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# Hedetniemi's conjecture for uncountable graphs

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**Abstract.** It is proved that in Gödel's constructible universe, for every infinite successor cardinal $\kappa$, there exist graphs $\mathcal{G}$ and $\mathcal{H}$ of size and chromatic number $\kappa$, for which the product graph $\mathcal{G} \times \mathcal{H}$ is countably chromatic.

In particular, this provides an affirmative answer to a question of Hajnal from 1985.

**Keywords.** Hedetniemi's conjecture, product graph, almost countably chromatic, incompactness, constructible universe, Ostaszewski square

## 1. Introduction

A graph $\mathcal{G}$ is a pair $(G, E)$, where $E \subseteq [G]^2 := \{\{x, y\} \mid x, y \in G \ \&\ x \neq y\}$. The *chromatic number* of $\mathcal{G}$, denoted $\mathrm{Chr}(\mathcal{G})$, is the least (finite or infinite) cardinal $\kappa$ such that $G$ is the union of $\kappa$ many $E$-independent sets. Equivalently, $\mathrm{Chr}(\mathcal{G})$ is the least cardinal $\kappa$ for which there exists an $E$-chromatic $\kappa$-coloring of $G$, that is, a coloring $\chi : G \to \kappa$ that satisfies $\chi(x) \neq \chi(y)$ whenever $x E y$.

Given graphs $\mathcal{G}_0 = (G_0, E_0)$ and $\mathcal{G}_1 = (G_1, E_1)$, the product graph $\mathcal{G}_0 \times \mathcal{G}_1$ is defined as $(G_0 \times G_1, E_0 * E_1)$, where

$$G_0 \times G_1 := \{(g_0, g_1) \mid g_0 \in G_0, g_1 \in G_1\},$$
$$E_0 * E_1 := \{\{(g_0, g_1), (g_0', g_1')\} \mid (g_0, g_0') \in E_0 \text{ and } (g_1, g_1') \in E_1\}.$$

Clearly, a chromatic $\kappa$-coloring of one of the two graphs induces a chromatic $\kappa$-coloring of their product, and hence $\mathrm{Chr}(\mathcal{G}_0 \times \mathcal{G}_1) \leq \min\{\mathrm{Chr}(\mathcal{G}_0), \mathrm{Chr}(\mathcal{G}_1)\}$. It is then natural to ask whether this inequality is best possible, and the following answer was conjectured by Hedetniemi fifty years ago:

**Hedetniemi's Conjecture** ([Hed66]). *For all finite graphs $\mathcal{G}_0$ and $\mathcal{G}_1$,*

$$\mathrm{Chr}(\mathcal{G}_0 \times \mathcal{G}_1) = \min\{\mathrm{Chr}(\mathcal{G}_0), \mathrm{Chr}(\mathcal{G}_1)\}.$$

In [BEL76], Burr, Erdős and Lovász rediscovered Hedetniemi's conjecture through the perspective of *Ramsey-type graphs*, and in his survey paper [Tar08], Tardif made explicit a well-known Ramsey-type consequence of Hedetniemi's conjecture:

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**The Weak Hedetniemi Conjecture** ([Tar08]). *For every positive integer $k$, there exists a positive integer $\varphi(k)$ such that if* $\mathrm{Chr}(\mathcal{G}_0) = \mathrm{Chr}(\mathcal{G}_1) = \varphi(k)$*, then* $\mathrm{Chr}(\mathcal{G}_0 \times \mathcal{G}_1) \geq k$.

The weak conjecture goes back to [PR81], but nonetheless it is still standing.

As sometimes happens, infinitary versions of problems in graph theory lead to problems in set theory of independent interest (see [Sou08] for a recent treatment). Hedetniemi's conjecture is a wonderful example of such a problem—indeed, in [JT95], the finitary version appeared as Problem 11.1, and the infinitary version appeared as Problem 16.13.

This paper is dedicated to the solution of the infinitary counterparts. In [Haj85], Hajnal proved that for every infinite cardinal $\kappa$, there exist graphs $\mathcal{G}_0, \mathcal{G}_1$ of chromatic number $\kappa^+$ such that $\mathrm{Chr}(\mathcal{G}_0 \times \mathcal{G}_1) = \kappa$. In [JJ74], [Tod81], [ASS87], [Dav90], [AS93] more structural counterexamples were constructed in the form of pairs of nonspecial $\kappa^+$-Aronszajn trees whose product is special. All of these show that a one-cardinal gap is possible, but does not refute the weak conjecture:

**Infinite Weak Hedetniemi Conjecture.** *For every infinite cardinal $\kappa$, there exists a cardinal $\varphi(\kappa)$ such that if* $\mathrm{Chr}(\mathcal{G}_0) = \mathrm{Chr}(\mathcal{G}_1) = \varphi(\kappa)$*, then* $\mathrm{Chr}(\mathcal{G}_0 \times \mathcal{G}_1) \geq \kappa$.

Note that since it is possible to get the consistency of $\varphi(\kappa) > 2^\kappa$, the nontrivial context to examine the infinite weak conjecture (and its instances) is that of GCH. In his original paper [Haj85], Hajnal asked about the consistency of GCH together with an infinite gap (and this was echoed in [JT95]), but as of now, the best known result in this vein is Soukup's model [Sou88] of GCH with a counterexample of gap 2.

Let us point out a central obstruction towards getting the consistency of GCH with larger gaps. Hajnal discovered (the proof may be found in [Haj04]) that if $\mathcal{G}_0, \mathcal{G}_1$ is a pair of graphs of size and chromatic number $\lambda$ whose product has chromatic number $\kappa$, then $\mathcal{G}_0$ and $\mathcal{G}_1$ are $(\kappa, \lambda)$-chromatic. That is, $\mathcal{G}_i$ has chromatic number $\lambda$, but all of its smaller subgraphs have chromatic number $\leq \kappa$. Thus, in simple words, if $\mathcal{G}_0, \mathcal{G}_1$ were to witness the failure of an instance of the weak conjecture, then $\mathcal{G}_0, \mathcal{G}_1$ are in particular witnesses to the incompactness of infinite chromatic numbers.

The question of the very existence of incompactness graphs is a difficult set-theoretic question that goes back to a paper by Erdős and Hajnal [EH66], which, incidently, is from the same year of Hedetniemi's paper [Hed66]. Moreover, unlike the non-GCH context, answers in the context of GCH are quite rare, as we shall now describe.

A model of ZFC + GCH in which there exists an $(\aleph_0, \aleph_2)$-chromatic graph of size $\aleph_2$ was obtained by Baumgartner [Bau84] via a very complicated notion of forcing, and, indeed, Soukup's model [Sou88] of GCH with $\varphi(\aleph_0) > \aleph_2$ is a further sophistication of Baumgartner's attack. Unfortunately, Baumgartner's approach does not seem to generalize to yield a model of an $(\aleph_0, \aleph_3)$-chromatic graph of size $\aleph_3$. In fact, at the time of writing his chapter for the *Handbook of Combinatorics*, Hajnal thought that the problem of getting the consistency of GCH together with an $(\aleph_0, \aleph_3)$-chromatic graph of size $\aleph_3$ "seems to be hopelessly difficult at present" (see page 2093 of [GGL95]).

So, if GCH is consistent with $\varphi(\aleph_0) > \aleph_3$, then this will require an alternative construction.

An alternative construction of incompactness graphs was finally given by Shelah [She90], and up to recently, this has been the only known method for getting the consistency of GCH together with $(\aleph_0, \lambda)$-chromatic graphs of size $\lambda$, for arbitrarily large $\lambda$. But, as Hajnal mentioned in a more recent paper [Haj04], there was no success in generalizing Shelah's result (from incompactness to Hedetniemi).

Recently, the author [Rin15a] found yet another construction of incompactness graphs—a construction which is inspired by the concept of *Ostaszewski square* from [Rin14]. He denoted these graphs by $G(\overrightarrow{C})$ and identified the features of $G$ and $\overrightarrow{C}$ that make $G(\overrightarrow{C})$ into $(\aleph_0, \lambda^+)$-chromatic graphs. Even more recently, answering a question of Magidor, he proved that, for an appropriate choice of $\overrightarrow{C}$, these highly chromatic graphs can be made countably chromatic in a certain "nice" forcing extension [Rin17]. In this paper, these new findings are combined with the basic idea of Hajnal's 1985 construction to obtain the desired pair of graphs, arbitrarily high:

**Main Theorem.** *If $\lambda$ is an uncountable cardinal, and $\boxtimes_\lambda$ holds, then there exist graphs $\mathcal{G}_0, \mathcal{G}_1$ of size $\lambda^+$ such that:*

- $\mathrm{Chr}(\mathcal{G}_0) = \mathrm{Chr}(\mathcal{G}_1) = \lambda^+$;
- $\mathrm{Chr}(\mathcal{G}_0 \times \mathcal{G}_1) = \aleph_0$.

**Remark 1.1.** It is a curious fact that while the graphs $\mathcal{G}_0, \mathcal{G}_1$ are derived directly from $\boxtimes_\lambda$, their analysis relies heavily on passing to forcing extensions of the universe. In fact, we do not know of a forcing-free proof.

Recalling that Gödel's constructible universe is a model of $\mathsf{ZFC} + \mathsf{GCH}$ in which the principle $\boxtimes_\lambda$ holds for every uncountable cardinal $\lambda$, we get:

**Corollary 1.** *In Gödel's constructible universe, GCH holds and all instances of the Infinite Weak Hedetniemi Conjecture fail. Indeed, for any infinite cardinals $\lambda \geq \kappa$, there exist graphs $\mathcal{G}_0, \mathcal{G}_1$ with $\mathrm{Chr}(\mathcal{G}_0 \times \mathcal{G}_1) = \kappa$ such that $\mathrm{Chr}(\mathcal{G}_0) = \mathrm{Chr}(\mathcal{G}_1) > \lambda$.*

Recalling that counterexamples to instances of the weak conjecture are in particular incompactness graphs, a straightforward generalization of the de Bruin–Erdős theorem [dBE51] then entails:

**Corollary 2.** *If there exist class many strongly-compact cardinals, then the Infinite Weak Hedetniemi Conjecture holds.*

Altogether, this establishes the independence of the Infinite Weak Hedetniemi Conjecture from $\mathsf{ZFC} + \mathsf{GCH}$.

In addition, Corollary 1 settles the (unnumbered) Problem from [Haj85], Problem 16.13 from [JT95], Problem 40 from [Haj04], and Problem 2.4 from [Sou08].

*Organization of this paper.* In Section 2 we prove the main result of this paper, and its subsequent corollaries. In Section 3, we also settle the generalized problem concerning the product of $n + 1$ graphs $(0 < n < \omega)$.

## 2. Proof of the Main Theorem

Suppose that $\lambda$ is an uncountable cardinal, and $\boxtimes_\lambda$ holds. By $\boxtimes_\lambda$ and a standard partitioning argument [Dev78], [Rin14], let us fix a sequence $\langle (D_\alpha, X_\alpha) \mid \alpha < \lambda^+\rangle$ along with a function $h : \lambda^+ \to \lambda^+$ such that:

1. for every limit $\alpha < \lambda^+$, $D_\alpha$ is a club in $\alpha$ of order-type $\leq \lambda$;
2. if $\beta \in \mathrm{acc}(D_\alpha)$, then $D_\beta = D_\alpha \cap \beta$, $X_\beta = X_\alpha \cap \beta$ and $h(\beta) = h(\alpha)$;[1]
3. for every $X \subseteq \lambda^+$, a club $E \subseteq \lambda^+$ and $\varsigma < \lambda^+$, there exists a limit $\alpha < \lambda^+$ with $\mathrm{otp}(D_\alpha) = \lambda$ such that $h(\alpha) = \varsigma$, $X_\alpha = X \cap \alpha$ and $\mathrm{acc}(D_\alpha) \subseteq E$.

Clearly, $\langle X_\alpha \mid \alpha \in G_i\rangle$ is a $\diamondsuit(G_i)$-sequence, where $G_i := \{\alpha < \lambda^+ \mid h(\alpha) = i$ & $\mathrm{otp}(D_\alpha) = \lambda\}$. Since $G_0$ and $G_1$ are nonreflecting and pairwise-disjoint stationary sets, it is then natural to use $G_0(\overrightarrow{D})$ and $G_1(\overrightarrow{D})$ as the building blocks of our graphs.[2] Loosely speaking, one of the features that we would need is the ability to kill (via forcing) the guessing feature of $\langle X_\alpha \mid \alpha \in G_0\rangle$, while preserving the features of $\langle X_\alpha \mid \alpha \in G_1\rangle$, and vice versa. For this, we shall borrow an idea from [She77, proof of Theorem 2.4], where a model of $\diamondsuit(\omega_1 \setminus S) + \neg\diamondsuit(S)$ was obtained for the first time.[3]

Fix a large enough regular cardinal $\theta \gg \lambda$ together with a well-ordering $\trianglelefteq_\theta$ of $\mathcal{H}_\theta$. Fix a bijection $\psi : ({}^{<\lambda^+}\omega) \times ({}^{<\lambda^+}({}^{\lambda+1}2)) \leftrightarrow \lambda^+$.

For every limit $\alpha < \lambda^+$ with $\sup(\mathrm{acc}(D_\alpha)) < \alpha$, let $d_\alpha$ be a cofinal subset of $\alpha$ of order-type $\omega$, consisting of successor ordinals. For $\alpha < \lambda^+$ with $\sup(\mathrm{acc}(D_\alpha)) = \alpha$, let $d_\alpha := \mathrm{acc}(D_\alpha)$.

Fix a limit ordinal $\alpha < \lambda^+$. We would like to determine a function $g_\alpha \in {}^{\leq\alpha}({}^{\lambda+1}2)$. For this, let $\{\alpha_i \mid i < \mathrm{otp}(d_\alpha)\}$ be the increasing enumeration of $d_\alpha$. Recursively define a sequence $\langle (p^\alpha_i, f^\alpha_i) \mid i < \mathrm{otp}(d_\alpha)\rangle$ as follows:

- ▶ Let $f_0 := \emptyset$ and $p_0 := \emptyset$.
- ▶ If $i < \mathrm{otp}(d_\alpha)$ and $\langle (p^\alpha_j, f^\alpha_j) \mid j \leq i\rangle$ is defined, let

$$\mathcal{P}^\alpha_i := \{p \in {}^{<\alpha_{i+1}}\omega \mid \psi(p,f) \in X_\alpha \cap \alpha_{i+1},\ p \supseteq p_i,\ f \supseteq f_i,\ \mathrm{dom}(p) > \mathrm{dom}(f) = \alpha_i\},$$
$$\mathcal{F}^\alpha_i := \left\{ f \in {}^{\alpha_i}({}^{\lambda+1}2) \;\middle|\; \psi(p,f) \in X_\alpha \cap \alpha_{i+1},\ p = \min_{\trianglelefteq_\theta} \mathcal{P}^\alpha_i,\ f \supseteq f_i \right\},$$

and set

$$(p^\alpha_{i+1}, f^\alpha_{i+1}) := \begin{cases} (\min_{\trianglelefteq_\theta} \mathcal{P}^\alpha_i, \min_{\trianglelefteq_\theta} \mathcal{F}^\alpha_i), & \mathcal{P}^\alpha_i \neq \emptyset, \\ (\emptyset, \emptyset), & \text{otherwise.} \end{cases}$$

- ▶ If $i < \mathrm{otp}(d_\alpha)$ is a limit ordinal, and $\langle (p^\alpha_j, f^\alpha_j) \mid j < i\rangle$ is defined, let $p^\alpha_i := \bigcup_{j<i} p^\alpha_j$ and $f^\alpha_i := \bigcup_{j<i} f^\alpha_j$.

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[1] Here, $\mathrm{acc}(A) := \{\alpha \in \sup(A) \mid \sup(A \cap \alpha) = \alpha > 0\}$.

[2] The graph $G(\overrightarrow{D})$ was introduced in [Rin15a], and it was proven there that if $\overrightarrow{D}$ is a $\square_\lambda$-sequence, and $G$ is a nonreflecting subset of $\lambda^+$, then $G(\overrightarrow{D})$ is $(\aleph_0, \kappa)$-chromatic for some cardinal $\kappa$.

[3] The proof is not given in [She77], rather, it is given as the proof of Theorem 2.4 from [She80]. Personally, I learned that proof from Juris Steprāns.

This completes the construction of $\langle (p_i^\alpha, f_i^\alpha) \mid i < \mathrm{otp}(d_\alpha)\rangle$. Define

$$\begin{aligned} g_\alpha &:= \bigcup\{f_i^\alpha \mid i < \mathrm{otp}(d_\alpha), \forall j < i\ (\mathcal{P}_j^\alpha \neq \emptyset)\},\\ A_\alpha^i &:= \{\beta < \mathrm{dom}(g_\alpha) \mid g_\alpha(\beta)(i) = 1,\ h(\beta) = h(\alpha)\} \text{ for all } i < \lambda,\\ K_\alpha &:= \{\beta < \mathrm{dom}(g_\alpha) \mid g_\alpha(\beta)(\lambda) = 1\}. \end{aligned}$$

For every $i < \mathrm{otp}(d_\alpha)$, set $\alpha_i' := \min((K_\alpha \cup \{\alpha_{i+1}\}) \setminus \alpha_i + 1)$, and $\alpha_i'' := \min((A_\alpha^i \cup \{\alpha_{i+1}\}) \setminus \alpha_i')$. Finally, set

$$C_\alpha := \begin{cases} d_\alpha \setminus \mathrm{dom}(g_\alpha), & \mathrm{dom}(g_\alpha) < \alpha,\\ \mathrm{acc}(d_\alpha) \cup \{\alpha_i'' \mid i < \mathrm{otp}(d_\alpha), \alpha_i < \alpha_i'' < \alpha_{i+1}\}, & \text{otherwise.} \end{cases}$$

It can be shown that $\langle C_\alpha \mid \alpha < \lambda^+\rangle$ is a relativized Ostaszewski square sequence [Rin14], but here we shall only need the following.

**Lemma 2.1.** *For every limit $\alpha < \lambda^+$:*

(1) *$C_\alpha$ is a club in $\alpha$ of order-type $\leq \lambda$;*
(2) *if $\beta \in \mathrm{acc}(C_\alpha)$, then $C_\beta = C_\alpha \cap \beta$;*
(3) *if $\mathrm{otp}(C_\alpha) = \lambda$, then $h(\beta) = h(\alpha)$ for all $\beta \in C_\alpha$.*

*Proof.* Fix a limit ordinal $\alpha < \lambda^+$.

(1) If $\mathrm{dom}(g_\alpha) < \alpha$, then $C_\alpha = d_\alpha \setminus \mathrm{dom}(g_\alpha)$ is a club in $\alpha$ of order-type $\leq \mathrm{otp}(d_\alpha) \leq \lambda$. Note that $\mathrm{acc}(C_\alpha) \subseteq \mathrm{acc}(d_\alpha)$.

If $\mathrm{dom}(g_\alpha) = \alpha$, then since $\alpha_i < \alpha_i'' \leq \alpha_{i+1}$ for all $i < \mathrm{otp}(d_\alpha)$, we have $\mathrm{acc}(C_\alpha) \subseteq \mathrm{acc}(d_\alpha)$ and $\mathrm{otp}(C_\alpha) \leq \mathrm{otp}(d_\alpha)$. In particular, $C_\alpha$ is a club in $\alpha$ of order-type $\leq \lambda$.

(2) Fix $\beta \in \mathrm{acc}(C_\alpha)$. From $\beta \in \mathrm{acc}(C_\alpha) \subseteq \mathrm{acc}(d_\alpha)$, we have $\mathrm{otp}(d_\alpha) > \omega$ and $d_\alpha = \mathrm{acc}(D_\alpha)$. In particular, $\beta \in \mathrm{acc}(D_\alpha)$, $X_\beta = X_\alpha \cap \beta$, $D_\beta = D_\alpha \cap \beta$, and $d_\beta = \mathrm{acc}(D_\beta)$. Consequently, the sequence $\langle (p_i^\beta, \mathcal{P}_i^\beta, f_i^\beta, \mathcal{F}_i^\beta) \mid i < \mathrm{otp}(d_\beta)\rangle$ is an initial segment of the sequence $\langle (p_i^\alpha, \mathcal{P}_i^\alpha, f_i^\alpha, \mathcal{F}_i^\alpha) \mid i < \mathrm{otp}(d_\alpha)\rangle$, and $g_\beta = g_\alpha \restriction \beta$.

If $\mathrm{dom}(g_\alpha) < \alpha$, then since $\beta \in \mathrm{acc}(C_\alpha) = \mathrm{acc}(d_\alpha \setminus \mathrm{dom}(g_\alpha))$, we get $g_\alpha = g_\beta$ and $C_\beta = d_\beta \setminus \mathrm{dom}(g_\beta) = d_\alpha \cap \beta \setminus \mathrm{dom}(g_\alpha) = C_\alpha \cap \beta$.

If $\mathrm{dom}(g_\alpha) = \alpha$, then as $g_\beta = g_\alpha \restriction \beta$, we get $\{\beta_i'' \mid i < \mathrm{otp}(d_\beta)\} = \{\alpha_i'' \mid i < \mathrm{otp}(d_\alpha)\} \cap \beta$, and $C_\beta = C_\alpha \cap \beta$.

(3) Clearly, if $\mathrm{otp}(C_\alpha) = \lambda$, then $d_\alpha = \mathrm{acc}(D_\alpha)$. So $h(\beta) = h(\alpha)$ for all $\beta \in \mathrm{acc}(C_\alpha)$. Now, if $\beta \in C_\alpha \setminus \mathrm{acc}(d_\alpha)$, then there exists some $i < \mathrm{otp}(d_\alpha)$ such that $\beta = \alpha_i'' \in A_\alpha^i \subseteq h^{-1}\{\alpha\}$. So $h(\beta) = h(\alpha)$. □

For $i < 2$, set

$$\begin{aligned} S_i &:= \{\alpha < \lambda^+ \mid h(\alpha) = i\}, \qquad G_i := \{\alpha \in S_i \mid \mathrm{otp}(C_\alpha) = \lambda\},\\ E_i &:= \{\{\alpha, \delta\} \in [G_i]^2 \mid \alpha \in C_\delta,\ \min(C_\alpha) > \sup(C_\delta \cap \alpha)\}. \end{aligned}$$

Finally, for $i < 2$, let

$$\begin{aligned} V_i &:= \{\chi : \beta \to \omega \mid \beta \in G_i, \chi \text{ is } E_{(1-i)}\text{-chromatic}\},\\ F_i &:= \{\{\chi, \chi'\} \in [V_i]^2 \mid \{\mathrm{dom}(\chi), \mathrm{dom}(\chi')\} \in E_i,\ \chi \subseteq \chi'\}. \end{aligned}$$

**Lemma 2.2.** $\mathrm{Chr}(V_0 \times V_1, F_0 * F_1) \leq \aleph_0$.

*Proof.* This is where Hajnal's idea [Haj85] comes into play. Define $c : V_0 \times V_1 \to \omega$ as follows. Given $(\chi, \eta) \in V_0 \times V_1$, as $G_0 \cap G_1 = \emptyset$, we have $\mathrm{dom}(\chi) \neq \mathrm{dom}(\eta)$; thus, let

$$c(\chi, \eta) := \begin{cases} 2 \cdot \chi(\mathrm{dom}(\eta)), & \mathrm{dom}(\chi) > \mathrm{dom}(\eta), \\ 2 \cdot \eta(\mathrm{dom}(\chi)) + 1, & \mathrm{dom}(\eta) > \mathrm{dom}(\chi). \end{cases}$$

Towards a contradiction, suppose that $\{(\chi, \eta), (\chi', \eta')\} \in F_0 * F_1$, while $c(\chi, \eta) = c(\chi', \eta') =: n$.

If $n$ is even, we let $\chi^* := \chi \cup \chi'$. Since $(\chi, \chi') \in F_0$, we know that $\chi^*$ is $E_1$-chromatic. Since $n$ is even, we have $\mathrm{dom}(\eta), \mathrm{dom}(\eta') \in \chi^*$. So $\chi^*(\mathrm{dom}(\eta)) = n/2 = \chi^*(\mathrm{dom}(\eta'))$. But then the fact that $\chi^*$ is $E_1$-chromatic entails that $\{\mathrm{dom}(\eta), \mathrm{dom}(\eta')\} \notin E_1$, contradicting the hypothesis that $\{\eta, \eta'\} \in F_1$.

If $n$ is odd, we let $\eta^* := \eta \cup \eta'$. As $(\eta, \eta') \in F_1$, $\eta^*$ is $E_0$-chromatic. Since $n$ is odd, we have $\eta^*(\mathrm{dom}(\chi)) = (n-1)/2 = \eta^*(\mathrm{dom}(\chi'))$. But then the fact that $\eta^*$ is $E_0$-chromatic entails that $\{\mathrm{dom}(\chi), \mathrm{dom}(\chi')\} \notin E_0$, contradicting the hypothesis that $\{\chi, \chi'\} \in F_0$. $\square$

**Definition 2.3.** For $i < 2$ and a limit $\delta < \lambda^+$, write

$$C^i_\delta := \{\alpha \in C_\delta \cap G_i \mid \min(C_\alpha) > \sup(C_\delta \cap \alpha)\}.$$

**Definition 2.4.** For $i < 2$ and $\gamma < \lambda^+$, we say that a coloring $\chi : \gamma \to \omega$ is *$i$-suitable* if:

- $\chi[C^i_\delta]$ is finite for all $\delta \leq \gamma$;
- $\chi(\alpha) \neq \chi(\delta)$ for all $\alpha < \delta \leq \gamma$ with $\{\alpha, \delta\} \in E_i$.

**Lemma 2.5.** *For every $i < 2$, $\beta < \gamma < \lambda^+$ with $\beta \notin G_i$, and an $i$-suitable coloring $\chi : \beta \to \omega$, there exists an $i$-suitable coloring $\chi' : \gamma \to \omega$ extending $\chi$.*

*Proof.* By virtually the same proof of Claim 3.1.3 from [Rin15a], building on Lemma 2.1(2) above. $\square$

**Lemma 2.6.** *For $i < 2$, the notion of forcing*

$$\mathbb{Q}_i := (\{\chi : \beta \to \omega \mid \beta \in \lambda^+ \setminus G_i, \chi \text{ is } i\text{-suitable}\}, \subseteq)$$

*is $(\leq\lambda)$-distributive.*

*Proof.* For concreteness, we work with $\mathbb{Q}_1$.

Suppose that $\langle \Omega_i \mid i < \lambda \rangle$ is a given sequence of dense open subsets of $\mathbb{Q}_1$, $p_0$ is an arbitrary condition, and let us show that there exists $p \in \bigcap_{i<\lambda} \Omega_i$ extending $p_0$. Let $\langle N_\alpha \mid \alpha < \lambda^+ \rangle$ be an increasing and continuous sequence of elementary submodels of $(\mathcal{H}(\theta), \in, \leq_\theta)$, each of size $\lambda$, such that $\langle D_\delta \mid \delta < \lambda^+ \rangle, \mathbb{Q}_1, \langle \Omega_i \mid i < \lambda \rangle, p_0 \in N_0$, and $\langle N_\beta \mid \beta \leq \alpha \rangle \in N_{\alpha+1}$ for all $\alpha < \lambda^+$.

Set $E := \{\delta < \lambda^+ \mid N_\delta \cap \lambda^+ = \delta\}$. By the choice of $\langle (D_\alpha, X_\alpha) \mid \alpha < \lambda^+ \rangle$, let us pick some $\alpha < \lambda^+$ with $\mathrm{otp}(D_\alpha) = \lambda$ such that $h(\alpha) = 0$ and $\mathrm{acc}(D_\alpha) \subseteq E$.

Let $\{\alpha_i \mid i \leq \lambda\}$ denote the increasing enumeration of $\mathrm{acc}(D_\alpha) \cup \{\alpha\}$. Write $M_i := N_{\alpha_i}$. Notice that for all $i < \lambda$, since $\langle N_\beta \mid \beta \leq \alpha_i \rangle \in N_{\alpha_i+1} \subseteq M_{i+1}$ and $\langle D_\delta \mid \delta < \lambda^+ \rangle$

$\in M_{i+1}$, we have $\langle M_j \mid j \leq i\rangle \in M_{i+1}$. Also notice that for all $i \leq \lambda$, we have $h(\alpha_i) = 0$ and $M_i \cap \lambda^+ = \alpha_i \in S_0$. In particular, $\alpha_i \in \lambda^+ \setminus G_1$.

We shall recursively define an increasing sequence $\langle p_i \mid i < \lambda\rangle$ of conditions that will satisfy the following for all $i < \lambda$:

- $p_{i+1} \in \Omega_i$;
- $\langle p_j \mid j \leq i\rangle \in M_{i+1}$;
- $\mathrm{dom}(p_i) = \alpha_i$ whenever $i > 0$.

By recursion on $i < \lambda$:

► $p_0$ was already given to us, and indeed $p_0 \in M_1$.

► Suppose that $i < \lambda$, and $\langle p_j \mid j \leq i\rangle$ has already been defined, and is an element of $M_{i+1}$. In particular, $p_i \in M_{i+1}$. We claim that the set $\Psi_i := \{q \in \Omega_i \mid q \supseteq p_i, \mathrm{dom}(q) = \alpha_{i+1}\}$ is nonempty. To see this, notice that since $p_i, \Omega_i \in M_{i+1}$, elementarity of $M_{i+1}$ yields some $p \in \Omega_i \cap M_{i+1}$ extending $p_i$. Then, from $M_{i+1} \cap \lambda^+ = \alpha_{i+1}$, we have $\mathrm{dom}(p) < \alpha_{i+1}$, and then by Lemma 2.5, we infer the existence of a 1-suitable coloring $q$ extending $p$ with $\mathrm{dom}(q) = \alpha_{i+1}$. As $\alpha_{i+1} \in S_0$, $q$ is a legitimate condition, and since $\Omega_i$ is open, we deduce that $q$ is in $\Omega_i$, testifying that $\Psi_i$ is nonempty.

Thus, we let $p_{i+1}$ be the $\leq_\theta$-least element of $\Psi_i$. Since $\Psi_i$ is defined from parameters within $M_{i+2}$, and by the canonical choice of $p_{i+1}$, we have $p_{i+1} \in M_{i+2}$. Altogether, $\langle p_j \mid j \leq i+1\rangle \in M_{i+2}$.

► Suppose that $i < \lambda$ is a nonzero limit ordinal, and $\langle p_j \mid j < i\rangle$ has already been defined by our canonical process. Set $p_i := \bigcup_{j<i} p_j$. Then $\mathrm{dom}(p_i) = \alpha_i$, and since $p_i$ is the limit of an increasing chain of 1-suitable colorings, $p_i$ is $E_1$-chromatic, and $p_i[C^1_\beta]$ is finite for every $\beta < \alpha_i$. Thus, to see that $p_i$ is 1-suitable, we are left with verifying that $p_i[C^1_{\alpha_i}]$ is finite. As $h(\alpha_i) = 0$, Lemma 2.1(2)&(3) shows that $h(\beta) \neq 1$ for all $\beta \in C_\alpha \supseteq C_{\alpha_i}$, so $C^1_{\alpha_i} = \emptyset$, which entails that $p_i[C^1_{\alpha_i}]$ is finite indeed. Thus, $p_i$ is a legitimate condition.

By the canonical process, and the fact that $\langle M_j \mid j \leq i\rangle \in M_{i+1}$, we have $\langle p_j \mid j < i\rangle \in M_{i+1}$, and hence $p_i = \bigcup_{j<i} p_j \in M_{i+1}$. So $\langle p_j \mid j \leq i\rangle \in M_{i+1}$.

This completes the construction.

Set $p := \bigcup_{i<\lambda} p_i$. Then $p$ is $E_1$-chromatic, and $p[C^1_\beta]$ is finite for every $\beta < \alpha$. As $\mathrm{dom}(p) = \alpha$ and $C^1_\alpha$ is empty, we find that $p$ is a legitimate condition. Consequently, $p$ is an element of $\bigcap_{i<\lambda} \Omega_i$ that extends $p_0$. □

It is clear that $|V_i| \leq 2^\lambda = \lambda^+$ for $i < 2$, so it remains to establish the following.

**Lemma 2.7.** $\mathrm{Chr}(V_i, F_i) = \lambda^+$ *for every* $i < 2$.

*Proof.* For concreteness, we prove that $\mathrm{Chr}(V_0, F_0) = \lambda^+$.

Towards a contradiction, suppose that $c : V_0 \to \lambda$ is $F_0$-chromatic. Let $\mathbb{G}$ be $\mathbb{Q}_1$-generic over $V$, and work in $V[\mathbb{G}]$.

Set $\chi^* := \bigcup \mathbb{G}$. Since $\mathbb{G}$ is directed, for every $\alpha, \delta \in \mathrm{dom}(\chi^*)$ there exists $\chi \in \mathbb{G}$ such that $\{\alpha, \delta\} \subseteq \mathrm{dom}(\chi)$, and hence $\chi^*(\alpha) \neq \chi^*(\delta)$ whenever $\alpha, \delta \in E_1$. By Lemma 2.5, we also know that $\mathrm{dom}(\chi^*) \geq \gamma$ for all $\gamma < \lambda^+$. Altogether, $\chi^* : \lambda^+ \to \omega$

is an $E_1$-chromatic coloring, and so are its initial segments. In particular, we may derive a coloring $c^* : G_0 \to \lambda$ by letting $c^*(\beta) := c(\chi^*\restriction\beta)$ for all $\beta \in G_0$. Since $c$ is $F_0$-chromatic, we infer that $c^*$ is $E_0$-chromatic. That is, $c^*$ witnesses that $\mathrm{Chr}(G_0, E_0) \leq \lambda$.

For all $i < \lambda$, set $H_i := \{\alpha \in G_0 \mid c^*(\alpha) = i\}$ and $M_i := \{\min(C_\alpha) \mid \alpha \in H_i\}$. Define a function $h_i : \lambda^+ \to \lambda^+$ by letting, for all $\tau < \lambda^+$,

$$h_i(\tau) := \begin{cases} \min\{\alpha \in H_i \mid \min(C_\alpha) > \tau\}, & \sup(M_i) = \lambda^+, \\ \sup(M_i), & \text{otherwise.} \end{cases}$$

Then, for all $i < \lambda$, set

$$A_i := \begin{cases} \mathrm{rng}(h_i), & \sup(M_i) = \lambda^+, \\ \lambda^+, & \sup(M_i) < \lambda^+, \end{cases}$$

and

$$K := \{\beta < \lambda^+ \mid \forall i < \lambda,\ h_i[\beta] \subseteq \beta\}.$$

Finally, define a function $g : \lambda^+ \to {}^{\lambda+1}2$ by letting $g(\alpha)(i) = 1$ iff ($i < \lambda$ and $\alpha \in A_i$) or ($i = \lambda$ and $\alpha \in K$). Note that by Lemma 2.6, any initial segment of $g$ belongs to the ground model.

Work back in $V$. Let $p_0 \in \mathbb{Q}_1$ be such that

$$p_0 \Vdash \dot{g} : \check{\lambda}^+ \to {}^{\check{\lambda}+1}2, \text{ and } c^* \text{ is } E_0\text{-chromatic.}$$

By possibly extending $p_0$, we may moreover assume that $p_0$ forces that $\{\alpha < \lambda^+ \mid g(\alpha)(i) = 1\}$ is unbounded in $\lambda^+$ for all $i \leq \lambda$, and knows about the interaction of $g$ with $c^*$.

As any initial segment of $g$ belongs to $V$, it makes sense to consider the set

$$Z := \{(p, f) \in \mathbb{Q}_1 \times {}^{<\lambda^+}({}^{\lambda+1}2) \mid p_0 \subseteq p \Vdash_{\mathbb{Q}_1} \dot{g}\restriction\mathrm{dom}(f) = \check{f}\}.$$

Let $\langle N_\alpha \mid \alpha < \lambda^+\rangle$ be an increasing and continuous sequence of elementary submodels of $(\mathcal{H}(\theta), \in, \leq_\theta)$, each of size $\lambda$, such that $\langle D_\delta \mid \delta < \lambda^+\rangle, \mathbb{Q}_1, \psi, \dot{g}, p_0 \in N_0$, and $\langle N_\beta \mid \beta \leq \alpha\rangle \in N_{\alpha+1}$ for all $\alpha < \lambda^+$.

Set $E := \{\delta < \lambda^+ \mid N_\delta \cap \lambda^+ = \delta\}$. By the choice of $\langle (D_\alpha, X_\alpha) \mid \alpha < \lambda^+\rangle$, let us pick some $\alpha < \lambda^+$ with $\mathrm{otp}(D_\alpha) = \lambda$ such that $h(\alpha) = 0$, $X_\alpha = \psi[Z] \cap \alpha$, and $\mathrm{acc}(D_\alpha) \subseteq E$.

Let $\{\alpha_i \mid i \leq \lambda\}$ denote the increasing enumeration of $\mathrm{acc}(D_\alpha) \cup \{\alpha\}$. Write $M_i := N_{\alpha_i}$. Notice that for all $i < \lambda$, we have $\langle M_j \mid j \leq i\rangle \in M_{i+1}$. Also, we have $h(\alpha_i) = 0$ and $M_i \cap \lambda^+ = \alpha_i \in S_0$ for all $i \leq \lambda$.

We shall recursively define a sequence $\langle (p_i, f_i) \mid i < \lambda\rangle$ of pairs that will satisfy the following for all $i < \lambda$:

- $p_{i+1} \Vdash \dot{g}\restriction\check{\alpha}_i = \check{f}_{i+1}$;
- $\alpha_i \leq \mathrm{dom}(p_i) < \alpha_{i+1}$;
- $\langle p_j \mid j \leq i\rangle$ is an increasing sequence of conditions that belongs to $M_{i+1}$.

By recursion on $i < \lambda$:

► $p_0$ was already given to us, and indeed $p_0 \in M_1$. Set $f_0 := \emptyset$.

► Suppose that $i < \lambda$, and $\langle p_j \mid j \leq i\rangle$ has already been defined, and is an element of $M_{i+1}$. In particular, $p_i \in M_{i+1}$. By Lemmas 2.5 and 2.6, the set $\Psi_i := \{q \in \mathbb{Q}_1 \mid q \supseteq p_i, \alpha_i < \mathrm{dom}(q) < \alpha_{i+1}, q \text{ decides } \dot{g}\restriction\alpha_i\}$ is nonempty. Thus, we let $p_{i+1}$ be the $\leq_\theta$-least element of $\Psi_i$, and let $f_{i+1}$ be such that $p_{i+1} \Vdash \dot{g}\restriction\check{\alpha}_i = \check{f}_{i+1}$.

As $\Psi_i$ is defined from parameters within $M_{i+2}$, and by the canonical choice of $p_{i+1}$, we have $p_{i+1} \in M_{i+2}$. Altogether, $\langle p_j \mid j \leq i+1\rangle \in M_{i+2}$.

► Suppose that $i < \lambda$ is a nonzero limit ordinal, and $\langle (p_j, f_j) \mid j < i\rangle$ has already been defined by our canonical process. Set $p_i := \bigcup_{j<i} p_j$ and $f_i := \bigcup_{j<i} p_j$. Then $\mathrm{dom}(p_i) = \alpha_i$, and since $p_i$ is the limit of an increasing chain of 1-suitable colorings, $p_i$ is chromatic, and $p_i[C^1_\beta]$ is finite for every $\beta < \alpha_i$. Thus, to see that $p_i$ is 1-suitable, it remains to verify that $p_i[C^1_{\alpha_i}]$ is finite. As $h(\alpha_i) = 0$, Lemma 2.1 shows that $h(\beta) \neq 1$ for all $\beta \in C_{\alpha_i}$, so $p_i[C^1_{\alpha_i}] = \emptyset$ is finite indeed, and $p_i$ is a legitimate condition.

By the canonical process, and as $\langle M_j \mid j \leq i\rangle \in M_{i+1}$, we have $\langle p_j \mid j < i\rangle \in M_{i+1}$, and hence $p_i = \bigcup_{j<i} p_j \in M_{i+1}$. So $\langle p_j \mid j \leq i\rangle \in M_{i+1}$.

This completes the construction. Set $p := \bigcup_{i<\lambda} p_i$. Then $p$ is a legitimate condition.

Clearly, $\{(p_i, f_i) \mid i < \lambda\} \subseteq Z$. Note that for all $i < \lambda$, as $\mathbb{Q}_1, p_i, \dot{g}, \alpha_i, \psi \in M_{i+1}$, we have $\psi(p_i, f_i) \in M_{i+1}$. That is, $\psi(p_i, f_i) \in \psi(Z) \cap \alpha_{i+1} = X_\alpha \cap \alpha_{i+1}$. It follows that $\langle (p_i, f_i) \mid 0 < i < \lambda\rangle = \langle (p^\alpha_i, f^\alpha_i) \mid 0 < i < \lambda\rangle$!

So, $p \Vdash \dot{g}\restriction\check{\alpha} = \check{g}_\alpha$. Consequently, $p$ forces that $A_i \cap \alpha = A^i_\alpha$ for all $i < \lambda$, and $K \cap \alpha = K_\alpha$. Also, since $p_0$ forces that $\{\alpha < \lambda^+ \mid g(\alpha)(i) = 1\}$ is unbounded in $\lambda^+$ for all $i \leq \lambda$, we find that $\sup(K_\alpha \cap \alpha_i) = \sup(A^i_\alpha \cap \alpha_i) = \alpha_i$ and $\alpha_i < \alpha''_i < \alpha_{i+1}$ for all $i < \lambda$. In particular, $\{\alpha''_i \mid i < \lambda\} \subseteq C_\alpha$, and $p \Vdash \min(C_\alpha) = \alpha''_0 \geq \min(K)$. Let $p^*$ be an extension of $p$ that decides $c^*(\alpha)$, say $p^* \Vdash c^*(\alpha) = \check{i}$, and decides $h_i\restriction\alpha$.

Then $p^*$ forces that $\sup(M_i) = \lambda^+$, because otherwise

$$\sup(M_i) < \min(K) \leq \min(C_\alpha),$$

contradicting the fact that $i = c^*(\alpha)$ entails $\sup(M_i) \geq \min(C_\alpha)$.

The upcoming considerations are all forced by $p^*$. We have $\alpha_i < \alpha'_i \leq \alpha''_i < \alpha_{i+1}$ with $\alpha'_i \in K$ and $\alpha''_i \in A_i \cap C_\alpha$. Since $\alpha''_i \in A_i$ and $\sup(M_i) = \lambda^+$, we have $\alpha''_i \in \mathrm{rng}(h_i)$. Fix $\tau < \alpha$ such that $h_i(\tau) = \alpha''_i$. Then $\min(C_{\alpha''_i}) > \tau$. As $h_i[\alpha'_i] \subseteq \alpha'_i \leq \alpha''_i = h_i(\tau)$, we have $\tau \geq \alpha'_i$, and hence $\min(C_{\alpha''_i}) > \tau \geq \alpha'_i > \sup(C_\alpha \cap \alpha''_i)$. It follows that $\{\alpha''_i, \alpha\} \in E_0$. Recalling that $\alpha''_i \in \mathrm{rng}(h_i) \subseteq H_i$, we conclude that $c^*(\alpha''_i) = i = c^*(\alpha)$. So $p^*$ forces that $c^*$ is not an $E_0$-chromatic coloring, contradicting the fact that $p^*$ extends $p_0$. □

**Remark 2.1.** Péter Komjáth pointed out that the above construction shows that $\boxtimes_\lambda$ yields a sequence $\langle \mathcal{G}_i \mid i < \lambda^+\rangle$ of graphs, each of size and chromatic number $\lambda^+$, such that $\mathrm{Chr}(\mathcal{G}_i \times \mathcal{G}_j) = \aleph_0$ for all $i < j < \lambda^+$.

*Proof of Corollary 1*

If $\lambda = \aleph_0$, then $\kappa = \aleph_0$, and Hajnal's example [Haj85] apply.[4] Otherwise, since $\boxtimes_\lambda$ holds in Gödel's constructible universe (see [ASS87]), let us invoke the main result of this paper and pick subsets $E_0, E_1$ of $[\lambda^+]^2$ with $\mathrm{Chr}(\lambda^+, E_0) = \mathrm{Chr}(\lambda^+, E_1) = \lambda^+$ and $\mathrm{Chr}(\lambda^+ \times \lambda^+, E_0 * E_1) \leq \aleph_0$ as witnessed by $c : \lambda^+ \times \lambda^+ \to \omega$. Set $F_0 := E_0 \cup [\kappa]^2$ and $F_1 := E_1 \cup [\kappa]^2$. Clearly, $\mathrm{Chr}(\lambda^+, F_0) = \mathrm{Chr}(\lambda^+, F_1) = \lambda^+$, and $\mathrm{Chr}(\lambda^+ \times \lambda^+, F_0 * F_1) \geq \mathrm{Chr}(\kappa, [\kappa]^2) = \kappa$. Finally, fix an injection $d : \kappa \times 2 \to \kappa \setminus \omega$, and define $c' : \lambda^+ \times \lambda^+ \to \kappa$ by letting

$$c'(\alpha, \beta) := \begin{cases} d(\alpha, 0), & \alpha < \kappa, \\ d(\beta, 1), & \beta < \kappa \leq \alpha, \\ c(\alpha, \beta), & \text{otherwise.} \end{cases}$$

Then $c'$ is $F_0 * F_1$-chromatic, and hence $\mathrm{Chr}(\lambda^+ \times \lambda^+, F_0 * F_1) = \kappa$.

*Proof of Corollary 2*

De Bruijn and Erdős [dBE51] proved that if $\mathcal{G}$ is a graph, $k < \omega$, and every subgraph of $\mathcal{G}$ of size $< \omega$ has chromatic number $\leq k$, then $\mathrm{Chr}(\mathcal{G}) \leq k$. The statement remains true after replacing $\omega$ in the above statement with a strongly-compact cardinal $\theta$.

Hajnal [Haj04] proved that if $\mathcal{G}_0, \mathcal{G}_1$ are graphs of infinite chromatic number, then every subgraph of $\mathcal{G}_0$ of size $< \mathrm{Chr}(\mathcal{G}_1)$ has chromatic number $\leq \mathrm{Chr}(\mathcal{G}_0 \times \mathcal{G}_1)$.

Thus, for a cardinal $\kappa$, let $\varphi(\kappa)$ be the least strongly-compact cardinal $\theta \geq \kappa$. Towards a contradiction, suppose that $\mathcal{G}_0, \mathcal{G}_1$ are graphs, each of chromatic number $\geq \theta$, while $\mathrm{Chr}(\mathcal{G}_0 \times \mathcal{G}_1) = \kappa' < \kappa$. Then, by Hajnal's finding, every subgraph of $\mathcal{G}_0$ of size $< \theta$ would have chromatic number $\leq \mathrm{Chr}(\mathcal{G}_0 \times \mathcal{G}_1) = \kappa'$. But then the generalized de Bruijn–Erdős theorem entails that $\mathrm{Chr}(\mathcal{G}) \leq \kappa' < \theta$. This is a contradiction.

## 3. A generalization

The main result of this paper generalizes as follows.

**Theorem B.** *Suppose that $\lambda \geq \kappa$ are infinite cardinals. If $\lambda > \aleph_0$, suppose in addition that $\boxtimes_\lambda$ holds. Then for every positive integer $n$, there exist graphs $\langle \mathcal{G}_i \mid i < n+1 \rangle$ of size $\lambda^+$ such that:*

- $\mathrm{Chr}(\bigtimes_{i \in I} \mathcal{G}_i) = \lambda^+$ *for every* $I \in [n+1]^n$*;*
- $\mathrm{Chr}(\bigtimes_{i < n+1} \mathcal{G}_i) = \kappa$.

*Proof.* We focus on the case $\lambda > \aleph_0 = \kappa$. All the ideas needed to modify the construction of [Haj85] to establish the case $\lambda = \aleph_0$ will appear in the proof.

[4] In fact, a minor modification to the proof of the main theorem allows one to derive the case $\lambda = \aleph_0$ as well.

Let $\langle (D_\alpha, X_\alpha) \mid \alpha < \lambda^+\rangle$, $h : \lambda^+ \to \lambda^+$, and $\langle C_\alpha \mid \alpha < \lambda^+\rangle$ be as in the proof from the previous section. For all $i < \omega$, set

$$S_i := \{\alpha < \lambda^+ \mid h(\alpha) = i\}, \quad G_i := \{\alpha \in S_i \mid \mathrm{otp}(C_\alpha) = \lambda\},$$
$$E_i := \{\{\alpha, \delta\} \in [G_i]^2 \mid \alpha \in C_\delta, \min(C_\alpha) > \sup(C_\delta \cap \alpha)\},$$
$$C^i_\delta := \{\alpha \in C_\delta \cap G_i \mid \min(C_\alpha) > \sup(C_\delta \cap \alpha)\} \text{ for all } \delta < \lambda^+.$$

For $i < \omega$ and $\gamma < \lambda^+$, we say that a coloring $\chi : \gamma \to \omega$ is *$i$-suitable* if:

- $\chi[C^i_\delta]$ is finite for all $\delta \leq \gamma$;
- $\chi(\alpha) \neq \chi(\delta)$ for all $\alpha < \delta \leq \gamma$ with $\{\alpha, \delta\} \in E_i$.

As in the previous section, for every $i < \omega$ and $\beta < \gamma < \lambda^+$ with $\beta \notin G_i$, and an $i$-suitable coloring $\chi : \beta \to \omega$, there exists an $i$-suitable coloring $\chi' : \gamma \to \omega$ extending $\chi$.

Set $\mathbb{Q}_i := (\{\chi : \beta \to \omega \mid \beta \in \lambda^+ \setminus G_i,\ \chi \text{ is } i\text{-suitable}\}, \subseteq)$. Then a straightforward variation of the proof of Lemma 2.6 shows that the product forcing $\bigtimes_{i\in I} \mathbb{Q}_i$ is $(\leq\lambda)$-distributive for every $I \in [\omega]^{<\omega}$. Moreover, for $I \in [\omega]^{<\omega}$, as $\langle G_i \mid i \in I\rangle$ are pairwise disjoint, the product forcing $\bigtimes_{i\in I} \mathbb{Q}_i$ is isomorphic to

$$\mathbb{Q}_I := \left(\left\{\chi : \beta \to \omega \;\middle|\; \beta < \lambda^+ \ \& \ \bigwedge_{i\in I}(\beta \notin G_i \ \& \ \chi \text{ is } i\text{-suitable})\right\}, \subseteq\right).$$

Finally, fix a positive integer $n < \omega$, and for all $i < n + 1$, set

$$V_i := \left\{\chi : \beta \to \omega \;\middle|\; \beta \in \biguplus\{G_j \mid j < n+1,\ j \neq i\},\ \chi \text{ is } E_i\text{-chromatic}\right\},$$
$$F_i := \left\{\{\chi, \chi'\} \in [V_i]^2 \;\middle|\; \{\mathrm{dom}(\chi), \mathrm{dom}(\chi')\} \in \biguplus_{j<n+1} E_j,\ \chi \subseteq \chi'\right\},$$
$$\mathcal{V}_i := (V_i, F_i).$$

**Lemma 3.1.** $\mathrm{Chr}(\mathcal{V}_0 \times \cdots \times \mathcal{V}_n) \leq \aleph_0$.

*Proof.* Define $c : V_0 \times \cdots \times V_n \to [\omega^3]^{<\omega}$ by

$$c(\chi_0, \ldots, \chi_n) := \{(\chi_i(\mathrm{dom}(\chi_j)), i, j) \mid i, j < n+1, \mathrm{dom}(\chi_j) \in G_i \cap \mathrm{dom}(\chi_i)\}.$$

Note that, by definition of $V_i$, $h(\mathrm{dom}(\chi_i)) \neq i$ for all $i \leq n$. Let us also point out that $c(\chi_0, \ldots, \chi_n)$ is nonempty. For this, define a sequence $\langle a_i \mid i < n+1\rangle$ by letting $a_0 := \chi_0$, and $a_{j+1} := \chi_{h(\mathrm{dom}(a_j))}$ for all $j < n$.

If there exists some $j < n$ such that $\mathrm{dom}(a_j) < \mathrm{dom}(a_{j+1})$, then clearly

$$\big(a_{j+1}(\mathrm{dom}(a_j)), h(\mathrm{dom}(a_{j+1})), h(\mathrm{dom}(a_j))\big) \in c(\chi_0, \ldots, \chi_n),$$

and we are done. Otherwise, we have $\mathrm{dom}(a_0) > \mathrm{dom}(a_1) > \cdots > \mathrm{dom}(a_n)$, so set $a_{n+1} := \chi_{h(\mathrm{dom}(a_n))}$. Let $i < n$ be such that $a_{n+1} = a_i$. Then $\mathrm{dom}(a_{n+1}) = \mathrm{dom}(a_i) > \mathrm{dom}(a_n)$, and hence

$$\big(a_{n+1}(\mathrm{dom}(a_n)), h(\mathrm{dom}(a_{n+1})), h(\mathrm{dom}(a_n))\big) \in c(\chi_0, \ldots, \chi_n).$$

Finally, suppose towards a contradiction that $\{(\chi_0,\dots,\chi_n),(\chi'_0,\dots,\chi'_n)\}\in F_0*\cdots*F_n$, while $c(\chi_0,\dots,\chi_n)=c(\chi'_0,\dots,\chi'_n)$. Pick $(m,i,j)\in c(\chi_0,\dots,\chi_n)$. By $(\chi_i,\chi'_i)\in F_i$, we know that $\chi^*:=\chi_i\cup\chi'_i$ is $E_i$-chromatic. So, as $\chi^*(\mathrm{dom}(\chi_j))=m=\chi^*(\mathrm{dom}(\chi'_j))$, we see that $\{\mathrm{dom}(\chi_j),\mathrm{dom}(\chi'_j)\}\notin E_i$, contradicting the fact that $\{\chi_j,\chi'_j\}\in F_j$ and $h(\mathrm{dom}(\chi_j))=i=h(\mathrm{dom}(\chi'_j))$. □

**Lemma 3.2.** $\mathrm{Chr}(\bigtimes_{i\in I}\mathcal{V}_i)=\lambda^+$ *for every* $I\in[n+1]^n$.

*Proof.* Fix $I\in[n+1]^n$. Let $k<n+1$ be such that $n+1=(I\uplus\{k\})$.

Towards a contradiction, suppose that $c:\bigtimes_{i\in I}V_i\to\lambda$ is $*_{i\in I}F_i$-chromatic. Let $\mathbb{G}$ be $\mathbb{Q}_I$-generic over $V$, and work in $V[\mathbb{G}]$. Set $\chi^*:=\bigcup\mathbb{G}$. Then $\chi^*:\lambda^+\to\omega$ is $E_i$-chromatic for all $i\in I$. Notice that for all $i\in I$ and $\beta\in G_k$, as $i\neq k$, we have $\chi^*\restriction\beta\in V_i$. Thus, we may derive a coloring $c^*:G_k\to\lambda$ by letting, for all $\beta\in G_k$,

$$c^*(\beta):=c\Big(\prod_{i\in I}\chi^*\restriction\beta\Big).$$

Since $c$ is $*_{i\in I}F_i$-chromatic, we find that $c^*$ is $E_k$-chromatic. That is, $c^*$ witnesses that $\mathrm{Chr}(G_k,E_k)\leq\lambda$.

For concreteness, let us assume that $k=0$. Define $H_i,M_i,h_i,A_i,K,g$ as in the proof of Lemma 2.7. Work back in $V$. Let $p_0\in\mathbb{Q}_I$ be such that

$$p_0\Vdash\dot g:\check\lambda^+\to{}^{\check\lambda+1}2,\ \text{and } c^* \text{ is } E_0\text{-chromatic}.$$

By possibly extending $p_0$, we may moreover assume that $p_0$ forces that $\{\alpha<\lambda^+\mid g(\alpha)(i)=1\}$ is unbounded in $\lambda^+$ for all $i\leq\lambda$, and knows about the interaction of $g$ with $c^*$.

As any initial segment of $g$ belongs to $V$, we shall consider the set

$$Z:=\{(p,f)\in\mathbb{Q}_I\times{}^{<\lambda^+}({}^{\lambda+1}2)\mid p_0\subseteq p\Vdash_{\mathbb{Q}_I}\dot g\restriction\mathrm{dom}(f)=\check f\}.$$

Let $\langle N_\alpha\mid\alpha<\lambda^+\rangle$ be an increasing and continuous sequence of elementary submodels of $(\mathcal{H}(\theta),\in,\leq_\theta)$, each of size $\lambda$, such that $\langle D_\delta\mid\delta<\lambda^+\rangle,\mathbb{Q}_I,\psi,\dot g,p_0\in N_0$ and $\langle N_\beta\mid\beta\leq\alpha\rangle\in N_{\alpha+1}$ for all $\alpha<\lambda^+$.

Pick some $\alpha<\lambda^+$ with $\mathrm{otp}(D_\alpha)=\lambda$ such that $h(\alpha)=0$, $X_\alpha=\psi[Z]\cap\alpha$, and $\mathrm{acc}(D_\alpha)\subseteq E:=\{\delta<\lambda^+\mid N_\delta\cap\lambda^+=\delta\}$.

Let $\{\alpha_i\mid i\leq\lambda\}$ denote the increasing enumeration of $\mathrm{acc}(D_\alpha)\cup\{\alpha\}$. We have $h(\alpha_i)=0$ and $M_i\cap\lambda^+=\alpha_i\in S_0$ for all $i\leq\lambda$. Write $M_i:=N_{\alpha_i}$.

Recursively and $\trianglelefteq_\theta$-canonically define a continuous sequence $\langle(p_i,f_i)\mid i<\lambda\rangle$ of pairs that will satisfy the following for all $i<\lambda$:

- $p_{i+1}\Vdash\dot g\restriction\check\alpha_i=\check f_{i+1}$;
- $\alpha_i\leq\mathrm{dom}(p_i)<\alpha_{i+1}$;
- $\langle p_j\mid j\leq i\rangle$ is an increasing sequence of conditions that belongs to $M_{i+1}$.

This process is feasible thanks to the fact that $C^j_{\alpha_i}$ is empty for every limit $i < \lambda$ and every $j \in I$.[5] Then $\langle (p_i, f_i) \mid 0 < i < \lambda\rangle = \langle (p^\alpha_i, f^\alpha_i) \mid 0 < i < \lambda\rangle$, and $p := \bigcup_{i<\lambda} p_i$ is a legitimate condition. Let $p^*$ be an extension of $p$ that decides $c^*(\alpha)$, say $p^* \Vdash c^*(\alpha) = \check{i}$, and decides $h_i \restriction \alpha$. Then $p^* \Vdash \{\alpha''_i, \alpha\} \in E_0 \;\&\; \alpha''_i \in \mathrm{rng}(h_i) \subseteq H_i$. So $p^*$ forces that $c^*$ is not an $E_0$-chromatic coloring, contradicting the fact that $p^*$ extends $p_0$. □

This completes the proof of Theorem B. □

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*Added in proof.* The main result of [Rin15b] implies the following generalization of Corollary 1: In any set-forcing extension of Gödel's constructible universe, all instances of the Infinite Weak Hedetniemi Conjecture fail.

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