [CL], the order relation $<$ on the value group sort should be part of the language $\mathcal{L}_\mathcal{A}$, in order for the quantifier elimination result of Theorem 4.5.15 to make sense.

Remark A.1.9. The alternative proof of quantifier elimination, based on compactness, that is given in Section 6 of [CL] (in the proof of Theorem 6.3.7) is not well-presented. In order to obtain a proof along these lines, the results used ([CLR1, Lemma 3.16] and [CL, Theorem 5.5.3]) should be made more uniform, allowing variables as well as constants to occur in the terms. Proposition A.1.10 below is the needed uniform version of [CL, Theorem 5.5.3].

Proposition A.1.10. *Let $\mathcal{A}$ be a separated Weierstrass system and let K be a valued field with separated analytic $\mathcal{A}$-structure. Let $\mathcal{L}_{\mathcal{A}(K)}$ be the language of valued fields, $\langle 0, 1, +, \cdot, (\cdot)^{-1}, |\cdot|\rangle$, augmented with function symbols for all the elements of $\bigcup_{m,n} A_{m,n}(K)$. (We extend functions $f \in A_{m,n}(K)$ by zero outside $(K_{\mathrm{alg}}^{\circ})^m \times (K_{\mathrm{alg}}^{\circ\circ})^n$.) Let x be several variables and y one variable, and let $\tau(x, y)$ be a term of $\mathcal{L}_{\mathcal{A}(K)}$. There is a finite set $\chi_\alpha(x)$ of quantifier-free $\mathcal{L}_{\mathcal{A}(K)}$-formulas such that $K_{\mathrm{alg}}^{\circ} \models \bigvee_\alpha \chi_\alpha$ (i.e. the sets $\{x : \chi_\alpha(x)\}$ cover $(K_{\mathrm{alg}}^{\circ})^n$) and for each α there is a finite family of quantifier-free formulas $\varphi_{\alpha,i}(x, y)$ that depend on y in a polynomial way (i.e. are boolean combinations of atomic formulas of the form $p(x, y) = 0$ and $|p(x, y)| \leq |q(x, y)|$, where the $p(x, y), q(x, y)$ are polynomials in y with coefficients $\mathcal{L}_{\mathcal{A}(K)}$-terms in x) such that, writing $\mathcal{U}_{\alpha,i} = \mathcal{U}_{\alpha,i}(x)$ for the set $\{y : \varphi_{\alpha,i}(x, y)\}$, for each α the $\mathcal{U}_{\alpha,i}(x)$ cover K_{alg}°, and for each α and i there are rational functions $R_{\alpha,i}(y)$ with coefficients $\mathcal{L}_{\mathcal{A}(K)}$-terms in x, and terms $E_{\alpha,i}(x, y)$ that are strong units on $\mathcal{U}_{\alpha,i}$ such that for x satisfying χ_α we have*

$$\tau(x, y)|_{\mathcal{U}_{\alpha,i}} = E_{\alpha,i} \cdot R_{\alpha,i}|_{\mathcal{U}_{\alpha,i}}. \tag{A.1}$$

[See [CL, Definition 5.1.4] for the definition of "strong unit". Here we slightly generalize that definition. The size of a strong unit $E_{\alpha,i}(x, y)$ is determined by a term $c_{\alpha,i}(x)$ satisfying $|E_{\alpha,i}(x, y)^\ell| = |c_{\alpha,i}(x)|$ for all $y \in \mathcal{U}_{\alpha,i}$ and there is a polynomial $P_{\alpha,i}$ with coefficients $\mathcal{L}_{\mathcal{A}(K)}$-terms in x such that $\|P_{\alpha,i}\| = 1$ and $P_{\alpha,i}(c_{\alpha,i}^{-1}(E_{\alpha,i})^\ell)^\sim = 0$ for all $y \in \mathcal{U}_{\alpha,i}(x)$. So $c_{\alpha,i}$ and $P_{\alpha,i}$ determine the finitely many possible angular components of $E_{\alpha,i}$.]

Proof. Let K^* be a $|\mathcal{L}_{\mathcal{A}(K)}|^+$-saturated $\mathcal{L}_{\mathcal{A}(K)}$-elementary extension of K_{alg}. Let $\mathcal{D}(x)$ be the algebra of $\mathcal{L}_{\mathcal{A}(K)}$-terms in the variables x. Then for each $\overline{x} \in (K^*)^n$, we know that $K \subset \mathcal{D}(\overline{x})$ is a field with analytic $\mathcal{A}(K)$-structure, so [CL, Theorem 5.5.3] applies and there is a finite set of annuli $\mathcal{U}_{\overline{x},i}$ defined by annulus formulas $\varphi_{\overline{x},i}$ [CL, Definition 5.1.1] and rational functions $R_{\overline{x},i}(y)$, strong units $E_{\overline{x},i}(y)$ and finite exceptional sets $S_{\overline{x},i}$ satisfying

$$\tau(\overline{x}, y)|_{\mathcal{U}_{\overline{x},i} \setminus S_{\overline{x},i}} = E_{\overline{x},i} \cdot R_{\overline{x},i}|_{\mathcal{U}_{\overline{x},i} \setminus S_{\overline{x},i}}. \tag{A.2}$$

Observe that the exceptional sets arise from common zeros of numerators and denominators (of subterms in τ) and at these points the term becomes simpler, so the same induction on terms establishes the analogous representation on the exceptional sets, which are defined by polynomial equations in y:

$$\tau(\overline{x}, y)|_{S_{\overline{x},i,j}} = E'_{\overline{x},i,j} \cdot R'_{\overline{x},i,j}|_{S_{\overline{x},i,j}}, \tag{A.3}$$

where $S_{\overline{x},i} = \bigcup_j S_{\overline{x},i,j}$. Hence, in the formalism of the proposition there is no need to consider exceptional sets separately, and we may assume that we have quantifier-free formulas $\varphi_i(\overline{x}, y)$ depending on y in a polynomial way, and terms $E_{\overline{x},i}$ that are strong units, and rational functions $R_{\overline{x},i}$ such that, for all i and all $y \in \mathcal{U}_{\overline{x},i}$ (the set defined by $\varphi_i(\overline{x}, y)$),

$$\tau(\overline{x}, y)|_{\mathcal{U}_{\overline{x},i}} = E_{\overline{x},i} \cdot R_{\overline{x},i}|_{\mathcal{U}_{\overline{x},i}}. \tag{A.4}$$

Let $E^*_{\overline{x},i}(x, y)$, $R^*_{\overline{x},i}(x, y)$ be terms in $\mathcal{D}(x, y)$ such that $E^*_{\overline{x},i}(\overline{x}, y) = E_{\overline{x},i}(y)$ and $R^*_{\overline{x},i}(\overline{x}, y) = R_{\overline{x},i}(y)$, and similarly for other terms such as the $c_{\overline{x},i}$ and $P_{\overline{x},i}$ and the terms occurring in $\varphi_{\overline{x},i}$ implicit in the description of A.4. Denote the resulting formula corresponding to $\varphi_{\overline{x},i}$ by $\varphi^*_{\overline{x},i}(x, y)$. Let $\Delta_{\overline{x}}(x)$ be the $\mathcal{L}_{\mathcal{A}(K)}$-formula that asserts that for all i,

$$\tau(x, y)|_{\mathcal{U}_{x,i}} = E^*_{\overline{x},i}(x, y) \cdot R^*_{\overline{x},i}(x, y)|_{\mathcal{U}_{x,i}}, \tag{A.5}$$

together with the implied background description about $E^*_{\overline{x},i}$ being strong units and $\mathcal{U}_{x,i}$ being the sets defined by the formulas $\varphi^*_{\overline{x},i}(x, y)$. Let $\chi_{\overline{x}}(x)$ be a quantifier-free $\mathcal{L}_{\mathcal{A}(K)}$-formula equivalent to $\Delta_{\overline{x}}(x)$ in K^*_{alg}. We know that $\chi_{\overline{x}}(x)$ exists by *algebraically closed* quantifier elimination (see Remark 3.2.12). Since $K^* \models \chi_{\overline{x}}(\overline{x})$, by the saturation of K^* the formulas $\chi_{\overline{x}}(x)$ cover the Stone space of $\mathcal{L}_{\mathcal{A}(K)}$-$n$-types, and by the compactness of that Stone space, finitely many of the $\chi_{\overline{x}}(x)$ cover, say the $\chi_\alpha(x)$ for α in some finite subset of $K^{*\circ}_{\text{alg}}$. The proposition now follows. $\square$

In the case of *algebraically closed* fields with separated analytic structure, one can say much more than Proposition 3.4.5 above about definable functions.

Proposition A.1.11. (i) (characteristic 0) *Let $\mathcal{A}$ be a separated Weierstrass system and let $x = (x_1, \ldots, x_n)$ be several variables and y one variable. Suppose that a formula $\varphi(x, y) \in \mathcal{L}_{\mathcal{A}}$ defines y as a function of x in all algebraically closed fields of characteristic 0 with analytic $\mathcal{A}$-structure. Then there is a term τ of $\mathcal{L}_{\mathcal{A}}$ such that $L \models \varphi(x, y) \leftrightarrow y = \tau(x)$ for all algebraically closed valued fields L of characteristic 0 with analytic $\mathcal{A}$-structure.*

(ii) (characteristic $p \neq 0$) *In characteristic p let $\mathcal{L}'_{\mathcal{A}}$ be the language $\mathcal{L}_{\mathcal{A}}$ with the p-th root function $(\cdot)^{1/p}$ adjoined. The analogous result to* (i) *holds, with $\mathcal{L}'_{\mathcal{A}}$ in place of $\mathcal{L}_{\mathcal{A}}$, for all algebraically closed valued fields L of characteristic p with analytic $\mathcal{A}$-structure.*

Proof. (i) We may assume that φ is quantifier-free. Suppose, for sake of contradiction, that there is no such term τ. Then, since we can combine terms defined by cases into one term, for any finite set of terms $\tau_i(x), i = 1, \ldots, k$, there is an x such that $\models \bigwedge_{i=1}^k \neg\varphi(x, \tau_i(x))$. Hence there is an algebraically closed field L with analytic $\mathcal{A}$-structure and $x \in L^k$ such that $\models_L \neg\varphi(x, \tau(x))$ for all terms $\tau(x)$. Let $\Delta(x) := \{\tau(x) : \tau$ an $\mathcal{L}_{\mathcal{A}}$-term$\} \subset L$. Let L' be the field of fractions of $\Delta(x)$. Then $L' \subset L$ is a field with analytic $\mathcal{A}$-structure and this structure extends uniquely to an analytic $\mathcal{A}$-structure on L'_{alg}. Hence there is a unique $y \in L'_{\text{alg}} \setminus L$ such that $\models_{L'} \varphi(x, y)$. But, since L' is perfect, there is an analytic $\mathcal{A}$-structure automorphism of L' that leaves the elements of L fixed but moves y, contradicting the uniqueness of y. (Indeed, since $\mathcal{A}$ is separated, L' is henselian, and by [CL, Theorem 4.5.11] every field automorphism of L' over L is an automorphism of fields with analytic $\mathcal{A}$-structure.)

(ii) Observe that every formula of $\mathcal{L}'_{\mathcal{A}}$ is equivalent to an $\mathcal{L}_{\mathcal{A}}$-formula, and if we define $\Delta'(x) := \{\tau(x) : \tau$ an $\mathcal{L}'_{\mathcal{A}}$-term$\}$, and L' as the field of fractions of $\Delta'(x)$, then L' is a perfect subfield of L, so the same argument works. $\square$

Remark A.1.12. We amend Lemmas 6.3.12 and 6.3.14 of [CL]. In fact, with notation of [CL, Lemma 6.3.12], its proof shows an even stronger conclusion holds: $rv_n(E^\sigma(x))$ only depends on $x \mod (n \cdot K^{\circ\circ}_{\mathrm{alg}})$ when x varies over K°_{alg}. It is in fact this stronger form of Lemma 6.3.12 that is used to prove [CL, Lemma 6.3.14]. We thank S. Rideau for pointing this out to us.

Remark A.1.13. In [CL, Section 4.3] we gave a two-case definition of *strictly convergent Weierstrass system* $\mathcal{A}$ and *(henselian) field with analytic $\mathcal{A}$-structure.* Case (i) (denoted $(\pi = 1)$) of that definition clearly fits into the framework of this paper as an analytic $\mathcal{A}$-structure with side conditions (namely that π be a prime element of K°)—see Definitions A.4.1 and A.4.2 below. In case (ii) (denoted $(\pi \ne 1)$ in [CL, Section 3.3]) we considered all rings C of fractions coming from a *separated* Weierstrass system containing $\mathcal{A}$. In general, some of the elements of such a C will not be definable in the language $\mathcal{L}_\mathcal{A}$. The treatment in this paper is more natural in that all the elements and functions that arise are existentially definable in $\mathcal{L}_\mathcal{A}$ (see Lemma 3.2.8), and the definition depends only on the pre-Weierstrass system $\mathcal{A} = \{A_m\}$, not on some (unspecified) separated Weierstrass system containing $\mathcal{A}$.

A.2. A discussion of strong noetherian properties

In this subsection we discuss variants of Definitions A.1.4 and A.1.6

Conditions (7)′ and (vii)′ below easily imply conditions (7) of Definition A.1.4 and (vii) of Definition A.1.6 and are easier to use in Theorem 3.5.1.

Definition A.2.1. Let $\mathcal{B} = \{B_{m,n}\}$ be a pre-Weierstrass system, let $m \le m'$, $n \le n'$, and let $\xi = (\xi_1, \ldots, \xi_m)$, $\rho = (\rho_1, \ldots, \rho_n)$, $\xi' = (\xi_1, \ldots, \xi_{m'})$, $\rho' = (\rho_1, \ldots, \rho_{n'})$, $\xi'' = (\xi_{m+1}, \ldots, \xi_{m'})$ and $\rho'' = (\rho_{n+1}, \ldots, \rho_{n'})$.

(7)′ Let

$$f = \sum_{\mu,\nu} \overline{f}_{\mu\nu}(\xi, \rho)(\xi'')^\mu (\rho'')^\nu \in B_{m',n'}$$

where the $\overline{f}_{\mu\nu}(\xi, \rho)$ are in $B_{m,n}$. There is a finite set $J \subset \mathbb{N}^{m'-m+n'-n}$ and for each $(\mu, \nu) \in J$ a function $g_{\mu\nu} \in (B_{m',n'})^\circ$ such that

$$f = \sum_{(\mu,\nu) \in J} \overline{f}_{\mu\nu}(\xi, \rho)(\xi'')^\mu (\rho'')^\nu (1 + g_{\mu\nu})$$

as an element of $B_{m',n'}$.

(vii)′ Let

$$f = \sum_{\mu,\nu} \overline{f}_{\mu\nu}(\xi, \rho)(\xi'')^\mu (\rho'')^\nu \in A^H_{m',n'}$$

where the $\overline{f}_{\mu\nu}(\xi, \rho)$ are in $A^H_{m,n}$. There is a finite set $J \subset \mathbb{N}^{m'-m+n'-n}$ and for each $(\mu, \nu) \in J$ a function $g_{\mu\nu} \in (A^H_{m',n'})^\circ$ such that

$$f = \sum_{(\mu,\nu) \in J} \overline{f}_{\mu\nu}(\xi, \rho)(\xi'')^\mu (\rho'')^\nu (1 + g_{\mu\nu})$$

as an element of $A^H_{m',n'}$.

The following is immediate.

Lemma A.2.2. (i) *If the separated pre-Weierstrass system $\mathcal{B}$ satisfies condition* (7)′ *above then $\mathcal{B}$ is a separated Weierstrass system.*

(ii) *If the good, strictly convergent pre-Weierstrass system $\mathcal{A}$ satisfies condition* (vii)′ *above then $\mathcal{A}$ is a strictly convergent Weierstrass system.*

A weaker strong noetherian property that is still sufficient for results on quantifier elimination, b-minimality and cell decomposition is (7)* below. See [CL, Section 4.2] for definitions. This definition is [CL, Theorem 4.2.15].

Definition A.2.3. (7)* Let $\mathcal{B} = \{B_{m,n}\}$ be a pre-Weierstrass system, let $m \leq m'$ and $n \leq n'$, let $\xi = (\xi_1, \ldots, \xi_m)$, $\rho = (\rho_1, \ldots, \rho_n)$, $\xi' = (\xi_1, \ldots, \xi_{m'})$, $\rho' = (\rho_1, \ldots, \rho_{n'})$, $\xi'' = (\xi_{m+1}, \ldots, \xi_{m'})$ and $\rho'' = (\rho_{n+1}, \ldots, \rho_{n'})$, and let

$$f = \sum_{\mu,\nu} \overline{f}_{\mu\nu}(\xi, \rho)(\xi'')^{\mu}(\rho'')^{\nu} \in B_{m',n'}$$

where the $\overline{f}_{\mu\nu}(\xi, \rho)$ are in $B_{m,n}$. (The $\overline{f}_{\mu\nu}(\xi, \rho)$ are well defined by Weierstrass Division in $B_{m',n'}$). There is a finite system $\mathcal{F}$ of rings of B-fractions, and for each $B' \in \mathcal{F}$ there is a finite, disjointly covering family of Laurent rings C over $A'_{m,n}$, and for each C there are a finite set J_C and C-units $u_{C\mu\nu} \in C$ and functions $h_{C\mu\nu} \in C^{\circ}_{m'-m,n'-n}$ for $(\mu, \nu) \in J_C$ such that

$$f = \sum_{(\mu,\nu) \in J_C} \overline{f}_{\mu\nu}(\xi, \rho)(\xi'')^{\mu}(\rho'')^{\nu} u_{C\mu\nu}(1 + h_{C\mu\nu})$$

as an element of $C_{m'-m,n'-n}$.

(vii)* The corresponding condition for a strictly convergent pre-Weierstrass system $\mathcal{A}$ is that the separated pre-Weierstrass system $\mathcal{A}^H$ satisfies condition (7)*.

Definition A.2.4. The arguments in [CL, Section 4.2] (Lemma 4.2.14 and Theorem 4.2.15) show that the following less cumbersome condition implies condition (7)*:

(7)″ For any

$$f = \sum_{\mu,\nu} \overline{c}_{\mu\nu}(\xi)^{\mu}(\rho)^{\nu} \in B_{m,n}$$

there is a system $\mathcal{F}$ of rings of $\mathcal{B}$-fractions, and for each $B' \in \mathcal{F}$ there is a finite set $J_{B'}$ and functions $g_{\mu\nu} \in (B'_{m,n})^{\circ}$ for $(\mu, \nu) \in J_{B'}$ such that

$$f = \sum_{(\mu,\nu) \in J_{B'}} \overline{c}_{\mu\nu} \xi^{\mu} \rho^{\nu}(1 + g_{\mu\nu})$$

as an element of $B'_{m,n}$.

We do not know an example of a good pre-Weierstrass system $\mathcal{A}$ such that $\mathcal{A}^H := \{A^H_{m,n}\}$ is a separated Weierstrass system but does not satisfy condition (vii)′ of Definition A.2.1. Indeed, we do not have examples that distinguish among the various strong noetherian properties discussed in this section.

A.3. Extension of parameters

Let $\mathcal{B}$ be a separated pre-Weierstrass system and let K be a field with analytic $\mathcal{B}$-structure via σ. As in [CL, Definition 4.5.6(i)], assume that $\sigma_0 : A \to K^\circ$ is an embedding, and define

$$B_{m,n}(K) := \{f^\sigma(\xi, \rho, c, d) : c \in (K^\circ)^M, \ d \in (K^{\circ\circ})^N \text{ and } f \in B_{m+M,n+N}\}.$$

Then $B_{m,n}(K)$ is a ring of functions $(K^\circ)^m \times (K^{\circ\circ})^n \to K$. If $\mathcal{B}$ satisfies condition (7)* above (for example if $\mathcal{B}$ is a Weierstrass system), it follows that the homomorphism $f^\sigma \mapsto \mathcal{S}(f^\sigma)$ is an isomorphism, so we may regard $B_{m,n}(K)$ as a ring of power series over K°. Then $\mathcal{B}(K) := \{B_{m,n}(K)\}$ is a Weierstrass system by [CL, Theorem 4.5.7(i)].

In [CL, Definition 4.5.6(ii)], a similar extension of parameters is given for some strictly convergent analytic pre-Weierstrass systems, i.e. those for which both I and $K^{\circ\circ}$ have a prime element. As a further example, complementing the ones in 3.6, we have the following extension of [CL, Theorem 4.5.7(ii)], where we no longer impose that condition on I or K.

Lemma A.3.1. *Let K be a henselian field with analytic $\mathcal{A}$-structure for a strictly convergent Weierstrass system $\mathcal{A}$. Then also the system $\mathcal{A}(K)$ obtained by extension of parameters is a strictly convergent Weierstrass system.*

Proof. Because K° is a valuation ring, no rings of fractions are needed and $\mathcal{A}(K)^H = \mathcal{A}^H(K)$. □

We will give another strong noetherian property ((7)‴ below) that has the advantage that it is expressed in terms of the fields with analytic $\mathcal{B}$-structure, rather than the somewhat awkward concept of systems of rings of fractions. It is useful in defining the concept of Weierstrass system with side conditions in Subsection A.4 below. Let $\mathcal{B}$ be a pre-Weierstrass system. For any (henselian) field K with analytic $\mathcal{B}$-structure, let

$$K_\mathcal{B}^\circ := \{\tau : \tau \text{ a variable-free term of } \mathcal{L}_\mathcal{B}^D\}$$

and let $K_\mathcal{B}$ be the field of fractions of $K_\mathcal{B}^\circ$.

Definition A.3.2. Let $\mathcal{B}$ be a pre-Weierstrass system and let $\mathcal{B}(K_\mathcal{B})$ be obtained by extending parameters.

(7)‴ If $f^\sigma \in B_{m,n}(K_\mathcal{B})$ and $\mathcal{S}(f^\sigma) = \sum_{\mu,\nu} \bar{c}_{\mu\nu}\xi^\mu\rho^\nu$ with the $\bar{c}_{\mu\nu} \in K_\mathcal{B}^\circ$, then there is a finite set $J \subset \mathbb{N}^{m+n}$ and for each $(\mu, \nu) \in J$ there is a $g_{\mu\nu}^\sigma \in (B_{m,n}(K_\mathcal{B}))^\circ$ such that

$$f^\sigma = \sum_{(\mu,\nu)\in J} \bar{c}_{\mu\nu}\xi^\mu\rho^\nu(1 + g_{\mu\nu}^\sigma). \tag{A.6}$$

Remark A.3.3. (a) When $B = K^\circ$ for K a (henselian) field all the rings of fractions are in fact just K°, so in that case the definitions become much simpler.

(b) Condition (7)‴ gives us for each f a (possibly different) strong noetherian property (equation (A.6)) in each field K with analytic $\mathcal{B}$-structure. A standard compactness

argument shows that there is a finite set of representations of the form (A.6) such that for every field K with analytic $\mathcal{B}$-structure (or in the case below of Weierstrass systems with side conditions, for every such K satisfying additional axioms $\mathcal{T}$), one of the finite set of equations (A.6) holds in K.

A.4. Weierstrass systems with side conditions

The following small extension of the above concepts may be useful. See the discussion preceding Theorem 3.5.1.

Definition A.4.1. A (strictly convergent, or separated) *pre-Weierstrass system with side conditions* is a (good strictly convergent, or separated) pre-Weierstrass system $\mathcal{A}$ together with some axioms $\mathcal{T}_{\mathcal{A}}$ in the language $\mathcal{L}_{\mathcal{A}}$.

Definition A.4.2. A *separated Weierstrass system with side conditions* is a separated pre-Weierstrass system $\mathcal{B}$ together with axioms $\mathcal{T}_{\mathcal{B}}$ in the language $\mathcal{L}_{\mathcal{A}}$ such that, for each henselian field K with analytic $\mathcal{B}$-structure that satisfies the axioms of $\mathcal{T}_{\mathcal{B}}$, the pre-Weierstrass system $\mathcal{B}(K_{\mathcal{B}})$ obtained by extension of parameters (cf. Subsection A.3 above) satisfies the condition

(7)″ If $f = \sum_{\mu,\nu} \overline{c}_{\mu\nu} \xi^{\mu} \rho^{\nu}$ in $B_{m,n}(K_{\mathcal{B}})$ with the $\overline{c}_{\mu\nu}$ in $K_{\mathcal{B}}^{\circ}$, then there is a finite set $J \subset \mathbb{N}^{m+n}$ and for each $(\mu, \nu) \in J$ there is a $g_{\mu\nu} \in (B_{m,n}(K_{\mathcal{B}}))^{\circ}$ such that

$$f = \sum_{(\mu,\nu)\in J} \overline{c}_{\mu\nu} \xi^{\mu} \rho^{\nu}(1 + g_{\mu\nu}).$$

In this situation we refer to the analytic structure on K as an *analytic structure with side conditions.* We call a good strictly convergent pre-Weierstrass system $\mathcal{A}$ a *strictly convergent Weierstrass system with side conditions* if the separated pre-Weierstrass system $\mathcal{A}^{H}$ satisfies the above condition for every henselian field K with analytic $\mathcal{A}$-structure that satisfies the side conditions.

The following is immediate.

Corollary A.4.3. *Let $\mathcal{A}$ be a strictly convergent Weierstrass system with side conditions $\mathcal{T}_{\mathcal{A}}$. There is a set of existential definitions in the language $\mathcal{L}_{\mathcal{A}}^{D}$ which, in every henselian field K with analytic $\mathcal{A}$-structure with side conditions $\mathcal{T}_{\mathcal{A}}$, define a separated analytic structure on K extending the analytic $\mathcal{A}$-structure. Indeed, the functions are given by terms of $\mathcal{L}_{\mathcal{A}}^{D,h}$.*

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