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Compact homogeneous Riemannian manifolds with low coindex of symmetry

Dedicated to the memory of Sergio Console (1965–2013)

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Abstract. We develop a general structure theory for compact homogeneous Riemannian manifolds in relation to the coindex of symmetry. We will then use these results to classify irreducible, simply connected, compact homogeneous Riemannian manifolds whose coindex of symmetry is less than or equal to three. We will also construct many examples which arise from the theory of polars and centrioles in Riemannian symmetric spaces of compact type.

Keywords. Compact homogeneous manifolds, symmetric spaces, index of symmetry, Killing fields, polars, centrioles

1. Introduction

A *homogeneous manifold* is a manifold M together with a Lie group G acting transitively on M. Homogeneous manifolds are of particular interest in geometry, topology, algebra and physics. In the context of Riemannian geometry one is interested in homogeneous Riemannian manifolds, where the group G acts transitively by isometries. *Killing fields* are vector fields preserving the metric on the manifold. Such vector fields are of interest in particle physics where they correspond to symmetries in theoretical models. On a homogeneous Riemannian manifold there are many Killing vector fields. More precisely, a connected complete Riemannian manifold M is homogeneous if and only if at every point $p \in M$ and for every $v \in T_pM$ there exists a Killing field X on M with $X_p = v$. This characterization of homogeneous Riemannian manifolds is very useful.

A Killing field is uniquely determined by its value and its covariant derivative at a point. Important classes of homogeneous Riemannian manifolds are obtained by imposing additional conditions on the covariant derivative of Killing fields. For example, a homogeneous Riemannian manifold M is a Riemannian symmetric space if and only if for every point $p \in M$ and every $v \in T_pM$, there exists a Killing field X on M with

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$X_p = v$ and $(\nabla X)_p = 0$. Riemannian symmetric spaces were classified by Élie Cartan and there is a beautiful theory relating such spaces to the algebraic theory of semisimple Lie algebras (see e.g. [3]).

Motivated by this characterization of symmetric spaces, the second and third author together with Tamaru introduced in [8] the index of symmetry of a Riemannian manifold. Let M be a Riemannian manifold and denote by $\mathfrak{K}(M)$ the Lie algebra of Killing fields on M. For $q \in M$ define the *symmetric subspace* $\mathfrak{s}_q$ of $T_q M$ by $\mathfrak{s}_q = \{X_q \in T_q M : X \in \mathfrak{K}(M) \text{ and } (\nabla X)_q = 0\}$. The *index of symmetry* $i_{\mathfrak{s}}(M)$ of M is defined as $i_{\mathfrak{s}}(M) = \inf\{\dim(\mathfrak{s}_q) : q \in M\}$, and the *coindex of symmetry* $ci_{\mathfrak{s}}(M)$ is defined by $ci_{\mathfrak{s}}(M) = \dim(M) - i_{\mathfrak{s}}(M)$. If M is a homogeneous Riemannian manifold, say $M = G/H$, then the symmetric subspaces form a G-invariant distribution $\mathfrak{s}$ on M, called the *distribution of symmetry* on M. In [8] it was shown that the distribution of symmetry is integrable and its maximal integral manifolds are Riemannian symmetric spaces which are embedded in M as totally geodesic submanifolds. For normal homogeneous Riemannian manifolds and a class of naturally reductive homogeneous Riemannian manifolds the distribution of symmetry was explicitly determined in [8].

As mentioned above, a homogeneous Riemannian manifold is a Riemannian symmetric space if and only if $ci_{\mathfrak{s}}(M) = 0$. Thus the coindex of symmetry can be regarded as a measure of how far a homogeneous Riemannian manifold fails to be a Riemannian symmetric space. The purpose of this paper is to develop some general structure theory for compact homogeneous Riemannian manifolds in relation to the coindex of symmetry. We will then use these results to classify irreducible, simply connected, compact homogeneous Riemannian manifolds whose coindex of symmetry is less than or equal to 3. We will also determine the coindex of symmetry for compact homogeneous Riemannian manifolds which arise as total spaces over polars in Riemannian symmetric spaces of compact type and whose fibres are centrioles.

The paper is organized as follows. In Section 2 we present some basic results about Riemannian symmetric spaces, to be used later.

In Section 3 we investigate G-invariant autoparallel distributions $\mathcal{D}$ on compact homogeneous Riemannian manifolds $M = G/H$. Such a distribution is said to be *strongly symmetric with respect* to G if every maximal integral manifold L of $\mathcal{D}$ is a Riemannian symmetric space and the transvection group of L is contained in $\{g|_L : g \in G \text{ and } g(L) = L\}$. The main result is Theorem 3.7 which says, roughly speaking, that if the corank k of a strongly symmetric G-invariant distribution on G/H satisfies $k \geq 2$, then M is a homogeneous space of a normal semisimple subgroup G' of G with $2\dim(G') \leq k(k+1)$.

In Section 4 we introduce the index and coindex of symmetry and review some results from [8].

In Section 5 we develop some general structure theory for compact homogeneous Riemannian manifolds in relation to the coindex of symmetry. The main result in this section is Theorem 5.3: Let M be a simply connected compact homogeneous Riemannian manifold and assume that M does not have a symmetric de Rham factor. Then $k = ci_{\mathfrak{s}}(M) \geq 2$ and there exists a transitive semisimple normal Lie subgroup G' of the isometry group of M such that $2\dim(G') \leq k(k+1)$. Equality holds if and only if the universal covering group of G' is Spin$(k+1)$. Moreover, if equality holds and $ci_{\mathfrak{s}}(M) \geq 3$, then the isotropy group of G' has positive dimension.

In Section 6 we investigate compact homogeneous Riemannian manifolds with $ci_{\mathfrak{s}}(M) = 3$. We will construct explicitly a 2-parameter family of non-homothetical SO(4)-invariant Riemannian metrics on $M = \mathrm{SO}(4)/\mathrm{SO}(2)$ with $ci_{\mathfrak{s}}(M) = 3$. The main result, Theorem 6.7, states that every irreducible, simply connected, compact homogeneous Riemannian manifold with $ci_{\mathfrak{s}}(M) = 3$ is homothetic to $M = \mathrm{SO}(4)/\mathrm{SO}(2)$ with such a Riemannian metric.

In Section 7 we investigate compact homogeneous Riemannian manifolds with $ci_{\mathfrak{s}}(M) = 2$. We will construct explicitly two 1-parameter families of non-homothetical left-invariant Riemannian metrics on $M = \mathrm{Spin}(3)$ with $ci_{\mathfrak{s}}(M) = 2$. The main result, Theorem 7.1, states that every irreducible, simply connected, compact homogeneous Riemannian manifold with $ci_{\mathfrak{s}}(M) = 2$ is homothetic to $M = \mathrm{Spin}(3)$ with such a left-invariant Riemannian metric.

In Section 8 we review the construction by Nagano and Tanaka [4] of certain fibrations $K^+/K^{++} \to K/K^{++} \to K/K^+$. Let $M = G/K$ be a simply connected Riemannian symmetric space of compact type such that K is the isotropy group of G at $o \in M$. Let p be an antipodal point of o in M. Then the orbit $B = K \cdot p = K/K^+$ of K through p is called a *polar* of M. Assume that $\dim(B) > 0$ and that B is irreducible. Let q be the midpoint of a distance minimizing geodesic joining o and p and assume that the orbit $S = K \cdot q = K/K^{++}$ is not a Riemannian symmetric space with respect to the induced metric from M. The fibres K^+/K^{++} of the fibration $K/K^{++} \to K/K^+$ are centrioles in M. We will show in Theorem 8.1 that the coindex of symmetry of the orbit $S = K/K^{++}$, with the induced Riemannian metric, is equal to the dimension of the polar $B = K/K^+$. This provides many examples of compact homogeneous Riemannian manifolds for which the coindex of symmetry can be calculated explicitly in a rather simple way.

2. Preliminaries and basic results

Let $M = G/K$ be an n-dimensional, connected, simply connected, Riemannian symmetric space, where $n \geq 2$ and (G, K) is an effective symmetric pair. We denote by $I(M)$ the full isometry group of M and by $I^o(M)$ the connected component of $I(M)$ containing the identity transformation of M. Note that $G = I^o(M)$ if the Riemannian universal covering space of M has no Euclidean de Rham factor, or equivalently, if M is a semisimple Riemannian symmetric space. The geodesic symmetry at $p \in M$ will be denoted by σ_p. A Riemannian symmetric space is said to be *inner* if $\sigma_p \in I^o(M)$ for one (and hence all) $p \in M$.

Lemma 2.1. *Let $\gamma \in Z_{I(M)}(G)$ be in the centralizer of G in $I(M)$ and assume that for every $q \in M$ with $\gamma(q) \neq q$ the isometry γ translates a minimizing geodesic in M joining q and $\gamma(q)$. Then $\sigma_p \gamma \sigma_p^{-1} = \gamma^{-1}$ for all $p \in M$. If, in particular, M is inner, then $\gamma^2 = \mathrm{id}_M$.*

Proof. Let $p \in M$ and set $\bar{\gamma} = \sigma_p \gamma \sigma_p^{-1}$. It is clear that $\bar{\gamma}$ and $\bar{\gamma}^{-1}$ satisfy the assumptions of this lemma. Let $q \in M$ with $\gamma(q) \neq q$. By assumption, there exists a geodesic $\beta : \mathbb{R} \to M$ through q and $\bar{\gamma}(q)$ which minimizes the distance between $q = \beta(0)$ and $\bar{\gamma}(q) = \beta(a)$ with $a > 0$ and is translated by $\bar{\gamma}$. Then $\bar{\gamma}(\beta(t)) = \beta(t + a)$ and $\bar{\gamma}^{-1}(\beta(t)) = \beta(t - a)$ for all $t \in \mathbb{R}$. Since $\gamma \in Z_{I(M)}(G)$ and $\sigma_q \sigma_p \in G$, we have $\gamma(q) = (\sigma_q \sigma_p)\gamma(\sigma_q \sigma_p)^{-1}(q)$, and therefore

$$\gamma(q) = \sigma_q \bar{\gamma}(q) = \sigma_q \beta(a) = \beta(-a) = \bar{\gamma}^{-1}(\beta(0)) = \bar{\gamma}^{-1}(q) = \sigma_p \gamma^{-1} \sigma_p^{-1}(q),$$

which implies $\sigma_p \gamma \sigma_p^{-1} = \gamma^{-1}$. □

Remark 2.2. A well-known result of Joseph Wolf states that in a homogeneous Riemannian manifold N any geodesic loop must be a closed geodesic. In fact, let $\beta : \mathbb{R} \to N$ be a unit-speed geodesic and let X be a Killing field on N with $X(\beta(0)) = \beta'(0)$. Then it follows from the Killing equation that the inner product between $X(\beta(t))$ and $\beta'(t)$ is a constant function. The value of the inner product at $t = 0$ is 1. Assume that $\beta(0) = \beta(a)$ with some $a \neq 0$. Then the inner product between $X(\beta(a)) = X(\beta(0)) = \beta'(0)$ and $\beta'(a)$ is equal to 1, and it follows from the Cauchy–Schwarz inequality that $\beta'(0) = \beta'(a)$, which shows that β is a closed geodesic.

Corollary 2.3. *Let $M = G/K$ be a Riemannian globally symmetric space, where (G, K) is an effective symmetric pair. Let $\pi : M \to N = G/\bar{K}$ be a G-equivariant local isometry, where the action of G on N is almost effective. Then N is a Riemannian globally symmetric space.*

Proof. Let $\Gamma \subset I(M)$ be the group of deck transformations of N. We can assume that π is not a global isometry, or equivalently that Γ is non-trivial. Since the action of G on M projects to an action on N, and since M is connected and Γ is discrete, it follows that G centralizes Γ. Let $\mathrm{id}_M \neq \gamma \in \Gamma$, $q \in M$, and let $\beta : \mathbb{R} \to M$ be any minimizing geodesic between $\beta(0) = q$ and $\beta(a) = \gamma(q)$. The geodesics $\gamma(\beta(t))$ and $\beta(t)$ in M project to the same geodesic $\bar{\beta}(t) = \pi(\beta(t)) = \pi(\gamma(\beta(t)))$ in N. Since $\bar{\beta}(0) = \bar{\beta}(a)$, it follows from Remark 2.2 that the geodesic $\bar{\beta}$ in N is periodic with period a (not necessarily the smallest period). This implies that $\gamma(\beta(t)) = \beta(t+a)$, and so γ translates the geodesic β. From Lemma 2.1 we get $\sigma_p \gamma \sigma_p^{-1} = \gamma^{-1}$ for all $p \in M$ and $\gamma \in \Gamma$. This implies that the geodesic symmetry σ_p on M descends to a geodesic symmetry of N. We now conclude that N is globally symmetric. □

Remark 2.4. Conjugation by σ_p defines a group automorphism of Γ, and the proof of Corollary 2.3 shows that this automorphism is given by $\gamma \mapsto \gamma^{-1}$. This implies that Γ is an abelian group, which reflects the well-known fact that the homotopy group of a globally symmetric space is an abelian group. From Lemma 2.1 we also see that Γ must be isomorphic to a direct product $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ if M is an inner Riemannian symmetric space.

Remark 2.5. In this remark we fill a gap in the proof of Lemma 5 on page 491 of [2] for the global symmetry and simplify the arguments. In fact, condition $(*)$ in [2, p. 493] need not be true a priori, since the equality only holds for the restriction of those groups to the flat. Let us keep the notation of [2] and prove Lemma 5.

For any maximal flat F in the globally symmetric space X let τ_F be the abelian subgroup of $I_0(X)$ which consists of the glide translations along geodesics in F. More precisely, $\tau_F = \{\mathrm{Exp}(X) : X \in \mathfrak{p}\}$, where $\mathfrak{p}$ is the Cartan subspace at some point $p \in F$. The abelian subgroup τ_F is a normal subgroup of $I_F(M)$, the subgroup of $I(X)$ which leaves F invariant. Since any element of $\Gamma_F \subset \Gamma$ acts as a translation on F (Sublemma 1 is correct!), it follows that τ_F commutes with Γ_F. In fact, for all $g \in \Gamma_F$ and $X \in \mathfrak{p}$ we have $g\,\mathrm{Exp}(X)g^{-1} = \mathrm{Exp}(g_*(X)) \in \tau_F$. Since g restricted to F is a translation, this implies $g_*(X) = X$.

From the assumptions of Lemma 5 one deduces that $\{\tilde{g}\tau_F\tilde{g}^{-1} : \tilde{g} \in \tilde{G}\}$ contains any geometric transvection subgroup $\{\mathrm{Exp}(tX) : t \in \mathbb{R}\}$ where X belongs to any Cartan subspace. Then $\tilde{G}$ and τ_F generate T, the full transvection group of X. Since $\tilde{G}$ and τ_F commute with any element of Γ_F, we conclude that T commutes with Γ_F. Since, as stated in Sublemma 1, Γ is the union of Γ_F, F an arbitrary flat, we find that T commutes with Γ. Since the geodesics in M have no self-intersection (since they lie in a globally symmetric 1-1 immersed flat), any $\gamma \in \Gamma$ satisfies the assumptions of Lemma 2.1. Then $\sigma_p(\Gamma) = \Gamma$, and so the geodesic symmetry σ_p descends from X to the quotient M, which implies that M is globally symmetric. This completes the proof of [2, Lemma 5].

Let $M = G/K$ be a connected, simply connected, Riemannian symmetric space, where (G, K) is an effective symmetric pair. Denote by $\mathfrak{g}$ the Lie algebra of G. Assume that every Killing field X on M, $X \in \mathfrak{g}$, is bounded. This is equivalent to saying that the de Rham decomposition of M does not contain a Riemannian symmetric space of noncompact type. Let $M = \mathbb{R}^k \times M_1 \times \cdots \times M_r$ be the de Rham decomposition of M ($k = 0$ is possible) and let us write

$$G/K = \mathbb{R}^k \times (G_1/K_1) \times \cdots \times (G_r/K_r),$$

where $M_i = G_i/K_i$ is a connected, simply connected, Riemannian symmetric space of compact type. If M_i is not of group type, then G_i is a compact simple Lie group. If M_i is of group type, then $G_i = \bar{G}_i \times \bar{G}_i$, where $\bar{G}_i$ is a compact simple Lie group and $K_i = \mathrm{diag}(\bar{G}_i \times \bar{G}_i)$. Moreover, $M_i \simeq \bar{G}_i$.

Choose $p = (p_0, \ldots, p_r) \in M$ so that $K = G_p$ is the isotropy subgroup of G at p. Then the isotropy representation of K on T_pM is, up to the trivial representation on $\mathbb{R}^k$, the direct sum of the irreducible representations of K_i on $T_{p_i}M_i$.

Definition 2.6. The Lie algebra $\mathfrak{g}_i$ of G_i ($i = 1, \ldots, r$), considered as a subalgebra of $\mathfrak{g}$, is called a *symmetric irreducible factor* of $\mathfrak{g}$.

Note that a symmetric irreducible factor of $\mathfrak{g}$ is either a simple Lie algebra or the direct sum of a simple Lie algebra with itself.

Let $N = G/\bar{K}$ be a Riemannian symmetric space which is not necessarily simply connected. We assume that N is equivariantly covered by $M = G/K$ (see Corollary 2.3). Then the autoparallel distributions on M corrresponding to the factors in the de Rham decomposition of M induce autoparallel distributions on N. In fact, any element γ in the group Γ of deck transformations of the projection $\pi : M \to N$ commutes with the transvection group G of M. This implies that γ preserves the autoparallel distribution on M associated to any of its de Rham factors. If $\bar{K}^o$ is the identity component of $\bar{K}$, then, as for the simply connected case, the isotropy representation of $\bar{K}^o$ decomposes, up to a trivial representation, as a direct sum of irreducible representations.

The following lemma is easy to prove.

Lemma 2.7. *Let $N = G/\bar{K}$ be a Riemannian globally symmetric space, where G is the group of transvections (N is not assumed to be simply connected). Let $\tilde{\mathfrak{g}}'$ be an ideal of $\mathfrak{g}$ that contains the abelian part of $\mathfrak{g}$. Assume that $\tilde{G}'$ does not act transitively on N, where $\tilde{G}'$ is the normal Lie subgroup of G with Lie algebra $\tilde{\mathfrak{g}}'$. Let $\hat{\mathfrak{g}}$ be a complementary ideal to $\tilde{\mathfrak{g}}'$. Then $\hat{\mathfrak{g}}$ contains an irreducible symmetric factor $\mathfrak{g}_i$ of $\mathfrak{g}$.*

Remark 2.8. If in the situation of Lemma 2.7 the symmetric space N is simply connected, and if $\hat{G}$ contains only one of the two factors of $G_i = \bar{G}_i \times \bar{G}_i$, where M_i is a de Rham factor of group type, then $\hat{G}/\hat{G}_p$ is not a symmetric presentation of the symmetric orbit $\hat{G} \cdot p$, $p \in N$.

3. Symmetric autoparallel distributions

Let $M = G/H$ be an n-dimensional compact homogeneous Riemannian manifold, where $n \geq 2$ and G is a connected Lie subgroup of $I(M)$. Let $\mathcal{D}$ be an autoparallel G-invariant distribution on M of rank $r > 0$. We denote by $k = n - r = \dim(M) - \mathrm{rk}(\mathcal{D})$ the *corank* of $\mathcal{D}$. The maximal integral manifold of $\mathcal{D}$ containing a point $p \in M$ will be denoted by $L(p)$. Note that $L(p)$ is a totally geodesic submanifold of M since $\mathcal{D}$ is autoparallel. For all $g \in G$ and $p \in M$ such that $g(L(p)) = L(p)$ we denote by $g|_{L(p)}$ the isometry on $L(p)$ which is obtained by restricting g to $L(p)$. If X is a Killing field on M which is induced by G, then we denote by $X|_{L(p)}$ the restriction of X to $L(p)$.

Definition 3.1. The G-invariant autoparallel distribution $\mathcal{D}$ is *strongly symmetric* with respect to G if every integral manifold $L(p)$ of $\mathcal{D}$ is a globally symmetric space and the identity component of $\{g|_{L(p)} : g(L(p)) = L(p),\ g \in G\}$ contains the transvection group of $L(p)$ (or equivalently, since the Killing fields associated to G are bounded, coincides with the transvection group of $L(p)$).

From Corollary 2.3 one has the following equivalent definition:

Definition 3.2. The G-invariant autoparallel distribution $\mathcal{D}$ is *strongly symmetric* with respect to G if for every $p \in M$ and every $v \in \mathcal{D}_p$ there exists a Killing field X on M which is induced by G such that $X_p = v$ and $X|_{L(p)}$ is parallel at p.

Let $M = G/H$ be a compact homogeneous Riemannian manifold and let $\mathcal{D}$ be a nontrivial G-invariant distribution on M which is strongly symmetric with respect to G. Let $\mathfrak{g} = T_eG$ be the Lie algebra of G, where any element X of $\mathfrak{g}$ is identified with the Killing field $p \mapsto X.p = \frac{d}{dt}\big|_{t=0} \operatorname{Exp}(tX)(p)$ of M. It is important to note that with this identification the brackets change sign, since the Killing field X is related via the isometry g to the right-invariant vector field of G with initial condition $X \in T_eG$. The subspace

$$\mathfrak{g}^{\mathcal{D}} := \{X \in \mathfrak{g} : X \text{ lies on } \mathcal{D}\}$$

of $\mathfrak{g}$ is an ideal of $\mathfrak{g}$ since $\mathcal{D}$ is G-invariant.

Lemma 3.3. *Let $\mathfrak{g}' \subset \mathfrak{g}$ be a complementary ideal of $\mathfrak{g}^{\mathcal{D}}$ and let G' be the normal subgroup of G with Lie algebra $\mathfrak{g}'$. Then $2\dim(G') \leq k(k+1)$, where $k = n - r = \dim(M) - \operatorname{rk}(\mathcal{D})$ is the corank of $\mathcal{D}$. Moreover, for $k \geq 2$, equality holds if and only if the universal covering group of G' is $\operatorname{Spin}(k+1)$. For $k = 1$, $\mathcal{D}$ is a parallel distribution and the Riemannian universal covering space of M has a line as a de Rham factor (and then M is locally symmetric).*

Proof. Since the integral manifolds of $\mathcal{D}$ are not necessarily closed submanifolds of M (when they have a Euclidean local factor), we will consider, locally, the quotient space of M by the foliation given by the maximal integral manifolds of $\mathcal{D}$. Let $p \in M$ and let U be the domain of a Frobenius chart of the distribution $\mathcal{D}$ in a neighbourhood of p. Let us denote by $\mathcal{F}$ the foliation of U given by the maximal integral manifolds of $\mathcal{D}|_U$ and by $\bar{U} = U/\mathcal{F}$ the quotient space. Let $\pi : U \to \bar{U}$ be the canonical projection. Any $Z \in \mathfrak{g}$, regarded as a Killing field on U, projects via π to a vector field $\bar{Z}$ on $\bar{U}$, since any $g \in G$ which is close to the identity preserves (locally) the foliation $\mathcal{F}$. We have $\bar{Z} = 0$ if and only if $Z|_U$ is tangent to the distribution $\mathcal{D}|_U$.

Let $p \in U$ be fixed and let $q = g(p) \in U$. Since $\mathcal{D}$ is G-invariant, we have $Z_q \in \mathcal{D}_q$ if and only if $\operatorname{Ad}(g)(Z)_p \in \mathcal{D}_p$. Let Ω be a neighbourhood of the identity e in G such that $\{g(p) : g \in \Omega\} \subset U$. Then, if $\bar{Z} = 0$, we have $\operatorname{Ad}(g)(Z)_p \in \mathcal{D}_p$ for all $g \in \Omega$. Since Ω generates G, this gives $\operatorname{Ad}(g)(Z)_p \in \mathcal{D}_p$ for all $g \in G$. This implies that the Killing field Z is tangent to $\mathcal{D}$. Therefore, $Z \in \mathfrak{g}^{\mathcal{D}}$ if and only if $\bar{Z} = 0$.

Let us now consider the normal homogeneous Riemannian metric on $M = G/H$. This metric, restricted to U, projects via π to a Riemannian metric on $\bar{U}$. In fact, if $p \in U$, then $\bar{U}$ can be locally regarded as $G/\tilde{H}$, where $\tilde{H} \supset H$ is the Lie subgroup of G which leaves $L(p)$ invariant. So any element $Z \neq 0$ in the complementary ideal $\mathfrak{g}'$ of $\mathfrak{g}^{\mathcal{D}}$ can be regarded as a non-trivial Killing field on $\bar{U}$. If $\bar{p} = \pi(p)$, then the map $j : \mathfrak{g}' \to T_{\bar{p}}\bar{U} \times \mathfrak{so}(T_{\bar{p}}\bar{U})$, $j(Z) = (\bar{Z}_{\bar{p}}, (\nabla\bar{Z})_{\bar{p}})$, which assigns to Z the initial conditions of the Killing field $\bar{Z}$ at $\bar{p}$, is injective. Then, since $k = \dim(\bar{U})$, we conclude that $2\dim(G') \leq k(k+1)$.

We now consider the injective Lie algebra homomorphism $\pi_* : \mathfrak{g}' \to \mathcal{K}(\bar{U})$, $Z \mapsto \bar{Z}$, where $\mathcal{K}(\bar{U})$ denotes the Lie algebra of Killing fields on $\bar{U}$ with the projection of the normal homogeneous metric on M and where the bracket on $\mathfrak{g}'$ is the bracket of Killing fields. Note that $2\dim(\mathfrak{g}') \leq 2\dim(\mathcal{K}(\bar{U})) \leq k(k+1)$. It follows that $\bar{U}$ has constant curvature when $2\dim(\mathfrak{g}') = k(k+1)$. In this case, since $\mathfrak{g}'$ admits a bi-invariant metric, we get $\mathfrak{g}' \simeq \mathcal{K}(\bar{U}) \simeq \mathfrak{so}(k+1)$. Then the universal covering group of G' is $\operatorname{Spin}(k+1)$ if $k \geq 2$.

For $k = 1$ we have $\dim(G') = 1$. If $G^{\mathcal{D}}$ is the normal Lie subgroup of G with Lie algebra $\mathfrak{g}^{\mathcal{D}}$, then the $G^{\mathcal{D}}$-orbits in M coincide with the integral manifolds $L(q)$ of $\mathcal{D}$. In fact, $G^{\mathcal{D}}$ cannot be transitive on M since the orbit $G^{\mathcal{D}} \cdot q$ is contained in $L(q)$ for all $q \in M$. Therefore, since G is transitive on M and $\dim(\mathfrak{g}^{\mathcal{D}}) = \dim(\mathfrak{g}) - 1$, any $G^{\mathcal{D}}$-orbit must have codimension one, and therefore $G^{\mathcal{D}} \cdot q = L(q)$. Thus $G^{\mathcal{D}}$ acts on M with cohomogeneity one and without singular orbits. In fact, since $G^{\mathcal{D}}$ is a normal subgroup of G, we have $G^{\mathcal{D}} \cdot g(q) = g(G^{\mathcal{D}} \cdot q)$ for all $g \in G$. Then the 1-dimensional distribution perpendicular to the $G^{\mathcal{D}}$-orbits (or equivalently orthogonal to $\mathcal{D}$) is autoparallel. Since $\mathcal{D}$ is also an autoparallel distribution, both distributions must be parallel. This implies that the Riemannian universal covering space of M has a line as a de Rham factor. □

Remark 3.4. The normal subgroup $G^{\mathcal{D}}$ of G with Lie algebra $\mathfrak{g}^{\mathcal{D}}$ acts effectively on every integral manifold $L(q)$ of $\mathcal{D}$. In fact, as in Lemma 3.3, let G' be the normal Lie subgroup of G associated with a complementary ideal of $\mathfrak{g}^{\mathcal{D}}$. This gives an almost direct product $G = G^{\mathcal{D}} \times G'$. Since G is transitive on M, the subgroup G' acts transitively on the family $\{L(q) : q \in M\}$. Let $g \in G^{\mathcal{D}}$ and $p \in M$ be such that $g|_{L(p)} : L(p) \to L(p)$ is the identity, and let $L(q)$ be another maximal integral manifold of $\mathcal{D}$. Then there exists $g' \in G'$ such that $g'(L(p)) = L(q)$. Let $q' = g'(p') \in L(q)$ with $p' \in L(p)$. Then $g(q') = g(g'(p')) = g'(g(p')) = g'(p') = q'$, and thus $g = e$.

We continue with the notation and assumptions of Lemma 3.3. Let $q \in M$ and define

$$\bar{G}^q = \{g|_{L(q)} : g(L(q)) = L(q),\ g \in G\}^o,$$

which coincides with the transvection group of $L(q)$ since $\mathcal{D}$ is strongly symmetric with respect to G. Let G_q be the isotropy group of G at q and define

$$\bar{K}^q = \{g|_{L(q)} : g \in G_q\}.$$

Then $\bar{G}^q/\bar{K}^q$ is a symmetric presentation of the symmetric space $L(q)$. Note that G_q and hence $\bar{K}^q$ are connected if M is simply connected.

The subgroup

$$\bar{G}'^q = \{g|_{L(q)} : g(L(q)) = L(q),\ g \in G'\}^o$$

is a normal subgroup of $\bar{G}^q$ and commutes with $G^{\mathcal{D}}$, and we have $\bar{G}^q = \{g_1 g_2 : g_1 \in \bar{G}'^q,\ g_2 \in G^{\mathcal{D}}\}$, where $G^{\mathcal{D}}$ is identified with its restriction to $L(q)$. In general $\bar{G}'^q$ and $G^{\mathcal{D}}$ intersect in a normal subgroup of $\bar{G}'^q$ with positive dimension. Let $\bar{\mathfrak{g}}'^q$ be the Lie algebra of $\bar{G}'^q$ and define $\mathfrak{u} = \bar{\mathfrak{g}}'^q \cap \mathfrak{g}^{\mathcal{D}}$. Let $\hat{\mathfrak{g}}$ be a complementary ideal to $\mathfrak{u}$ in $\mathfrak{g}^{\mathcal{D}}$. Note that $\hat{\mathfrak{g}}$ is an ideal of the Lie algebra $\bar{\mathfrak{g}}^q$ of $\bar{G}^q$ which can be identified with an ideal of $\mathfrak{g}$ which does not depend on the choice of $q \in M$. If $\hat{G} \subset G^{\mathcal{D}}$ denotes the normal Lie subgroup of G associated with $\hat{\mathfrak{g}}$, we have

$$\bar{G}^q = \bar{G}'^q \times \hat{G} \quad \text{(almost direct product)}.$$

Recall that $\bar{G}^q/\bar{K}^q$ is a symmetric presentation of $L(q)$ and that $\bar{\mathfrak{g}}'^q$ is an ideal of $\bar{\mathfrak{g}}^q$. Since G' is transitive on the family $\{L(q) : q \in M\}$ (see Remark 3.4), we see that G' is transitive on M if and only if $\bar{G}'^q$ is transitive on $L(q)$ for some (or equivalently all) $q \in M$. Let $\hat{\mathfrak{g}}_0$ be the abelian part of $\hat{\mathfrak{g}}$ (which is regarded, depending on the context, as an ideal of $\mathfrak{g}$ or of $\bar{\mathfrak{g}}^q$). Moreover, let $\check{\mathfrak{g}}' = \mathfrak{g}' \oplus \hat{\mathfrak{g}}_0$, and let $\check{G}'$ be the Lie subgroup of G with

Lie algebra $\check{\mathfrak{g}}'$. Since M is simply connected, we observe from Remark 3.9 that G' acts transitively on M if and only if $\check{G}'$ acts transitively on M. But this is equivalent to the fact that $\bar{H}'^q$ acts transitively on $L(q)$, where $\bar{H}'^q$ is the (normal) Lie subgroup of $\bar{G}^q$ which is associated with the ideal $\bar{\mathfrak{g}}'^q \oplus \hat{\mathfrak{g}}_0$ of $\bar{\mathfrak{g}}^q$. Note that this ideal always contains the abelian part of $\bar{\mathfrak{g}}^q$. If G' is not transitive on M, then $\bar{H}'^q$ is not transitive on $L(q)$. It follows from Lemma 2.7 that the semisimple part of $\hat{\mathfrak{g}}$, which is a complementary ideal of $\bar{\mathfrak{g}}'^q \oplus \hat{\mathfrak{g}}_0$, has an irreducible symmetric factor $\mathfrak{g}_{\text{irr}}$. Thus $\hat{\mathfrak{g}}$ has an irreducible symmetric factor $\mathfrak{g}_{\text{irr}}$. Note that $\mathfrak{g}_{\text{irr}}$ is an ideal of $\mathfrak{g}^{\mathcal{D}}$ which does not depend on $q \in M$. Thus we have proved the following lemma:

Lemma 3.5. *If G' is not transitive on M, then $\hat{\mathfrak{g}}$ has a non-trivial irreducible symmetric factor $\hat{\mathfrak{g}}_{\text{irr}}$.*

Remark 3.6. Here we will present an example where $\mathfrak{u}$ is non-trivial. Let $M = G/H$ be a normal homogeneous space and decompose $\mathfrak{g} = \mathfrak{g}_{\text{ss}} \oplus \mathfrak{g}_{\text{ab}}$ as a direct sum of ideals, where $\mathfrak{g}_{\text{ss}}$ is semisimple and $\mathfrak{g}_{\text{ab}}$ is abelian. Assume that $\dim(\mathfrak{g}_{\text{ab}}) \geq 2$. Let $p = [e]$ and let $\mathbb{V} \subset T_pM$ be the subspace of fixed vectors of H (via the isotropy representation). Let $\mathbb{W} \subset \mathbb{V}$ be a subspace of codimension one. We choose $X_1, \dots, X_{k-1} \in \mathfrak{g}_{\text{ab}}$ such that $X_1.p, \dots, X_{k-1}.p$ is a basis of $\mathbb{W}$. Let $\mathcal{D}$ be the G-invariant distribution on M with $\mathcal{D}_p = \mathbb{W}$. Note that $\mathcal{D}$ is strongly symmetric with respect to G (see [8]). Let $X_k \in \mathfrak{g}_{\text{ab}}$ be such that $X_k.p \in \mathbb{V} - \mathbb{W}$. Then $\mathfrak{g}^{\mathcal{D}}$ is the linear span of $X_1, \dots, X_{k-1}$. Moreover, if we define $\mathfrak{g}' = \mathfrak{g}_{\text{ss}} \oplus \mathbb{R}(X_{k-1} + X_k)$, then we obtain $\mathfrak{u} = \bar{\mathfrak{g}}'^p \cap \mathfrak{g}^{\mathcal{D}} = \mathbb{R}X_{k-1|L(p)}$.

Theorem 3.7. *Let $M = G/H$ be an n-dimensional compact simply connected homogeneous Riemannian manifold with $n \geq 2$. Let $\mathcal{D}$ be a G-invariant distribution on M of rank $r > 0$ which is strongly symmetric with respect to G, and denote by $k = n - r$ the corank of $\mathcal{D}$. Assume that M does not have a symmetric de Rham factor whose associated parallel distribution on M is contained in $\mathcal{D}$. Then $k \geq 2$ and there exists a normal semisimple Lie subgroup G' of G with $2\dim(G') \leq k(k+1)$ and acting transitively on M such that its Lie algebra $\mathfrak{g}'$ is a complementary ideal of $\mathfrak{g}^{\mathcal{D}} := \{X \in \mathfrak{g} : X \text{ lies on } \mathcal{D}\}$. Moreover, equality holds if and only if the universal covering group of G' is* $\mathrm{Spin}(k+1)$.

Proof. The fact that $k \geq 2$ follows from Lemma 3.3, since M is compact and simply connected. Let G' be given as in Lemma 3.3 and assume that G' does not act transitively on M. Then, by Lemma 3.5, $\hat{\mathfrak{g}}$ has an irreducible symmetric factor $\hat{\mathfrak{g}}_{\text{irr}}$ which is an ideal of $\mathfrak{g}^{\mathcal{D}}$. Observe that $\mathfrak{g}_{\text{irr}}$ does not intersect $\bar{\mathfrak{g}}'^q$. Let $\tilde{\mathfrak{g}}$ be a complementary ideal of $\mathfrak{g}_{\text{irr}}$ in $\mathfrak{g}^{\mathcal{D}}$. Let us consider the ideal $\tilde{\mathfrak{g}}' = \mathfrak{g}' \oplus \tilde{\mathfrak{g}}$ and its associated normal Lie subgroup $\tilde{G}'$ of G. Then we have the direct sum decomposition $\mathfrak{g} = \tilde{\mathfrak{g}}' \oplus \mathfrak{g}_{\text{irr}}$ of $\mathfrak{g}$ into two ideals.

Let G_{irr} be the normal Lie subgroup of G with Lie algebra $\mathfrak{g}_{\text{irr}}$. Then $\tilde{G}'$ commutes with G_{irr} and $G = \tilde{G}' \times G_{\text{irr}}$ (almost direct product). Every orbit $G_{\text{irr}} \cdot q$ is a totally geodesic symmetric submanifold of $L(q)$ and of M. Let K^q_{irr} be the isotropy group of G_{irr} at q. Then $G_{\text{irr}}/K^q_{\text{irr}}$ is a symmetric presentation of $G_{\text{irr}} \cdot q$ and $(K^q_{\text{irr}})^o$ acts irreducibly, via the isotropy representation, on $T_q(G_{\text{irr}} \cdot q)$. Since $(K^q_{\text{irr}})^o$ commutes with $\tilde{G}'$, it acts trivially on the orbit $\tilde{G}' \cdot q$. Then, since $G \cdot q = (\tilde{G}' \times G_{\text{irr}}) \cdot q = M$, we get

$$T_qM = T_q(\tilde{G}' \cdot q) \oplus T_q(G_{\text{irr}} \cdot q) \quad \text{(orthogonal direct sum)}$$

and $\tilde{G}' \cdot q$ must coincide with the connected component of the fixed point set of $(K^q_{\mathrm{irr}})^o$ containing q. Then $\tilde{G}' \cdot q$ is a totally geodesic submanifold of M, and so the distribution $\tilde{\mathcal{D}}'$ on M, given by the tangent spaces of the $\tilde{G}'$-orbits, is autoparallel. Moreover, this distribution is orthogonal and complementary to the autoparallel distribution $\mathcal{D}_{\mathrm{irr}}$, which is tangent to the G_{irr}-orbits. Then $\tilde{\mathcal{D}}'$ and $\mathcal{D}_{\mathrm{irr}}$ are parallel distributions and $\mathcal{D}_{\mathrm{irr}}$ is contained in $\mathcal{D}$. This contradicts the assumptions of this theorem, and therefore G' acts transitively on M. The other statements follow from Lemma 3.3. □

Remark 3.8. We recall here a well-known fact. Let M be a complete and simply connected Riemannian manifold. Let H be a connected Lie subgroup of $I(M)$ which admits a bi-invariant Riemannian metric. Assume that all H-orbits have codimension one in M, that is, H acts with cohomogeneity one on M and there are no singular orbits. Then M splits as $M = N \times \mathbb{R}$ (generally not a Riemannian product). For the sake of completeness we will sketch the proof.

Let us change the Riemannian metric $(\, , \,)$ on M along the distribution $\mathcal{T}$ given by the tangent spaces of the H-orbits. The new metric at $q \in M$, restricted to $\mathcal{T}_q$, is the normal homogeneous metric on the orbit $H \cdot q$ at q (this is a local construction and it does not depend on whether the orbit is exceptional or not). The group H also acts by isometries on M with this new Riemannian metric. If $\gamma(t)$ is a geodesic which is perpendicular at $\gamma(0) = p$ to the orbit $H \cdot p$, then it is always perpendicular to the H-orbits (since a Killing field projects constantly on any geodesic). So the distribution ν perpendicular to the H-orbits is an autoparallel distribution of rank one. Moreover, the 1-parameter perpendicular variation of orbits $H \cdot \gamma(t)$ (we consider these orbits only locally around $\gamma(t)$) is by isometries. Then the H-orbits are totally geodesic, and hence $\mathcal{T} = \nu^{\perp}$ is also an autoparallel distribution. It follows that ν is a parallel distribution and then, by the de Rham decomposition theorem, M has a line as a de Rham factor.

Remark 3.9. Let M be a compact and simply connected Riemannian homogeneous space. Let G be a Lie subgroup of $I(M)$ which acts transitively on M. Then the semisimple part G_{ss} of G acts also transitively on M. In fact, let $\mathfrak{g} = \mathfrak{g}_{\mathrm{ss}} \times \mathbb{R}^k$, where $\mathfrak{g}_{\mathrm{ss}}$ is semisimple. We always have such a decomposition since $I(M)$ is compact and therefore G admits a bi-invariant metric. Let $0 \leq d \leq k$ be the smallest integer such that the Lie subgroup of G with Lie algebra $\mathfrak{g}_{\mathrm{ss}} \times \mathbb{R}^d$ is transitive on M. If $d \geq 1$, let $\bar{G}$ be the Lie subgroup of G with Lie algebra $\mathfrak{g}_{\mathrm{ss}} \times \mathbb{R}^{d-1}$. Then all orbits of $\bar{G}$ have codimension one in M. This is a contradiction since M is compact and simply connected (see Remark 3.8). Therefore we must have $d = 0$.

We will need the following result from [6] (see also [1, Chapter 9]) for the proof of the next lemma which we will use later.

Proposition 3.10 (see [6, Lemma 5.1]). *Let $M = G/H$ be a homogeneous Riemannian manifold (where G is not necessarily connected), $p = [e]$ and Φ be a normal subgroup (possibly finite) of the isotropy group H at p. Let $\mathcal{D}^{\Phi}$ be the G-invariant distribution on M such that $\mathcal{D}^{\Phi}_{g(p)} \subset T_{g(p)}M$ is the subspace of fixed vectors of $g\Phi g^{-1}$. Then $\mathcal{D}^{\Phi}$ is an autoparallel distribution.*

Lemma 3.11. *Let $M = G/H$ be a compact homogeneous Riemannian manifold and $\mathcal{D}^1$ be an autoparallel G-invariant distribution on M which is strongly symmetric with respect to G. Let $\mathcal{D}^2$ be an autoparallel G-invariant distribution on M such that $\mathcal{D}^1 \subset \mathcal{D}^2$ and $\mathrm{rk}(\mathcal{D}^2) = \mathrm{rk}(\mathcal{D}^1) + 1$. Then $\mathcal{D}^2$ is strongly symmetric with respect to G.*

Proof. Let $q \in M$ and $L^i(q)$ be the maximal integral manifold of $\mathcal{D}^i$ containing q, $i = 1, 2$. Let $v \in T_q(L^1(q))$. Since $\mathcal{D}^1$ is strongly symmetric, there exists $X \in \mathfrak{g}$, regarded as a Killing field, such that $X.q = v$ and $\langle \nabla_w X, z\rangle = 0$ for all $w, z \in T_q(L^1(q))$. Since $\mathcal{D}^i$ is G-invariant, $X|_{L^i(q)}$ must always be tangent to $L^i(q)$, $i = 1, 2$.

Let $\xi \in \mathfrak{g}$ be such that $0 \neq \xi.q \in \mathcal{D}^2_q$ and $\xi.q$ is orthogonal to $\mathcal{D}^1_q$. Since the projection of $\xi|_{L^1(q)}$ onto the tangent space of $L^1(q)$ is a bounded Killing field, it lies in the Lie algebra of the transvection group of $L^1(q)$. Since $\mathcal{D}^1$ is strongly symmetric, there exists $Y \in \mathfrak{g}$ such that $Y|_{L^1(q)}$ is always tangent to $L^1(q)$ and coincides with the projection of $\xi|_{L^1(q)}$ onto the tangent spaces of $L^1(q)$. So, by replacing ξ by $\xi - Y$, we may assume that $\xi|_{L^1(q)}$ is always perpendicular to $L^1(q)$. Note that $\xi|_{L^2(q)}$ must always be tangent to $L^2(q)$.

If $\eta \in \mathfrak{g}$ is tangent to $L^2(q)$ and perpendicular to $L^1(q)$, then η must be a scalar multiple of ξ. In fact, let $\lambda \in \mathbb{R}$ be such that $\lambda(\xi.q) = \eta.q$. Then $\psi = \eta - \lambda\xi$ vanishes at q and so $\psi.q \in T_q(L^1(q))$. Since $L^1(q)$ is G-invariant, ψ must always be tangent to $L^1(q)$. However, ψ is always perpendicular to $L^1(q)$, and therefore ψ is identically zero on $L^1(q)$. Since the totally geodesic submanifold $L^1(q)$ of $L^2(q)$ has codimension one, we get $\eta|_{L^2(q)} = 0$. We may have chosen, by making use of a bi-invariant metric on $\mathfrak{g}$, $\xi \in (\mathfrak{g}_0)^\perp$, where $\mathfrak{g}_0 = \{X \in \mathfrak{g} : X|_{L^2(q)} = 0\}$. Let G^1 be the connected component of the subgroup of G that leaves $L^1(q)$ invariant. If $g \in G^1$, then $g_*\xi = \mathrm{Ad}(g)\xi \in (\mathfrak{g}_0)^\perp$ is tangent to $L^2(q)$ and perpendicular to $L^1(q)$. Then $\mathrm{Ad}(g)\xi$ is a scalar multiple of ξ. Since $\mathrm{Ad}(g) : (\mathfrak{g}_0)^\perp \to (\mathfrak{g}_0)^\perp$ is an isometry and G^1 is connected, we get $\mathrm{Ad}(g)\xi = \xi$, and so ξ commutes with $\mathfrak{g}^1$.

Now observe that for all $z \in T_q(L^1(q))$ we have $\langle \nabla_{\xi.q} X, z\rangle = -\langle \nabla_z X, \xi.q\rangle = 0$, since X is tangent to the totally geodesic submanifold $L^1(q)$ of M. As $\langle \nabla_{\xi.q} X, \xi.q\rangle = 0$, we conclude that $X|_{L^2(q)}$ is a transvection at q.

Let us now prove that $\xi|_{L^2(q)}$ is also a transvection at q. Let X be as above. Since $[X, \xi] = 0$ we obtain $\nabla_{X.q}\xi = \nabla_{\xi.q} X = 0$. Observe also that $\langle \nabla_{\xi.q}\xi, v\rangle = -\langle \nabla_v \xi, \xi.q\rangle = 0$ for all $v \in T_q(L^1(q))$. Since $\langle \nabla_{\xi.q}\xi, \xi.q\rangle = 0$, we conclude that $\xi|_{L^1(q)}$ is a transvection at q. It follows that $\mathcal{D}^2$ is strongly symmetric. □

The proof was rather involved, since we had to use the fact that $\mathfrak{g}$ admits a bi-invariant metric. Otherwise, if we consider for example the hyperbolic plane H^2 as a solvable Lie group S, the distribution tangent to the lines that meet at infinity is S-strongly symmetric, but the distribution TH^2 is not S-strongly symmetric.

4. The index of symmetry

In this section we present the definition and some basic facts about the index of symmetry; for details we refer to [8]. Let $(M, \langle\cdot,\cdot\rangle)$ be an n-dimensional Riemannian manifold with

Riemannian metric $\langle\cdot,\cdot\rangle$. We denote by $\mathfrak{K}(M)$ the Lie algebra of global Killing fields on M. The *Cartan subspace* $\mathfrak{p}^q$ at $q \in M$ is

$$\mathfrak{p}^q := \{X \in \mathfrak{K}(M) : (\nabla X)_q = 0\},$$

where ∇ is the Levi-Civita connection of M. The elements of $\mathfrak{p}^q$ are called *transvections at* q. The *symmetric isotropy subalgebra* at q is

$$\mathfrak{k}^q := \text{linear span of } \{[X, Y] : X, Y \in \mathfrak{p}^q\}.$$

For $X, Y \in \mathfrak{p}^q$ we have $[X, Y]_q = (\nabla_X Y)_q - (\nabla_Y X)_q = 0$. Thus $\mathfrak{k}^q$ is contained in the full isotropy algebra $\mathfrak{K}_q(M) = \{X \in \mathfrak{K}(M) : X_q = 0\}$. Moreover, since $\mathfrak{p}^q$ is invariant under the action of the isotropy algebra at q,

$$\mathfrak{g}^q := \mathfrak{k}^q \oplus \mathfrak{p}^q$$

is an involutive Lie algebra. Let G^q and K^q be the Lie subgroups of $I(M)$ with Lie algebras $\mathfrak{g}^q$ and $\mathfrak{k}^q$, respectively.

The *symmetric subspace* $\mathfrak{s}_q$ of $T_q M$ at $q \in M$ is defined by

$$\mathfrak{s}_q := \{X_q : X \in \mathfrak{p}^q\}.$$

The *index of symmetry* $i_{\mathfrak{s}}(M)$ of M is the infimum of $\{\dim(\mathfrak{s}_q) : q \in M\}$. Note that $\dim(\mathfrak{s}_q) = \dim(\mathfrak{p}^q) = \dim(L(q))$, where $L(q) := G^q \cdot q$ is the so-called *leaf of symmetry* containing q. The *coindex of symmetry* $ci_{\mathfrak{s}}(M)$ of M is defined by $ci_{\mathfrak{s}}(M) = n - i_{\mathfrak{s}}(M)$.

Facts 4.1 (see [8, Section 3]). *Let* $q \in M$.

(a) $G^{h(q)} = hG^q h^{-1}$ *and* $d_q h(\mathfrak{s}_q) = \mathfrak{s}_{h(q)}$ *for all* $h \in I(M)$.
(b) $L(q)$ *is a totally geodesic submanifold of* M *and a globally symmetric space.*
(c) G^q *is a normal subgroup of* $\{g \in I(M) : g(L(q)) = L(q)\}$*, and* K^q *is a normal subgroup of the full isotropy group* $I(M)_q$.
(d) *If* $X \in \mathfrak{p}^q$*, then* $\gamma(t) = \text{Exp}(tX)(q)$ *is a geodesic in* M*. Moreover, the parallel transport along* γ *from* $q = \gamma(0)$ *to* $\gamma(t)$ *is given by* $d_q \text{Exp}(tX)$.
(e) *For every* $I^o(M)$*-invariant tensor field* T *on* M *we have* $\nabla_{X_q} T = 0$ *for all* $X \in \mathfrak{p}^q$*. In particular,* $\nabla_{X_q} R = 0$*, where* R *is the Riemannian curvature tensor of* M.
(f) *If* $X \in \mathfrak{p}^q$ *and* Z *is any vector field on* M*, then* $\nabla_{X_q} Z = [X, Z]_q$.
(g) *If* M *is compact, then* G^q *acts almost effectively on* $L(q)$.

In this paper we will only deal with compact homogeneous Riemannian manifolds $M = G/H$. In this case $q \mapsto \mathfrak{s}_q$ is a G-invariant, and hence smooth, distribution which is called the *distribution of symmetry* of M. The distribution $\mathfrak{s}$ on M is autoparallel and the leaves of symmetry $L(q)$ are the maximal integral manifolds of $\mathfrak{s}$. Note that the distribution of symmetry is a strongly symmetric distribution with respect to $I^o(M)$. Let $\mathcal{K}(M)^{\mathfrak{s}}$ be the ideal of $\mathcal{K}(M)$ which consists of those Killing fields that are tangent to $\mathfrak{s}$.

Remark 4.2. G^q is a Lie subgroup of $I(M)$ but it is not necessarily contained in the presentation group G of M. In the notation of Section 3, if $\mathcal{D} = \mathfrak{s}$ and $G = I^o(M)$, then $\bar{G}^q = G^q|_{L(q)}$.

5. Structure results for spaces with non-trivial index of symmetry

In this section we develop some general structure theory in relation to the index and coindex of symmetry. These results are useful for understanding the geometry of (irreducible) compact homogeneous spaces with a non-trivial index of symmetry. Our main theorem is crucial for classifying compact homogeneous spaces M^n with low coindex of symmetry $k = ci_{\mathfrak{s}}(M)$, since it gives a bound on the dimension of a transitive group, and hence on n, in terms of k.

Remark 5.1 (The Jacobi operator in directions of the distribution of symmetry). If $X \in \mathfrak{p}^q$ then, from Facts 4.1(d)&(e), $\nabla_{\gamma'(t)} R = 0$, where $\gamma(t) = \mathrm{Exp}(tX)(q)$ is the geodesic with initial condition $\gamma'(0) = X_q$. Let $e_1 = X_q, e_2, \ldots, e_n$ be an orthonormal basis of $T_q M$ which diagonalizes the Jacobi operator $R_{\cdot, X_q} X_q$ at q with corresponding eigenvalues $a_1 = 0, a_2, \ldots, a_n$. Then $e_1(t), \ldots, e_n(t)$ diagonalizes $R_{\cdot,\gamma'(t)}\gamma'(t)$ with the same corresponding eigenvalues, where $e_i(t)$ denotes the parallel transport of e_i along $\gamma(t)$. For $\kappa \in \mathbb{R}$ we define

$$\sin_\kappa(t) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}\, t) & \text{if } \kappa > 0, \\ t & \text{if } \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa}\, t) & \text{if } \kappa < 0, \end{cases}$$

$$\cos_\kappa(t) = \begin{cases} \cos(\sqrt{\kappa}\, t) & \text{if } \kappa > 0, \\ 1 & \text{if } \kappa = 0, \\ \cosh(\sqrt{-\kappa}\, t) & \text{if } \kappa < 0. \end{cases}$$

Let $v = v_1 e_1 + \cdots + v_n e_n$ and $w = w_1 e_1 + \cdots + w_n e_n$. Then the Jacobi field $J(t)$ along $\gamma(t)$ with initial conditions $J(0) = v$ and $J'(0) = w$ is given by

$$J(t) = \sum_{i=1}^{n} v_i \cos_{a_i}(t) e_i(t) + \sum_{i=1}^{n} w_i \sin_{a_i}(t) e_i(t).$$

Let now $Y \in \mathfrak{K}(M)$ be a Killing field with $Y_q = e_i$. Then $J_Y(t) = Y_{\gamma(t)}$ is a Jacobi field along $\gamma(t)$ with $J_Y(0) = e_i$. Since M is compact, $Y(t)$ is bounded, and thus also $J_Y(t)$ is bounded for $t \in \mathbb{R}$. From the above description of the Jacobi fields along γ it follows that $a_i \geq 0$ for all $i = 1, \ldots, n$. Therefore the Jacobi operator $R_{\cdot, X_q} X_q$ is positive semidefinite.

Proposition 5.2. *Let M be a homogeneous compact Riemannian manifold with a non-trivial index of symmetry. Let $I^q(M)$ be the Lie subgroup of $I(M)$ that leaves invariant the leaf of symmetry $L(q)$. We identify $\mathfrak{K}(M)$ with the Lie algebra of $I(M)$ and define*

$$\mathfrak{m}^q = \{\xi \in \mathfrak{K}(M) : \xi|_{L(q)} \text{ is always perpendicular to } L(q)\}.$$

Then:

(i) $\mathfrak{m}^q$ *is an* $\mathrm{Ad}(I^q(M))$*-invariant subspace of* $\mathfrak{K}(M)$.

(ii) *The linear map* $\mathrm{Ev}^q : \mathfrak{m}^q \to (T_q L(q))^\perp$, $\xi \mapsto \xi_q$, *is surjective and*

$$\ker(\mathrm{Ev}^q) = \{\xi \in \mathfrak{K}(M) : \xi|_{L(q)} = 0\}.$$

(iii) *Let* $0 \neq X \in \mathfrak{p}^q$ *be a transvection at* q *and let* $\gamma(t) = \mathrm{Exp}(tX)(q)$. *Decompose* $T_{\gamma(t)}M = E_0(t) \oplus \cdots \oplus E_r(t)$ (E_0 *may be trivial*) *into the eigenspaces associated to the different* (*constant*) *eigenvalues* $0 = \lambda_0 < \cdots < \lambda_r$ *of the Jacobi operator* $R_{\cdot,\gamma'(t)}\gamma'(t)$. *Let* $\xi \in \mathfrak{K}(M)$ *and let* $(\xi_{\gamma(t)})^i$ *be the orthogonal projection of* $\xi_{\gamma(t)}$ *onto* $E_i(t)$. *Then there exists* $\eta \in \mathfrak{K}(M)$ *such that* $\eta_{\gamma(t)} = (\xi_{\gamma(t)})^i$.

Proof. (i) For every $g \in I(M)$ the adjoint transformation $\mathrm{Ad}(g)$ maps Killing fields to Killing fields. If moreover $g \in I^q(M)$, then $g(L(q)) = L(q)$, and thus $\mathrm{Ad}(g)$ maps any Killing field which is perpendicular to $L(q)$ to a Killing field which is perpendicular to $L(q)$. This proves (i).

(ii) Let $w \in (T_qL(q))^\perp$ and choose $Z \in \mathfrak{K}(M)$ with $Z_q = w$. The orthogonal projection $\bar{Z}^T$ of $Z|_{L(q)}$ to $TL(q)$ is an intrinsic transvection of $L(q)$ since $\bar{Z}^T$ is bounded. Thus there exists $Y \in \mathfrak{g}^q$ such that $Y|_{L(q)} = \bar{Z}^T$. Then $Z - Y$ is always perpendicular to $L(q)$ and $\mathrm{Ev}^q(Z - Y) = (Z - Y)_q = w$. This shows that Ev^q is surjective. Let $\xi \in \mathfrak{m}^q$ with $\xi_q = 0$. Then $\xi_q \in T_qL(q)$. Hence, since the foliation of symmetry $\mathcal{L} = \{L(q) : q \in M\}$ is invariant under isometries, $\xi|_{L(q)}$ must always be tangent to $L(q)$. Therefore $\xi|_{L(q)} = 0$, which implies the second statement in (ii).

(iii) Since $X \in \mathfrak{p}^{\gamma(t)}$, we have $\nabla_{X_{\gamma(t)}}\xi = \nabla_{X_{\gamma(t)}}\xi - \nabla_{\xi_{\gamma(t)}}X = [X, \xi]_{\gamma(t)}$, and therefore

$$[X, [X, \xi]]_{\gamma(t)} = \frac{D^2}{dt^2}(\xi_{\gamma(t)}) = -R_{\xi_{\gamma(t)},\gamma'(t)}\gamma'(t)$$

by the Jacobi equation. Let $J_i(t)$ be the orthogonal projection onto $E_i(t)$ of the Jacobi field $J^\xi(t)) = \xi_{\gamma(t)}$, $i = 0, \ldots, r$. Observe that $J_i(t)$ is a Jacobi field. Let $L : \mathfrak{K}(M) \to \mathfrak{K}(M)$ be the linear map defined by $L(\eta) = [X, [X, \eta]]$. Then

$$L(\xi)_{\gamma(t)} = \lambda_0 J_0(t) + \cdots + \lambda_r J_r(t),$$

where $-\lambda_i \geq 0$ is the eigenvalue of the Jacobi operator $R_{\cdot,\gamma'(0)}\gamma'(0)$ associated to $E_i(0)$ ($\lambda_0 = 0$). Let us write

$$L^j(\xi)_{\gamma(t)} = \lambda_0^j J_0(t) + \cdots + \lambda_r^j J_r(t)$$

for $j = 0, \ldots, r - 1$, where $L^0(\xi) = \xi$. The vectors $v_0, \ldots, v_r \in \mathbb{R}^{r+1}$ are linearly independent, where $v_j = (\lambda_j^0, \lambda_j^1, \ldots, \lambda_j^r)$, $j = 0, \ldots, r$ (since the Vandermonde determinant is not zero). It is not hard to see that for every $i \in \{0, \ldots, r\}$ there exist scalars $c(i)_0, \ldots, c(i)_r$ such that

$$c(i)_0\xi_{\gamma(t)} + c(i)_1L^1(\xi)_{\gamma(t)} + \cdots + c(i)_rL^r(\xi)_{\gamma(t)} = L^i(\xi)_{\gamma(t)} = J_i(t).$$

Then $\eta = L^i(\xi)$ has the desired properties. □

We have the following stronger version of Theorem 3.7 for the distribution of symmetry, which is a consequence of Theorem 3.7, except for the last assertion which follows from Lemma 5.4.

Theorem 5.3. *Let M be a compact, simply connected, Riemannian homogeneous manifold with coindex of symmetry k. Assume that M does not have a symmetric de Rham factor. Then $k \geq 2$ and there exists a transitive semisimple normal Lie subgroup G' of $I(M)$, whose Lie algebra is a complementary ideal to $\mathfrak{K}(M)^{\mathfrak{s}}$, such that $2\dim(G') \leq k(k+1)$. Equality holds if and only if the universal covering group of G' is $\mathrm{Spin}(k+1)$. Moreover, if equality holds and $k \geq 3$, then the isotropy group of G' has positive dimension.*

Lemma 5.4. *Assume that in Theorem 5.3 equality holds and so $G' = \mathrm{Spin}(k+1)$ acts transitively by isometries on M (an almost effective action). Then, if $k \geq 3$, the isotropy group $\mathrm{Spin}(k+1)_q$ at $q \in M$ has positive dimension (or equivalently, since M is simply connected, $\mathrm{Spin}(k+1)_q$ is not trivial).*

Proof. Assume that the isotropy group $\mathrm{Spin}(k+1)_q$ is trivial. Let $\mathfrak{s}$ be the distribution of symmetry, which has dimension $\frac{1}{2}k(k-1)$, since $\dim(\mathrm{Spin}(k+1)) = \frac{1}{2}k(k+1)$. Let $q \in M$ and define

$$\mathrm{Spin}(k+1)^q = \{g \in \mathrm{Spin}(k+1) : g(L(q)) = L(q)\}.$$

Since the isotropy group $\mathrm{Spin}(k+1)_q$ is trivial, the group $\mathrm{Spin}(k+1)^q$ acts effectively on $L(q)$, and so it can be identified with the group

$$\overline{\mathrm{Spin}}(k+1)^q = \{g|_{L(q)} \in \mathrm{Spin}(k+1) : g(L(q)) = L(q)\}^o.$$

From Theorem 5.3 the isometry algebra is given by the following sum of ideals:

$$\mathfrak{K}(M) = \mathfrak{so}(k+1) \oplus \mathfrak{K}(M)^{\mathfrak{s}}. \tag{5.1}$$

In the notation of this section, since $\mathrm{Spin}(k+1)$ is a normal subgroup of $I(M)$, $\overline{\mathrm{Spin}}(k+1)^q$ is a normal subgroup of $\bar{G}^q$, where $\bar{G}^q$ is the transvection group at q, restricted to $L(q)$. Then, since $\overline{\mathrm{Spin}}(k+1)^q$ acts simply transitively on $L(q)$, $L(q)$ must be a Lie group with a bi-invariant Riemannian metric (see Lemma 2.7). In general, $L(q)$ could be non-simply connected. Observe that no element $g \in I(M)^{\mathfrak{s}}$, the subgroup of $I(M)$ associated with the ideal $\mathfrak{K}(M)^{\mathfrak{s}}$, can belong to the full isotropy group $I(M)_q$. In fact, since g commutes with $\mathrm{Spin}(k+1)^q$, which is transitive on $L(q)$, g must be the identity on $L(q)$, and therefore $g = e$ (see Remark 3.4). Note also that $\mathrm{Spin}(k+1)^q$ is semisimple, since the quotient $\mathrm{Spin}(k+1)/\mathrm{Spin}(k+1)^q$ is (equivariantly) isomorphic to $\mathrm{SO}(k+1)/\mathrm{SO}(k)$ (see the proof of Lemma 3.3). Then $L(q)$ has no flat factor locally. By (5.1) this implies that $\dim(\mathfrak{K}(M)^{\mathfrak{s}}) = \dim(L(q)) = \dim(\mathrm{Spin}(k+1)^q)$ and $\mathfrak{g}^q = \mathfrak{so}(k) \oplus \mathfrak{K}(M)^{\mathfrak{s}} \simeq \mathfrak{so}(k) \oplus \mathfrak{so}(k)$.

Then $I^o(M) = \mathrm{Spin}(k+1) \times \mathrm{Spin}'(k)$, where the second factor is the subgroup $\mathrm{Spin}(k) \subset \mathrm{Spin}(k+1)$, but acting from the right on $M \simeq \mathrm{Spin}(k+1)$, that is, if $g \in \mathrm{Spin}'(k)$ then $g(q) = qg^{-1}$. Note that $I(M)^{\mathfrak{s}}$ must be transitive on $L(q)$ and so on any maximal integral manifold of $\mathfrak{s}$. This implies that the Riemannian metric on $M = \mathrm{Spin}(k+1)$ induces a Riemannian submersion onto the quotient

$$\mathrm{Spin}(k+1)/\mathrm{Spin}(k+1)^q \simeq \mathrm{SO}(k+1)/\mathrm{SO}(k),$$

which is a sphere. We are now in the following situation:

(a) $M = \mathrm{Spin}(k+1)$.
(b) $I^o(M) = \mathrm{Spin}(k+1) \times \mathrm{Spin}'(k)$.
(c) The distribution of symmetry is

$$g \mapsto \mathfrak{so}'(k)g = \mathrm{Ad}(g)(\mathfrak{so}(k)g), \qquad g \in M \simeq \mathrm{Spin}(k+1).$$

(d) The maximal integral manifolds of the distribution of symmetry are

$$L(g) = \mathrm{Spin}'(k)g = g\,\mathrm{Spin}(k).$$

(e) The isotropy group at e is

$$(I^o(M))_e = \mathrm{diag}(\mathrm{Spin}(k)) = \{(h,h) \in \mathrm{Spin}(k) \times \mathrm{Spin}'(k) : h \in \mathrm{Spin}(k)\}.$$

(f) $K^e = (I^o(M))_e$, $\mathfrak{k}^e = \mathrm{diag}(\mathfrak{so}(k))$, $\mathfrak{p}^e = \{(v,-v) \in \mathfrak{so}(k) \times \mathfrak{so}'(k)\}$, $G^e = \mathrm{Spin}(k) \times \mathrm{Spin}'(k)$, $\mathfrak{g}^e = \mathfrak{so}(k) \oplus \mathfrak{so}'(k)$. Recall that K^e acts almost effectively on $L(e)$ (see Facts 4.1).

Let $X \in \mathfrak{so}(k+1) \subset \mathfrak{so}(k+1) \oplus \mathfrak{so}'(k) \simeq \mathfrak{K}(M)$. Then the orthogonal projection $\bar{X}$ of $X|_{L(e)}$ onto $TL(e)$ is a bounded Killing field on $L(e)$, and so it belongs to $\mathfrak{g}^e|_{L(e)}$. Since X commutes with any Killing field induced by $\mathfrak{so}'(k)$, and $\mathrm{Spin}'(k)$ preserves the distribution of symmetry, we see that $\mathfrak{so}'(k)|_{L(e)}$ commutes with $\bar{X}$. Then there must exist $Z \in \mathfrak{so}(k)$ such that $\bar{X} = \bar{Z}$, where $\bar{Z}$ denotes the restriction of Z to $L(e)$. Hence $Y = X - Z \in \mathfrak{so}(k+1)$ is a Killing field whose restriction to $L(e)$ is always perpendicular to $L(e)$. Note that in this way we can construct such a Killing field Y with an arbitrary initial condition $Y_e \in \mathfrak{s}^\perp$.

Let

$$\mathfrak{m} = \{Y \in \mathfrak{so}(k+1) : Y|_{L(e)} \text{ is perpendicular to } L(e)\}.$$

Then $\mathfrak{m}$ is an $\mathrm{Ad}(\mathrm{Spin}(k))$-invariant complementary subspace of $\mathfrak{so}(k)$ in $\mathfrak{so}(k+1)$. By Lemma 5.5, if $k \neq 3$, then $\mathfrak{m} = \mathfrak{so}(k)^\perp$, where the orthogonal complement is with respect to the Killing form of $\mathfrak{so}(k+1)$. We equip $M \simeq \mathrm{Spin}(k+1)$ with the bi-invariant Riemannian metric $(\cdot,\cdot)$. Note that $I^o(M) = \mathrm{Spin}(k+1) \times \mathrm{Spin}(k) \subset I^o(M,(\cdot,\cdot)) = \mathrm{Spin}(k+1) \times \mathrm{Spin}'(k+1)$.

If $\xi, \eta \in \mathfrak{m} = \mathfrak{so}(k)^\perp$, then these two Killing fields are perpendicular to $L(e) = \mathrm{Spin}(k) \cdot e$ with respect to both Riemannian metrics $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ (the given one). Moreover, if $X \in \mathfrak{p}^e$, then X is a parallel vector field at e with respect to both metrics. Note that the canonical projection onto $S^k = \mathrm{Spin}(k+1)/\mathrm{Spin}(k)$ is a Riemannian submersion (up to rescaling) with respect to any of the two metrics on M. So, up to rescaling, $(\cdot,\cdot)$ coincides with $\langle\cdot,\cdot\rangle$ on $\mathfrak{so}(k)^\perp \simeq (\mathfrak{s}_e)^\perp$. Unless $(\cdot,\cdot) = \langle\cdot,\cdot\rangle$, this contradicts the so-called bracket formula of [8, Proposition 3.6]:

$$2\langle[\xi,X],\eta\rangle_e = -\langle X,[\xi,\eta]\rangle_e, \qquad 2([\xi,X],\eta)_e = -(X,[\xi,\eta])_e, \tag{5.2}$$

taking into account that $[\mathfrak{so}(k)^\perp, \mathfrak{so}(k)^\perp] = \mathfrak{so}(k)$. Then, if $k \neq 3$, $M \simeq \mathrm{Spin}(k+1)$ has a bi-invariant metric and thus M is a symmetric space, which is a contradiction, since the coindex of symmetry is k. Therefore the isotropy group is non-trivial if $k \neq 3$.

The case $k = 3$ is more involved since SO(4) is not simple. Since Spin(4) acts almost effectively on the quotient $\mathrm{Spin}(4)/\mathrm{Spin}(4)^e$ of M by the leaves of symmetry (see the proof of Lemma 3.3), we see that $\mathrm{Spin}(4)^e$ cannot be a factor of Spin(4). Then, according to Remark 5.6, $\mathrm{Spin}(4)^e \simeq \mathrm{Spin}(3)$ is the subgroup of Spin(4), which is equivalent to the diagonal inclusion of Spin(3) in $\mathrm{Spin}(4) = \mathrm{Spin}(3) \times \mathrm{Spin}(3)$. As remarked above, $\mathfrak{m} = \{Y \in \mathfrak{so}(4) : Y|_{L(e)}$ is perpendicular to $L(e)\}$ is an Ad(Spin(3))-invariant complementary subspace of $\mathfrak{so}(3)$ in $\mathfrak{so}(4)$ and gives a reductive decomposition of $\mathrm{Spin}(3) \times \mathrm{Spin}(3)/\mathrm{diag}(\mathrm{Spin}(3))$.

We still have to deal with cases (1) and (2) of Remark 5.5. In the first case $\mathfrak{m}$ is the orthogonal complement with respect to an Ad(SO(4))-invariant bilinear form Q. Such a form Q is equal to B on the first ideal of $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ and equal to λB on the second ideal, where $0 \neq \lambda \neq -1$ and $-B$ is the Killing form of $\mathfrak{so}(3)$. The bilinear form Q induces on $M = \mathrm{Spin}(4)$ a bi-invariant pseudo-Riemannian metric. Then M is a pseudo-Riemannian product of Spin(3) with a bi-invariant Riemannian metric and Spin(3) with a Riemannian or anti-Riemannian metric (depending on the sign of λ). If $\langle\cdot,\cdot\rangle = Q$, we get the same contradiction as in (5.2) unless $\langle\cdot,\cdot\rangle$ is proportional to Q. Thus Q is positive definite and M is a symmetric space, which gives a contradiction. Therefore the isotropy group cannot be trivial.

Let us now consider case (2) of Remark 5.5, where $\mathfrak{m} \simeq (\mathfrak{so}(3), 0) \subset \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ (the other case $\mathfrak{m} \simeq (0, \mathfrak{so}(3))$ is analogous). In this case, the distribution perpendicular to $\mathfrak{s}$ is integrable with maximal integral manifolds $H \cdot q$, where H is the first factor of Spin(4). Since the projection of M onto the quotient of M by the leaves of symmetry is a Riemannian submersion, the orbit $H \cdot q$ is a totally geodesic submanifold of M for every $q \in M$. Thus $(\mathfrak{s})^{\perp}$ and $\mathfrak{s}$ are autoparallel distributions, and hence both are parallel distributions. This implies that M is a Riemannian product, which is a contradiction. Altogether we conclude that the isotropy group of Spin(4) is not trivial. □

Remark 5.5. The second and third authors observed in [7, Remark 2.1] (see also [1, Chapter 9]) that there is only one naturally reductive decomposition on the homogeneous space $\mathrm{SO}(n+1)/\mathrm{SO}(n)$ if $n \neq 3$. The assumption that the reductive decomposition is naturally reductive is not necessary. In fact, let ∇ be the Levi-Civita connection on $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$ and ∇^c be the canonical connection associated with a reductive decomposition on the homogeneous space $\mathrm{SO}(n+1)/\mathrm{SO}(n)$, and define $D = \nabla - \nabla^c$. We will show that D is totally skew. Since ∇^c is a metric connection, we have $\langle D_X Y, Y\rangle = 0$ for all vector fields X, Y on S^n. So we only need to show that $\langle D_X X, Z\rangle = 0$ for perpendicular vector fields X, Z on S^n. Since for $n = 1$ there is no isotropy group, we have $D = 0$. If $n = 2$ then there is only one reductive decomposition $\mathfrak{so}(3) = \mathfrak{so}(2) + \mathbb{V}$, where $\mathbb{V}$ is the orthogonal complement to $\mathfrak{so}(2)$ with respect to the Killing form of $\mathfrak{so}(3)$. This is because $\mathbb{V}$ is the only irreducible SO(2)-invariant subspace.

Thus we may assume that $n \geq 3$. Let $h \in \mathrm{SO}(n+1)_q \simeq \mathrm{SO}(n)$ be such that $h(q) = q$, $dh(x) = x$ and $dh(z) = -z$. Since D is $\mathrm{SO}(n+1)$-invariant, we have $\langle D_x x, z\rangle = 0$. Thus D is totally skew and ∇^c is associated with a naturally reductive decomposition. Moreover, D is parallel (since it is invariant under the transvections of S^n). Hence $\langle D_{\cdot}\cdot, \cdot\rangle$ is a harmonic 3-form which represents a 3-cohomology class on S^n. Thus $D = 0$ if $n \neq 3$.

Observe that for $n = 3$ the above argument implies that D is also totally skew. So a reductive decomposition on $\mathrm{SO}(4)/\mathrm{SO}(3)$ must be naturally reductive. It is well-known that there is a 1-parameter family of naturally reductive decompositions on the Lie group $S^3 \simeq \mathrm{Spin}(3)$.

The only reductive decomposition on $\mathrm{SO}(n+1)/\mathrm{SO}(n)$ is the orthogonal complement to $\mathfrak{so}(n)$ in $\mathfrak{so}(n+1)$, with respect to minus the Killing form of $\mathfrak{so}(n+1)$. The reductive decompositions on $\mathrm{SO}(4)/\mathrm{SO}(3) \simeq \mathrm{Spin}(3) \times \mathrm{Spin}(3)/\operatorname{diag}(\mathrm{Spin}(3))$ are of one of the following two types (cf. [8, Section 5]):

(1) The orthogonal complement to $\operatorname{diag}(\mathfrak{so}(3))$ with respect to a bi-invariant pseudo-Riemannian (non-degenerate) scalar product on $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$. Such an inner product has to be a multiple of minus the Killing form on each factor of $\mathfrak{so}(4)$. These multiples, up to rescaling, are $\lambda_1 = 1$, $\lambda_2 \in \mathbb{R}$, $0 \neq \lambda_2 \neq -1$. In this case the transvection group associated with the canonical connection is $\mathrm{Spin}(4)$.
(2) The reductive complement of $\operatorname{diag}(\mathfrak{so}(3))$ is either $(\mathfrak{so}(3), 0)$ or $(0, \mathfrak{so}(3))$. The transvection group is either $\mathrm{Spin}(3)$, regarded as the left factor of $\mathrm{Spin}(4)$, or $\mathrm{Spin}(3)$, regarded as the right factor of $\mathrm{Spin}(4)$. In both cases the canonical connection is flat.

Remark 5.6. Let H be a connected Lie subgroup of $\mathrm{Spin}(k+1)$ of codimension $k \geq 2$.

(i) If $k \neq 3$, then $\mathrm{Spin}(k+1)/H$ is equivariantly isomorphic to $S^k = \mathrm{SO}(k+1)/\mathrm{SO}(k)$.
(ii) If $k = 3$, then either H is one factor of $\mathrm{Spin}(4) = \mathrm{Spin}(3) \times \mathrm{Spin}(3)$, or $\mathrm{Spin}(4)/H$ is equivariantly isomorphic to $S^3 = \mathrm{SO}(4)/\mathrm{SO}(3)$.

In fact, assume that no normal subgroup of $\mathrm{Spin}(k+1)$ with positive dimension is contained in the closure $\bar{H}$ of H. This is always the case if $k \neq 3$, since $\mathrm{Spin}(k+1)$ is a simple Lie group for $k \neq 3$. Note that $\bar{H} \neq \mathrm{Spin}(k+1)$, because otherwise the Lie algebra of $\mathrm{Spin}(k+1)$ would have a flat factor. Then $\mathrm{Spin}(k+1)$ acts almost effectively on the k'-dimensional compact quotient $M = \mathrm{Spin}(k+1)/\bar{H}$, where $0 \leq k' \leq k$. The manifold M is simply connected since $\mathrm{Spin}(k+1)$ is simply connected and $\bar{H}$ is connected. Since the dimension of the isometry group of M is at least $k(k+1)/2$, it follows that M is isometric to a sphere, $k' = k$ and $\bar{H} = H$. Moreover, the effectivized action of $\mathrm{Spin}(k+1)$ gives the identity component of the full isometry group of the sphere, which is isomorphic to $\mathrm{SO}(k+1)$.

6. Classification for coindex of symmetry equal to 3

Let $M = G/H$ be an $(r+3)$-dimensional ($r \geq 1$) compact simply connected homogeneous Riemannian manifold with coindex of symmetry $k = 3$. By Theorem 3.7 there exists a compact semisimple normal subgroup G' of G with $\dim(G') \leq 6$ which acts transitively on M. We may assume that G' is simply connected and that the action of G' on M is almost effective. The only possibilities for such a group are $G' = \mathrm{Spin}(4) = \mathrm{Spin}(3) \times \mathrm{Spin}(3)$ and $G' = \mathrm{Spin}(3)$. However, since M has a positive index of symmetry, we cannot have $G' = \mathrm{Spin}(3)$. Therefore $G' = \mathrm{Spin}(4)$, which has dimension 6, and so the dimension d of the isotropy group must satisfy $d \in \{0, 1, 2\}$. The case $d = 0$ can be excluded by the last statement of Theorem 5.3. If $d = 2$, then the isotropy group is, up to conjugation, the standard torus $S^1 \times S^1 \subset \mathrm{Spin}(3) \times \mathrm{Spin}(3)$. Such a quotient space, with any G'-invariant Riemannian metric, is the Riemannian product of two 2-spheres. This implies that M is symmetric, and so this case can also be disregarded.

We can therefore assume that the dimension d of the isotropy group T is 1. Thus M is 5-dimensional and its index of symmetry is 2. For such a subgroup there are infinitely many possibilities, depending on the different velocities of the projections of this subgroup onto the two factors. However, this is never the case when the index of symmetry is 2, in which case we have the following lemma which uses the results of the general theory we developed in Section 5.

Lemma 6.1. *Let $M = \mathrm{Spin}(4)/T$ be a 5-dimensional compact simply connected homogeneous Riemannian manifold with coindex of symmetry $k = 3$. Then, up to conjugation, $T = \mathrm{diag}(S^1) = \{(u, u) \in \mathrm{Spin}(3) \times \mathrm{Spin}(3) : u \in S^1\}$. Moreover, after making the action effective, we have $M = \mathrm{SO}(4)/\mathrm{SO}(2)$, which is isometric to the unit tangent bundle of the 3-sphere with an $\mathrm{SO}(4)$-invariant Riemannian metric.*

Proof. We choose $p \in M$ such that T is the isotropy group of Spin(4) at p. Note that T is connected since M is simply connected. We consider T as a subgroup of $\mathrm{SO}(T_pM)$ via the isotropy representation of $M = \mathrm{Spin}(4)/T$ at p. Since the distribution of symmetry $\mathfrak{s}$ is invariant under the action of Spin(4), we see that $\mathfrak{s}_p$ is a T-invariant 2-dimensional subspace of T_pM. We decompose T_pM orthogonally into T-invariant subspaces,

$$T_pM = \mathfrak{s}_p \oplus \mathbb{V} \oplus \mathbb{L},$$

where $\dim(\mathbb{V}) = 2$ and $\dim(\mathbb{L}) = 1$. Note that the action of T on $\mathfrak{s}_p$ or on $\mathbb{V}$ may be trivial. Let $\rho_1 : T \to \mathfrak{so}(\mathfrak{s}_p)$, $\rho_1(h) = h_{|\mathfrak{s}_p}$ and let $\rho_2 : T \to \mathfrak{so}(\mathbb{V})$, $\rho_2(h) = h_{|\mathbb{V}}$. It is not hard to see the following: *if ρ_1 and ρ_2 are both (Lie group) isomorphisms, then T is standard*. Namely, T is conjugate to $\mathrm{diag}(S^1) = \{(h, h) \in \mathrm{Spin}(3) \times \mathrm{Spin}(3) : h \in S^1\}$, where S^1 is any 1-dimensional Lie subgroup of Spin(3).

Let us show that both ρ_1 and ρ_2 are isomorphisms. Let Φ_i be the kernel of ρ_i, $i = 1, 2$. Since T is abelian, Φ_1 and Φ_2 are normal subgroups of the isotropy group T at p.

We first assume that Φ_1 is not trivial. Then, in the notation of Proposition 3.10, $\mathcal{D}^{\Phi_1}$ is the (unique) Spin(4)-invariant autoparallel distribution with $\mathcal{D}_p^{\Phi_1} = \mathfrak{s}_p \oplus \mathbb{L}$. Due to Lemma 3.11 this distribution is strongly symmetric with respect to Spin(4). Moreover, $\mathfrak{s}$ restricted to any integral manifold $F^{\Phi_1}(q)$ is a parallel distribution. Observe that the corank of $\mathcal{D}^{\Phi_1}$ is 2. Then, by Theorem 3.7, if M does not have a symmetric de Rham factor, we have $\dim(M) \leq 3$ (since there is a 3-dimensional group which is transitive on M). This is a contradiction, and hence Φ_1 is trivial.

We next assume that Φ_2 is not trivial. Then, in the notation of Proposition 3.10, $\mathcal{D}^{\Phi_2}$ is the (unique) Spin(4)-invariant autoparallel distribution with $\mathcal{D}_p^{\Phi_2} = \mathbb{V} \oplus \mathbb{L}$. Observe that $\mathcal{D}^{\Phi_2} = \mathfrak{s}^{\perp}$. Since $\mathfrak{s}$ is also autoparallel, both distributions must be parallel, and so M has a symmetric de Rham factor. This is a contradiction, and hence Φ_2 is trivial.

It now follows that T is standard, and so $M = \mathrm{Spin}(3) \times \mathrm{Spin}(3)/\mathrm{diag}(S^1)$. After making the action effective, this homogeneous space becomes $\mathrm{SO}(4)/\mathrm{SO}(2)$, where SO(2) is naturally included in SO(4). So $M = \mathrm{SO}(4)/\mathrm{SO}(2)$, which is isometric to the unit tangent bundle of the 3-sphere with a suitable SO(4)-invariant Riemannian metric. □

We have proved that $M = \mathrm{SO}(4)/\mathrm{SO}(2)$. Let us determine the leaf of symmetry at $p = [e]$. The subspace of vectors of T_pM which are fixed by the isotropy group SO(2) has dimension 1. So the 2-dimensional leaf of symmetry $L(p)$ has non-trivial isotropy group. Thus $L(p)$ is covered by a 2-dimensional sphere, and so the transvection group G^p is 3-dimensional (with Lie algebra isomorphic to $\mathfrak{so}(3)$ and $K^p = \mathrm{SO}(2)$). Since $\mathrm{SO}(2) \subset G^p$, G^p cannot be contained in a local factor of SO(4) (i.e., a factor corresponding to the decomposition of $\mathrm{Spin}(4) = \mathrm{Spin}(3) \times \mathrm{Spin}(3)$). Then, by Remark 5.6(ii), $\mathrm{SO}(4)/G^p$ is equivariantly isomorphic to SO(4)/SO(3). This isomorphism maps SO(2) into a 1-dimensional subgroup of SO(3). Such a group is conjugate in SO(3) to the standard SO(2). Thus we may assume that $M = \mathrm{SO}(4)/\mathrm{SO}(2)$ and the leaf of symmetry at p is given by

$$L(p) = \mathrm{SO}(3)/\mathrm{SO}(2) \subset \mathrm{SO}(4)/\mathrm{SO}(2) = M.$$

We have to determine the SO(4)-invariant metrics on $M = \mathrm{SO}(4)/\mathrm{SO}(2)$ for which the index of symmetry is 2. As observed above, the isotropy group SO(2) coincides with the isotropy group K^p of the transvection group $G^p = \mathrm{SO}(3)$. In particular, since K^p acts almost effectively on $L(p) = \mathrm{SO}(3) \cdot p$ (see Facts 4.1), we deduce that

$$H^p := \{g \in G : g|_{L(p)} = \mathrm{Id}\,|_{L(p)}\}^o$$

is trivial.

As noted before, if $\xi \in \mathfrak{so}(4)$, regarded as a Killing field of M, then there is $Z \in \mathfrak{g}^q$ such that $\xi - Z$, restricted to $L(p)$, is always perpendicular to $L(p)$ (since the projection of $\xi|_{L(p)}$ to $L(p)$ is an intrinsic transvection of $L(p)$). Then, since M is homogeneous, for any $u \in (T_pL(p))^\perp$ there exists $\xi \in \mathfrak{so}(4)$ such that $\xi.p = u$ and ξ, restricted to $L(p)$, is always perpendicular to $L(p)$. Moreover, such a ξ is unique. In fact, assume that $\eta \in \mathfrak{so}(4)$ is always perpendicular to $L(p)$ and $\eta.p = 0$. Then η belongs to the isotropy algebra, which coincides, as previously observed, with $\mathfrak{k}^p$. Therefore η is always tangent to $L(p)$. It follows that $\eta|_{L(p)} = 0$, and so η belongs to the Lie algebra $\mathfrak{h}^p$ of H^p. This Lie algebra is trivial, and thus we have $\eta = 0$.

Let

$$\mathfrak{m} = \{\xi \in \mathfrak{so}(4) : \xi|_{L(e)} \text{ is always perpendicular to } L(p)\}.$$

Since $L(p)$ is invariant under the action of SO(3), $\mathfrak{m}$ is an Ad(SO(3))-invariant subspace of $\mathfrak{so}(4)$. Since the evaluation at p, from $\mathfrak{m}$ into $(T_p(L(p)))^\perp$, is an isomorphism, we find that

$$\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{m}$$

is a reductive decomposition on SO(4)/SO(3) (the quotient space of M by the leaves of symmetry) and

$$\mathfrak{m}.p = (T_p(L(p)))^\perp = (\mathfrak{so}(3).p)^\perp.$$

From Remark 5.5 we see that the above reductive decomposition is naturally reductive (i.e., the canonical geodesics in $S^3 = \mathrm{SO}(4)/\mathrm{SO}(3)$, associated to $\mathfrak{m}$, coincide with the geodesics of the round sphere S^3) and of one of the following forms:

(i) $\mathfrak{m} = \mathfrak{m}^\lambda$, where $\mathfrak{m}^\lambda$ is the orthogonal complement of $\mathfrak{so}(3)$ with respect to the (pseudo-Riemannian) inner product $(\ ,\ )_\lambda = (B, \lambda B)$ on $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, where $-B$ is the Killing form of $\mathfrak{so}(3)$ and $0 \neq \lambda \in \mathbb{R}$.

(ii) $\mathfrak{m} = \mathfrak{m}^0$, where $\mathfrak{m}^0 \simeq \mathfrak{so}(3)$ is the Lie algebra of one of the factors of Spin(4) (and so $\mathfrak{m}^0$ is a Lie algebra).

We will now show that case (ii) cannot occur. Recall that, for arbitrary Killing fields ξ, η, X, the Levi-Civita connection is given by

$$2\langle \nabla_\xi X, \eta\rangle = \langle [\xi, X], \eta\rangle + \langle [\xi, \eta], X\rangle + \langle [X, \eta], \xi\rangle \tag{6.1}$$

(see [8, (3.4)]). If $X \in \mathfrak{p}^p$ is a transvection at $p = [e]$ and $\xi, \eta \in \mathfrak{m}^0$, then $0 = \langle [\xi, X], \eta\rangle_p + \langle [X, \eta], \xi\rangle_p$, or equivalently

$$\langle [X, \xi], \eta\rangle_p = \langle [X, \eta], \xi\rangle_p. \tag{6.2}$$

There exists $X \in \mathfrak{p}^p$ such that $[X, \mathfrak{m}^0] \neq \{0\}$. Otherwise, $[\mathfrak{p}^p, \mathfrak{m}^0] = \{0\}$ and so $[[\mathfrak{p}^p, \mathfrak{p}^p], \mathfrak{m}^0] = \{0\}$ and hence $[\mathfrak{g}^p, \mathfrak{m}^0] = \{0\}$, which is a contradiction (recall that $\mathfrak{g}^p = \mathfrak{so}(3)$, the Lie algebra of the standard $SO(3) \subset SO(4)$, which is not an ideal of $\mathfrak{so}(4)$). If we equip $\mathfrak{so}(4)$ with a bi-invariant (positive definite) metric, then $[X, \cdot\,] : \mathfrak{m}^0 \to \mathfrak{m}^0$ is skew-symmetric. Then there exist linearly independent vectors $\xi, \eta \in \mathfrak{m}^0$ such that $[X, \xi] = \eta$ and $[X, \eta] = -\xi$. Inserting this into (6.2) leads to $\|\eta(p)\|^2 = -\|\xi(p)\|^2$, which implies $\xi = 0 = \eta$ because, as previously observed, the evaluation at p is an isomorphism from $\mathfrak{m}^0$ onto $(T_p(L(p))^p$. This is a contradiction, and therefore case (ii) cannot occur.

We will now deal with case (i). For this we will use the construction given in [8, Section 6].

Case (a): $\lambda > 0$, that is, the bi-invariant metric $(\cdot, \cdot)_\lambda = (B, \lambda B)$ of $\mathfrak{so}(4)$ is Riemannian. In the notation of [8], $G = SO(4)$, $G' = SO(3)$ and $K' = SO(2)$ (and so $G \supset G' \supset K'$). Moreover, the general assumptions in this reference are satisfied, i.e., (SO(4), SO(3)) and (SO(3), SO(2)) are irreducible symmetric pairs and SO(3) is a simple (compact) Lie group. Let $\mathfrak{so}(3) = \mathfrak{so}(2) + \mathfrak{p}'$ be the Cartan decomposition on $S^2 = SO(3)/SO(2)$. Since $\mathfrak{so}(3)$ is simple, the restriction of $(\cdot, \cdot)_\lambda$ to $\mathfrak{so}(3)$ is a multiple of the Killing form of $\mathfrak{so}(3)$. So $\mathfrak{p}' \subset \mathfrak{so}(2)^\perp$ (the orthogonal complement in $\mathfrak{so}(4)$ with respect to $(\cdot, \cdot)_\lambda$), and thus

$$\mathfrak{so}(2)^\perp = \mathfrak{m}^\lambda \oplus \mathfrak{p}'.$$

We will first define a Riemannian metric on $M = SO(4)/SO(2)$ such that the canonical projection onto the sphere SO(4)/SO(3) is a Riemannian submersion, with index of symmetry 2 (and such that the orthogonal complement to the subspace of symmetry is given by $\mathfrak{m}^\lambda.p$). Then we will deform this metric to obtain all the invariant metrics with index of symmetry 2 and such that the subspace which is orthogonal to the subspace of symmetry at $p = [e]$ is given by $\mathfrak{m}^\lambda.p$.

Following [8], we equip $T_p(SO(4)/SO(2)) \simeq \mathfrak{so}(2)^\perp = \mathfrak{m}^\lambda \oplus \mathfrak{p}'$ with the positive definite inner product $\langle \cdot, \cdot\rangle_\lambda$ which is defined by the following three properties:

(i) $\langle \mathfrak{m}^\lambda, \mathfrak{p}'\rangle_\lambda = 0$;

(ii) the restrictions of $(\cdot, \cdot)_\lambda$ and $\langle \cdot, \cdot\rangle_\lambda$ to $\mathfrak{m}^\lambda$ coincide;

(iii) $\langle \cdot, \cdot\rangle_\lambda = 2(\cdot, \cdot)_\lambda$ on $\mathfrak{p}' \times \mathfrak{p}'$.

We then equip $M = \mathrm{SO}(4)/\mathrm{SO}(2)$ with the SO(4)-invariant metric, also denoted by $\langle\cdot,\cdot\rangle_\lambda$, which coincides at p with the above defined inner product. Then, by [8, Lemma 6.2], the subspace of symmetry at p is $\mathfrak{p}'.p$, unless $(M, \langle\cdot,\cdot\rangle_\lambda)$ is symmetric (observe that M is simply connected).

Since the fixed set of the isotropy representation of SO(2) on T_pM has dimension 1, it follows that the action of SO(2) on $\mathfrak{m}^\lambda$ is non-trivial. Let e_1, e_2, e_3 be an orthonormal basis of $\mathfrak{m}^\lambda \simeq \mathfrak{m}^\lambda.p$ with respect to $\langle\cdot,\cdot\rangle_\lambda$. We may assume that if $\mathbb{R}X_0 = \mathfrak{so}(2)$ then $[X_0, e_1] = 0$, $[X_0, e_2] = e_3$ and $[X_0, e_3] = -e_2$. Observe that the isotropy group SO(2) acts trivially on $\mathbb{R}e_1$ and irreducibly on the linear span $\mathbb{V}$ of e_2 and e_3. Let $\langle\cdot,\cdot\rangle$ be an SO(4)-invariant metric on $M = \mathrm{SO}(4)/\mathrm{SO}(2)$ such that $\mathfrak{m}^\lambda.p$ is perpendicular to the subspace of symmetry $\mathfrak{p}'.p = \mathfrak{so}(3).p$. Then, up to rescaling, $\langle\cdot,\cdot\rangle$ has the following four properties:

(i) $\langle\cdot,\cdot\rangle$ coincides with $\langle\cdot,\cdot\rangle_\lambda$ on $\mathfrak{p}'.p$;
(ii) $\langle e_1, \mathbb{V}\rangle = 0$;
(iii) $\langle e_1, e_1\rangle = s$ for some $s > 0$;
(iv) $\langle\cdot,\cdot\rangle = t\langle\cdot,\cdot\rangle_\lambda$ on $\mathbb{V}$ for some $t > 0$.

We will now prove that $s + t = 2$. Let $X \in \mathfrak{p}'$. Then $\mathrm{SO}(3)\cdot p$ is a totally geodesic submanifold of $(M, \langle\cdot,\cdot\rangle)$ and $X|_{\mathrm{SO}(3)\cdot p}$ is an intrinsic transvection of $\mathrm{SO}(3)\cdot p$ at p. From (6.1) we know that X is a transvection at p if and only if

$$\langle[\xi, X], \eta\rangle_p + \langle[\xi, \eta], X\rangle_p + \langle[X, \eta], \xi\rangle_p = 0 \tag{6.3}$$

for all $\xi, \eta \in \mathfrak{m}^\lambda$. First of all, note that the orthogonal projection of $[e_2, e_3]$ onto $\mathfrak{so}(3)$ is a multiple of X_0. In fact, $[X_0, [e_2, e_3]] = [[X_0, e_2], e_3] + [e_2, [X_0, e_3]] = 0$. Now decompose $[e_2, e_3] = Z + \psi$ with $Z \in \mathfrak{so}(3)$ and $\psi \in \mathfrak{m}^\lambda$. Then $[X_0, Z] = 0$, and hence $Z = aX_0$, since $\mathfrak{so}(3)$ has rank one (and so $Z.p = 0$). Next, we have

$$\begin{aligned} 2\langle\nabla_{e_1}X, e_2\rangle &= \langle[e_1, X], e_2\rangle_p + \langle[e_1, e_2], X\rangle_p + \langle[X, e_2], e_1\rangle_p \\ &= t\langle[e_1, X], e_2\rangle_{\lambda|p} + \langle[e_1, e_2], X\rangle_{\lambda|p} + s\langle[X, e_2], e_1\rangle_{\lambda|p}. \end{aligned} \tag{6.4}$$

The projection $\pi : (M, \langle\cdot,\cdot\rangle_\lambda) \to \mathrm{SO}(4)/\mathrm{SO}(3) = S^3$ is a Riemannian submersion, up to a rescaling of the metric. We denote by ∇^λ the Levi-Civita connection of M with respect to $\langle\cdot,\cdot\rangle_\lambda$. Since e_1 and e_2 are projectable vector fields, which are horizontal along $\mathrm{SO}(3)\cdot p$, we obtain

$$0 = (X\langle e_1, e_2\rangle_\lambda)_p = \langle\nabla^\lambda_X e_1, e_2\rangle_{\lambda|p} + \langle e_1, \nabla^\lambda_X e_2\rangle_{\lambda|p} = \langle[X, e_1], e_2\rangle_{\lambda|p} + \langle e_1, [X, e_2]\rangle_{\lambda|p},$$

because $[X, e_i]_p = (\nabla^\lambda_X e_i)_p$ and $(\nabla^\lambda_{e_i} X)_p = 0$. Inserting this into (6.4) yields

$$2\langle\nabla_{e_1}X, e_2\rangle = (t + s)\langle[e_1, X], e_2\rangle_{\lambda|p} + \langle[e_1, e_2], X\rangle_{\lambda|p}. \tag{6.5}$$

If $s = t = 1$ we have $\langle\nabla_{e_1}X, e_2\rangle = 0$ since X is parallel at p because $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_\lambda$ in this case. From (6.5) we then get $2\langle[e_1, X], e_2\rangle_{\lambda|p} = -\langle[e_1, e_2], X\rangle_{\lambda|p}$ in this case. We have $[\mathfrak{m}^\lambda, \mathfrak{m}^\lambda]^{\mathfrak{so}(3)} = \mathfrak{so}(3)$, where $(\ )^{\mathfrak{so}(3)}$ denotes the projection onto $\mathfrak{so}(3)$. In fact, this projection is not trivial, since $\mathfrak{m}^\lambda$ is not a Lie algebra and is Ad(SO(3))-invariant. Recall that, as we have shown, $[e_2, e_3]^{\mathfrak{so}(3)} \subset \mathfrak{so}(2)$. Then $[e_1, e_2]$ projects non-trivially into $\mathfrak{p}'$. If X were parallel at p, for any X in $\mathfrak{p}'$, then we would also have $(s+t)\langle [e_1, X], e_2\rangle_{\lambda|p} = -\langle [e_1, e_2], X\rangle_{\lambda|p}$ for any $X \in \mathfrak{p}'$, which implies that $-\langle [e_1, e_2], X\rangle_{\lambda|p} = 0$. In particular, for X equal to the projection of $[e_1, e_2]$ onto $\mathfrak{p}'$, this gives a contradiction. This implies that X is a transvection of $(M, \langle\cdot,\cdot\rangle)$ at p if and only if $t = 2 - s$, $0 < s < 2$.

We denote this metric by $\langle\cdot,\cdot\rangle_{(\lambda,s)}$ with $\lambda > 0$ and $0 < s < 2$. If we replace λ by $1/\lambda$, the metrics are homothetical, so we may assume that $0 < \lambda \leq 1$ (see Remark 6.2).

Case (b): $\lambda < 0$, that is, $(\cdot,\cdot)_\lambda = (B, \lambda B)$ is a pseudo-Riemannian bi-invariant metric on $\mathfrak{so}(4)$. By making the same construction as in Case (a), possibly changing the sign of the metric, we obtain a pseudo-Riemannian metric $\langle\cdot,\cdot\rangle_\lambda$ on M that is positive definite on $\mathfrak{so}(3).p$ and negative definite on its orthogonal complement $\mathfrak{m}^\lambda.p$. Moreover, if $X \in \mathfrak{p}'.p$, then $(\nabla^\lambda X)_p = 0$. As in Case (a), such a metric can only be deformed when rescaling by s on $\mathbb{R}e_1$ and by $2 - s$ on $\mathbb{V}$ (in order that X be a transvection at p). But s and $2 - s$ cannot be both negative if the metric $\langle\cdot,\cdot\rangle_{(\lambda,s)}$ is to be Riemannian. So this case can be excluded.

We conclude that if the index of symmetry of SO(4)/SO(2) is 2, then the Riemannian metric has to be of the form $\langle\cdot,\cdot\rangle_{(\lambda,s)}$ with $\lambda > 0$, $0 < s < 2$.

Conversely, such metrics have index of symmetry 2, unless the space is globally symmetric. In fact, the distribution of symmetry on SO(4)/SO(2) descends to an SO(4)-invariant (and therefore parallel) distribution on the irreducible symmetric space $S^3 =$ SO(4)/SO(3). Such a distribution must be trivial, and if the rank is zero the index of symmetry of SO(4)/SO(2) is 2, while if the rank is maximal then SO(4)/SO(2) has index of symmetry 5 and so it is a symmetric space.

Remark 6.2. Let us consider the bi-invariant inner product $(B, \lambda B)$, $\lambda > 0$, on $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, where $-B$ is the Killing form of $\mathfrak{so}(3)$. The involution τ of Spin(4) $=$ Spin(3) $\times$ Spin(3) that permutes the factors maps both diag(SO(3)) and diag(SO(2)) into itself. So τ induces an isomorphism $\bar{\tau}$ of $M =$ Spin(4)/diag(Spin(2)) into itself. The map $\bar{\tau}$ is an isometry from $(M, \langle\ ,\ \rangle)$ into $(M, \langle\ ,\ \rangle')$, where $\langle\ ,\ \rangle$ is the normal homogeneous metric with respect to $(B, \lambda B)$, and $\langle\ ,\ \rangle'$ is the normal homogeneous metric with respect to $(\lambda B, B)$. The same is true if we rescale the metrics by a factor 2, as in our construction, on the tangent space of diag(Spin(3))/diag(Spin(2)) at $[e]$. Now observe that the normal homogeneous metric on M with respect to $(\lambda B, B)$, or that modified as before, is homothetical to the normal homogeneous metric induced by $(B, \lambda^{-1}B)$.

Remark 6.3. A compact, simply connected, Riemannian symmetric space of dimension 5 is isometric to one of the following spaces: $S^2 \times S^3$, S^5 or SU(3)/SO(3). The last space is irreducible and of rank 2.

The homogeneous space SO(4)/SO(2) is not homeomorphic to S^5. In fact, from the long exact homotopy sequence of the fibration SO(2) $\to$ SO(4) $\to$ SO(4)/SO(2) it follows that $\pi_3(\mathrm{SO}(4)/\mathrm{SO}(2)) = \mathbb{Z} \oplus \mathbb{Z} \neq \pi_3(S^5)$.

The space $M^5 =$ SO(4)/SO(2), with any SO(4)-invariant metric, can never be isometric to an irreducible symmetric space of higher rank. In fact, if $p = [e]$, the isotropy representation of SO(2) on T_pM is the direct sum of two copies of the standard representation of SO(2) on $\mathbb{R}^2$, plus a trivial 1-dimensional representation. If $\phi \in$ SO(2) is the rotation of angle π (with the standard representation), then ϕ represents an element of the isotropy group of M which has eigenvalue -1 with multiplicity 4 and eigenvalue 1 with multiplicity 1. If M is a symmetric space, then the decomposition of ϕ with respect to the symmetry σ at p, via the isotropy representation, has eigenvalue 1 with multiplicity 4 and eigenvalue -1 with multiplicity 1. Then the connected component containing p of the fixed set of $\sigma \circ \phi$ would be a totally geodesic hypersurface N of M. Let K' be the full connected isotropy group of N at p. We may regard K' as a subset of K, the full connected isotropy group of the symmetric space M. Observe that K', via the isotropy representation, acts trivially on the 1-dimensional normal space $\nu_p(N) \simeq \mathbb{R}$ of N at p. Let $\bar{R}$ be the direct product of R' and the zero tensor on $\nu_p(N)$, where R' is the curvature tensor of N at p. Then $\bar{R}_{x,y} \in \mathfrak{k}$ and so, by Simons' Theorem [5, 9], if M is of rank at least 2, then $\bar{R}$ must be a scalar multiple of R, the curvature tensor of M at p. This is a contradiction if M is an irreducible symmetric space. Thus M cannot be isometric to the irreducible rank 2 symmetric space $\mathrm{SU}(3)/\mathrm{SO}(3)$.

Note that $\mathrm{SO}(4)/\mathrm{SO}(2)$ is diffeomorphic to $S^2 \times S^3$, since the first space is diffeormorphic to the unit tangent bundle of the (parallelizable) sphere S^3.

Example 6.4 (Product of spheres). We denote by S^2 the sphere of dimension 2 and radius ρ and by S^3 the sphere of dimension 3 and radius 1, and set $M = S^2 \times S^3$. Observe that any product of a round 2-sphere and a round 3-sphere is homothetic to M with a suitable ρ.

The group $\mathrm{Spin}(4) = \mathrm{Spin}(3) \times \mathrm{Spin}(3)$ acts transitively by isometries on $M = S^2 \times S^3 \simeq S^2 \times \mathrm{Spin}(3)$ in the following way:

$$(g, h)((q, k)) = (\pi(g)(q), gkh^{-1}),$$

where $(g, h) \in \mathrm{Spin}(3) \times \mathrm{Spin}(3)$, $q \in S^2$, $k \in \mathrm{Spin}(3) \simeq S^3$, and π is the canonical projection from $\mathrm{Spin}(3)$ onto $\mathrm{SO}(3)$. The isotropy group at $p = (\rho e_1, e) \in S^2 \times \mathrm{Spin}(3)$ is $\mathrm{diag}(\mathrm{SO}(2)) \subset \mathrm{Spin}(3) \times \mathrm{Spin}(3)$. After making this action effective, one finds that $\mathrm{SO}(4)$ acts transitively on M and the isotropy group is conjugate to $\mathrm{SO}(2)$, where $\mathrm{SO}(2) \subset \mathrm{SO}(4)$ is the standard inclusion. Recall that for $\mathfrak{so}(n)$ the Killing form $-B$ is given by

$$B(X, Y) = -(n-2)\,\mathrm{trace}(X \circ Y).$$

For $n = 3$ the Killing form coincides with the negative of the usual inner product of matrices.

Let $p = (\rho e_1, e) \in M = S^2 \times \mathrm{Spin}(3)$, where $e_1 = (1, 0, 0)$. The parallel Killing fields at the identity e of $\mathrm{Spin}(3) = S^3$ are the elements of $\mathfrak{so}(3) \times \mathfrak{so}(3)$ of the form $Z = (X, -X)$ (regarded as a Killing field on $\mathrm{Spin}(3)$). The parallel Killing fields on S^2 at ρe_1 are elements in the Cartan subspace

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & a & b \\ -a & 0 & 0 \\ -b & 0 & 0 \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

associated with the symmetric pair $(\mathrm{SO}(3), \mathrm{SO}(2))$. Therefore an element $Z \in \mathfrak{so}(3) \times \mathfrak{so}(3)$ is parallel at $(\rho e_1, e)$ if and only if $Z = (Y, -Y)$ with $Y \in \mathfrak{p}$. Observe that the subspace $\mathfrak{p}^{(\rho e_1,e)} = \{(Y, -Y) : Y \in \mathfrak{p}\}$ of parallel Killing fields at $(\rho e_1, e) \in S^2 \times \mathrm{Spin}(3)$ belonging to $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ has dimension 2. We use here the general notation of the paper, but take into account that the Cartan subspace is relative to the presentation group (i.e., the parallel Killing fields at a given point that lie in the Lie algebra $\mathfrak{so}(3) \times \mathfrak{so}(3)$). The (relative) Cartan subspace is given by $\mathfrak{p}^{(\rho e_1,e)}$, which spans the involutive Lie algebra

$$\mathfrak{g}^{(\rho e_1,e)} = \mathrm{diag}(\mathfrak{so}(2)) \oplus \mathfrak{p}^{(\rho e_1,e)},$$

where $\mathfrak{so}(2) = \{u \in \mathfrak{so}(3) : u.e_1 = 0\}$.

Up to homothety, $S^2 \times S^3$ must carry a metric $\langle \cdot, \cdot \rangle_{(\lambda,s)}$ as described above (recall that ρ is the radius of S^2 and 1 is the radius of S^3). We will now determine λ. Observe that $G^{(\rho e_1,e)}$, the group which is generated by the transvections at $(\rho e_1, e)$, is not the canonical $\mathrm{diag}(\mathrm{Spin}(3)) \subset \mathrm{Spin}(3) \times \mathrm{Spin}(3)$ (but it must be conjugate to it). So the reductive complement, associated to the Killing fields in $\mathfrak{so}(3) \times \mathfrak{so}(3)$ that are always perpendicular to $L((\rho e_1, e)) = G^{(\rho e_1,e)} \cdot (\rho e_1, e)$, is conjugate to $\mathfrak{m}_\lambda = \{(Z, -\lambda^{-1} Z) : Z \in \mathfrak{so}(3)\}$.

We will find $h \in \mathrm{Spin}(3)$ such that $G^{(\rho e_1,h)} = \mathrm{diag}(\mathrm{Spin}(3))$. In order to simplify the calculations, we will use the quaternions. Identify Spin(3) with the unit sphere of the quaternionic space $\mathbb{H} = \{a + ib + cj + dk : a, b, c, d \in \mathbb{R}\}$, $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. Let $\pi : \mathrm{Spin}(3) \to \mathrm{SO}(3)$ be the canonical projection. By identifying $\mathbb{R}^3$ with the purely imaginary quaternions $\Im(\mathbb{H}) = \{q \in H : \bar{q} = -q\}$ we obtain

$$\pi(g)(x) = gxg^{-1} = gx\bar{g}.$$

The Lie algebra $\mathfrak{so}(3)$ of Spin(3) is identified with $\Im(\mathbb{H})$ with the bracket $[x, y] = xy - yx$. Observe that, with these identifications, $i = e_1$, $1 = e$. The exponential map is given by $\mathrm{Exp}(x) = \cos(\|x\|) + \sin(\|x\|)\frac{1}{\|x\|}x$. If $x \in \Im(\mathbb{H})$, then $\frac{d}{dt}\big|_{t=0}\pi(\mathrm{Exp}(tx))(z) = xz - zx$. So x defines the Killing field of $\Im(\mathbb{H})$ given by $z \mapsto x.z = xz - zx$. Observe that

$$\mathfrak{so}(2) = \{U \in \mathfrak{so}(3) : U.e_1 = 0\} = \{w \in \Im(\mathbb{H}) : wi - iw = 0\} = \mathbb{R}i.$$

With these identifications the (relative) Cartan subspace $\mathfrak{p}$ is given by the linear span of j and k. It is not hard to see that $(1, -i)G^{(\rho i,1)}(1, -i)^{-1} = \mathrm{diag}(\mathrm{Spin}(3))$, and thus

$$G^{(\rho i,i)} = G^{(1,-i)\cdot(\rho i,1)} = (1, -i)G^{(\rho i,1)}(1, -i)^{-1} = \mathrm{diag}(\mathrm{Spin}(3)).$$

Moreover, $\mathfrak{k}^{(\rho i,i)} = \mathbb{R}i$ and

$$\mathfrak{p}^{(\rho e_1,i)} = \mathrm{diag}(\mathfrak{p}) = \{(Y, Y) : Y \in \mathfrak{p}\} = \{(v, v) : v \in \text{linear span of } \{j, k\}\}.$$

If $v \in \mathfrak{p}$, then

$$(v, v).(\rho i, i) = (v.\rho i, v.i) = (\rho(vi - iv), vi - iv) = (2\rho vi, 2vi).$$

Observe that $vi \in \mathfrak{p}$, and therefore

$$\mathfrak{s}^{(\rho i,i)} = \mathfrak{p}^{(\rho i,i)}.(\rho i, i) = \{(\rho v, v) : v \in \mathfrak{p}\}.$$

This subspace must be perpendicular to $\mathfrak{m}^\lambda.(\rho i, i)$, where

$$\mathfrak{m}^\lambda = \{(Z, -\lambda^{-1}Z) : Z \in \mathfrak{so}(3) = \Im(\mathbb{H})\}.$$

Take $Z = k, Y = j \in \mathfrak{p}$. Then $(k, -\lambda^{-1}k).(\rho i, i) = (2\rho j, (1-\lambda^{-1})j)$. This must be perpendicular to $(\rho j, j)$. Then $2\rho^2 = \lambda^{-1} - 1$, and therefore

$$\lambda = \frac{1}{1+2\rho^2}.$$

The fixed vectors in $\mathfrak{m}^{(1+2\rho^2)^{-1}}$ are $\mathbb{R}(i, -(1+2\rho^2)i) \in \mathfrak{so}(3) \oplus \mathfrak{so}(3)$. Let us compare the metric on the product of spheres with the one given by the bi-invariant inner product $(B, (1+2\rho^2)^{-1}B)$. The norm of $(i, -(1+2\rho^2)i)$ with the given metric is

$$\begin{aligned}\|(i, -(1+2\rho^2)i).(\rho i, i)\|^2 &= \|([i, \rho i], ii + i(1+2\rho^2)i)\|^2 \\ &= \|(0, -2(1+\rho^2)\|^2 = 4(1+\rho^2)^2,\end{aligned}$$

and the norm using $(B, (1+2\rho^2)^{-1}B)$ is

$$\begin{aligned}\|(i, -(1+2\rho^2)i)\|^2 &= B(i,i) + \frac{1}{1+2\rho^2}B(-(1+2\rho^2)i, -(1+2\rho^2)i) \\ &= 8 + 8(1+2\rho^2) = 16(1+\rho^2),\end{aligned}$$

since $B(i,i) = 8$. So the quotient is $s' = \frac{1}{4}(1+\rho^2)$.

Let us choose the element $(j, -(1+2\rho^2)j) \in \mathfrak{m}^{(1+2\rho^2)^{-1}}$ that is perpendicular to the fixed vectors $\mathbb{R}(i, -(1+2\rho^2)i)$ of the isotropy group. The norm with the given metric is

$$\begin{aligned}\|(j, -(1+2\rho^2)j).(\rho i, i)\|^2 &= \|([j, \rho i], ji + i(1+2\rho^2)j)\|^2 = \|(-2\rho k, 2\rho^2 k)\|^2 \\ &= 4\rho^2 + 4\rho^4 = 4\rho^2(1+\rho^2),\end{aligned}$$

and the norm using $(B, (1+2\rho^2)^{-1}B)$ gives, as before,

$$\|(j, -(1+2\rho^2)j)\|^2 = 16(1+\rho^2).$$

The quotient is $t' = \frac{1}{4}\rho^2$.

We have $s' + t' \neq 2$ because we need to rescale the metric in line with our classification. So, define $s = 2s'/(s'+t')$, and the metric $\langle\cdot,\cdot\rangle_{((1+2\rho^2)^{-1}, s)}$ is the metric in the family. An explicit calculation gives

$$s = 2\frac{1+\rho^2}{1+2\rho^2} \quad \text{and} \quad t = 2\frac{\rho^2}{1+2\rho^2}.$$

For instance, if $\rho = 1$, then $\lambda = 1/3$, $s = 4/3$ and $t = 2/3$.

Remark 6.5. Recall that, in the above examples of products of spheres, $\lambda = (1+2\rho^2)^{-1}$ and $s = 2(1+\rho^2)/(1+2\rho^2)$. Then $s = \lambda + 1$. Therefore the family of examples of products of spheres as previously discussed corresponds to the family of metrics $\langle\cdot,\cdot\rangle_{(\lambda,\lambda+1)}$, where $0 < \lambda < 1$ (and the quotient of the radius of the 2-sphere by the radius of the 3-sphere is given by $\rho = \sqrt{(1-\lambda)/(2\lambda)}$). In particular, the reductive complement is never the standard one, i.e., $\lambda \neq 1$. Observe also that $0 < t < s < 2$ (recall that $s + t = 2$). Then the metric does not project down, as a Riemannian submersion, to the quotient $\mathrm{SO}(4)/\mathrm{SO}(3)$ of M by the leaves of symmetry (relative to $\mathrm{SO}(4)$).

Remark 6.6. Any transitive action of $\mathrm{Spin}(3) \times \mathrm{Spin}(3)$ on $S^2 \times S^3 \simeq S^2 \times \mathrm{Spin}(3)$ is equivalent to the previously described action or to the action given by

$$(g,h)((u,d)) = (\pi(g)(u), h(d)).$$

However, the isotropy group of the latter action is $\mathrm{SO}(2) \times \{e\}$ and fixes the 3-dimensional space $T_d(\mathrm{Spin}(3))$. So this homogeneous space is not (equivariantly) isomorphic to the canonical $\mathrm{SO}(4)/\mathrm{SO}(2)$.

We can now state the main result of this section.

Theorem 6.7. *Let M be an n-dimensional, simply connected, compact, irreducible Riemannian homogeneous manifold with $n > 3$. Then the coindex of symmetry of M is equal to 3 if and only if M is homothetic to $M = \mathrm{SO}(4)/\mathrm{SO}(2)$ with a metric of the family $\langle\cdot,\cdot\rangle_{(\lambda,s)}$, where $0 < \lambda \leq 1$, $0 < s < 2$ and $s \neq \lambda + 1$. (If $s = \lambda + 1$, then, up to homothety, M is a product of spheres $S^2_\rho \times S^3$ with $\rho = \sqrt{(1-\lambda)/(2\lambda)}$.)*

Proof. It only remains to prove that different pairs (λ, s) correspond to non-homothetical metrics. First of all, we note that $\mathrm{SO}(4)$ is the (connected) full isometry group of $M = \mathrm{Spin}(4)/\mathrm{diag}(\mathrm{SO}(2)) = \mathrm{SO}(4)/\mathrm{SO}(2)$ with any of the metrics of the family $\langle\cdot,\cdot\rangle_{(\lambda,s)}$. (Note that M is not symmetric.) Otherwise, by Remark 6.3, it would be a product of spheres. But such a product of spheres corresponds to $s = \lambda + 1$ (see Remark 6.6). So, by the paragraph before Remark 6.2, the index of symmetry of M is 2. So, in the 3-dimensional quotient N of M by the leaves of symmetry, the group $\mathrm{SO}(4)$ acts by isometries (with the normal homogeneous metric). Then, up to a cover, N is a sphere, and hence $\mathrm{SO}(4)$ must be the full (connected) isometry group of N. Therefore, if the isometry group $I^o(M)$ of M is bigger than $\mathrm{SO}(4)$, then $I^o(M)$ has a proper (connected) normal subgroup H acting trivially on N. If $L([e]) = \mathrm{SO}(3)/\mathrm{SO}(2) \simeq S^2$ is the leaf of symmetry at $[e]$, then $H \cdot L([e]) = L([e])$ and H commutes with $\mathrm{SO}(3)$, which is a contradiction. Hence we must have $I^o(M) = \mathrm{SO}(4)$.

Assume that the pairs (λ, s) and (λ', s') correspond to homothetical metrics (and do not correspond to the exceptions that are products of spheres). Assume that $\lambda \neq \lambda'$, say $\lambda < \lambda'$. If h is the homothety between the metrics, then it induces a Lie algebra isomorphism $\rho = h_*$ of $\mathfrak{so}(4)$ (the Lie algebra of the full isometry group) that maps $\mathrm{diag}(\mathfrak{so}(3))$ into itself (since it corresponds to the group of transvections at $[e]$) and ρ maps $\mathrm{diag}(\mathrm{SO}(2))$ into itself (the Lie algebra of the isotropy at $[e]$). Moreover, $\rho(\mathfrak{m}^\lambda) = \mathfrak{m}^{\lambda'}$. In fact, these subspaces are given by the geometry as the Killing fields which are always perpendicular to the leaves of symmetry $\mathrm{SO}(3)/\mathrm{SO}(2) = \mathrm{diag}(\mathrm{SO}(3))/\mathrm{diag}(\mathrm{SO}(2))$, with the respective metrics. Observe that ρ must preserve (B, B), where $-B$ is the Killing form of $\mathfrak{so}(3)$. Let $(u, 0) \in \mathfrak{so}(3) \oplus \mathfrak{so}(3) = \mathfrak{so}(4)$. Then

$$(u, 0) = \frac{1}{1+\lambda}(u, u) + \frac{\lambda}{1+\lambda}\left(u, -\frac{1}{\lambda}u\right),$$

which gives the decomposition of $(u, 0)$ in terms of the direct sum

$$\mathfrak{so}(3) \oplus \mathfrak{so}(3) = \mathrm{diag}(\mathfrak{so}(3)) \oplus \mathfrak{m}^{\lambda}.$$

Then the projection onto $\mathrm{diag}(\mathfrak{so}(3))$ is given by

$$\pi^{\lambda}((u, 0)) = \frac{1}{1+\lambda}(u, u).$$

We also have

$$(0, v) = \frac{\lambda}{1+\lambda}(v, v) - \frac{\lambda}{1+\lambda}\left(v, -\frac{1}{\lambda}v\right),$$

and so

$$\pi^{\lambda}((0, v)) = \frac{\lambda}{1+\lambda}(v, v).$$

Since $\rho(\mathrm{diag}(\mathfrak{so}(3))) = \mathrm{diag}(\mathfrak{so}(3))$ and $\rho(\mathfrak{m}^{\lambda}) = \mathfrak{m}^{\lambda'}$, we obtain

$$\rho \circ \pi^{\lambda} = \pi^{\lambda'}.$$

Since $\rho : \mathfrak{so}(3) \oplus \mathfrak{so}(3) \to \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ is a Lie algebra isomorphism, $\rho((u, 0))$ is of the form either $(u', 0)$ or $(0, u')$. Moreover, since ρ preserves the Killing form, $B(u, u) = B(u', u')$. Also,

$$\begin{aligned} B\big(\pi^{\lambda'}(\rho((u, 0))), \pi^{\lambda'}(\rho((u, 0)))\big) &= B\big(\rho(\pi^{\lambda}((u, 0))), \rho(\pi^{\lambda}((u, 0)))\big) \\ &= B\big(\pi^{\lambda}((u, 0)), \pi^{\lambda}((u, 0))\big). \end{aligned}$$

Let us choose $u \neq 0$. If $\rho((u, 0)) = (u', 0)$, the above equality yields

$$\frac{1}{1+\lambda'}B(u', u') = \frac{1}{1+\lambda}B(u, u),$$

and so $1 + \lambda' = 1 + \lambda$. This contradicts $\lambda \neq \lambda'$. If $\rho((u, 0)) = (0, u')$, then the previous equality implies $\frac{\lambda'}{1+\lambda'} = \frac{1}{1+\lambda}$, which also gives a contradiction, since $0 < \lambda < \lambda' \leq 1$. It follows that $\lambda = \lambda'$.

Since the curvature of the leaf of symmetry $\mathrm{SO}(3)/\mathrm{SO}(2)$ of $\mathrm{SO}(4)/\mathrm{SO}(2)$ with respect to the metric $\langle \cdot, \cdot \rangle_{(\lambda, t)}$ depends only on λ (and B), and since the homothety h maps leaves of symmetry onto leaves of symmetry, we see that the homothety must be an isometry. We choose v in $\mathfrak{m}^{\lambda}$ of unit length and fixed by the isotropy group. Then the length of the closed geodesic $\gamma_v(t)$ determined by v is equal to as, where a is a constant. Since h maps $\mathfrak{m}^{\lambda}$ onto $\mathfrak{m}^{\lambda'}$ and fixed vectors of the isotropy group to fixed vectors of the isotropy group, we have $h(\gamma_v(t)) = \gamma_{v'}(t)$, where $dh(v) = v'$. Since the second geodesic has length as', it follows that $s = s'$. □

7. Classification for coindex of symmetry equal to 2

The main result of this section is the following classification:

Theorem 7.1. *Let M be an n-dimensional ($n > 2$), simply connected, compact, irreducible Riemannian homogeneous manifold with coindex of symmetry $k = 2$. Then $M = \mathrm{Spin}(3)$ with a left-invariant Riemannian metric that belongs to one of the two families $\langle\cdot,\cdot\rangle_s$ ($0 < s < 1$) and $\langle\cdot,\cdot\rangle^t$ ($0 < t \neq 2$) which are described below. No two of these metrics are homothetic to each other. The second family of metrics corresponds to Berger sphere metrics.*

The rest of this section is devoted to the proof of Theorem 7.1. If M is a homogeneous irreducible Riemannian manifold with coindex of symmetry $k = 2$, then $M = \mathrm{Spin}(3)$ with a left-invariant Riemannian metric by Theorem 5.3.

Let us first describe the left-invariant Riemannian metrics on $\mathrm{Spin}(3) \simeq S^3$. As usual, we will identify a left-invariant Riemannian metric on Spin(3) with a positive definite inner product on $T_e(\mathrm{Spin}(3)) \simeq \mathfrak{so}(3)$. Let B be the positive definite inner product on $\mathfrak{so}(3)$ given by $B(X, Y) = -\mathrm{trace}(XY)$ (so $-B$ is the Killing form of $\mathfrak{so}(3)$). Any positive definite inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{so}(3)$ is of the form $\langle X, Y\rangle = B(AX, Y)$, where A is a positive definite symmetric endomorphism of $\mathfrak{so}(3)$ with respect to B. Observe that any positive definite inner product $\langle X, Y\rangle = B(AX, Y)$ is isometric to the inner product

$$B\big(A(\mathrm{Ad}(g)(X)), \mathrm{Ad}(g)(Y)\big) = B\big((\mathrm{Ad}(g))^{-1}A(\mathrm{Ad}(g))(X), Y\big)$$

for any $g \in \mathrm{Spin}(3)$ (the isometry between the corresponding two left-invariant Riemannian metrics is given by conjugation with g in Spin(3)). Note that Ad(Spin(3)) coincides with the full special orthogonal group $\mathrm{SO}(\mathfrak{so}(3), B)$. Then, to prescribe an arbitrary left-invariant Riemannian metric on Spin(3) (modulo isometries), one only needs to know the eigenvalues of A.

We identify $X \in \mathfrak{so}(3)$ with the Killing field $q \mapsto X.q = \frac{d}{dt}\big|_{t=0} \mathrm{Exp}(tX)(q)$. The Lie algebra structure on $\mathfrak{so}(3)$ will be that of Killing fields. So the Lie bracket is given by $[X, Y] = XY - YX$, which is minus the bracket of left-invariant vector fields, since a Killing field may be regarded as a right-invariant vector field.

Let $\mathfrak{s}$ be the 1-dimensional distribution of symmetry on Spin(3). Since $\mathfrak{s}$ is a left-invariant distribution, we may assume that $\mathfrak{s}_1 = \mathbb{R}i$, where we are using the quaternions as before. We identify Spin(3) with the unit sphere of $\mathbb{H}$ and $\mathfrak{so}(3)$ with $\mathrm{Im}(\mathbb{H})$. With this identification the bracket of $q_1, q_2 \in \mathrm{Im}(\mathbb{H})$ is given by $q_1q_2 - q_2q_1$, which coincides with $-[q_1, q_2]$, where $[\cdot,\cdot]$ is the bracket between Killing fields of $(\mathrm{Spin}(3)\langle\cdot,\cdot\rangle)$ (identifying $q \in \mathrm{Im}(\mathbb{H})$ with the Killing field $x \mapsto q.x$). The Killing form $-B$ is given by $B(q, q) = 8|q|^2$, $q \in \mathrm{Im}(\mathbb{H})$.

Just as for the case $k = 3$, we define

$$\mathfrak{m} = \{q \in \mathrm{Im}(\mathbb{H}) : q \text{ is always perpendicular to } L(1) = \mathrm{e}^{ti}\}.$$

Then $\mathfrak{m}$ is an $\mathrm{Ad}(S^1)$-invariant subspace of $\mathrm{Im}(\mathbb{H}) \simeq \mathfrak{so}(3)$, where $S^1 = \{\mathrm{e}^{ti} : t \in \mathbb{R}\}$. Thus, by Remark 5.5, $\mathfrak{m}$ is unique, and so it coincides with the linear span of $\{j, k\}$. This implies that the vectors $j = j.1$ and $k = k.1$ of $T_1(\mathrm{Spin}(3))$ are perpendicular to $\mathfrak{s}_1 = \mathbb{R}i$. So $\langle i, j\rangle = 0 = \langle i, k\rangle$. Thus, if $\langle q, q'\rangle = B(Aq, q')$, then i is an eigenvector of A. By conjugating Spin(3) with some e^{ti}, we may assume that j and k are also eigenvectors of A. By rescaling the metric $\langle\cdot,\cdot\rangle$ we may assume that $Ai = 2i$ (in order to use a similar construction as for the case $k = 3$, where the normal homogeneous metric was at the first step perturbed by a factor of 2 on the distribution of symmetry). Let $Aj = sj$ and $Ak = tk$. We may assume that $0 < s \leq t$ (by conjugating Spin(3) with i if necessary). We will now consider i, j and k as Killing fields $I : q \mapsto i.q$, $J : q \mapsto j.q$ and $K : q \mapsto k.q$.

We first assume that $I^o(\mathrm{Spin}(3), \langle\cdot,\cdot\rangle) = \mathrm{Spin}(3)$. In this case we have $(\nabla I)_1 = 0$, since there are no more Killing fields than those induced by $\mathfrak{so}(3)$. Recall that for any homogeneous Riemannian manifold, if X, Y, Z are Killing fields, then the Levi-Civita connection is given by

$$2\langle\nabla_X Y, Z\rangle = \langle[X, Y], Z\rangle + \langle[X, Z], Y\rangle + \langle[Y, Z], X\rangle.$$

In fact, this equation comes from the well-known Koszul formula for the Levi-Civita connection, by observing that the Lie derivative of the metric along any Killing field is zero. So we have

$$0 = \langle[J, I], K\rangle + \langle[J, K], I\rangle + \langle[I, K], J\rangle.$$

Since $[J, I]_1 = ij - ji = 2k$, $[J, K]_1 = kj - jk = -2i$ and $[I, K]_1 = ki - ik = 2j$, we get $0 = 2tB(k, k) - 4B(i, i) + 2sB(j, j)$. Since $B(i, i) = B(j, j) = B(k, k) \neq 0$, this implies $s + t = 2$. Conversely, if $s + t = 2$, a direct calculation shows that $(\nabla I)_1 = 0$. We conclude that $\langle\cdot,\cdot\rangle_s$, $0 < s \leq 1$, are the Spin(3)-invariant Riemannian metrics on Spin(3) such that the Killing field I is parallel at 1. So the index of symmetry is at least 1.

Remark 7.2. (i) The manifold $M = (\mathrm{Spin}(3), \langle\cdot,\cdot\rangle_s)$ is not a product. Otherwise, it would have a line as a de Rham factor. Assume that $0 < s < 1$. If the index of symmetry is greater than 1, then by Theorem 5.3, M would be symmetric. A direct computation shows that $(\nabla_J J)_1 = 0$. So $x \mapsto \mathrm{e}^{jx}$ is a closed geodesic of M with period $2\pi\sqrt{s}$. This period is different from the period $2\pi\sqrt{2}$ of the geodesic $x \mapsto \mathrm{e}^{ix}$ (recall that $\langle i, i\rangle = 2$ and $s < 1$). Thus M is not symmetric, as otherwise it would be isometric to a sphere, and hence all geodesics would have the same length. So the index of symmetry of M is 1.

(ii) Let $S^2 = \mathrm{Spin}(3)/S^1$ be the quotient of $M = (\mathrm{Spin}(3), \langle\cdot,\cdot\rangle_s)$ by the leaves of symmetry, where $S^1 = \{\mathrm{e}^{xi} : x \in \mathbb{R}\}$. It is not difficult to show that the projection $\pi : (\mathrm{Spin}(3), \langle\ ,\ \rangle_s) \to S^2 = \mathrm{Spin}(3)/S^1$ is a Riemannian submersion (possibly after rescaling the metric of S^2) if and only if $s = 1$ (and so $t = 1$). Assume that the full (connected) isometry group $I^o(M)$ of M with any left-invariant Riemannian metric with $k = 2$ satisfies $\dim(I^o(M)) > 3$. The compact group $I^o(M)$ acts on the quotient space S^2 (since any isometry preserves the foliation of symmetry). Thus, if S^2 has the normal homogeneous metric, then $I^o(M)$ acts by isometries, and so $I^o(M)$ must have a normal subgroup of positive dimension which acts trivially on S^2. If $X \neq 0$ belongs to the Lie algebra of this normal subgroup, then X defines a Killing field on M which must be tangent to the 1-dimensional distribution of symmetry $\mathfrak{s}$. This implies that for any two points p, q in a leaf of symmetry, there exists $h \in I^o(M)$ with $h(p) = q$ and such that h projects trivially to the quotient S^2. Then the projection $\pi : M \to S^2$ must be a Riemannian submersion (for some Spin(3)-invariant metric on S^2, which is unique up to scaling). This implies $s = t = 1$.

Assume that Spin(3) together with a left-invariant Riemannian metric has index of symmetry equal to 1. If there exists a point $g \in \mathrm{Spin}(3)$ such that $Z \in \mathfrak{so}(3)$ is tangent to the 1-dimensional leaf of symmetry $L(g)$ of M at g, then it must always be tangent to $L(g)$ (since the distribution of symmetry is invariant under isometries). This implies $L(g) = \mathrm{Exp}(tZ)(g)$ $(t \in \mathbb{R})$, and so $L(g)$ is closed (since all the 1-parameter subgroups of Spin(3) are closed).

In order to describe all left-invariant Riemannian metrics on $M = \mathrm{Spin}(3)$ it only remains to analyze the case where there is no parallel Killing field at 1 which belongs to $\mathfrak{so}(3)$. This implies that $\dim I^o(M) = 4$. In fact, observe that the dimension of the full isotropy group has to be 1, 2 or 3. In the last case M must be a round sphere and hence symmetric. The dimension of the isotropy group at $p \in M$ cannot be 2 because it would, via the isotropy representation, be an abelian 2-dimensional subgroup of $\mathrm{SO}(T_p(M)) \simeq \mathrm{SO}(3)$. Thus the dimension of the full isotropy group must be 1.

In this case there exists a non-trivial ideal $\mathfrak{a}$ of the Lie algebra $\mathfrak{g}$ of $G = I^o(M)$. Such an ideal must have dimension 1. In fact, this ideal must be complementary to $\mathfrak{so}(3)$, which must also be an ideal, since it has codimension 1 (and $\mathfrak{g}$ admits a bi-invariant metric). Moreover, since any $X \in \mathfrak{a}$ projects trivially to the quotient of M over the leaves of symmetry, X must always be tangent to $\mathfrak{s}$. Observe that X must be a left-invariant vector field since it commutes with $\mathfrak{so}(3)$. So, as previously observed, we may assume that $X = \hat{i}$, the left-invariant vector field with initial condition i at $1 \in \mathrm{Spin}(3)$ (i.e. $X_g = gi$). Recall that a Killing field associated with an element in $\mathfrak{so}(3)$ may be regarded as a right-invariant vector field. In particular, I is a right-invariant vector field ($I_g = ig$). Then the left-invariant Riemannian metric $\langle \cdot, \cdot \rangle$ of $M = \mathrm{Spin}(3)$ is $\mathrm{Ad}(\mathrm{Exp}(ti))$-invariant. This implies that i is an eigenvector of A at 1, and the eigenvalues of A in the orthogonal complement of i are equal, where $\langle x, y \rangle = B(Ax, y)$.

So the left-invariant Riemannian metric must be associated to a triple of numbers (t, t, a) corresponding to the eigenvalues associated to the eigenvectors j, k and i, respectively. By rescaling the metric we may assume that $a = 2$ (in order to be coherent with the first family of metrics $\langle \cdot, \cdot \rangle_s$). Conversely, a metric described by such a triple $(t, t, 2)$ has a parallel Killing field at 1. In fact, consider the two Killing fields $\hat{i}$ and I, which cannot be proportional, because no vector field of Spin(3) can be both left- and right-invariant. Since the integral curves of both Killing fields coincide at 1 and give a geodesic, we have $\nabla_i \hat{i} = 0 = \nabla_i \hat{I}$. Then the skew-symmetric endomorphisms $(\nabla \hat{i})_1$ and $(\nabla I)_1$ of $T_1 M$ must be proportional (since $\dim(M) = 3$). Thus there is a linear combination $\alpha \hat{i} + \beta I$ which is parallel at 1 (and it is non-zero, since $\hat{i}$ and I are not proportional). Observe that if $t = 1$, then I is parallel at 1, and so $\alpha = 0$ (the associated metric is the same as $\langle \cdot, \cdot \rangle_1$, previously described). If $t \neq 2$, then M cannot be symmetric, since the integral curves of I and J, starting at 1, have different length. If $t = 2$, then Spin(3) has the bi-invariant Riemannian metric, and so it is a symmetric space. We denote the left-invariant Riemannian metrics associated to $(t, t, 2)$ by $\langle \cdot, \cdot \rangle^t$, $0 < t \neq 2$.

Remark 7.3. (i) Any homothety between two different metrics in the union of the families $\langle\cdot,\cdot\rangle_s$, $0 < s < 1$, and $\langle\cdot,\cdot\rangle^t$, $0 < t \neq 2$ must be an isometry, since the lengths of the respective circles of symmetry are equal to $2\pi\sqrt{2}$.

(ii) No metric $\langle\cdot,\cdot\rangle_s$, $0 < s < 1$, is isometric to a metric $\langle\cdot,\cdot\rangle^t$, $t > 0$. In fact, the first family of metrics never define a Riemannian submersion onto S^2, the quotient of M by the leaves of symmetry, whereas the second family always does.

(iii) Let $M_s = (\mathrm{Spin}(3), \langle\cdot,\cdot\rangle_s)$. Then, from Remark 7.2(ii), $I^o(M_s) = \mathrm{Spin}(3)$ $(0 < s < 1)$. Observe that $s < 2 - s < 2$ are the eigenvalues of the symmetric tensor A_s that relates $\langle\cdot,\cdot\rangle_s$ to $\langle\cdot,\cdot\rangle = -B$, where B is the Killing form of $\mathfrak{so}(3)$. If $h : M_s \to M_{s'}$ is an isometry, then h induces a group isomorphism from $\mathrm{Spin}(3) = I^o(M_s)$ onto $\mathrm{Spin}(3) = I^o(M_{s'})$. This implies that the eigenvalues of A_s are the same as those of $A_{s'}$, and hence $s = s'$.

(iv) If $t \neq t'$, then $\langle\cdot,\cdot\rangle^t$ is not isometric to $\langle\cdot,\cdot\rangle^{t'}$. In fact, $t/2$ is the radius of the sphere, obtained as the quotient of M by the leaves of symmetry, such that the projection is a Riemannian submersion.

The above remark finishes the proof of Theorem 7.1.

8. Examples from fibre bundles over polars

In this section we review the construction of certain fibre bundles by Nagano and Tanaka [4], and show how to get examples of compact simply connected Riemannian homogeneous manifolds with non-trivial index of symmetry.

Let $M = G/K$ be an irreducible simply connected symmetric space of compact type and choose $o \in M$ such that $K \cdot o = o$. Let $B \neq \{o\}$ be a connected component of the set of fixed points of σ_o, where σ_o is the geodesic symmetry of M at o. Note that B is a totally geodesic submanifold, since it is a connected component of the fixed point set of an isometry. There always exists such a totally geodesic submanifold B since the midpoint of a closed geodesic through o is fixed by σ_o.

Let d be the distance between o and B, and choose $q \in B$ such that the distance from o to q is equal to d. Let γ be a unit speed geodesic through o and q such that $\gamma(0) = o$ and $\gamma(d) = q$. Then γ is a closed geodesic of period $2d$. In fact, $q = \gamma(d) = \sigma_o(\gamma(d)) = \gamma(-d)$. It then follows from Remark 2.2 that γ is a closed geodesic. This implies that o is fixed by σ_q, the symmetry at q. Also, the symmetries σ_o and σ_q commute, since they both fix o and their differentials commute.

Since M is simply connected, the isotropy group K is connected. One can show that $B = K \cdot q$. In particular, all the points in B are equidistant to o. In fact, $d_q\sigma_o$ is the identity when restricted to T_qB and minus the identity when restricted to $(T_qB)^\perp$. Moreover, this holds at any point of B. So any $g \in G$ which leaves B invariant commutes with σ_o. Conversely, it is obvious that K maps fixed points of σ_o into fixed points of σ_o. We have thus proved that the subgroup of G which leaves B invariant coincides with K.

Note that the involution σ_q leaves B invariant (since B is totally geodesic), and so it maps K into K. Thus, (K, K^+) is a symmetric pair, where K^+ is the isotropy group of K at q. Moreover, $K^+ = K \cap K'$, where K' is the isotropy group of G at q. Such a symmetric pair is not, in general, effective (as one can see from the tables in [4]).

The totally geodesic submanifold B is called a *polar* of M. The normal space to $T_q B$ at q is a Lie triple system and hence induces, via the exponential map, a totally geodesic submanifold of M which is called a *meridian*. This follows from the fact that $\exp_q((T_q B)^\perp)$ coincides with the set of fixed points of $\sigma_q \circ \sigma_o$ (the connected component through q). In fact, if $w \in (T_q B)^\perp$ and $\beta(t)$ is a geodesic with $\beta'(0) = w$, then $(\sigma_q \circ \sigma_o)(\beta(t)) = \beta(t)$, since $d_q(\sigma_q \circ \sigma_o)$ is the identity when restricted to $(T_q B)^\perp$. This shows that $\exp_q((T_q B)^\perp)$ is contained in the fixed point set of $\sigma_q \circ \sigma_o$. The other inclusion holds since q is an isolated fixed point of σ_q.

We now construct the so-called centrioles. Let p be the midpoint of the geodesic γ joining o and q. In line with our notation above we have $p = \gamma(d/2)$. The *centriole* through p is the orbit $K^+ \cdot p$. Such an orbit is totally geodesic. In fact, the symmetry σ_p interchanges o and q, and so K with K'. So σ_p leaves $K^+ = K \cap K'$ invariant, and since it fixes p, it leaves the centriole $K^+ \cdot p$ invariant. Thus, σ_p leaves the second fundamental form of $K^+ \cdot p$ invariant, but on the other hand it reverses its sign. So the centriole $K^+ \cdot p$ must be totally geodesic. Moreover, it is contained in the meridian containing q, since K^+ commutes with both σ_q and σ_o and $\sigma_q \circ \sigma_o(p) = p$. Furthermore, (K^+, K^{++}), where K^{++} is the isotropy subgroup of K^+ at p, is a symmetric pair (not effective in general).

We now define $S = K \cdot p$, which is a fibre bundle over B whose fibres are the centrioles. In fact, since γ is minimizing in $[0, d]$, γ is the unique (unit speed) geodesic from o to $p = \gamma(d/2)$. So, the isotropy K_p of K at p must fix γ, since it fixes o and p. Then $K \cdot q = K \cdot \gamma(d) = q$, and therefore $K_p \subset K^+$, which implies $K_p = K^{++}$. So, we get the fibre bundle

$$K^+/K^{++} \to K/K^{++} \to K/K^+.$$

Moreover, $K \cdot p$ turns out to be diffeomorphic, via the exponential map at o, to the R-space $K \cdot v \subset T_o M$, where $v = \gamma'(0)$ (or equivalently $K \cdot p$ is diffeomorphic to an orbit of an s-representation).

The submanifold $S = K \cdot p$ has parallel Killing fields in any direction of the centriole $K^+ \cdot p$. In fact, if $\mathfrak{p}^+$ is the Cartan subspace associated with (K^+, K^{++}), then $\mathfrak{p}^+ \subset \mathfrak{p}$, where $\mathfrak{p}$ is the Cartan subspace associated to (G, K) (and elements of $\mathfrak{p}^+$ are parallel at p on M, and so on S with the induced metric). With the same arguments as in [8, Lemma 6.2], one can prove the following result:

Theorem 8.1. *Let $M = G/K$ be an irreducible simply connected Riemannian symmetric space of compact type. Assume that the polar $B = K/K^+$ is irreducible and $S = K/K^{++}$, with the induced Riemannian metric, is not a symmetric space. Then the coindex of symmetry of K/K^{++} is equal to the dimension of the polar $B = K/K^+$ and the leaves of symmetry coincide with the fibres of the fibration $K^+/K^{++} \to K/K^{++} \to K/K^+$ (which are centrioles in M).*

Proof. We have already proved that the centrioles are tangent to the distribution of symmetry $\mathfrak{s}$. Note that $\mathfrak{s}$ projects down to a distribution $\bar{\mathfrak{s}}$ on the symmetric space $B = K/K^+$, which must be K-invariant (since isometries preserve the distribution of symmetry). So, since B is irreducible, we have $\bar{\mathfrak{s}} = 0$ or $\bar{\mathfrak{s}} = TB$. However, $\bar{\mathfrak{s}} = TB$ implies $\mathfrak{s} = TS$, which cannot happen since S is not a symmetric space by assumption. Thus we have $\bar{\mathfrak{s}} = 0$, and therefore $\mathfrak{s}$ coincides with the distribution given by the tangent spaces to the centrioles. □

Example 8.2. Consider the complex projective plane $M = \mathbb{C}P^2 = \mathrm{SU}(3)/S(\mathrm{U}(1)\,\mathrm{U}(2)) = G/K$. There is only one polar in this situation, namely

$$B = \mathbb{C}P^1 = S(\mathrm{U}(1)\,\mathrm{U}(2))/S(\mathrm{U}(1)\,\mathrm{U}(1)\,\mathrm{U}(1)) = K/K^+ \cong \mathrm{U}(2)/\mathrm{U}(1)\,\mathrm{U}(1).$$

The orbit of K through the midpoint of a geodesic from o to a point in B is a distance sphere $S^3 = K/K^{++} \cong \mathrm{U}(2)/\mathrm{U}(1)$ in $\mathbb{C}P^2$ and the fibres of the projection $K/K^{++} \to K/K^+$ are circles $S^1 = K^+/K^{++} \cong \mathrm{U}(1)\,\mathrm{U}(1)/\mathrm{U}(1) \cong \mathrm{U}(1)$. These circles are centrioles in $\mathbb{C}P^2$. The induced metric from $\mathbb{C}P^2$ on the distance sphere S^3 gives a Berger sphere and its coindex of symmetry is equal to 2. Up to homothety, it is one of the metrics $\langle\cdot,\cdot\rangle^t$ in our classification for $k = 2$. By rescaling the metric on $\mathbb{C}P^2$ one obtains other metrics in this family. The remaining Berger sphere metrics can be obtained by considering distance spheres in the complex hyperbolic plane $\mathbb{C}H^2 = \mathrm{SU}(1,2)/S(\mathrm{U}(1)\,\mathrm{U}(2))$ which are not covered by the construction method in Theorem 8.1.

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