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Qualitative behaviour for flux-saturated mechanisms: travelling waves, waiting time and smoothing effects

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Abstract. This paper is devoted to the analysis of qualitative properties of flux-saturated type operators in dimension one. Specifically, we study regularity properties and smoothing effects, discontinuous interfaces, the existence of travelling wave profiles, sub- and supersolutions and waiting time features. The aim of the paper is to better understand these phenomena through two prototypic operators: the relativistic heat equation and the porous media flux-limited equation. As an important consequence of our results we deduce that solutions to the one-dimensional relativistic heat equation become smooth inside their support in the long time run.

Keywords. Flux limitation, flux saturation, porous media equations, waiting time, pattern formation, travelling waves, optimal mass transportation, entropy solutions, relativistic heat equation, complex systems

1. Introduction, preliminaries and main results

The aim of this paper is to investigate some qualitative properties for a couple of models arising in flux-saturated processes. First we have

$$\frac{\partial u}{\partial t} = \nu \left(\frac{|u|(u^m)_x}{\sqrt{1 + \frac{\nu^2}{c^2}|(u^m)_x|^2}} \right)_x, \qquad \nu, c > 0,\ m \geq 1, \tag{1.1}$$

which combines flux-saturation effects with those of porous media flow. We will refer to this equation as *flux-limited porous medium equation* (FLPME) as it was introduced in [21] (see also [23]). Here u^m inside $(u^m)_x$ is meant to stand for $|u|^m \operatorname{sign}(u)$.

The second problem concerns the so-called *relativistic heat equation* (RHE) [10, 31]:

$$\frac{\partial u}{\partial t} = \nu \left(\frac{|u|u_x}{\sqrt{u^2 + \frac{\nu^2}{c^2}|u_x|^2}} \right)_x, \qquad \nu, c > 0. \tag{1.2}$$

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Both systems can be deduced from optimal mass transportation arguments [1, 22]. They exhibit differences and similarities in their qualitative properties, which makes them prototypes to analyze, compare and better understand the dynamics of flux-saturated mechanisms.

Specifically, this paper deals with different smoothing effects of these flux-saturated mechanisms as well as with finite time extinction of discontinuous interfaces of solutions to the FLPME (while this kind of interfaces are preserved along the evolution of the RHE). Another interesting aspect reported in this paper is a waiting time phenomena for the FLPME. Under some circumstances the support will not spread until a sharp interface is formed by means of a mass redistribution process taking place inside the support. Once this happens, the support will grow at a rate that depends on the parameters of the system. Moreover, there is a family of travelling wave solutions to FLPME that can be used to get accurate information about the aforementioned features.

Several aspects concerning the mathematical theory of flux-saturated mechanisms were introduced in the pioneering works [23, 27, 31]. The theory for the existence of entropy solutions associated to flux-saturated equations has been widely developed in the framework of bounded variation functions [2, 3, 4]. The fact that the propagation speed of discontinuous interfaces is generically given by c has been remarked in [6, 11, 15, 24]; the precise Rankine–Hugoniot characterization of travelling jump discontinuities can be found in [20, 21]. The problem of regularity has been previously treated in [4, 7, 18], while diverse aspects of the waiting time phenomenon are addressed in [7, 18, 24]. Travelling waves associated with various flux-saturated operators have been thoroughly studied in [12, 16, 17]. Applications of these ideas to diverse contexts such as physics, astronomy or biology can be found for instance in [15, 29, 31, 32]. See [13] for a survey of the above topics.

The idea to analyze the regularity of solutions is to transfer the problem to an auxiliary dual problem by using a transformation called "the mass coordinate" of Lagrange [30]. This dual problem has some regularity properties that are typical of uniformly elliptic operators of second order. Lagrange transformations of this type are of relevance in dealing with free boundary problems for nonlinear parabolic PDEs because the support transforms into a known domain (see [9, 26, 30] for references). This change of variables was applied to the RHE in [18]. We are able to extend some of the results in [18], taking advantage of the fact that jump discontinuities determine dynamic regions where the quantity of mass is preserved. This enables us to apply local regularity arguments for each of these regions separately and ultimately to show that there is a global smoothing effect for (1.2) in the long time run that dissolves all singularities of the solution but those at its interface. This program applies to (1.1) only partially, as the use of the dual formulation breaks down when interfaces become continuous. In fact, as shown in Section 3, jump discontinuities (and particularly discontinuous interfaces) disappear in finite time for FLPME.

The paper is structured as follows. Section 2 introduces a family of dual problems that serve as a tool to analyze regularity properties; important differences between (1.1) and (1.2) will become clear at this point. In Section 3 we construct travelling wave solutions to the FLPME, which we use right away to prove that jump discontinuities vanish in finite time. This implies in particular that initially discontinuous interfaces will eventually become continuous, as opposed to the case of the RHE. Section 4 is devoted to constructing sub- and supersolutions of the FLPME which, in particular, imply that waiting time phenomena for support growth are present in many cases. Section 5 concerns smoothing effects for the RHE with a single singularity inside the support of the solution. This study is then used in Section 6 to discuss regularity issues in the case of a finite number of singularities. Finally, Section 7 establishes some regularity properties for the FLPME before interfaces become continuous.

2. The dual problem for the inverse distribution function

In this section we associate to equations (1.1) and (1.2) dual problems that will allow us later on to study local-in-time regularity properties in the interior of the support for both systems. As we proceed, we will compare solutions launched by compactly supported initial conditions that are strictly positive inside their support; we will realize that there are several fundamental differences between their qualitative behaviours. The well-posedness theory of the problems we are interested in was developed in [2, 3] (see also [19, 21] and references therein). This theory develops the concept of *entropy solutions* and shows that this is a well-suited class (in the functional context of *bounded variation* and L^∞ functions) in order to obtain well-posedness results. Although this is the concept of solution we will use in order to deal with (1.1) and (1.2), we refrain from giving details on this here, inviting the interested reader to consult [2, 3, 19, 21] instead. Let us just mention that the aforementioned theory establishes the existence of a unique entropy solution $u : (0,T)\times\mathbb{R}^N \to \mathbb{R}^N$ of (1.1) (resp. (1.2)) for every $T > 0$, for any initial datum $0 \le u_0 \in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$. The functional framework where the solution lives is $0 \le u(t,\cdot) \in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ and suitable truncations of it are bounded variation functions. We prove that under certain conditions the regularity of the solution cannot be worse than that of u_0 and in fact it can acquire some regularity during evolution.

To proceed, we introduce a change of variables that was previously used in [18] to study regularity properties of solutions to (1.2). Let $u(t)$ be an entropy solution of the Cauchy problem for (1.2) which is smooth inside its support, which we assume to be connected (later on we will relax these conditions). Define $(a(t), b(t)) :=$ $(\min \operatorname{supp} u(t), \max \operatorname{supp} u(t))$. Provided that $M := \int_{\mathbb{R}} u(0)\,dx$ (note that the total mass is preserved during evolution) we introduce $\varphi(t,\cdot) : (0,M) \to (a(t), b(t))$ defined by

$$\int_{a(t)}^{\varphi(t,\eta)} u(t,x)\,dx = \eta, \qquad \eta \in (0,M). \tag{2.1}$$

Note that $\varphi(t,\cdot)$ is a bijection as long as $u(t) \ge 0$ has only isolated zeros inside its support. We will use this fact freely when displaying some formulas regarding sets of points in $(a(t), b(t))$, which can be seen as images of sets in $(0,M)$. Now we let $v(t,\eta) :=$ $\frac{\partial\varphi}{\partial\eta}(t,\eta)$, which relates to $u(t,x)$ by means of

$$v(t,\eta) = \frac{1}{u(t,\varphi(t,\eta))}. \tag{2.2}$$

Note also that

$$\frac{\partial v}{\partial \eta}(t,\eta) = -v(t,\eta)^3 \frac{\partial u}{\partial x}(t,\varphi(t,\eta)). \tag{2.3}$$

This function v satisfies the following equation:

$$v_t = \left(\frac{\nu v_\eta}{\sqrt{v^4 + \frac{\nu^2}{c^2}(v_\eta)^2}} \right)_\eta, \qquad t > 0,\ \eta \in (0, M). \tag{2.4}$$

Moreover, as $\operatorname{supp} u(t)$ is connected,

$$\int_0^M v(t,\eta)\, d\eta = |\operatorname{supp} u(t)| = b(t) - a(t). \tag{2.5}$$

We know from [6] that if u_0 is compactly supported in a connected set then the interfaces move exactly at speed c. Therefore, taking into account (2.5), the corresponding boundary conditions for (2.4) must be

$$\frac{\nu v_\eta}{\sqrt{v^4 + \frac{\nu^2}{c^2}(v_\eta)^2}} n = c \quad \text{at } \eta \in \partial(0, M), \tag{2.6}$$

with $\partial(0, M) = \{0, M\}$ and n denoting the outer unit normal to $(0, M)$, that is, $n(0) = -1$ and $n(M) = 1$.

The aim of this section is to construct solutions v to (2.4)–(2.6) that are regular in the interior of their support. As long as any such v is a bounded function satisfying $\operatorname{ess\,inf} v > 0$ we will prove the possibility of recovering u through (2.2) as the unique entropy solution of (1.2); this permits one to show that u inherits some regularity properties from v.

The same rules to pass to the dual formulation apply for any equation of the form $u_t = [\mathbf{a}(u, u_x)]_x$. In the particular case of (1.1), we get

$$\varphi_t = \frac{\nu m \varphi_{\eta\eta}}{\sqrt{(\varphi_\eta)^{4+2m} + \frac{\nu^2 m^2}{c^2}(\varphi_{\eta\eta})^2}}$$

and

$$v_t = \left(\frac{\nu m v_\eta}{\sqrt{v^{4+2m} + \frac{\nu^2 m^2}{c^2}(v_\eta)^2}} \right)_\eta, \qquad t > 0,\ \eta \in (0, M). \tag{2.7}$$

Relation (2.5) holds also in this case. However, the evolution of the support of $u(t)$ may depend strongly on its behaviour at the interface, and so do the boundary conditions for v at $\partial(0, M)$. One of the main goals of our analysis is to prove the existence of jumps at the boundaries during a finite time interval for solutions to the FLPM system with initial conditions originally compactly supported and strictly positive. In this case, by the analysis of the Rankine–Hugoniot conditions given in [21] we know that interfaces will move at speed c as long as they remain discontinuous (but how long will this last is unclear; at the present stage we cannot discard that it breaks down instantaneously). In order to show that these discontinuous solutions exist at least during some finite time interval we propose to study (2.7) with the following boundary conditions (after (2.5) and our belief that interfaces will move with speed c for some time):

$$\frac{\nu m v_\eta}{\sqrt{v^{4+2m} + \frac{\nu^2 m^2}{c^2}(v_\eta)^2}} n = c \quad \text{at } \eta \in \partial(0, M). \tag{2.8}$$

The study of the equivalence between the problems determining u and v is different for RHE and FLPM systems. Solutions of FLPM equations touch zero in finite time, but they still make sense afterwards as entropy solutions of the original problem. However, to deal with the equivalence of the systems, a new concept of solution to (2.7) taking into account the possible occurrence of asymptotes for v is needed. This is not the aim of this paper and for our study it is enough to consider bounded functions u and v such that $u(t) \geq \kappa(t) > 0$ and $\operatorname{ess\,inf} v > 0$ in $[0, M]$.

We may now normalize the solutions to (2.4)–(2.6) and (2.7)–(2.8). Let

$$\bar{v}(t, \eta) := v\left(\nu t, \frac{\nu}{c}\eta\right) \tag{2.9}$$

for the case of (2.4)–(2.6), and

$$\bar{v}(t, \eta) := v\left(\nu m t, \frac{\nu m}{c}\eta\right) \tag{2.10}$$

for (2.7)–(2.8). Then, irrespective of the case, $\bar{v}$ satisfies the following general dual formulation:

$$\bar{v}_t = \left(\frac{\bar{v}_\eta}{\sqrt{\bar{v}^{4+2m} + (\bar{v}_\eta)^2}}\right)_\eta, \qquad t > 0,\ \eta \in (0, M), \tag{2.11}$$

with boundary conditions

$$\frac{\bar{v}_\eta}{\sqrt{\bar{v}^{4+2m} + (\bar{v}_\eta)^2}} n = 1 \quad \text{at } \eta \in \partial(0, M), \tag{2.12}$$

where $m \geq 1$ for FLPM and $m = 0$ for RHE. We maintain the notation M for the rescaled mass and we will work with the rescaled systems from now on.

In general, solutions of (2.11)–(2.12) do not fulfill the previous boundary conditions in the classical sense. The notion of weak trace as introduced in [5] should be used to give a meaning to (2.12), which is the meaning that should be attached to it—and also to (2.6), (2.8)—during Section 2. We will refrain from doing so here though, since we will not require this weak form of boundary conditions for future sections. In fact, we will be able to show that the boundary conditions (2.12) can be given a more tractable formulation as traces of functions of bounded variation in some particular circumstances (see Lemma 5.2 and Corollary 5.1 below); for practical purposes this will be enough, as we will always be working in this easier setting. That being said, the first step in our analysis is to prove a regularity result for (2.11)–(2.12):

Theorem 2.1. *Let $m \geq 0$. Assume that $\bar{v}_0 \in W^{1,\infty}(0, M)$, $\bar{v}_0 \geq \alpha_1 > 0$. Then there exists some $0 < T^* < \infty$ (depending on $\bar{v}_0$) and a smooth solution $\bar{v}$ of (2.11) in $(0, T^*) \times (0, M)$ with $\bar{v}(0, \eta) = \bar{v}_0(\eta)$ and satisfying the boundary conditions (2.12).*

Proof. The proof follows the lines of that of [18, Theorem 2.1]. Therefore we will only give here a brief sketch of how to proceed and point out what are the main differences to be taken into account.

The following regularized Cauchy problem is considered for any $T > 0$, with $\epsilon > 0$:

$$v_t = \left(\frac{v_\eta}{\sqrt{v^{4+2m} + (v_\eta)^2}} \right)_\eta + \epsilon v_{\eta\eta}, \qquad t \in (0, T), \eta \in (0, M), \tag{2.13}$$

$$\left(\frac{v_\eta}{\sqrt{v^{4+2m} + (v_\eta)^2}} + \epsilon v_\eta \right) n = 1 - \epsilon^{1/3}, \quad t \in (0, T),\ \eta \in \partial(0, M), \tag{2.14}$$

Here we use v instead of $\bar{v}$ for simplicity. Set

$$\mathbf{a}(z, \xi) = \frac{\xi}{\sqrt{z^{4+2m} + (\xi)^2}}, \qquad z \geq 0,\ \xi \in \mathbb{R}.$$

We start by showing L^∞ bounds on v from above and below which are independent of ϵ. First we notice that the constant function $\bar{V} = \alpha_1$ is a subsolution. Then any solution v to the Cauchy problem (2.13)–(2.14) is bounded from below by α_1, globally in time. Next we look for a supersolution of (2.13)–(2.14) having the following form:

$$V(t, \eta) = B(t) - \sqrt{\epsilon^{2/3} + \eta(M - \eta)}.$$

Here $B(\cdot)$ is an increasing function of time to be determined. Since v_0 is bounded above, we can choose $B(0) > 1$ and such that $V(0, \eta) \geq B(0) - \sqrt{\epsilon^{2/3} + M/4} \geq v_0(\eta)$. This is the same ansatz that was used in [18, proof of Theorem 2.1]; however, we will display the associated computations here, as at this point only local-in-time estimates can be derived, which is a crucial issue in what follows. It is readily seen that

$$(a(V, V_\eta))_\eta = \frac{1}{D} + \frac{(\eta - M/2)^2}{D^3} \left(V^{4+2m} - 1 - (2 + m) V^{3+2m} \sqrt{\epsilon^{2/3} + \eta(M - \eta)} \right).$$

Here $D = D(t, \eta) = \sqrt{V^{4+2m}(\epsilon^{2/3} + \eta(M - \eta)) + (\eta - M/2)^2}$ is bounded from below, since

$$D(t, \eta) \geq D(0, \eta) > \sqrt{\alpha_1^{4+2m}(\eta(M - \eta)) + (\eta - M/2)^2} > \min\{1, \alpha_1^{2+m}\} M/2.$$

Then it can be shown that

$$|(a(V, V_\eta))_\eta| \leq C_2 + C_1 B(t)^{4+2m}, \qquad |(a(V, V_\eta))_\eta| \leq C_3(\epsilon), \tag{2.15}$$

where C_1 and C_2 are positive constants independent of ϵ, C_3 blows up as $\epsilon \to 0$, and $\eta \in (0, M)$. On the other hand,

$$|\epsilon V_{\eta\eta}| \leq \epsilon \frac{\epsilon^{2/3} + M/4}{\epsilon} \leq C_4$$

for bounded values of ϵ. Thus, if we use the second estimate in (2.15) we easily get a global-in-time supersolution. This provides a global L^∞ bound that is not uniform in ϵ. If we want to get a uniform bound we must use the first estimate in (2.15). The price to pay is that we can only construct a local-in-time supersolution: According to that estimate, we must have

$$(a(V, V_\eta))_\eta + \epsilon V_{\eta\eta} \leq C_2 + C_4 + C_1 B(t)^{4+2m} \leq B'(t) = V_t.$$

Such a function $B(t)$ exists only in a finite time interval $(0, T^*)$ for a certain $T^* < \infty$ (depending on m, $B(0)$, C_1 and C_2). In order to conclude that the function V determined in this way is a supersolution we have to check that $(a(V, V_\eta) + \epsilon V_\eta)n \geq 1 - \epsilon^{1/3}$ for $t \in (0, T^*)$. This is easily seen for ϵ small enough, as in [18]. Then we conclude that any solution v to the Cauchy problem (2.13)–(2.14) is bounded from above by $V(t, \eta)$ for $t < T^*$.

Some integral estimates can be obtained easily as in [18]. Namely,

$$v(t, \cdot) \in L^p[0, M] \quad \text{and} \quad \int_0^t \int_0^M |(v^p)_\eta| \, d\eta \, dt \leq C(t, p)$$

for any $p \in [1, \infty]$ and $t \in [0, T^*)$. All estimates so far allow us to show the existence of solutions v_ϵ for the Cauchy problem (2.13)–(2.14) thanks to standard results in the theory of parabolic equations (see [18] for more details). The first derivatives of v_ϵ are Hölder continuous up to the boundary and standard interior regularity results hold. In particular, the solution is infinitely smooth in $(0, M) \times (0, T)$. Note that here the smoothness bounds depend on ϵ; nevertheless, this enables us to use Bernstein's method to derive uniform regularity estimates as we explain next.

For simplicity of notation, we write v instead of v_ϵ in this paragraph. Let $w = |v_\eta|^2 \phi^2$ where $\phi(\eta) \geq 0$ is smooth with compact support. Then similar computations to those in [18, proof of Theorem 2.1, Step 5] enable us to show that

$$w_t \leq A(t, \eta) w_{\eta\eta} + B(t, \eta) w_\eta + C(t, \eta) w + f(t, \eta) \tag{2.16}$$

with

$$\begin{aligned}
A(t, \eta) &= \tfrac{1}{2}(a_\xi + \epsilon),\\
B(t, \eta) &= a_{z\xi} v_\eta + \tfrac{1}{2} a_z + \tfrac{1}{2} a_{\xi\xi} v_{\eta\eta},\\
C(t, \eta) &= 3(2+m)^2 v^m + \tfrac{1}{2}\epsilon,\\
f(t, \eta) &= 7(2+m) v^{3+2m} \phi |\phi_\eta| + v^{2+m}\big(\tfrac{27}{2}\phi_\eta^2 + \phi|\phi_{\eta\eta}|\big) + \epsilon v_\eta^2\big(\tfrac{1}{2}\phi_{\eta\eta}^2 + 3\phi_\eta^2\big).
\end{aligned}$$

The supersolution previously introduced provides us with a uniform bound for $C(t, \eta)$ in $(0, T^*)$ that does not depend on ϵ. Uniform bounds for $f(t, \eta)$ independent of ϵ in

$(0, T^*) \times (0, M)$ follow as in [18]. This ensures L^∞ bounds on w in $(0, T^*)$ which are independent of ϵ thanks to the maximum principle. Local Lipschitz bounds on v_η which are uniform in ϵ and hold for $t \in (0, T^*)$ follow.

Thanks to all the previous estimates we obtain uniform (in ϵ) interior bounds for any space and time derivative of v_ϵ (see [28, Chapter V, Theorem 3.1] for instance). These regularity bounds allow one to pass to the limit and obtain a solution v of

$$v_t = \left(\frac{v_x}{\sqrt{v^{4+2m} + (v_x)^2}} \right)_x \qquad \text{in } \mathcal{D}'((0, T^*) \times (0, M))$$

(plus boundary conditions). This is done in the same fashion as in [18, proof of Theorem 2.1, Step 7]. The only important difference is that uniform bounds on

$$\mathbf{a}_\epsilon = \frac{v_{\epsilon x}}{\sqrt{v_\epsilon^{4+2m} + (v_{\epsilon x})^2}} + \epsilon v_{\epsilon x}$$

independent of ϵ are obtained just in $(0, T^*) \times (0, M)$, instead of $(0, T) \times (0, M)$ for any $T > 0$. The boundary conditions (2.12) in weak form are recovered using the convergence result of [5, Lemma 10]. □

The relevance of this result lies in the fact that it allows one to construct an entropy solution for either (1.1) or (1.2) that enjoys certain nice properties. To see how, let $u_0 \in L^\infty(\mathbb{R})$ with $u_0(x) \geq \kappa > 0$ for $x \in [a, b]$, and $u_0(x) = 0$ for $x \notin [a, b]$. Assume that $u_0 \in W^{1,\infty}([a, b])$. Let $v_0(\eta)$ be defined in $(0, M)$ according to (2.2). Then we let $u(t, x)$ be defined in $[a - ct, b + ct]$ by (2.1)–(2.2) and (2.10) or (2.9) depending on the case, while we set $u(t, x) = 0$, $x \notin [a - t, b + t]$, $t \in (0, T^*)$. Notice that $u(t, x) \geq \kappa(t) > 0$ for any $x \in (a - t, b + t)$ and any $t < T^*$. Under those circumstances, a straightforward adaptation of [18, Proposition 2.5] yields the following result.

Proposition 2.1. *Let $m > 1$ or $m = 0$ and let $\bar{v}$ be a solution given by Theorem 2.1. Let u be defined by (2.2)–(2.10) if $m > 1$ or by (2.2)–(2.9) if $m = 0$. Then $u \in C([0, T^*), L^1(\mathbb{R}))$, $u(0) = u_0$ and satisfies*

(i) *$u(t) \in BV(\mathbb{R})$, $u(t) \in W^{1,1}(a - ct, b + ct)$ a.e. in $t \in (0, T^*)$ and $u(t)$ is smooth inside its support.*
(ii) *$u_t = \mathbf{z}_x$ in $\mathcal{D}'((0, T^*) \times \mathbb{R})$, where*

$$\mathbf{z}(t) = \frac{\nu u(t) (u^m)_x(t)}{\sqrt{1 + \frac{\nu^2}{c^2}((u^m)_x(t))^2}} \qquad (\textit{case } m > 1),$$

$$\mathbf{z}(t) = \frac{\nu u(t) u_x(t)}{\sqrt{u(t)^2 + \frac{\nu^2}{c^2}((u_x(t))^2}} \qquad (\textit{case } m = 0).$$

(iii) *$u(t, x)$ is the entropy solution of (1.1) (resp. (1.2)) with initial data u_0 in $(0, T^*)$.*
(iv) *$u(t)$ is strictly positive inside its support.*

2.1. Global statements for the relativistic heat equation

Recall that Theorem 2.1 holds only in a finite time interval $(0, T)$, due to the fact that we have not been able to obtain global-in-time uniform bounds on v. This cannot be helped in the case of (2.7)–(2.8), because if such a global bound is to exist, then u would be strictly positive in its support for all time, and this would contradict Corollary 3.1 below. On the contrary, we know that solutions of the relativistic heat equation which are initially strictly positive everywhere in their support remain so during evolution. Thus, switching back to (2.4), we are able to prove a global uniform bound on the associated solutions.

Proposition 2.2. *Assume that* $v_0 \in W^{1,\infty}(0, M)$ *and* $v_0 \geq \alpha_1 > 0$. *Given any* $T > 0$, *there exists a smooth solution* v *of* (2.4) *in* $(0, T) \times (0, M)$ *with* $v(0, \eta) = v_0(\eta)$ *and satisfying the boundary conditions* (2.6).

Proof. Apply Theorem 2.1 to v_0, obtaining a smooth solution v^1 defined on some time interval $(0, T^1)$. Then, we use Proposition 2.1 to deduce the estimate

$$\|v^1(t)\|_\infty \leq 1/ \inf_{\operatorname{supp} u(t)} u(t) \quad \text{for any } t < T^1.$$

Thanks to [6, Proposition 2] we obtain

$$\inf_{\operatorname{supp} u(t)} u(t) \geq e^{-\beta_1 t - \beta_2 t^2} \inf_{\operatorname{supp} u_0} u_0/2$$

for some constants $\beta_1, \beta_2 > 0$ (to be precise these constants get larger as $|\operatorname{supp} u_0|$ does, but given that $T < \infty$ has been fixed, the measure of $\operatorname{supp} u(t)$ is controlled for any $t < T$ and we can neglect this dependence in what follows). Hence, $\sup_{t\in(0,T^1)} \|v^1(t)\|_\infty < \infty$ and v^1 can be extended smoothly to a solution of (2.4) in $[0, T^1]$. Let

$$I_1 := e^{-\beta_1 T^1 - \beta_2 (T^1)^2} \inf_{\operatorname{supp} u_0} u_0/2.$$

This allows us to use Theorem 2.1 again with $v^1(T^1)$ as initial condition, obtaining a new solution v^2 defined on some interval $[T^1, T^1 + T^2)$, with T^2 depending only on I_1. As before,

$$\inf_{\operatorname{supp} u(t)} u(T^1 + t) \geq I_1 e^{-\beta_1 t - \beta_2 t^2}/2$$

for any $t \in (0, T^2)$. We can extend v^2 to $T^1 + T^2$ with finite uniform bounds. Proceeding as before, we set

$$I_2 := e^{-\beta_1 T^2 - \beta_2 (T^2)^2} I_1/2.$$

We may repeat this at will. To prove our statement we must show that $T^1 + T^2 + \cdots$ diverges. To obtain a contradiction, let $T^* = \sum_{i=1}^{\infty} T^i$. Superposing the various solutions v^i we define a solution v in $(0, T^*)$. Using Proposition 2.1 we obtain a solution of defined in $(0, T^*)$. Resorting again to [6, Proposition 2], we get

$$\inf_{\operatorname{supp} u(t)} u(t) \geq e^{-\beta_1 t - \beta_2 t^2} \inf_{\operatorname{supp} u_0} u_0/2$$

for any $t \in (0, T^*)$, and thus

$$\sup_{t\in(0,T^*)} \|v(t)\|_\infty < \infty.$$

Then v can be extended smoothly to $[0, T^*]$. This allows us to use Theorem 2.1 one more time and extend the definition of v beyond T^*, obtaining the desired contradiction. □

Corollary 2.1. *Let $\bar{v}$ be a global-in-time solution given by Theorem 2.1 and Proposition 2.1 for the case $m = 0$. Now consider u to be defined by (2.2)–(2.9). Then $u \in C([0, T], L^1(\mathbb{R}))$ for any $T < \infty$, hence $u(0) = u_0$. Moreover, statements* (i)–(iv) *of Proposition 2.1 hold in* $(0, T)$.

In order to perform our regularity analysis in Section 5.1 below, we need to sharpen the statement of Theorem 2.1. First we need a definition.

Definition 2.1. Let v be a (weak) solution of (2.4) with suitable boundary conditions. Given $0 \le t < T$, we say that $x \in (0, M)$ is a *singular point* for $v(t)$ if $v(t, \cdot)$ is not Lipschitz continuous at x. We write $S_v(t)$ for the set of singular points of $v(t)$.

Hereafter we will use m as the "spatial" variable for (2.4), in order to stress that we are dealing only with (1.2) this time. Our improvement on Proposition 2.2 goes as follows:

Theorem 2.2. *Assume that $v_0 \in BV(0, M)$, $v_0 \ge \alpha_1 > 0$. Assume also that $S_v(0)$ is finite and $v_0 \in W^{1,\infty}_{\mathrm{loc}}((0, M) \setminus S_v(0))$. Then for any $T > 0$ there exists a weak solution of (2.4)–(2.6) in $(0, T) \times (0, M)$ with $v(0, x) = v_0(x)$. Moreover:*

(1) *$S_v(t_2) \subset S_v(t_1)$ for any $t_2 > t_1 \ge 0$. Thus, $v(t) \in W^{1,\infty}_{\mathrm{loc}}((0, M) \setminus S_v(0))$ for every $0 < t < T$.*
(2) *$v(t)$ is smooth in $(0, M) \setminus S_v(t)$ for every $0 < t < T$ (in fact, v is smooth in $\bigcup_{0<t<T}(\{t\} \times ((0, M) \setminus S_v(t)))$).*
(3) *$v(t) \in BV(0, M)$ for a.e. $0 < t < T$.*

Proof. In order to show this result we approximate the initial datum by Lipschitz functions, to which we apply Proposition 2.2, up to renormalization (2.9). Let $\{v_{0,\epsilon}\} \subset W^{1,\infty}(0, M)$ be a sequence of functions satisfying (2.6) such that $v_{0,\epsilon} \ge \alpha_1$ and $v_{0,\epsilon} \to v_0$ in $W^{1,\infty}_{\mathrm{loc}}((0, M) \setminus S_v(0)) \cap BV(0, M)$ as $\epsilon \to 0$. Then Proposition 2.2 ensures that for each $\epsilon > 0$ there exists a smooth solution v_ϵ of (2.4) in $(0, T) \times (0, M)$ with $v_\epsilon(0, m) = v_{0,\epsilon}(m)$ and satisfying the boundary conditions (2.8). As $\epsilon \to 0$ the derivatives of $v_{0,\epsilon}$ will blow up in the vicinity of $S_v(0)$, but keep in mind that $v_{0,\epsilon}$ is locally Lipschitz inside $(0, M) \setminus S_v(0)$ with bounds independent of ϵ. In the following we skip the subscript ϵ except where we find useful to keep it.

Step 1: Integral bounds. To begin with, using the comparison principles in [18, proof of Theorem 2.1, Step 1] together with Proposition 2.2, we deduce that $\alpha_1 \le v(t, m) \le C$, $(t, m) \in [0, T] \times [0, M]$, $C > 0$ being some positive constant depending only on u_0 and T. Next, it is easily seen that

$$\int_0^M v(t, m)\, dm = \int_0^M v_0(m)\, dm + 2ct.$$

Thus, $v_\epsilon \in L^\infty(0, T, L^p(0, M))$ for any $1 \le p \le \infty$ with bounds not depending on ϵ. Now let us prove estimate (3). As in the proof of Theorem 2.1 we have

$$\int_0^T \int_0^M |(v^p)_m|\, dm\, dt \le C(T, p), \qquad \forall p \in (1, \infty), \tag{2.17}$$

where the constant $C(T, p)$ does not depend on ϵ. Note that

$$\int_0^M |(v^p)_m|\, dm = \int_0^M p v^{p-1} |v_m|\, dm \ge \int_0^M p \alpha_1^{p-1} |v_m|\, dm,$$

and hence we get $v \in L^1(0, T, BV(0, M))$.

Step 2: Local Lipschitz bounds and consequences. Recall that each approximation v_ϵ is smooth inside $(0, M)$. This allows us to perform Bernstein-type estimates. We can repeat the computations in the proof of Theorem 2.1 that lead to (2.16) (which are even simpler this time, as we have no extra term coming from a Laplacian regularization) to learn that

$$\sup_{t \in [0,T]} \|w(t)\|_\infty \le C(T, \phi, \|w(0)\|_\infty). \tag{2.18}$$

Here $w = |v_m|^2 \phi^2$ where $\phi \ge 0$ is smooth with compact support $[\phi_1, \phi_2] \subset (0, M)$. Now we observe the following: if $[\phi_1, \phi_2] \cap S_v(0) = \emptyset$ then $\|w(0)\|_\infty$ can be bounded independently of ϵ (as we have already argued that $v_{0,\epsilon}$ is locally Lipschitz inside $(0, M) \setminus S_v(0)$ with bounds independent of ϵ). $S_v(0)$ being a discrete set of points, consequences are twofold:

- $S_{v_\epsilon}(t_2) \subset S_{v_\epsilon}(t_1)$ for any $t_2 > t_1 \ge 0$ (and in particular for $t_1 = 0$, so that $S_{v_\epsilon}(t) \subset S_v(0)$ for all $t > 0$).
- v_ϵ is locally Lipschitz inside $(0, M) \setminus S_v(0)$ with bounds independent of ϵ, for each $t \in [0, T]$. Each v_ϵ has uniform (in ϵ) interior bounds for any space and time derivative in $(0, T) \times ((0, M) \setminus S_v(0))$ (as a consequence of the Lipschitz bounds together with [28, Chapter V, Theorem 3.1]).

Step 3: Passing to the limit as $\epsilon \to 0^+$. We observe that the regularity bounds on v_ϵ derived in the previous step allow us to pass to the limit $\epsilon \to 0^+$, to obtain some function v. In fact, the convergence of v_ϵ to v is locally uniform on $(0, T) \times ((0, M) \setminus S_v(0))$ and the same holds for any derivative of the solution. Thus, v satisfies the estimates of points (1)–(3). Moreover, as every v_ϵ satisfies the boundary conditions (2.6), so does v thanks to [5, Lemma 10]. We may show that it satisfies (2.4) also arguing as in [18, proof of Theorem 2.1, Step 7]. □

Now we can pass again to the original formulation to recover the entropy solution. In fact, we are able to show that, loosely speaking, the regularity of the solution u cannot be worse than that of the initial datum (i.e. the number of "singularities" cannot increase). A regularization effect also takes place, turning Lipschitz corners into smooth points. These are consequences of the following result.

Proposition 2.3. *Let $u_0 \in BV(\mathbb{R})$ with $u_0(x) \geq \kappa > 0$ for $x \in [a,b]$, and $u_0(x) = 0$ for $x \notin [a,b]$. Assume that u_0 is locally Lipschitz in its support outside a finite set $\varphi(0, S_v(0))$. Then the entropy solution u of (1.2) is recovered in terms of the function v constructed in Theorem 2.2 by virtue of (2.1)–(2.2)—extending by zero off $[a(t), b(t)] := [a - ct, b + ct]$. This solution has the following additional properties:*

- *$u(t) \in W^{1,\infty}_{\mathrm{loc}}((a(t), b(t)) \setminus \varphi(t, S_v(t)))$ for every $t \in (0, T)$.*
- *$u(t)$ is smooth in $(a(t), b(t)) \setminus \varphi(t, S_v(t))$ (in fact, it is smooth in the set $\bigcup_{0<t<T}(\{t\} \times ((a(t), b(t)) \setminus \varphi(t, S_v(t))))$).*
- *$u(t) \in BV(\mathbb{R})$ for all $t \in (0, T)$. Moreover, if $u_0 \in W^{1,1}(0, M)$, then $u(t) \in W^{1,1}(\mathbb{R})$ for all $t \in (0, T)$.*

Proof. We can show that formula (2.1) produces an entropy solution (hence unique) of (1.2) in terms of the solution v of (2.4) just constructed. This can be done as in [18, proof of Proposition 2.5]; having estimate (3) of Theorem 2.2 available is crucial in order to do so. Smoothness properties are transferred from v to u by means of (2.1)–(2.2). Note that according to Theorem 2.2 we would get $u(t) \in BV(\mathbb{R})$ for a.e. $t \in (0, T)$, but this holds in fact for every $t \in (0, T)$ thanks to the contractivity of the operator [20]. □

Corollary 2.2. *Let u_0 be as in Proposition 2.3. Then the function v constructed in Theorem 2.2 is such that $v(t) \in BV(0, M)$ for every $0 < t < T$.*

There is also an important consequence of what was done so far, which sheds some light on the nature of singular points. We state it in the form of a corollary.

Corollary 2.3. *Let $u_0 \in BV(\mathbb{R})$ with $u_0(x) \geq \kappa > 0$ for $x \in [a,b]$, and $u_0(x) = 0$ for $x \notin [a,b]$. Assume that u_0 is locally Lipschitz in its support outside a finite set $\varphi(0, S_v(0))$ and let v_0 be defined by (2.1). Pick $m^* \in S_v(0)$ and define $x = x(t) := \varphi(t, m^*) \in (a(t), b(t))$. Then*

$$\int_{a(t)}^{x(t)} u(t)\, dx = m^* \quad \text{and} \quad \int_{x(t)}^{b(t)} u(t)\, dx = M - m^*$$

for as long as the singularity at m^* stands.

Proof. This is a direct consequence of (2.1) and point (1) of Theorem 2.2. □

3. Travelling waves: discontinuity fronts expire in finite time

In this section we analyze some qualitative properties of solutions to (1.1) through comparison with a class of specific travelling wave solutions (a similar idea was used in [25] to characterize degenerate parabolic equations having the property of finite propagation speed). In this way we deduce that jump discontinuities are dissolved in finite time (see Figure 1), no matter if they are inside the support or at the interface. In particular, initially discontinuous interfaces become continuous after a finite time. Hence the dual mass distribution formulation introduced in Section 2 for (1.2) does only make sense for a finite time interval.

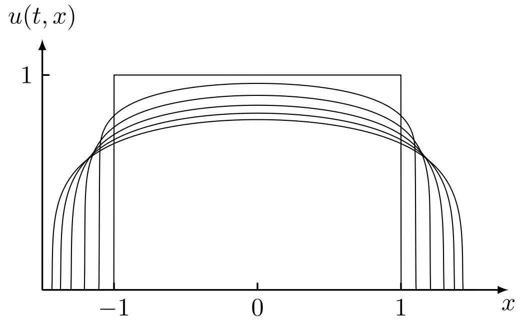

Fig. 1. Finite time dissolution of a discontinuous interface: Numerical time evolution by (1.1) for a compactly supported initial condition with $m = 4$, $\nu = c = 1$ and $t \in [0, 0.5]$ (the smaller the height, the more advanced the time). The time step between different profiles is 0.1. The initial data is the characteristic function of an interval. Two observations deserve to be made here: regularization in finite time (both in the interior and at the interfaces) as well as the different velocities for support growth. In fact, the spreading rate is c until the jump discontinuity at the interface disappears. Then this spreading rate decreases progressively, slowed down by the supersolutions constructed in Proposition 3.1.

Proposition 3.1. *Let $\sigma \in (-c, c)$ and $\xi := x - \sigma t$. Then the continuous function given by $u(\xi) = \left(\frac{\sigma(\xi_0-\xi)}{\nu\sqrt{1-(\sigma/c)^2}}\right)^{1/m}$ if $\sigma(\xi_0 - \xi) \geq 0$ and $u(\xi) = 0$ elsewhere is a distributional solution of travelling wave type to (1.1), for any $\xi_0 \in \mathbb{R}$.*

Proof. A profile $u(\xi)$ is a classical travelling wave solution $u(t, x) = u(x - \sigma t)$ to (1.1) if it satisfies

$$\nu \left(\frac{u(u^m)'}{\sqrt{1 + \frac{\nu^2}{c^2}(u^m)'^2}} \right)' + \sigma u' = 0. \tag{3.1}$$

This implies that

$$\nu \frac{u(u^m)'}{\sqrt{1 + \frac{\nu^2}{c^2}(u^m)'^2}} + \sigma u = K \tag{3.2}$$

for some $K \in \mathbb{R}$. When $K = 0$ we readily check that $u(\xi) = \left(\frac{\sigma(\xi_0-\xi)}{\nu\sqrt{1-(\sigma/c)^2}}\right)^{1/m}$ is a positive solution if $\sigma(\xi_0 - \xi) \geq 0$. The matching of this positive branch with the zero solution for $\sigma(\xi_0 - \xi) \geq 0$ constitutes a distributional solution to (1.1). □

Remark 3.1. It can be shown that $K = 0$ in (3.2) is the only choice that yields non-negative distributional solutions. If we consider $r = -\frac{(u^m)'}{\sqrt{1+\frac{\nu^2}{c^2}(u^m)'^2}}$, then the planar system

$$\begin{cases} u' = -\dfrac{1}{mu^{m-1}} \dfrac{r}{\sqrt{1 - \frac{\nu^2}{c^2} r^2}}, \\ r' = \dfrac{u'}{u}\left(\dfrac{\sigma}{\nu} - r \right), \end{cases}$$

is equivalent to (3.1). Observe that by definition the signs of r and u' coincide, and also $|r| < c/\nu$. Then, by considering a graph formulation of this system, valid as long as u is monotone, we have $\frac{dr}{du} = \frac{1}{u}\left(\frac{\sigma}{\nu} - r\right)$. This equation has as solutions the constant $r(u) = -\sigma/\nu$ and also $r(u) = \sigma/\nu - k/u$ for certain values $k > 0$, which are related to the choices $K = 0$ and $K = \nu k$ respectively. The latter does not yield distributional solutions of (1.1), since the associated function u would become negative. Hence the extension of its positive part by zero does not comply with Rankine–Hugoniot's conditions.

Remark 3.2. There are no travelling wave solutions for (1.2) in the absence of reaction terms. See [16] for a study of travelling wave solutions when (1.2) is coupled with a reaction term of FKPP type.

The existence of this kind of solutions implies interesting consequences on the qualitative behaviour of arbitrary time dependent solutions. Note in particular that any bounded, compactly supported solution for (1.1) can be placed under an appropriate travelling wave for any $\sigma \in (-c, c)$ choosing ξ_0 large enough thanks to the comparison criteria proposed in [6, 24]. Let us develop this idea in the following results.

Lemma 3.1. *Let $0 \le u_0 \in BV(\mathbb{R})$ be compactly supported in $[a, b]$ and let u be the associated entropy solution of (1.1). If $d \in J_{u_0}$, then the ensuing discontinuous travelling front is dissolved in finite time.*

Remark 3.3. This result does not prevent the spontaneous appearance of jump discontinuities. It only states that the life span of any such jump discontinuity is finite.

Proof. To fix ideas, assume that the velocity of the discontinuity front is positive. Assume that $\|u_0\|_\infty = \alpha$. Given any $\sigma \in (0, c)$, we let $\upsilon := \alpha^m \nu\sqrt{1-(\sigma/c)^2}/\sigma + b$. Then, according to Proposition 3.1 (use $\xi_0 = b + \alpha^m \nu\sqrt{1-(\sigma/c)^2}/\sigma$), the travelling wave profile

$$u_\sigma(t, x) = \left(\alpha^m + \frac{\sigma\big(b - x + \sigma t\big)}{\nu\sqrt{1-(\sigma/c)^2}}\right)^{1/m} \chi_{(-\infty, \upsilon + ct)}$$

qualifies as supersolution in the extended sense introduced in [24]; note that this is a decreasing profile such that $u_\sigma(0, b) = \alpha$ and hence $u_\sigma(0) \ge u_0$. Thus, by a comparison principle (see [24]) the support of u must be contained in the support of any of these travelling waves for every $t > 0$. Apart from this, we can use the Rankine–Hugoniot conditions [21] to deduce that if the discontinuity persists forever then the support of u contains the interval $(a, d + ct)$ for any $t \ge 0$. The vanishing of the discontinuity follows from the previous considerations, since $\sigma < c$ and

$$d + ct \le \frac{\alpha^m \nu\sqrt{1-(\sigma/c)^2}}{\sigma} + (b + \sigma t). \tag{3.3}$$

In fact (3.3) determines an upper bound on the time of existence for the discontinuity front, namely

$$t(\sigma) := \inf_{0<\sigma<c} \frac{\frac{\alpha^m \nu\sqrt{1-(\sigma/c)^2}}{\sigma} + b - d}{c - \sigma}.$$

□

Corollary 3.1. *Let u_0 be compactly supported in $[a, b]$ and such that $u_0 \in BV(\mathbb{R})$. Assume that $u_0(x) \geq \alpha > 0$ for every $x \in [a, b]$. Let u be the associated entropy solution of* (1.1)*. Then there exists some $T^* > 0$ such that $u(T^*, (a - ct)^+) = 0$ and/or $u(T^*, (b + ct)^-) = 0$.*

4. Sub- and supersolutions: waiting time for support growth

This section is devoted to proving the existence of a waiting time for the support growth of certain compactly supported solutions to (1.1). In agreement with the numerical results shown in Figures 2 & 3, this effect can be justified if certain decay conditions at the interface (depending on m) are satisfied by the initial datum. This will be a consequence of the existence of certain supersolutions with constant support, as we state below. Some of the results in this section have been independently discovered in [24].

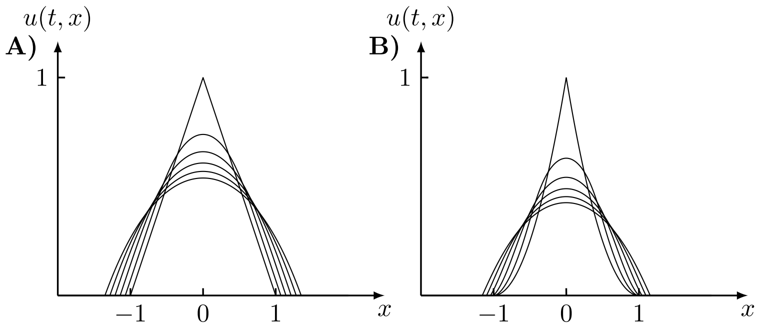

Fig. 2. Waiting time: Numerical time evolution by (1.1) for a compactly supported initial condition with $m = 1$, $\nu = c = 1$ and $t \in [0, 0.5]$ (the smaller the height, the more advanced the time). The time step between different profiles is 0.1. u_0 in A) is a triangle and in B) is u_0^2. Note that in B) there is a waiting time in which the mass is reorganized before the support starts to spread; this fact does not occur in A).

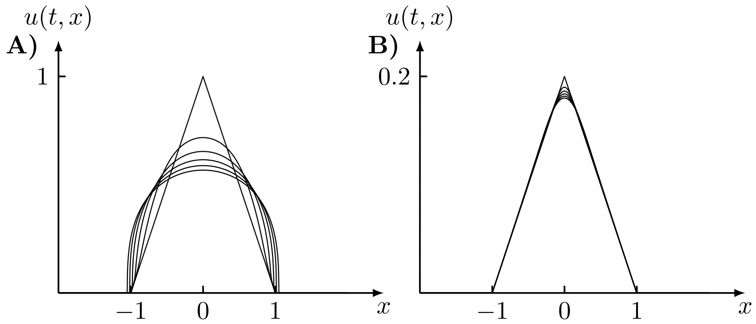

Fig. 3. Waiting time: Numerical time evolution by (1.1) for a compactly supported initial condition with $m = 3$, $\nu = c = 1$ and $t \in [0, 0.5]$ (the smaller the height, the more advanced the time). The time step between different profiles is 0.1. The initial data are triangles of different height. Note that in B) the waiting time is longer than in A).

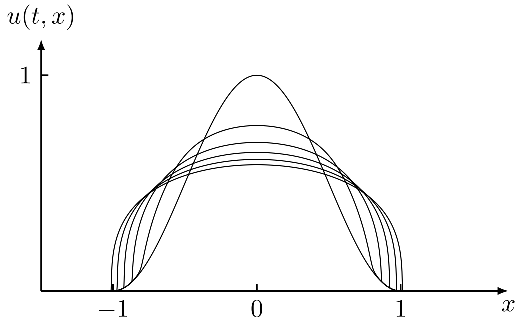

Fig. 4. Waiting time: Numerical time evolution by (1.1) for a compactly supported initial condition with $m = 3$, $\nu = c = 1$ and $t \in [0, 0.5]$ (the smaller the height, the more advanced the time). The time step between different profiles is 0.1. The initial data is $u_0(x) = \cos^2(x)$ for $x \in [0, \pi]$ and zero elsewhere. Note that in the case of initial conditions with continuous interfaces, singularities in the first derivative may be developed in the interior of the support. This effect is probably justified by the fact that the solution can be sandwiched by sub- and supersolutions of the types defined in Sections 3 and 4.

Proposition 4.1. *Let u_0 be such that* $\operatorname{supp} u_0 \subset [-\delta, \delta]$ *and* $0 \le u_0 \le v(x)^{1/m}$ *for* $x \in (-\delta, \delta)$*, where* $v \in C^2(-\delta, \delta)$ *is a non-negative function satisfying*

(i) $\lim_{x\to\delta} v(x) = \lim_{x\to\delta} v'(x) = 0$,
(ii) $\lim_{x\to-\delta} v(x) = \lim_{x\to-\delta} v'(x) = 0$,
(iii) $xv'(x) < 0$ *for* $x \in (-\delta, \delta) \setminus \{0\}$,
(iv) $v''(x) \le k$ *for* $x \in (-\delta, \delta)$.

Then $\tilde{u}(t, x) = \left(\frac{v(x)}{1-(2+m)kt}\right)^{1/m}$ *is a supersolution of* (1.1) *with initial datum* u_0 *for* $t \in \left[0, \frac{1}{(2+m)k}\right)$*, and its support satisfies* $\operatorname{supp}(\tilde{u}(t, x)) = \operatorname{supp}(\tilde{u}(0, x))$.

Proof. Note that the proposed supersolution is an ansatz constructed by separation of variables, where the temporal part $\alpha(t) = (1 - (2 + m)kt)^{-1/m}$ is a solution to the initial value problem

$$\alpha'(t) = \frac{(2+m)k}{m}\alpha(t)^{m+1}, \qquad \alpha(0) = 1, \tag{4.1}$$

and the spatial part is $v(x)^{1/m}$. We claim that assumptions on v allow one to prove that such a function satisfies

$$v'(x)^2/v(x) \le 2k \tag{4.2}$$

for any $x \in (-\delta, \delta)$. Inequality (4.2) is obviously valid for $x = 0$ by (iii). In the rest of the argument we will assume $x \in (0, \delta)$. The same ideas can be analogously applied to the case $x \in (-\delta, 0)$. By using Cauchy's Mean Value Theorem we see that $(v'(x)^2 - v'(y)^2)v'(\xi) = 2v'(\xi)v''(\xi)(v(x) - v(y))$ for any $y \in (x, \delta)$ and $\xi = \xi(x, y)$. This implies that $(v'(x)^2 - v'(y)^2) \le 2k(v(x) - v(y))$ for any $y \in (x, \delta)$, where we have used (iv) and the fact that $v'(\xi) \ne 0$ due to (iii). Then (4.2) holds by letting $y \to \delta$ and using (i).

Set $\Phi(s) = s/\sqrt{1+s^2}$. Then we can prove the estimate

$$(\tilde{u}\Phi\big((\tilde{u})^m\big))' = (\alpha v^{1/m}\Phi(\alpha^m v'))' = \frac{\alpha}{m} v^{1/m-1} v' \Phi(\alpha^m v') + \alpha^{m+1} v^{1/m} v'' \Phi'\big(\alpha^m v'\big)$$

$$\leq \frac{\alpha^{m+1}}{m} v^{1/m-1} v'^2 + \alpha^{m+1} v^{1/m} k \leq \frac{(2+m)k}{m} \alpha(t)^{m+1} v^{1/m} = \alpha' v^{1/m} = \tilde{u}_t, \qquad (4.3)$$

where we have used $s\Phi(s) \leq s^2$, $\Phi'(s) \leq 1$, (iv), (4.2) and (4.1). This concludes the proof. □

A result similar to Proposition 4.1 has been obtained recently and independently in [24]. The next result will be of interest in order to ensure local separation from zero.

Proposition 4.2. *Let v be as in Proposition 4.1. Then*

$$W(t,x) = \left(\frac{1}{(2+m)kt+1}\right)^{1/m} v(x)^{1/m}$$

is a subsolution with static support for any $t > 0$.

Proof. The proof follows the same lines as that of Proposition 4.1, but with reversed signs and inequalities in the chain of estimates (4.3). □

Remark 4.1. Any function v satisfying (i), (ii) and inequality (4.2) can be bounded by quadratic polynomials in the following way:

$$v(x) \leq k^2(x-\delta)^2 \quad \text{if } x \in [0,\delta), \quad \text{and} \quad v(x) \leq k^2(x+\delta)^2 \quad \text{if } x \in (-\delta, 0].$$

Given any function $u_0 \in L^\infty([-\delta,\delta])$ such that

$$\frac{u_0(x)}{(x-\delta)^{2/m}}, \frac{u_0(x)}{(x+\delta)^{2/m}} \in L^\infty([-\delta,\delta]),$$

this allows us to ensure the existence of a (maybe non-optimal) function v such that Proposition 4.1 applies. Just note that the function $v(x) = \tilde{k}(\delta - x)^2(x+\delta)^2$ satisfies the hypotheses of Proposition 4.1 for some constant $\tilde{k}$ large enough.

Now, a simple application of our previous result to any compactly supported initial condition with appropriate decay estimates at the interface allows us to conclude that the spatial support is confined to a fixed spatial interval during a certain time period. In those cases in which the initial support coincides with this spatial interval, we conclude that the support does not grow for a while. That is, we are dealing with a waiting time mechanism.

Corollary 4.1. *Let $0 \leq u_0 \in L^\infty(\mathbb{R})$ be supported in $[a,b]$ and such that*

$$\frac{u_0(x)}{(x-a)^{2/m}}, \frac{u_0(x)}{(b-x)^{2/m}} \in L^\infty(a,b).$$

Let u be the associated entropy solution of (1.1). Then there exists some positive constant $\tilde{k}$ such that

$$\operatorname{supp}(u(t,\cdot)) \subset [a,b] \quad \textit{for any } t \leq \frac{1}{2(2+m)\tilde{k}(b-a)^2}.$$

Proof. We can deduce easily from the hypothesis on u_0 the existence of a constant $\tilde{k}$ such that $u_0 \leq (\tilde{k}(a-x)^2(b-x)^2)^{1/m}$. Now, we apply Proposition 4.1 to $u_0(x+(a+b)/2)$, which is compactly supported on $[-(b-a)/2, (b-a)/2]$ and bounded by $v(x) = (\tilde{k}(x+(b-a)/2)^2(x-(b-a)/2)^2)^{1/m}$. Note that v satisfies all the required hypotheses with $k = 2\tilde{k}(b-a)^2$. This clearly concludes the proof of the result, given the translation invariant character of (1.1) and the comparison principle in [6]. □

5. Smoothing effects for the RHE: analysis of a model case

5.1. Analysis of a model case

The aim of this section is to show the following: Given an initial datum u_0 with a single jump discontinuity inside its support, we can ensure under some technical conditions that there is some $t^* < \infty$ such that the associated entropy solution $u(t)$ of (1.2) is smooth inside its support for every $t > t^*$. This means that an isolated jump discontinuity is dissolved in finite time and after that the solution is smooth everywhere inside the support. The analysis of this simple case will allow us to show in Section 6, via reduction to simpler cases, that the regularizing effect of (1.2) is indeed more general than what we will discuss here. To be more precise, in this section we will track the evolution of initial data which are compactly supported in an interval, having discontinuous interfaces at both ends and another jump discontinuity inside their support.

Definition 5.1. Let $u_0 \in L^\infty(\mathbb{R})$. We say that $u_0 \in \mathcal{J}_0$ if the following conditions hold:

(1) u_0 is supported in $[a, b]$.
(2) $u_0 \in BV(\mathbb{R})$.
(3) $u_0 \geq \kappa > 0$ for $x \in [a, b]$.
(4) The jump set of the initial datum is $J_{u_0} = \{a, \delta, b\}$, with $a < \delta < b$. Assume that the discontinuity at δ will travel to the right (say), i.e. we choose $\nu^\delta = +1$ and so $u^+(\delta) < u^-(\delta)$.
(5) $u_0 \in W^{2,1}(\mathbb{R} \setminus J_{u_0})$ (hence $u_0 \in W^{1,\infty}(\mathbb{R} \setminus J_{u_0})$).
(6) $(u_0)_x(\delta)^-, (u_0)_x(\delta)^+ \leq 0$.

Some comments are in order here. Given $T > 0$, set $Q_T := (0, T) \times \mathbb{R}$. First, it is mandatory to ensure that $u \in BV_{\text{loc}}(Q_T)$ in order to use the Rankine–Hugoniot conditions and the characterization of entropy conditions at jump points proved in [20], which are crucial in what follows. To achieve this, Proposition 4.2 in [20] is the only tool so far. That is why we require (5)–(6). And second, (1), (3) and (4) are assumed just for the sake of technical convenience and a clearer exposition; we will remove these assumptions in Section 6. We let $u(t) = u(t, \cdot)$ and $u_t(t) = \frac{\partial u}{\partial t}(t, \cdot)$. Our aim is to prove the following result:

Theorem 5.1. *Let $u_0 \in \mathcal{J}_0$ and let u be the associated entropy solution of (1.2). Then:*

(1) *$u(t) \in BV(\mathbb{R})$ for each $t > 0$ and $u \in BV((0, T) \times \mathbb{R})$ for every $T > 0$.*
(2) *$u(t)$ is supported on $[a - ct, b + ct]$ and $u(t) \geq \kappa(t) > 0$ in the support.*
(3) *There exists some $0 < T^* < \infty$ such that $u(t) \in W^{1,1}(a - ct, b + ct)$ and $u(t)$ is smooth inside its support, for every $t \geq T^*$.*

The rest of the section constitutes a proof for the third statement of this theorem (the others being already a consequence of previous results in the literature, see [18] for instance). To begin with, let us state some properties of entropy solutions with initial data in $\mathcal{J}_0$ that can be derived from [6, 20].

Lemma 5.1. *Let $u_0 \in \mathcal{J}_0$ and let u be the associated entropy solution of (1.2). Then:*

(1) $(a(t), b(t)) = (a - ct, b + ct)$.
(2) $u(t) > \kappa(t) > 0$ *for* $x \in (a(t), b(t))$.
(3) *Jump discontinuities at the interfaces* $x = a(t)$ *and* $x = b(t)$ *do not dissolve in finite time.*
(4) $u \in BV([\tau, T] \times \mathbb{R})$ *for all* $\tau > 0$ *and* $u_t(t)$ *is a finite Radon measure in* $\mathbb{R}$ *for all* $t > 0$*; in particular*

$$\frac{uu_x}{\sqrt{u^2 + \frac{\nu^2}{c^2}(u_x)^2}} \in BV(\mathbb{R}) \quad \text{for any } t > 0.$$

Using the previous result we can show that the traces of the flux can be computed in a stronger sense than the one in [20]. This is the content of the next statement, which we formulate in a broader context.

Lemma 5.2. *Let $u_0 \in BV(\mathbb{R})^+$ and let u be the associated entropy solution of (1.2) in Q_T. Assume that u_0 is supported in (a, b) and $u_0(x) > \kappa > 0$ for every $x \in (a, b)$. Assume further that $u_t(t)$ is a finite Radon measure in $\mathbb{R}$ for any $t > 0$. Then:*

(i) $\mathbf{a}(u, u_x), \mathbf{b}(u, u_x) \in BV(\mathbb{R})$, *where* $z\mathbf{b}(z, \xi) = \mathbf{a}(z, \xi)$, *for every* $t > 0$.
(ii) $[\mathbf{a}(u, u_x) \cdot \nu^\Omega] = u_{|\partial\Omega}[\mathbf{b}(u, u_x) \cdot \nu^\Omega]$ *for all* $x \in \partial\Omega$, *for every subdomain* $\Omega \subset (a(t), b(t))$ *and all* $t > 0$, *where the square brackets denote weak traces in the sense of Anzellotti* [8].
(iii) *For almost any* $t \in (0, T)$,

$$[\mathbf{z} \cdot \nu^{J_{u(t)}}]_+ = u^+ \quad \text{and} \quad [\mathbf{z} \cdot \nu^{J_{u(t)}}]_- = u^- \tag{5.1}$$

at each point of $J_{u(t)}$—where $[\mathbf{z} \cdot \nu^{J_{u(t)}}]_+$ and $[\mathbf{z} \cdot \nu^{J_{u(t)}}]_-$ are the lateral traces of the flux—and the speed of any discontinuity front is c. In fact, for every $(t_0, x_0) \in J_u$ and for every spatiotemporal ball B about (t_0, x_0) which is contained in $\bigcup_{t>0}(a(t), b(t))$, we have

$$[\mathbf{b} \cdot \nu^{J_{u(t)}}]_+ = c \quad \text{and} \quad [\mathbf{b} \cdot \nu^{J_{u(t)}}]_- = c \quad \text{for every } (t, x) \in J_u \cap B.$$

Proof. The fact that $\mathbf{a}(u, u_x) \in BV(\mathbb{R})$ for every $t > 0$ is given in the previous lemma, while $\mathbf{b}(u, u_x) \in BV(\mathbb{R})$ for every $t > 0$ follows from [20, Lemma 5.5]. The factorization of the trace follows from [20, Lemma 5.6]. Proposition 8.1 in [20] together with the remarks about vertical contact angles that are stated afterwards constitute a proof of the third statement. □

Corollary 5.1. *Let u_0 be as in Lemma 5.2. Consider the associated function v_0 defined by (2.1)–(2.2) and assume that v_0 is regular enough so that Theorem 2.2 applies. Then:*

- $\dfrac{\nu v_m}{\sqrt{v^4 + \frac{\nu^2}{c^2}(v_m)^2}} \in BV(0, M)$ *for every* $t > 0$.

- *For every $t > 0$,*

$$\frac{\nu v_m}{\sqrt{v^4 + \frac{\nu^2}{c^2}(v_m)^2}}(t, 0^+) = \frac{\nu v_m}{\sqrt{v^4 + \frac{\nu^2}{c^2}(v_m)^2}}(t, M^-) = c.$$

The following two results are also easy consequences of Lemma 5.2.

Lemma 5.3. *Let $T > 0$ and $\lambda, \mu \in \mathbb{R}$. Assume that u solves (1.2) in Q_T and is smooth in $\bigcup_{0<t<T}(a - \lambda t, b + \mu t)$. Then, for any $0 < t < T$,*

$$\frac{d}{dt}\int_{a-\lambda t}^{b+\mu t} u\,dx = u(t, b+\mu t)(\mu + \mathbf{b}(t, b+\mu t)) - u(t, a-\lambda t)(\lambda + \mathbf{b}(t, a-\lambda t)). \tag{5.2}$$

Lemma 5.4. *Assume that $u(t)$ is smooth in $(a(t), \tilde{\delta}(t))$ and in $(\tilde{\delta}(t), b(t))$ for $0 \le t < T^*$. Provided that*

$$t \mapsto \int_{a(t)}^{\tilde{\delta}(t)} u(t)\,dx \quad \text{and} \quad t \mapsto \int_{\tilde{\delta}(t)}^{b(t)} u(t)\,dx$$

are constant functions for $0 \le t < T^*$, the following assertions hold:

- *$t \mapsto \tilde{\delta}(t)$ is differentable for any $0 < t < T^*$.*
- *$\mathbf{b}(u, u_x)(t, \tilde{\delta}(t)^-)$ and $\mathbf{b}(u, u_x)(t, \tilde{\delta}(t)^+)$ agree, for every $t \in (0, T^*)$.*

Thanks to Proposition 2.3 we have an alternative description of u in terms of a globally defined function $v : (0, T) \times (0, M) \to \mathbb{R}^+$. In such a way, we know that no new singularities will appear. Let $\delta = \tilde{\delta}(0)$. Then $u(t)$ is smooth in $(a(t), b(t)) \setminus \varphi(t, \varphi^{-1}(0, \delta))$ (that is, everywhere in its support except maybe on the trajectory traced out by the jump discontinuity). Thus, our first step in order to prove Theorem 5.1 is to analyze the behaviour of the jump discontinuity at $x = \delta$ more closely. We have some information already coming from the Rankine–Hugoniot and entropy conditions:

Lemma and Definition 5.1. *Let $u_0 \in \mathcal{J}_0$ and let u be the associated entropy solution. Then the jump discontinuity at $x = \delta$ is not dissolved instantaneously, i.e., there exists some $t_1 > 0$ such that $\mathcal{J}_{u(t)}$ contains precisely three elements for every $t < t_1$. Let t_1 be maximal with that property (that is, $t_1 := \min\{t \mid \#\mathcal{J}_{u(t)} \neq 3\}$). The initial jump discontinuity at $x = \delta$ will be travelling to the right with speed c for $t < t_1$. Let $\delta(t) := \delta + ct$ be the virtual trajectory of the base point of the discontinuity. Then condition (5.1) is satisfied for $0 < t < t_1$.*

Proof. Since $u \in C([0, T], L^1(\mathbb{R}))$, we can find a sequence $t_n \searrow 0$ such that $\{u(t_n)\}$ converges a.e. $x \in \mathbb{R}$. This is not compatible with instantaneous dissolution of the jump discontinuity. The trajectory of its base point is given by the Rankine–Hugoniot conditions, while the last statement follows from Lemma 5.2(iii). □

Let us introduce

$$m_l := \int_a^l u_0\,dx \quad \text{and} \quad m_r := \int_l^b u_0\,dx = M - m_l.$$

Recall that $v(t) \in BV(0, M)$ for every $t > 0$; this allows us to compute traces at m_l for any $t > 0$. A variant of Corollary 5.1 shows that

$$\begin{cases} \dfrac{\nu v_m}{\sqrt{(v)^4 + \frac{\nu^2}{c^2}(v_m)^2}}(t, m_l^-) = c, \\ \dfrac{\nu v_m}{\sqrt{(v)^4 + \frac{\nu^2}{c^2}(v_m)^2}}(t, m_l^+) = c, \end{cases} \quad \forall 0 < t < t_1. \tag{5.3}$$

Nothing precludes that (5.3) may hold true past t_1.

Lemma and Definition 5.2. *The following statements hold true:*

(1) $S_v(t) = S_v(0)$ *for every* $t < t_1$. *Let* $t^* \in [t_1, \infty]$ *be maximal with this property (i.e. the first time at which the singularity vanishes,* $t^* := \min\{t \mid S_v(t) = \emptyset\}$*, and* $t^* = +\infty$ *if the latter set is empty).*
(2) *If* $t^* > t_1$ *we extend* $\delta(t)$ *to* $(0, t^*)$ *as* $\delta(t) = \varphi(t, \varphi^{-1}(0, \delta))$. *Then both*

$$t \mapsto \int_{a(t)}^{\delta(t)} u(t)\,dx \quad \textit{and} \quad t \mapsto \int_{\delta(t)}^{b(t)} u(t)\,dx$$

are constant functions for $t < t^*$.
(3) *Let* t_2 *be the maximal time such that* (5.3) *holds true (that is,* $t_2 := \min\{t \mid (5.3)$ *does not hold*$\}$*, and* $t_2 = +\infty$ *if the latter set is empty). Then* $t_2 = t^*$.

Proof. The first statement is clearly deduced from (2.2). The second is just Corollary 2.3. To prove the third, we notice that $t_2 \le t^*$ by definition. Now let us consider what happens with (5.3) at $t = t_2$. As $v_t(t_2)$ is a finite Radon measure on $(0, M)$, spatial traces of the flux are defined for $t = t_2$ and any $m \in [0, M]$. Then either one of the lateral traces in (5.3) becomes different from c, or both lateral traces differ from c at the same time. Given that $v(t_2)$ is smooth and bounded in $(0, m_l) \cup (m_l, M)$, we deduce in the second case that $v_m \in L^\infty_{\text{loc}}(0, M)$. Hence $m \mapsto v(t_2, m)$ is Lipschitz continuous or even smooth at $m = m_l$. Thus $S_v(t^*) = \emptyset$ and $t_2 = t^*$ in this case.

Let us show that the first case leads to a contradiction. In that case, we would have $S_v(t_2) \neq \emptyset$, thus $t_2 < t^*$. By point (2) of the present result, no mass flow is allowed across m_l for any $t \in [t_2, t^*)$. Then Lemma 5.4 applies, giving a contradiction that concludes the proof. ◻

We are now ready to apply the change of variables studied in Section 2 in the regions $(a(t), \delta(t))$ and $(\delta(t), b(t))$ separately. To that end, we consider a pair of functions $v^l(t, m)$, $v^r(t, m)$ defined for $t < t^*$,

$$v^l(t, \cdot) : (0, m_l) \to (a(t), \delta(t)), \qquad v^r(t, \cdot) : (0, m_r) \to (\delta(t), b(t)),$$

together with the following problems:

$$v_t^l = \left(\frac{\nu v_m^l}{\sqrt{(v^l)^4 + \frac{\nu^2}{c^2}(v_m^l)^2}} \right)_m, \qquad m \in (0, m_l),\ t \in (t, t^*), \tag{5.4}$$

with boundary conditions

$$\frac{\nu v^l_m}{\sqrt{(v^l)^4+\frac{\nu^2}{c^2}(v^l_m)^2}}n=c, \qquad m\in\partial(0,m_l), \qquad n(0)=-1,\ n(m_l)=1, \tag{5.5}$$

and

$$v^r_t=\left(\frac{\nu v^r_m}{\sqrt{(v^r)^4+\frac{\nu^2}{c^2}(v^r_m)^2}}\right)_m, \qquad m\in(0,m_r),\ t\in(t,t^*), \tag{5.6}$$

with boundary conditions

$$\frac{\nu v^r_m}{\sqrt{(v^r)^4+\frac{\nu^2}{c^2}(v^r_m)^2}}n=c \text{ at } m=m_r \quad\text{and}\quad \frac{\nu v^r_m}{\sqrt{(v^r)^4+\frac{\nu^2}{c^2}(v^r_m)^2}}n=c \text{ at } m=0, \tag{5.7}$$

with $n(0)=+1$ and $n(m_r)=1$. The boundary conditions that we impose here are the natural ones after Lemma 5.1 and Lemma and Definition 5.1 (compare with Section 2). Following Corollary 5.1, we see that (5.5), (5.7) can be interpreted as relations between traces of functions of bounded variation and there is no need to use weak traces to describe the behaviour at the boundary.

Arguing as in Section 2 we get the following results:

Proposition 5.1. *There exists a smooth solution v^l of (5.4) in $(0,t^*)\times(0,m_l)$ with $v^l(0,m)=v^l_0(m)$ and satisfying the boundary conditions (5.5). A similar result holds for (5.6)–(5.7). We have $v(t)=v^l\chi_{(0,m_l)}+v^r\chi_{(m_l,M)}$ for every $0<t<t^*$.*

Proposition 5.2. *Let us decompose $u(t,x):=u^l\chi_{(a(t),\delta(t))}(x)+u^r\chi_{(\delta(t),r(t))}(x)$ for any $t<t^*$. Then*

(1) *u^l is related to v^l by means of the change of variables φ restricted to $(0,m_l)$; u^r is related to v^r by means of the change of variables φ restricted to (m_l,M).*
(2) *$u^l(t)\in W^{1,1}(a(t),\delta(t))$ and $u^r(t)\in W^{1,1}(\delta(t),b(t))$ for every $t\in(0,t^*)$.*
(3) *$u^l(t)\in W^{1,\infty}_{\text{loc}}(a(t),\delta(t))$ and $u^r(t)\in W^{1,\infty}_{\text{loc}}(\delta(t),b(t))$ for every $t\in(0,t^*)$.*
(4) *Both u^l and u^r are smooth in their domains of definition.*

Using this parallel formulation, we can show that the size of the inner jump at $x=\delta(t)$ cannot increase with time. More precisely:

Proposition 5.3. *Let $t<t^*$. Then:*

- *$u^r(t+h,\delta(t+h)^+)\ge u^r(t,\delta(t)^+)$ for any $0<t<t+h<t^*$.*
- *$u^r(t+h,b(t+h)^-)\le u^r(t,b(t)^-)$ for any $0<t<t+h<t^*$.*
- *$u^l(t+h,\delta(t+h)^-)\le u^l(t,\delta(t)^-)$ for any $0<t<t+h<t^*$.*
- *$u^l(t+h,a(t+h)^-)\le u^l(t,a(t)^-)$ for any $0<t<t+h<t^*$.*

Proof. Let us show the first statement, the proof of the rest being similar. Let $t \in (0, t^*)$ be fixed. Since v^r is smooth at $(0, t^*) \times (0, m_r)$, we compute, for any $\lambda \in (0, m_r)$,

$$\frac{d}{dt}\int_0^\lambda v^r(t,m)\,dm = \int_0^\lambda \frac{d}{dm}\left(\frac{\nu v^r_m(t)}{\sqrt{(v^r(t))^4 + \frac{\nu^2}{c^2}(v^r_m(t))^2}}\right) dm$$
$$= \left.\frac{-\nu v^r_m(t)}{\sqrt{(v^r(t))^4 + \frac{\nu^2}{c^2}(v^r_m(t))^2}}\right|_{m=\lambda^-} - c.$$

Thanks to the uniform estimates for v^r provided by Steps 1 and 2 of the proof of Theorem 2.2, we arrive at

$$\frac{1}{\lambda}\frac{d}{dt}\int_0^\lambda v^r(t,m)\,dm < 0$$

for any $\lambda \leq m_r$ and $t < t^*$. Then, for any $h > 0$ such that $t + h < t^*$, we can integrate in time to get

$$\frac{1}{\lambda}\int_0^\lambda (v^r(t+h,m) - v^r(t,m))\,dm < 0.$$

Now, we take traces at $m = 0^+$ letting $\lambda \to 0$. We conclude that $v^r(t+h, 0^+) - v^r(t, 0^+) \leq 0$ for any $0 < t < t + h < t^*$. This implies the final result. □

The previous statement shows that the size of the jump discontinuity cannot increase with time. Let us show next that it vanishes in finite time. For that we will need an auxiliary result:

Proposition 5.4. *Let u_0 be an even, compactly supported initial condition which is non-negative, bounded and log-concave. Let $[-R, R]$ be its support and assume that $u_0(x) \geq \alpha > 0$ for $x \in (-R, R)$. Let $\mathcal{SC}(r) = [-r - ct, r + ct]$ be the sound cone about $[-r, r]$. Then for any $\epsilon \in (0, R)$ the associated entropy solution of (1.2) satisfies*

$$\|u(t)\|_{L^\infty(\mathcal{SC}(R)\setminus\mathcal{SC}(\epsilon))} \leq \frac{M}{2(\epsilon + ct)} \qquad \forall t \geq 0.$$

Proof. This is just the combination of mass conservation, log-concavity and symmetry. All these are preserved during evolution [4]. The point is that any log-concave profile which is even is decreasing as a function of $|r|$. Thus, a geometric argument shows that $M \geq 2u(t, \epsilon + ct)(\epsilon + ct)$ and the result follows.[1] □

Lemma 5.5. *We have $t^* < \infty$ and $u(t^*, \delta(t^*)^-) = u(t^*, \delta(t^*)^+)$. In fact, $u_x(t^*) \in L^\infty_{\mathrm{loc}}(a(t^*), b(t^*))$. Thus $u(t^*) \in W^{1,\infty}_{\mathrm{loc}}(a(t^*), b(t^*))$.*

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[1] We get the following estimate in arbitrary dimension:

$$\|u(t)\|_{L^\infty(\mathcal{SC}(R)\setminus\mathcal{SC}(\epsilon))} \leq \frac{M}{|\mathbb{S}^{N-1}|(\epsilon + ct)^N} \qquad \forall t \geq 0.$$

Proof. Assume first that $u(t,\delta(t)^-) > u(t,\delta(t)^+)$ for every $t > 0$. Then $u(t,\delta(t)^+) \geq u_0(\delta^+)$ for every $t > 0$, in particular as a consequence of Proposition 5.3. This contradicts Proposition 5.4.

Thus, there exists some $t_3 < \infty$ such that $u(t_3,\delta(t_3)^-) \leq u(t_3,\delta(t_3)^+)$. Set $t_3 := \min\{t \mid u(t,\delta(t)^-) \leq u(t,\delta(t)^+)\}$. Note that $t_3 \leq t^*$ as t^* is defined. In fact $u(t_3,\delta(t_3)^-) = u(t_3,\delta(t_3)^+)$, otherwise the condition $u \in C([0,T],L^1(\mathbb{R}))$ would be violated.

Hence, there exists some $t_3 \leq t^*$ such that $u(t_3,\delta(t_3)^-) = u(t_3,\delta(t_3)^+)$. As long as $t < t^*$, the boundary conditions (5.3) hold true. Thus, Proposition 5.3 applies and we deduce that $\mathcal{J}_{u(t)} = \{a(t), b(t)\}$ for $t_3 \leq t < t^*$. Thanks to Theorem 2.2 and Proposition 2.3 we deduce that our solution has $W^{1,1}$ spatial regularity inside the support from t_3 on and moreover it is smooth outside $x = \varphi(t,\varphi^{-1}(0,\delta))$.

Let us show next that $t^* < \infty$: Given that the boundary conditions (5.3) hold true for $t < t^*$, Proposition 5.3 applies and $\|u(t)\|_\infty \geq u_0(\delta^+)$ for $t < t^*$ as a consequence. But this would contradict Proposition 5.4 if $t^* = +\infty$. Altogether, the first statement of the lemma is proved.

The remaining statements follow as in the proof of Lemma and Definition 5.2 (recall that $S_v(t^*) = \emptyset$ by definition). □

The previous result does not preclude the possibility of having $t_3 < t^*$. Were that the case, Lemma 5.4 would show that $\delta(t) = \delta + ct$ for $t < t^*$. One way or another, once we have Lemma 5.5 at our disposal we may apply Proposition 2.3 with $u(t^*)$ as initial datum. We conclude that $u(t)$ is smooth inside its support for every $t > t^*$. Combining all the results so far completes the proof of Theorem 5.1.

5.2. Analysis of Hölder cusps, continuous interfaces and isolated zeros

The purpose of this paragraph is to extend the ideas involved in the proof of Theorem 5.1 in order to treat a number of other distinctive features that may be present during the evolution given by (1.2). We will state and prove here several partial statements treating separately the evolution of an initial datum with a single Hölder cusp, with continuous interfaces or with an isolated zero inside its support. These results will be blended together with that of Theorem 5.1 to form a completely general statement in Section 6 below.

Non-Lipschitz continuity points inside the support

We can show that there is a regularization effect which dissolves continuity points for which Lipschitz continuity does not hold (including the case of Hölder cusps):

Proposition 5.5. *Let $u_0 \in L^\infty(\mathbb{R})$ be such that the following conditions hold:*

- *u_0 is supported in $[a,b]$ and $u_0 \geq \kappa > 0$ for $x \in [a,b]$.*
- *$u_0 \in BV(\mathbb{R})$ and $J_{u_0} = \{a,b\}$.*
- *$u_0 \in W^{1,\infty}_{\text{loc}}(\mathbb{R} \setminus (J_{u_0} \cup \{\delta\}))$ for some $a < \delta < b$.*

Let u be the entropy solution of (1.2) with initial datum u_0. Assume that $u_t(t)$ is a finite Radon measure for any $t > 0$. Then:

(1) *$u(t) \in BV(\mathbb{R})$ for each $t > 0$, and $u \in BV((0, T) \times \mathbb{R})$ for every $T > 0$.*
(2) *$u(t)$ is supported on $[a - ct, b + ct]$ and $u(t) \geq \kappa(t) > 0$ in the support.*
(3) *$u(t) \in W^{1,1}(a - ct, b + ct)$ for every $t > 0$. Moreover, there exists $T^* \geq 0$ such that $u(t)$ is smooth inside its support for every $t > T^*$.*

Proof. Thanks to our hypothesis both lateral traces of u_0 at $x = \delta$ coincide. Hence u_0 is continuous at $x = \delta$ and so $u_0 \in W^{1,1}(a, b)$. Then Proposition 2.3 ensures that $u(t) \in W^{1,1}(a(t), b(t))$ for every $t > 0$.

Using Theorem 2.2 we are able to pass to the inverse distribution formulation (2.4)–(2.6). Then either $v(t)$ is smoothed out instantaneously, or there is some $t_1 \in (0, \infty]$ such that $S_v(t)$ is not empty for every $t < t_1$. We are in the first case if, for instance, u_0 has a Hölder cusp at $x = \delta$ (combining Corollary 2.3, Lemma 5.4 and the fact that $u \in C([0, T], L^1(\mathbb{R}))$).

Assume now that we are in the second case; we pick t_1 maximal with this property. We notice that $S_v(0) = \{\varphi(0, \delta)\}$. Let $\delta(t) := \varphi(t, \varphi^{-1}(0, \delta))$. Then Corollary 2.3 ensures that mass transfer across $\delta(t)$ is prevented as long as $\varphi(0, \delta)$ lies in the singularity set of $v(t)$. This can be combined with Lemma 5.4 to argue that $\mathbf{b}(u, u_x)(t, \delta(t)^-) = \mathbf{b}(u, u_x)(t, \delta(t)^+)$ for any $t \in (0, t_1)$, and that both assume either the value $+c$ or $-c$. Thus, (5.3) is satisfied in $(0, t_1)$ with $m_l = \varphi(0, \delta)$. Then we can argue exactly as in Lemma and Definition 5.1. Our situation here is even simpler, as we can assume that $t_3 = 0$. There is just one minor change: We do not know a priori if $\delta(t) = \delta + ct$ or $\delta(t) = \delta - ct$. Apart from that, mimicking those arguments we show that there is a regularizing effect in the long time run. ◻

Continuous interfaces

Now we show that the statement of Theorem 5.1 remains true if we replace discontinuous interfaces by continuous ones. In fact, we can argue as in [18, Proposition 3.2], as long as we are separated from zero inside the support. Let us assume for instance that both interfaces are continuous.

Definition 5.2. Let $u_0 \in L^\infty(\mathbb{R})$. We say that $u_0 \in \mathcal{J}_C$ if the following conditions hold:

(1) u_0 is supported in $[a, b]$ and $u_0 > 0$ for $x \in (a, b)$.
(2) $u_0 \in BV(\mathbb{R})$.
(3) The jump set of the initial datum is $J_{u_0} = \{\delta\}$ with $a < \delta < b$. Assume that the discontinuity at δ will travel to the right (for instance), i.e. we choose $\nu^\delta = +1$ and so $u^+(\delta) < u^-(\delta)$.
(4) $u_0 \in W^{1,\infty}(\mathbb{R} \setminus J_{u_0})$ and $u_0(x) \to 0$ as $x \to a, b$.
(5) $u_0 \in W^{2,1}(\mathbb{R} \setminus J_{u_0})$ and $(u_0)_x(\delta)^-, (u_0)_x(\delta)^+ \leq 0$.

Proposition 5.6. *The results in Theorem 5.1 hold true for $u_0 \in \mathcal{J}_C$, with the following exception: the property $u(t) > 0$ holds only in the interior of the support. Moreover, if $u_0(x) \leq A(b-x)^\alpha (x-a)^\alpha$ for some $A, \alpha > 0$, then $u(t, x) \leq A(t)(b(t)-x)^\alpha (x-a(t))^\alpha$ for any $x \in (a(t), b(t))$, $t > 0$ and some $A(t)$. In that case, $u(t, x)$ is continuous in a neighborhood of the interface and tends to 0 as $x \to a(t), b(t)$.*

Proof. We know that the support evolves as $[a - ct, b + ct]$ and that we are separated from zero inside it thanks to [6]. Then we can obtain a number of statements resembling those in Section 2 but only of local nature. The point here is that $v_0 \notin L^\infty$, because it diverges at $\partial(0, M)$. This can be bypassed as in [18, proof of Proposition 3.2]: Modify the initial datum adding a constant δ that will converge to zero afterwards. In this way we obtain regularized solutions v_δ which are bounded, on which we can perform estimates like those in Theorem 2.2. This time also the integral estimates will be local (in order to avoid the lack of integrability at the boundary). But such local bounds suffice to pass to the limit as $\delta \to 0$ and construct a suitable entropy solution, as explained in [18, proof of Proposition 3.2]. This means that we can handle inverse distribution formulations in terms of v, v^l and v^r as we did in Subsection 5.1.

Let us detail what would be the minor changes. First, we must replace $BV(0, M)$ by $BV_{\text{loc}}(0, M)$ in point (3) of Theorem 2.2. And second, Corollary 2.2 only asserts $BV_{\text{loc}}(0, M)$ regularity this time; we cannot compute traces on $\partial(0, M)$. We would have (say) $v^l \in BV(m_l/3, m_l) \cap BV_{\text{loc}}(0, 2m_l/3)$ and something similar for v^r; this is more than enough in order to proceed. If we take these remarks into account, everything goes as in Subsection 5.1. Moreover, the supersolutions given in [6, Proposition 2] provide us with information on the behaviour of the interfaces, as quoted in the statement. □

It is clear that when there is just one continuous interface, the arguments can be performed in the same way; the only significant difference is maybe that the supersolutions in [6] can be used to control only one end of the support. It is also clear that we can get a variant of Proposition 5.5 with continuous interfaces at one or both ends.

Analysis of isolated zeros

It would seem that the presence of isolated zeros inside the support could spoil the passage to the inverse distribution formulation. Let us examine more closely the dynamical behaviour of such isolated zeros. The following statement is our main tool in that regard.

Proposition 5.7. *Given $R_0, \alpha_0, l, \kappa > 0$, there are values $\beta_1, \beta_2 > 0$ large enough such that $w(t,x) = \exp\{-\beta_1 t - \beta_2 t^2\}\alpha_0(c/\nu)\Theta(t,x)$ is an entropy subsolution of (1.2), where $\Theta(t,x)$ is defined by*

$$\sqrt{(\kappa + ct)^2 - |x + l|^2}\, \chi_{(-l-\kappa-ct, \min\{0, -l+\kappa+ct\}]} + \sqrt{(\kappa + ct)^2 - |x - l|^2}\, \chi_{(\max\{0, l-\kappa-ct\}, l+\kappa+ct)}.$$

Proof. The above profile represents two configurations like the one in [6, Proposition 2], each with initial radius κ and centred at $\pm l$, so that the arrangement is symmetric around the origin. Thus, as long as $l - \kappa - ct > 0$ the proof given in [6] works. We only have to modify it slightly for $t_0 \geq (l - \kappa)/c$ in order to get our statement. For that, let

$$D_l(t) := -\exp\{-\beta_1 t - \beta_2 t^2\}\alpha_0 \frac{c}{\nu} \frac{x + l}{\sqrt{(\kappa + ct)^2 - |x + l|^2}} \chi_{(-l-\kappa-ct, \min\{0, -l+\kappa+ct\}]} \mathcal{L}^1,$$

$$D_r(t) := -\exp\{-\beta_1 t - \beta_2 t^2\}\alpha_0 \frac{c}{\nu} \frac{x - l}{\sqrt{(\kappa + ct)^2 - |x - l|^2}} \chi_{(\max\{0, l-\kappa-ct\}, l+\kappa+ct)} \mathcal{L}^1,$$

where $\mathcal{L}^1$ denotes the 1-dimensional Lebesgue measure. If $t_0 = (l-\kappa)/c$ we get $D_x\mathbf{z} = D_l(t_0) + D_r(t_0) + 2c\delta(0)$ (δ being the Dirac measure). The extra term comes from the fact that $\mathbf{a}(w, w_x)(t_0, 0^-) = -c$ and $\mathbf{a}(w, w_x)(t_0, 0^+) = +c$. Similarly, when $t > t_0$ we get $D_x\mathbf{z} = D_l(t) + D_r(t) + 2c\theta(t)\delta(0)$ with $0 < \theta(t) < c$ depending on the (finite) contact angle.

Having that information we track [6, proof of Proposition 2] to learn that our result will be proved if we are able to show that

$$\int_{t_0}^{T}\int_{-l-\kappa-ct}^{l+\kappa+ct} \phi(t) w_t T(w) S(w)\, dx\, dt \geq \int_{t_0}^{T}\int_{-l-\kappa-ct}^{l+\kappa+ct} D_x\mathbf{a}(w, w_x)\phi(t) T(w) S(w)\, dt$$

for any $0 \leq \phi \in \mathcal{D}((t_0, T) \times \mathbb{R})$ and any $T \in \mathcal{T}^+$, $S \in \mathcal{T}^-$, where $\mathcal{T}^+, \mathcal{T}^-$ are defined in [6]. In fact, Step 2 in [6, proof of Proposition 2] already shows that

$$\int_{t_0}^{T}\int_{-l-\kappa-ct}^{l+\kappa+ct} \phi(t) w_t T(w) S(w)\, dx\, dt \geq \int_{t_0}^{T}\int_{-l-\kappa-ct}^{l+\kappa+ct} D_x^{ac}\mathbf{a}(w, w_x)\phi(t) T(w) S(w)\, dt.$$

As

$$\int_{t_0}^{T}\int_{-l-\kappa-ct}^{l+\kappa+ct} D_x^{s}\mathbf{a}(w, w_x)\phi(t) T(w) S(w)\, dt = 2c\int_{t_0}^{T} \theta(t)\phi(t,0) T(w(t,0)) S(w(t,0))\, dt \leq 0$$

in our particular case, the proof is complete. □

Corollary 5.2. *Let $u_0 \in BV(\mathbb{R})$ with connected support and let u be the associated entropy solution of (1.2). The following statements hold true:*

(1) *Assume that u_0 is continuous at $x_0 \in \operatorname{int}(\operatorname{supp} u_0)$ and $u_0(x_0) = 0$. Assume further that $u_0(x) > 0$ for $x \in \operatorname{int}(\operatorname{supp} u_0) \setminus \{x_0\}$. Then $u(t, x) > 0$ for every $x \in \operatorname{int}(\operatorname{supp} u(t))$ and every $t > 0$.*
(2) *Assume that $u_0(x) > 0$ for $x \in \operatorname{int}(\operatorname{supp} u_0) \setminus \{x_0\}$, where x_0 is such that $u(x_0^-) = 0$ and $u(x_0^+) > 0$ (resp. $u(x_0^-) > 0$ and $u(x_0^+) = 0$). Then $u(t, x) > 0$ for every $x \in \operatorname{int}(\operatorname{supp} u(t))$ and every $t > 0$.*
(3) *A similar statement holds true if we replace x_0 by a finite collection of points falling into any combination of cases (1) and (2).*

Proof. There is no loss of generality in assuming that $x_0 = 0$. To prove the first point, let $\epsilon > 0$ be given. Then we can find suitable parameters so that the profile w constructed in Proposition 5.7 satisfies $w(0, x) \leq u_0(x)$ and $w(\epsilon, x_0) > 0$. As we can do this for any value of $\epsilon > 0$, we deduce that $u(t, x_0) > 0$ for any $t > 0$. The rest is a consequence of the results in [6].

The proof of the second point is similar: We are able to find parameters such that $w(0, x) \leq u_0(x)$ and $w(\epsilon, x_0) > 0$, thus $u(t, x_0^-) > 0$ and $u(t, x_0^+) > 0$ for any $t > 0$. Finally, the last statement is a consequence of the local character of Proposition 5.7 and the results in [6] concerning positivity and support size. □

Provided that $u_t(t)$ is a finite Radon measure for any $t > 0$, this result shows that any initial datum falling under points (1) or (2) of the previous results falls immediately under the assumptions of either Theorem 5.1 or Proposition 5.5 (or any suitable modification of those with continuous interfaces). Hence the associated solution becomes eventually smooth.

6. Smoothing effects for the RHE: the general situation

Let us discuss now what happens when we consider an initial condition with a finite number of jump discontinuities. Keep in mind that a jump discontinuity could evolve into a point of continuity which is not Lipschitz, and that a point where u_0 vanishes could evolve into a point of continuity which is not Lipschitz and also into a jump discontinuity. From the point of view of our analysis in Section 5, the common trait that these singular points share is that they allow no mass flux through them as long as they stand—the only noticeable difference is that zeros of u_0 disappear instantaneously, while non-Lipschitz continuity points and jump discontinuities may take some time to dissolve.

We have discussed in Section 5 what would be the dynamics of an isolated singular point: it will eventually disappear. This will also be the case if we have an array of singular points initially, as long as the trajectories that they trace out during evolution do not cross or do not meet those of the interfaces. In such a case we would be able to treat them one by one as isolated singular points. If this is true, the analysis of the evolution would be reduced to label and track carefully each trajectory traced out by a singular point as long as it is not dissolved. The following statement gives shape to these ideas.

Proposition 6.1. *Let $0 \le u_0 \in BV(\mathbb{R})$. Assume that u_0 is supported in $[a, b]$. Consider a finite set $\mathcal{S}_{u_0} = \{s_i\} \subset [a, b]$, in which each s_i is one of the following:*

- *a point at which u_0 has a jump discontinuity,*
- *a point at which u_0 is continuous but not Lipschitz continuous,*
- *a point at which u_0 has a zero.*

Assume also that $u_0 \in (W^{1,\infty}_{\mathrm{loc}} \cap W^{1,1})(\mathbb{R} \setminus \mathcal{S}_{u_0})$ and $u_0(x) > 0$ in $(a, b) \setminus \mathcal{S}_{u_0}$. Assume finally that $u_t(t)$ is a finite Radon measure for any $t > 0$. Then:

(1) *$u(t) \in BV(\mathbb{R})$ for each $t > 0$.*
(2) *$u(t)$ is supported on $[a - ct, b + ct]$.*
(3) *There exists some $0 < T^* < \infty$ such that $u(t) \in W^{1,1}(a - ct, b + ct)$ and $u(t)$ is smooth inside its support, for every $t \ge T^*$. Moreover, $u(t) > 0$ for all $x \in (a - ct, b + ct)$.*

Proof. To start with, we notice that no singularity overlap can take place during the dynamical evolution, due to the fact that mass flux is not allowed through any such singular point. If the trajectories traced out by two singular points happen to cross, a Dirac measure would appear at the crossing location due to mass preservation. But this is not possible, since $u_0 \in L^\infty(\mathbb{R})$. Now, based on our previous results, there is some $t_1 > 0$ such that the cardinality of the set $S_v(t)$ is constant for all $0 < t < t_1$. Choose t_1 to be maximal with this property. Then we define $\mathcal{S}_{\text{ess}}(u_0) = \varphi(0, S_v(t_1/2))$. This is the set of points that are associated with singularities that are not dissolved instantaneously, the only ones we have to worry about. In fact, as a consequence of the results in Section 5, members of $\mathcal{S}_{\text{ess}}(u_0)$ fall at most into one of two categories: jump discontinuities or points of continuity such that both lateral traces of $\mathbf{b}$ happen to be $+c$ or $-c$.

Say $\mathcal{S}_{\text{ess}}(u_0) = \{p_i\}_i, i = 1, \ldots, n$. Let $P_i(t) = (p_i \pm ct, p_{i+1} \pm ct) := (p_i(t), p_{i+1}(t))$, $i = 1, \ldots, n-1$, be the corresponding virtual evolution of the connected components of $[a, b] \setminus J_{u_0}$ for $t > 0$. We choose $\pm$ according to the Rankine–Hugoniot relations when we are tracking a jump discontinuity. When dealing with points of continuity at which Lipschitz continuity does not hold, we choose "+" if both lateral traces of $\mathbf{b}$ are $-c$, and "−" if both lateral traces of $\mathbf{b}$ are $+c$ (note that this can be regarded as a limiting case of the Rankine–Hugoniot relations). Now we may define $m_i = \int_{P_i(0)} u_0\, dx > 0$, $i = 1, \ldots, n-1$. Then, since none of the trajectories given by $p_i(t)$ cross, we see that $\int_{P_i(t)} u(t)\, dx = m_i$, $i = 1, \ldots, n-1$, as long as no singularity is dissolved. Thus, what we do is to consider the set of maps

$$\varphi_i(t, m) = p_i(t) + \int_0^m v^i(t, r)\, dr, \qquad u(t, \varphi_i(t, m)) = \frac{1}{v^i(t, m)}, \qquad i = 1, \ldots, n,$$

which define a set of functions $v^i : (t_1/2, t_1) \times (0, m_i) \to \mathbb{R}^+$, $i = 1, \ldots, n-1$. Each v^i falls under the hypothesis of Theorem 2.2—and moreover $S_{v^i}(t_1/2) = \emptyset$. In that way we get a description of the evolution of $u(t)$ in terms of the functions $v^i(t)$ as long as there is no breakdown of singularities.

Thus, there is a first time t_1 for which a singularity (meaning a jump discontinuity or a continuity point at which Lipschitz continuity does not hold) is dissolved, say that at $p_2(t_1)$. Then we merge $P_1(t_1)$ and $P_2(t_1)$ into one single component $\tilde{P}_1(t)$, $t \geq t_1$, enclosing a quantity of mass $\tilde{m}_1 := m_1 + m_2$, while we relabel the remaining $P_i(t_1)$ accordingly and reset the inverse distribution formulation for each $\tilde{P}_i(t)$, $t \geq t_1$, in terms of a reduced set of functions v^i, $i = 1, \ldots, n-2$. We modify this procedure accordingly if two or more singularities happen to vanish at the same time. This new description can be used until another singularity vanishes at a time t_2, when we repeat the relabelling operation and we reset again the inverse distribution formulation for each separated piece. We proceed similarly until every singularity which was initially present has vanished, which happens thanks to the results in Section 5. □

Once the connected compact support case is done, we address the general case:

Theorem 6.1. *Let $0 \leq u_0 \in BV(\mathbb{R})$ and let* supp u_0 *be a disjoint union of closed intervals. Consider $\mathcal{S}_{u_0} = \{s_i\} \subset$* supp u_0 *such that $\mathcal{S}_{u_0}$ is finite on each connected component of* supp u_0*, with each s_i being one of the following:*

- *a point at which u_0 has a jump discontinuity,*
- *a point at which u_0 is continuous but not Lipschitz continuous,*
- *a point at which u_0 has a zero.*

Assume also that $u_0 \in (W^{1,\infty}_{\text{loc}} \cap W^{1,1})(\mathbb{R} \setminus \mathcal{S}_{u_0})$ and $u_0(x) > 0$ for every $x \in$ int(supp $u_0) \setminus \mathcal{S}_{u_0}$*. Assume finally that $u_t(t)$ is a finite Radon measure for any $t > 0$. Then:*

(1) *$u(t) \in BV(\mathbb{R})$ for each $t > 0$.*
(2) *There exists some $0 < T^* < \infty$ such that $u(t) \in W^{1,1}(\mathrm{int}(\mathrm{supp}\, u(t)))$ and $u(t)$ is smooth inside its support, for every $t \geq T^*$. Moreover, $u(t) > 0$ in the support.*

Proof. It is mostly straightforward: we apply Proposition 6.1 to each connected component in the initial support. In fact, an obvious modification of Proposition 6.1 applies to the case of a connected support which is not compact (if any such component is present): The support is no longer $[a - ct, b + ct]$ and we have to replace it with the Minkowski sum $\mathrm{supp}\, u_0 \oplus B(0, ct)$. Once this is done, the result applies mutatis mutandis. Thus, this procedure describes what happens as long as no pair of connected components interact. When two (or more) connected components meet, we consider their union as a new connected component of the support. At the merging time $t = t_m$, the solution may have a singularity at each contact point, depending on what the meeting interfaces were. (More specifically, we may get a jump discontinuity, a continuous zero—maybe not Lipschitz continuous—or a continuity point of strict positivity, where we may lack Lipschitz regularity.) These are all instances that we met previously, so we consider the solution at $t = t_m$ as a new initial datum and we apply Proposition 6.1—more precisely, a variant of it allowing for unbounded supports—to each of the connected components. We repeat the procedure until no more connected components merge (which is a finite time that we can estimate in terms of the initial configuration of connected components), and in this way the result is proved. □

7. Regularity for the FLPME before contact time

We can state a local regularity result:

Proposition 7.1. *Let $u_0 \in BV(\mathbb{R})$ with $u_0(x) \geq \kappa > 0$ for $x \in [a, b]$, and $u_0(x) = 0$ for $x \notin [a, b]$. Assume that u_0 is locally Lipschitz in its support outside a finite set $\varphi(0, S_v(0))$. Let T^* be defined by Corollary 3.1. Then the entropy solution u of (1.1) has the following additional properties:*

- *$u(t) \in W^{1,\infty}_{\mathrm{loc}}((a(t), b(t)) \setminus \varphi(t, S_v(t)))$ for all $t \in (0, T^*)$.*
- *$u(t)$ is smooth in $(a(t), b(t)) \setminus \varphi(t, S_v(t))$ for $t < T^*$ (in fact, u is smooth in $\bigcup_{0<t<T^*}(\{t\} \times ((a(t), b(t)) \setminus \varphi(t, S_v(t)))))$.*
- *$u(t) \in BV(\mathbb{R})$ for all $t \in (0, T^*)$. Moreover, if $u_0 \in W^{1,1}(0, M)$ then $u(t) \in W^{1,1}(\mathbb{R})$ for all $t \in (0, T^*)$.*

Roughly speaking, this result shows that, up to the time at which (some) interfaces become continuous, the solution undergoes some regularizing effect. In fact, Lipschitz cusps are regularized instantaneously, while no new jump discontinuities and/or points with Hölder continuity appear. This can be shown by a careful adaptation of the arguments in Section 2.1. Moreover, arguing as in Lemma 5.4 we see that Hölder cusps vanish instantaneously, and arguing as Proposition 5.3 we find that the size of any jump discontinuity does not increase. The main technical difficulty that we face in order to try to extend this result beyond T^* is that we do not know how to make sense of the inverse distribution formulation in that case. Were this possible, the arguments in Sections 5 and 6 would likely imply a complete smoothing effect in the long time run as the one in Theorem 6.1 (replacing Proposition 5.4 by Proposition 3.1 this time).

Regarding the case of initial data with continuous interfaces, local-in-time regularity results were shown in [14] for initial data having global Lipschitz regularity. The local character of these results, together with heuristic arguments and numerical simulations like that in Fig. 4 and those in [23], suggest that there will be a loss of regularity which is connected with a waiting time phenomenon. Anyhow, after the support starts to spread we expect smoothing effects to operate on the solution.

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