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A classification of Nichols algebras of semisimple Yetter–Drinfeld modules over non-abelian groups

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Abstract. Over fields of arbitrary characteristic we classify all braid-indecomposable tuples of at least two absolutely simple Yetter–Drinfeld modules over non-abelian groups such that the group is generated by the support of the tuple and the Nichols algebra of the tuple is finite-dimensional. Such tuples are classified in terms of analogs of Dynkin diagrams which encode much information about the Yetter–Drinfeld modules. We also compute the dimensions of these finite-dimensional Nichols algebras. Our proof uses essentially the Weyl groupoid of a tuple of simple Yetter–Drinfeld modules and our previous result on pairs.

Keywords. Nichols algebra, Weyl groupoid, Hopf algebra

Contents

Introduction

Let $\mathbb{K}$ be a field and let G be a group. The G-graded $\mathbb{K}G$-modules (also known as *Yetter–Drinfeld modules*) form a braided vector space. For any braided vector space V there exists up to isomorphism a unique connected graded braided Hopf algebra $\mathcal{B}(V)$ gener-

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ated by V, such that the generators have degree 1 and all primitive elements are in V. This braided Hopf algebra is known as the *Nichols algebra* of V. It is a fundamental problem in Hopf algebra theory to understand the structure of Nichols algebras (see for example [10] and [2]). Besides their applications to quantum groups and Hopf algebras, Nichols algebras have many other interesting applications such as Schubert calculus [15], Lie superalgebras [1, Example 5.2], and logarithmic conformal field theories [38, 39, 40].

First definitions and structural results on Nichols algebras were obtained by Nichols [36]. Nichols algebras were rediscovered later by Woronowicz [41, 42] and Majid [35], and they were used as a basic object in the lifting method of Andruskiewitsch and Schneider [8] to classify (finite-dimensional) pointed Hopf algebras [9, 11, 12]. Nowadays there exist generalizations of this method to other classes of Hopf algebras [4]. A common step in these methods is to determine all Nichols algebras satisfying a finiteness or a moderate growth condition. While Nichols was only able to determine Nichols algebras over very small abelian groups, the theory of Weyl groupoids [20] leads to satisfactory classification results for arbitrary finite abelian groups. Among the results in this direction we mention the classification of finite-dimensional Nichols algebras of diagonal type [19, 21, 22, 23, 24, 31], and the results related to presentations of such Nichols algebras [13, 14].

Based on the successful experience with Weyl groupoids related to Yetter–Drinfeld modules over abelian groups, the theory was extended to arbitrary Hopf algebras with bijective antipode and semisimple Yetter–Drinfeld modules over them [7]. It turned out that Weyl groupoids provide very strong information on the growth and on combinatorial properties of Nichols algebras in the case of several direct summands. It is remarkable that this theory is also very useful for studying Nichols algebras of simple Yetter–Drinfeld modules. Indeed, the only known tool today to study Nichols algebras over such Yetter–Drinfeld modules is to look at braided subspaces which can be viewed as semisimple Yetter–Drinfeld modules over another Hopf algebra (see for example [5, 6]). From this point of view, the study of Nichols algebras of semisimple Yetter–Drinfeld modules is also crucial and has several potential applications.

Let us explain the main results of this paper and the strategy of the proof. Let G_0 be a group and let V be a finite-dimensional Yetter–Drinfeld module over G_0. By restriction of the module structure, one can view V as a Yetter–Drinfeld module over the subgroup G of G_0 generated by the support of V. Moreover, under some assumptions on G and the field $\mathbb{K}$ one can decompose V into the direct sum of absolutely simples. Motivated by this setting, we study tuples $M = (M_1, \dots, M_\theta)$ of absolutely simple Yetter–Drinfeld modules over a non-abelian group G, where $\theta \in \mathbb{N}$, such that G is generated by the support of $V = \bigoplus_{i=1}^{\theta} M_i$.

Let us add here a side remark. The reflection theory and the Weyl groupoid exist for tuples of simples. However, allowing simple Yetter–Drinfeld modules would lead to a discussion of group representations depending heavily on the field. Further, one would loose essential parts of the combinatorics of the Weyl groupoid: In the worst case one has only one simple summand over the base field instead of several absolutely simples over an extended field. On the other hand, field extensions of Nichols algebras are well understood. Therefore, in general it is more promising to extend the field appropriately before studying a Nichols algebra.

Since V is a braided vector space, it admits a Nichols algebra $\mathcal{B}(V)$. The general theory implies that in some cases, containing those with $\dim \mathcal{B}(V) < \infty$, one can attach to M a connected finite Cartan graph of rank θ (see Section 1 for the definitions). If $\theta = 1$, then the Cartan graph contains no information about $\mathcal{B}(V)$. Therefore we restrict our attention to the case $\theta \geq 2$. Our aim is now to provide a classification of θ-tuples M of absolutely simple Yetter–Drinfeld modules over non-abelian groups such that the group is generated by the support of $\bigoplus_{i=1}^{\theta} M_i$, and M has an indecomposable finite Cartan graph. We record that the indecomposability assumption on the finite Cartan graph is merely of technical nature. It allows us to exclude the components of the Cartan graph of rank one. Since the classification in the case of two simple summands was performed in [29, 30, 28], here we consider the case $\theta \geq 3$. To write our classification theorem, we introduce two concepts:

Braid-indecomposability. The braid-indecomposability of the tuple of Yetter–Drinfeld modules records the indecomposability assumption on the finite Cartan graph (see Definition 2.1).

Skeletons. To describe the structure of the Yetter–Drinfeld modules involved, we make use of diagrams which are analogs of Dynkin diagrams of finite type. We call them *skeletons of finite type*. See Definition 2.2 for the definition of a skeleton and Figure 2.1 for skeletons of finite type.

A consequence of our main result is the following classification (see Theorem 2.5).

Classification theorem. *Let $\theta \geq 3$ and let $M = (M_1, \dots, M_\theta)$ be a braid-indecomposable tuple of Yetter–Drinfeld modules over a non-abelian group G such that the support of $M_1 \oplus \cdots \oplus M_\theta$ generates G. Then the Nichols algebra $\mathcal{B}(M_1 \oplus \cdots \oplus M_\theta)$ is finite-dimensional if and only if M has a skeleton of finite type.*

We remark that no assumption on the characteristic of the field $\mathbb{K}$ is needed. The theorem is the culmination of several theorems stated in Section 2. These theorems contain the dimensions and the Hilbert series for the Nichols algebras of the classification. Almost all of the examples appearing in our classification admit a standard (classical) root system. The dimensions of the Nichols algebras admiting a standard root system are shown in Table 1.

Table 1. Finite-dimensional Nichols algebras with a standard root system.

Dimension	$2^{\theta(\theta+1)}$	$4^{\theta(\theta-1)}3^{2\theta}$	$2^{2\theta^2-\theta}$	$4^{\theta(\theta-1)}$	4^{36}	4^{63}	4^{120}	4^{18}
Root system	A_θ	B_θ	C_θ	D_θ	E_6	E_7	E_8	F_4
char $\mathbb{K}$		3	$\neq 2$					$\neq 2$

It is remarkable that in characteristic zero and in the case where $\theta \geq 4$, these finite-dimensional Nichols algebras appear in the work of Lentner [34] related to coverings of Nichols algebras.

Table 2. Finite-dimensional Nichols algebras with a non-standard root system of rank three.

Dimension	$3^6\,12^7$	$2^{12}\,12^4$	$6^6\,12^7$
char $\mathbb{K}$	2	3	$\neq 2, 3$

In the case of three simple summands, one has an additional family of examples which admit a non-standard root system. The dimensions of these Nichols algebras are shown in Table 2.

The proof of Theorem 2.5 is based on a general PBW-type theorem on certain Nichols algebras from [25, Thm. 2.6] (see Theorem 1.2), on the classification in the case $\theta = 2$ [28], and on the classification of connected indecomposable finite Cartan graphs of rank three [17]. In fact, we only need Lemma 3.1 from [17], for the proof of which we had to use the main result in [17]. In order to simplify our approach further, we prove the following theorem (see Theorem 4.2).

Theorem. *Any connected indecomposable finite Cartan graph has an object with a Cartan matrix of finite type.*

This result is of independent interest and its proof does not use the classification of finite Cartan graphs [17, 18]. At an early stage of our work we had a proof of our main classification theorem without using the classification in [17], but it was much more technical than the present work.

The paper is organized as follows. Notation, terminology and a review of the theory of Weyl groupoids of tuples of simple Yetter–Drinfeld modules over groups is given in Section 1. In Section 2 we state the main result of the paper, a classification of finite-dimensional Nichols algebras of semisimple Yetter–Drinfeld modules over groups in terms of skeletons of finite type. This section also contains the Hilbert series of each of the Nichols algebras appearing in our classification (see Theorems 2.6–2.10). In Sections 3 and 4 we collect useful facts about finite Cartan graphs. In particular, in Theorem 4.2 we prove that every finite connected indecomposable Cartan graph contains a point with a Cartan matrix of finite type. Section 5 contains several useful lemmas related to the structure of Yetter–Drinfeld modules over arbitrary groups. Sections 6–9 are devoted to the proofs of the structure theorems in cases ADE, C, B and F_4. The main theorem, Theorem 2.5, is then proved in Section 10. The paper contains two appendices. Appendix A is devoted to the structure theory of $(\operatorname{ad} V)(W)$ for particular Yetter–Drinfeld modules V and W. Some known results are cited and some new results, which are needed in the paper, are obtained using known methods. Appendix B reviews the main results of [28], where finite-dimensional Nichols algebras of direct sums of two absolutely simple Yetter–Drinfeld modules were studied.

1. Preliminaries

1.1. As usual, $\mathbb{N}$ is the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ is the set of integers, $\mathbb{K}$ is an arbitrary field of characteristic char $\mathbb{K}$ and $\mathbb{K}^\times = \mathbb{K} \setminus \{0\}$.

For a set X we write $|X|$ for the cardinality of X.

For a group G we write $\widehat{G}$ for the set of linear characters of G, and $Z(G)$ for the center of G. For $g \in G$, we write G^g for the centralizer of g in G. The conjugacy class of g will be denoted by g^G. For any $g, h \in G$ we sometimes write $g \triangleright h$ for ghg^{-1}. If $X \subseteq G$ is a subset, then $\langle X \rangle$ denotes the subgroup of G generated by X.

The category of Yetter–Drinfeld modules over G will be denoted by ${}^G_G\mathcal{YD}$. Recall that a *Yetter–Drinfeld module* over G, also called a *G-graded $\mathbb{K}G$-module*, is a $\mathbb{K}G$-*module* $V = \bigoplus_{g\in G} V_g$ such that $hV_g \subseteq V_{hgh^{-1}}$ for all $g, h \in G$. It is a braided vector space with braiding $c: V \otimes V \to V \otimes V$ defined by $c(u \otimes v) = gv \otimes u$ for all $u \in V_g$, $v \in V$. The *support* of V is

$$\operatorname{supp} V = \{g \in G : V_g \neq 0\}.$$

We say that V is *absolutely simple* if $V \neq 0$ and if for any field extension $\mathbb{L}$ of $\mathbb{K}$ the only Yetter–Drinfeld submodules of $\mathbb{L} \otimes_{\mathbb{K}} V$ over $\mathbb{L}G$ are $\{0\}$ and $\mathbb{L} \otimes_{\mathbb{K}} V$. (Absolutely) simple Yetter–Drinfeld modules over G are parametrized by pairs (g^G, ρ), where g^G is a conjugacy class of G and $\rho : \mathbb{K}G^g \to \operatorname{End}(W)$ is an (absolutely) irreducible representation of the centralizer G^g. The (absolutely) simple Yetter–Drinfeld modules over G are

$$M(g^G, \rho) = \operatorname{Ind}_{G^g}^G \rho$$

with the induced action $y(x \otimes w) = yx \otimes w$ for $x, y \in G$ and $w \in W$, and the coaction $\delta : M(g^G, \rho) \to \mathbb{K}G \otimes M(g^G, \rho)$ is given by $\delta(x \otimes w) = xgx^{-1} \otimes (x \otimes w)$ for all $w \in W$ and $x \in G$. One also says that $x \otimes w$ has *G-degree* xgx^{-1}.

For a G-graded $\mathbb{K}G$-module V we write $\mathcal{B}(V)$ for the Nichols algebra of V. Nichols algebras are connected strictly $\mathbb{N}_0$-graded braided Hopf algebras with V as degree one part. The *Hilbert series* of an $\mathbb{N}_0$-graded algebra $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ is $\sum_{n\geq 0}(\dim R_n)t^n \in \mathbb{Z}[[t]]$. For all $k \in \mathbb{N}_0$ and $t \in \mathbb{K}$ let $(k)_t = 1 + t + \cdots + t^{k-1}$ be the usual t-number.

Many examples of finite-dimensional Nichols algebras of pairs of absolutely simple Yetter–Drinfeld modules are related to the groups Γ_n for $n \in \{2, 3, 4\}$ defined in [25]: For all $n \in \mathbb{N}_{\geq 2}$, the group Γ_n is defined by the generators a, b, ν and relations

$$ba = \nu ab, \qquad \nu a = a\nu^{-1}, \qquad \nu b = b\nu, \qquad \nu^n = 1.$$

1.2. Weyl groupoids and root systems. We review the basics of the theory of Weyl groupoids of tuples of simple Yetter–Drinfeld modules over groups. We refer to [7] and [26, 25] for details and proofs. We use the terminology introduced in [31] after several discussions with Andruskiewitsch and Schneider.

Let $\theta \in \mathbb{N}$ and let $I = \{1, \dots, \theta\}$. Let $\mathcal{X}$ be a non-empty set and for each $X \in \mathcal{X}$ let $A^X = (a_{ij}^X)_{1\leq i,j\leq\theta}$ be a generalized Cartan matrix. For all $i \in I$ let $r_i : \mathcal{X} \to \mathcal{X}$ be a map. The quadruple

$$\mathcal{C} = \mathcal{C}(I, \mathcal{X}, r, A),$$

where $r = (r_i)_{i\in I}$ and $A = (A^X)_{X\in\mathcal{X}}$, is called a *semi-Cartan graph* if $r_i^2 = \mathrm{id}_X$ for all $i \in I$, and $a_{ij}^X = a_{ij}^{r_i(X)}$ for all $X \in \mathcal{X}$ and $i, j \in I$. We say that a semi-Cartan graph $\mathcal{C}$ is *connected* if there is no proper non-empty subset $\mathcal{Y} \subset \mathcal{X}$ such that $r_i(Y) \in \mathcal{Y}$ for all $i \in I$ and $Y \in \mathcal{Y}$.

Let $\mathcal{C}=\mathcal{C}(I,\mathcal{X},r,A)$ be a semi-Cartan graph. There exists a unique category $\mathcal{D}(\mathcal{X},I)$ with $\mathcal{X}$ as its set of objects and with morphisms

$$\operatorname{Hom}(X,Y)=\{(Y,f,X):f\in\operatorname{End}(\mathbb{Z}^\theta)\}$$

for $X,Y\in\mathcal{X}$ with composition defined by

$$(Z,g,Y)\circ(Y,f,X)=(Z,gf,X)$$

for all $X,Y,Z\in\mathcal{X}$ and $f,g\in\operatorname{End}(\mathbb{Z}^\theta)$.

We write $\alpha_1,\dots,\alpha_\theta$ for the standard basis of $\mathbb{Z}^\theta$.

For each $X\in\mathcal{X}$ and $i\in I$ let

$$s_i^X\in\operatorname{Aut}(\mathbb{Z}^\theta),\qquad s_i^X\alpha_j=\alpha_j-a_{ij}^X\alpha_i\quad\text{for all } j\in I.$$

Let $\mathcal{W}(\mathcal{C})$ be the subcategory of $\mathcal{D}(\mathcal{X},I)$ generated by the morphisms $(r_i(X),s_i^X,X)$, where $i\in I$ and $X\in\mathcal{X}$. Then $\mathcal{W}(\mathcal{C})$ is a groupoid. For any $X,Y\in\mathcal{X}$ and $f\in\operatorname{Aut}(\mathbb{Z}^\theta)$ with $w=(Y,f,X)\in\operatorname{Hom}(X,Y)$ and for any $\alpha\in\mathbb{Z}^\theta$ we also write $w\alpha$ for $f\alpha$. For all $k\in\mathbb{N}_0$, $i_1,\dots,i_k\in I$ and $X_0,X_1,\dots,X_k\in\mathcal{X}$ with $r_{i_m}(X_m)=X_{m-1}$ for all $1\le m\le k$ let

$$\operatorname{id}_{X_0}s_{i_1}\cdots s_{i_k}=s_{i_1}^{X_1}\cdots s_{i_k}^{X_k}\in\operatorname{Hom}(X_k,X_0).$$

For each $X\in\mathcal{X}$ the set of *real roots* of $\mathcal{C}$ at X is

$$\boldsymbol{\Delta}^{\mathrm{re}\,X}=\Big\{w\alpha_i : w\in\bigcup_{Y\in\mathcal{X}}\operatorname{Hom}(Y,X)\Big\}\subseteq\mathbb{Z}^\theta.$$

The sets of *positive real roots* and *negative real roots* are

$$\boldsymbol{\Delta}_+^{\mathrm{re}\,X}=\boldsymbol{\Delta}^{\mathrm{re}\,X}\cap\mathbb{N}_0^I,\qquad \boldsymbol{\Delta}_-^{\mathrm{re}\,X}=\boldsymbol{\Delta}^{\mathrm{re}\,X}\cap-\mathbb{N}_0^I,$$

respectively. The semi-Cartan graph $\mathcal{C}$ is *finite* if its set of real roots at X is finite for all $X\in\mathcal{X}$. The semi-Cartan graph $\mathcal{C}$ is a *Cartan graph* if:

(1) For each $X\in\mathcal{X}$ the set $\boldsymbol{\Delta}^{\mathrm{re}\,X}$ consists of positive and negative roots.
(2) Let $X\in\mathcal{X}$ and $i,j\in I$. If $t_{ij}^X=|\boldsymbol{\Delta}^{\mathrm{re}\,X}\cap(\mathbb{N}_0\alpha_i+\mathbb{N}_0\alpha_j)|<\infty$ then $(r_ir_j)^{t_{ij}^X}(X)=X$.

If $\mathcal{C}$ is a Cartan graph, the groupoid $\mathcal{W}(\mathcal{C})$ is the *Weyl groupoid* of $\mathcal{C}$.

For all points $X\in\mathcal{X}$ of the semi-Cartan graph $\mathcal{C}$ let $\boldsymbol{\Delta}^X\subseteq\mathbb{Z}^\theta$. We say that $\mathcal{R}=\mathcal{R}(\mathcal{C},(\boldsymbol{\Delta}^X)_{X\in\mathcal{X}})$ is a *root system* of type $\mathcal{C}$ if:

(1) $\boldsymbol{\Delta}^X=(\boldsymbol{\Delta}^X\cap\mathbb{N}_0^I)\cup-(\boldsymbol{\Delta}^X\cap\mathbb{N}_0^I)$ for all $X\in\mathcal{X}$.
(2) $\boldsymbol{\Delta}^X\cap\mathbb{Z}\alpha_i=\{\alpha_i,-\alpha_i\}$ for all $i\in I$, $X\in\mathcal{X}$.
(3) $s_i^X(\boldsymbol{\Delta}^X)=\boldsymbol{\Delta}^{r_i(X)}$ for all $i\in I$, $X\in\mathcal{X}$.
(4) $(r_ir_j)^{m_{ij}^X}(X)=X$ for all $i,j\in I$ with $i\ne j$ and all $X\in\mathcal{X}$, where $m_{ij}^X=|\boldsymbol{\Delta}^X\cap(\mathbb{N}_0\alpha_i+\mathbb{N}_0\alpha_j)|$ is finite.

Note that (4) is similar to the condition (2) of a Cartan graph, but here $\boldsymbol{\Delta}$ is involved instead of the set of real roots. Axiom (4) is necessary for the Coxeter relations of the Weyl groupoid. For any finite Cartan graph $\mathcal{C}$ the family $(\boldsymbol{\Delta}^{\mathrm{re}\,X})_{X\in\mathcal{X}}$ defines the unique root system of type $\mathcal{C}$ (see [16, Props. 2.9 and 2.12]).

A connected semi-Cartan graph is *indecomposable* if there exists $X \in \mathcal{X}$ such that the Cartan matrix A^X is indecomposable, that is, there are no disjoint subsets $I_1, I_2 \subset I$ such that $I_1, I_2 \neq \emptyset$, $I_1 \cup I_2 = I$, and $a_{ij}^X = 0$ for all $i \in I_1$ and $j \in I_2$. It is known by [16, Prop. 4.6] that if a connected finite Cartan graph $\mathcal{C}$ is indecomposable, then A^X is an indecomposable Cartan matrix for all points X of $\mathcal{C}$.

A semi-Cartan graph $\mathcal{C}$ is *standard* if $A^X = A^Y$ for all $X, Y \in \mathcal{X}$. In this case the real roots form the set of real roots of the Weyl group attached to the Cartan matrix, and hence $\mathcal{C}$ is a Cartan graph. We then say that the Weyl groupoid $\mathcal{W}(\mathcal{C})$ is standard. If $\mathcal{R}$ is a root system of a standard Cartan graph $\mathcal{C}$, then we say that $\mathcal{R}$ is standard. The terminology is based on [3].

Let us review the connections between Cartan graphs and Nichols algebras. Let G be a group and ${}_G^G\mathcal{YD}$ be the category of Yetter–Drinfeld modules over G. We write $\mathcal{F}_\theta^G$ for the set of θ-tuples of finite-dimensional absolutely simple objects in ${}_G^G\mathcal{YD}$, and $\mathcal{X}_\theta$ for the θ-tuples of isomorphism classes of finite-dimensional absolutely simple objects in ${}_G^G\mathcal{YD}$.

For any Yetter–Drinfeld module U over G and any $x \in U$, $y \in \mathcal{B}(U)$ we write $(\operatorname{ad} x)(y)$ for $\operatorname{mult}(\operatorname{id} - c)(x \otimes y)$, where mult denotes the multiplication map in $\mathcal{B}(U)$ and c is the braiding of $\mathcal{B}(U)$. Then for any two subsets $U' \subseteq U$ and $U'' \subseteq \mathcal{B}(U)$ we write $(\operatorname{ad} U')(U'')$ for the linear span of the elements $(\operatorname{ad} x)(y)$ with $x \in U'$ and $y \in U''$.

Let $\theta \in \mathbb{N}$ and let $I = \{1, \dots, \theta\}$. For $M = (M_1, \dots, M_\theta) \in \mathcal{F}_\theta^G$ let $[M] = ([M_1], \dots, [M_\theta]) \in \mathcal{X}_\theta$ be the corresponding θ-tuple of isomorphism classes. For all $i \in I$ and $j \in I \setminus \{i\}$ let

$$a_{ij}^M = \begin{cases} -\infty & \text{if } (\operatorname{ad} M_i)^m(M_j) \neq 0 \text{ for all } m \geq 0, \\ -\sup\{m \in \mathbb{N}_0 : (\operatorname{ad} M_i)^m(M_j) \neq 0\} & \text{otherwise.} \end{cases}$$

Moreover, let $a_{ii}^M = 2$ for all $i \in I$. Then $A^M = (a_{ij}^M)_{i,j\in I}$ is called the *Cartan matrix* of M. Clearly, A^M depends only on the isomorphism class of M and hence we also write $A^{[M]}$ for A^M.

For all $i \in I$ the reflection map $R_i : \mathcal{F}_\theta^G \to \mathcal{F}_\theta^G$ is defined by $R_i(N) = N$ if $a_{ij}^N = -\infty$ for some $j \in I$, and otherwise $R_i(N) = (N_1', \dots, N_\theta')$, where

$$N_j' = \begin{cases} (\operatorname{ad} N_i)^{-a_{ij}^N}(N_j) & \text{if } j \neq i, \\ N_i^* & \text{if } j = i. \end{cases}$$

Since $[R_i(M)] = [R_i(N)]$ in $\mathcal{X}_\theta$ for all $M, N \in \mathcal{F}_\theta^G$ with $[M] = [N]$ and all $i \in I$, we may define $r_i : \mathcal{X}_\theta \to \mathcal{X}_\theta$ by $r_i([N]) = [R_i(N)]$ for all $i \in I$. We then define

$$\mathcal{F}_\theta^G(M) = \{R_{i_1} \cdots R_{i_k}(M) \in \mathcal{F}_\theta^G : k \in \mathbb{N}_0,\ i_1, \dots, i_k \in I\},$$
$$\mathcal{X}_\theta(M) = \{r_{i_1} \cdots r_{i_k}([M]) \in \mathcal{X}_\theta : k \in \mathbb{N}_0,\ i_1, \dots, i_k \in I\}.$$

A tuple $M \in \mathcal{F}_\theta^G$ *admits all reflections* if $a_{ij}^N \in \mathbb{Z}$ for all $N \in \mathcal{F}_\theta^G(M)$ and all $i, j \in I$.

For all $M \in \mathcal{F}_\theta^G$ let $\mathcal{B}(M) = \mathcal{B}(M_1 \oplus \cdots \oplus M_\theta)$. Following the terminology in [26] we say that a Nichols algebra $\mathcal{B}(M)$ is *decomposable* if there exists a totally ordered set L and a sequence $(W_l)_{l \in L}$ of finite-dimensional absolutely simple $\mathbb{N}_0^\theta$-graded objects in ${}_G^G\mathcal{YD}$ such that

$$\mathcal{B}(M) \simeq \bigotimes_{l \in L} \mathcal{B}(W_l).$$

In this case, the isomorphism classes of the W_l and the $\mathbb{Z}^\theta$-degrees are uniquely determined and hence one may define the set $\boldsymbol{\Delta}_+^{[M]}$ of *positive roots* and the set $\boldsymbol{\Delta}^{[M]}$ of *roots* of $[M]$:

$$\boldsymbol{\Delta}_+^{[M]} = \{\deg W_l : l \in L\}, \qquad \boldsymbol{\Delta}^{[M]} = \boldsymbol{\Delta}_+^{[M]} \cup -\boldsymbol{\Delta}_-^{[M]}.$$

There are several results that imply the decomposability of a Nichols algebra. For example, Kharchenko [33, Thm. 2] proved that $\mathcal{B}(M)$ is decomposable if G is abelian and $\dim M_i = 1$ for all $i \in I$. In [26] it is proved that if all finite tensor powers of $M_1 \oplus \cdots \oplus M_\theta$ are direct sums of absolutely simple objects in ${}_G^G\mathcal{YD}$ then $\mathcal{B}(M)$ is decomposable.

Suppose that M admits all reflections. Then

$$\mathcal{C}(M) = (I, \mathcal{X}_\theta(M), (r_i)_{i \in I}, (A^X)_{X \in \mathcal{X}_\theta(M)})$$

is a connected semi-Cartan graph and hence the groupoid $\mathcal{W}(M) = \mathcal{W}(\mathcal{C}(M))$ is defined.

We stress that the above reflection theory works more generally for tuples of simple Yetter–Drinfeld modules. From [25, Cor. 2.4 and Thm. 2.3] one obtains the following theorem.

Theorem 1.1. *Let $\theta \in \mathbb{N}$, let G be a group, and let $M = (M_1, \dots, M_\theta)$, where each M_i is a simple Yetter–Drinfeld module over G. Assume that M admits all reflections and that $\mathcal{W}(M)$ is finite. Then $\mathcal{B}(M)$ is decomposable and $\mathcal{C}(M)$ is a finite Cartan graph.*

Clearly, the same theorem holds if one starts with a tuple of absolutely simple Yetter–Drinfeld modules. Since extension of the base field of a Nichols algebra is compatible with the grading and with taking coinvariants, any reflection of a tuple of absolutely simples is again a tuple of absolutely simples.

As in the case of Coxeter groups, on morphisms of Weyl groupoids one defines a length function (see [26, §1]). The following theorem is an analog of a PBW-decomposition for the Nichols algebras $\mathcal{B}(M)$ of tuples M of absolutely simple Yetter–Drinfeld modules.

Theorem 1.2 ([25, Thm. 2.6]). *Let $\theta \geq 2$ and $M \in \mathcal{F}_\theta^G$. Suppose that M admits all reflections and that $\mathcal{W}(M)$ is finite. Let $w = \mathrm{id}_{[M]} s_{i_1} \cdots s_{i_l}$ be a reduced decomposition of a longest element of $\bigcup_{[N] \in \mathcal{X}_\theta(M)} \mathrm{Hom}([N], [M])$. Let*

$$\beta_m = \mathrm{id}_{[M]} s_{i_1} \cdots s_{i_{m-1}} \alpha_{i_m}$$

for all $m \in \{1, \dots, l\}$, where l is the length of w. Then $\boldsymbol{\Delta}_+^{[M]} = \{\beta_1, \dots, \beta_l\}$ and $\beta_k \neq \beta_m$ for all $k, m \in \{1, \dots, l\}$ with $k \neq m$. There exist finite-dimensional absolutely simple *subobjects* $M_{\beta_m} \subseteq \mathcal{B}(M)$ *in* ${}^G_G\mathcal{YD}$ *of degree* β_m *for all* $m \in \{1, \dots, l\}$ *with* $M_{\beta_m} \simeq R_{i_{m-1}} \cdots R_{i_2} R_{i_1}(M)_{i_m}$ *in* ${}^G_G\mathcal{YD}$. *Moreover, the multiplication map*

$$\mathcal{B}(M_{\beta_l}) \otimes \cdots \mathcal{B}(M_{\beta_2}) \otimes \mathcal{B}(M_{\beta_1}) \to \mathcal{B}(M)$$

is an isomorphism of $\mathbb{N}_0^\theta$*-graded objects in* ${}^G_G\mathcal{YD}$.

In this theorem, as everywhere else, we write N_i for the i-th entry of a tuple $N \in \mathcal{F}_\theta^G$ (here $N = R_{i_{m-1}} \cdots R_{i_2} R_{i_1}(M)$), where $1 \le i \le \theta$.

For all $\alpha = \sum_{i=1}^\theta n_i\alpha_i \in \mathbb{Z}^\theta$, we write t^α for $t_1^{n_1} \cdots t_\theta^{n_\theta} \in \mathbb{Z}[[t_1, \dots, t_\theta]]$. For any $\mathbb{N}_0^\theta$-graded object $X = \bigoplus_{\alpha \in \mathbb{N}_0^\theta} X_\alpha$ in ${}^G_G\mathcal{YD}$, the (multivariate) Hilbert series of X is

$$\sum_{\alpha \in \mathbb{N}_0^\theta} (\dim X_\alpha) t^\alpha \in \mathbb{Z}[[t_1, \dots, t_\theta]].$$

The Yetter–Drinfeld modules $(\operatorname{ad} V)^k(W) \subseteq \mathcal{B}(V \oplus W)$ for $k \in \mathbb{N}_0$ and $V, W \in {}^G_G\mathcal{YD}$ can be computed as a certain subobject of $V^{\otimes k} \otimes W$ using Lemma 1.3 below, and hence the M_{β_m} in Theorem 1.2 can be computed effectively. This allows us to compute the Hilbert series of $\mathcal{B}(M)$.

Lemma 1.3 ([25, Thm. 1.1]). *Let* V *and* W *be Yetter–Drinfeld modules over a group. Let* $\varphi_0 = 0$, $X_0^{V,W} = W$, *and*

$$\varphi_{m+1} = \mathrm{id} - c_{V^{\otimes m} \otimes W, V}\, c_{V, V^{\otimes m} \otimes W} + (\mathrm{id} \otimes \varphi_m) c_{1,2},$$
$$X_{m+1}^{V,W} = \varphi_{m+1}(V \otimes X_m^{V,W}) \subseteq V^{\otimes(m+1)} \otimes W$$

for all $m \ge 0$. *Then* $(\operatorname{ad} V)^n(W) \simeq X_n^{V,W}$ *for all* $n \in \mathbb{N}_0$.

The following important fact on $(\operatorname{ad} M_i)^m(M_j)$ for any $\theta \in \mathbb{N}$, $M \in \mathcal{F}_\theta^G$, $m \in \mathbb{N}$, and $i, j \in \{1, \dots, \theta\}$ is also used heavily for explicit calculations. It is a variant of [26, Thm. 7.2(3)].

Theorem 1.4. *Let* $\theta \in \mathbb{N}$ *and* $M \in \mathcal{F}_\theta^G$. *Assume that* M *admits all reflections and* $\mathcal{W}(M)$ *is finite. Let* $i, j \in \{1, \dots, \theta\}$ *with* $i \ne j$. *Then* $(\operatorname{ad} M_i)^m(M_j) \in {}^G_G\mathcal{YD}$ *is absolutely simple for all* $0 \le m \le -a_{ij}^M$ *and zero for all* $m > -a_{ij}^M$.

2. Main results

We will need q-numbers and q-factorials in rings in different contexts. For any ring R, any $m \in \mathbb{N}_0$ and any $q \in R$ let

$$(m)_q = \sum_{i=0}^{m-1} q^i, \qquad (m)_q^! = \prod_{i=1}^{m} (i)_q.$$

Let G be a group. For all $M \in \mathcal{F}_\theta^G$ let

$$\operatorname{supp} M = \operatorname{supp} M_1 \cup \cdots \cup \operatorname{supp} M_\theta.$$

Let $\mathcal{E}_\theta^G$ denote the subclass of $\mathcal{F}_\theta^G$ consisting of all tuples M such that G is generated by $\operatorname{supp} M$.

Definition 2.1. Let $\theta \in \mathbb{N}$. Then $M \in \mathcal{F}_\theta^G$ is called *braid-indecomposable* if there exists no decomposition $M' \oplus M''$ of $\bigoplus_{i=1}^\theta M_i$ in ${}_G^G\mathcal{YD}$ with $M', M'' \neq 0$ and such that $(\mathrm{id} - c^2)(M' \otimes M'') = 0$.

In this work we will attach a skeleton (a kind of decorated Dynkin diagram) to some tuples in $\mathcal{F}_\theta^G$.

Definition 2.2. Let $\theta \in \mathbb{N}_{\geq 2}$ and $M = (M_1, \dots, M_\theta) \in \mathcal{F}_\theta^G$. Let $A = (a_{ij})_{1 \leq i,j \leq \theta}$ be the Cartan matrix of M. We say that M *has a skeleton* if:

- for all $1 \leq i \leq \theta$ there exist $s_i \in \operatorname{supp} M_i$ and $\sigma_i \in \widehat{G^{s_i}}$ such that $M_i \simeq M(s_i, \sigma_i)$,
- for all $1 \leq i < j \leq \theta$ with $a_{ij} \neq 0$ at least one of a_{ij}, a_{ji} is -1.

In this case the *skeleton* of M is a partially oriented partially labeled loopless graph with θ vertices with the following properties:

- For all $1 \leq i \leq \theta$, the i-th vertex is symbolized by $|\operatorname{supp} M_i| = \dim M_i$ points. If $\dim M_i = 1$, then the vertex is labeled by $\sigma_i(s_i)$. If $\dim M_i = 2$ and there is an additional restriction on $p = \sigma_i(s_i' s_i^{-1})$, where $\operatorname{supp} M_i = \{s_i, s_i'\}$, then the i-th vertex is labeled by (p). Otherwise there is no label.
- For all $i, j \in \{1, \dots, \theta\}$ with $i \neq j$ there are $a_{ij}a_{ji}$ edges between the i-th and j-th vertex. The edge is oriented towards j if and only if $a_{ij} = -1$, $a_{ji} < -1$.
- Let $1 \leq i < j \leq \theta$ with $a_{ij} < 0$. If $\operatorname{supp} M_i$ and $\operatorname{supp} M_j$ commute, then the connection between the i-th and j-th vertex consists of continuous lines. Otherwise the connection consists of dashed lines. The connection is labeled with $\sigma_i(s_j)\sigma_j(s_i)$ if $\dim M_i = 1$ or $\dim M_j = 1$, and otherwise it is not labeled.

Remark 2.3. Let $i \in \{1, \dots, \theta\}$ with $\dim M_i = 1$ in Definition 2.2. Since the Yetter–Drinfeld modules M_j are absolutely simple for all j, the support of each M_j is a conjugacy class of G and the central element s_i acts by a scalar on each M_j. Thus $\sigma_i(s_j)$ and $\sigma_j(s_i)$ do not depend on the choice of $s_j \in \operatorname{supp} M_j$.

We will show in Lemma 5.3 that the label (p) of a vertex with two points in Definition 2.2 is well-defined. Therefore all labels of the skeleton of M are well-defined.

Definition 2.4. A skeleton is called *simply-laced* if any two vertices are connected by at most one edge. A skeleton is called *connected* if the underlying graph is connected. A connected skeleton with at least three vertices is said to be of *finite type* if it appears in Figure 2.1. For technical reasons we say that a skeleton of type α_2 is of finite type.

The main result of this paper is the following theorem.

Theorem 2.5. *Let $\theta \in \mathbb{N}_{\geq 3}$. Let G be a non-abelian group and $M \in \mathcal{E}_\theta^G$. Assume that M is braid-indecomposable. Then the following are equivalent:*

(1) *M has a skeleton of finite type.*
(2) *$\mathcal{B}(M)$ is finite-dimensional.*
(3) *M admits all reflections and the Weyl groupoid $\mathcal{W}(M)$ of M is finite.*

We record that the third property of M in the theorem is also equivalent to the finiteness of the set of $\mathbb{N}_0$-graded right coideal subalgebras of $\mathcal{B}(M)$ by [27, Thm. 6.15].

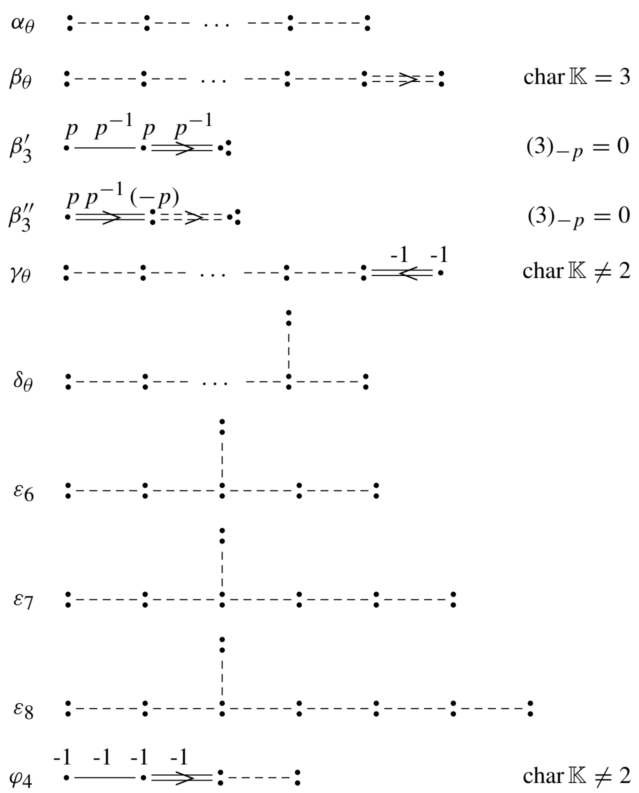


Fig. 2.1. Skeletons of finite type with at least three vertices.

Theorem 2.5 will be proved in Section 10. The Hilbert series of the Nichols algebras of Theorem 2.5 will be given in Subsections 2.1–2.4. In Sections 6–9 we give a description of all tuples in $\mathcal{F}_\theta^G$ which have a skeleton of finite type.

2.1. The ADE series. The following theorem will be proved in Section 6.

Theorem 2.6. *Let $\theta \in \mathbb{N}_{\geq 2}$. Let G be a non-abelian group and $M \in \mathcal{E}_\theta^G$. Assume that the Cartan matrix A^M is of finite type and the Dynkin diagram of A^M is connected and simply-laced. Then:*

(1) *M has a simply-laced skeleton of finite type.*
(2) *M admits all reflections and the Weyl groupoid $\mathcal{W}(M)$ is finite with a finite root system of standard type A_θ with $\theta \geq 2$, D_θ with $\theta \geq 4$, E_6, E_7 or E_8.*
(3) *$\mathcal{B}(M)$ is finite-dimensional and its Hilbert series is*

$$\mathcal{H}(t) = \prod_{\alpha \in \boldsymbol{\Delta}_+} (1 + t^\alpha)^2,$$

where $\boldsymbol{\Delta}_+$ denotes the set of positive roots of the root system associated with the Cartan matrix A^M. The dimensions of these Nichols algebras are listed in Table 1.

2.2. The C series. The following theorem will be proved in Section 7.

Theorem 2.7. *Let $\theta \in \mathbb{N}_{\geq 3}$, G be a non-abelian group and $M \in \mathcal{E}_\theta^G$. Assume that the Cartan matrix A^M is of type C_θ. Then the following are equivalent:*

(1) $\operatorname{char} \mathbb{K} \neq 2$ *and M has a skeleton of type γ_θ.*
(2) *M admits all reflections and $\mathcal{W}(M)$ is finite.*
(3) *M admits all reflections and $\mathcal{W}(M)$ is standard.*
(4) *$\mathcal{B}(M)$ is finite-dimensional.*

In this case, the Hilbert series of $\mathcal{B}(M)$ is

$$\mathcal{H}(t) = \prod_{\alpha \in \mathbf{\Delta}_+^{\text{short}}} (1+t^\alpha)^2 \prod_{\alpha \in \mathbf{\Delta}_+^{\text{long}}} (1+t^\alpha),$$

where $\mathbf{\Delta}_+^{\text{short}}$ and $\mathbf{\Delta}_+^{\text{long}}$ denote the set of short positive roots and long positive roots of the root system associated with $\mathcal{W}(M)$, respectively. In particular

$$\dim \mathcal{B}(M) = 2^{2\theta^2 - \theta}.$$

2.3. The B series. Our main results in this subsection are the following two theorems.

Theorem 2.8. *Let $\theta \in \mathbb{N}_{\geq 3}$. Let G be a non-abelian group and $M \in \mathcal{E}_\theta^G$. Assume that* $\dim M_1 = 1$ *and the Cartan matrix A^M is of type B_θ. Then the following are equivalent:*

(1) *$\theta = 3$ and M has a skeleton of type β_3'.*
(2) *M admits all reflections and $\mathcal{W}(M)$ is finite.*
(3) *$\mathcal{B}(M)$ is finite-dimensional.*

Let $h = 3$ if $\operatorname{char} \mathbb{K} = 2$, *$h = 2$ if* $\operatorname{char} \mathbb{K} = 3$, *and $h = 6$ otherwise, and let $h' = 2$ if* $\operatorname{char} \mathbb{K} = 3$ *and $h' = 6$ otherwise. Then in the above cases the Hilbert series of $\mathcal{B}(M)$ is*

$$\mathcal{H}(t) = \prod_{\alpha \in \mathcal{O}_1} (h)_{t^\alpha} \prod_{\alpha \in \mathcal{O}_3} (2)_{t^\alpha}^2 (3)_{t^\alpha} \prod_{\alpha \in \mathcal{O}_{233}} (2)_{t^\alpha} (h')_{t^\alpha},$$

where $\mathcal{O}_1, \mathcal{O}_3$, and $\mathcal{O}_{233}$ are the sets of positive roots in the orbits of α_1, α_3, and $\alpha_2 + 2\alpha_3$, respectively, under the action of the automorphism group of the skeleton of M in its Cartan graph (see Lemma 8.8). In particular,

$$\dim \mathcal{B}(M) = h^6 12^4 (2h')^3.$$

Theorem 2.9. *Let $\theta \in \mathbb{N}_{\geq 3}$. Let G be a non-abelian group and $M \in \mathcal{E}_\theta^G$. Assume that* $\dim M_1 > 1$, *and the Cartan matrix A^M is of type B_θ. Then the following are equivalent:*

(1) $\operatorname{char} \mathbb{K} = 3$ *and M has a skeleton of type β_θ.*
(2) *M admits all reflections and $\mathcal{W}(M)$ is finite.*
(3) *M admits all reflections and $\mathcal{W}(M)$ is standard.*
(4) *$\mathcal{B}(M)$ is finite-dimensional.*

In this case the Hilbert series of $\mathcal{B}(M)$ *is*

$$\mathcal{H}(t)=\prod_{\alpha\in\boldsymbol{\Delta}_+^{\mathrm{short}}}(1+t^\alpha+t^{2\alpha})^2\prod_{\alpha\in\boldsymbol{\Delta}_+^{\mathrm{long}}}(1+t^\alpha)^2,$$

where $\boldsymbol{\Delta}_+^{\mathrm{short}}$ *and* $\boldsymbol{\Delta}_+^{\mathrm{long}}$ *denote respectively the sets of short positive roots and of long positive roots of the root system associated with* $\mathcal{W}(M)$*. In particular*

$$\dim\mathcal{B}(M)=2^{2\theta(\theta-1)}3^{2\theta}.$$

Theorems 2.8 and 2.9 will be proved in Section 8.

2.4. The exceptional case F_4**.** The following theorem will be proved in Section 9.

Theorem 2.10. *Let* G *be a non-abelian group and* $M\in\mathcal{E}_\theta^G$*. Assume that the Cartan matrix* A^M *is of type* F_4*. Then the following are equivalent:*

(1) $\operatorname{char}\mathbb{K}\neq 2$ *and* M *has a skeleton of type* φ_4*.*
(2) M *admits all reflections and* $\mathcal{W}(M)$ *is finite.*
(3) M *admits all reflections and* $\mathcal{W}(M)$ *is standard.*
(4) $\mathcal{B}(M)$ *is finite-dimensional.*

In this case, the Hilbert series of $\mathcal{B}(M)$ *is*

$$\mathcal{H}(t)=(2)_t^6(2)_{t^2}^5(2)_{t^3}^5(2)_{t^4}^5(2)_{t^5}^4(2)_{t^6}^3(2)_{t^7}^3(2)_{t^8}^2(2)_{t^9}(2)_{t^{10}}(2)_{t^{11}}.$$

In particular $\dim\mathcal{B}(M)=2^{36}$.

3. Finite Cartan graphs of rank three

In this section we collect some facts about finite Cartan graphs of rank three which will be used for our classification. Our main reference is [17].

Lemma 3.1. *Let* $\mathcal{C}=\mathcal{C}(I,\mathcal{X},r,A)$ *be a connected indecomposable finite Cartan graph with* $|I|=3$*. If* A^X *is of type* A_3 *for no* $X\in\mathcal{X}$*, then up to a permutation of* I *one of the following holds:*

(1) $\mathcal{C}$ *is standard of type* C_3*.*
(2) $\mathcal{C}$ *is standard of type* B_3*.*
(3) *For each point* X *of* $\mathcal{C}$*,* A^X *is one of the matrices*

$$\begin{pmatrix}2&-1&0\\-1&2&-2\\0&-1&2\end{pmatrix},\quad\begin{pmatrix}2&-1&0\\-2&2&-2\\0&-1&2\end{pmatrix}.$$

(4) *For each point* X *of* $\mathcal{C}$*,* A^X *is one of the matrices*

$$\begin{pmatrix}2&-1&0\\-1&2&-1\\0&-2&2\end{pmatrix},\quad\begin{pmatrix}2&-1&0\\-2&2&-1\\0&-2&2\end{pmatrix}.$$

(5) *For each point X of $\mathcal{C}$, A^X is one of the matrices*

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -4 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix},$$
$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -2 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -2 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix}.$$

(6) *For each point X of $\mathcal{C}$, A^X is one of the matrices*

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix},$$
$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}.$$

The six cases correspond to the set of positive roots in [17, *Appendix A*] *with number* 3, 4*,* 13*,* 14*,* 25*, and* 28*, respectively.*

Remark 3.2. The Cartan graphs in cases Lemma 3.1(3), (4) also appeared in [16, Thm. 5.4].

Proof. Consider the list of all possible sets of positive roots in [17, Appendix A]. There are precisely 55 such sets up to permutation of I and up to the choice of a point of $\mathcal{C}$. By [17, Cor. 2.9], the Cartan matrix of the point X can be obtained from the set $\boldsymbol{\Delta}_+^X$ of its positive roots: $\alpha_j + m\alpha_i \in \boldsymbol{\Delta}^X$ for $m \in \mathbb{Z}$, $i, j \in I$ with $i \neq j$, if and only if $0 \le m \le -a_{ij}^X$. Since the reflection s_i^X for $i \in I$ maps $\boldsymbol{\Delta}_+^X \setminus \{\alpha_i\}$ bijectively onto $\boldsymbol{\Delta}_+^{r_i(X)} \setminus \{\alpha_i\}$, one can calculate the Cartan matrices and the sets of positive roots at all points of $\mathcal{C}$. The elementary calculations are done most efficiently by a computer program. □

For later reference we extract two easy corollaries of the lemma.

Corollary 3.3. *Let $\mathcal{C} = \mathcal{C}(I, \mathcal{X}, r, A)$ be a connected indecomposable finite Cartan graph with $|I| = 3$. If A^X is of type A_3 for no $X \in \mathcal{X}$, then for all $X \in \mathcal{X}$ and for each column of A^X there is at most one entry which is strictly smaller than -1.*

Remark 3.4. The conclusion of Corollary 3.3 holds without the assumption in the second sentence, but we will not need this.

Corollary 3.5. *Let $\mathcal{C} = \mathcal{C}(I, \mathcal{X}, r, A)$ be a connected indecomposable finite Cartan graph with $|I| = 3$. If A^X is of type A_3 or C_3 for no $X \in \mathcal{X}$, then either $\mathcal{C}$ is standard of type B_3, or there is a permutation of I such that for all X the Cartan matrix A^X is one of the matrices in Lemma* 3.1(4).

4. Cartan matrices of finite type

Recall from [32, Thm. 4.3] the classification of a class of indecomposable real matrices. One says that a matrix $A = (a_{ij})_{i,j\in\{1,\dots,n\}}$ is *indecomposable* if there are no proper subsets I, J of $\{1,\dots,n\}$ such that $I \cap J = \emptyset$, $I \cup J = \{1,\dots,n\}$, and $a_{ij} = a_{ji} = 0$ for all $i \in I$ and $j \in J$. For $x, y \in \mathbb{R}^n$ we write $x > y$ ($x \geq y$, respectively) if $x - y$ has only positive (non-negative, respectively) entries.

Theorem 4.1. *Let $n \in \mathbb{N}$ and let A be an indecomposable real $n \times n$-matrix such that $a_{ij} \leq 0$ for all $i, j \in \{1,\dots,n\}$ with $i \neq j$, and $a_{ij} = 0$ whenever $a_{ji} = 0$. Then A has precisely one of the following properties:*

(Fin) $\det A \neq 0$*; there exists* $u > 0$ *such that* $Au > 0$*;* $Av \geq 0$ *implies* $v > 0$ *or* $v = 0$.
(Aff) $\operatorname{corank} A = 1$*; there exists* $u > 0$ *such that* $Au = 0$*;* $Av \geq 0$ *implies* $Av = 0$.
(Ind) *There exists* $u > 0$ *such that* $Au < 0$*;* $Av \geq 0$ *with* $v \geq 0$ *implies that* $v = 0$.

Then A is called of finite, affine, or indefinite type, respectively. Moreover, A^t has the same type as A.

Now we apply this theorem to prove that any connected indecomposable finite Cartan graph has a point with Cartan matrix of finite type. The classification of indecomposable Cartan matrices of finite type is well-known and can be found for example in [32].

Theorem 4.2. *Let $\mathcal{C} = \mathcal{C}(I, \mathcal{X}, r, A)$ be a connected indecomposable finite Cartan graph. Then there exists $X \in \mathcal{X}$ such that A^X is of finite type.*

Proof. The indecomposability of $\mathcal{C}$ implies that A^X is indecomposable for all $X \in \mathcal{X}$ (see [16, Prop. 4.6]). Assume to the contrary that for all $X \in \mathcal{X}$ the Cartan matrix A^X is of affine or indefinite type.

Since $\mathcal{C}$ is finite and connected, it follows that $\mathcal{X}$ is a finite set and $\boldsymbol{\Delta}^{\mathrm{re}\,X}$ is finite for all $X \in \mathcal{X}$. Among all real roots of $\mathcal{C}$ in all objects, choose $\alpha = \sum_{i\in I} x_i\alpha_i$ which is maximal with respect to $>$. Let $x = (x_i)_{i\in I}$ and let $X \in \mathcal{X}$ be such that $\alpha \in \boldsymbol{\Delta}^{\mathrm{re}\,X}$. Let

$$B = \{s_{j_1}\cdots s_{j_k}^X(\alpha) : k \geq 0,\ j_1,\dots,j_k \in I\}.$$

Observe that $s_j(\alpha) = \alpha - \sum_{i\in I} a_{ji}x_i\alpha_j$ for all $j \in I$. Thus the maximality of α implies that $Ax \geq 0$. Since $x \geq 0$ and $x \neq 0$, A is not of indefinite type. Then A is of affine type and $Ax = 0$. Consequently, $s_j^X(\alpha) = \alpha$ for all $j \in I$. Since α is maximal, by induction on k we conclude that $s_{j_1}\cdots s_{j_k}^X(\alpha) = \alpha$ for any $k \in \mathbb{N}_0$ and $j_1,\dots,j_k \in I$. Therefore $B = \{\alpha\}$. On the other hand, α is a real root, which implies that $s_{i_1}\cdots s_{i_k}^X(\alpha) = \alpha_i$ for some $k \in \mathbb{N}_0$ and $i_1,\dots,i_k \in I$, and so $s_i s_{i_1}\cdots s_{i_k}^X(\alpha) = -\alpha_i \neq \alpha$, a contradiction. □

5. Auxiliary lemmas

In this section, let G be a group.

5.1. We first extend some results of [29]. We start with considerations in a general setting.

Lemma 5.1. *Let $s \in G$. Assume that $|s^G| = 2$. Let $r, \epsilon \in G$ be such that $rs = \epsilon sr$, $\epsilon \neq 1$. Then:*

(1) $s^G = \{s, \epsilon s\}$, $r\epsilon = \epsilon^{-1} r$, *and* $g\epsilon = \epsilon g$, $g\epsilon s = \epsilon sg$ *for all* $g \in G^s$.
(2) $r^{-1}sr = rsr^{-1} = \epsilon s$ *and* $r^2, r^{-1}gr, rgr^{-1} \in G^s$ *for all* $g \in G^s$.
(3) $(\epsilon^m s^n)^G = \{\epsilon^m s^n, \epsilon^{n-m} s^n\}$ *for all* $m, n \in \mathbb{Z}$.
(4) *Let H be a subgroup of G containing r and s. Then H is generated by $(H \cap G^s) \cup \{r\}$.*

Proof. Since $rsr^{-1} = \epsilon s$ and $|s^G| = 2$, we conclude that $s^G = \{s, \epsilon s\}$. Then $r\epsilon sr^{-1} = s$, and therefore $r\epsilon r^{-1} = \epsilon^{-1}$. Moreover, $s^G = \{s, \epsilon s\}$ implies that $g\epsilon sg^{-1} = \epsilon s$ for all $g \in G^s$, and hence $g\epsilon = \epsilon g$ for all $g \in G^s$. In particular, (1) is proven.

(2) and (3) follow by similar arguments.

(4) Since $|s^G| = 2$, G^s has index 2 in G. Therefore $H \cap G^s$ has index at most 2 in H. Since $r \in H \setminus G^s$, we get the claim. □

Lemma 5.2. *Let $r, s, \epsilon \in G$. Assume that $|r^G| = |s^G| = 2$, $rs = \epsilon sr$, and $\epsilon \neq 1$. Then:*

(1) $r^G = \{r, \epsilon r\}$, $s^G = \{s, \epsilon s\}$, $\epsilon^2 = 1$ *and* $\epsilon \in Z(G)$.
(2) *Let $t \in G$. Assume that $|t^G| = 2$, $rt = tr$, and $st \neq ts$. Then $t^G = \{t, \epsilon t\}$ and $st = \epsilon ts$.*

Proof. (1) Lemma 5.1(1) implies that $s^G = \{s, \epsilon s\}$, $r^G = \{r, \epsilon^{-1} r\}$, G^r and G^s commute with ϵ, and $r\epsilon = \epsilon^{-1} r$. Thus $\epsilon^2 = 1$. Since G^s and r generate G, we conclude that $\epsilon \in Z(G)$.

(2) Since $s^G = \{s, \epsilon s\}$ by (1) and since $st \neq ts$, we obtain $ts = \epsilon st$. Thus (1) with $r = t$ implies that $t^G = \{t, \epsilon t\}$ and $st = \epsilon ts$. □

We shall also need the following lemmas.

Lemma 5.3. *Let $s_1, s_2 \in G$ be such that $s_1 \neq s_2$, and let $V \in {}^G_G\mathcal{YD}$. Assume that* $\dim V = 2$ *and* $\operatorname{supp} V = s_1^G = \{s_1, s_2\}$. *Then there exist unique $p_1, p_2 \in \mathbb{K} \setminus \{0\}$ such that $s_1 v = p_1 v$ and $s_2 v = p_2 v$ for all $v \in V_{s_1}$. Moreover,*

$$s_2 s_1^{-1} v = p_2 p_1^{-1} v, \qquad s_1 w = p_2 w, \qquad s_2 w = p_1 w, \qquad s_1 s_2^{-1} w = p_2 p_1^{-1} w$$

for all $v \in V_{s_1}$ and $w \in V_{s_2}$.

Proof. Since $s_1^G = \{s_1, s_2\}$ by assumption, there exists $r \in G$ such that $rs_1 = s_2 r$ and $rs_2 = s_1 r$. Moreover, p_1, p_2 exist since $s_1 s_2 = s_2 s_1$ by Lemma 5.1 and $\dim V_{s_i} = 1$ for all $i \in \{1, 2\}$. Then $s_1 rv = rs_2 v = p_2 rv$ and $s_2 rv = rs_1 v = p_1 rv$ for all $v \in V_{s_1}$. This implies the claim since $V_{s_2} = rV_{s_1}$. □

Lemma 5.4. *Let $V, W \in {}^G_G\mathcal{YD}$ be non-zero Yetter–Drinfeld modules with the property that* $(\mathrm{id} - c_{W,V} c_{V,W})(V \otimes W) = 0$. *Then* $\operatorname{supp} V$ *and* $\operatorname{supp} W$ *commute, and for any* $s \in \operatorname{supp} V$ *and* $t \in \operatorname{supp} W$ *there exists $\lambda_{st} \in \mathbb{K} \setminus \{0\}$ such that $sw = \lambda_{st} w$ for all $w \in W_t$ and $tv = \lambda_{st}^{-1} v$ for all $v \in V_s$.*

Proof. Let $s \in \operatorname{supp} V$, $t \in \operatorname{supp} W$, $v \in V_s \setminus \{0\}$, and $w \in W_t \setminus \{0\}$. Then

$$(\mathrm{id} - c_{W,V}c_{V,W})(v \otimes w) = v \otimes w - sts^{-1}v \otimes sw.$$

Since $sw \in W_{sts^{-1}}$, the latter is zero if and only if $st = ts$ and $sw = \lambda_{st}w$, $tv = \lambda_{st}^{-1}v$ for some $\lambda_{st} \in \mathbb{K} \setminus \{0\}$. These conditions are independent of the choice of v and w, and the lemma follows. □

We will also need a stronger claim in a more specific context.

Lemma 5.5. *Let $s, t, \epsilon \in G$, $\sigma \in \widehat{G^s}$, $\tau \in \widehat{G^t}$, and let $V, W \in {}^G_G\mathcal{YD}$. Assume that $\epsilon \neq 1$, $s^G = \{s, \epsilon s\}$, $t^G = \{t, \epsilon t\}$, and $V \simeq M(s, \sigma)$, $W \simeq M(t, \tau)$. Then:*

(1) $\epsilon \in G^s \cup G^t$.
(2) *If $G^s \neq G^t$ and $st = ts$ then $\sigma(\epsilon) = \tau(\epsilon) = 1$.*
(3) *The following are equivalent:*
   (a) $(\mathrm{id} - c_{W,V}c_{V,W})(V \otimes W) = 0$,
   (b) $st = ts$, $\sigma(t)\tau(s) = 1$, *and* $\sigma(\epsilon)\tau(\epsilon) = 1$.

Proof. Since $s^G = \{s, \epsilon s\}$ and $t^G = \{t, \epsilon t\}$, Lemma 5.1(1) tells us that $\epsilon \in G^s \cup G^t$. Note that ϵ is possibly not central if s and t commute.

(2) Assume that $G^s \neq G^t$. Since both G^s and G^t have index two in G, there exists $r \in G^t$ with $rs = \epsilon sr$. If $st = ts$, then $s, \epsilon \in G^t$ and hence $\tau(rs) = \tau(\epsilon)\tau(sr)$. Thus $\tau(\epsilon) = 1$ and similarly $\sigma(\epsilon) = 1$.

(3) Let $v \in V_s \setminus \{0\}$, $w \in W_t \setminus \{0\}$, and let $r \in G$ be such that $rs = \epsilon sr$. Since $\mathbb{K}Gw = W$ and the braiding commutes with the action of G, we conclude that $(\mathrm{id} - c_{W,V}c_{V,W})(V \otimes W) = 0$ if and only if $(\mathrm{id} - c_{W,V}c_{V,W})(v' \otimes w) = 0$ for all $v' \in V_s \cup V_{\epsilon s}$. Since $V = \mathbb{K}v + \mathbb{K}rv$, by Lemma 5.4 the latter claim is equivalent to

$$st = ts, \qquad v \otimes w = tv \otimes sw, \qquad rv \otimes w = trv \otimes \epsilon sw. \tag{5.1}$$

The second equality in (5.1) is equivalent to $\sigma(t)\tau(s) = 1$. If $G^s = G^t$, then r and t do not commute. Hence $tr = r(\epsilon t)$, and the third equality in (5.1) is equivalent to $\sigma(\epsilon t)\tau(\epsilon s) = 1$. This implies (2). On the other hand, if $G^s \neq G^t$, then we may assume that $r \in G^t$. In that case the last equality in (5.1) is equivalent to the second, and the last equality in (b) is a tautology because of (1). Thus again (2) holds. □

The following lemma is partially contained in [29, Lemmas 5.13, 5.15].

Lemma 5.6. *Let $V, W \in {}^G_G\mathcal{YD}$ be non-zero finite-dimensional objects such that $(\mathrm{ad}\, V)^2(W) = 0$ in $\mathcal{B}(V \oplus W)$.*

(1) *If $(\mathrm{ad}\, V)(W) \neq 0$ then $\operatorname{supp} V$ is commutative.*
(2) *Let $s \in \operatorname{supp} V$ and $t \in \operatorname{supp} W$. Assume that $(\mathrm{id} - c^2)(V_s \otimes W_t) \neq 0$, $st = ts$, and there exists $\lambda \in \mathbb{K}$ such that $sw = \lambda w$ for all $w \in W_t$. Then $G^t \subseteq G^s$.*
(3) *Let $s \in \operatorname{supp} V$ and $t \in \operatorname{supp} W$. Assume that $(\mathrm{id} - c^2)(V_s \otimes W_t) \neq 0$, $st = ts$, and there exist $\lambda, \lambda' \in \mathbb{K}$ such that $sw = \lambda w$ and $tv = \lambda' v$ for all $w \in W_t$, $v \in V_s$. Then* $\dim V_s = 1$.
(4) *If $s \in \operatorname{supp} V$ and $t \in \operatorname{supp} W$ with $st \neq ts$, then $(\mathrm{ad}\, V)(W) \neq 0$, $\varphi_t|_{\operatorname{supp} V}$ is the transposition $(s\ t \triangleright s)$, $\dim V_s = 1$, and $sv = -v$ for all $v \in V_s$.*

Proof. (1) Let $s \in \operatorname{supp} V$ and $t \in \operatorname{supp} W$ be such that $(\operatorname{ad} V_s)(W_t) \neq 0$. Assume that $\operatorname{supp} V$ is not commutative. Since it is a union of conjugacy classes of G, there exists $r \in \operatorname{supp} V \setminus \{s, t^{-1} \triangleright s\}$ such that $rs \neq sr$. Then $(\operatorname{ad} V_r)(\operatorname{ad} V_s)(W_t) \neq 0$ by [29, Prop. 5.5], contradicting $(\operatorname{ad} V)^2(W) = 0$.

(2) Let $u \in V_s$, $w \in W_t \setminus \{0\}$, and $\lambda \in \mathbb{K}^\times$ such that $sw = \lambda w$. Then

$$(\mathrm{id} - c^2)(u \otimes w) = u \otimes w - tu \otimes sw = (u - \lambda tu) \otimes w.$$

Thus, by assumption, there exists $v \in V_s$ such that $tv \neq \lambda^{-1} v$.

Let $g \in G^t$, $s' = gsg^{-1}$, and $v' = gv$. Then $v' \in V_{s'}$ and $s't = ts'$. Moreover,

$$(\mathrm{id} - c^2)(v' \otimes w) = v' \otimes w - tv' \otimes gsg^{-1} w = g(v - \lambda tv) \otimes w,$$

and hence $(\mathrm{id} - c^2)(v' \otimes w) \neq 0$. Assume that $g \notin G^s$, that is, $s' \neq s$. Recall that $(\operatorname{ad} V)^2(W) \simeq X_2^{V,W}$ in ${}^G_G\mathcal{YD}$, and $X_2^{V,W} = \varphi_2(\mathrm{id} \otimes \varphi_1)(V \otimes V \otimes W)$. Then

$$\begin{aligned}
&\varphi_2(\mathrm{id} \otimes \varphi_1)(v' \otimes v \otimes w)\\
&\quad = (\mathrm{id} + c_{12} - c_{23}^2 c_{12} - c_{12} c_{23}^2 c_{12})(v' \otimes (v - \lambda tv) \otimes w)\\
&\quad = (\mathrm{id} - c_{12} c_{23}^2 c_{12})(v' \otimes (v - \lambda tv) \otimes w) + s'(v - \lambda tv) \otimes (\mathrm{id} - c^2)(v' \otimes w).
\end{aligned}$$

Since s and s' commute by (1), the first summand of the last expression is in $V_{s'} \otimes V_s \otimes W$, and the second is non-zero in $V_s \otimes V_{s'} \otimes W$. This contradicts $(\operatorname{ad} V)^2(W) = 0$.

(3) Assume to the contrary that $v, v' \in V_s$ are linearly independent. By a computation similar to one in the proof of (2), we obtain

$$(\mathrm{id} - c_{12} c_{23}^2 c_{12})(v' \otimes (v - \lambda tv) \otimes w) + s(v - \lambda tv) \otimes (\mathrm{id} - c^2)(v' \otimes w) = 0.$$

Since $tv = \lambda' v$ and $tv' = \lambda' v'$, we conclude that $\lambda\lambda' \neq 1$ and

$$(1 - \lambda\lambda')\big(v' \otimes v - \lambda\lambda' sv' \otimes sv + (1 - \lambda\lambda') sv \otimes v'\big) \otimes w = 0.$$

Applying to the second tensor factor a functional $v'^* \in V_s^*$ with $v'^*(v) = 0$, $v'^*(v') = 1$ implies that $sv \in \mathbb{K} sv'$, which yields the desired contradiction.

(4) Since $(\operatorname{ad} V_s)(W_t) \simeq (\mathrm{id} - c_{W_t, V_s} c_{V_s, W_t})(V_s \otimes W_t)$ in ${}^G_G\mathcal{YD}$ and $st \neq ts$, we conclude from [29, Prop. 5.5] that $(\operatorname{ad} V_s)(W_t) \neq 0$. If $\operatorname{supp} V = \{s, t \triangleright s\}$, then $\varphi_t|_{\operatorname{supp} V} = (s\ (t \triangleright s))$. So assume that $|\operatorname{supp} V| \geq 3$. Let $r \in \operatorname{supp} V$ be such that $r \notin \{s, t^{-1} \triangleright s\}$. Since $(\operatorname{ad} V_r)(\operatorname{ad} V_s)(W_t) = 0$ by assumption, [29, Prop. 5.5] implies that $rt = tr$. Hence $\varphi_t|_{\operatorname{supp} V} = (s\ t^{-1} \triangleright s)$. This implies the claim on $\varphi_t|_{\operatorname{supp} V}$.

Let now $v_1, v_2 \in V_s$ and $w \in W_t$. Then

$$\begin{aligned}
\varphi_2(\mathrm{id} \otimes \varphi_1)(v_1 \otimes v_2 \otimes w) &= \varphi_2(v_1 \otimes v_2 \otimes w - v_1 \otimes sts^{-1} v_2 \otimes sw)\\
&= (v_1 \otimes v_2 + sv_2 \otimes v_1) \otimes w\\
&\quad - (sv_2 \otimes sts^{-1} v_1 + s^2 ts^{-1} v_1 \otimes sv_2) \otimes sw\\
&\quad - (v_1 \otimes sts^{-1} v_2 + s^2 ts^{-1} v_2 \otimes v_1) \otimes sw\\
&\quad + (s^2 ts^{-1} v_2 \otimes s^2 ts^{-2} v_1 + s^2 ts^{-1} v_1 \otimes s^2 ts^{-1} v_2) \otimes s^2 w.
\end{aligned}$$

Since $sw \in W_{sts^{-1}}$ and $w, s^2w \notin W_{sts^{-1}}$, if $(\operatorname{ad} V)^2(W) = 0$ then the second and third lines in the last expression have to cancel each other. Since

$$s^2ts^{-1}V_s = V_{s^2tst^{-1}s^{-2}}$$

and $s^2tst^{-1}s^{-2} \neq s$, we conclude that

$$sv_2 \otimes sts^{-1}v_1 + v_1 \otimes sts^{-1}v_2 = 0.$$

In particular, $\dim V_s = 1$ and $sv + v = 0$ for all $v \in V_s$. □

Lemma 5.7. *Let $\theta \in \mathbb{N}$ and let $V_1, \dots, V_\theta$ be Yetter–Drinfeld modules over G. Let $i \in \{1, \dots, \theta\}$ and $J \subseteq \{1, \dots, \theta\} \setminus \{i\}$ be such that $\operatorname{supp} V_j, \operatorname{supp} V_k$ commute for all $j, k \in J \cup \{i\}$. Assume that G is generated by $\bigcup_{j=1}^{\theta} \operatorname{supp} V_j$, V_i is absolutely simple, $\dim V_i < \infty$, and $(\mathrm{id} - c_{V_j,V_i}c_{V_i,V_j})(V_i \otimes V_j) = 0$ for all $j \in \{1, \dots, \theta\} \setminus (J \cup \{i\})$. Then $\dim V_i = 1$.*

Proof. Lemma 5.4 and the conditions on $\operatorname{supp} V_i$ imply that $\operatorname{supp} V_i$ commutes with $\operatorname{supp} V_j$ for all $1 \le j \le \theta$. Since $\operatorname{supp} V_i$ is a conjugacy class of G and G is generated by $\bigcup_{j=1}^{\theta} \operatorname{supp} V_j$, we conclude that $|\operatorname{supp} V_i| = 1$. Let $t \in \operatorname{supp} V_i$ and set $J' = J \cup \{i\}$. By assumption, $rs = sr$ for all $r, s \in \bigcup_{j \in J'} \operatorname{supp} V_j$, and hence all elements of that union have a common eigenspace $\tilde{V}$ in $\overline{\mathbb{K}} \otimes_{\mathbb{K}} V_i$ for some field extension $\overline{\mathbb{K}}$ of $\mathbb{K}$. Further, for each $r \in \operatorname{supp} V_j$ with $j \in \{1, \dots, \theta\} \setminus J'$ there exists $\lambda_r \in \mathbb{K}$ such that $rv = \lambda_r v$ for all $v \in V_i$ by Lemma 5.4. Since G is generated by $\bigcup_{j=1}^{\theta} \operatorname{supp} V_j$, we conclude that all elements of G act as multiplication by a constant on $\tilde{V}$. Since V_i is absolutely simple, it follows that $\dim V_i = 1$. □

Similar calculations to the proof of Lemma 5.6 prove the following claim on braided vector spaces of diagonal type, which will be needed in the proof of Lemma 5.14.

Lemma 5.8. *Let $g_1, g_2, g_3 \in G$ and let $V \in {}^G_G\mathcal{YD}$. Assume that $g_ig_j = g_jg_i$ for all $1 \le i < j \le 3$, and there exist $(q_{ij})_{1 \le i,j \le 3} \in (\mathbb{K}^\times)^{3\times 3}$ and linearly independent $v_i \in V_{g_i}$ for $i \in \{1, 2, 3\}$ such that $g_iv_j = q_{ij}v_j$ for all $i, j \in \{1, 2, 3\}$. Then $(\operatorname{ad} v_1)(\operatorname{ad} v_2)(v_3) = 0$ if and only if $q_{23}q_{32} = 1$ or $q_{13}q_{31} = q_{12}q_{21} = 1$.*

Proof. In the proof of [25, Thm. 1.1] it was shown that $(\operatorname{ad} v_1)(\operatorname{ad} v_2)(v_3) = 0$ if and only if $\varphi_2(\mathrm{id} \otimes \varphi_1)(v_1 \otimes v_2 \otimes v_3) = 0$. Since

$$\varphi_1(v_2 \otimes v_3) = (1 - q_{23}q_{32})v_2 \otimes v_3,$$
$$\varphi_2(v_1 \otimes v_2 \otimes v_3) = v_1 \otimes (1 - q_{12}q_{21}q_{13}q_{31})v_2 \otimes v_3 + q_{12}(1 - q_{13}q_{31})v_2 \otimes v_1 \otimes v_3,$$

the claim follows from the linear independence of v_1, v_2, v_3. □

Finally, we make an important observation on tuples with certain Cartan matrices.

Proposition 5.9. *Let $\theta \in \mathbb{N}_{\ge 2}$, $M \in \mathcal{F}_\theta^G$, and $i, j \in \{1, \dots, \theta\}$ be such that $i \neq j$. Assume that $\{-a_{ij}^M, -a_{ji}^M\} \in \{\{0\}, \{1\}, \{1, 2\}\}$. Then $(\operatorname{ad} M_i)^m(M_j)$ is absolutely simple or zero for all $m \in \mathbb{N}_0$.*

Proof. Since $M \in \mathcal{F}_\theta^G$, $(\operatorname{ad} M_i)^0(M_j) = M_j$ is absolutely simple. On the other hand, $(\operatorname{ad} M_i)^a(M_j) = R_1(M_i, M_j)_2$ for $a = -a_{ij}^M$ is absolutely simple by [7, Thm. 3.8], and $(\operatorname{ad} M_i)^m(M_j) = 0$ for all $m > a$. Thus the claim holds whenever $a_{ij} \in \{0, -1\}$. The only remaining case is when $a_{ij}^M = -2$, $a_{ji}^M = -1$, and $m = 1$. In this case

$$(\operatorname{ad} M_i)(M_j) \simeq (\operatorname{id} - c_{M_j,M_i} c_{M_i,M_j})(M_i \otimes M_j) \simeq (\operatorname{ad} M_j)(M_i),$$

which is absolutely simple by the previous argument since $a_{ji}^M = -1$. □

5.2. Cartan matrices and restrictions. Let $H \subseteq G$ be a subgroup and let $V \in {}^G_G\mathcal{YD}$. If $\operatorname{supp} V \subseteq H$, then by restricting the G-module structure of V to H one obtains a unique Yetter–Drinfeld module $V' \in {}^H_H\mathcal{YD}$, which we will denote by $\operatorname{Res}^G_H V$.

Lemma 5.10. *Let H be a subgroup of G. Let $X \subset G$ be a union of conjugacy classes of G such that $X \cup H$ generates G. Then*

$$G = \langle X \rangle H = H \langle X \rangle.$$

Proof. This follows from $hx = (hxh^{-1})h$ for all $h \in H$ and $x \in X$, since G is generated by $X \cup H$. □

Lemma 5.11. *Let H be a subgroup of G. Let $X \subset G$ be a union of conjugacy classes of G such that $X \cup H$ generates G.*

(1) *Let V be a simple $\mathbb{K}G$-module. If $xv \in \mathbb{K}v$ for all $v \in V$ and all $x \in X$, then V is a simple $\mathbb{K}H$-module by restriction.*
(2) *Let V be a simple Yetter–Drinfeld module over G. Assume that* $\operatorname{supp} V \subseteq H$. *Let $h \in \operatorname{supp} V$. If $xv \in \mathbb{K}v$ for all $x \in X$, $v \in V_h$, then $\operatorname{Res}^G_H V \in {}^H_H\mathcal{YD}$ is simple.*

Proof. (1) By Lemma 5.10, $G = H\langle X \rangle$. Hence

$$V = \mathbb{K}Gv = \mathbb{K}H\langle X \rangle v = \mathbb{K}Hv \tag{5.2}$$

for all $v \in V \setminus \{0\}$. Therefore V is a simple $\mathbb{K}H$-module by restriction.

(2) Lemma 5.10 implies that $G = H\langle X \rangle$. Since V is simple and $xv \in \mathbb{K}v$ for all $x \in X$ and $v \in V_h$, we conclude from (5.2) that $\mathbb{K}Hv = V$ for all $v \in V_h \setminus \{0\}$. Thus $\operatorname{Res}^G_H V$ is simple. □

The last three lemmas in this subsection will be used for induction arguments.

Lemma 5.12. *Let $\theta \in \mathbb{N}_{\geq 2}$ and $M \in \mathcal{E}_\theta^G$. Assume that $a_{12}^M = a_{21}^M = -1$ and $a_{1j}^M = 0$ for all $j \in \{3, \dots, \theta\}$. Assume further that* $\operatorname{supp} M_1$ *and* $\operatorname{supp} M_2$ *commute. Then* $\dim M_1 = 1$ *and* $\dim M_2 = 1$.

Proof. From Lemma 5.6(1) we deduce that $\operatorname{supp} M_1$ and $\operatorname{supp} M_2$ are commutative since $a_{12}^M = a_{21}^M = -1$. Hence $\dim M_1 = 1$ by Lemma 5.7 with $i = 1$ and $J = \{2\}$. Let $r_1 \in Z(G)$ with $\operatorname{supp} M_1 = \{r_1\}$ and let $r_2 \in \operatorname{supp} M_2$. Since any $s_2 \in \operatorname{supp} M_2$ acts as multiplication by a constant on M_1, Lemma 5.6(2) with $V = M_2$ and $W = M_1$ implies that $G^{r_1} \subseteq G^{r_2}$. Hence $G^{r_2} = G$, that is, $r_2 \in Z(G)$ and $\operatorname{supp} M_2 = \{r_2\}$. Since $r_1 \in Z(G)$ and M_2 is absolutely simple, there exists $\lambda' \in \mathbb{K}^\times$ such that $r_1 v_2 = \lambda' v_2$ for all $v_2 \in M_2$. Then Lemma 5.6(3) with $V = M_2$, $W = M_1$ implies that $\dim M_2 = 1$. □

Lemma 5.13. *Let $\theta \in \mathbb{N}_{\geq 2}$ and $M \in \mathcal{E}_\theta^G$. Assume that $a_{12}^M = a_{21}^M = -1$ and $a_{1j}^M = 0$ for all $j \in \{3, \dots, \theta\}$. Assume further that* $\operatorname{supp} M_1$ *and* $\operatorname{supp} M_2$ *do not commute. Then* $|\operatorname{supp} M_1| = |\operatorname{supp} M_2| = 2$ *and* $\dim M_1 = \dim M_2 = 2$.

Proof. Since $a_{12}^M = a_{21}^M = -1$, Lemma 5.6(1) tells us that $\operatorname{supp} M_1$ and $\operatorname{supp} M_2$ are commutative. Moreover, since they do not commute, Lemma 5.6(4) implies that $\varphi_r|_{\operatorname{supp} M_2}$ and $\varphi_s|_{\operatorname{supp} M_1}$ are transpositions for all $r \in \operatorname{supp} M_1$ and $s \in \operatorname{supp} M_2$. Let $r \in \operatorname{supp} M_1$ and $s \in \operatorname{supp} M_2$. Then r commutes with $\operatorname{supp} M_i$ for all $3 \leq i \leq \theta$ by Lemma 5.4. It follows that $\operatorname{supp} M_1 = r^G = \{r, s \triangleright r\}$. Moreover, $\dim (M_1)_r = 1$ and $\dim (M_2)_s = 1$ by Lemma 5.6(4). Since s does not commute with any element of r^G, the same holds for all $s' \in s^G$. Then $|\operatorname{supp} M_2| = 2$ since $\varphi_r|_{\operatorname{supp} M_2}$ is a transposition. □

Lemma 5.14. *Let $\theta \in \mathbb{N}_{\geq 3}$ and $M \in \mathcal{E}_\theta^G$. Assume that $a_{12}^M = a_{21}^M = a_{23}^M = -1$ and $a_{1j}^M = 0$ for all $j \in \{3, \dots, \theta\}$. Let $H = \langle \bigcup_{j=2}^\theta \operatorname{supp} M_j \rangle$ and $M' = (\operatorname{Res}_H^G M_j)_{2 \leq j \leq \theta}$. Then $M' \in \mathcal{E}_{\theta-1}^H$. If H is abelian, then G is abelian.*

Proof. If $\operatorname{supp} M_1$ and $\operatorname{supp} M_2$ commute, then $\dim M_1 = 1$ by Lemma 5.12. Hence $\operatorname{supp} M_1$ consists of a central element of G, and the claim follows from Lemma 5.11(2).

Assume that $\operatorname{supp} M_1$ and $\operatorname{supp} M_2$ do not commute. Then $\dim M_1 = \dim M_2 = 2$ and $|\operatorname{supp} M_1| = |\operatorname{supp} M_2| = 2$ by Lemma 5.13. In particular, either $\operatorname{supp} M_2 \subseteq Z(H)$, or $\operatorname{Res}_H^G M_2 \in {}_H^H\mathcal{YD}$ is absolutely simple. Assume first that $\operatorname{supp} M_2$ does not commute with $\operatorname{supp} M_i$ for some $3 \leq i \leq \theta$. Then $\operatorname{Res}_H^G M_2 \in {}_H^H\mathcal{YD}$ is absolutely simple. Further, $\operatorname{Res}_H^G M_i \in {}_H^H\mathcal{YD}$ is absolutely simple for all $i \geq 3$ by Lemmas 5.4 and 5.11(2). Then $M' \in \mathcal{E}_{\theta-1}^H$ and H is non-abelian.

Assume that $\operatorname{supp} M_1$ and $\operatorname{supp} M_2$ do not commute, and $\operatorname{supp} M_2$ commutes with $\operatorname{supp} M_3$. Let $r, r', s, s' \in G$ be such that $r \neq r'$, $s \neq s'$, and $\operatorname{supp} M_1 = \{r, r'\}$, $\operatorname{supp} M_2 = \{s, s'\}$. Let $t \in \operatorname{supp} M_3$ be such that $(\mathrm{id} - c^2)((M_2)_s \otimes (M_3)_t) \neq 0$. By Lemma 5.4, there exists $\lambda \in \mathbb{K}^\times$ such that $rw = \lambda w$ for all $w \in (M_3)_t$. Assume that $\mathbb{K}$ contains all eigenvalues of the action of s and s' on $(M_3)_t$. Since G^t is generated by $(G^t \cap G^s) \cup \{r\}$ by Lemma 5.1(4), a joint eigenspace W of s and s' in $(M_3)_t$ is then invariant under the action of G^t. Since M_3 is absolutely simple, we conclude that s and s' each act as multiplication by a constant on $(M_3)_t$. Since $rsr^{-1} = s'$, these two constants coincide. For the same reason, t acts as multiplication by a constant on $M_2 = (M_2)_s \oplus (M_2)_{s'}$. Since $a_{23}^M = -1$ and $(\mathrm{id} - c^2)((M_2)_s \otimes (M_3)_t) \neq 0$, Lemma 5.8 implies that $\operatorname{ad} (M_2)_s \operatorname{ad} (M_2)_{s'}((M_3)_t) \neq 0$, contrary to $a_{23}^M = -1$. □

5.3. Skeletons of finite type. Here we collect two basic lemmas about skeletons and their reflections.

Lemma 5.15. *Let $J, K \subseteq \{1, \dots, \theta\}$ be disjoint non-empty subsets and let $i \in J$. Let $M \in \mathcal{F}_\theta^G$ be such that $a_{ij}^M \in \mathbb{Z}$ for all $j \in \{1, \dots, \theta\}$. If $a_{jk}^M = 0$ for all $j \in J$ and $k \in K$ then $a_{jk}^{R_i(M)} = 0$ for all $j \in J$ and $k \in K$.*

Proof. Suppose that $j \neq i$. Recall that $R_i(M)_j = (\operatorname{ad} M_i)^m(M_j)$, where $m = -a_{ij}^M$, and $(\operatorname{ad} M_i)^m(M_j) \simeq \varphi(M_i^{\otimes m} \otimes M_j)$ for some morphism φ in ${}_G^G\mathcal{YD}$ (see Lemma 1.3). In particular, $R_i(M)_k = M_k$ for all $k \in K$. Moreover, $a^M_{jk} = 0$ if and only if $c_{M_k,M_j}c_{M_j,M_k} = \mathrm{id}_{M_j\otimes M_k}$. Since c^2 is a natural isomorphism, it commutes with $\varphi \otimes \mathrm{id}$. This implies the claim of the lemma for $j \neq i$. The case $j = i$ means that $(\mathrm{id} - c_{W,V}c_{V,W})(V \otimes W) = 0$ implies $(\mathrm{id} - c_{W,V^*}c_{V^*,W})(V^* \otimes W) = 0$ for $V = M_i$ and $W = M_k$, where $k \in K$. The latter is well-known. □

The following lemma and the remark below will be used to simplify the calculations of the skeletons of reflections of tuples.

Lemma 5.16. *Let $\theta \geq 3$, $i \in \{1, \dots, \theta\}$ and let $M \in \mathcal{F}^G_\theta$. Suppose that M has a skeleton and for all $j, k \in \{1, \dots, \theta\} \setminus \{i\}$ with $j \neq k$, the triple $R_1(M_i, M_j, M_k)$ has a skeleton $\mathcal{S}'_{jk}$. Then $R_i(M)$ has a skeleton $\mathcal{S}'$. Moreover, $\mathcal{S}'$ is uniquely determined, and restricts to $\mathcal{S}'_{jk}$ when considering only the vertices i, j, and k.*

Note that $R_1(M_i, M_j, M_k)$ means reflection on the first entry of the triple, that is, on M_i.

Proof. The definition of a skeleton of $R_i(M)$ and its existence consist of a family of conditions in each of which at most two entries $R_i(M)_j$, $R_i(M)_k$ with $j, k \in \{1, \dots, \theta\}$ are involved. Thus these conditions can be obtained from $R_1(M_i, M_j, M_k)$. This implies the claim. □

Remark 5.17. Let $\theta \geq 3$, $i \in \{1, \dots, \theta\}$ and let $M = (M_1, \dots, M_\theta) \in \mathcal{F}^G_\theta$. Suppose that M has a connected skeleton $\mathcal{S}$. Lemma 5.16 can be used to quickly obtain the skeleton of $R_i(M)$ for some $M \in \mathcal{F}^G_\theta$ (if it exists).

Assume that for all $j, k \in \{1, \dots, \theta\} \setminus \{i\}$ such that $j \neq k$ and the skeleton of (M_i, M_j, M_k) is connected, the triple $R_1(M_i, M_j, M_k)$ has a skeleton $\mathcal{S}'_{jk}$. We show that then the conditions of Lemma 5.16 are fulfilled, and hence $R_i(M)$ has a skeleton.

Indeed, for any triple (i, j, k) with $|\{i, j, k\}| = 3$ one of the following possibilities occurs:

(1) j and k are not connected to i in $\mathcal{S}$. Then $R_i(M)_j = M_j$, $R_i(M)_k = M_k$, and hence $R_1(M_i, M_j, M_k)$ has a skeleton $\mathcal{S}'_{jk}$. In this skeleton, j and k are not connected to i by Lemma 5.15. Hence $\mathcal{S}'_{jk}$ coincides with the skeleton of (M_i, M_j, M_k).
(2) (M_i, M_j, M_k) has a connected skeleton. Then $R_1(M_i, M_j, M_k)$ has a connected skeleton by assumption.
(3) Precisely one of j and k (say j) is connected to the vertex i and the other is connected neither to i nor to j. Then $R_i(M)_k = M_k$. Moreover, there exists $l \in \{1, \dots, \theta\} \setminus \{i, j, k\}$ such that (M_i, M_j, M_l) has a connected skeleton. Then $R_1(M_i, M_j, M_l)$ has a connected skeleton by assumption. Then $R_1(M_i, M_j, M_k)$ has a skeleton with two connected components by Lemma 5.15.

This leads to the claim on the existence (and shape) of the skeleton of $R_i(M)$.

6. Proof of Theorem 2.6: The case ADE

In this section we require that all assumptions in Theorem 2.6 hold. Thus let $\theta \in \mathbb{N}_{\geq 2}$ and let G be a non-abelian group and $M \in \mathcal{E}^G_\theta$. Assume that the Cartan matrix A^M of M is a Cartan matrix of type A_θ with $\theta \geq 2$, or D_θ with $\theta \geq 4$, or E_θ with $\theta \in \{6, 7, 8\}$.

Lemma 6.1.

(1) $|\operatorname{supp} M_i| = 2 = \dim M_i$ *for all* $i \in \{1, \dots, \theta\}$.
(2) $\operatorname{supp} M_i$ *does not commute with* $\operatorname{supp} M_j$ *whenever* $a_{ij}^M = -1$.

Proof. We proceed by induction on θ. If $\theta = 2$, then A^M is of type A_2. If $\operatorname{supp} M_1$ and $\operatorname{supp} M_2$ commute, then Lemma 5.12 implies that G is commutative, contrary to our assumption. Hence $\operatorname{supp} M_1$ and $\operatorname{supp} M_2$ do not commute, and the lemma follows from Lemma 5.13. Assume that $\theta \geq 3$. Let $I = \{1, \dots, \theta\}$. By the assumptions on A^M there exist $i, j, k \in I$ such that $a_{ij}^M = a_{ji}^M = a_{jk}^M = -1$, and $a_{il}^M = 0$ for all $l \in I \setminus \{i, j\}$. Let H be the subgroup of G generated by $\bigcup_{l \in I \setminus \{i\}} \operatorname{supp} M_l$. Then $M' = (\operatorname{Res}_H^G M_l)_{l \in I \setminus \{i\}} \in \mathcal{E}_{\theta-1}^H$ by Lemma 5.14, and $a_{lm}^{M'} = a_{lm}^M$ for all $l, m \in I \setminus \{i\}$. Hence, by induction hypothesis, the lemma holds for the tuple M'. In particular, $\dim M_j = 2$. Then $\operatorname{supp} M_i$ and $\operatorname{supp} M_j$ do not commute and $|\operatorname{supp} M_i| = 2 = \dim M_i$ by Lemmas 5.12 and 5.13. □

The following lemma describes the structure of the Yetter–Drinfeld modules encoded in a skeleton of types $\alpha_\theta, \delta_\theta, \varepsilon_6, \varepsilon_7$ and ε_8.

Lemma 6.2. *Let* $N \in \mathcal{F}_\theta^G$. *The following are equivalent:*

(1) N *has a connected simply-laced skeleton of finite type.*
(2) *There exist*

- *a symmetric indecomposable Cartan matrix* $A \in \mathbb{Z}^{\theta \times \theta}$ *of finite type,*
- *an element* $\epsilon \in Z(G)$ *with* $\epsilon^2 = 1$, *and*
- *for all* $i \in \{1, \dots, \theta\}$ *and* $s_i \in \operatorname{supp} N_i$ *a unique character* σ_i *of* G^{s_i},

such that $\operatorname{supp} N_i = \{s_i, \epsilon s_i\}$, $N_i \simeq M(s_i, \sigma_i)$ *for all* $i \in \{1, \dots, \theta\}$, *and*

$$\sigma_i(s_j)\sigma_j(s_i) = \sigma_i(\epsilon)\sigma_j(\epsilon) = 1 \quad \text{for all } i, j \text{ such that } a_{ij} = 0, \tag{6.1}$$

$$\sigma_i(\epsilon s_j^2)\sigma_j(\epsilon s_i^2) = 1 \quad \text{for all } i, j \text{ such that } a_{ij} = -1, \tag{6.2}$$

$$\sigma_i(s_i) = -1 \quad \text{for all } i \in \{1, \dots, \theta\}, \tag{6.3}$$

$$s_i s_j = \epsilon s_j s_i \quad \text{for all } i, j \text{ such that } a_{ij} = -1, \tag{6.4}$$

$$s_i s_j = s_j s_i \quad \text{for all } i, j \text{ such that } a_{ij} = 0. \tag{6.5}$$

(3) *Let* $P = (\operatorname{Res}_H^G N_1, \dots, \operatorname{Res}_H^G N_\theta)$, *where* $H \subseteq G$ *is the subgroup generated by* $\bigcup_{i=1}^\theta \operatorname{supp} N_i$. *Then* H *is non-abelian,* $P \in \mathcal{E}_\theta^H$, *and* A^P *is of type* A_θ *with* $\theta \geq 2$, D_θ *with* $\theta \geq 4$, *or* E_θ *with* $\theta \in \{6, 7, 8\}$.

Proof. The implication (1)⇒(3) follows from the definition of a simply-laced skeleton.

To prove that (3) implies (2), let $A = A^P$ $(= A^N)$. Then A is a symmetric indecomposable Cartan matrix of finite type and $|\operatorname{supp} N_i| = \dim N_i = 2$ for all $i \in \{1, \dots, \theta\}$ by Lemma 6.1. Moreover, Lemmas 6.1 and 5.4 imply that for $i \neq j$, $\operatorname{supp} N_i$ commutes with $\operatorname{supp} N_j$ if and only if $a_{ij} = 0$. Let $s_i \in \operatorname{supp} N_i$ for all $i \in \{1, \dots, \theta\}$. Then for all $i \in \{1, \dots, \theta\}$ there exists a unique character σ_i of G^{s_i} such that $N_i \simeq M(s_i, \sigma_i)$. Lemma 5.2 implies that there exists $\epsilon \in Z(G)$ such that $\epsilon^2 = 1$, $\operatorname{supp} N_i = \{s_i, \epsilon s_i\}$ for all $i \in \{1, \dots, \theta\}$, and (6.4) holds. Now (6.3) holds by Lemma 5.6(4), and (6.1) follows from Lemma 5.5. Finally, if $a_{ij} = -1$ then $(\operatorname{ad} N_i)(N_j)$ is absolutely simple. Therefore (6.2) follows from Lemma A.3.

Finally, we prove (2)⇒(1). Let $i, j \in \{1, \dots, \theta\}$ be distinct. Since $\epsilon \in Z(G)$, we conclude from Lemma 5.5 and from (6.1) and (6.5) that $(\operatorname{ad} N_i)(N_j) = 0$ if $a_{ij} = 0$. Finally, if $a_{ij} = -1$ then (6.2)–(6.4) and Corollary A.7 imply $a_{ij}^N = -1$. This proves (1). □

We now study some reflections. In the case of rank three one has the following lemma.

Lemma 6.3. *Let $N \in \mathcal{F}_3^G$. Assume that N has a skeleton $\mathcal{S}$ of type α_3. Then $\mathcal{S}$ is a skeleton of $R_k(N)$ for each $k \in \{1, 2, 3\}$.*

Proof. By symmetry of the skeleton of type α_3, it suffices to prove the lemma for the reflections R_1 and R_2. Let $s_i \in G$ and $\sigma_i \in \widehat{G^{s_i}}$ be as in Lemma 6.2(2). Let $(U, V, W) = R_1(M)$. Then Lemma A.8 implies that $U \simeq M(s_1^{-1}, \sigma_1^*)$, $V \simeq M(s_1 s_2, \sigma')$ and $W = M_3$, where $\sigma' \in \widehat{G^{s_1 s_2}}$ with $\sigma'(s_1 s_2) = -1$ and $\sigma'(h) = \sigma_1(h)\sigma_2(h)$ for all $h \in G^{s_1} \cap G^{s_2}$. For the proof of the claim we use Lemma 6.2. For (U, V, W), conditions (6.1) and (6.5) follow from Lemmas 5.15 and 5.5. Conditions (6.2) and (6.4) for $\{i, j\} = \{1, 2\}$ and (6.3) for $i \in \{1, 2\}$ hold by Lemma A.8. Condition (6.3) for $i = 3$ holds since $R_1(M)_3 = M_3$. Thus we need to prove (6.2) and (6.4) for $i = 2$, $j = 3$.

Clearly, (6.4) follows easily, since $s_1 s_3 = s_3 s_1$ and $s_2 s_3 = \epsilon s_3 s_2$ imply that $s_1 s_2 s_3 = \epsilon s_3 s_1 s_2$. Regarding (6.2) we obtain the following:

$$\sigma'(\epsilon s_3^2)\sigma_3(\epsilon(s_1 s_2)^2) = \sigma_1(\epsilon s_3^2)\sigma_2(\epsilon s_3^2)\sigma_3(s_1^2 s_2^2) = \sigma_1(\epsilon)\sigma_3(\epsilon) = 1,$$

where the last equality follows from Lemma 5.5.

Let now $(U', V', W') = R_2(M)$. By Lemma A.8,

$$U' \simeq M(s_2 s_1, \rho), \qquad V' \simeq M(s_2^{-1}, \sigma_2^*), \qquad W' \simeq M(s_2 s_3, \tau),$$

where $\rho \in \widehat{G^{s_2 s_1}}$ with $\rho(s_2 s_1) = -1$, $\rho(h) = \sigma_1(h)\sigma_2(h)$ for all $h \in G^{s_1} \cap G^{s_2}$, and $\tau \in \widehat{G^{s_2 s_3}}$ with $\tau(s_2 s_3) = -1$, $\tau(h) = \sigma_2(h)\sigma_3(h)$ for all $h \in G^{s_2} \cap G^{s_3}$. As in the first part of the proof of the lemma, one needs to check the conditions of Lemma 6.2 for $R_2(M)$.

Conditions (6.2)–(6.4) follow from Lemma A.8. For (6.5) we record that

$$(s_2 s_1)(s_2 s_3) = s_2 \epsilon s_2 s_1 s_3 = s_2 \epsilon s_2 s_3 s_1 = s_2 s_3 s_2 s_1$$

since $\epsilon^2 = 1$. Finally, $s_1^{-1} s_3 \in G^{s_1} \cap G^{s_2} \cap G^{s_3}$ and hence we get (6.1) from the calculations

$$\begin{aligned}\rho(s_2 s_3)\tau(s_2 s_1) &= \rho(s_2 s_1 s_1^{-1} s_3)\tau(s_2 s_3 s_3^{-1} s_1)\\ &= (-1)\rho(s_1^{-1} s_3)(-1)\tau(s_3^{-1} s_1)\\ &= \sigma_1(s_1^{-1} s_3)\sigma_2(s_1^{-1} s_3 s_3^{-1} s_1)\sigma_3(s_3^{-1} s_1) = 1\end{aligned}$$

and

$$\rho(\epsilon)\tau(\epsilon) = \sigma_1(\epsilon)\sigma_2(\epsilon)^2\sigma_3(\epsilon) = 1.$$ □

The reflections are studied in the following proposition.

Proposition 6.4. *Let $N \in \mathcal{F}_\theta^G$. Suppose that N has a skeleton $\mathcal{S}$ of type α_θ, δ_θ (with $\theta \geq 4$), ε_6, ε_7, or ε_8. Then $\mathcal{S}$ is a skeleton of $R_k(N)$ for all $k \in \{1, \ldots, \theta\}$.*

Proof. For $\theta = 2$ the claim follows from Lemmas 6.2 and A.8.

Assume that $\theta \geq 3$. By Remark 5.17, it is enough to prove that for all pairwise distinct $i, j, k \in \{1, \ldots, \theta\}$ such that the skeleton $\mathcal{S}_{ijk}$ of (M_i, M_j, M_k) is connected, $\mathcal{S}_{ijk}$ is a skeleton of $R_1(M_i, M_j, M_k)$. All such skeletons are of type α_3. Hence the claim follows from Lemma 6.3. □

Proof of Theorem 2.6. (1) holds by Lemma 6.2(3)⇒(1), and (2) follows from (1) and Proposition 6.4.

(3) Theorem 1.2 applies because of (2). Since the Cartan graph of M is standard, the root system of M coincides with the root system associated with the Cartan matrix A^M. Hence

$$\mathcal{H}(t) = \prod_{\alpha \in \boldsymbol{\Delta}_+} \mathcal{H}_{\mathcal{B}(M_\alpha)}(t^\alpha).$$

The Nichols algebras $\mathcal{B}(M_i)$ are quantum linear spaces with Hilbert series $(1 + t)^2$ (see also [25, Thm. 4.6(2)]). Then the claim on the Hilbert series of $\mathcal{B}(M)$ follows from Theorem 1.2. □

7. Proof of Theorem 2.7: The case C

In this section we require that all assumptions in Theorem 2.7 hold. Let $\theta \in \mathbb{N}_{\geq 3}$ and let G be a non-abelian group. Assume that $M \in \mathcal{E}_\theta^G$ and that A^M is a Cartan matrix of type C_θ, where $a^M_{\theta-1,\theta} = -2$ and $a^M_{ij} = -1$ for $|i - j| = 1$, $(i, j) \neq (\theta - 1, \theta)$.

7.1. We first study some particular aspects for triples.

Lemma 7.1. *Assume that $\theta = 3$. Then:*

(1) $|\mathrm{supp}\, M_1| = |\mathrm{supp}\, M_2| = \dim M_1 = \dim M_2 = 2$ *and* $\dim M_3 = 1$.
(2) $\mathrm{supp}\, M_1$ *does not commute with* $\mathrm{supp}\, M_2$.

Proof. Suppose that $\mathrm{supp}\, M_1$ and $\mathrm{supp}\, M_2$ commute. Then $\dim M_i = 1$ for all $i \in \{1, 2\}$ by Lemma 5.12. Since $a^M_{32} = -1$, Lemma 5.6(1) implies that $\mathrm{supp}\, M_3$ is commutative. Then G is abelian, a contradiction. Hence $\mathrm{supp}\, M_1$ and $\mathrm{supp}\, M_2$ do not commute. Then Lemma 5.13 implies that

$$|\mathrm{supp}\, M_1| = |\mathrm{supp}\, M_2| = \dim M_1 = \dim M_2 = 2.$$

Let $r \in \mathrm{supp}\, M_1$, $s_1, s_2 \in \mathrm{supp}\, M_2$ with $s_1 \neq s_2$, and $t \in \mathrm{supp}\, M_3$. Then $rt = tr$ because $a^M_{13} = 0$. Hence

$$\mathrm{supp}\, M_3 \ni s_1 t s_1^{-1} = r(s_1 t s_1^{-1})r^{-1} = s_2 r t r^{-1} s_2^{-1} = s_2 t s_2^{-1}.$$

Assume that $\mathrm{supp}\, M_2$ and $\mathrm{supp}\, M_3$ do not commute. Then s_1, s_2 act on $\mathrm{supp}\, M_3$ via conjugation by the same transposition in view of Lemma 5.6. Since $\mathrm{supp}\, M_3$ is a conjugacy class of G, we conclude that $|\operatorname{supp} M_3| = 2$. Moreover, $\dim (M_3)_t = 1$ by Lemma 5.6(4). Let now σ be a character of G^{s_1} such that $M_2 \simeq M(s_1, \sigma)$. Then Corollary A.7 for (M_1, M_2) and (M_2, M_3) implies that $\sigma(s_1) = -1$ and $\sigma(s_1) = 1$, $\operatorname{char} \mathbb{K} = 3$, respectively. This is clearly impossible. Hence $\operatorname{supp} M_2$ and $\operatorname{supp} M_3$ commute.

Since $a_{32}^M = -1$ and $\operatorname{supp} M_3$ commutes with $\operatorname{supp} M_1$ and $\operatorname{supp} M_2$, we conclude from Lemma 5.7 for $\theta = 3$, $V_1 = M_1$, $V_2 = M_2$, $V_3 = M_3$, $i = 3$, $J = \{2\}$ that $\dim M_3 = 1$. □

In the following two lemmas we consider a slightly more general context, which is motivated by Lemma 7.1 and will be used crucially in the proof of Lemma 7.4.

Let $N \in \mathcal{F}_3^G$ and let $r \in \operatorname{supp} N_1$, $s \in \operatorname{supp} N_2$, $t \in \operatorname{supp} N_3$. Assume that $|r^G| = |s^G| = 2$, $t \in Z(G)$, and $rs \neq sr$. Let $\epsilon \in G$ be such that $rs = \epsilon sr$. Then $\epsilon \neq 1$. Moreover, $r^G = \{r, \epsilon r\}$, $s^G = \{s, \epsilon s\}$, $\epsilon^2 = 1$, and $\epsilon \in Z(G)$ by Lemma 5.2(1). Assume further that $N_1 \simeq M(r, \rho)$, $N_2 \simeq M(s, \sigma)$ and $N_3 \simeq M(t, \tau)$, where $\rho \in \widehat{G^r}$, $\sigma \in \widehat{G^s}$, and $\tau \in \widehat{G}$.

Lemma 7.2. *The following are equivalent:*

(1) *A^N is of type C_3.*

(2) $$\begin{aligned} &\rho(\epsilon s^2)\sigma(\epsilon r^2) = \rho(t)\tau(r) = 1, && \rho(r) = \sigma(s) = -1,\\ &(\tau(t)+1)(\sigma(t)\tau(st)-1) = 0, && \sigma(t)\tau(s) \neq 1. \end{aligned}$$

Proof. We first prove that (1) implies (2). Since $N \in \mathcal{F}_3^G$ and A^N is of type C_3, Proposition 5.9 implies that $(\operatorname{ad} N_i)^m(N_j)$ is absolutely simple or zero for all $m \in \mathbb{N}_0$ and all $i, j \in \{1, 2, 3\}$ with $i \neq j$. By Corollary A.7, $a_{12}^N = a_{21}^N = -1$ implies that $\rho(\epsilon s^2)\sigma(\epsilon r^2) = 1$ and $\rho(r) = \sigma(s) = -1$. Further, from Lemma A.14 and from $a_{13}^N = 0$, $a_{23}^N \neq 0$ we deduce that $\rho(t)\tau(r) = 1$, $\sigma(t)\tau(s) \neq 1$. Finally, since $a_{32}^N = -1$, Lemma A.2 implies that $(\tau(t)+1)(\sigma(t)\tau(st)-1) = 0$.

Now assume that (2) holds. Then $a_{12}^N = a_{21}^N = -1$ by Corollary A.7, $a_{13}^N = a_{31}^N = 0$ by Lemma A.14, and $a_{23}^N = -2$ by Lemmas A.15 and A.16(1). Finally, $a_{32}^N = -1$ by Lemma A.2. This proves (1). □

The classes $\wp_0^G$ and $\wp_1^G$ of pairs are introduced in Definition A.17.

Lemma 7.3. *Suppose that N admits all reflections and the Weyl groupoid of N is finite. Then $(N_2, N_3) \in \wp_0^G$ or $(N_2, N_3) \in \wp_1^G$.*

Proof. Regard N_1 and N_2 as absolutely simple Yetter–Drinfeld modules over the group $H = \langle \operatorname{supp}(N_1 \oplus N_2)\rangle$. Then H is a non-abelian epimorphic image of Γ_2. By Theorem 1.4, the Yetter–Drinfeld modules $(\operatorname{ad} N_1)^m(N_2)$ and $(\operatorname{ad} N_2)^m(N_1)$ are absolutely simple or zero for all $m \geq 0$, and they are zero for some $m \in \mathbb{N}$. Thus Lemma A.6 implies that

$$\begin{aligned} &\rho(r)^2 = \sigma(s)^2 = 1, && \rho(\epsilon s^2)\sigma(\epsilon r^2) = 1,\\ &\rho(r) = \sigma(s) = -1 && \text{if } \operatorname{char} \mathbb{K} = 0. \end{aligned} \tag{7.1}$$

Moreover, Corollary A.24 applied to (N_2, N_3) implies that

$$(N_2, N_3) \in \wp_i^G \quad \text{for some } i \in \{0, 1, 2, 3, 4\},$$

since $\epsilon^2 = 1$ — see also Definition A.17 and Table 3. But $\sigma(s)^2 = 1$ implies that

$$(N_2, N_3) \notin \wp_4^G.$$

Consider $R_3(N) = (U, V, W)$. Since $\operatorname{supp} N_3 = \{t\}$ and $t \in Z(G)$, Lemma A.2 implies that (U, V, W) satisfies the assumptions of the lemma. In particular, we have $(V, W) \notin \wp_4^G$, and hence

$$(N_2, N_3) \notin \wp_2^G$$

by Remark A.23.

Consider $R_2(N) = (U', V', W')$. Then (7.1) and Lemma A.8 for (N_2, N_1) imply that

$$\dim U' = \dim V' = |\operatorname{supp} U'| = |\operatorname{supp} V'| = 2$$

and $\operatorname{supp} U'$, $\operatorname{supp} V'$ do not commute. Moreover, Remark A.23 for (N_2, N_3) implies that $\dim W' = 1$. In particular, (U', V', W') satisfies the assumptions of the lemma. Therefore $(V', W') \notin \wp_2^G$, and hence

$$(N_2, N_3) \notin \wp_3^G$$

by Remark A.23. □

Now we look again at our main tuple M.

Lemma 7.4. *Suppose that $\theta = 3$, M admits all reflections, and the Weyl groupoid of M is finite. Then $(M_2, M_3) \in \wp_1^G(2)$ and $\operatorname{char} \mathbb{K} \neq 2$.*

Proof. By Lemma 7.1, M satisfies the assumptions on N stated before Lemma 7.2. Let $r, s, t \in G$ and ρ, σ, τ as there. Since $a_{23}^M \neq 0$, we infer from Lemma 7.3 that $(M_2, M_3) \in \wp_1^G$. In particular, $\sigma(t)\tau(st) = 1$ and $\tau(t) \neq 1$. Moreover, since A^M is of type C_3, the formulas in Lemma 7.2(2) hold. Let $M' = R_2(M)$. Since $a_{21}^M = -1$, Lemma A.8 implies that $M_1' \simeq M(sr, \rho')$, where $\rho' \in \widehat{G^{sr}}$ with $\rho'(sr) = -1$, $\rho'(h) = \rho(h)\sigma(h)$ for all $h \in G^r \cap G^s$. Further, $M_2' \simeq M(s^{-1}, \sigma^*)$ and $M_3' \simeq M(\epsilon s^2 t, \tau_2)$ by Lemma A.15, since $a_{23}^M = -2$. Since $\epsilon^2 = 1$ and $\sigma(s) = -1$, we obtain

$$\tau_2(r) = -\sigma(\epsilon r^2)\tau(r), \qquad \tau_2(s) = \sigma(\epsilon s^2)\tau(s), \qquad \tau_2(t) = \sigma(t^2)\tau(t).$$

Then

$$\rho'(\epsilon s^2 t)\tau_2(sr) = \rho(\epsilon s^2 t)\sigma(\epsilon s^2 t)\sigma(\epsilon s^2)\tau(s)(-\sigma(\epsilon r^2)\tau(r)) = -\sigma(t)\tau(s). \tag{7.2}$$

Now Lemma 7.3 for $N = (M_2', M_1', M_3')$ implies that $(M_1', M_3') \in \wp_0^G$ or $(M_1', M_3') \in \wp_1^G$. In the first case $\sigma(t)\tau(s) = -1$, and hence $\tau(t) = -1$ and

$$(M_2, M_3) \in \wp_1^G(2).$$

Moreover, $\operatorname{char} \mathbb{K} \neq 2$ since $\tau(t) \neq 1$ by the first paragraph.

Assume that $(M_1', M_3') \in \wp_1^G$. Since $(M_2', M_3') \in \wp_1^G$, Remark A.23 implies that $a_{13}^{M'} = a_{23}^{M'} = -2$. Moreover, since A^M is of type C_3, the Cartan graph of M has no point with Cartan matrix of type A_3 by Theorem 2.6. This contradicts $a_{13}^{M'} = a_{23}^{M'} = -2$ in view of Corollary 3.3. □

7.2. Recall the assumptions of Section 7: $\theta \in \mathbb{N}_{\geq 3}$, G is a non-abelian group, and $M \in \mathcal{E}_\theta^G$ such that A^M is of type C_θ.

Lemma 7.5.

(1) $|\operatorname{supp} M_i| = 2 = \dim M_i$ *for all* $1 \leq i \leq \theta - 1$, *and* $\dim M_\theta = 1$.
(2) $\operatorname{supp} M_i$ *does not commute with* $\operatorname{supp} M_{i+1}$ *for* $1 \leq i \leq \theta - 2$.

Proof. We proceed by induction on θ. For $\theta = 3$ the claim holds by Lemma 7.1.

Assume that $\theta > 3$. Let H be the subgroup of G generated by $\bigcup_{i=2}^{\theta} \operatorname{supp} M_i$. Then

$$M' = (\operatorname{Res}_H^G M_2, \ldots, \operatorname{Res}_H^G M_\theta) \in \mathcal{E}_{\theta-1}^H$$

by Lemma 5.14, and H is non-abelian. Clearly, $A^{M'}$ is of type $C_{\theta-1}$. Then the induction hypothesis yields the claim except for $i = 1$. In particular, $\dim M_2 = |\operatorname{supp} M_2| = 2$. Then $\operatorname{supp} M_1$ and $\operatorname{supp} M_2$ do not commute by Lemma 5.12, and $|\operatorname{supp} M_1| = 2 = \dim M_1$ by Lemma 5.13. □

Before we prove Theorem 2.7, we have to study skeletons of type γ_θ.

Lemma 7.6. *Assume that* $\operatorname{char} \mathbb{K} \neq 2$. *Let* $\theta \in \mathbb{N}_{\geq 3}$ *and let* $N \in \mathcal{F}_\theta^G$. *The following are equivalent:*

(1) *N has a skeleton of type* γ_θ.
(2) *For any tuple* $(s_1, \ldots, s_\theta)$, *where* $s_i \in \operatorname{supp} N_i$ *for all* i, *there exists a unique tuple* $(\epsilon, \sigma_1, \ldots, \sigma_\theta)$ *with the following properties:* $\epsilon \in Z(G) \setminus 1$, $\epsilon^2 = 1$, $\sigma_i \in \widehat{G^{s_i}}$ *for all* i, $\operatorname{supp} N_i = \{s_i, \epsilon s_i\}$ *for all* $i \in \{1, \ldots, \theta - 1\}$, $\operatorname{supp} N_\theta = \{s_\theta\}$, $N_i \simeq M(s_i, \sigma_i)$ *for all* i, *and*

$$\sigma_i(s_j)\sigma_j(s_i) = 1 \quad \text{if } |i - j| \geq 2 \text{ and } 1 \leq i, j \leq \theta, \tag{7.3}$$
$$\sigma_i(\epsilon)\sigma_j(\epsilon) = 1 \quad \text{if } |i - j| \geq 2 \text{ and } i, j < \theta, \tag{7.4}$$
$$\sigma_{\theta-1}(s_\theta)\sigma_\theta(s_{\theta-1}) = -1, \tag{7.5}$$
$$\sigma_i(\epsilon s_{i+1}^2)\sigma_{i+1}(\epsilon s_i^2) = 1 \quad \text{for all } i \in \{1, \ldots, \theta - 2\}, \tag{7.6}$$
$$\sigma_i(s_i) = -1 \quad \text{for all } i \in \{1, \ldots, \theta\}, \tag{7.7}$$
$$s_i s_{i+1} = \epsilon s_{i+1} s_i \quad \text{for all } i \in \{1, \ldots, \theta - 2\}, \tag{7.8}$$
$$s_i s_j = s_j s_i \quad \text{if } j \geq i + 2 \text{ or } j = \theta. \tag{7.9}$$

Proof. We first prove that (2) implies (1). For this, the only non-trivial task is to show that the Cartan matrix A^N is of type C_θ. Now $a_{i,i+1}^N = a_{i+1,i}^N = -1$ for all $i \in \{1, \ldots, \theta - 2\}$ by Corollary A.7, $a_{i\theta}^N = a_{\theta i}^N = 0$ for $i \in \{1, \ldots, \theta-2\}$ by Lemma A.14, $a_{ij}^N = a_{ji}^N = 0$ for $i, j \in \{1, \ldots, \theta - 1\}$ with $|i - j| > 1$ by Lemma 5.5, and $a_{\theta-1,\theta}^N = -2$ by Lemmas A.15 and A.16. Finally, $a_{\theta,\theta-1}^N = -1$ by Lemma A.2. This proves (1).

Assume now that (1) holds. Then the claims in (2) on ϵ and $\operatorname{supp} N_i$ for $i \in \{1, \ldots, \theta\}$ including (7.8) and (7.9) follow from Lemma 5.2(1). Moreover, A^N is of type C_θ, and hence Proposition 5.9 implies that $(\operatorname{ad} N_i)^m(N_j)$ is absolutely simple or zero for all distinct $i, j \in \{1, \ldots, \theta\}$. Then (7.5) holds by assumption on the skeleton, and (7.3)–(7.7) follow from Lemmas 5.5, A.14, A.2, and Corollary A.7. □

For the reflections one needs the following lemmas.

Lemma 7.7. *Let $N \in \mathcal{F}_3^G$. Assume that* $\operatorname{char}\mathbb{K} \neq 2$ *and N has a skeleton $\mathcal{S}$ of type γ_3. Then $\mathcal{S}$ is a skeleton of $R_k(N)$ for all $k \in \{1, 2, 3\}$.*

Proof. Since N has a skeleton of type γ_3, by Lemma 7.6 there exist $r, s, t, \epsilon \in G$ and $\rho \in \widehat{G^r}$, $\sigma \in \widehat{G^s}$, $\tau \in \widehat{G}$ as specified before Lemma 7.2. Moreover, ρ, σ, τ satisfy the equations in Lemma 7.6(2) with $s_1 = r$, $s_2 = s$, $s_3 = t$, $\sigma_1 = \rho$, $\sigma_2 = \sigma$, $\sigma_3 = \tau$. In particular, $\sigma(t)\tau(s) = \tau(t) = -1$.

Let $(U, V, W) = R_1(N)$. Then Lemma A.8 implies that $U \simeq M(r^{-1}, \rho^*)$, $V \simeq M(rs, \sigma')$ and $W' = W$, where $\sigma' \in \widehat{G^{rs}}$ with $\sigma'(rs) = -1$ and $\sigma'(h) = \rho(h)\sigma(h)$ for all $h \in G^r \cap G^s$. Now we use Lemma 7.6 to prove that $\mathcal{S}$ is a skeleton of $R_1(N)$. Lemma 7.6(2) for N, especially (7.3) and (7.7), imply that $\rho(t)\tau(r) = 1$ and $\rho(r) = -1$. Hence $\rho^*(t)\tau(r^{-1}) = 1$ and $\rho^*(r^{-1}) = -1$. Further, $\rho^*(\epsilon(rs)^2)\sigma'(\epsilon r^{-2}) = 1$ by Lemma A.8. Finally,

$$\sigma'(t)\tau(rs) = \rho(t)\sigma(t)\tau(r)\tau(s) = -1.$$

Let now $(U', V', W') = R_2(N)$. Lemmas A.8 and A.22(1) imply that $U' \simeq M(sr, \rho')$, $V' \simeq M(s^{-1}, \sigma^*)$ and $W' \simeq M(\epsilon s^2 t, \tau')$, where $\rho' \in \widehat{G^{sr}}$ with $\rho'(sr) = -1$, $\rho'(h) = \rho(h)\sigma(h)$ for all $h \in G^r \cap G^s$, and $\tau' \in \widehat{G}$ with $\tau'(r) = -\sigma(\epsilon r^2)\tau(r)$, $\tau'(z) = \sigma(zr^{-1}zr)\tau(z)$ for all $z \in G^s$. Again we use Lemma 7.6 to prove that $\mathcal{S}$ is a skeleton of $R_2(N)$. Lemmas A.8 and A.22(1) imply that $\rho'(sr) = -1$, $\sigma^*(s^{-1}) = -1$, $\tau'(\epsilon s^2 t) = -1$, and

$$\rho'(\epsilon s^{-2})\sigma^*(\epsilon(rs)^2) = 1, \qquad \sigma^*(\epsilon s^2 t)\tau_2(s^{-1}) = -1.$$

Finally,

$$\begin{aligned}\rho'(\epsilon s^2 t)\tau'(sr) &= \rho(\epsilon s^2 t)\sigma(\epsilon s^2 t)(-\sigma(\epsilon r^2)\tau(r))\sigma(sr^{-1}sr)\tau(s)\\ &= -\rho(\epsilon s^2)\sigma(\epsilon r^2)\rho(t)\tau(r)\sigma(\epsilon^2 s^4)\sigma(t)\tau(s) = 1.\end{aligned}$$

Thus $\mathcal{S}$ is a skeleton of $R_2(N)$.

Now let $(U'', V'', W'') = R_3(N)$. Then Lemmas A.22(5) and A.2 imply that $U'' = U$, $V'' \simeq M(st, \sigma'')$ and $W'' \simeq M(t^{-1}, \tau^*)$ where $\sigma'' \in \widehat{G^{st}}$ with $\sigma''(z) = \sigma(z)\tau(z)$ for all $z \in G^s$. Lemmas A.22(5) and A.2 imply all conditions in Lemma 7.6(2) for (U'', V'', W'') except (7.6) and (7.3) for $i = 1$ and $j = 3$. These two we obtain as follows:

$$\begin{aligned}&\rho(\epsilon s^2 t^2)\sigma''(\epsilon r^2) = \rho(\epsilon s^2)\sigma(\epsilon r^2)\rho(t^2)\tau(\epsilon r^2) = 1,\\ &\rho(t^{-1})\tau^*(r) = \rho(t)^{-1}\tau(r)^{-1} = 1.\end{aligned}$$

Thus $\mathcal{S}$ is a skeleton of $R_3(N)$. □

Proposition 7.8. *Let $\theta \geq 3$ and $N \in \mathcal{F}_\theta^G$. If N has a skeleton $\mathcal{S}$ of type γ_θ, then A^N is of type C_θ, and $\mathcal{S}$ is a skeleton of $R_k(N)$ for all $k \in \{1, \dots, \theta\}$.*

Proof. Proceed as in the proof of Proposition 6.4 and apply Lemmas 7.7 and 6.3. □

Proof of Theorem 2.7. We prove (1)⇒(4)⇒(2)⇒(1) and (1)⇒(3)⇒(2).

(1)⇒(4). Since $M \in \mathcal{E}_\theta^G$ has a skeleton of type γ_θ, Proposition 7.8 implies that M admits all reflections and $\mathcal{W}(M)$ is standard of type C_θ. Moreover, from Lemma 7.6 we conclude that $\mathcal{B}(M_i)$ is finite-dimensional for all $i \in \{1, \dots, \theta\}$. More precisely,

$$\mathcal{H}_{\mathcal{B}(M_i)}(t) = (2)_t^2, \qquad \mathcal{H}_{\mathcal{B}(M_\theta)}(t) = (2)_t$$

for all $i \in \{1, \dots, \theta - 1\}$. Since the long roots are on the orbit of α_θ and the short roots on the orbit of (any) α_i with $i < \theta$, Theorem 1.2 implies that $\mathcal{B}(M)$ is finite-dimensional with the claimed Hilbert series.

(4)⇒(2). Since $\dim \mathcal{B}(M) < \infty$, the tuple M admits all reflections by [7, Cor. 3.18] and the Weyl groupoid is finite by [7, Prop. 3.23].

(2)⇒(1). It is assumed that M admits all reflections, A^M is of type C_θ, and $\mathcal{W}(M)$ is finite. Thus $\mathcal{C}(M)$ is a connected indecomposable finite Cartan graph by Theorem 1.1. For the proof of (1) we just have to verify the conditions in Lemma 7.6(2) and that $\operatorname{char} \mathbb{K} \neq 2$. Now Lemma 7.5 tells that $\operatorname{supp} M_i = \dim M_i = 2$ for all $i \in \{1, \dots, \theta - 1\}$, $\dim M_\theta = 1$, and $\operatorname{supp} M_i$ and $\operatorname{supp} M_{i+1}$ do not commute for $1 \leq i \leq \theta - 2$. Moreover, an iterated application of Lemma 5.14 implies that

$$(\operatorname{Res}^G_H M_{\theta-2}, \operatorname{Res}^G_H M_{\theta-1}, \operatorname{Res}^G_H M_\theta) \in \mathcal{E}^H_3,$$

where H is the (non-abelian) subgroup of G generated by $\cup_{i=\theta-2}^{\theta} \operatorname{supp} M_i$. Therefore Lemma 7.4 implies that $\operatorname{char} \mathbb{K} = 2$ and that the conditions in Lemma 7.6(2) hold whenever $i, j \in \{\theta - 2, \theta - 1, \theta\}$. The remaining claims in Lemma 7.6(2) follow from Lemmas 5.5, A.2, and Corollary A.7.

(1)⇒(3). Since $M \in \mathcal{E}^G_\theta$ has a skeleton of type γ_θ, Proposition 7.8 implies that M admits all reflections and $\mathcal{W}(M)$ is standard of type C_θ.

(3)⇒(2). This is clear: see e.g. [16, Thm. 3.3]. □

8. Proof of Theorems 2.8 and 2.9: The case B

In the whole section let G be a non-abelian group. In order to prove Theorems 2.8 and 2.9, we first collect some information on skeletons of type β_θ, β'_θ and β''_θ for $\theta \geq 3$, on tuples in $\mathcal{F}^G_\theta$ with such skeletons, and on a particular Cartan graph.

Extending the definition of a skeleton of type β'_3 and β''_3, we say that the skeletons in Figure 8.1 *are of type* β'_θ and β''_θ, respectively. We will need them for the proof of Theorem 2.8. We want to stress that skeletons of type β'_θ and β''_θ are of finite type if and only if $\theta = 3$.

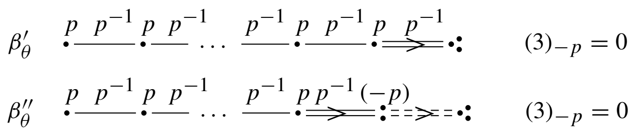


Fig. 8.1. Skeletons of type β'_θ and β''_θ.

For tuples with skeletons of type β_θ, β'_θ, and β''_θ, respectively, where $\theta \geq 3$, one obtains the following.

Lemma 8.1. *Suppose that* char $\mathbb{K} = 3$. *Let* $\theta \in \mathbb{N}_{\geq 3}$ *and* $M \in \mathcal{F}_\theta^G$. *The following are equivalent:*

(1) *M has a skeleton of type β_θ.*
(2) *There exists $\epsilon \in Z(G)$ with $\epsilon^2 = 1$, $\epsilon \neq 1$, and for all $i \in \{1,\dots,\theta\}$ and $s_i \in$* supp M_i *a unique character σ_i of G^{s_i} such that* supp $M_i = \{s_i, \epsilon s_i\}$, $M_i \simeq M(s_i, \sigma_i)$ *for all $i \in \{1,\dots,\theta\}$, and*

$$\sigma_i(s_j)\sigma_j(s_i) = \sigma_i(\epsilon)\sigma_j(\epsilon) = 1 \quad \text{for all } i, j \text{ such that } |i-j| \geq 2, \tag{8.1}$$

$$\sigma_i(\epsilon s_{i+1}^2)\sigma_{i+1}(\epsilon s_i^2) = 1 \quad \text{for all } i \in \{1,\dots,\theta-1\}, \tag{8.2}$$

$$\sigma_i(s_i) = -1 \quad \text{for all } i \in \{1,\dots,\theta-1\}, \tag{8.3}$$

$$\sigma_\theta(s_\theta) = 1, \tag{8.4}$$

$$s_i s_{i+1} = \epsilon s_{i+1} s_i \quad \text{for all } i \in \{1,\dots,\theta-1\}, \tag{8.5}$$

$$s_i s_j = s_j s_i \quad \text{for all } i, j \text{ such that } |i-j| \geq 2. \tag{8.6}$$

Proof. We first prove that (2) implies (1). By Definition 2.2 and the assumptions in (2), we just have to prove that the Cartan matrix A^M is of type B_θ. Now $a^M_{i,i+1} = a^M_{i+1,i} = -1$ for all $i \in \{1,\dots,\theta-2\}$ by Corollary A.7(1), $a^M_{ij} = a^M_{ji} = 0$ for $i, j \in \{1,\dots,\theta\}$ with $|i-j| > 1$ by Lemma 5.5, and $a^M_{\theta-1,\theta} = -1$, $a^M_{\theta,\theta-1} = -2$ by Corollary A.7(2). This proves (1).

Assume now that (1) holds. Then the claims in (2) on supp M_i for all $i \in \{1,\dots,\theta\}$ including (8.5) and (8.6) follow from Lemma 5.2(1) and the shape of the skeleton of M. Moreover, A^M is of type B_θ by (1) and the definition of a skeleton. Then (8.1)–(8.4) follow from Lemma 5.5 and Corollary A.7. □

Lemma 8.2. *Let $\theta \in \mathbb{N}_{\geq 3}$ and $M \in \mathcal{F}_\theta^G$. The following are equivalent:*

(1) *M has a skeleton of type β'_θ, and there exist $t_1, t_2 \in$* supp M_θ *such that $t_1 t_2 \neq t_2 t_1$.*
(2) *For any tuple $(s_1,\dots,s_\theta)$, where $s_i \in$* supp M_i *for all i, there exist $\epsilon \in G \setminus \{1\}$ and a unique tuple $(\sigma_1,\dots,\sigma_\theta)$ with the following properties: $\epsilon^3 = 1$, $\sigma_i \in \widehat{G^{s_i}}$ for all i,* supp $M_i = \{s_i\}$ *for all $i \in \{1,\dots,\theta-1\}$,* supp $M_\theta = \{s_\theta, \epsilon s_\theta, \epsilon^2 s_\theta\}$, $M_i \simeq M(s_i, \sigma_i)$ *for all i, and*

$$\sigma_i(s_j)\sigma_j(s_i) = 1 \quad \text{if } |i-j| \geq 2, \tag{8.7}$$

$$\sigma_i(s_{i+1})\sigma_{i+1}(s_i) = p^{-1} \quad \text{for all } i \in \{1,\dots,\theta-1\}, \tag{8.8}$$

$$\sigma_i(s_i) = p \quad \text{for all } i \in \{1,\dots,\theta-1\}, \tag{8.9}$$

$$\sigma_\theta(s_\theta) = -1, \tag{8.10}$$

$$\epsilon s_\theta = s_\theta \epsilon^{-1}, \tag{8.11}$$

where $p \in \mathbb{K}$ with $1 - p + p^2 = 0$.

Proof. We first prove that (2) implies (1). According to Definition 2.2 and the assumptions in (2), one has only to prove that the Cartan matrix A^M is of type B_θ. Now $a^M_{i,i+1} = a^M_{i+1,i} = -1$ and $a^M_{ij} = 0$ for all $i \in \{1,\dots,\theta-2\}$ and all $j > i+1$ with $j \neq \theta$ by

Lemma A.1. Further, $a^M_{\theta-1,\theta} = -1$ and $a^M_{i\theta} = 0$ (and hence $a^M_{\theta i} = 0$) for all $i < \theta - 1$ in view of Lemma A.2. Finally, $a^M_{\theta,\theta-1} = -2$ according to Lemma A.10. This proves (1).

Assume now that (1) holds. In particular, A^M is of type B_θ by the definition of a skeleton and a skeleton of type β'_θ. Let $s_\theta \in \operatorname{supp} M_\theta$. Since $|\operatorname{supp} M_\theta| = 3$, (1) and Lemma A.9 imply that there exists $\epsilon \in G$ such that $\epsilon^3 = 1$, $\epsilon \neq 1$, $\epsilon s_\theta = s_\theta \epsilon^{-1}$, and $\operatorname{supp} M_\theta = \{s_\theta, \epsilon s_\theta, \epsilon^2 s_\theta\}$. Then (8.7) follows from Lemma A.2, since $a^M_{ij} = 0$ whenever $|i-j| \geq 2$. Equations (8.8) and (8.9) hold by the definition of the skeleton. Since $a^M_{\theta,\theta-1} = -2$, (8.10) follows from Lemma A.10. □

Lemma 8.3. *Let $M \in \mathcal{F}^G_\theta$. The following are equivalent:*

(1) *M has a skeleton of type β''_θ.*

(2) *For any tuple $(s_1, \dots, s_\theta)$, where $s_i \in \operatorname{supp} M_i$ for all i, there exist $\epsilon \in G \setminus \{1\}$ and a unique tuple $(\sigma_1, \dots, \sigma_\theta)$ with the following properties: $\epsilon^3 = 1$, $\sigma_i \in \widehat{G^{s_i}}$ for all i, $\operatorname{supp} M_i = \{s_i\}$ for all $i \in \{1, \dots, \theta-2\}$, $\operatorname{supp} M_{\theta-1} = \{s_{\theta-1}, \epsilon s_{\theta-1}\}$, $\operatorname{supp} M_\theta = \{s_\theta, \epsilon s_\theta, \epsilon^2 s_\theta\}$, $M_i \simeq M(s_i, \sigma_i)$ for all i, and*

$$\sigma_i(s_j)\sigma_j(s_i) = 1 \quad \text{if } |i-j| \geq 2 \text{ with } i, j \leq \theta, \tag{8.12}$$

$$\sigma_i(s_{i+1})\sigma_{i+1}(s_i) = p^{-1} \quad \text{for all } i \in \{1, \dots, \theta-2\}, \tag{8.13}$$

$$\sigma_i(s_i) = p \quad \text{for all } i \in \{1, \dots, \theta-2\}, \tag{8.14}$$

$$\sigma_{\theta-1}(s_{\theta-1}) = \sigma_\theta(s_\theta) = -1, \tag{8.15}$$

$$\sigma_{\theta-1}(\epsilon) = -p, \tag{8.16}$$

$$\sigma_{\theta-1}(\epsilon s_\theta^2)\sigma_\theta(\epsilon s_{\theta-1}^2) = 1, \tag{8.17}$$

$$s_\theta s_{\theta-1} = \epsilon s_{\theta-1} s_\theta, \tag{8.18}$$

$$\epsilon s_\theta = s_\theta \epsilon^{-1}, \tag{8.19}$$

where $p \in \mathbb{K}$ with $1 - p + p^2 = 0$.

Proof. Again we first prove that (2) implies (1). According to Definition 2.2 and the assumptions in (2), one has only to prove that the off-diagonal entries of A^M correspond to the integers obtained from the skeleton of type β''_θ. Now $a^M_{i,i+1} = a^M_{i+1,i} = -1$ for all $i \in \{1, \dots, \theta-3\}$ by Lemma A.1 and $a^M_{ij} = 0$ for all $1 \leq i, j \leq \theta$ with $j \geq i+2$ by Lemma A.2. Also, $a^M_{\theta-2,\theta-1} = -1$ by Lemma A.2. Moreover, $a^M_{\theta-1,\theta-2} = -2$ by Lemmas A.15 and A.16(1). Finally, $a^M_{\theta-1,\theta} = -1$ and $a^M_{\theta,\theta-1} = -2$ in view of Lemma A.13. This proves (1).

Assume now that (1) holds. Since $s^G_{\theta-1}$ and s^G_θ do not commute and since $|s^G_{\theta-1}| = 2$, we see that $s_\theta s_{\theta-1} \neq s_{\theta-1} s_\theta$. Let $\epsilon \in G$ be such that $s^G_{\theta-1} = \{s_{\theta-1}, \epsilon s_{\theta-1}\}$. Then $\epsilon^3 = 1$, $\operatorname{supp} M_\theta = \{s_\theta, \epsilon s_\theta, \epsilon^2 s_\theta\}$, and (8.18), (8.19) hold by Lemma A.12. It remains to prove (8.12)–(8.17).

Now (8.12) follows from Lemma A.2, since $a^M_{ij} = 0$ whenever $1 \leq i < j-1$. By Proposition 5.9, all $(\operatorname{ad} M_i)^m(M_j)$ for $i \neq j$ and $m \geq 0$ are absolutely simple or zero because of (1). Since $a^M_{\theta,\theta-1} = -1$ and $a^M_{\theta,\theta-1} = -2$, (8.15) and (8.17) follow from

Lemma A.13. Finally, conditions (8.13), (8.14), and (8.16) come from the definition of the skeleton. □

In the following three propositions we study reflections of skeletons of type β_θ, β'_θ, and β''_θ with $\theta \geq 3$.

Proposition 8.4. *Let $\theta \in \mathbb{N}$ with $\theta \geq 3$ and let $M \in \mathcal{F}^G_\theta$. Assume that M has a skeleton $\mathcal{S}$ of type β_θ. Then the Cartan matrix of M is of type B_θ, and $\mathcal{S}$ is a skeleton of $R_k(M)$ for all $k \in \{1, \dots, \theta\}$.*

Proof. Following the arguments in the proof of Proposition 6.4 and using Lemma 6.3, it suffices to prove the claim for $\theta = 3$. In this case, one obtains the claim following the proof of Lemma 6.3 and using Lemma 8.1. □

Proposition 8.5. *Let $M \in \mathcal{F}^G_\theta$. Assume that M has a skeleton $\mathcal{S}$ of type β'_θ. Then $\mathcal{S}$ is a skeleton of $R_k(M)$ for $1 \leq k \leq \theta - 1$, and $R_\theta(M)$ has a skeleton of type β''_θ.*

Proof. By Remark 5.17, it is enough to consider connected subgraphs of $\mathcal{S}$ with three vertices i_1, i_2, i_3. If $\theta \notin \{i_1, i_2, i_3\}$ and $k \in \{i_1, i_2, i_3\}$, then Lemma 8.2 implies that $M_{i_1} \oplus M_{i_2} \oplus M_{i_3}$ is a braided vector space of Cartan type with Cartan matrix of type A_3, and hence the tuple $R_j(M_{i_1}, M_{i_2}, M_{i_3})$ for $j \in \{1, 2, 3\}$ has the same skeleton as $(M_{i_1}, M_{i_2}, M_{i_3})$. Thus it remains to prove the proposition for $\theta = 3$ and $k \in \{1, 2, 3\}$.

Assume first that $k = 1$. Then $\dim M_k = 1$, $a^M_{12} = -1$, and $a^M_{13} = 0$. Hence $R_1(M)_1 = M_1^*$, $R_1(M)_2 \simeq M_1 \otimes M_2$ by Lemma A.1, and $R_1(M)_3 = M_3$. We now verify the conditions in Lemma 8.2 for $R_1(M)$. The only non-trivial condition is (8.8) for $i = 2$. For this we obtain

$$\sigma_1\sigma_2(s_3)\sigma_3(s_1s_2) = \sigma_1(s_3)\sigma_3(s_1)\sigma_2(s_3)\sigma_3(s_2) = p^{-1},$$

and hence $\mathcal{S}$ is a skeleton of $R_1(M)$.

Assume now that $k = 2$. Then $\dim M_k = 1$ and $a^M_{21} = a^M_{23} = -1$. Hence $R_2(M)_1 \simeq M_2 \otimes M_1$, $R_2(M)_2 \simeq M_2^*$, and $R_2(M)_3 \simeq M_2 \otimes M_3$ by Lemma A.2. We verify the conditions in Lemma 8.2 for $R_2(M)$. We obtain

$$\begin{aligned}
&\sigma_1\sigma_2(s_2s_3)\sigma_2\sigma_3(s_2s_1) = \sigma_1(s_3)\sigma_3(s_1)\sigma_1(s_2)\sigma_2(s_1s_2^2s_3)\sigma_3(s_2) = p^{-1}p^2p^{-1} = 1,\\
&\sigma_1\sigma_2(s_2^{-1})\sigma_2^*(s_2s_1) = (\sigma_1(s_2)\sigma_2(s_1))^{-1}\sigma_2(s_2)^{-2} = pp^{-2} = p^{-1},\\
&\sigma_2^*(s_2s_3)\sigma_2\sigma_3(s_2^{-1}) = (\sigma_2(s_3)\sigma_3(s_2))^{-1}\sigma_2(s_2)^{-2} = pp^{-2} = p^{-1},\\
&\sigma_1\sigma_2(s_1s_2) = pp^{-1}p = p,\\
&\sigma_2^*(s_2^{-1}) = p, \qquad \sigma_2\sigma_3(s_2s_3) = pp^{-1}\sigma_3(s_3) = \sigma_3(s_3).
\end{aligned}$$

Condition (8.11) for $R_2(M)$ is clear. Therefore $\mathcal{S}$ is a skeleton of $R_2(M)$.

Finally, assume that $k = 3$. Then $R_3(M) = (M_1, (\operatorname{ad} M_3)^2(M_2), M_3^*)$. We have to show that $R_3(M)$ has a skeleton of type β''_3. To do so we apply Lemma 8.3. By Proposition A.11, $R_3(M)_2 \simeq M(s', \sigma')$ and $M_3^* \simeq M(s_3^{-1}, \sigma_3^*)$, where $s' = \epsilon s_2 s_3^2$, $\sigma'(\epsilon) = p^{-2} = -p$ (which proves (8.16)), and $\sigma'(h) = \tau(h)^2\sigma(h)$ for all $h \in G^\epsilon \cap G^{s_3}$.

Now conditions (8.12), (8.14), (8.18) and (8.19) are clear. Moreover, (8.15) and (8.17) follow from the last claim of Proposition A.11. We now verify (8.13):

$$\sigma_1(s')\sigma'(s_1) = \sigma_1(\epsilon s_2 s_3^2)\sigma_2(s_1)\sigma_3(s_1)^2 = \sigma_1(s_2)\sigma_2(s_1)(\sigma_1(s_3)\sigma_3(s_1))^2 = p^{-1}. \qquad \square$$

For the proof of the third of the three propositions mentioned above we will use the following lemma, which will also play a role in the proof of Proposition 9.3.

Lemma 8.6. *Let $M \in \mathcal{F}_3^G$. Assume that M has a skeleton $\mathcal{S}$ as in Figure 8.2, where $p = -1$ if $q^2 = 1$. Then $\mathcal{S}$ is a skeleton of $R_k(M)$ for all $k \in \{1, 2, 3\}$.*

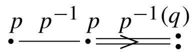


Fig. 8.2. The skeleton in Lemma 8.6.

Proof. By assumption, there exist $r, t \in Z(G)$, $s, \epsilon \in G$, and $\rho, \tau \in \widehat{G}$, $\sigma \in \widehat{G^s}$ such that $M_1 \simeq M(r, \rho)$, $M_2 \simeq M(t, \tau)$, $M_3 \simeq M(s, \sigma)$, $s^G = \{s, \epsilon s\}$, and $\epsilon \neq 1$. By Lemma 5.1, there exists $x \in G$ such that $xs = \epsilon sx$ and $x\epsilon = \epsilon^{-1}x$. The skeleton contains the following additional information (see Lemmas A.1 and A.14):

$$\rho(r) = p, \qquad \tau(t) = p, \qquad \sigma(\epsilon) = q,$$
$$\rho(t)\tau(r) = p^{-1}, \qquad \rho(s)\sigma(r) = 1, \qquad \tau(s)\sigma(t) = p^{-1},$$

and $a_{32}^M = -2$. Since $a_{32}^M = -2$, Lemma A.14 implies that $p \neq 1$. Since $a_{ij}^M a_{ji}^M \in \{0, 1, 2\}$ for all $i, j \in \{1, 2, 3\}$ with $i \neq j$, Proposition 5.9 and Lemmas A.15, A.16 imply that the relations in one of the following three lines hold:

$$\sigma(s) = -1, \qquad \sigma(\epsilon^2) \neq 1, \qquad \sigma(\epsilon^2 t^2)\tau(s^2) = 1,$$
$$\sigma(s) = -1, \qquad \sigma(\epsilon^2) = 1,$$
$$\sigma(s) \neq -1, \qquad \sigma(\epsilon^2) \neq 1, \qquad \sigma(st)\tau(s) = 1, \qquad \sigma(\epsilon^2 s^2) = 1.$$

Since $\sigma(t)\tau(st) = 1$, we conclude that $(M_3, M_2) \in \wp_5 \cup \wp_1 \cup \wp_7$.

Let now $(U, V, W) = R_1(M)$. Then $U \simeq M(r^{-1}, \rho^*)$, $V \simeq M(rt, \rho\tau)$, and $W = M_3 \simeq M(s, \sigma)$. In particular,

$$\rho\tau(s)\sigma(rt) = \tau(s)\sigma(t) = p^{-1}.$$

Using the above formulas and the definition of a skeleton, we conclude that $\mathcal{S}$ is a skeleton of $R_1(M)$.

Let $(U', V', W') = R_2(M)$. Then $U' \simeq M(tr, \tau\rho)$, $V' \simeq M(t^{-1}, \tau^*)$, and $W' \simeq M(ts, \tau\sigma)$. Consequently,

$$\tau\rho(ts)\tau\sigma(tr) = \tau(t^2)\tau(s)\sigma(t)\rho(t)\tau(r)\rho(s)\sigma(r) = p^2 p^{-1} p^{-1} = 1$$

and $\tau\sigma(\epsilon ts) = \tau(ts)\sigma(t)\sigma(\epsilon s) = q$, so Lemma A.22(5) implies that $\mathcal{S}$ is a skeleton of $R_2(M)$.

Now let $(U'', V'', W'') = R_3(M)$. Then $U'' = M_1 \simeq M(r, \rho)$, $V'' \simeq M(\epsilon s^2 t, \tau_2)$, and $W'' \simeq M(s^{-1}, \sigma^*)$, where $\tau_2 \in \widehat{G}$ as in Lemma A.15. We record that $(s^{-1})^G = \{s^{-1}, \epsilon^{-1}s^{-1}\}$ and $\sigma^*(\epsilon^{-1}) = \sigma(\epsilon)$. Moreover,

$$\tau_2(r)\rho(\epsilon s^2 t) = \sigma(r^2)\tau(r)\rho(s^2 t) = \rho(t)\tau(r) = p^{-1}.$$

Thus, if $(M_3, M_2) \in \wp_5, \wp_1$, or $\wp_7$, then Lemma A.22(2), (1), or (3), respectively, implies that $\mathcal{S}$ is a skeleton of $R_3(M)$. Here, in the case of $\sigma(\epsilon^2) = 1$ we have used (and needed) that $p = -1$ in order to identify $\mathcal{S}$ as a skeleton of $R_3(M)$. □

Proposition 8.7. *Let $M \in \mathcal{F}_\theta^G$. Assume that M has a skeleton $\mathcal{S}$ of type β''_θ. Then $\mathcal{S}$ is a skeleton of $R_k(M)$ for $1 \le k \le \theta - 1$, and $R_\theta(M)$ has a skeleton of type β'_θ.*

Proof. By Remark 5.17, it is enough to consider connected subgraphs of $\mathcal{S}$ with three vertices i_1, i_2, i_3 and their reflections. If $i_1, i_2, i_3 \le \theta - 2$, then $M_{i_1} \oplus M_{i_2} \oplus M_{i_3}$ is a braided vector space of Cartan type and their reflections have the same skeleton. If $\{i_1, i_2, i_3\} = \{\theta - 3, \theta - 2, \theta - 1\}$, then the reflections of $M_{i_1} \oplus M_{i_2} \oplus M_{i_3}$ have the same skeleton as $M_{i_1} \oplus M_{i_2} \oplus M_{i_3}$. Indeed, $(3)_{-p} = 0$ by assumption and hence $p \ne 1$. Therefore $(-p)^2 = 1$ implies that $p = -1$, and so Lemma 8.6 applies.

It remains to determine the skeleton of $R_k(M_{\theta-2}, M_{\theta-1}, M_\theta)$ for all $k \in \{1, 2, 3\}$, that is, to prove the claim for $\theta = 3$. To do so, assume that $\theta = 3$, and let $s_1, s_2, s_3, \epsilon \in G$ and $\sigma_1, \sigma_2, \sigma_3$ be as in Lemma 8.3.

Let $(U, V, W) = R_1(M)$. Then $U \simeq M(s_1^{-1}, \sigma^*)$, $V \simeq M(s_1 s_2, \sigma_1\sigma_2)$, and $W \simeq M(s_3, \sigma_3)$ by Lemma A.2. Consequently,

$$\sigma_1\sigma_2(\epsilon s_3^2)\sigma_3(\epsilon s_1^2 s_2^2) = (\sigma_1(s_3)\sigma_3(s_1))^2\sigma_2(\epsilon s_3^2)\sigma_3(\epsilon s_2^2) = 1,$$

and hence $\mathcal{S}$ is a skeleton of $R_1(M)$ by Lemmas 8.3 and A.22(5).

Let $(U', V', W') = R_2(M)$. Then

$$V' \simeq M(s_2^{-1}, \sigma^*), \qquad (s_2^{-1})^G = \{s_2^{-1}, \epsilon^{-1}s_2^{-1}\},$$

and $U' \simeq M(\epsilon s_2^2 s_1, \rho')$ for some $\rho' \in \widehat{G}$ by Lemmas A.15 and A.16. Moreover, the skeleton of (M_1, M_2) is a skeleton of (U', V') by Lemma A.22(1) (if $p \ne -1$) and by Lemma A.22(2) (if $p = -1$), since $\sigma_2(s_2) = -1$. Further, since $(3)_{\sigma_2(\epsilon)} = 0$ by (8.16), we conclude from Lemma A.13 that $W' \simeq M(\epsilon^{-1}s_2 s_3, \tau')$, where $\tau' \in \widehat{G^{s_3}}$ with $\tau'(s_3) = \sigma_3(\epsilon s_2^{-1})\sigma_2(\epsilon)$ and $\tau'(h) = \sigma_2(h)\sigma_3(h)$ for all $h \in G^{s_2} \cap G^{s_3}$. Thus

$$\begin{aligned}
&\sigma_2^*(s_2^{-1}) = \sigma_2(s_2) = -1,\\
&\tau'(\epsilon^{-1}s_2 s_3) = \sigma_2(\epsilon^{-1}s_2)\sigma_3(\epsilon^{-1}s_2)\sigma_3(\epsilon s_2^{-1})\sigma_2(\epsilon) = \sigma_2(s_2) = -1,\\
&\sigma_2^*(\epsilon^{-1}) = \sigma_2(\epsilon) = -p,\\
&\sigma_2^*(\epsilon^{-1}(\epsilon^{-1}s_2 s_3)^2)\tau'(\epsilon^{-1}s_2^{-2}) = \sigma_2(s_2^2 s_3^2)^{-1}\sigma_2\sigma_3(\epsilon s_2^2)^{-1} = 1,\\
&\epsilon^{-1}s_2 s_3\, s_2^{-1} = \epsilon^{-2}s_3 s_2 s_2^{-1} = \epsilon^{-1}s_2^{-1}(\epsilon^{-1}s_2 s_3).
\end{aligned}$$

Therefore $\mathcal{S}$ is a skeleton of $R_2(M)$ by Lemma 8.3.

Let $(U'', V'', W'') = R_3(M)$. Then $U'' = M_1$ and $W'' \simeq M(s_3^{-1}, \sigma_3^*)$. Lemma A.13 implies that $V'' \simeq M(\epsilon^{-1}s_3^2 s_2, \sigma'')$, where $\sigma'' \in \widehat{G}$ is such that

$$\sigma''(\epsilon) = 1, \qquad \sigma''(s_3) = -\sigma_3(\epsilon s_2^{-1})\sigma_2(\epsilon), \qquad \sigma''(h) = \sigma_3(h)^2\sigma_2(h)$$

for all $h \in G^{s_2} \cap G^{s_3}$. Now we verify the conditions in Lemma 8.2 for $R_3(M) \in \mathcal{F}_3^G$. Except (8.9) for $i = 2$ and except (8.8), everything is clear or can be seen directly. Since $\epsilon^{-1}s_2 \in Z(G)$ by Lemma A.12, for (8.9), $i = 2$, we obtain

$$\sigma''(\epsilon^{-1}s_3^2 s_2) = \sigma_3(\epsilon s_2^{-1})^2\sigma_2(\epsilon)^2\sigma_3(\epsilon^{-1}s_2)^2\sigma_2(\epsilon^{-1}s_2) = \sigma_2(\epsilon s_2) = p.$$

Finally, for (8.8) we calculate as follows:

$$\sigma_1(\epsilon^{-1}s_3^2 s_2)\sigma''(s_1) = \sigma_1(s_3)^2\sigma_1(s_2)\sigma_3(s_1)^2\sigma_2(s_1) = p^{-1},$$
$$\sigma''(s_3^{-1})\sigma_3^*(\epsilon^{-1}s_3^2 s_2) = -\sigma_3(\epsilon^{-1}s_2)\sigma_2(\epsilon^{-1})\sigma_3(\epsilon s_2^{-1}) = -\sigma_2(\epsilon^{-1}) = p^{-1}.$$

Thus $R_3(M)$ has a skeleton of type β_3'. □

Before proving Theorem 2.8 we also need more information on the finite Cartan graph in Lemma 3.1(4).

Lemma 8.8. *Let $\mathcal{C} = \mathcal{C}(I, \mathcal{X}, r, A)$ be a Cartan graph with $I = \{1, 2, 3\}$, $\mathcal{X} = \{X, Y\}$, such that $r_1 = r_2 = \mathrm{id}$, r_3 is the transposition $(X\ Y)$ and*

$$A^X = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}, \qquad A^Y = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}.$$

Let $W_0 \subset W(\mathcal{C})$ be the automorphism group of X. Then

$$\mathbf{\Delta}_+^X = \{1, 2, 3, 12, 23, 123, 23^2, 123^2, 12^23^2, 12^23^3, 12^23^4, 12^33^4, 1^22^33^4\},$$
$$\mathbf{\Delta}_+^Y = \{1, 2, 3, 12, 23, 12^2, 123, 23^2, 12^23, 123^2, 12^23^2, 12^33^2, 1^22^33^2\},$$

and the orbits of $\mathbf{\Delta}^X$ with respect to the action of W_0 are

$$\{\pm 1, \pm 2, \pm 12, \pm 12^23^4, \pm 12^33^4, \pm 1^22^33^4\},$$
$$\{\pm 3, \pm 23, \pm 123, \pm 12^23^3\}, \qquad \{\pm 23^2, \pm 123^2, \pm 12^23^2\},$$

where $1^a2^b3^c$ and $-1^a2^b3^c$ mean $a\alpha_1 + b\alpha_2 + c\alpha_3$ and $-a\alpha_1 - b\alpha_2 - c\alpha_3$, respectively, for all $a, b, c \in \mathbb{Z}$.

Proof. It is clear from the definition that $\mathcal{C}$ is a semi-Cartan graph. It is a Cartan graph by [16, Thm. 5.4]. The root system with number 14 in [17, Appendix A], where one interchanges α_1 and α_2, has Cartan graph $\mathcal{C}$ and corresponds to the point Y (see also the proof of Lemma 3.1). From this one easily obtains the set $\mathbf{\Delta}^X = s_3^Y(\mathbf{\Delta}^Y)$.

By the proof of [16, Thm. 5.4] (see [16, eqs. (5.4), (5.5)]), W_0 is generated as a group by s_1^X, s_2^X, and $t = s_3 s_2 s_3^X$. (Observe that in [16] the roles of 1 and 3 in I are interchanged.) We record that

$$t(\alpha_1) = \alpha_1 + 2\alpha_2 + 4\alpha_3, \qquad t(\alpha_2) = \alpha_2, \qquad t(\alpha_3) = -(\alpha_2 + \alpha_3).$$

Applying successively these generators of W_0 to the elements of $\mathbf{\Delta}^X$ one obtains the last claim of the lemma. □

Proof of Theorem 2.8: $\dim M_1 = 1$.

(1)⇒(3). Since $\theta = 3$ and M has a skeleton of type β_3', Propositions 8.5 and 8.7 imply that M admits all reflections, and the skeletons of M and of $R_3(M)$ form the points of the semi-Cartan graph $\mathcal{C}$ in Lemma 8.8. This graph is a finite Cartan graph, and the positive roots of its points are given in Lemma 8.8. Since M has a skeleton of type β_3', Lemma 8.2 implies that $\mathcal{B}(M_i)$ is finite-dimensional for all $i \in \{1,2,3\}$. More precisely,

$$\mathcal{H}_{\mathcal{B}(M_1)}(t) = \mathcal{H}_{\mathcal{B}(M_2)}(t) = (h)_t, \qquad \mathcal{H}_{\mathcal{B}(M_3)}(t) = (2)_t^2(3)_t,$$

where $h = 3$ if $\operatorname{char}\mathbb{K} = 2$, $h = 2$ if $\operatorname{char}\mathbb{K} = 3$, and $h = 6$ otherwise. Similarly, Lemma 8.3 implies that $R_3(M)_2$ is a braided vector space of diagonal type with braiding matrix

$$\begin{pmatrix} -1 & -\zeta \\ -\zeta & -1 \end{pmatrix}$$

where $\zeta = \sigma(\epsilon)$ in the notation of Lemma 8.3. Therefore

$$\mathcal{H}_{\mathcal{B}(R_3(M)_2)}(t) = (2)_t(h')_t = \begin{cases} (2)_t^2 & \text{if } \operatorname{char}\mathbb{K} = 3, \\ (2)_t^2(3)_{t^2} & \text{if } \operatorname{char}\mathbb{K} \neq 3, \end{cases}$$

where $h' = 6$ if $\operatorname{char}\mathbb{K} \neq 3$ and $h' = 2$ if $\operatorname{char}\mathbb{K} = 3$. Now Theorem 1.2, using the decomposition of $\mathbf{\Delta}_+^X$ into W_0-orbits in Lemma 8.8, implies that $\mathcal{B}(M)$ is finite-dimensional with the claimed Hilbert series.

(3)⇒(2). Since $\dim \mathcal{B}(M) < \infty$, the tuple M admits all reflections by [7, Cor. 3.18] and the Weyl groupoid is finite by [7, Prop. 3.23].

(2)⇒(1). It is assumed that $\dim M_1 = 1$, M admits all reflections, A^M is of type B_θ, and $\mathcal{W}(M)$ is finite. Thus Theorem 1.1 tells us that $\mathcal{C}(M)$ is a connected indecomposable finite Cartan graph.

Assume first that $\theta = 3$. If $\mathcal{C}(M)$ has a point with Cartan matrix of type A_3 or C_3, then M is standard of type A_3 and C_3, respectively, by Theorems 2.6 and 2.7. Since the Cartan matrix A^M is of type B_3, from Corollary 3.5 we conclude that either M is standard of type B_3, or each point of $\mathcal{C}(M)$ has one of the two Cartan matrices in Lemma 3.1(4).

Since $\dim M_1 = 1$, Lemma 5.12 implies that $\dim M_2 = 1$. Let H be the subgroup generated by $\operatorname{supp} M_2 \cup \operatorname{supp} M_3$. Then H is non-abelian, $M' = (\operatorname{Res}_H^G M_2, \operatorname{Res}_H^G M_3) \in \mathcal{E}_2^H$, M' admits all reflections, and $\mathcal{W}(M')$ is standard of type B_2 by Corollary 3.5. Now [28, Thm. 2.1, Table 1], especially the claim on the support of M', implies immediately that $\operatorname{supp} M_3$ is non-abelian and $|\operatorname{supp} M_3| \in \{3,4\}$. Moreover, the only possible example with $|\operatorname{supp} M_3| = 4$ would be [28, Ex. 1.7]. However, this example has a root system which is standard of type G_2, and hence a Cartan matrix of type B_2 is impossible if $|\operatorname{supp} M_3| = 4$. On the other hand, M' being standard implies that $M' \notin \wp_5$ in the notation of [28, 7.1,8.4]. The only remaining possibility is discussed in [28, Thm. 8.2]: There exist $r, s \in Z(G)$, $t, \epsilon \in G$, and characters ρ, σ of G and τ of G^t such that

$$M_1 \simeq M(r, \rho), \qquad M_2 \simeq M(s, \sigma), \qquad M_3 \simeq M(t, \tau),$$

and G is generated by r, s, t, ϵ, the relations $t\epsilon = \epsilon^{-1}t$ and $\epsilon^3 = 1$ hold in G, and

$$(3)_{-\sigma(s)} = 0, \qquad \sigma(st)\tau(s) = 1, \qquad \tau(t) = -1. \tag{8.20}$$

Moreover, the condition $a_{13}^M = 0$ is equivalent to $\rho(t)\tau(r) = 1$.

Both if M is standard and if $\mathbf{\Delta}_+^M$ is the root system of X as in Lemma 8.8, we obtain

$$\mathbf{\Delta}_+^M = \mathbf{\Delta}_+^{R_1(M)} = \mathbf{\Delta}_+^{R_2(M)}, \qquad A^M = A^{R_1(M)} = A^{R_2(M)}.$$

Since $R_1(M) \simeq (M_1^*, M_1 \otimes M_2, M_3)$ and $M_1 \otimes M_2 \simeq M(rs, \rho\sigma)$, the above arguments for M applied to $R_1(M)$ imply that

$$(3)_{-\rho(rs)\sigma(rs)} = 0, \qquad \rho(rst)\sigma(rst)\tau(rs) = 1,$$

and hence $\rho(rs)\sigma(r) = 1$. Similarly, $R_2(M) \simeq (M_1 \otimes M_2, M_2^*, M_2 \otimes M_3)$. Then $a_{13}^{R_2(M)} = 0$ implies that

$$\rho\sigma(st)\sigma\tau(rs) = 1,$$

and therefore $\rho(s)\sigma(rs) = 1$. Thus M has a skeleton of type β_3' by Lemma 8.2.

Assume now that $\theta \geq 4$. Since $\dim M_1 = 1$, Lemma 5.12 implies that $\dim M_2 = 1$. Let H be the subgroup generated by $\bigcup_{i=2}^{\theta} \operatorname{supp} M_i$. Then H is non-abelian, $M' = (\operatorname{Res}_H^G M_i)_{2\le i\le\theta} \in \mathcal{E}_{\theta-1}^H$, M' admits all reflections, $\mathcal{W}(M')$ is finite, and $A^{M'}$ is of type $B_{\theta-1}$. Thus it suffices to show that these assumptions lead to a contradiction in the case $\theta = 4$.

Assume that $\theta = 4$. By the claim for $\theta = 3$ we conclude that there exist $r', r, s \in Z(G)$, $t, \epsilon \in G$, and characters ρ', ρ, σ of G and τ of G^t such that r', r, s, t, ϵ generate G, and the relations $\epsilon^3 = 1$, $t\epsilon = \epsilon^{-1}t$ hold in G. Moreover,

$$M_1 \simeq M(r', \rho'), \qquad M_2 \simeq M(r, \rho), \qquad M_3 \simeq M(s, \sigma), \qquad M_4 \simeq M(t, \tau),$$

and the characters satisfy the relations

$$\begin{aligned} &\rho'(s)\sigma(r') = 1, \quad \rho'(t)\tau(r') = 1, \quad \rho(rs)\sigma(r) = 1, \quad \rho(t)\tau(r) = 1,\\ &\rho(s)\sigma(rs) = 1, \quad (3)_{-\sigma(s)} = 0, \quad \sigma(st)\tau(s) = 1, \quad \tau(t) = -1. \end{aligned}$$

Since $R_1(M) \in \mathcal{E}_4^G$ and $\dim R_1(M)_1 = 1$, we conclude that

$$M' = (R_1(M)_i)_{i\in\{2,3,4\}} \in \mathcal{E}_3^H,$$

where H is the subgroup of G generated by $\bigcup_{i=2}^4 \operatorname{supp} R_1(M)_i$. We record that

$$M_1' \simeq M_1 \otimes M_2, \qquad M_2' \simeq M_3, \qquad M_3' \simeq M_4.$$

We now apply Theorem 2.5 for $\theta = 3$. This is possible since the proof uses no results on tuples in $\mathcal{F}_n^G$, $n \geq 4$. Since $c_{M'_3,M'_2} c_{M'_2,M'_3} \neq \mathrm{id}_{M'_2 \otimes M'_3}$, according to Theorem 2.5 and the equalities $\dim M'_1 = \dim M'_2 = 1$ we conclude that either $c_{M'_2,M'_1} c_{M'_1,M'_2} = \mathrm{id}_{M'_1 \otimes M'_2}$ or (M'_1, M'_2, M'_3) has a skeleton of type β'_3. This implies that

$$\rho'\rho(s)\sigma(rr') = 1 \quad \text{or} \quad \rho'\rho(r'r)\rho'\rho(s)\sigma(rr') = 1.$$

The first case is impossible since $\rho(s)\sigma(r) \neq 1$ and $\rho'(s)\sigma(r') = 1$. Thus $\rho'(r'r)\rho(r') = 1$.

Since $a_{21}^M = -1$, we know that $\rho(r) = -1$ or $\rho(rr')\rho'(r) = 1$. Assume first that $\rho(rr')\rho'(r) = 1$. Then Propositions 8.5 and 8.7 imply that there is a finite Cartan graph with two points corresponding to the skeletons of M and $R_4(M)$, respectively, such that the Cartan matrices of these points are

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -2 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -2 & 2 \end{pmatrix}.$$

However, by [16, Thm. 5.4] there is no such finite Cartan graph, which establishes the desired contradiction.

Assume now that $\rho'(r)\rho(r'r) \neq 1$ and $\rho(r) = -1$. Since $(3)_{-\rho(r)} = 0$, this implies that $\operatorname{char} \mathbb{K} = 3$. Let $M'' = (R_2(M)_1, R_2(M)_3, R_2(M)_4)$ and let now H be the subgroup of G generated by $\operatorname{supp} M''$. Since $R_2(M) \in \mathcal{E}_4^G$ and $\dim R_2(M)_2 = 1$, we conclude that $M'' \in \mathcal{E}_3^H$. Moreover,

$$M''_1 \simeq M_1 \otimes M_2, \quad M''_2 \simeq M_2 \otimes M_3, \quad M''_3 \simeq M_4.$$

Since

$$\rho'\rho(rs)\rho\sigma(r'r) = \rho'(r)\rho(r'r) \neq 1,$$

the tuple M'' is braid-indecomposable. From Theorem 2.5 for $\theta = 3$ and from the facts that $\dim M''_1 = \dim M''_2 = 1$ and $\rho\sigma(rs) = -1$ we conclude that $\rho'\rho(rs)\rho\sigma(r'r) = -1$. This immediately implies that $\rho'(r)\rho(r') = 1$, which contradicts $a_{12}^M \neq 0$. Thus $\theta \neq 4$ and the proof is complete. □

Proof of Theorem 2.9: $\dim M_1 > 1$.

(1)⇒(3),(4). Since $M \in \mathcal{E}_\theta^G$ has a skeleton of type β_θ, Proposition 8.4 implies that M admits all reflections and $\mathcal{W}(M)$ is standard of type B_θ. Lemma 8.1 implies that $\mathcal{B}(M_i)$ is finite-dimensional for all $i \in \{1, \dots, \theta\}$. More precisely,

$$\mathcal{H}_{\mathcal{B}(M_i)}(t) = (2)_t^2, \quad \mathcal{H}_{\mathcal{B}(M_\theta)}(t) = (3)_t^2.$$

Now Theorem 1.2 implies that $\mathcal{B}(M)$ is finite-dimensional with the claimed Hilbert series.

(4)⇒(2). Since $\dim \mathcal{B}(M) < \infty$, the tuple M admits all reflections by [7, Cor. 3.18] and the Weyl groupoid is finite by [7, Prop. 3.23].

(2)⇒(1). It is assumed that $\dim M_1 > 1$, M admits all reflections, A^M is of type B_θ, and $\mathcal{W}(M)$ is finite. Thus Theorem 1.1 shows that $\mathcal{C}(M)$ is a connected indecomposable finite Cartan graph.

Since $\theta \geq 3$ and $\dim M_1 > 1$, it follows from Lemmas 5.12 and 5.13 that $\operatorname{supp} M_1$ and $\operatorname{supp} M_2$ do not commute and

$$\dim M_1 = \dim M_2 = |\operatorname{supp} M_1| = |\operatorname{supp} M_2| = 2.$$

Let H be the subgroup generated by $\bigcup_{i=2}^{\theta} \operatorname{supp} M_i$. Then Lemma 5.14 implies that $M' = (\operatorname{Res}_H^G M_i)_{2\leq i\leq \theta} \in \mathcal{E}_{\theta-1}^H$.

Assume first that $\theta = 3$. Let $r \in \operatorname{supp} M_1$, $s \in \operatorname{supp} M_2$, and $t \in \operatorname{supp} M_3$. Since $a_{13}^M = 0$, we conclude that

$$s \triangleright t = r \triangleright (s \triangleright t) = (r \triangleright s) \triangleright (r \triangleright t) = (r \triangleright s) \triangleright t$$

(recall that $\triangleright$ means conjugation: $s \triangleright t = sts^{-1}$). Since $s \neq r \triangleright s$, this means that both elements of $\operatorname{supp} M_2$ act in the same way on $\operatorname{supp} M_3$. Then [28, Thm. 2.1] implies that $\operatorname{char} \mathbb{K} = 3$, $\dim M_3 = |\operatorname{supp} M_3| = 2$, and that the conditions in Lemma 8.1(2) hold. Consequently, Lemma 8.1 implies (1).

Assume now that $\theta > 3$. Since $\dim M_2 > 1$, the claim for $\theta - 1$ implies that $\operatorname{char} \mathbb{K} = 3$ and M' has a skeleton of type $\beta_{\theta-1}$. In particular, by Lemma 8.1 there exist $s_2, \dots, s_\theta, \epsilon \in G$ such that $\epsilon s_i = s_i \epsilon$ and $s_i^H = \{s_i, \epsilon s_i\}$ for $2 \leq i \leq \theta$, where $H \subseteq G$ is the subgroup generated by $s_2, \dots, s_\theta, \epsilon$. Let $s_1 \in \operatorname{supp} M_1$. Since $s_1 s_2 \neq s_2 s_1$ and $\epsilon^2 = 1$, we infer from Lemma 5.1 that $\operatorname{supp} M_1 = \{s_1, \epsilon s_1\}$ and $s_1 \epsilon = \epsilon s_1$. Since G is generated by $s_1, \dots, s_\theta, \epsilon$, we conclude that $\epsilon \in Z(G)$. In order to prove that M has a skeleton of type β_θ, one has to check conditions (8.1)–(8.6) in Lemma 8.1 for $i = 1$. These follow from Lemmas A.3, A.4, and 5.5.

(3)⇒(2) is clear. □

9. Proof of Theorem 2.10: The case F_4

In this section we require all the assumptions of Theorem 2.10 to hold. Thus let G be a non-abelian group and let $M = (M_1, M_2, M_3, M_4) \in \mathcal{E}_4^G$. Assume that A^M is a Cartan matrix of type F_4. More precisely,

$$a_{12}^M = a_{21}^M = a_{23}^M = a_{34}^M = a_{43}^M = -1, \qquad a_{32}^M = -2,$$

and other a_{ij}^M with $i \neq j$ are zero.

Lemma 9.1. *Let $H = \langle \bigcup_{i=2}^4 \operatorname{supp} M_i \rangle$ and $M' = (\operatorname{Res}_H^G M_i)_{2\leq i\leq 4}$. Then H is non-abelian, $M' \in \mathcal{E}_3^H$ and $A^{M'}$ is of type C_3. Moreover, $\dim M_1 = 1$.*

Proof. Lemma 5.14 implies that $M' \in \mathcal{E}_3^H$ and that H is non-abelian. Since A^M is of type F_4, we conclude that $A^{M'}$ is of type C_3.

Since H is non-abelian, Lemma 7.5 for M' implies that $\dim M_2 = 1$. Therefore $\operatorname{supp} M_2$ commutes with $\operatorname{supp} M_1$, and hence $\dim M_1 = 1$ by Lemma 5.12. □

The skeleton of type φ_4 is described in the following lemma.

Lemma 9.2. *Assume that* char $\mathbb{K} \neq 2$. *Let* $N \in \mathcal{F}_4^G$. *The following are equivalent:*

(1) *N has a skeleton of type* φ_4.
(2) *There exists* $\epsilon \in Z(G)$ *with* $\epsilon^2 = 1$ *and for all* $i \in \{1, \dots, 4\}$ *and all* $s_i \in \operatorname{supp} M_i$ *there exists a unique character* σ_i *of* G^{s_i} *such that* $\operatorname{supp} M_i = \{s_i\}$ *for* $i \in \{1, 2\}$, $\operatorname{supp} M_i = \{s_i, \epsilon s_i\}$ *for* $i \in \{3, 4\}$, $M_i \simeq M(s_i, \sigma_i)$ *for all* $i \in \{1, \dots, 4\}$, *and*

$$\sigma_1(s_1) = \sigma_2(s_2) = \sigma_3(s_3) = \sigma_4(s_4) = -1, \tag{9.1}$$
$$\sigma_4(\epsilon s_3^2)\sigma_3(\epsilon s_4^2) = 1, \tag{9.2}$$
$$\sigma_4(s_1)\sigma_1(s_4) = \sigma_3(s_1)\sigma_1(s_3) = \sigma_4(s_2)\sigma_2(s_4) = 1, \tag{9.3}$$
$$\sigma_3(s_2)\sigma_2(s_3) = -1, \tag{9.4}$$
$$\sigma_1(s_2)\sigma_2(s_1) = -1, \tag{9.5}$$
$$s_3 s_4 = \epsilon s_4 s_3. \tag{9.6}$$

Proof. Suppose that N has a skeleton of type φ_4. Then A^N is of type F_4. Lemma 5.2(1) implies now the existence of ϵ such that (9.6) holds and the supports of M_3, M_4 are of the given form. Since $A^{(N_3,N_4)}$ is of type A_2, Corollary A.7 implies (9.2) and $\sigma_4(s_4) = \sigma_3(s_3) = -1$. The remaining conditions in (9.1) and (9.4), (9.5) hold by the definition of the skeleton. Now (9.3) follows from Lemma A.2 since $a_{14}^M = a_{13}^M = a_{24}^M = 0$.

The converse follows immediately from the definition of a skeleton of type φ_4 by using Lemmas A.2, A.15, A.16, and Corollary A.7. □

Reflections of the skeleton of type φ_4 are considered in the following lemma.

Proposition 9.3. *Let* $M \in \mathcal{F}_4^G$. *Assume that M has a skeleton $\mathcal{S}$ of type φ_4. Then $\mathcal{S}$ is a skeleton of $R_k(M)$ for all* $k \in \{1, 2, 3, 4\}$.

Proof. By Remark 5.17 it suffices to determine the skeletons of $R_k(M_{i_1}, M_{i_2}, M_{i_3})$, where i_1, i_2, i_3 correspond to three vertices of a connected subgraph of $\mathcal{S}$ and $k \in \{1, 2, 3\}$. There are only two such subgraphs, and hence the proposition follows from Lemmas 8.6 and 7.7. □

Proof of Theorem 2.10. We prove (1)⇒(4)⇒(2)⇒(1) and (1)⇒(3)⇒(2).

(3)⇒(2). This is clear: see e.g. [16, Thm. 3.3].

(1)⇒(3),(4). Since $M \in \mathcal{E}_4^G$ has a skeleton of type φ_4, Proposition 9.3 implies that M admits all reflections and $\mathcal{W}(M)$ is standard of type F_4. The longest element of the Weyl group of type F_4 is

$$s_1 s_2 s_1 s_3 s_2 s_1 s_3 s_2 s_3 s_4 s_3 s_2 s_1 s_3 s_2 s_3 s_4 s_3 s_2 s_1 s_3 s_2 s_3 s_4.$$

The Nichols algebras $\mathcal{B}(M_i)$ are finite-dimensional for $i \in \{1, \dots, 4\}$ and

$$\mathcal{H}_{\mathcal{B}(M_i)}(t) = \begin{cases} (2)_t & \text{if } i \in \{1, 2\}, \\ (2)_t^2 & \text{if } i \in \{3, 4\}. \end{cases}$$

With respect to the Cartan matrix of type F_4 one computes

$$\begin{aligned}
\beta_1 &= \alpha_1, & \beta_2 &= \alpha_1+\alpha_2,\\
\beta_3 &= \alpha_2, & \beta_4 &= \alpha_1+\alpha_2+\alpha_3,\\
\beta_5 &= \alpha_1+2\alpha_2+2\alpha_3, & \beta_6 &= \alpha_1+\alpha_2+2\alpha_3,\\
\beta_7 &= \alpha_2+\alpha_3, & \beta_8 &= \alpha_2+2\alpha_3,\\
\beta_9 &= \alpha_3, & \beta_{10} &= \alpha_1+2\alpha_2+3\alpha_3+\alpha_4,\\
\beta_{11} &= \alpha_1+2\alpha_2+2\alpha_3+\alpha_4, & \beta_{12} &= 2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4,\\
\beta_{13} &= \alpha_1+3\alpha_2+4\alpha_3+2\alpha_4, & \beta_{14} &= \alpha_1+\alpha_2+2\alpha_3+\alpha_4,\\
\beta_{15} &= \alpha_1+2\alpha_2+4\alpha_3+2\alpha_4, & \beta_{16} &= \alpha_2+2\alpha_3+\alpha_4,\\
\beta_{17} &= \alpha_1+2\alpha_2+3\alpha_3+2\alpha_4, & \beta_{18} &= \alpha_1+\alpha_2+\alpha_3+\alpha_4,\\
\beta_{19} &= \alpha_1+2\alpha_2+2\alpha_3+2\alpha_4, & \beta_{20} &= \alpha_1+\alpha_2+2\alpha_3+2\alpha_4,\\
\beta_{21} &= \alpha_2+\alpha_3+\alpha_4, & \beta_{22} &= \alpha_2+2\alpha_3+2\alpha_4,\\
\beta_{23} &= \alpha_3+\alpha_4, & \beta_{24} &= \alpha_4.
\end{aligned}$$

The long and short roots are β_j with

$$j \in \{1,2,3,5,6,8,12,13,15,19,20,22\}$$

and

$$j \in \{4,7,9,10,11,14,16,17,18,21,23,24\},$$

respectively. By Theorem 1.2,

$$\mathcal{B}(M) \simeq \mathcal{B}(M_{\beta_{24}}) \otimes \cdots \otimes \mathcal{B}(M_{\beta_1})$$

as $\mathbb{N}_0^4$-graded objects in ${}^G_G\mathcal{YD}$. Thus a direct calculation shows that $\mathcal{B}(M)$ is finite-dimensional with the claimed Hilbert series.

(4)⇒(2). Since $\dim \mathcal{B}(M) < \infty$, the tuple M admits all reflections by [7, Cor. 3.18] and the Weyl groupoid is finite by [7, Prop. 3.23].

(2)⇒(1). Let H be the subgroup of G generated by $\bigcup_{i=2}^4 \operatorname{supp} M_i$. Define $N = (\operatorname{Res}^G_H M_i)_{i\in\{2,3,4\}}$. Lemma 9.1 implies that $N \in \mathcal{E}_3^H$ and A^N is of type C_3. Therefore, by Theorem 2.7(2)⇒(1), N has a skeleton of type γ_3 and $\operatorname{char}\mathbb{K} \neq 2$. Moreover, $\dim M_1 = 1$ by Lemma 9.1. For all $i \in \{1,2,3,4\}$ let $s_i \in G$ and $\sigma_i \in \widehat{G^{s_i}}$ be such that $M_i \simeq M(s_i,\sigma_i)$. Then $\sigma_2(s_2) = -1$, $\sigma_2(s_3)\sigma_3(s_2) = -1$, and

$$(\sigma_1(s_1)+1)(\sigma_1(s_1s_2)\sigma_2(s_1)-1) = 0, \qquad \sigma_1(s_3)\sigma_3(s_1) = 1 \tag{9.7}$$

by Lemma A.1, since $a_{12}^M = -1$ and $a_{13}^M = 0$. It remains to show that $\sigma_1(s_1) = \sigma_1(s_2)\sigma_2(s_1) = -1$.

Let $M' = R_2(M)$. Then $M'_1 \simeq M(s_2s_1, \sigma_2\sigma_1)$, $M'_3 \simeq M(s_2s_3, \sigma_2\sigma_3)$, and $M'_4 = M_4$ by Lemma A.2. In particular, $\dim M'_1 = 1$, $\dim M'_3 = \dim M'_4 = 2$, and $\operatorname{supp} M'_3$ and $\operatorname{supp} M'_4$ do not commute. Moreover, $a_{14}^{M'} = 0$ by Lemma 5.15. Then $(M'_1, M'_3, M'_4) \in \mathcal{F}_3^{H'}$, where $H' \subseteq G$ is the subgroup generated by $\operatorname{supp} M'_1 \cup \operatorname{supp} M'_3 \cup \operatorname{supp} M'_4$.

The Weyl groupoid of (M_1', M_3', M_4') is finite by assumption. We apply Theorem 2.5 for $\theta = 3$, which is possible, since its proof for $\theta = 3$ does not use anything about θ-tuples with $\theta \geq 4$. We deduce that either $a_{13}^{M'} = 0$, or the triple (M_1', M_3', M_4') has a skeleton of type γ_3. In the second case, necessarily $\sigma_2\sigma_1(s_2s_3)\sigma_2\sigma_3(s_2s_1) = -1$. The equalities $\sigma_2(s_2) = \sigma_2(s_3)\sigma_3(s_2) = -1$ and (9.7) imply that

$$\sigma_2\sigma_1(s_2s_3)\sigma_2\sigma_3(s_2s_1) = -\sigma_1(s_2)\sigma_2(s_1),$$

and hence in the second of the above two cases we necessarily have $\sigma_1(s_2)\sigma_2(s_1) = 1$. Since $\sigma_1(s_2)\sigma_2(s_1) \neq 1$ from $a_{12}^M \neq 0$ and Lemma A.2, we conclude that the second case is impossible, and hence $a_{13}^{M'} = 0$. Then $\sigma_1(s_2)\sigma_2(s_1) = -1$, in which case $\sigma_1(s_1) = -1$ by (9.7). Thus we are done, as said at the end of the previous paragraph. □

10. Proof of Theorem 2.5: The classification

Recall that $\theta \in \mathbb{N}_{\geq 3}$, G is a non-abelian group and $M \in \mathcal{E}_\theta^G$ is a braid-indecomposable tuple.

Proof of Theorem 2.5. (1)⇒(2). Assume that M has a skeleton $\mathcal{S}$ of finite type. If M has a skeleton of type α_θ, δ_θ or ε_θ, then $\dim \mathcal{B}(M) < \infty$ by Theorem 2.6. If M has a skeleton of type γ_θ or φ_4, then $\dim \mathcal{B}(M) < \infty$ by Theorem 2.7 or 2.10, respectively. If M has a skeleton of type β_θ, then $\dim M_1 > 1$ by Lemma 8.1, and hence $\dim \mathcal{B}(M) < \infty$ by Theorem 2.9. If M has a skeleton of type β_3', then $\dim M_1 = 1$ by Lemma 8.2, and so $\dim \mathcal{B}(M) < \infty$ by Theorem 2.8. Finally, if M has a skeleton of type β_3'', then $R_3(M)$ has a skeleton of type β_3' by Proposition 8.7. Hence $\dim \mathcal{B}(R_3(M)) < \infty$. Since $R_3(R_3(M)) \simeq M$ by [7, Thm. 3.12], we conclude from [7, Thm. 1] that $\dim \mathcal{B}(M) = \dim \mathcal{B}(R_3(M)) < \infty$.

(2)⇒(3). Since $\dim \mathcal{B}(M) < \infty$, the tuple M admits all reflections by [7, Cor. 3.18] and the Weyl groupoid is finite by [7, Prop. 3.23].

(3)⇒(1). Recall that M is braid-indecomposable. Suppose that M admits all reflections and $\mathcal{W}(M)$ is finite. Then $\mathcal{C}(M)$ is a connected indecomposable finite Cartan graph by Theorem 1.1. Therefore by Theorem 4.2 there exist $k \in \mathbb{N}_0$ and $i_1, \dots, i_k \in \{1, \dots, \theta\}$ such that A^N is an indecomposable Cartan matrix of finite type for $N = R_{i_1} \cdots R_{i_k}(M)$. The set of all indecomposable Cartan matrices of finite type is well-known: They are of types ADE, B_θ, C_θ, or F_4. By Theorems 2.6–2.10 the tuple N has a skeleton of finite type. Since $M \simeq R_{i_k} \cdots R_{i_1}(N)$, from Propositions 6.4, 7.8, 9.3, 8.4, 8.5, and 8.7 we conclude that M has a skeleton of finite type. □

Appendix A. Reflections of a pair

A.1. For one-dimensional Yetter–Drinfeld modules U, V over a group H, the Yetter–Drinfeld modules $(\operatorname{ad} U)^m(V)$ and $(\operatorname{ad} V)^m(U)$ for $m \geq 1$ are well-known by the theory of Nichols algebras of diagonal type. The following lemma goes back to Rosso [37, Lemma 14].

Lemma A.1 (Rosso). *Let H be a group and let $U, V \in {}^H_H\mathcal{YD}$. Assume that $U \simeq M(r,\rho)$ and $V \simeq M(s,\sigma)$, where $r, s \in Z(H)$ and ρ, σ are characters of H. Then $(\operatorname{ad} U)^m(V) \neq 0$ for a given $m \in \mathbb{N}$ if and only if*

$$(m)^!_{\rho(r)} \prod_{i=0}^{m-1} (\rho(r^i s)\sigma(r) - 1) \neq 0.$$

In that case, $(\operatorname{ad} U)^m(V) \simeq M(r^m s, \sigma_m)$, where σ_m is the character of H given by $\sigma_m(h) = \rho(h)^m \sigma(h)$ for all $h \in H$.

Rosso's lemma is a special case of a more general statement which we prove here.

Lemma A.2. *Let H be a group and let $U, V \in {}^H_H\mathcal{YD}$. Assume that $U \simeq M(r,\rho)$ and $V \simeq M(s,\sigma)$, where $r \in Z(H)$, $s \in H$, $\rho \in \widehat{H}$, and σ is a representation of H^s. Assume also that $\sigma(r)$ is a constant automorphism of V. Then $(\operatorname{ad} U)^m(V) \neq 0$ for a given $m \in \mathbb{N}$ if and only if*

$$(m)^!_{\rho(r)} \prod_{i=0}^{m-1} (1 - \rho(r^i s)\sigma(r)) \neq 0.$$

In that case, $(\operatorname{ad} U)^m(V) \simeq M(r^m s, \sigma_m)$, where σ_m is the representation of H^s given by $\sigma_m(h) = \rho(h)^m \sigma(h)$ for all $h \in H^s$.

Proof. By Lemma 1.3, it suffices to prove the claim for $X_m^{U,V}$ instead of $(\operatorname{ad} U)^m(V)$.

Let $u \in U \setminus \{0\}$ and $v \in V_s \setminus \{0\}$. For all $m \geq 1$ let

$$\gamma_m = (m)_{\rho(r)}\big(1 - \rho(r^{m-1}s)\sigma(r)\big).$$

We will prove that

$$X_m^{U,V} = \gamma_1 \cdots \gamma_m U^{\otimes m} \otimes V \tag{A.1}$$

for all $m \geq 1$. Then $X_m^{U,V} = 0$ if $\gamma_i = 0$ for some $i \in \{1, \dots, m\}$, and otherwise $X_m^{U,V} \simeq M(r^m s, \sigma_m)$. Indeed,

$$X_m^{U,V} = \bigoplus_{t \in \operatorname{supp} V} (U^{\otimes m} \otimes V_t)$$

in the latter case and

$$h(u^{\otimes m} \otimes w) = \rho(h)^m u^{\otimes m} \otimes hw$$

for all $w \in V_s$.

We will prove by induction on m that

$$\varphi_m(u^{\otimes m} \otimes v) = \gamma_m u^{\otimes m} \otimes v \tag{A.2}$$

for all $m \geq 1$ and all $v \in V_s$. This clearly implies (A.1).

Let $v \in V_s$. For $m = 1$ we have $\varphi_1(u \otimes v) = (\operatorname{id} - c^2)(u \otimes v)$ and

$$c^2(u \otimes v) = c(rv \otimes u) = \sigma(r)su \otimes v = \rho(s)\sigma(r)u \otimes v.$$

Therefore $\varphi_1(u \otimes v) = \gamma_1 u \otimes v$. Assume now that (A.2) holds for some $m \geq 1$. Then

$$\begin{aligned}\varphi_{m+1}(u^{\otimes m+1} \otimes v) &= u^{\otimes m+1} \otimes v - c^2(u \otimes (u^{\otimes m} \otimes v))\\ &\quad + (\mathrm{id} \otimes \varphi_m)c_{12}(u \otimes u \otimes (u^{\otimes m-1} \otimes v))\\ &= \big((1-\rho(r)^m\sigma(r)\rho(r^m s)) + \rho(r)\gamma_m\big)u^{\otimes m+1} \otimes v\\ &= \gamma_{m+1}u^{\otimes m+1} \otimes v.\end{aligned}$$

This proves the lemma. □

A.2. In this section we collect some auxiliary results of [25, §4] regarding reflections. Let G be a non-abelian group.

Let $g, h, \epsilon \in G$. Assume that $|g^G| = |h^G| = 2$, $gh \neq hg$, and $gh = \epsilon hg$. By Lemma 5.2 the subgroup $\langle g, h, \epsilon\rangle$ of G is an epimorphic image of Γ_2. Let $V, W \in {}^G_G\mathcal{YD}$ with $V \simeq M(g,\rho)$ and $W \simeq M(h,\sigma)$, where $\rho \in \widehat{G^g}$ and $\sigma \in \widehat{G^h}$. Let $v \in V_g \setminus \{0\}$. Then $\{v, hv\}$ is a basis of V. The degrees of these basis vectors are g and ϵg, respectively. Similarly let $w \in W_h \setminus \{0\}$. Then $\{w, gw\}$ is a basis of W and the degrees of these basis vectors are h and ϵh, respectively. In particular, $\mathrm{Res}^G_{\langle g,h,\epsilon\rangle} V$ and $\mathrm{Res}^G_{\langle g,h,\epsilon\rangle} W$ are absolutely simple Yetter–Drinfeld modules over $\langle g, h, \epsilon\rangle$. Since z acts on $V^{\otimes m} \otimes W^{\otimes n}$ for $z \in G^g \cap G^h$ and $m, n \in \mathbb{N}_0$ by $\rho(z)^m\sigma(z)^n\mathrm{id}$, the following claims follow directly from the corresponding results in [25].

Lemma A.3 ([25, Lemma 4.1]).

(1) $X_1^{V,W} \neq 0$. *Moreover,* $X_1^{V,W}$ *is absolutely simple if and only if* $\rho(\epsilon h^2)\sigma(\epsilon g^2) = 1$.
(2) *Assume that* $\rho(\epsilon h^2)\sigma(\epsilon g^2) = 1$. *Then* $X_1^{V,W} \simeq M(gh, \widetilde{\sigma})$, *where* $\widetilde{\sigma} \in \widehat{G^{gh}} = \langle\{gh\} \cup (G^g \cap G^h)\rangle$ *with* $\widetilde{\sigma}(gh) = -\rho(g)\sigma(h)$, *and* $\widetilde{\sigma}(z) = \rho(z)\sigma(z)$ *for all* $z \in G^g \cap G^h$.

Lemma A.4 ([25, Lemma 4.2]). *Assume that* $\rho(\epsilon h^2)\sigma(\epsilon g^2) = 1$.

(1) $X_2^{V,W} = 0$ *if and only if* $\rho(g) = -1$.
(2) $X_2^{V,W}$ *is absolutely simple if and only if* $\rho(g) = 1$ *and* $\mathrm{char}\,\mathbb{K} \neq 2$.

Lemma A.5 ([25, Lemma 4.3]). *Assume that* $\rho(\epsilon h^2)\sigma(\epsilon g^2) = 1$, $\rho(g) = 1$ *and* $\mathrm{char}\,\mathbb{K} \neq 2$. *Let* $n \in \mathbb{N}$.

(1) *If* $n \geq 3$ *then* $X_n^{V,W} = 0$ *if and only if* $0 < \mathrm{char}\,\mathbb{K} \leq n$.
(2) *If* $n \geq 1$ *and* $X_n^{V,W} \neq 0$ *then* $X_n^{V,W} \simeq M(g^n h, \widetilde{\sigma})$, *where* $\widetilde{\sigma}$ *is a character of* $G^{g^n h} = \langle\{g^n h\} \cup (G^g \cap G^h)\rangle$ *with* $\widetilde{\sigma}(g^n h) = (-1)^n\sigma(h)$ *and* $\widetilde{\sigma}(z) = \rho(z)^n\sigma(z)$ *for all* $z \in G^g \cap G^h$.

With the previous calculations and exchanging V and W one immediately obtains the following lemma (see [25, Prop. 4.4]).

Lemma A.6. *The Yetter–Drinfeld modules* $(\mathrm{ad}\,V)^m(W)$ *and* $(\mathrm{ad}\,W)^m(V)$ *are absolutely simple or zero for all* $m \geq 0$ *if and only if* $\rho(\epsilon h^2)\sigma(\epsilon g^2) = 1$ *and* $\rho(g)^2 = \sigma(h)^2 = 1$. *In this case, the non-diagonal entries of the Cartan matrix* $A^{(V,W)}$ *are*

$$a_{12}^{(V,W)} = \begin{cases} -1 & \text{if } \rho(g) = -1, \\ 1-p & \text{if } \rho(g) = 1 \text{ and } \operatorname{char} \mathbb{K} = p > 2, \end{cases}$$

and otherwise $(\mathrm{ad}\,V)^m(W) \neq 0$ *for all* $m \geq 0$, *and similarly*

$$a_{21}^{(V,W)} = \begin{cases} -1 & \text{if } \sigma(h) = -1, \\ 1-p & \text{if } \sigma(h) = 1 \text{ and } \operatorname{char} \mathbb{K} = p > 2, \end{cases}$$

and otherwise $(\mathrm{ad}\,W)^m(V) \neq 0$ *for all* $m \geq 0$.

Corollary A.7. *Let* V, W *be as above.*

(1) $a_{12}^{(V,W)} = a_{21}^{(V,W)} = -1$ *if and only if* $\rho(\epsilon h^2)\sigma(\epsilon g^2) = 1$ *and* $\rho(g) = \sigma(h) = -1$.
(2) $a_{12}^{(V,W)} = -1$, $a_{21}^{(V,W)} = -2$ *if and only if* $\rho(\epsilon h^2)\sigma(\epsilon g^2) = 1$, $\rho(g) = -1$, $\sigma(h) = 1$, *and* $\operatorname{char} \mathbb{K} = 3$.

Proof. The "if" parts follow directly from Lemma A.6.

For the "only if" parts observe first that $a_{12}^{(V,W)} = -1$, $a_{21}^{(V,W)} \geq -2$ imply that $(\mathrm{ad}\,V)(W)$ and $(\mathrm{ad}\,W)^m(V)$ with $0 \leq m \leq -a_{21}^{(V,W)}$ are absolutely simple by Proposition 5.9. Then $\rho(\epsilon h^2)\sigma(\epsilon g^2) = 1$ by Lemma A.3, and the "only if" parts of (1) and (2) follow from Lemmas A.4 and A.5. □

Finally, to compute the reflections of the pair (V, W) one has the following lemma.

Lemma A.8 ([25, Lemma 4.5]). *Assume that*

$$\rho(\epsilon h^2)\sigma(\epsilon g^2) = 1, \qquad \rho(g)^2 = \sigma(h)^2 = 1,$$

and $\rho(g) = -1$ *if* $\operatorname{char} \mathbb{K} = 0$. *Let* $m = 1$ *if* $\rho(g) = -1$, *and let* $m = p - 1$ *if* $\rho(g) = 1$ *and* $\operatorname{char} \mathbb{K} = p > 0$. *Let* $g' = g^{-1}$ *and* $h' = g^m h$. *Then*

$$|g'^G| = |h'^G| = 2, \qquad g'h' \neq h'g', \qquad g'h' = \epsilon h'g', \qquad G^g \cap G^h = G^{g'} \cap G^{h'}.$$

Moreover, $R_1(V, W) = (V', W')$ *with* $V' \simeq M(g', \rho')$ *and* $W' \simeq M(h', \sigma')$, *where* $\rho' \in \widehat{G^{g'}}$ *and* $\sigma' \in \widehat{G^{h'}}$ *with*

$$\rho'(\epsilon h'^2)\sigma'(\epsilon g'^2) = 1, \qquad \rho'(g') = \rho(g), \qquad \sigma'(h') = \sigma(h),$$

and $\rho'(z) = \rho(z)^{-1}$, $\sigma'(z) = \rho(z)^m \sigma(z)$ *for all* $z \in G^g \cap G^h$.

A.3. Here we recall results on particular pairs of Yetter–Drinfeld modules which play an important role in the study of skeletons of type β'_θ and β''_θ.

By Proposition 5.9, for any pair $(U, V) \in \mathcal{F}_2^G$ the Yetter–Drinfeld modules $(\mathrm{ad}\,U)^m(V)$ and $(\mathrm{ad}\,V)^m(U)$ are absolutely simple or zero if $a_{12}^{(U,V)} a_{21}^{(U,V)}$ is one of 0, 1, 2. Therefore Lemmas A.10 and A.13 below are special cases of [28, Prop. 6.6] and [28, Prop. 4.12], respectively.

Lemma A.9. *Let $t, t' \in G$. Assume that $tt' \neq t't$, $|t^G| = 3$, and $t^G = t'^G$. Let $\epsilon \in G$ be such that $t' = \epsilon t$. Then $\epsilon^3 = 1$, $t\epsilon = \epsilon^{-1}t$, and $t^G = \{t, \epsilon t, \epsilon^2 t\}$.*

Proof. Since $tt' \neq t't$, we conclude that $t\epsilon \neq \epsilon t$. Therefore ϵ commutes neither with t nor with ϵt. Let $t'' \in t^G$ be such that $t^G = \{t, \epsilon t, t''\}$. Then $\epsilon t \epsilon^{-1} \notin \{t, \epsilon t\}$, and hence $\epsilon t \epsilon^{-1} = t''$. Thus conjugation by ϵ permutes t^G via $t \mapsto t''$, $t'' \mapsto t'$, $t' \mapsto t$. Hence $\epsilon^2 t \epsilon^{-1} = \epsilon t' \epsilon^{-1} = t$. Then $t'' = \epsilon t \epsilon^{-1} = \epsilon^{-1} t$ and $\epsilon t = \epsilon t'' \epsilon^{-1} = t\epsilon^{-1} = \epsilon^{-2} t$. Thus $\epsilon^3 = 1$, which implies the rest. □

Lemma A.10. *Let $s \in Z(G)$ and $t, \epsilon \in G$ be such that $\epsilon^3 = 1$, $\epsilon \neq 1$, $t\epsilon = \epsilon^{-1}t$, and $|t^G| = 3$. Let $\sigma \in \widehat{G}$ and $\tau \in \widehat{G^t}$, and let $U, V \in {}_G^G\mathcal{YD}$ be such that $U \simeq M(s, \sigma)$ and $V \simeq M(t, \tau)$. Then $a_{12}^{(U,V)} = -1$ and $a_{21}^{(U,V)} = -2$ if and only if*

$$\tau(t) = -1, \quad (3)_{-\sigma(t)\tau(s)} = 0, \quad (1 + \sigma(s))(1 - \sigma(st)\tau(s)) = 0.$$

Proof. The assumptions imply that $\langle t, \epsilon, s\rangle$ is a non-abelian epimorphic image of Γ_3. By Lemma A.2, $a_{12}^{(U,V)} = -1$ if and only if $\sigma(s)\tau(t) \neq 1$ and $(1+\sigma(s))(1-\sigma(st)\tau(s)) = 0$. The rest follows from [28, Lemmas 6.2 and 6.3] since $a_{21}^{(U,V)} = -2$ implies that $(\operatorname{ad} V)^2(U) = R_2(U, V)_1$ is absolutely simple. □

Proposition A.11. *Let $s \in Z(G)$ and $t, \epsilon \in G$ be such that $\epsilon^3 = 1$, $\epsilon \neq 1$, $t\epsilon = \epsilon^{-1}t$, and $|t^G| = 3$. Let $\sigma \in \widehat{G}$ and $\tau \in \widehat{G^t}$ be such that*

$$\tau(t) = -1, \quad (3)_{-\sigma(t)\tau(s)} = 0, \quad \sigma(st)\tau(s) = 1,$$

and let $U, V, U', V' \in {}_G^G\mathcal{YD}$ such that $U \simeq M(s, \sigma)$, $V \simeq M(t, \tau)$, and $(U', V') = R_2(U, V)$. Then $U' \simeq M(s', \sigma')$ and $V' \simeq M(t^{-1}, \tau^*)$, where $s' = \epsilon s t^2$ and $\sigma' \in \widehat{G^\epsilon}$ is such that $\sigma'(\epsilon) = (\sigma(t)\tau(s))^2$ and $\sigma'(h) = \tau(h)^2\sigma(h)$ for all $h \in G^t \cap G^\epsilon$. Moreover, $\epsilon^G = \{\epsilon, \epsilon^{-1}\}$, $t^2 \in Z(G)$, and

$$\sigma'(\epsilon t^{-2})\tau^*(\epsilon s'^2) = 1, \quad \sigma'(s') = -1, \quad \tau^*(t^{-1}) = -1.$$

Proof. First we prove that $\epsilon^G = \{\epsilon, \epsilon^{-1}\}$ and $t^2 \in Z(G)$. Indeed, the assumptions imply that $t^G = \{t, \epsilon t, \epsilon^2 t\}$, and hence $G = \langle t, \epsilon, G^t \cap G^\epsilon\rangle$. Let H be the subgroup of G generated by s, t, and ϵ. Then $\operatorname{Res}_H^G V$, $\operatorname{Res}_H^G U \in {}_H^H\mathcal{YD}$ are absolutely simple. The calculation of $V^* = R_2(U, V)_2$ is standard. We conclude from [28, Lemmas 6.2 and 6.3] that $a_{21}^{(U,V)} = -2$. From [28, Lemma 6.2] we deduce that $R_2(U, V)_1 \simeq M(s', \sigma')$ and that the remaining claims hold. □

Lemma A.12. *Let $s, t, \epsilon \in G$ be such that $\epsilon \neq 1$, $st \neq ts$, $s^G = \{s, \epsilon s\}$, and $|t^G| = 3$. Then $\epsilon^3 = 1$, $s\epsilon = \epsilon s$, $t\epsilon = \epsilon^{-1}t$, $ts = \epsilon st$, and $t^G = \{t, \epsilon t, \epsilon^2 t\}$. Moreover, $\epsilon^{-1}s \in Z(G)$.*

Proof. We have assumed that $st \neq ts$ and $s^G = \{s, \epsilon s\}$, and hence $ts = \epsilon st$. Thus $\epsilon s = s\epsilon$ and $t\epsilon = \epsilon^{-1}t$ by Lemma 5.1(1). Therefore $s^k t s^{-k} = \epsilon^{-k} t \in t^G$ for all $k \geq 1$, that is, $\epsilon^2 = 1$ or $\epsilon^3 = 1$ because $|t^G| = 3$. To conclude the proof it suffices to show that $\epsilon^3 = 1$ and $\epsilon^{-1}s \in Z(G)$.

Assume to the contrary that $\epsilon^2 = 1$. Let $t' \in t^G \setminus \{t, \epsilon t\}$. Then $st' = t's$ and $\epsilon t' = t'\epsilon$. In particular, t' commutes with s^G, which is a contradiction, since $t' \in t^G$ and t does not commute with s^G.

Finally, Lemma 5.1(3) implies that $(\epsilon^{-1}s)^G = \{\epsilon^{-1}s\}$. □

Lemma A.13. *Let $s, t, \epsilon \in G$ be as in Lemma A.12. Let $\sigma \in \widehat{G^s}$ and $\tau \in \widehat{G^t}$, and let $U, V \in {}^G_G\mathcal{YD}$ be such that $U \simeq M(s, \sigma)$ and $V \simeq M(t, \tau)$. Then $a_{12}^{(U,V)} = -1$ and $a_{21}^{(U,V)} = -2$ if and only if*

$$\sigma(\epsilon t^2)\tau(\epsilon s^2) = 1, \quad \sigma(s) = -1, \quad \tau(t) = -1.$$

In that case, if $(3)_{\sigma(\epsilon)} = 0$ then $(\operatorname{ad} U)(V) \simeq M(\epsilon^{-1}st, \tau')$ and $(\operatorname{ad} V)^2(U) \simeq M(\epsilon^{-1}t^2s, \sigma')$, where $\tau' \in \widehat{G^t}$ with $\tau'(t) = \tau(\epsilon s^{-1})\sigma(\epsilon)$ and $\tau'(h) = \sigma(h)\tau(h)$ for all $h \in G^s \cap G^t$, and $\sigma' \in \widehat{G}$ with $\sigma'(\epsilon) = 1$, $\sigma'(t) = -\tau(\epsilon s^{-1})\sigma(\epsilon)$, and $\sigma'(h) = \tau(h)^2\sigma(h)$ for all $h \in G^s \cap G^t$.

Proof. By Lemma A.12, the subgroup $\langle s, t\rangle \subseteq G$ is a non-abelian epimorphic image of Γ_3. Hence U and V satisfy the assumptions of [28, Prop. 4.12] when viewed as Yetter–Drinfeld modules over $\langle s, t\rangle$. This leads to the claim. □

A.4. In this section we study reflections of a particular pair of Yetter–Drinfeld modules. Let G be a group and let $s \in G$. Assume that $|s^G| = 2$. Let $r, \epsilon \in G$ be such that $rs = \epsilon sr$ and $\epsilon \neq 1$.

Let $t \in Z(G)$, $\sigma \in \widehat{G^s}$, and $\tau \in \widehat{G}$. In particular, $\tau(\epsilon) = 1$. Let $V, W \in {}^G_G\mathcal{YD}$ be such that $V \simeq M(s, \sigma)$ and $W \simeq M(t, \tau)$. We determine the Yetter–Drinfeld modules $X_m^{V,W}$ for all $m \geq 1$.

Lemma A.14. *The Yetter–Drinfeld module $X_1^{V,W}$ is non-zero if and only if $\sigma(t)\tau(s) \neq 1$. In this case, $X_1^{V,W} \simeq M(st, \tau_1)$, where τ_1 is the character of $G^s = G^{st}$ with $\tau_1(h) = \sigma(h)\tau(h)$ for all $h \in G^s$.*

Proof. Let $v \in V_s$ and $w \in W$ be non-zero. Since $G \triangleright (s, t) = s^G \times \{t\}$, the element $(\mathrm{id} - c_{W,V}c_{V,W})(v \otimes w) = (1 - \sigma(t)\tau(s))v \otimes w$ generates $X_1^{V,W}$ as a $\mathbb{K}G$-module. This implies the claim. □

Lemma A.15. *Assume that $\sigma(t)\tau(s) \neq 1$. Then $X_2^{V,W} \neq 0$. Moreover, $X_2^{V,W}$ is absolutely simple if and only if one of the following holds:*

(1) $\sigma(\epsilon^2) = 1$, $(1 + \sigma(s))(1 - \sigma(st)\tau(s)) = 0$.
(2) $\sigma(s) = -1$, $\sigma(\epsilon^2 t^2)\tau(s^2) = 1$.
(3) $\sigma(st)\tau(s) = 1$, $\sigma(\epsilon^2 s^2) = 1$.

In that case, let $\lambda = -\sigma(\epsilon)$ if (1) holds, $\lambda = \sigma(\epsilon t)\tau(s)$ if (2) holds, and $\lambda = \sigma(\epsilon s)$ if (3) holds. Then $X_2^{V,W} \simeq M(\epsilon s^2 t, \tau_2)$, where $\tau_2 \in \widehat{G}$ with

$$\tau_2(r) = \lambda\sigma(r^2)\tau(r), \qquad \tau_2(g) = \sigma(g r^{-1} g r)\tau(g)$$

for all $g \in G^s$, and $w_2 = v \otimes rv \otimes w + \lambda rv \otimes v \otimes w$ is a basis of $X_2^{V,W}$.

Proof. Let $w_1 = v \otimes w$. By the proof of Lemma A.14, $w_1 \in (X_1^{V,W})_{st}$ generates $X_1^{V,W}$ as a $\mathbb{K}G$-module. Since $s^G \times (st)^G = G \triangleright (s, st) \cup G \triangleright (s, \epsilon st)$, the vectors $\varphi_2(v \otimes w_1)$ and $\varphi_2(v \otimes rw_1)$ generate the $\mathbb{K}G$-module $X_2^{V,W}$.

Let $w_2' = \varphi_2(v \otimes rw_1)$. Since

$$\begin{aligned}\varphi_2(v \otimes rw_1) &= v \otimes rw_1 - \epsilon stv \otimes srw_1 + \tau(r)(\mathrm{id} \otimes \varphi_1)(srv \otimes v \otimes w)\\ &= (1 - \sigma(\epsilon^2 s^2 t)\tau(s))v \otimes rw_1 + \sigma(\epsilon s)\tau(r)(1 - \sigma(t)\tau(s))rv \otimes w_1,\end{aligned}$$

we conclude that $w_2' \neq 0$, and hence $X_2^{V,W} \neq 0$.

Assume that $X_2^{V,W}$ is absolutely simple. Since

$$\begin{aligned}\varphi_2(v \otimes w_1) &= v \otimes w_1 - stv \otimes sw_1 + (\mathrm{id} \otimes \varphi_1)(sv \otimes v \otimes w)\\ &= (1 + \sigma(s))(1 - \sigma(st)\tau(s))v \otimes w_1,\end{aligned}$$

and $\varphi_2(v \otimes w_1) \in (X_2^{V,W})_{s^2 t}$, $w_2' \in (X_2^{V,W})_{\epsilon s^2 t}$, and $(s^2 t)^G \neq (\epsilon s^2 t)^G$ by Lemma 5.1(3), we conclude that

$$(1 + \sigma(s))(1 - \sigma(st)\tau(s)) = 0. \tag{A.3}$$

Also, the tensors $v \otimes rv \otimes w$, $rv \otimes v \otimes w$ form a basis of $(V \otimes V \otimes W)_{\epsilon s^2 t}$, and hence

$$gu = \sigma(g r^{-1} g r)\tau(g)u \quad \text{for all } u \in (V \otimes V \otimes W)_{\epsilon s^2 t},\ g \in G^s.$$

Since $G = G^s \cup rG^s$, the modules

$$\mathbb{K}(v \otimes rv + rv \otimes v) \otimes w, \qquad \mathbb{K}(v \otimes rv - rv \otimes v) \otimes w$$

are the only simple Yetter–Drinfeld submodules of $(V \otimes V \otimes W)_{\epsilon s^2 t}$. Thus, w_2' has to span one of these submodules, that is,

$$1 - \sigma(\epsilon^2 s^2 t)\tau(s) = \lambda\sigma(\epsilon s)(1 - \sigma(t)\tau(s))$$

for some $\lambda \in \{1, -1\}$. Equivalently,

$$(1 - \lambda\sigma(\epsilon s))(1 + \lambda\sigma(\epsilon st)\tau(s)) = 0 \tag{A.4}$$

for some $\lambda \in \{1, -1\}$. This and (A.3) imply that (1), (2) or (3) holds, and $X_2^{V,W} = \mathbb{K}(v \otimes rv \otimes w + \lambda rv \otimes v \otimes w)$.

Conversely, if one of (1), (2), (3) holds, then $X_2^{V,W} = \mathbb{K}w_2$ by the above calculations, and hence $X_2^{V,W}$ is absolutely simple. The remaining claims also follow similarly. □

Lemma A.16. *Assume that $\sigma(t)\tau(s) \neq 1$ and $X_2^{V,W}$ is absolutely simple. Let τ_3 be the character of G^s and τ_4 be the character of G with*

$$\tau_3(g) = \sigma(g^2 r^{-1} g r)\tau(g), \quad \tau_4(g) = \sigma(g^2 (r^{-1} g r)^2)\tau(g), \quad \tau_4(r) = \sigma(r^4)\tau(r)$$

for all $g \in G^s$. Then:

(1) $X_3^{V,W} = 0$ *if and only if* $\sigma(s) = -1$ *or* $\sigma(\epsilon^2) \neq 1$.

(2) $X_3^{V,W}$ *is absolutely simple if and only if* $\sigma(s) \neq -1$ *and* $\sigma(\epsilon^2) = 1$. *In that case,* $X_3^{V,W} \simeq M(\epsilon s^3 t, \tau_3)$ *and* $X_4^{V,W} \neq 0$.

(3) *Assume that* $\sigma(s) \neq -1$ *and* $\sigma(\epsilon^2) = 1$. *Then* $X_4^{V,W}$ *is absolutely simple if and only if* $(3)_{\sigma(s)} = 0$. *In that case,* $X_4^{V,W} \simeq M(\epsilon^2 s^4 t, \tau_4)$ *and* $X_5^{V,W} = 0$.

(4) *Assume that* $\sigma(\epsilon^2) = 1$ *and* $(3)_{\sigma(s)} = 0$. *Let* w_2 *be as in Lemma* A.15, $w_3 = v \otimes w_2$, *and*

$$w_4 = v \otimes r w_3 + \sigma(r^2)\tau(r) r v \otimes w_3.$$

Then $w_3 \in (X_3^{V,W})_{\epsilon s^3 t}$ *and* $w_4 \in (X_4^{V,W})_{\epsilon^2 s^4 t}$.

Proof. First we calculate that

$$\varphi_3(v \otimes w_2) = (1 + \sigma(s))(1 - \sigma(\epsilon^2 s^3 t)\tau(s)) v \otimes w_2.$$

Hence $\varphi_3(v \otimes w_2) = 0$ if and only if $\sigma(s) = -1$ or $\sigma(\epsilon^2 s^3 t)\tau(s) = 1$. Assume that $\sigma(s) \neq -1$. Since $X_2^{V,W}$ is absolutely simple, Lemma A.15 implies that $\sigma(st)\tau(s) = 1$. Thus $X_3^{V,W} = 0$ if and only if $\sigma(\epsilon^2 s^2) = 1$. Since $\sigma(s)^{-1} = \sigma(t)\tau(s) \neq 1$ and $\sigma(s) \neq -1$ by assumption, Lemma A.15 implies that $\sigma(\epsilon^2 s^2) = 1$ if and only if $\sigma(\epsilon^2) \neq 1$.

Assume now that $\sigma(\epsilon^2) = 1$ and $\sigma(s) \neq -1$. Then $\sigma(st)\tau(s) = 1$ by Lemma A.15. Let $w_3 = v \otimes w_2$. Then $w_3 \in (V^{\otimes 3} \otimes W)_{\epsilon s^3 t}$ and

$$X_3^{V,W} = \mathbb{K} w_3 + \mathbb{K} r w_3 \simeq M(\epsilon s^3 t, \tau_3),$$

since $g w_2 = \sigma(g r^{-1} g r)\tau(g) w_2$ for all $g \in G^s$ by Lemma A.15. Moreover,

$$\begin{aligned}\varphi_4(v \otimes w_3) &= (3)_{\sigma(s)}(1 - \sigma(s^3)) v \otimes w_3,\\ \varphi_4(v \otimes r w_3) &= (1 - \sigma(s^5)) v \otimes r w_3\\ &\quad - \sigma(s r^2)\tau(r)(1 + \sigma(s))(1 - \sigma(s^2)) r v \otimes w_3.\end{aligned}$$

Since $V \otimes V \otimes X_2^{V,W} = X_4' \oplus X_4''$ in ${}^G_G\mathcal{YD}$, where

$$\begin{aligned}X_4' &= v \otimes v \otimes X_2^{V,W} + r v \otimes r v \otimes X_2^{V,W},\\ X_4'' &= v \otimes r v \otimes X_2^{V,W} + r v \otimes v \otimes X_2^{V,W},\end{aligned}$$

similarly to an argument in the proof of Lemma A.15 we conclude that $X_4^{V,W}$ is absolutely simple if and only if $\varphi_4(v \otimes w_3) = 0$ and

$$\varphi_4(v \otimes r w_3) \in \mathbb{K}(v \otimes r v + \lambda r v \otimes v) \otimes w_2$$

for some $\lambda \in \mathbb{K}$ with $\lambda^2 = 1$. This is equivalent to $(3)_{\sigma(s)} = 0$, since then $\varphi_4(v \otimes r w_3) = (1 - \sigma(s)^{-1}) w_4$ and $r w_4 = \sigma(r^4)\tau(r) w_4$. The rest follows easily. □

Now we introduce classes of pairs of absolutely simple Yetter–Drinfeld modules over any group H. They will appear naturally in Corollary A.24 in the classification of specific pairs admitting all reflections.

Definition A.17. Let H be a group. For $i \in \{0, 1\}$ let $\wp_{22,i}^H$ be the class of pairs (V, W) of Yetter–Drinfeld modules over H such that:

(1) $|\operatorname{supp} V| = 2$ and $|\operatorname{supp} W| = 2$.

(2) There exist $s \in \operatorname{supp} V$, $t \in \operatorname{supp} W$, $\sigma \in \widehat{H^s}$, and $\tau \in \widehat{H^t}$ such that $V \simeq M(s, \sigma)$, $W \simeq M(t, \tau)$, and:
   (a) If $i = 0$, then $(\mathrm{id} - c_{W,V} c_{V,W})(V \otimes W) = 0$.
   (b) If $i = 1$, then $\sigma(\epsilon t^2)\tau(\epsilon s^2) = 1$ and $\sigma(s) = \tau(t) = -1$, where $\epsilon \in H$ with $st = \epsilon ts$ and $\epsilon \neq 1$.

Let $\wp_i^H$ for $0 \le i \le 8$ be the class of pairs (V, W) of Yetter–Drinfeld modules over H such that:

(1) $|\operatorname{supp} V| = 2$ and $|\operatorname{supp} W| = 1$.
(2) There exist $s \in \operatorname{supp} V$, $t \in \operatorname{supp} W$, $\sigma \in \widehat{H^s}$, and $\tau \in \widehat{H}$ such that $V \simeq M(s, \sigma)$, $W \simeq M(t, \tau)$, and σ and τ satisfy the conditions in Table 3.

For all $n \in \mathbb{N}$ with $n \ge 2$ let $\wp_1^H(n)$ be the subclass of $\wp_1^H$ of those pairs (V, W) for which additionally $\tau(t)$ is a primitive n-th root of 1.

Table 3. The classes $\wp_i^H$, $0 \le i \le 8$.

i	Conditions on σ and τ
0	$\sigma(t)\tau(s) = 1$
1	$\sigma(\epsilon^2) = 1, \sigma(s) = -1, \sigma(t)\tau(st) = 1, \tau(t) \neq 1$
2	$\sigma(\epsilon^2) = 1, \sigma(s) = -1, \tau(t) = -1, (3)_{\sigma(t)\tau(s)} = 0, \sigma(t)\tau(s) \neq 1$
3	$\sigma(\epsilon^2) = 1, \sigma(s) = -1, (3)_{\sigma(t)\tau(s)} = 0, \tau(t) = -\sigma(t)\tau(s), \sigma(t)\tau(s) \neq 1$
4	$\sigma(\epsilon^2) = 1, (3)_{\sigma(s)} = 0, \sigma(st)\tau(s) = 1, \tau(t) = -1, \sigma(s) \neq 1$
5	$\sigma(\epsilon^2) \neq 1, \sigma(s) = -1, \sigma(\epsilon^2 t^2)\tau(s^2) = 1, \sigma(t)\tau(st) = 1$
6	$\sigma(\epsilon^2) \neq 1, \sigma(s) = -1, \sigma(\epsilon^2 t^2)\tau(s^2) = 1, \tau(t) = -1$
7	$\sigma(\epsilon^2) \neq 1, \sigma(\epsilon^2 s^2) = 1, \sigma(st)\tau(s) = 1, \sigma(t)\tau(st) = 1$
8	$\sigma(\epsilon^2) \neq 1, \sigma(\epsilon^2 s^2) = 1, \sigma(st)\tau(s) = 1, \tau(t) = -1$

We point out that Lemma 5.5 gives a characterization of pairs in $\wp_{22,0}^H$. A characterization of $\wp_{22,1}^H$ was given in Corollary A.7.

The pairs (V, W) in $\wp_{22,j}^H$ for $j \in \{0, 1\}$ and $\wp_i^H$ for $0 \le i \le 8$ satisfy stronger properties. To prove them we need a lemma.

For any group H and any representation ρ of H we write $\mathrm{const}_\rho(H)$ for the normal subgroup of H consisting of those $g \in H$ such that $\rho(g)$ is constant. In particular, $\mathrm{const}_\rho(H) = H$ if $\deg \rho = 1$. The following lemma is probably well-known. It follows directly from the structure theory of Yetter–Drinfeld modules over groups.

Lemma A.18. *Let H be a group and let $V \in {}^H_H\mathcal{YD}$.*

(1) *For all $r \in \operatorname{supp} V$ there exists a representation ρ_r of H^r such that $\bigoplus_{s \in r^H} V_s \simeq M(r, \rho_r)$. These representations are unique up to isomorphism, and $\deg \rho_r = \deg \rho_s$ for all $r, s \in \operatorname{supp} V$ with $s \in r^H$.*
(2) *Let $r \in \operatorname{supp} V$, $h \in \mathrm{const}_{\rho_r}(H^r)$, and $g \in H$. Let $r' = grg^{-1}$ and $h' = ghg^{-1}$. Then $h' \in \mathrm{const}_{\rho_{r'}}(H^{h'})$ and $\rho_r(h) = \rho_{r'}(h')$.*

In the following two propositions we show that the presentation of the pairs in the classes $\wp_{22}^H$ and $\wp_i^H$, $0 \le i \le 8$, in terms of elements of the group H and representations of their centralizers is essentially independent of various choices. This simplifies the discussion of skeletons of tuples considerably.

Proposition A.19. *Let H be a group, $(V, W) \in \wp_{22,1}^H$, $s \in \operatorname{supp} V$, and $t \in \operatorname{supp} W$. Let $\epsilon \in H$ be such that $st = \epsilon ts$.*

(1) *There exist unique characters σ of H^s and τ of H^t such that $V \simeq M(s, \sigma)$ and $W \simeq M(t, \tau)$.*
(2) $s^H = \{s, \epsilon s\}$, $t^H = \{t, \epsilon t\}$, $\epsilon^2 = 1$, $\epsilon \in Z(H)$, $\epsilon \ne 1$.
(3) $\sigma(\epsilon t^2)\tau(\epsilon s^2) = 1$, $\sigma(s) = \tau(t) = -1$.

Proof. By assumption, there exist $s' \in \operatorname{supp} V$, $t' \in \operatorname{supp} W$, and $\epsilon' \in H$ such that $s't' = \epsilon' t's'$ and $\epsilon' \ne 1$. Since $|\operatorname{supp} V| = |\operatorname{supp} W| = 2$ and since $\operatorname{supp} V$, $\operatorname{supp} W$ are conjugacy classes of H, (2) follows from Lemma 5.2(1). In particular, there exists $x \in \langle s, t\rangle$ such that $x \triangleright s' = s$ and $x \triangleright t' = t$. Then $x \triangleright \epsilon' = \epsilon$.

Again by assumption, there exist characters σ' of $H^{s'}$ and τ' of $H^{t'}$ such that $V \simeq M(s', \sigma')$, $W \simeq M(t', \tau')$, and

$$\sigma'(\epsilon' t'^2)\tau'(\epsilon' s'^2) = 1, \qquad \sigma'(s') = \tau'(t') = -1.$$

Then (1) holds by Lemma A.18(1), and the two parts of (3) follow from Lemma A.18(2) with $r = s'$, $g = x$ and $r = t'$, $g = x$, respectively. □

Proposition A.20. *Let H be a group, $i \in \mathbb{Z}$ with $0 \le i \le 8$, $(V, W) \in \wp_i^H$, $s \in \operatorname{supp} V$, and $t \in \operatorname{supp} W$. Let $\epsilon \in H$ be such that $s^H = \{s, \epsilon s\}$.*

(1) *There exist unique characters σ of H^s and τ of H such that $V \simeq M(s, \sigma)$ and $W \simeq M(t, \tau)$.*
(2) *σ and τ satisfy the conditions in Table 3.*
(3) *If $n \in \mathbb{N}$ and $(V, W) \in \wp_1^H(n)$, then $\tau(t)$ is a primitive n-th root of 1.*

Proof. Similar to the proof of Proposition A.19. □

As before, let G be a group, $V, W \in {}^G_G\mathcal{YD}$ with $|\operatorname{supp} V| = 2$ and $|\operatorname{supp} W| = 1$, $s \in \operatorname{supp} V$, $t \in \operatorname{supp} W$, $\epsilon \in G$ with $s^G = \{s, \epsilon s\}$, σ a character of G^s, and τ a character of G. Assume that $V \simeq M(s, \sigma)$ and $W \simeq M(t, \tau)$. Then $\epsilon \ne 1$.

Proposition A.21. *Assume that $\sigma(t)\tau(s) \ne 1$. Then $(\operatorname{ad} V)^m(W)$ and $(\operatorname{ad} W)^m(V)$ are absolutely simple or zero for all $m \in \mathbb{N}$ if and only if:*

(1) $\sigma(\epsilon^2) = 1$, $\sigma(s) = -1$, *or*
$\sigma(\epsilon^2 t^2)\tau(s^2) = 1$, $\sigma(s) = -1$, $\sigma(\epsilon^2) \ne 1$, *or*
$\sigma(\epsilon^2 s^2) = \sigma(st)\tau(s) = 1$, $\sigma(\epsilon^2) \ne 1$, *or*
$\sigma(\epsilon^2) = \sigma(st)\tau(s) = 1$, $(3)_{\sigma(s)} = 0$.
(2) $(n+1)_{\tau(t)}(1 - \sigma(t)\tau(st^n)) = 0$ *for some* $n \ge 1$.

Moreover, the four possibilities in (1) *are mutually exclusive.*

Proof. This follows from Lemmas A.14–A.16 and A.2. □

Proposition A.21 leads to a characterization of those pairs (V, W) which have a finite Weyl groupoid. To obtain this characterization, we need some technicalities. For the definitions of τ_2, τ_4, and σ_n we refer to Lemmas A.15, A.16, and A.2, respectively.

Lemma A.22.

(1) *Assume that* $\sigma(t)\tau(s) \neq 1$, $\sigma(\epsilon^2) = 1$, *and* $\sigma(s) = -1$. *Then* $R_1(V, W) \simeq (M(s^{-1}, \sigma^*), M(\epsilon s^2 t, \tau_2))$ *and*

$$\sigma^*(s^{-1}) = -1, \qquad \sigma^*(\epsilon^{-2}) = 1,$$
$$\sigma^*(\epsilon s^2 t)\tau_2(s^{-1}) = \sigma(t^{-1})\tau(s^{-1}), \qquad \tau_2(\epsilon s^2 t) = \sigma(t^2)\tau(s^2 t).$$

(2) *Assume that* $\sigma(\epsilon^2 t^2)\tau(s^2) = 1$, $\sigma(s) = -1$, *and* $\sigma(\epsilon^2) \neq 1$. *Then* $R_1(V, W) \simeq (M(s^{-1}, \sigma^*), M(\epsilon s^2 t, \tau_2))$ *and*

$$\sigma^*(s^{-1}) = -1, \qquad \sigma^*(\epsilon^{-1}) = \sigma(\epsilon),$$
$$\sigma^*(\epsilon s^2 t)\tau_2(s^{-1}) = \sigma(t)\tau(s), \qquad \tau_2(\epsilon s^2 t) = \tau(t).$$

(3) *Assume that* $\sigma(\epsilon^2 s^2) = 1$, $\sigma(st)\tau(s) = 1$, *and* $\sigma(\epsilon^2) \neq 1$. *Then* $R_1(V, W) \simeq (M(s^{-1}, \sigma^*), M(\epsilon s^2 t, \tau_2))$ *and*

$$\sigma^*(s^{-1}) = \sigma(s), \qquad \sigma^*(\epsilon^{-1}) = \sigma(\epsilon),$$
$$\sigma^*(\epsilon s^2 t)\tau_2(s^{-1}) = \sigma(t)\tau(s), \qquad \tau_2(\epsilon s^2 t) = \tau(t).$$

(4) *Assume that* $\sigma(t)\tau(s) \neq 1$, $\sigma(\epsilon^2) = 1$, $\sigma(st)\tau(s) = 1$, *and* $(3)_{\sigma(s)} = 0$. *Then* $R_1(V, W) \simeq (M(s^{-1}, \sigma^*), M(\epsilon^2 s^4 t, \tau_4))$ *and*

$$\sigma^*(s^{-1}) = \sigma(s), \qquad \sigma^*(\epsilon^{-2}) = 1,$$
$$\sigma^*(\epsilon^2 s^4 t)\tau_4(s^{-1}) = \sigma(t)\tau(s), \qquad \tau_4(\epsilon^2 s^4 t) = \tau(t).$$

(5) *Let* $n \in \mathbb{N}$. *Assume that* $\sigma(t)\tau(st^n) = 1$ *and* $\tau(t^k) \neq 1$ *for all* $1 \leq k \leq n$. *Then* $R_2(V, W) \simeq (M(st^n, \sigma_n), M(t^{-1}, \tau^*))$ *and*

$$\sigma_n(st^n) = \sigma(s), \qquad \sigma_n(\epsilon) = \sigma(\epsilon),$$
$$\sigma_n(t^{-1})\tau^*(st^n) = \sigma(t)\tau(s), \qquad \tau^*(t^{-1}) = \tau(t).$$

(6) *Assume that* $(\sigma(t)\tau(s))^2 \neq 1$ *and* $\tau(t) = -1$. *Then* $R_2(V, W) \simeq (M(st, \sigma_1), M(t^{-1}, \tau^*))$ *and*

$$\sigma_1(st) = -\sigma(st)\tau(s), \qquad \sigma_1(\epsilon^2) = \sigma(\epsilon^2),$$
$$\sigma_1(t^{-1})\tau^*(st) = \sigma(t^{-1})\tau(s^{-1}), \qquad \tau^*(t^{-1}) = -1.$$

Proof. The claims follow from Lemmas A.15, A.16, and A.2. For example, in the first three cases one obtains $X_2^{V,W} \neq 0$, $X_3^{V,W} = 0$, and

$$\sigma^*(s^{-1}) = \sigma(s), \qquad \sigma^*(\epsilon^{-2}) = \sigma(\epsilon^2),$$
$$\sigma^*(\epsilon s^2 t)\tau_2(s^{-1}) = \sigma(\epsilon^{-2}s^{-4}t^{-1})\tau(s^{-1}),$$
$$\tau_2(\epsilon s^2 t) = \sigma(\epsilon^2 s^4 t^2)\tau(s^2 t).$$

The additional assumptions then imply the stated formulas. □

Remark A.23. From Lemmas A.14 and A.22 we obtain the Cartan matrix entries and reflections of the pairs in the classes $\wp_n^G$ for $0 \leq n \leq 8$. We collect these data in Table 4.

Table 4. Reflections of pairs $(V, W) \in \wp_n$.

(V, W)	$a_{12}^{(V,W)}$	$a_{21}^{(V,W)}$	$R_1(V, W)$	$R_2(V, W)$
$\wp_0^G$	0	0	$\wp_0^G$	$\wp_0^G$
$\wp_1^G$	−2	−1	$\wp_1^G$	$\wp_1^G$
$\wp_2^G$	−2	−1	$\wp_3^G$	$\wp_4^G$
$\wp_3^G$	−2	−2	$\wp_2^G$	$\wp_3^G$
$\wp_4^G$	−4	−1	$\wp_4^G$	$\wp_2^G$
$\wp_5^G$	−2	−1	$\wp_5^G$	$\wp_5^G$
$\wp_6^G$	−2	−1	$\wp_6^G$	$\wp_8^G$
$\wp_7^G$	−2	−1	$\wp_7^G$	$\wp_7^G$
$\wp_8^G$	−2	−1	$\wp_8^G$	$\wp_6^G$

Corollary A.24. *The following are equivalent:*

(1) *The pair (V, W) admits all reflections and $\mathcal{W}(V, W)$ is finite.*
(2) *$(V, W) \in \wp_i^G$ for some $0 \leq i \leq 8$.*

If $(V, W) \in \wp_0^G$, then (V, W) is standard of type $A_1 \times A_1$. If $(V, W) \in \wp_i^G$ with $i \in \{1, 5, 6, 7, 8\}$, then (V, W) is standard of type C_2. If $(V, W) \in \wp_i^G$ with $2 \leq i \leq 4$, then $\boldsymbol{\Delta}_+^{\mathrm{re}\,(V,W)}$ can be obtained from [29, Lemma 8.5].

Proof. (2)⇒(1). Since $(V, W) \in \wp_i^G$ for some $0 \leq i \leq 8$, the pair (V, W) admits all reflections by Remark A.23. Moreover, the Weyl groupoid $\mathcal{W}(V, W)$ is finite since the set of roots of (V, W) is finite.

(1)⇒(2). Assume that (V, W) admits all reflections and $\mathcal{W}(V, W)$ is finite. Then $(\operatorname{ad} V)^m(W)$ and $(\operatorname{ad} W)^m(V)$ are absolutely simple or zero for all $m \geq 1$ by Theorem 1.4. Lemmas A.15 and A.16, A.2 imply that all reflections of (V, W) are pairs (V', W') of absolutely simple Yetter–Drinfeld modules such that there exist $s', \epsilon' \in G$, $t' \in Z(G)$, and characters σ' of $G^{s'}$ and τ' of G with $\epsilon' \neq 1$, $s'^G = \{s', \epsilon' s'\}$, $V \simeq M(s', \sigma')$, and $W \simeq M(t', \tau')$. By Theorem 4.2, there exists an object (V', W') of $\mathcal{W}(V, W)$ with Cartan matrix of finite type. By Remark A.23, the reflections R_1 and R_2 induce permutations of the classes $\wp_i^G$ with $0 \le i \le 8$. Hence it suffices to show that $(V, W) \in \wp_i^G$ for some $0 \le i \le 8$ if the Cartan matrix $A^{(V,W)}$ is of finite type.

Assume that $A^{(V,W)}$ is of finite type different from $A_1 \times A_1$. Then $\sigma(t)\tau(s) \neq 1$, and we find that $a_{12}^{(V,W)} \le -2$ by Lemma A.15. Further, $a_{12}^{(V,W)} \in \{-2, -4\}$ by Lemma A.16. Hence $a_{12}^{(V,W)} = -2$ and $a_{21}^{(V,W)} = -1$. Then

$$\sigma(\epsilon^2) = 1, \qquad \sigma(s) = -1$$

or

$$\sigma(\epsilon^2 t^2)\tau(s^2) = 1, \qquad \sigma(s) = -1, \qquad \sigma(\epsilon^2) \neq 1$$

or

$$\sigma(\epsilon^2 s^2) = 1, \qquad \sigma(st)\tau(s) = 1, \qquad \sigma(\epsilon^2) \neq 1$$

by Lemma A.16, and

$$(\tau(t) + 1)(1 - \sigma(t)\tau(st)) = 0$$

by Lemma A.2. By the same lemmas, $R_1(V, W) \simeq (M(s^{-1}, \sigma^*), M(\epsilon s^2 t, \tau_2))$ and $R_2(V, W) \simeq (M(st, \sigma_1), M(t^{-1}, \tau^*))$.

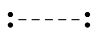
If $\sigma(\epsilon^2) \neq 1$, then $(V, W) \in \wp_i$ for some $5 \le i \le 8$. So assume that $\sigma(\epsilon^2) = 1$ and $\sigma(s) = -1$.

If $\sigma(t)\tau(st) = 1$, then $(V, W) \in \wp_1^G$. Assume now that $\tau(t) = -1$ and $(\sigma(t)\tau(s))^2 \neq 1$. Then Lemma A.22(6) for (V, W) and Proposition A.21 for $R_2(V, W)$ implies that $(3)_{\sigma_1(st)} = (3)_{\sigma(t)\tau(s)} = 0$, since $\sigma_1(st) = \sigma(t)\tau(s) \neq -1$. Thus $(V, W) \in \wp_2^G$. □

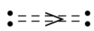
Appendix B. Rank two classification

In this appendix we collect the main results of [29, 30, 28]. The results are presented in the terminology of this paper. Many of the examples will be described using Definition 2.2. However, to include all the Nichols algebras found in [29, 30, 28], one needs to add some additional diagrams.

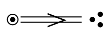
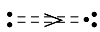
B.1. We first describe the examples related to the group Γ_2 of [28, §1.1]. For the Nichols algebras of dimension 64 one has the following skeleton:



In characteristic three, the pair of Yetter–Drinfeld modules which yields Nichols algebras of dimension 1296 has the following skeleton:

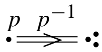
 $\operatorname{char} \mathbb{K} = 3$

B.2. Let us review the examples related to the group Γ_3, see [28, §1.4]. For the Nichols algebras of dimension 2304 related to the group Γ_3, [28, Example 1.11, §1.4], one has the following diagrams related by reflections:

We remark that the diagram on the left is not a skeleton in the sense of Definition 2.2 because the simple Yetter–Drinfeld module $M(s_1, \sigma_1)$ is constructed with a two-dimensional representation σ_1. This situation is described with a double circle at the left vertex of the diagram.

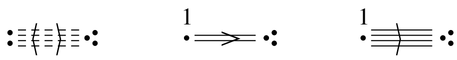
The examples of dimensions 10368, 5184 or 1152 can be described with the following skeleton:



$(3)_{-p} = 0$

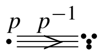
We remark that in this case, an extra assumption on the value of $p = \sigma_1(s_1)$ is needed.

The examples of dimension 2239488 related to the group Γ_3 of [28, Example 1.9, §1.4] can be described with the following diagrams related by reflections:



The diagram on the left is not a skeleton in the sense of Definition 2.2 since it has a double arrow. This double arrow means that the Cartan matrix of the pair satisfies $a_{12}^{(M_1,M_2)} = a_{21}^{(M_1,M_2)} = -2$.

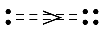
B.3. Nichols algebras related to the group T have dimension 1259712 over fields of characteristic two and 80621568 otherwise (see [28, §1.3]). In this case one has the following skeleton:



$(3)_{-p} = 0$

The dots on the right vertex describe the structure of the support of $M(s_2, \sigma_2)$ which is isomorphic (as a quandle) to the tetrahedron quandle. Further, the assumption $(3)_{-p} = 0$, where $p = \sigma_1(s_1)$, is needed.

B.4. Nichols algebras related to the group Γ_4 have dimension 65536 over fields of characteristic two and 262144 otherwise (see [28, §1.2]). In this case one has the following skeleton:



The four dots in the right vertex mean that the support of $M_2(s_2, \sigma_2)$ is isomorphic (as a quandle) to the dihedral quandle $\mathbb{D}_4$.

B.5. With these diagrams, the classification of finite-dimensional Nichols algebras admitting a finite root system of rank two, [28, Theorem 2.1], can be reformulated as follows.

Theorem. *Let G be a non-abelian group and M in $\mathcal{E}_2^G$. Assume that M is braid-indecomposable. The following are equivalent:*

(1) *M has a skeleton appearing in (B.1)–(B.4).*
(2) *$\mathcal{B}(M)$ is finite-dimensional.*
(3) *M admits all reflections and $\mathcal{W}(M)$ is finite.*

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References

[1] Andruskiewitsch, N.: On finite-dimensional Hopf algebras. Accepted for publication in Proceedings of the International Congress of Mathematicians. arXiv:1403.7838

[2] Andruskiewitsch, N.: About finite dimensional Hopf algebras. In: Quantum Symmetries in Theoretical Physics and Mathematics (Bariloche, 2000), Contemp. Math. 294, Amer. Math. Soc., Providence, RI, 1–57 (2002) Zbl 1135.16306 MR 1907185

[3] Andruskiewitsch, N., Angiono, I. E.: On Nichols algebras with generic braiding. In: Modules and Comodules, Trends Math., Birkhäuser, Basel, 47–64 (2008) Zbl 1227.16022 MR 2742620

[4] Andruskiewitsch, N., Cuadra, J.: On the structure of (co-Frobenius) Hopf algebras. J. Noncommut. Geom. **7**, 83–104 (2013) Zbl 1279.16024 MR 3032811

[5] Andruskiewitsch, N., Fantino, F., Graña, M., Vendramin, L.: Finite-dimensional pointed Hopf algebras with alternating groups are trivial. Ann. Mat. Pura Appl. (4) **190**, 225–245 (2011) Zbl 1234.16019 MR 2786171

[6] Andruskiewitsch, N., Fantino, F., Graña, M., Vendramin, L.: Pointed Hopf algebras over the sporadic simple groups. J. Algebra **325**, 305–320 (2011) Zbl 1217.16026 MR 2745542

[7] Andruskiewitsch, N., Heckenberger, I., Schneider, H.-J.: The Nichols algebra of a semisimple Yetter–Drinfeld module. Amer. J. Math. **132**, 1493–1547 (2010) Zbl 1214.16024 MR 2766176

[8] Andruskiewitsch, N., Schneider, H.-J.: Lifting of quantum linear spaces and pointed Hopf algebras of order p^3. J. Algebra **209**, 658–691 (1998) Zbl 0919.16027 MR 1659895

[9] Andruskiewitsch, N., Schneider, H.-J.: Finite quantum groups and Cartan matrices. Adv. Math. **154**, 1–45 (2000) Zbl 1007.16027 MR 1780094

[10] Andruskiewitsch, N., Schneider, H.-J.: Pointed Hopf algebras. In: New Directions in Hopf Algebras, Math. Sci. Res. Inst. Publ. 43, Cambridge Univ. Press, Cambridge, 1–68 (2002) Zbl 1011.16025 MR 1913436

[11] Andruskiewitsch, N., Schneider, H.-J.: A characterization of quantum groups. J. Reine Angew. Math. **577**, 81–104 (2004) Zbl 1084.16027 MR 2108213

[12] Andruskiewitsch, N., Schneider, H.-J.: On the classification of finite-dimensional pointed Hopf algebras. Ann. of Math. (2) **171**, 375–417 (2010) Zbl 1208.16028 MR 2630042

[13] Angiono, I.: On Nichols algebras of diagonal type. J. Reine Angew. Math. **683**, 189–251 (2013) Zbl 1331.16023 MR 3181554

[14] Angiono, I. E.: A presentation by generators and relations of Nichols algebras of diagonal type and convex orders on root systems. J. Eur. Math. Soc. **17**, 2643–2671 (2015) Zbl 1343.16022 MR 3420518

[15] Bazlov, Y.: Nichols–Woronowicz algebra model for Schubert calculus on Coxeter groups. J. Algebra **297**, 372–399 (2006) Zbl 1101.16027 MR 2209265

[16] Cuntz, M., Heckenberger, I.: Weyl groupoids with at most three objects. J. Pure Appl. Algebra **213**, 1112–1128 (2009) Zbl 1169.20020 MR 2498801

[17] Cuntz, M., Heckenberger, I.: Finite Weyl groupoids of rank three. Trans. Amer. Math. Soc. **364**, 1369–1393 (2012) Zbl 1246.20037 MR 2869179

[18] Cuntz, M., Heckenberger, I.: Finite Weyl groupoids. J. Reine Angew. Math. **702**, 77–108 (2015) Zbl 06435243 MR 3341467

[19] Heckenberger, I.: Weyl equivalence for rank 2 Nichols algebras of diagonal type. Ann. Univ. Ferrara Sez. VII (N.S.) **51**, 281–289 (2005) Zbl 1121.16032 MR 2294771

[20] Heckenberger, I.: The Weyl groupoid of a Nichols algebra of diagonal type. Invent. Math. **164**, 175–188 (2006) Zbl 1174.17011 MR 2207786

[21] Heckenberger, I.: Classification of arithmetic root systems of rank 3. In: Proc. XVIth Latin American Algebra Colloquium (Spanish), Bibl. Rev. Mat. Iberoamericana, Rev. Mat. Iberoamericana, Madrid, 227–252 (2007) Zbl 1197.17003 MR 2500361

[22] Heckenberger, I.: Examples of finite-dimensional rank 2 Nichols algebras of diagonal type. Compos. Math. **143**, 165–190 (2007) Zbl 1169.16027 MR 2295200

[23] Heckenberger, I.: Rank 2 Nichols algebras with finite arithmetic root system. Algebr. Represent. Theory **11**, 115–132 (2008) Zbl 1175.17003 MR 2379892

[24] Heckenberger, I.: Classification of arithmetic root systems. Adv. Math. **220**, 59–124 (2009) Zbl 1176.17011 MR 2462836

[25] Heckenberger, I., Schneider, H.-J.: Nichols algebras over groups with finite root system of rank two I. J. Algebra **324**, 3090–3114 (2010) Zbl 1219.16032 MR 2732989

[26] Heckenberger, I., Schneider, H.-J.: Root systems and Weyl groupoids for Nichols algebras. Proc. London Math. Soc. (3) **101**, 623–654 (2010) Zbl 1210.17014 MR 2734956

[27] Heckenberger, I., Schneider, H.-J.: Right coideal subalgebras of Nichols algebras and the Duflo order on the Weyl groupoid. Israel J. Math. **197**, 139–187 (2013) Zbl 1301.16033 MR 3096611

[28] Heckenberger, I., Vendramin, L.: The classification of Nichols algebras with finite root system of rank two. J. Eur. Math. Soc., to appear; arXiv:1311.2881 (2013)

[29] Heckenberger, I., Vendramin, L.: Nichols algebras over groups with finite root system of rank two II. J. Group Theory **17**, 1009–1034 (2014) Zbl 1305.16026 MR 3276225

[30] Heckenberger, I., Vendramin, L.: Nichols algebras over groups with finite root system of rank two III. J. Algebra **422**, 223–256 (2015) Zbl 1306.16028 MR 3272075

[31] Heckenberger, I., Wang, J.: Rank 2 Nichols algebras of diagonal type over fields of positive characteristic. SIGMA Symmetry Integrability Geom. Methods Appl. **11**, paper 011, 24 pp. (2015) Zbl 1318.16031 MR 3313687

[32] Kac, V. G.: Infinite-Dimensional Lie Algebras. 2nd ed., Cambridge Univ. Press, Cambridge (1985) Zbl 0574.17010 MR 823672

[33] Kharchenko, V. K.: A quantum analogue of the Poincaré–Birkhoff–Witt theorem. Algebra Logika **38**, 476–507, 509 (1999) (in Russian) Zbl 0936.16034 MR 1763385

[34] Lentner, S.: New large-rank Nichols algebras over nonabelian groups with commutator subgroup $\mathbb{Z}_2$. J. Algebra **419**, 1–33 (2014) Zbl 1306.16029 MR 3253277

[35] Majid, S.: Noncommutative differentials and Yang–Mills on permutation groups S_n. In: Hopf Algebras in Noncommutative Geometry and Physics, Lecture Notes in Pure Appl. Math. 239, Dekker, New York, 189–213 (2005) Zbl 1076.58004 MR 2106930

[36] Nichols, W. D.: Bialgebras of type one. Comm. Algebra **6**, 1521–1552 (1978) Zbl 0408.16007 MR 0506406

[37] Rosso, M.: Quantum groups and quantum shuffles. Invent. Math. **133**, 399–416 (1998) Zbl 0912.17005 MR 1632802

[38] Semikhatov, A. M.: Fusion in the entwined category of Yetter–Drinfeld modules of a rank-1 Nichols algebra. Theoret. Math. Phys. **173**, 1329–1358 (2012); Russian version: Teoret. Mat. Fiz. **173**, 3–37 (2012) Zbl 1280.81130 MR 3171534

[39] Semikhatov, A. M., Tipunin, I. Y.: The Nichols algebra of screenings. Comm. Contemp. Math. **14**, 1250029, 66 pp. (2012) Zbl 1264.81285 MR 2965674

[40] Semikhatov, A. M., Tipunin, I. Y.: Logarithmic $\widehat{s\ell}(2)$ CFT models from Nichols algebras: I. J. Phys. A **46**, 494011, 53 pp. (2013) Zbl 1283.81113 MR 3146017

[41] Woronowicz, S. L.: Compact matrix pseudogroups. Comm. Math. Phys. **111**, 613–665 (1987) Zbl 0627.58034 MR 901157

[42] Woronowicz, S. L.: Differential calculus on compact matrix pseudogroups (quantum groups). Comm. Math. Phys. **122**, 125–170 (1989) Zbl 0751.58042 MR 994499