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Forms in many variables and differing degrees

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Abstract. We generalise Birch's seminal work on forms in many variables to handle a system of forms in which the degrees need not all be the same. This allows us to prove the Hasse principle, weak approximation, and the Manin–Peyre conjecture for a smooth and geometrically integral variety $X \subseteq \mathbb{P}^m$, provided only that its dimension is large enough in terms of its degree.

Keywords. Hardy–Littlewood circle method, complete intersections, Hasse principle, weak approximation, rational points, Manin conjecture, forms in many variables

1. Introduction and statement of results

This paper will be concerned primarily with integral solutions to general systems of homogeneous equations

$$F_1(x_1, \dots, x_n) = \cdots = F_R(x_1, \dots, x_n) = 0, \tag{1.1}$$

where each form F_i has coefficients in $\mathbb{Z}$. Later in the paper we will specialize our results to "nonsingular systems", and make deductions about the Hasse principle, weak approximation and the distribution of rational points of bounded height for completely general smooth varieties.

Before describing the contents of the paper in detail, we would like to state one particularly succinct result.

Theorem 1.1. *Let $X \subseteq \mathbb{P}^m$ be a smooth and geometrically integral variety defined over $\mathbb{Q}$. Then X satisfies the Hasse principle and weak approximation provided only that*

$$\dim(X) \geq (\deg(X) - 1)2^{\deg(X)} - 1.$$

Moreover there is an asymptotic formula for the counting function for $\mathbb{Q}$-rational points of bounded height on X which agrees with the Manin–Peyre conjecture.

The meaning of the final sentence will be made clear later in this introduction.

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When X is a hypersurface this theorem essentially reduces to a well-known result of Birch [4]. However, we are able to handle varieties of arbitrary codimension. We would like to emphasize indeed that our hypotheses make no reference to the shape of the defining equations for X. In particular we have not required X to be a complete intersection.

It is rather striking that Theorem 1.1 provides such fine arithmetic information about the set $X(\mathbb{Q})$ of $\mathbb{Q}$-rational points on X with such little geometric input. In the setting of hypersurfaces, for example, Harris, Mazur and Pandharipande [13, § 1.2.2] have asked whether the above inequality already implies that X is *unirational*, meaning that there is a dominant rational map $\mathbb{P}^{m-1} \to X$ defined over $\overline{\mathbb{Q}}$. In fact one of the main results in [13] shows that there is an integer $M(d)$ such that for $m \geq M(d)$ any smooth hypersurface $X \subseteq \mathbb{P}^m$ of degree d is indeed unirational. The value of $M(d)$ obtained is extremely large, and grows much faster than a d-fold iterated exponential of d. It would be interesting to determine whether the methods of [13] could be generalised to prove an analogous result for general smooth varieties.

Our principal tool will be the Hardy–Littlewood circle method, so that we will be interested in the case in which the number of variables is large. Our general problem has been considered by Schmidt [22], whose main result establishes the Hardy–Littlewood formula when the number of variables is sufficiently large in terms of certain "h-invariants". Schmidt's work allowed him to deduce, for example, that the system always has non-trivial solutions when the forms all have odd degrees, provided only that the number of variables is large enough in terms of the degrees. The number required is very large, but not as large as in the original elementary proof of this result by Birch [3]. In general, while Schmidt's lower bound on the number of variables required is explicit, the bound is quite awkward to compute, grows rapidly, and depends on h-invariants which are very hard to calculate. However, Schmidt also establishes a result (see [22, Corollary, p. 262]) which is tolerably efficient for nonsingular systems, and which we will describe in a little more detail later. In the context of Theorem 1.1 it would produce a result when n is very roughly of size $2^{3\deg(X)}$ or more.

It is this second type of result that we wish to explore. Many of the ideas go back to work of Birch [4]. The method requires the system not to be too singular, but then gives relatively good lower bounds for the number of variables required. However, Birch's original result needed the forms all to have the same degree, and there is a significant technical problem in extending the method to the general case. Schmidt showed how this might be overcome, but his approach is somewhat wasteful, and does not recover Birch's theorem in the case in which the forms all have the same degree. One of the main purposes of this paper is to show how forms of unequal degrees can be handled in an efficient manner, so as to give results in the spirit of Birch [4] for arbitrary systems.

In order to describe Birch's result we introduce the *singular locus* for the system of forms (1.1), which is the set

$$\{\mathbf{x} \in \mathbb{A}^n : \operatorname{rank}(J(\mathbf{x})) < R\},$$

where $J(\mathbf{x})$ is the Jacobian matrix of size $R \times n$ formed from the gradient vectors $\nabla F_1(\mathbf{x}), \ldots, \nabla F_R(\mathbf{x})$. We note that the system (1.1) defines an algebraic variety $V \subseteq \mathbb{A}^n$. However, points of Birch's singular locus are not necessarily singular points of V, since they are not required to lie on V. If we write B for the dimension (in $\mathbb{A}^n$) of Birch's singular locus then his theorem is that the usual Hardy–Littlewood formula holds as soon as

$$n > B + R(R+1)(D-1)2^{D-1}, \tag{1.2}$$

where D is the common degree of the forms F_i.

For our main result we will need a little more notation. We will re-number the forms F_i in (1.1), grouping together those of equal degree. Let $D \in \mathbb{N}$ such that $D \geq 2$ and let $r_d \in \mathbb{N} \cup \{0\}$ for $1 \leq d \leq D$, with $r_1 = 0$ and $r_D \geq 1$. Suppose then that for every $d \leq D$ we have forms

$$F_{1,d}(x_1, \dots, x_n), \dots, F_{r_d,d}(x_1, \dots, x_n) \in \mathbb{Z}[x_1, \dots, x_n] \tag{1.3}$$

of degree d, so that the total number of forms is

$$R = r_1 + \cdots + r_D.$$

In practice, if one had any forms of degree 1 it would be natural to use them to eliminate appropriate variables, leaving a system of forms of degrees at least 2 but involving fewer variables than originally.

It will be convenient to write

$$\Delta := \{d \in \mathbb{N} : r_d \geq 1\} \subseteq \{2, \dots, D\}.$$

For each degree $d \in \Delta$ we define the matrix

$$J_d(\mathbf{x}) := \begin{pmatrix} \nabla F_{1,d}(\mathbf{x}) \\ \vdots \\ \nabla F_{r_d,d}(\mathbf{x}) \end{pmatrix}$$

and we set

$$S_d := \{\mathbf{x} \in \mathbb{A}^n : \operatorname{rank}(J_d(\mathbf{x})) < r_d\}.$$

This defines an affine algebraic variety and we henceforth set

$$B_d := \dim(S_d). \tag{1.4}$$

When $r_d = 0$ we shall take $B_d = 0$. It will also be convenient to set $B_0 = 0$. Our method breaks down if there is any degree d for which $B_d = n$, and so we impose the condition that $B_d < n$ for every $d \in \Delta$. For example, this rules out the case in which the forms (1.3) are linearly dependent.

At this point we should observe that independent work of Dietmann [10] and Schindler [21] allows one to replace B_d by an alternative invariant, which we denote temporarily by B'_d. One can show in complete generality that $B'_d \leq B_d$, but that B'_d can be strictly less than B_d in appropriate cases. However, we will work with Birch's invariant B_d throughout this paper.

We wish to count integral vectors in a fixed congruence class, and which lie in the dilation of a fixed box. We therefore choose an n-dimensional box $\mathcal{B} \subseteq [-1, 1]^n$, with sides aligned to the coordinate axes. We also give ourselves a modulus $M \in \mathbb{N}$ and a vector $\mathbf{m}_0 \in \mathbb{Z}^n$ with coordinates in $[0, M-1]$. The box $\mathcal{B}$, the modulus M and the vector $\mathbf{m}_0$ will be considered fixed. For any (large) positive real P we then write

$$N(P) := \#\{\mathbf{x} = \mathbf{m}_0 + M\mathbf{y} : \mathbf{y} \in \mathbb{Z}^n,\ \mathbf{x} \in P\mathcal{B},\ F_{i,d}(\mathbf{x}) = 0\ \forall i, d\}.$$

The vectors $\mathbf{x}$ which occur here all satisfy $\mathbf{x} \equiv \mathbf{m}_0 \pmod{M}$. Typically we will want to choose the box $\mathcal{B}$ so that the vectors $\mathbf{x}$ lie close (in a projective sense) to a given real point. Suppose we have chosen a nonzero vector $\mathbf{x}_0 \in (-1, 1)^n$ and a small positive constant η. Taking $|\mathbf{x}|$ to denote the sup-norm of the vector $\mathbf{x}$ and setting

$$\mathcal{B} := \{\mathbf{u} \in \mathbb{R}^n : |\mathbf{u} - \mathbf{x}_0| < \eta\},$$

we see that $P^{-1}\mathbf{x}$ will be close $\mathbf{x}_0$ whenever $\mathbf{x}$ is counted by $N(P)$.

Unfortunately the condition for n occurring in our first result is rather complicated. We define

$$\mathcal{D}_j := r_1 + 2r_2 + \cdots + jr_j, \tag{1.5}$$

for $1 \le j \le D$, and we set $\mathcal{D}_0 := 0$ and $\mathcal{D} := \mathcal{D}_D$. Finally, we write

$$s_d := \sum_{k=d}^{D} \frac{2^{k-1}(k-1)r_k}{n - B_k}. \tag{1.6}$$

With these conventions we now have the following.

Theorem 1.2. *Suppose that*

$$\mathcal{D}_d\left(\frac{2^{d-1}}{n - B_d} + s_{d+1}\right) + s_{d+1} + \sum_{j=d+1}^{D} s_j r_j < 1$$

for $d = 0$ and for every $d \in \Delta$. Then there is a positive δ such that

$$N(P) = \sigma_\infty\Big(\prod_p \sigma_p\Big) P^{n-\mathcal{D}} + O(P^{n-\mathcal{D}-\delta}),$$

where σ_∞ and σ_p are the usual local densities, given by (2.3) and (2.5), respectively.

Here, and for the rest of the paper, the implied constant is allowed to depend on the forms $F_{i,d}$ (and hence on n, R and $\mathcal{D}$) and also on the box $\mathcal{B}$, the modulus M and the vector $\mathbf{m}_0$.

We observe at this point that the entire analysis may be applied to systems of polynomials $f_{i,d}$, rather than systems of forms. For each such polynomial one defines the form $F_{i,d}$ to be the homogeneous part of $f_{i,d}$ of degree d. One then uses the various $F_{i,d}$ to define the numbers B_d as before. The entire argument now goes through with only minor modifications.

Although our condition on n is somewhat complicated, the reader may readily verify that if $r_1 = \cdots = r_{D-1} = 0$ and $r_D = R$, then it is equivalent to Birch's constraint in (1.2). In order to understand better our condition we give the following corollary of Theorem 1.2, which is simpler but potentially weaker.

Corollary 1.3. *Set*

$$B := \max\{B_d : d \in \Delta\},$$
$$t_d := \sum_{k=d}^{D} 2^{k-1}(k-1)r_k \quad (1 \le d \le D+1),$$
$$n_0(d) := \mathcal{D}_d(2^{d-1} + t_{d+1}) + t_{d+1} + \sum_{j=d+1}^{D} t_j r_j,$$
$$n_0 := \max\{n_0(d) : d \in \Delta \cup \{0\}\}.$$

Then the conclusion of Theorem 1.2 holds whenever $n > B + n_0$.

For comparison, the result of Schmidt [22, Corollary, p. 262] mentioned before would establish the same conclusion as Theorem 1.2 as soon as

$$n > \max_{d \le D}\big(B_d + (d-1)(1+2^{1-d})^{-1}2^{3d-5}r_d D\mathcal{D}\big).$$

As examples of Corollary 1.3 we proceed to consider some test cases.

Corollary 1.4. *For a system consisting of $r \ge 1$ quadratic forms and a single form of degree $D \ge 3$ we have $n_0 = (2+r)(D-1)2^{D-1} + 2r(r+1)$ when $r > (D-1)2^{D-2}$, and $n_0 = (2+2r)(D-1)2^{D-1} + 4r$ otherwise.*

Thus if D is fixed and r tends to infinity, our bound is asymptotic to the value $2r(r+1)$ we would have for a system consisting solely of quadratic forms. On the other hand, when r is fixed and D grows, we do not get a bound asymptotic to the value $(D-1)2^D$ we would have for a single form of degree D.

The proof of Corollary 1.4 is a straightforward calculation. We find that

$$n_0(D) = (D+2r)2^{D-1},$$
$$n_0(2) = (2+2r)(D-1)2^{D-1} + 4r,$$
$$n_0(0) = (2+r)(D-1)2^{D-1} + 2r(1+r).$$

Hence $n_0(D) \le n_0(0)$ for every value of r and moreover $n_0(0) \ge n_0(2)$ if and only if $r > (D-1)2^{D-2}$.

Corollary 1.5. *For a system consisting of one form of degree D and one of degree E, where $D > E \ge 2$, we have*

$$n_0 = (2+E)(D-1)2^{D-1} + E2^{E-1}.$$

In particular, if $E \ge 4$ then we have a larger value for n_0 than for a system consisting of two forms of degree D. This is slightly disappointing, since one would expect that it is "easier" to handle a pair of forms of degrees 4 and 5, say, than two forms of degree 5.

Again the proof of Corollary 1.5 is a straightforward calculation. This time we find that

$$\begin{aligned} n_0(D) &= (D+E)2^{D-1}, \\ n_0(E) &= (2+E)(D-1)2^{D-1} + E2^{E-1}, \\ n_0(0) &= 3(D-1)2^{D-1} + 2(E-1)2^{E-1}, \end{aligned}$$

and one readily checks that $n_0(E)$ is at least as large as $n_0(D)$ or $n_0(0)$.

In general we can give the following crude upper bound for n_0.

Theorem 1.6. *We have*

$$n_0 + R - 1 \le \mathcal{D}^2 2^{D-1} \le R^2 D^2 2^{D-1} \quad \textit{and} \quad n_0 + R - 1 \le (\mathcal{D}-1)2^{\mathcal{D}}.$$

Many variants of this are possible. We have chosen to give an estimate with a term $R-1$ on the left because there is a significant case in which one has $\max B_d \le R-1$, as we shall see below.

The first bound shows in particular that for any system of R forms of degrees at most D one has $n_0 \ll_D R^2$. A result of this type, with a somewhat worse dependence on D, was first proved by Schmidt [22, Corollary, p. 262].

In order to give more information about the dimensions B_d of Birch's singular loci we shall now investigate what happens if we impose a nonsingularity condition. This will also enable us to describe conditions under which the constant $\sigma_\infty \prod_p \sigma_p$ is positive in Theorem 1.2. We shall say that the collection of forms $F_{i,d}$ is a *nonsingular system* if $\operatorname{rank}(J(\mathbf{x})) = R$ for every nonzero $\mathbf{x} \in \overline{\mathbb{Q}}^n$ satisfying the equations

$$F_{i,d}(\mathbf{x}) = 0 \quad (1 \le i \le r_d,\ 1 \le d \le D), \tag{1.7}$$

where $J(\mathbf{x})$ is the $R \times n$ Jacobian matrix defined above.

In order to get good bounds on B_d we replace our system of forms by an "equivalent optimal system". We shall say that two systems $\{F_{i,d}\}$ and $\{G_{i,d}\}$ of integral forms (with $\deg(F_{i,d}) = \deg(G_{i,d}) = d$) are *equivalent* if for every pair i, d the form $F_{i,d} - G_{i,d}$ is a linear combination

$$\sum_{j<i} H_{j,d}(\mathbf{x})F_{j,d}(\mathbf{x}) + \sum_{e<d}\sum_{j\le r_e} H_{j,e}(\mathbf{x})F_{j,e}(\mathbf{x})$$

where $H_{j,e}$ is an integral form of degree $d-e$. One sees at once that this does indeed produce an equivalence relation, and that the forms $G_{i,d}$ have the same set of zeros as the original system $F_{i,d}$.

We shall prove in Section 3 that if one has a nonsingular system $\{F_{i,d}\}$ of forms, then there is an equivalent system $\{G_{i,d}\}$ with the property that for any value of i and d the subsystem

$$\{G_{j,d} : j \ge i\} \cup \{G_{j,e} : j \le r_e,\ d < e \le D\}$$

is itself a nonsingular system. We call such a system an *optimal system*. For example, if our original nonsingular system consists of a cubic form C and a quadratic form Q, then there will be a linear form L such that $C + LQ$ is a nonsingular form. The pair $\{C + LQ, Q\}$ is then an optimal system.

For an optimal system we shall show in Lemma 3.1 that

$$B_d \le r_d + \cdots + r_D - 1 \quad (1 \le d \le D). \tag{1.8}$$

It follows that $\max B_d \le R - 1$ for an optimal nonsingular system. Since equivalent systems have the same counting function $N(P)$, we deduce the following result.

Theorem 1.7. *Suppose we have a nonsingular system of forms such that $n > (\mathcal{D}-1)2^{\mathcal{D}}$. Then there is a positive δ such that*

$$N(P) = \sigma_\infty \Big(\prod_p \sigma_p\Big) P^{n-\mathcal{D}} + O(P^{n-\mathcal{D}-\delta}),$$

where σ_∞ and σ_p are the usual local densities, given by (2.3) and (2.5), respectively. Moreover σ_∞ is positive provided that the system of equations (1.7) has a real solution in $\mathcal{B}$. Similarly $\prod_p \sigma_p$ is positive provided that for each prime p there is a solution $\mathbf{x}_p \in \mathbb{Z}_p^n$ satisfying $\mathbf{x}_p \equiv \mathbf{m}_0 \pmod{M}$.

We show in Section 8 that the singular series and singular integral are absolutely convergent under the conditions of Theorem 1.2. Thus standard arguments, such as those used by Davenport [9, Chapters 16 & 17], show that they are positive whenever suitable nonsingular local solutions exist. The details are left to the reader.

The bound (1.8) also enables us to establish the following variant of Corollary 1.5.

Corollary 1.8. *For a nonsingular system consisting of one form of degree D and one of degree E, where $D > E \ge 2$, the conclusion of Theorem 1.7 holds whenever*

$$n > (2 + E)(D - 1)2^{D-1} + E2^{E-1}.$$

In the case of one quadratic and one cubic we find that $n \ge 37$ suffices. This reproduces one of the results from the work of Browning, Dietmann and Heath-Brown [5]. However, in this special case one can do better. Indeed, it is shown in [5, Theorem 1.3] that one can handle smooth intersections of one quadratic and one cubic as soon as $n \ge 29$.

To prove the corollary one has merely to interpret the condition of Theorem 1.2 subject to the information in (1.8). One therefore needs

$$\frac{(D + E)2^{D-1}}{n} < 1,$$
$$\frac{(2 + E)(D - 1)2^{D-1}}{n} + \frac{E2^{E-1}}{n-1} < 1,$$
$$\frac{3(D - 1)2^{D-1}}{n} + \frac{2(E - 1)2^{E-1}}{n-1} < 1,$$

corresponding to $d = D, E, 0$, respectively. It is easy to see that $(2+E)(D-1) \ge D+E$ whenever $D > E \ge 2$, so that the second condition implies the first. In general, if α and β

are positive integers one has

$$\frac{\alpha}{n}+\frac{\beta}{n}<\frac{\alpha}{n}+\frac{\beta}{n-1}<\frac{\alpha}{n-1}+\frac{\beta}{n-1},$$

so that the inequality

$$\frac{\alpha}{n}+\frac{\beta}{n-1}<1$$

will hold for $n=\alpha+\beta+1$, but not for $n=\alpha+\beta$. Since

$$(2+E)(D-1)2^{D-1}+E2^{E-1}\geq 3(D-1)2^{D-1}+2(E-1)2^{E-1},$$

we see that the condition in Theorem 1.2 holds if and only if

$$n\geq (2+E)(D-1)2^{D-1}+E2^{E-1}+1,$$

and the result follows.

Up to this point we have described our results in terms of zeros of systems of forms. We now turn to the related question of rational points on projective varieties. Recall that a family of projective algebraic varieties X, each defined over $\mathbb{Q}$, is said to satisfy the *Hasse principle* if X has a point over $\mathbb{Q}$ whenever it has a point over each completion of $\mathbb{Q}$. If in addition the set $X(\mathbb{Q})$ of $\mathbb{Q}$-points of X is dense in the adelic points then we say that *weak approximation* holds. When X is Fano (i.e. it is a nonsingular projective variety with ample anticanonical bundle ω_X^{-1}) and $X(\mathbb{Q})$ is dense in X under the Zariski topology, it is natural to study the counting function

$$N(U,H,P):=\#\{x\in U(\mathbb{Q}):H(x)\leq P\}$$

as $P\to\infty$. Here $U\subseteq X$ is any Zariski open subset and H is any anticanonical height function on X. The *Manin–Peyre conjecture* (see [11] and [19]) predicts the existence of an open subset $U\subseteq X$ such that for any anticanonical height function H on X there is a (precisely described) constant $c_{U,H}>0$ such that

$$N(U,H,P)\sim c_{U,H}P(\log P)^{\operatorname{rank}\operatorname{Pic}(X)-1}\quad (P\to\infty). \tag{1.9}$$

We will be interested in this when $U=X$ and $\operatorname{Pic}(X)\cong\mathbb{Z}$.

Any smooth complete intersection in $\mathbb{P}^{n-1}$ is the zero-set of a nonsingular system of forms. Conversely, the equations (1.7) define a variety, X say, in $\mathbb{P}^{n-1}$. We shall prove in Lemma 3.2 that if one has a nonsingular system, then the corresponding variety X is geometrically integral, and indeed the ideal in $\overline{\mathbb{Q}}[\mathbf{x}]$ which annihilates $X(\overline{\mathbb{Q}})$ is generated by the forms $F_{i,d}$. In particular X is smooth. Moreover we will show that X has codimension R in $\mathbb{P}^{n-1}$, and that its degree is

$$\deg(X)=\prod_{d\leq D}d^{r_d}.$$

Recall that $X\subseteq\mathbb{P}^{n-1}$ is said to be *nondegenerate* if it is not contained in any proper linear subspace of $\mathbb{P}^{n-1}$. In this case we must have $r_1=0$, whence one easily finds that

$\deg(X) \geq \mathcal{D}$. In view of Theorem 1.7 we can therefore handle any smooth non-degenerate complete intersection $X \subseteq \mathbb{P}^{n-1}$ for which

$$n > (\deg(X) - 1)2^{\deg(X)}. \tag{1.10}$$

We claim that the Hasse principle and weak approximation hold for such varieties, together with the Manin–Peyre conjecture with $U = X$. Since we have the lower bound

$$\deg(X) \geq \mathcal{D} \geq 2R,$$

the inequality (1.10) implies that $\dim(X) = n - 1 - R \geq 3$. In particular the natural map $\mathrm{Br}(\mathbb{Q}) \to \mathrm{Br}(X)$ is an isomorphism (see Proposition A.1 in Colliot-Thélène's appendix to [20]), where $\mathrm{Br}(X) = H^2_{\text{ét}}(X, \mathbb{G}_{\mathrm{m}})$ is the Brauer group of X. Hence this is compatible with the conjecture of Colliot-Thélène that the *Brauer–Manin obstruction* controls the Hasse principle and weak approximation for the varieties under consideration here (see [7] for the most general statement of this conjecture).

To see the claim, we observe that the Hasse principle and weak approximation follow on choosing $\mathcal{B}$ so that the vectors counted by $N(P)$ lie close to a given real point on X and letting P run through large positive integers. For the Manin–Peyre conjecture with $U = X$, we may assume that $X(\mathbb{Q}) \neq \emptyset$. It follows from [16, §II, Exercise 8.4] that $\omega_X^{-1} = \mathcal{O}(n - \mathcal{D})$. Moreover, the inequality (1.10) ensures that X is Fano. Noether's theorem (see [14, Corollary 3.3, p. 180]) implies that $\mathrm{Pic}\, X \cong \mathbb{Z}$. We work with the height function

$$H(x) := \|\mathbf{x}\|^{n-\mathcal{D}},$$

where $\|\cdot\|$ is an arbitrary norm on $\mathbb{R}^n$, on choosing a representative $x = [\mathbf{x}]$ such that $\mathbf{x} \in \mathbb{Z}^n$ is primitive. Set $C := \sup_{\mathbf{x}\in[-1,1]^n} \|\mathbf{x}\|$ and

$$\mathcal{R} := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq C\} \subseteq [-1, 1]^n.$$

In order to establish (1.9), it turns out that it is enough to estimate $N(P)$ with $M = 1$ and the box $\mathcal{B}$ replaced by the region $\mathcal{R}$. In effect one counts integral points of bounded height on the universal torsor over X. (Note that the affine cone over X in $\mathbb{A}^n \setminus \{\mathbf{0}\}$ is the unique universal torsor over X up to isomorphism since $\dim(X) \geq 3$.) Although $\mathcal{R}$ is not necessarily a box, it can be approximated arbitrarily closely, both from above and below, by a disjoint union of admissible boxes. The desired asymptotic formula for $N(P)$ now follows from Theorem 1.7.

It has been observed that there are no examples in the literature in which the Hardy–Littlewood circle method has been used for varieties which are not complete intersections. Indeed, there has been speculation that the circle method is incapable of handling such varieties. Of course, it is not easy to formalize such a claim.

However, one reason that the circle method has been applied only to complete intersections is that it requires the dimension to be large relative to the degree, as one sees in Birch's result (1.2) for example. In contrast, varieties which are not complete intersections tend to have dimension which is at most of size comparable with the degree. Indeed, Hartshorne [15] has conjectured that a smooth variety $X \subseteq \mathbb{P}^m$ is a complete intersection as soon as $\dim(X) > 2m/3$. According to Harris [12, Corollary 18.12] any variety

$X \subseteq \mathbb{P}^m$ lies in a linear subspace of dimension at most $\dim(X)+\deg(X)-1$, and if X is defined over $\mathbb{Q}$ we can take the subspace also to be defined over $\mathbb{Q}$. Thus in our context we may assume that $m \le \dim(X)+\deg(X)-1$, so that Hartshorne's conjecture implies that X is a complete intersection as soon as

$$\dim(X) > \tfrac{2}{3}(\dim(X)+\deg(X)-1),$$

or equivalently whenever

$$\dim(X) \ge 2\deg(X)-1. \tag{1.11}$$

If this were true, it would certainly explain why we have no examples where the circle method has handled a variety which is not a complete intersection.

Hartshorne's conjecture is still largely wide open. However, it has been shown by Bertram, Ein and Lazarsfeld [2, Corollary 3] that if $X \subseteq \mathbb{P}^m$ is smooth then it is a complete intersection as soon as

$$\deg(X) \le \frac{m}{2(m-\dim(X))}.$$

We may assume as above that $m \le \dim(X)+\deg(X)-1$. Inserting this information into the above inequality and rearranging we conclude that X is a complete intersection provided only that

$$\dim(X) > \deg(X)(2\deg(X)-3).$$

This enables us to deduce Theorem 1.1 from Theorem 1.7. We observe firstly that the result is trivial if X is linear. Otherwise, if X is as in Theorem 1.1, then it lies in a minimal linear space, L say, defined over $\mathbb{Q}$. If we write $n-1=\dim(L)>\dim(X)$, then X is a smooth, nondegenerate, geometrically integral subvariety of $L \cong \mathbb{P}^{n-1}$. Moreover, we have $n-1 > (\deg(X)-1)2^{\deg(X)}-1$. Under the hypothesis of Theorem 1.1, X will be a complete intersection, by the result of Bertram, Ein and Lazarsfeld, since we have

$$(\deg(X)-1)2^{\deg(X)}-1 > \deg(X)(2\deg(X)-3)$$

for $\deg(X)\ge 2$. Moreover, we shall prove in Lemma 3.3 that the annihilating ideal of X is generated by integral forms. The result then follows since we have already observed that (1.10) suffices for smooth nondegenerate complete intersections defined over $\mathbb{Q}$.

We conclude this introduction by discussing the extent to which one might relax the conditions of Theorem 1.1.

Conjecture 1.9. *Let $X \subseteq \mathbb{P}^m$ be a smooth and geometrically integral variety defined over $\mathbb{Q}$. Then X satisfies the Hasse principle and weak approximation provided only that* $\dim(X) \ge 2\deg(X)-1$. *Moreover, if $X(\mathbb{Q}) \ne \emptyset$, the Manin–Peyre conjecture holds with $U=X$.*

The conclusion of the conjecture is trivial if $\deg(X)=1$ and it is well-known for $\deg(X)=2$. Thus we may assume that $\deg(X)\ge 3$. In particular $\dim(X)\ge 5$. In this case the first part of the conjecture is based on combining the conjectures of Hartshorne and Colliot-Thélène that we mentioned above. According to the former, the inequality

(1.11) is enough to ensure that any X in the statement of Conjecture 1.9 is a complete intersection in L, for some linear subspace $L \cong \mathbb{P}^{n-1} \subset \mathbb{P}^m$. Assuming that X is defined by a system of R equations (1.1), we deduce that X is Fano since

$$n > \dim(X) + 1 \geq 2\deg(X) \geq 2\mathcal{D}. \tag{1.12}$$

Hence Colliot-Thélène's conjecture implies that X satisfies the Hasse principle and weak approximation (see [20, Conjecture 3.2 and Proposition A.1]). Finally, the inequality (1.12) is precisely what arises from the "square-root barrier" in the circle method, with the general expectation then being that the usual Hardy–Littlewood formula ought to hold, provided that X is smooth and geometrically integral. As above, this would lead to a resolution of the Manin–Peyre conjecture with $U = X$.

We close by discussing two examples to illustrate Theorem 1.1 and Conjecture 1.9. Suppose that $m = 2d - 1$ and consider the Fermat hypersurface

$$X: \quad x_0^d + \cdots + x_{d-1}^d = x_d^d + \cdots + x_{2d-1}^d$$

in $\mathbb{P}^m$. Note that X contains the $(d-1)$-plane given by the equations

$$x_i = x_{i+d} \quad \text{for } i = 0, \ldots, d-1.$$

It was shown by Hooley [17] that this variety has more points than the circle method leads one to expect. Indeed, it follows from work of Browning and Loughran [6, Example 3.2] that there is at least one choice of anticanonical height function for which the Manin–Peyre conjecture fails when $U = X$. This example shows that we cannot have a result like Theorem 1.1 in which the condition is relaxed to $\dim(X) \geq 2\deg(X) - 2$. Thus the lower bound in Conjecture 1.9 is optimal, from the point of view of the Manin–Peyre conjecture.

Turning to the question of the Hasse principle, for any $k \in \mathbb{N}$ we consider the variety $X \subseteq \mathbb{P}^{3k+2}$ defined as follows. Let $C \subseteq \mathbb{P}^2$ be the curve given by $3x_1^3 + 4x_2^3 + 5x_3^3 = 0$, and let $\varphi : \mathbb{P}^2 \times \mathbb{P}^k \to \mathbb{P}^{3k+2}$ be the Segre embedding. Then we take X to be $\varphi(C \times \mathbb{P}^k)$. It is easy to see that X fails the Hasse principle since C fails the Hasse principle. Moreover $\deg(X) = 3(k+1)$, as in Harris [12, pp. 239 & 240], and $\dim(X) = k + 1$. Finally, X is smooth, as in Hartshorne [16, Proposition III.10.1(d)]. Thus Theorem 1.1 would be false if the lower bound on $\dim(X)$ were replaced by $\frac{1}{3}\deg(X)$. It would be interesting to have examples of the failure of the Hasse principle in which $\dim(X)$ grows faster than $\frac{1}{3}\deg(X)$.

Notation. For any $\alpha \in \mathbb{R}$, we will follow common convention and write $e(\alpha) := e^{2\pi i\alpha}$ and $e_q(\alpha) := e^{2\pi i\alpha/q}$. We will allow all of our implied constants to depend on ε, in addition to the forms $F_{i,d}$ and the objects $\mathcal{B}$, M and $\mathbf{m}_0$ occurring in the definition of $N(P)$. We shall write $|\mathbf{x}|$ for the sup-norm of a vector $\mathbf{x} \in \mathbb{C}^n$ and we use $\|\theta\|$ for the distance from a real number θ to the nearest integer. Finally, we shall often write $\mathbf{a} = (a_{i,d})$ to denote the vector whose R entries are indexed by i, d satisfying $1 \leq i \leq r_d$ and $1 \leq d \leq D$.

2. Overview of the paper

The aim of the present section is to present the main ideas in the proof of Theorem 1.2, which is the principal result in this paper. The starting point in the circle method is the identity

$$N(P) = \int_{(0,1]^R} S(\boldsymbol{\alpha})\, d\boldsymbol{\alpha},$$

where $\boldsymbol{\alpha} = (\alpha_{i,d})$ for $1 \le i \le r_d$ and $1 \le d \le D$, and

$$S(\boldsymbol{\alpha}) := \sum_{\substack{\mathbf{x}\in\mathbb{Z}^n \\ \mathbf{m}_0+M\mathbf{x}\in P\mathcal{B}}} e\Big(\sum_{d=1}^{D}\sum_{i=1}^{r_d} \alpha_{i,d} F_{i,d}(\mathbf{m}_0 + M\mathbf{x})\Big).$$

The idea is then to divide the region $(0,1]^R$ into a set of major arcs $\mathfrak{M}$ and minor arcs $\mathfrak{m}$. In the usual way we wish to prove an asymptotic formula

$$\int_{\mathfrak{M}} S(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} = \sigma_\infty\Big(\prod_p \sigma_p\Big) P^{n-\mathcal{D}} + O(P^{n-\mathcal{D}-\delta}) \tag{2.1}$$

for some $\delta > 0$, together with a satisfactory bound on the minor arcs

$$\int_{\mathfrak{m}} S(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} = O(P^{n-\mathcal{D}-\delta}). \tag{2.2}$$

In the above formula the real density associated to the counting problem described by $N(P)$ is defined to be

$$\sigma_\infty := \frac{1}{M^n}\int_{\mathbb{R}^R} J(\boldsymbol{\gamma})\, d\boldsymbol{\gamma}, \tag{2.3}$$

where

$$J(\boldsymbol{\gamma}) := \int_{\mathcal{B}} e\Big(\sum_{d=1}^{D}\sum_{i=1}^{r_d} \gamma_{i,d} F_{i,d}(\mathbf{x})\Big)\, d\mathbf{x}. \tag{2.4}$$

The corresponding p-adic density is

$$\sigma_p := \lim_{k\to\infty} p^{-(n-R)k}\mathcal{N}(p^k) \tag{2.5}$$

where

$$\mathcal{N}(q) := \#\{\mathbf{x} \in (\mathbb{Z}/q\mathbb{Z})^n : F_{i,d}(\mathbf{m}_0 + M\mathbf{x}) \equiv 0 \pmod{q}\ \forall i, d\}.$$

Let $\varpi \in (0, 1/3)$ be a parameter to be decided upon later (see (8.3)). We will take as major arcs

$$\mathfrak{M} := \bigcup_{q\le P^{\varpi}} \bigcup_{\substack{\mathbf{a} \,(\mathrm{mod}\, q) \\ \gcd(q,\mathbf{a})=1}} \mathfrak{M}_{q,\mathbf{a}},$$

where $\mathbf{a} = (a_{i,d})$ and

$$\mathfrak{M}_{q,\mathbf{a}} := \left\{\boldsymbol{\alpha} \pmod{1} : \begin{array}{l} |\alpha_{i,d} - a_{i,d}/q| \le P^{-d+\varpi} \text{ for} \\ 1 \le i \le r_d \text{ and } d \in \Delta \end{array}\right\}. \tag{2.6}$$

We have $\mathfrak{M}_{q,\mathbf{a}} \cap \mathfrak{M}_{q',\mathbf{a}'} = \emptyset$ whenever $\mathbf{a}/q \ne \mathbf{a}'/q'$, provided that P is taken to be sufficiently large.

The minor arcs are defined to be $\mathfrak{m} := (0,1]^R \setminus \mathfrak{M}$. Our estimation of $S(\boldsymbol{\alpha})$ for $\boldsymbol{\alpha} \in \mathfrak{m}$ is based on a version of Weyl differencing, which is inspired by the work of Birch [4], but which is specially adapted to systems of forms of differing degree.

For each $d \in \Delta$ let $F_{i,d}(\mathbf{x}_1,\ldots,\mathbf{x}_d)$ be the d-multilinear polar form attached to $F_{i,d}(\mathbf{x})$. After multiplying $F_{i,d}$ by $d!$ we may assume that $F_{i,d}(\mathbf{x}_1,\ldots,\mathbf{x}_d)$ has integer coefficients. We take $\underline{F}_{i,d}(\mathbf{x}_1,\ldots,\mathbf{x}_{d-1})$ to be the row vector for which

$$F_{i,d}(\mathbf{x}_1,\ldots,\mathbf{x}_d) = \underline{F}_{i,d}(\mathbf{x}_1,\ldots,\mathbf{x}_{d-1}).\mathbf{x}_d, \tag{2.7}$$

and we set

$$\widehat{J}_d(\mathbf{x}_1,\ldots,\mathbf{x}_{d-1}) := \begin{pmatrix} \underline{F}_{1,d}(\mathbf{x}_1,\ldots,\mathbf{x}_{d-1}) \\ \vdots \\ \underline{F}_{r_d,d}(\mathbf{x}_1,\ldots,\mathbf{x}_{d-1}) \end{pmatrix}$$

and

$$\widehat{S}_d := \{(\mathbf{x}_1,\ldots,\mathbf{x}_{d-1}) \in (\mathbb{A}^n)^{d-1} : \operatorname{rank}(\widehat{J}_d(\mathbf{x}_1,\ldots,\mathbf{x}_{d-1})) < r_d\}. \tag{2.8}$$

Thus $\widehat{S}_d$ is an affine algebraic variety.

Using $D-1$ successive applications of Weyl differencing, as in Birch's work, we can relate the size of the exponential sum $S(\boldsymbol{\alpha})$ to the locus of integral points on the affine variety $\widehat{S}_D$. In this way we shall be able to get good control over $S(\boldsymbol{\alpha})$ unless $\alpha_{1,D},\ldots,\alpha_{r_D,D}$ all happen to be close to a rational number with small denominator. If this occurs then we shall modify the final Weyl squaring, in a way suggested by the "q-analogue" of van der Corput's method, so as to remove the effect of the degree D terms. This process is then iterated for the terms of degrees $d \in \Delta$, in decreasing order, ultimately obtaining a suitable estimate unless all of the coefficients $\alpha_{i,d}$ have good rational approximations.

We should comment here on two other approaches to these questions involving exponential sums. Parsell, Prendiville and Wooley [18] give estimates for general multidimensional sums based on a multidimensional version of Vinogradov's mean value theorem. However, the bounds obtained save only a small power of P in our notation, whereas our results require a saving in excess of $P^{\mathcal{D}}$. Baker [1, Theorem 5.1] gives a strong result for exponential sums for a one-variable polynomial, taking account of the Diophantine approximation properties of all the coefficients. It would be very useful if such a result were available in our situation. However, Baker's proof ultimately depends on estimates for complete exponential sums in one variable. Although Baker only requires a relatively weak bound for such complete sums, there appear to be no corresponding estimates available in the higher-dimensional setting.

Our modified version of Weyl differencing is the subject of Section 4. We shall apply it in Section 5 to the leading forms $F_{1,D},\ldots,F_{r_D,D}$ of degree D. The iteration process is then described in Section 6, producing our final bound for the exponential sum $S(\boldsymbol{\alpha})$ in Lemma 6.2. Next, in Section 7, we will show how this suffices to prove (2.2) under the hypothesis in the statement of Theorem 1.2. To complete the proof of the theorem we will establish (2.1) in Section 8. We begin with Section 3, which is concerned with the facts from algebraic geometry alluded to in the introduction, and conclude with Section 9, which provides the proof of Theorem 1.6.

3. Geometric considerations

We commence this section by showing that, given any nonsingular system $\{F_{i,d}\}$ of forms, there is an equivalent optimal system $\{G_{i,d}\}$. An inspection of the proof of [5, Lemma 3.1] easily confirms this fact. Specifically, it shows that one can take

$$G_{i,d} := F_{i,d} + \sum_{1\le k<i} \lambda_k^{(i,d)} F_{k,d} + \sum_{\substack{1\le j\le n\\ 1\le e<d\\ 1\le \ell\le r_e}} \lambda_{j,\ell,e}^{(i,d)} x_j^{d-e} F_{\ell,e}$$

for $1 \le i \le r_d$, $1 \le d \le D$ and appropriate integers $\lambda_k^{(i,d)}, \lambda_{j,\ell,e}^{(i,d)}$.

Recall from (1.4) that $B_d = \dim(S_d)$ with

$$S_d = \{\mathbf{x} \in \mathbb{A}^n : \operatorname{rank}(J_d(\mathbf{x})) < r_d\}.$$

For an optimal system we can establish the following estimate for B_d, as claimed in (1.8).

Lemma 3.1. *Suppose that $\{F_{i,d}\}$ is an optimal system of forms. Let $d \in \Delta$. Then we have $B_d \le r_d + \cdots + r_D - 1$.*

Proof. In what follows let us write $R_d := r_d + \cdots + r_D$. It will be convenient to work projectively. Let $d \in \Delta$ and write

$$T_d := \{[\mathbf{x}] \in \mathbb{P}^{n-1} : \operatorname{rank}(J_d(\mathbf{x})) < r_d\}.$$

In order to establish the lemma it suffices to show that $\dim(T_d) \le R_d - 2$.

We introduce the varieties $V_d, \tilde{V}_d \subseteq \mathbb{P}^{n-1}$, given by

$$V_d: \quad F_{1,d} = \cdots = F_{r_d,d} = 0 \quad \text{and} \quad \tilde{V}_d: \quad F_{2,d} = \cdots = F_{r_d,d} = 0.$$

Note that only $r_d - 1$ forms appear in the definition of $\tilde{V}_d$. Since $\{F_{i,d}\}$ is an optimal system it follows that the varieties

$$W_d := V_D \cap \cdots \cap V_d \quad \text{and} \quad \tilde{W}_d := V_D \cap \cdots \cap V_{d+1} \cap \tilde{V}_d$$

are smooth. Note that $\tilde{W}_d$ has codimension at most

$$r_d - 1 + r_{d+1} + \cdots + r_D = R_d - 1$$

in $\mathbb{P}^{n-1}$, since $r_d \ge 1$.

We are now ready to estimate the dimension of T_d. To do so we note that T_d is the set of $[\mathbf{x}] \in \mathbb{P}^{n-1}$ for which there exists a point $[\lambda_1, \ldots, \lambda_{r_d}] \in \mathbb{P}^{r_d-1}$ such that

$$\lambda_1 \nabla F_{1,d}(\mathbf{x}) + \cdots + \lambda_{r_d} \nabla F_{r_d,d}(\mathbf{x}) = \mathbf{0}. \tag{3.1}$$

Consider the intersection $I_d = T_d \cap \tilde{W}_d$. We claim that I_d is empty. Any point $[\mathbf{x}] \in I_d$ for which (3.1) occurs with $\lambda_1 \ne 0$ must have $F_{1,d}(\mathbf{x}) = 0$, by Euler's identity. But then $[\mathbf{x}]$

must be a point in W_d for which the matrix

$$\begin{pmatrix} J_{r_d}(\mathbf{x}) \\ \vdots \\ J_D(\mathbf{x}) \end{pmatrix}$$

has rank strictly less than R_d. This contradicts the fact that W_d is smooth. Alternatively, any point $[\mathbf{x}] \in I_d$ for which (3.1) occurs with $\lambda_1 = 0$ must produce a singular point on $\tilde{W}_d$, which is also impossible. This shows that I_d is empty, whence

$$\dim(T_d) < \mathrm{codim}(\tilde{W}_d) \le R_d - 1.$$

This concludes the proof of the lemma. □

Our remaining results deal with complete intersections. Recall that a variety $X \subseteq \mathbb{P}^{n-1}$ of codimension R is said to be a *complete intersection* if its annihilating ideal is generated by R forms. The following result shows that any nonsingular system of forms produces a smooth complete intersection of the appropriate degree, which is geometrically integral.

Lemma 3.2. *Let $\{F_1, \dots, F_R\}$ be a nonsingular system of integral forms, defining a variety X in $\mathbb{P}^{n-1}$. Then the annihilating ideal of X is generated by $\{F_1, \dots, F_R\}$, and X is a smooth complete intersection of codimension R. Moreover, X is geometrically integral and has degree*

$$\deg(X) = \deg(F_1) \dots \deg(F_R).$$

Proof. It follows from [16, Exercise II.8.4] that X is a complete intersection (as a scheme) of codimension R, whose annihilating ideal is generated by $\{F_1, \dots, F_R\}$. The smoothness of X follows from the fact that the system $\{F_1, \dots, F_R\}$ of forms is nonsingular.

Now the local rings of any smooth scheme are regular. Moreover, a regular local ring is an integral domain. Thus every local ring of a smooth scheme must be an integral domain. Moreover, X is connected by [16, Exercise III.5.5]. It follows that X is geometrically reduced and irreducible, as required. Indeed, if it failed to be geometrically integral, then it would have two components with a nonempty intersection, since X is connected. But this is impossible since the local ring of any point lying in the intersection would not be an integral domain.

Let $d_i := \deg F_i$ for $1 \le i \le R$. Since X is a complete intersection of codimension R in $\mathbb{P}^{n-1}$, the degree of X can be computed via its Hilbert polynomial. Now $\{F_1, \dots, F_R\}$ forms a "regular sequence" of homogeneous elements of $\mathbb{Q}[\mathbf{x}]$, since X is a complete intersection. According to Harris [12, Example 13.16], the Koszul complex associated to the regular sequence $\{F_1, \dots, F_R\}$ is a free resolution of the coordinate ring $\mathbb{Q}[\mathbf{x}]/(F_1, \dots, F_R)$. This enables us to compute the Hilbert polynomial of X and we find that it has $d_1 \dots d_R/(n+1-R)!$ for its leading coefficient. Hence $\deg(X) = d_1 \dots d_R$, as claimed. □

Our final result in this section shows that any complete intersection which is globally defined over $\mathbb{Q}$ is cut out by integral forms.

Lemma 3.3. *Let X be a smooth complete intersection of codimension R which is globally defined over $\mathbb{Q}$. Then there exist forms $F_1, \dots, F_R$, with coefficients in $\mathbb{Z}$, such that the annihilating ideal of X is generated by $\{F_1, \dots, F_R\}$.*

Proof. Suppose that $X \subset \mathbb{P}^{n-1}$ is defined by a system of R equations (1.1). We claim that there exist forms $G_i \in \mathbb{Q}[x_1, \dots, x_n]$ such that $\deg(F_i) = \deg(G_i)$ for $1 \le i \le R$, and such that the annihilating ideal of X is generated by $\{G_1, \dots, G_R\}$. This will establish the lemma on rescaling the forms appropriately.

Let $\deg(F_i) = d_i$ for $d_1 \le \cdots \le d_R$. The annihilating ideal of X is defined to be $\mathrm{Ann}(X) := \langle F_1, \dots, F_R \rangle$. We will argue by induction, the claim being obvious in the case $R = 1$ of hypersurfaces. We suppose that we have found $G_1, \dots, G_r$ such that $\mathrm{Ann}(X) = \langle G_1, \dots, G_r, F_{r+1}, \dots, F_R \rangle$. Since X is defined over $\mathbb{Q}$ and $F_{r+1} \in \mathrm{Ann}(X)$, we must have $F_{r+1}^\sigma \in \mathrm{Ann}(X)$ for every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Thus

$$F_{r+1}^\sigma \in \langle G_1, \dots, G_r, F_{r+1}, \dots, F_R \rangle$$

for any σ, whence

$$\mathrm{Tr}_{K/\mathbb{Q}}(cF_{r+1}) \in \langle G_1, \dots, G_r, F_{r+1}, \dots, F_R \rangle$$

for any $c \in \overline{\mathbb{Q}}$, where K is the field of definition of cF_{r+1}. We choose c such that $\mathrm{Tr}_{K/\mathbb{Q}}(cF_{r+1})$ is nonzero and call it G_{r+1}, so that it has the correct degree. Thus there exists forms H_i defined over $\overline{\mathbb{Q}}$ and constants $e_i \in \overline{\mathbb{Q}}$ such that

$$G_{r+1} = G_1 H_1 + \cdots + G_r H_r + \sum_i e_i F_i, \tag{3.2}$$

where the sum is only over those i for which $r + 1 \le i \le R$ and $d_i = d_{r+1}$. If there is any choice of c for which one of the e_i is nonzero, we can use (3.2) to swap G_{r+1} for the corresponding F_i in the basis $\langle G_1, \dots, G_r, F_{r+1}, \dots, F_R \rangle$ of $\mathrm{Ann}(X)$, thereby completing the induction step. Alternatively, if we just have $G_{r+1} \in \langle G_1, \dots, G_r \rangle$, irrespective of the choice of c, then $F_{r+1} \in \langle G_1, \dots, G_r \rangle$, which is impossible. $\square$

4. Exponential sums

In this section we consider a quite general situation, independent of the setup described in Section 2. Let

$$f(x_1, \dots, x_n), g(x_1, \dots, x_n) \in \mathbb{R}[x_1, \dots, x_n]$$

be polynomials, and let $P \ge 1$ be given. Suppose that f has degree at most d, and let F be the leading form of degree d. (We shall not rule out the possibility that F vanishes identically.) We write $F(\mathbf{x}_1, \dots, \mathbf{x}_d)$ for the d-linear polar form, and we set $F(\mathbf{x}_1, \dots, \mathbf{x}_d) = \underline{F}(\mathbf{x}_1, \dots, \mathbf{x}_{d-1}).\mathbf{x}_d$ in analogy to (2.7). We then take $F^{(i)}$ to be the i-th component of the row vector $\underline{F}(\mathbf{x}_1, \dots, \mathbf{x}_{d-1})$.

Suppose also that g takes the shape

$$g = q^{-1}g_1 + g_2 \quad \text{with } q \in \mathbb{N} \text{ and } g_1 \in \mathbb{Z}[x_1, \ldots, x_n],$$

where g_2 is a polynomial over $\mathbb{R}$ satisfying

$$\frac{\partial^{i_1+\cdots+i_n}}{\partial^{i_1}x_1 \ldots \partial^{i_n}x_n} g_2(x_1, \ldots, x_n) \ll_{i_i,\ldots,i_n} \varphi P^{-i_1-\cdots-i_n} \tag{4.1}$$

for some parameter $\varphi \geq 1$, uniformly on $[-P, P]^n$.

We give ourselves an n-dimensional box $\mathcal{B}' \subseteq [-P, P]^n$ with sides aligned to the coordinate axes. We then proceed to consider the exponential sum

$$\Sigma := \sum_{\mathbf{x}\in\mathcal{B}'} e(f(\mathbf{x}) + g(\mathbf{x})),$$

in which f is the polynomial which mainly concerns us, and g is regarded as an inconvenient perturbation. Our estimate for Σ will be expressed in terms of the number $L \ll 1$ defined by

$$|\Sigma| = P^n L.$$

We now proceed to establish the following bound.

Lemma 4.1. *Let $d \geq 2$ and $K \geq 1$. Then*

$$L^{2^{d-1}} \ll P^{-(d-1)n}(q\varphi K)^{(d-1)n}(\log P)^n \mathcal{M},$$

where $\mathcal{M}$ counts $(d-1)$-tuples $(\mathbf{x}_1, \ldots, \mathbf{x}_{d-1})$ of integer vectors satisfying

$$|\mathbf{x}_i| < \frac{P}{q\varphi K} \quad (1 \leq i \leq d-1)$$

such that

$$\|qF^{(i)}(\mathbf{x}_1, \ldots, \mathbf{x}_{d-1})\| \leq \frac{1}{P(q\varphi)^{d-2}K^{d-1}} \quad (1 \leq i \leq n).$$

Notice that $\mathcal{M} \geq 1$ since the $(d-1)$-tuple $(\mathbf{0}, \ldots, \mathbf{0})$ is always counted. The conclusion of the lemma is therefore trivial unless

$$q\varphi \leq P,$$

as we henceforth suppose.

We start our argument by using $d-2$ standard Weyl differencing steps to get

$$L^{2^{d-2}} \ll P^{-(d-1)n} \sum_{|\mathbf{x}_1|<P} \cdots \sum_{|\mathbf{x}_{d-2}|<P} \Big|\sum_{\mathbf{x}\in I} \psi(\mathbf{x})\Big| \tag{4.2}$$

with

$$\psi(\mathbf{x}) := e\big(\Delta_{\mathbf{x}_1,\ldots,\mathbf{x}_{d-2}}(f+g)(\mathbf{x})\big),$$

and where $I \subseteq [-P, P]^n$ is a box with sides parallel to the coordinate axes, depending on $\mathbf{x}_1, \dots, \mathbf{x}_{d-2}$. Here $\Delta_{\mathbf{x}_1,\dots,\mathbf{x}_{d-2}}$ is the usual forward-difference operator. Normally, since f potentially has degree d, one would want to perform $d-1$ Weyl differencing steps. However, we will modify the final step in a way suggested by the van der Corput argument and by its q-analogue. This will enable us to eliminate the effect of the polynomial g.

We now set

$$H := \left[\frac{P}{q\varphi}\right], \tag{4.3}$$

whence $qH \le P/\varphi \le P$. We then have

$$\sum_{\mathbf{x}\in I} \psi(\mathbf{x}) = \sum_{\mathbf{x}\in\mathbb{Z}^n} \psi(\mathbf{x})\chi_I(\mathbf{x})$$

where χ_I is the indicator function for I, and hence

$$\begin{aligned} H^n \sum_{\mathbf{x}\in I} \psi(\mathbf{x}) &= \sum_{1\le\mathbf{u}\le H} \sum_{\mathbf{x}\in\mathbb{Z}^n} \psi(\mathbf{x}+q\mathbf{u})\chi_I(\mathbf{x}+q\mathbf{u}) \\ &= \sum_{|\mathbf{x}|\le 2P} \sum_{1\le\mathbf{u}\le H} \psi(\mathbf{x}+q\mathbf{u})\chi_I(\mathbf{x}+q\mathbf{u}), \end{aligned}$$

where the notation $1 \le \mathbf{u} \le H$ is short for $1 \le u_1, \dots, u_n \le H$. Here we have used the fact that $qH \le P$ in order to bound $|\mathbf{x}|$. Cauchy's inequality now yields

$$\begin{aligned} H^{2n}\Big|\sum_{\mathbf{x}\in I}\psi(\mathbf{x})\Big|^2 &\ll P^n \sum_{|\mathbf{x}|\le 2P}\Big|\sum_{1\le\mathbf{u}\le H} \psi(\mathbf{x}+q\mathbf{u})\chi_I(\mathbf{x}+q\mathbf{u})\Big|^2 \\ &= P^n \sum_{1\le\mathbf{u},\mathbf{v}\le H} \sum_{\mathbf{x}\in\mathbb{Z}^n} \psi(\mathbf{x}+q\mathbf{v})\chi_I(\mathbf{x}+q\mathbf{v})\overline{\psi(\mathbf{x}+q\mathbf{u})}\chi_I(\mathbf{x}+q\mathbf{u}) \\ &= P^n \sum_{|\mathbf{w}|<H} n(\mathbf{w}) \sum_{\mathbf{y}\in\mathbb{Z}^n} \psi(\mathbf{y}+q\mathbf{w})\chi_I(\mathbf{y}+q\mathbf{w})\overline{\psi(\mathbf{y})}\chi_I(\mathbf{y}), \end{aligned}$$

where

$$n(\mathbf{w}) := \#\{(\mathbf{u},\mathbf{v}) \in \mathbb{Z}^n \cap (0, H]^{2n} : \mathbf{w} = \mathbf{v}-\mathbf{u}\} \le H^n.$$

We therefore deduce that

$$\Big|\sum_{\mathbf{x}\in I}\psi(\mathbf{x})\Big|^2 \ll P^n H^{-n} \sum_{|\mathbf{w}|<H}\Big|\sum_{\mathbf{y}\in I'}\psi(\mathbf{y}+q\mathbf{w})\overline{\psi(\mathbf{y})}\Big| \ll q^n\varphi^n \sum_{|\mathbf{w}|<H}\Big|\sum_{\mathbf{y}\in I'}\psi(\mathbf{y}+q\mathbf{w})\overline{\psi(\mathbf{y})}\Big|$$

with some new box $I' \subseteq I \subseteq [-P, P]^n$. On applying Cauchy's inequality to (4.2) we thus find that

$$L^{2^{d-1}} \ll P^{-dn} q^n \varphi^n \sum_{|\mathbf{x}_1|<P} \cdots \sum_{|\mathbf{x}_{d-2}|<P} \sum_{|\mathbf{w}|<H} \Big|\sum_{\mathbf{y}\in I'} \psi(\mathbf{y}+q\mathbf{w})\overline{\psi(\mathbf{y})}\Big|. \tag{4.4}$$

Referring to the definition of the function ψ we see that

$$\psi(\mathbf{y}+q\mathbf{w})\overline{\psi(\mathbf{y})} = e\big(\Delta_{\mathbf{x}_1,\dots,\mathbf{x}_{d-2},q\mathbf{w}}(f+g)(\mathbf{y})\big).$$

Since f is a polynomial of degree d with leading form F, we see that

$$\Delta_{\mathbf{x}_1,\ldots,\mathbf{x}_{d-2},q\mathbf{w}}(f)(\mathbf{y})$$

is a linear polynomial in $\mathbf{y}$ with leading homogeneous part

$$F(\mathbf{x}_1,\ldots,\mathbf{x}_{d-2},q\mathbf{w},\mathbf{y}) = qF(\mathbf{x}_1,\ldots,\mathbf{x}_{d-2},\mathbf{w},\mathbf{y}),$$

where $F(\mathbf{x}_1,\ldots,\mathbf{x}_d)$ is the polar form for F, described above. Moreover

$$\Delta_{\mathbf{x}_1,\ldots,\mathbf{x}_{d-2},q\mathbf{w}}(g_1)(\mathbf{y})$$

will be an integral polynomial identically divisible by q, so that

$$e\big(\Delta_{\mathbf{x}_1,\ldots,\mathbf{x}_{d-2},q\mathbf{w}}(q^{-1}g_1)(\mathbf{y})\big) = 1$$

for every $\mathbf{y} \in \mathbb{Z}^n$. Finally, we consider the exponential factor involving g_2. Using (4.1), for any nonnegative integer k each of the k-th order partial derivatives of

$$\Delta_{\mathbf{x}_1,\ldots,\mathbf{x}_{d-2},q\mathbf{w}}(g_2)(\mathbf{y})$$

will be

$$\ll_k \Big(\prod_{i=1}^{d-2}|\mathbf{x}_i|\Big)q|\mathbf{w}|\varphi P^{-(d-1)-k} \ll_k qH\varphi P^{-1-k} \ll_k P^{-k}$$

for $\mathbf{y} \in I'$, in view of our choice (4.3) of H. We may therefore remove the exponential factor involving g_2, using multi-dimensional partial summation, so as to produce

$$\sum_{\mathbf{y}\in I'}\psi(\mathbf{y}+q\mathbf{w})\overline{\psi(\mathbf{y})} \ll \Big|\sum_{\mathbf{y}\in I''} e(qF(\mathbf{x}_1,\ldots,\mathbf{x}_{d-2},\mathbf{w},\mathbf{y}))\Big| \tag{4.5}$$

for a further box I''. (To be precise, partial summation produces a bound involving sums over various boxes, and we take I'' to be the box for which the sum is maximal.)

We proceed to sum over $\mathbf{y}$ to get

$$\sum_{\mathbf{y}\in I''} e(qF(\mathbf{x}_1,\ldots,\mathbf{x}_{d-2},\mathbf{w},\mathbf{y})) \ll E$$

with

$$E := \prod_{i=1}^{n}\frac{P}{1+P\|qF^{(i)}(\mathbf{x}_1,\ldots,\mathbf{x}_{d-2},\mathbf{w})\|}.$$

Combining the above estimate with (4.4) and (4.5) leads to the bound

$$L^{2^{d-1}} \ll P^{-dn}q^n\varphi^n \sum_{|\mathbf{x}_1|<P}\cdots\sum_{|\mathbf{x}_{d-2}|<P}\sum_{|\mathbf{w}|<H} E.$$

We now follow the strategy used by Davenport in his proof of [8, Lemma 3.2]. We write, temporarily, $\{\theta\} := \theta - [\theta]$ for any real θ, and define $N(\mathbf{x}_1, \ldots, \mathbf{x}_{d-2}; \mathbf{r})$ as the number of integer vectors $\mathbf{w}$ for which $|\mathbf{w}| < H$ and

$$\{qF^{(i)}(\mathbf{x}_1, \ldots, \mathbf{x}_{d-2}, \mathbf{w})\} \in (r_i/P, (1+r_i)/P] \quad \text{for } 1 \leq i \leq n.$$

We also write $n(\mathbf{x}_1, \ldots, \mathbf{x}_{d-2})$ similarly for the number of integer vectors $\mathbf{w}$ for which $|\mathbf{w}| < H$ and

$$\|qF^{(i)}(\mathbf{x}_1, \ldots, \mathbf{x}_{d-2}, \mathbf{w})\| \leq P^{-1} \quad \text{for } 1 \leq i \leq n.$$

Now if $\mathbf{w}_1, \mathbf{w}_2$ are counted by $N(\mathbf{x}_1, \ldots, \mathbf{x}_{d-2}; \mathbf{r})$ then the vector $\mathbf{w}_2 - \mathbf{w}_1$ is counted by $n(\mathbf{x}_1, \ldots, \mathbf{x}_{d-2})$, whence $N(\mathbf{x}_1, \ldots, \mathbf{x}_{d-2}; \mathbf{r}) \leq n(\mathbf{x}_1, \ldots, \mathbf{x}_{d-2})$ for any $\mathbf{r} \in \mathbb{R}^n$. Thus

$$\begin{aligned}
\sum_{|\mathbf{x}_1|<P} \cdots \sum_{|\mathbf{x}_{d-2}|<P} \sum_{|\mathbf{w}|<H} \prod_{i=1}^{n} \left(1 + P\|qF^{(i)}(\mathbf{x}_1, \ldots, \mathbf{x}_{d-2}, \mathbf{w})\|\right)^{-1} \\
&\ll \sum_{\substack{\mathbf{r}\in\mathbb{Z}^n \\ |\mathbf{r}|\leq P}} \prod_{i=1}^{n} (1+|r_i|)^{-1} \sum_{|\mathbf{x}_1|<P} \cdots \sum_{|\mathbf{x}_{d-2}|<P} N(\mathbf{x}_1, \ldots, \mathbf{x}_{d-2}; \mathbf{r}) \\
&\ll \sum_{\substack{\mathbf{r}\in\mathbb{Z}^n \\ |\mathbf{r}|\leq P}} \prod_{i=1}^{n} (1+|r_i|)^{-1} \sum_{|\mathbf{x}_1|<P} \cdots \sum_{|\mathbf{x}_{d-2}|<P} n(\mathbf{x}_1, \ldots, \mathbf{x}_{d-2}) \\
&\ll (\log P)^n \sum_{|\mathbf{x}_1|<P} \cdots \sum_{|\mathbf{x}_{d-2}|<P} n(\mathbf{x}_1, \ldots, \mathbf{x}_{d-2}).
\end{aligned}$$

We therefore conclude that

$$L^{2^{d-1}} \ll P^{-(d-1)n} q^n \varphi^n (\log P)^n \mathcal{N}, \tag{4.6}$$

where $\mathcal{N}$ counts $(d-1)$-tuples of integer vectors $(\mathbf{x}_1, \ldots, \mathbf{x}_{d-2}, \mathbf{w})$ satisfying

$$|\mathbf{x}_i| < P \quad (1 \leq i \leq d-2) \quad \text{and} \quad |\mathbf{w}| < H,$$

such that

$$\|qF^{(i)}(\mathbf{x}_1, \ldots, \mathbf{x}_{d-2}, \mathbf{w})\| \leq P^{-1} \quad \text{for } 1 \leq i \leq n.$$

To estimate $\mathcal{N}$ we apply the following result, which is Lemma 3.3 of Davenport [8].

Lemma 4.2. *Let $L \in M_n(\mathbb{R})$ be a real symmetric $n \times n$ matrix. Let $a > 1$ and let*

$$N(Z) := \#\{\mathbf{u} \in \mathbb{Z}^n : |\mathbf{u}| < aZ,\ \|(L\mathbf{u})_i\| < a^{-1}Z\ \forall i \leq n\}.$$

Then, if $0 < Z_1 \leq Z_2 \leq 1$, we have

$$N(Z_2) \ll (Z_2/Z_1)^n N(Z_1).$$

We proceed to choose a parameter $K \geq 1$, as in Lemma 4.1. It follows in particular that $q\varphi K \geq 1$, since q and φ are at least 1. We then apply Lemma 4.2 to each of the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_{d-2}$ in succession. At the i-th step we use

$$a = P(q\varphi K)^{(i-1)/2}, \quad Z_1 = (q\varphi K)^{-(i+1)/2}, \quad Z_2 = (q\varphi K)^{-(i-1)/2}.$$

Finally, we apply Lemma 4.2 to $\mathbf{w}$ with

$$a = (HP)^{1/2}(q\varphi K)^{(d-2)/2},$$
$$Z_1 = K^{-1}H^{1/2}P^{-1/2}(q\varphi K)^{-(d-2)/2}, \quad Z_2 = H^{1/2}P^{-1/2}(q\varphi K)^{-(d-2)/2}.$$

One readily verifies that these choices satisfy the conditions for the lemma, and concludes that

$$\mathcal{N} \ll (q\varphi)^{(d-2)n} K^{(d-1)n} \mathcal{M},$$

where $\mathcal{M}$ is as in the statement of Lemma 4.1. The required estimate then follows on inserting this into (4.6).

5. The degree D case

We now return to the situation in Section 2. Suppose that we have a parameter $\alpha_{i,d} \in \mathbb{R}$ corresponding to each form $F_{i,d}$ for $1 \leq i \leq r_d$ and each $1 \leq d \leq D$. Recall that a box $\mathcal{B} \subseteq [-1, 1]^n$, a modulus $M \in \mathbb{N}$ and an integer vector $\mathbf{m}_0$ are given, and are fixed once for all.

We apply the work of the previous section with

$$f(\mathbf{x}) := \sum_{j=1}^{D} \sum_{i=1}^{r_j} \alpha_{i,j} F_{i,j}(M\mathbf{x} + \mathbf{m}_0) \quad \text{and} \quad g(\mathbf{x}) := 0.$$

If we take $\mathcal{B}' := \{\mathbf{x} : M\mathbf{x} + \mathbf{m}_0 \in P\mathcal{B}\}$ then $\mathcal{B}' \subseteq [-P, P]^n$ for large enough P (since $\mathbf{m}_0 = \mathbf{0}$ for $M = 1$). We may set $q = 1$ and $\varphi = 1$ in the notation of Section 4. Moreover the leading form of f has degree D and is given by

$$F(\mathbf{x}) := M^D \sum_{i=1}^{\rho} \alpha_{i,D} F_{i,D}(\mathbf{x}), \tag{5.1}$$

where we have written $r_D = \rho$ for brevity. Our problem now corresponds closely to that encountered by Birch [4], and we shall follow his line of attack. The outcome will be that either the exponential sum is small, or the coefficients $\alpha_{i,D}$ are all close to rationals with a small denominator. This denominator will be denoted by q, and is not to be confused with the number $q = 1$ above, which is related to the polynomial $g(\mathbf{x}) = 0$.

The analysis of the previous section shows that we have a bound of the shape in Lemma 4.1, in which the parameter K is at our disposal. We will take $K := \max\{1, K_1\}$ with

$$K_1 := P\left(\frac{L^{2^{D-1}}}{(\log P)^{n+1}}\right)^{1/(n-B_D)},$$

where B_D is given by (1.4). The reader should observe that it is perfectly permissible to use a value for K which depends on L. We now examine $\mathcal{M}$, considering three different cases. The first of these is that in which $K_1 \le 1$, so that

$$L^{2^{D-1}} \le P^{B_D - n}(\log P)^{n+1}.$$

This is satisfactory for our purposes (see Lemma 5.2). We will therefore assume henceforth that $K = K_1 > 1$.

The second case is that in which all the $(D-1)$-tuples counted by $\mathcal{M}$ correspond to elements of the set $\widehat{S}_D$ given by (2.8). In this situation we will apply the following estimate.

Lemma 5.1. *Let $d \le D$, let $P \ge 1$ and let $\mathcal{M}_0(P)$ be the number of $(d-1)$-tuples of vectors $(\mathbf{x}_1, \dots, \mathbf{x}_{d-1}) \in \widehat{S}_d(\mathbb{Z})$ having* $\max |\mathbf{x}_i| \le P$. *Then*

$$\mathcal{M}_0(P) \ll P^{B_d + n(d-2)}.$$

Proof. Since S_d is the intersection of $\widehat{S}_d$ with the diagonal $\mathrm{Diag} := \{(\mathbf{x}, \dots, \mathbf{x}) \in (\mathbb{A}^n)^{d-1}\}$, we see that

$$\dim(\widehat{S}_d) \le B_d + \mathrm{codim}(\mathrm{Diag}) = B_d + n(d-2).$$

We now apply Lemma 3.1 of Birch [4] to conclude the proof. □

Now, with the above notation, one has

$$\mathcal{M} \le \mathcal{M}_0(P/K) \ll (P/K)^{B_D + n(D-2)}.$$

In this case Lemma 4.1 yields

$$L^{2^{D-1}} \ll (K/P)^{n - B_D}(\log P)^n,$$

Since $K = K_1$ we deduce that

$$L^{2^{D-1}} \ll L^{2^{D-1}}(\log P)^{-1}.$$

Thus this second case cannot occur if P is sufficiently large.

This takes us to the third case, in which $K = K_1 > 1$ and there is some $(D-1)$-tuple counted by $\mathcal{M}$ for which

$$\mathrm{rank}(\widehat{J}_D(\mathbf{x}_1, \dots, \mathbf{x}_{D-1})) = r_D = \rho.$$

Suppose the matrix corresponding to columns $j_1, \dots, j_\rho$ has nonzero determinant. Calling the matrix W, we have

$$W_{ik} = F_{i,D}^{(j_k)}(\mathbf{x}_1, \dots, \mathbf{x}_{D-1}) \quad (1 \le i, k \le \rho),$$

where $F_{i,D}^{(j_k)}(\mathbf{x}_1, \ldots, \mathbf{x}_{D-1})$ is the j_k-th component of the row vector $\underline{F}_{i,D}(\mathbf{x}_1, \ldots, \mathbf{x}_{D-1})$. But then (5.1) yields

$$F^{(j_k)}(\mathbf{x}_1, \ldots, \mathbf{x}_{D-1}) = M^D \sum_{i=1}^{\rho} \alpha_{i,D} F_{i,D}^{(j_k)}(\mathbf{x}_1, \ldots, \mathbf{x}_{D-1}) = M^D \sum_{j=1}^{\rho} \alpha_{i,D} W_{ik}.$$

We record for future reference the fact that

$$H(W) \ll (\max |\mathbf{x}_h|)^{D-1} \ll (P/K_1)^{D-1}, \tag{5.2}$$

where we use $H(W)$ to denote the maximum of $|W_{jk}|$.

Since $(\mathbf{x}_1, \ldots, \mathbf{x}_{D-1})$ is counted by $\mathcal{M}$ it follows that

$$\left\| M^D \sum_{i=1}^{\rho} \alpha_{i,D} W_{ik} \right\| \leq \frac{1}{P K_1^{D-1}} \quad (1 \leq k \leq \rho).$$

We therefore write

$$M^D \sum_{i=1}^{\rho} \alpha_{i,D} W_{ik} = n_k + \xi_k \tag{5.3}$$

for $k = 1, \ldots, \rho$ with $n_k \in \mathbb{Z}$ and

$$|\xi_k| \leq \frac{1}{P K_1^{D-1}}.$$

We proceed to abbreviate the system (5.3) by writing

$$M^D W \underline{\alpha} = \underline{n} + \underline{\xi},$$

and then multiply by the adjoint, W' say, of W to see that

$$M^D \det(W) \underline{\alpha} = W' \underline{n} + W' \underline{\xi}.$$

However, W' is an integer matrix, with

$$H(W') \ll H(W)^{\rho-1} \ll (P/K_1)^{(D-1)(\rho-1)},$$

by (5.2). It follows that

$$\|M^D \det(W) \alpha_{i,D}\| \ll \left(\frac{P}{K_1}\right)^{(D-1)(\rho-1)} \frac{1}{P K_1^{D-1}}$$

for $i = 1, \ldots, \rho$. If we now write $q = M^D |\det(W)| \ll H(W)^{\rho}$, then q will be a positive integer, since we chose W to have nonzero determinant. Moreover for large enough P we will have $q \leq Q$, where

$$Q := (P/K_1)^{(D-1)\rho} \log P \quad \text{and} \quad \|q \alpha_{i,D}\| \leq Q P^{-D}.$$

We may now summarize all these conclusions as follows.

Lemma 5.2. *Let* $|S(\boldsymbol{\alpha})| = P^n L$ *and write* $\rho = r_D$. *Then if* P *is large enough, either*

$$L^{2^{D-1}} \le P^{B_D - n}(\log P)^{n+1}, \tag{5.4}$$

or there is a $q \le Q$ *with*

$$Q \le ((\log P)^{n+1} L^{-2^{D-1}})^{(D-1)\rho/(n-B_D)} \log P$$

such that

$$\|q\alpha_{i,D}\| \le QP^{-D} \quad (1 \le i \le \rho).$$

We now ask what one can say about the minor arc integral using Lemma 5.2. For any $L_0 > 0$ we write $\mathcal{A}(L_0)$ for the set of R-tuples of values $\alpha_{i,d}$ with $d \le D$, $i \le r_d$ such that $L_0 < L \le 2L_0$. Then if L_0 is such that (5.4) holds, the contribution to the minor arc integral will be

$$\ll P^{n+\varepsilon-(n-B_D)/2^{D-1}},$$

for any fixed $\varepsilon > 0$. This is satisfactory if $(n - B_D)/2^{D-1} > \mathcal{D}$, or in other words, if $n > B_D + 2^{D-1}\mathcal{D}$.

In the alternative case we see that there is an integer $q \le Q$ such that every $\alpha_{i,D}$, for $1 \le i \le \rho$, has an approximation

$$\alpha_{i,D} = a_{i,D}/q + O(QP^{-D}q^{-1})$$

with $a_{i,D} \in \mathbb{Z}$ and $0 \le a_{i,D} \le q$. Hence

$$\begin{aligned}\text{meas}(\mathcal{A}(L_0)) &\ll \sum_{q \le Q} q^\rho (QP^{-D}q^{-1})^\rho \ll Q^{1+\rho}P^{-D\rho} \\ &\ll L_0^{-2^{D-1}(D-1)\rho(1+\rho)/(n-B_D)} P^{\varepsilon - D\rho}.\end{aligned}$$

The corresponding contribution to the minor arc integral will therefore be

$$\ll L_0^{1-2^{D-1}(D-1)\rho(1+\rho)/(n-B_D)} P^{n+\varepsilon-D\rho}.$$

Hence, for example, if our system has forms of degree D only, then $\mathcal{D} = D\rho$ and we have a satisfactory bound when

$$n > B_D + \rho(\rho+1)2^{D-1}(D-1),$$

providing that L_0^{-1} exceeds some small fixed power of P. This corresponds precisely to the condition on n in (1.2).

6. Exponential sums—the iterative argument

In the previous section we showed that either $S(\boldsymbol{\alpha})$ (or equivalently L) is small, as expressed by (5.4), or the coefficients $\alpha_{i,D}$ all have good rational approximations with the same small denominator q. In this section we iterate this idea, assuming that we have good approximations for $\alpha_{i,j}$ for $d < j \le D$ and $1 \le i \le r_j$, and deducing either that L is small, or that the values $\alpha_{i,d}$ also have good rational approximations for $1 \le i \le r_d$.

Thus we suppose we have a degree $d < D$ in Δ, and we suppose that there is a positive integer $q \le Q$ such that

$$\|q\alpha_{i,j}\| \le QP^{-j} \quad \text{for } d < j \le D \text{ and } 1 \le i \le r_j.$$

We then define

$$f(\mathbf{x}) := \sum_{j=1}^{d}\sum_{i=1}^{r_j}\alpha_{i,j}F_{i,j}(M\mathbf{x}+\mathbf{m}_0), \qquad g(\mathbf{x}) := \sum_{j=d+1}^{D}\sum_{i=1}^{r_j}\alpha_{i,j}F_{i,j}(M\mathbf{x}+\mathbf{m}_0),$$

and we write $r_d =: \rho$ for brevity. Then the polynomial f has degree at most d and the leading form of degree d is now

$$F(\mathbf{x}) := M^d\sum_{i=1}^{\rho}\alpha_{i,d}F_{i,d}(\mathbf{x}).$$

We also write

$$\alpha_{i,j} := a_{i,j}/q + \theta_{i,j} \quad \text{for } d < j \le D \text{ and } 1 \le i \le r_j,$$

so that

$$|\theta_{i,j}| \le QP^{-j}q^{-1}.$$

To complete the setup we define

$$g_1(\mathbf{x}) := \sum_{j=d+1}^{D}\sum_{i=1}^{r_j}a_{i,j}F_{i,j}(M\mathbf{x}+\mathbf{m}_0), \qquad g_2(\mathbf{x}) := \sum_{j=d+1}^{D}\sum_{i=1}^{r_j}\theta_{i,j}F_{i,j}(M\mathbf{x}+\mathbf{m}_0).$$

Then $g = q^{-1}g_1 + g_2$ is in the required shape to apply the work of Section 4, and in particular we see that (4.1) holds with $\varphi = Q/q$.

We now proceed exactly as in the previous section, taking $K := \max\{1, K_1\}$ with

$$K_1 := PQ^{-1}\left(\frac{L^{2^{d-1}}}{(\log P)^{n+1}}\right)^{1/(n-B_d)}.$$

Then, if $K_1 \le 1$ as in the first case of the argument, we will have

$$L^{2^{d-1}} \le (P/Q)^{B_d-n}(\log P)^{n+1},$$

which will be satisfactory. The second case will be that in which all the $(d-1)$-tuples counted by $\mathcal{M}$ correspond to elements of the set $\widehat{S}_d$. Since $q\varphi = Q$ we then have

$$\mathcal{M} \le \mathcal{M}_0(P/QK) \ll \left(\frac{P}{QK}\right)^{B_d+n(d-2)}$$

by Lemma 5.1, after which Lemma 4.1 yields

$$L^{2^{d-1}} \ll (QK/P)^{n-B_d}(\log P)^n.$$

Since $K = K_1$ we deduce that

$$L^{2^{d-1}} \ll L^{2^{d-1}}(\log P)^{-1},$$

and as before we conclude that this second case cannot occur if P is sufficiently large.

The third case is that in which some $(d-1)$-tuple counted by $\mathcal{M}$ has

$$\mathrm{rank}(\widehat{J}_d(\mathbf{x}_1, \ldots, \mathbf{x}_{d-1})) = r_d = \rho.$$

Here the argument again follows that in the previous section, but now

$$H(W) \ll (\max |\mathbf{x}_h|)^{d-1} \ll \left(\frac{P}{QK_1}\right)^{d-1}$$

and

$$\left\| qM^d \sum_{i=1}^{\rho} \alpha_{i,d} W_{ik} \right\| \le \frac{1}{PQ^{d-2}K_1^{d-1}} \quad (1 \le k \le \rho).$$

This time we write

$$qM^d \sum_{i=1}^{\rho} \alpha_{i,d} W_{ik} = n_k + \xi_k$$

with $n_k \in \mathbb{Z}$ and

$$|\xi_k| \le \frac{1}{PQ^{d-2}K_1^{d-1}}.$$

We will then have

$$H(W') \ll H(W)^{\rho-1} \ll \left(\frac{P}{QK_1}\right)^{(d-1)(\rho-1)},$$

whence

$$\|qM^d \det(W)\alpha_{i,d}\| \ll \left(\frac{P}{QK_1}\right)^{(d-1)(\rho-1)} \frac{1}{PQ^{d-2}K_1^{d-1}}$$

for $i = 1, \ldots, \rho$. We therefore set $q^* := M^d|\det(W)| \ll H(W)^\rho$, so that $q^* \le Q^*$ with

$$Q^* := \left(\frac{P}{QK_1}\right)^{(d-1)\rho} \log P \quad \text{and} \quad \|qq^*\alpha_{i,d}\| \le QQ^*P^{-d}.$$

We may now summarize all these conclusions as follows.

Lemma 6.1. *Let* $|S(\boldsymbol{\alpha})| = P^n L$. *Suppose that* $d \in \Delta$ *and*

$$\|q\alpha_{i,j}\| \le QP^{-j} \quad \text{for } d < j \le D \text{ and } 1 \le i \le r_j$$

with $q \le Q$. *Then if* P *is large enough, either*

$$L^{2^{d-1}} \le (Q/P)^{n-B_d}(\log P)^{n+1},$$

or there is a $q^* \le Q^*$ *with*

$$Q^* := ((\log P)^{n+1}L^{-2^{d-1}})^{(d-1)r_d/(n-B_d)}\log P$$

such that

$$\|qq^*\alpha_{i,d}\| \le QQ^*P^{-d} \quad (1 \le i \le r_d).$$

We may of course interpret Lemma 5.2 as a special case of Lemma 6.1, corresponding to $d = D$ and $Q = 1$.

Our plan is to use Lemma 5.2 followed by repeated applications of Lemma 6.1 for the successively smaller values of $d \in \Delta$. Thus in Lemma 5.2 either

$$L^{2^{D-1}} \le P^{B_D - n}(\log P)^{n+1},$$

or there is a $q_D \le Q_D$ with

$$Q_D := ((\log P)^{n+1}L^{-2^{D-1}})^{(D-1)r_D/(n-B_D)}\log P$$

such that

$$\|q_D\alpha_{i,D}\| \le Q_D P^{-D} \quad (1 \le i \le r_D).$$

If the second case holds we may then apply Lemma 6.1 for degree

$$D' := \max\{d \in \Delta : d < D\}.$$

We then deduce that either

$$L^{2^{D'-1}} \le (Q_D/P)^{n-B_{D'}}(\log P)^{n+1},$$

or there is a $q_{D'} = q_D q^* \le Q_{D'} = Q_D Q^*$ with

$$Q^* := ((\log P)^{n+1}L^{-2^{D'-1}})^{(D'-1)r_{D'}/(n-B_{D'})}\log P$$

such that

$$\|q_{D'}\alpha_{i,D'}\| \le Q_{D'}P^{-D'} \quad (1 \le i \le r_{D'}).$$

Continuing in this manner we produce a succession of values Q_d for decreasing values of d in Δ, taking the form

$$Q_d := (\log P)^{e(d)}L^{-s_d} \quad (d \in \Delta), \tag{6.1}$$

where $e(d)$ is some easily computed but unimportant exponent, and s_d is given by (1.6).

When $1 \le j \le D$ but $j \notin \Delta$ it will be convenient to set $Q_j = Q_k$, where k is the smallest integer in Δ with $k > j$. We will also define $Q_{D+1} = 1$. In view of (1.6) we have $s_j = s_k$ so that (6.1) extends to give

$$Q_d = (\log P)^{e(d)} L^{-s_d} \quad (1 \le d \le D) \tag{6.2}$$

for appropriate exponents $e(d)$. Now, for a general exponent $j \in \Delta$, as we iterate we will either obtain a bound

$$L^{2^{j-1}} \le (Q_{j+1}/P)^{n-B_j} (\log P)^{n+1} \quad (j \in \Delta), \tag{6.3}$$

or find a positive integer q_j satisfying

$$q_k \mid q_j \quad (k > j,\ k \in \Delta), \tag{6.4}$$

$$q_j \le Q_j, \tag{6.5}$$

$$\|q_j \alpha_{i,j}\| \le Q_j P^{-j} \quad (1 \le i \le r_j). \tag{6.6}$$

When $1 \le j \le D$ but $j \notin \Delta$ it will also be convenient to set $q_j = q_k$, where k is the smallest integer in Δ with $k > j$. With this convention we then have $q_j \le Q_j$ and $q_j \mid q_{j+1}$ in general.

We can now partition the R-tuples $\boldsymbol{\alpha}$ into sets $I_d^{(1)}$ (for $d \in \Delta$) and $I^{(2)}$, as follows. The set $I_d^{(1)}$ consists of those $\boldsymbol{\alpha}$ for which (6.3) fails for $j > d$, but holds for $j = d$. The set $I^{(2)}$ then consists of the remaining R-tuples $\boldsymbol{\alpha}$, for which (6.3) fails for all $j \in \Delta$.

It follows from (6.1) that if (6.3) holds one has

$$L^{2^{j-1}+(n-B_j)s_{j+1}} \ll P^{B_j - n + \varepsilon}$$

for any fixed $\varepsilon > 0$. We therefore draw the following conclusion.

Lemma 6.2. *Let $d \in \Delta$ and $\boldsymbol{\alpha} \in I_d^{(1)}$. Then*

$$L^{2^{d-1}+(n-B_d)s_{d+1}} \ll P^{B_d - n + \varepsilon}. \tag{6.7}$$

Moreover there are positive integers q_j such that the conditions (6.4)–(6.6) hold for all relevant values of $j > d$.

Similarly, if $\boldsymbol{\alpha} \in I^{(2)}$ then there are positive integers q_j such that the conditions (6.4)–(6.6) hold for all values of $j \in \Delta$.

We conclude this section by remarking that it may be possible to improve on the above estimates in certain cases. The numbers q_d are built up from a sequence of factors. This would allow one to replace the argument in Section 4 by one in which there were several van der Corput steps, using various factors of q_d. In our present treatment, when one uses Lemma 4.2, the ratio Z_2/Z_1 is $q\varphi K$ for the first $d-2$ steps, and K for the final step. In the proposed variant these values become more equal, which should be to our advantage. However, this can only be of use when Δ contains at least three values $d \ge 3$, since the number q in our argument would need to have at least two factors, and so the number $d-1$ of squaring steps would have to be at least two.

7. The minor arc contribution

As in Section 5, for any $L_0 > 0$ we write $\mathcal{A}(L_0)$ for the set of R-tuples of values $\alpha_{i,d}$ with $d \le D$, $i \le r_d$ such that $L_0 < L \le 2L_0$. We also write $\mathcal{A}(L_0; I_d^{(1)}) := I_d^{(1)} \cap \mathcal{A}(L_0) \cap \mathfrak{m}$, and similarly for $\mathcal{A}(L_0; I^{(2)})$. In order to establish the required minor arc estimate (2.2) we begin by examining

$$T(I_d^{(1)}) := \int_{\mathcal{A}(L_0; I_d^{(1)})} |S(\boldsymbol{\alpha})|\, d\boldsymbol{\alpha}$$

for $d \in \Delta$.

When $d = D$ we have

$$L^{2^{D-1}} \le P^{B_D - n + \varepsilon},$$

by (6.7). Since $\mathrm{meas}(\mathcal{A}(L_0; I_D^{(1)})) \le 1$ and $|S(\boldsymbol{\alpha})| = P^n L$ it follows that

$$T(I_D^{(1)}) \ll P^n L_0 \ll P^{n-(n-B_D)/2^{D-1}+\varepsilon}.$$

Thus we will have a satisfactory estimate $T(I_D^{(1)}) \ll P^{n-\mathcal{D}-\delta}$, for some $\delta > 0$, provided that

$$\mathcal{D}\frac{2^{D-1}}{n - B_D} < 1. \tag{7.1}$$

We now consider the general case, in which $\boldsymbol{\alpha} \in I_d^{(1)}$ for some $d < D$ in Δ. Thus (6.7) holds, so that

$$L_0^{2^{d-1}/(n-B_d)+s_{d+1}} \ll P^{-1+\varepsilon}. \tag{7.2}$$

When $d = D$ we estimated $\mathrm{meas}(\mathcal{A}(L_0; I_D^{(1)}))$ trivially, but when $d < D$ we have useful information on the numbers $\alpha_{i,j}$ for $j > d$, since we know that (6.6) applies for these. Thus there are positive integers $q_D, q_{D-1}, \dots, q_{d+1}$ depending on $\boldsymbol{\alpha}$ and satisfying (6.4) and (6.5), such that

$$\alpha_{i,j} = a_{i,j}/q_j + O(q_j^{-1} Q_j P^{-j}) \qquad (d < j \le D,\ 1 \le i \le r_j)$$

with $0 \le a_{i,j} \le q_j$. Thus, given q_j, each $\alpha_{i,j}$ takes values in a set of measure $O(Q_j P^{-j})$, and the r_j-tuple $(\alpha_{1,j}, \dots, \alpha_{r_j,j})$ has values in a set of measure $O((Q_j P^{-j})^{r_j})$. At this point we recall our convention concerning the values of q_j and Q_j when $j \notin \Delta$. With this in mind we see that q_{d+1} determines $O(P^{\varepsilon})$ possibilities for $q_{d+2}, \dots, q_D$, by (6.4), and we conclude that

$$\mathrm{meas}(\mathcal{A}(L_0; I_d^{(1)})) \ll P^{\varepsilon} Q_{d+1} \prod_{j=d+1}^{D} (Q_j P^{-j})^{r_j}.$$

Hence, using (6.2), we obtain

$$\mathrm{meas}(\mathcal{A}(L_0; I_d^{(1)})) \ll P^{\varepsilon-(d+1)r_{d+1}-\cdots-Dr_D} L_0^{-(s_{d+1}+s_{d+1}r_{d+1}+\cdots+s_D r_D)}.$$

Recalling the notation (1.5) for $\mathcal{D}_j$ and that $|S(\boldsymbol{\alpha})| = P^n L$, it now follows that

$$T(I_d^{(1)}) \ll P^{n-\mathcal{D}+\mathcal{D}_d+\varepsilon} L_0^{1-(s_{d+1}+s_{d+1}r_{d+1}+\cdots+s_D r_D)}$$

with L_0 subject to (7.2). Thus we will have a satisfactory estimate $T(I_d^{(1)}) \ll P^{n-\mathcal{D}-\delta}$ for some $\delta > 0$ provided that

$$\mathcal{D}_d\left(\frac{2^{d-1}}{n-B_d} + s_{d+1}\right) + s_{d+1} + \sum_{j=d+1}^{D} s_j r_j < 1. \tag{7.3}$$

It is clear now that the corresponding condition (7.1) for $d = D$ is just a special case of this.

For the integral

$$T(I^{(2)}) := \int_{\mathcal{A}(L_0; I^{(2)})} |S(\boldsymbol{\alpha})|\, d\boldsymbol{\alpha}$$

we will provide an estimate for L by using the fact that our R-tuple $\boldsymbol{\alpha}$ belongs to $\mathfrak{m}$. It follows from (6.4)–(6.6) that

$$\|q_1\alpha_{i,d}\| \le q_1 q_d^{-1} Q_d P^{-d} \le Q_1 Q_d P^{-d}$$

for $1 \le d \le D$ and $i \le r_d$. If we write $s_{\max} = \max s_d$ and $e_{\max} = \max e(d)$, then (6.2) yields

$$\|q_1\alpha_{i,d}\| \le L^{-2s_{\max}} P^{-d} (\log P)^{2e_{\max}} \quad \text{with } q_1 \le Q_1 \le L^{-s_{\max}} (\log P)^{e_{\max}}.$$

Let ϖ be as in Section 2. Then if P is large enough one would have

$$\|q_1\alpha_{i,d}\| \le P^{-d+\varpi} \quad (1 \le d \le D,\ i \le r_d) \quad \text{with } q_1 \le P^{\varpi}$$

so long as

$$L \ge P^{-\varpi/4s_{\max}}.$$

However, this would place $\boldsymbol{\alpha}$ in the major arcs, in view of the definition (2.6). We therefore conclude that

$$L_0 \ll P^{-\varpi/4s_{\max}} \tag{7.4}$$

for $\boldsymbol{\alpha} \in I^{(2)}$.

We can now use (6.4)–(6.6) as before to show that

$$\operatorname{meas}(\mathcal{A}(L_0; I^{(2)})) \ll P^{\varepsilon} Q_1 \prod_{j=1}^{D} (Q_j P^{-j})^{r_j} \ll P^{\varepsilon - r_1 - 2r_2 - \cdots - D r_D} L_0^{-(s_1+s_1r_1+\cdots+s_D r_D)}.$$

It follows that

$$T(I^{(2)}) \ll P^{n-\mathcal{D}+\varepsilon} L_0^{1-(s_1+s_1r_1+\cdots+s_D r_D)}.$$

In view of (7.4) we obtain a satisfactory bound $T(I^{(2)}) \ll P^{n-\mathcal{D}-\delta}$, for some $\delta > 0$, under the constraint

$$s_1 + \sum_{j=1}^{D} s_j r_j < 1.$$

The reader should notice that this condition is the case $d = 0$ of (7.3), since we have defined $\mathcal{D}_0 = 0$ in connection with (1.5).

We therefore see that we have a satisfactory minor arc estimate provided that (7.3) holds for all $d \in \Delta \cup \{0\}$, as required for Theorem 1.2.

8. The major arc contribution

We now turn to the major arc analysis, with the goal of establishing (2.1) under suitable hypotheses on $\mathfrak{M}$ and the forms $(F_{i,d})$. Let us define

$$S(\mathbf{a}, q) := \sum_{\mathbf{x} \,(\mathrm{mod}\, q)} e_q\Big(\sum_{d=1}^{D} \sum_{i=1}^{r_d} a_{i,d} F_{i,d}(M\mathbf{x} + \mathbf{m}_0)\Big)$$

for $\mathbf{a} = (a_{i,d}) \in (\mathbb{Z}/q\mathbb{Z})^R$ with $\gcd(q, \mathbf{a}) = 1$. Next, define the truncated singular series

$$\mathfrak{S}(H) := \sum_{q \le H} \frac{1}{q^n} \sum_{\substack{\mathbf{a} \,(\mathrm{mod}\, q) \\ \gcd(q,\mathbf{a})=1}} S(\mathbf{a}, q)$$

for any $H > 0$. We set $\mathfrak{S} = \lim_{H\to\infty} \mathfrak{S}(H)$ whenever this limit exists. We will also need to study the integral

$$\mathfrak{I}(H) := \frac{1}{M^n} \int_{[-H,H]^R} \int_{\mathcal{B}} e\Big(\sum_{d=1}^{D} \sum_{i=1}^{r_d} \gamma_{i,d} F_{i,d}(\mathbf{x})\Big)\, d\mathbf{x}\, d\boldsymbol{\gamma} \tag{8.1}$$

for any $H > 0$, where $\boldsymbol{\gamma} = (\gamma_{i,d})$. Recalling (2.3), we have $\sigma_\infty = \lim_{H\to\infty} \mathfrak{I}(H)$ whenever the limit exists. The main aim of this section is to establish the following result.

Lemma 8.1. *Assume that*

$$s_1 + \sum_{j=1}^{D} s_j r_j < 1. \tag{8.2}$$

Then the singular series $\mathfrak{S}$ and the singular integral $\mathfrak{I}$ are absolutely convergent. Moreover, if we choose

$$\varpi = \frac{1}{2R+4} \tag{8.3}$$

then there is a positive constant δ such that

$$\int_{\mathfrak{M}} S(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} = \mathfrak{S}\mathfrak{I} P^{n-\mathcal{D}} + O(P^{n-\mathcal{D}-\delta}).$$

The condition in the lemma is the case $d = 0$ of the condition in Theorem 1.2. Once the lemma is established we will have $\mathfrak{S} = \prod_p \sigma_p$ by the argument of Davenport [9, Chapter 17], for example. We leave the details to the reader. Theorem 1.2 then follows.

Recall the definition of the major arcs $\mathfrak{M}$ from Section 2, defined in terms of the parameter $\varpi \in (0, 1/3)$. Any $\boldsymbol{\alpha} \in \mathfrak{M}_{q,\mathbf{a}}$ can be written

$$\alpha_{i,d} = a_{i,d}/q + \theta_{i,d}$$

for $1 \le i \le r_d$ and $d \in \Delta$. Our first step in the analysis of $S(\boldsymbol{\alpha})$ on $\mathfrak{M}$ is an analogue of [4, Lemma 5.1]. The argument is well-known and we leave the details to the reader. It leads to the conclusion that

$$S(\boldsymbol{\alpha}) - (qM)^{-n} P^n S(\mathbf{a}, q) J(\boldsymbol{\gamma}) \ll q \sum_{1\le d\le D} \sum_{1\le i\le r_d} |\theta_{i,d}| P^{n+d-1} + qP^{n-1}, \tag{8.4}$$

where $J(\boldsymbol{\gamma})$ is given by (2.4), and $\boldsymbol{\gamma}$ is the vector whose i, d entry is $\theta_{i,d} P^d$. But then

$$S(\boldsymbol{\alpha}) = (qM)^{-n} P^n S(\mathbf{a}, q) J(\boldsymbol{\gamma}) + O(P^{n-1+2\varpi})$$

for any $\boldsymbol{\alpha} \in \mathfrak{M}$. The major arcs are easily seen to have measure $O(P^{-\mathcal{D}+(2R+1)\varpi})$. Hence

$$\int_{\mathfrak{M}} S(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} = P^{n-\mathcal{D}} \mathfrak{S}(P^{\varpi}) \mathfrak{J}(P^{\varpi}) + O(P^{n-\mathcal{D}-1+(2R+1)\varpi+2\varpi}).$$

This error term is satisfactory for Lemma 8.1 if ϖ is taken as in (8.3).

In order to complete the proof of the lemma, it remains to show that $\mathfrak{S}$ and $\mathfrak{J}$ are absolutely convergent when (8.2) holds, and that there is a positive constant δ such that

$$\mathfrak{S} - \mathfrak{S}(H) \ll H^{-\delta} \tag{8.5}$$

and

$$\mathfrak{J} - \mathfrak{J}(H) \ll H^{-\delta}, \tag{8.6}$$

for any $H > 0$.

Beginning with the singular series, we proceed to use (8.4) and our Weyl estimate Lemma 6.2 to estimate the complete exponential sum $S(\mathbf{a}, q)$, as follows.

Lemma 8.2. *Let $\varepsilon > 0$ be given. Then*

$$S(\mathbf{a}, q) \ll q^{n+\varepsilon} \min_{j\in\Delta} \left(\frac{\gcd(q, \mathbf{a}^{(j)}, \dots, \mathbf{a}^{(D)})}{q} \right)^{1/s_j},$$

where $\mathbf{a}^{(j)} = (a_{1,j}, \dots, a_{r_j,j})$ for any $j \in \Delta$.

Proof. Noting that $J(\mathbf{0}) \gg 1$, we may take $(\theta_{i,d}) = \mathbf{0}$ in (8.4) to conclude that

$$S(\mathbf{a}, q) \ll \frac{q^n |S(\boldsymbol{\alpha})|}{P^n} + \frac{q^{n+1}}{P} \quad \text{with } \boldsymbol{\alpha} := a^{-1}\mathbf{a}.$$

In what follows we will take $P = q^A$ for some large value of A to be specified during the course of the proof. Assuming that $A > n + 1$, in the first instance, it follows from the previous bound that

$$S(\mathbf{a}, q) \ll 1 + q^n L, \tag{8.7}$$

where L is defined via $|S(\boldsymbol{\alpha})| = P^n L$. We now apply Lemma 6.2.

If there exists $d \in \Delta$ such that $\boldsymbol{\alpha} \in I_d^{(1)}$ then L satisfies (6.7). Once combined with (8.7), this gives

$$S(\mathbf{a}, q) \ll 1 + \frac{q^n P^\varepsilon}{P^{(n-B_d)/(2^{d-1}+(n-B_d)s_{d+1})}}.$$

This is $O(1)$ provided A satisfies $A(n - B_d) > (2^{d-1} + (n - B_d)s_{d+1})n$.

Suppose next that $\boldsymbol{\alpha} \in I^{(2)}$. Then Lemma 6.2 produces a sequence of positive integers q_j, for $j \in \Delta$, which satisfy the conditions (6.4)–(6.6). For each $j \in \Delta$ and $i \le r_j$ we may choose $b_{i,j} \in \mathbb{Z}$ and $z_{i,j} \in \mathbb{R}$ such that

$$q_j a_{i,j}/q = b_{i,j} + z_{i,j}$$

with $|z_{i,j}| \le Q_j P^{-j}$. If there is a choice of i, j for which $q_j a_{i,j} \neq q b_{i,j}$, then we would be able to conclude that

$$\frac{1}{q q_j} \le \frac{|z_{i,j}|}{q_j} \le \frac{Q_j P^{-j}}{q_j} \ll \frac{L^{-s_j} P^{-j+\varepsilon}}{q_j},$$

by (6.2). But then $L^{s_j} \ll q P^{-j+\varepsilon}$, which once substituted into (8.7), would show that $S(\mathbf{a}, q) \ll 1$ provided A satisfies $jA - 1 > ns_j$.

We may therefore proceed under the assumption that $q_j a_{i,j} = q b_{i,j}$ for every $j \in \Delta$ and every $i \le r_j$, or in other words, that $q_j \mathbf{a}^{(j)} = q \mathbf{b}^{(j)}$. This implies that

$$q \mid \gcd(q q_j, q \mathbf{b}^{(j)}) = \gcd(q q_j, q_j \mathbf{a}^{(j)}) = q_j \gcd(q, \mathbf{a}^{(j)}).$$

Moreover, in view of (6.4), we have $\gcd(q, \mathbf{a}^{(j)}) \mid \gcd(q, \mathbf{a}^{(k)})$ when $k > j$ and $k \in \Delta$. Thus

$$q \mid q_j \gcd(q, \mathbf{a}^{(j)}, \dots, \mathbf{a}^{(D)})$$

for every $j \in \Delta$. Applying (6.5) and (6.2) we are therefore led to the conclusion that

$$\frac{q}{\gcd(q, \mathbf{a}^{(j)}, \dots, \mathbf{a}^{(D)})} \le q_j \le Q_j = (\log P)^{e(j)} L^{-s_j}$$

for every $j \in \Delta$. Since $(\log P)^{e(j)/s_j} \ll q^\varepsilon$, this produces an upper bound for L which we substitute into (8.7) to arrive at the statement of the lemma. □

Using this result we may now handle the singular series. Let

$$A(q) := \sum_{\substack{\mathbf{a} \,(\mathrm{mod}\, q) \\ \gcd(q,\mathbf{a})=1}} |S(\mathbf{a}, q)|.$$

Set $d_j = \gcd(q, \mathbf{a}^{(j)}, \dots, \mathbf{a}^{(D)})$ for each $j \in \Delta$. Suppose that j_0 is the least index $j \in \Delta$. Then $d_{j_0} = 1$ since $\gcd(q, \mathbf{a}) = 1$. Moreover, we have $d_j \mid q$ for every $j \in \Delta$. The number of $\mathbf{a}^{(j)}$ (mod q) associated to a given d_j is $(q/d_j)^{r_j}$. Moreover the total number of $d_1, \dots, d_D$ associated to a given q is at most $\tau(q)^D = O(q^{\varepsilon})$. Next we note that

$$\min_{j\in\Delta} \left(\frac{d_j}{q}\right)^{1/s_j} \le \prod_{j\in\Delta} \left(\frac{d_j}{q}\right)^{\lambda_j/s_j}$$

for any real numbers $\lambda_j \ge 0$ such that $\sum_{j\in\Delta} \lambda_j = 1$. We will apply this with

$$\lambda_j := \begin{cases} \theta + r_{j_0} s_{j_0} & \text{if } j = j_0, \\ r_j s_j & \text{if } j \in \Delta \setminus \{j_0\}, \end{cases} \tag{8.8}$$

where $\theta := 1 - (s_1 r_1 + \cdots + s_D r_D)$. In view of our assumption (8.2), such a choice is possible with $\theta \in (0, 1)$. It therefore follows from Lemma 8.2 that

$$A(q) \ll q^{n+\varepsilon/2} \sum_{d_1,\dots,d_D \mid q} \left(\frac{1}{q}\right)^{\theta/s_{j_0}} \prod_{j\in\Delta} \left(\frac{q}{d_j}\right)^{r_j} \cdot \left(\frac{d_j}{q}\right)^{r_j} \ll q^{n-\theta/s_{j_0}+\varepsilon}.$$

Assuming that $\theta/s_{j_0} > 1$, which is evidently implied by (8.2), this shows that the singular series is absolutely convergent and that (8.5) holds for an appropriate $\delta > 0$.

We now turn to the exponential integral $J(\boldsymbol{\gamma})$ in (2.4) for general values of $\boldsymbol{\gamma}$.

Lemma 8.3. *We have $J(\boldsymbol{\gamma}) \ll 1$ for any $\boldsymbol{\gamma}$. Moreover, for given $\varepsilon > 0$, we have*

$$J(\boldsymbol{\gamma}) \ll |\boldsymbol{\gamma}|^{\varepsilon} \min_{j\in\Delta} |\boldsymbol{\gamma}^{(j)}|^{-1/s_j},$$

where $\boldsymbol{\gamma}^{(j)} = (\gamma_{1,j}, \dots, \gamma_{r_j,j})$.

Proof. The estimate $J(\boldsymbol{\gamma}) \ll 1$ is trivial. We proceed to establish the bound

$$J(\boldsymbol{\gamma}) \ll |\boldsymbol{\gamma}^{(j)}|^{-1/s_j} |\boldsymbol{\gamma}|^{\varepsilon}$$

for any $j \in \Delta$. In doing so we may assume that $|\boldsymbol{\gamma}^{(j)}| > 1$, since otherwise it follows from the trivial bound.

Our proof is analogous to the proof of Lemma 8.2. The starting point is (8.4), which we apply with $\boldsymbol{\alpha} = (P^{-d}\gamma_{i,d})$, $\mathbf{a} = \mathbf{0}$ and $q = 1$. This gives

$$|J(\boldsymbol{\gamma})| \le P^{-n} |S(\boldsymbol{\alpha})| + O(|\boldsymbol{\gamma}| P^{-1}) = L + O(|\boldsymbol{\gamma}| P^{-1}). \tag{8.9}$$

We take $P = |\boldsymbol{\gamma}|^A$ for some large value of A to be specified during the course of the proof. Our key ingredient is Lemma 6.2. The case in which $\boldsymbol{\alpha} \in I_d^{(1)}$ for some $d \in \Delta$ is easily dispatched on taking A to satisfy $A-1 > 1/s_j$ and $A(n-B_d) > (2^{d-1}+(n-B_d)s_{d+1})/s_j$.

It remains to consider the possibility that $\boldsymbol{\alpha} \in I^{(2)}$. Then Lemma 6.2 produces a positive integer q_j which satisfies the conditions (6.5) and (6.6). For each $i \leq r_j$ we may choose $b_{i,j} \in \mathbb{Z}$ and $z_{i,j} \in \mathbb{R}$ such that

$$q_j \alpha_{i,j} = b_{i,j} + z_{i,j} \tag{8.10}$$

with $\gcd(q_j, \mathbf{b}^{(j)}) = 1$ and $|z_{i,j}| \leq Q_j P^{-j}$. If there is a choice of $i \leq r_j$ for which $b_{i,j} \neq 0$, then we would be able to conclude that

$$1 \leq |b_{i,j}| \leq q_j |\alpha_{i,j}| + |z_{i,j}| \leq q_j P^{-j} |\gamma_{i,j}| + Q_j P^{-j},$$

whence

$$1 \leq Q_j P^{-j} |\gamma_{i,j}| + Q_j P^{-j} \ll Q_j P^{-j} |\boldsymbol{\gamma}^{(j)}| \ll L^{-s_j} P^{-j+\varepsilon} |\boldsymbol{\gamma}^{(j)}|.$$

This provides an upper bound for L, which once substituted into (8.9), produces a satisfactory estimate for $J(\boldsymbol{\gamma})$ provided that A is chosen to satisfy $A - 1 > 1/s_j$ and $A > 2/j$.

We proceed under the assumption that $b_{i,j} = 0$ in (8.10) for every $i \leq r_j$. But then $q_j = 1$ and it follows that

$$P^{-j} |\gamma_{i,j}| = |\alpha_{i,j}| = |z_{i,j}| \leq Q_j P^{-j} = (\log P)^{e(j)} L^{-s_j} P^{-j}$$

for every $i \leq r_j$. Hence $L \ll |\boldsymbol{\gamma}^{(j)}|^{-1/s_j} (\log P)^{e(j)/s_j}$. Substituting this into (8.9), we easily conclude the proof of the lemma. □

We now have everything in place to show that the singular integral converges. Recalling (8.1) and appealing to Lemma 8.3, we find that

$$|\mathfrak{J} - \mathfrak{J}(H)| \leq \int_{|\boldsymbol{\gamma}| > H} |J(\boldsymbol{\gamma})| \, d\boldsymbol{\gamma} \ll \int_{|\boldsymbol{\gamma}| > H} |\boldsymbol{\gamma}|^{\varepsilon/2} \min_{j \in \Delta} |\boldsymbol{\gamma}^{(j)}|^{-1/s_j} \, d\boldsymbol{\gamma}.$$

Let $N := \#\Delta$ and let $\mathbf{t} \in \mathbb{R}^N_{>0}$. For given $j \in \Delta$, the set of $\boldsymbol{\gamma}^{(j)} \in \mathbb{R}^{r_j}$ satisfying $|\boldsymbol{\gamma}^{(j)}| = t_j$ has $(r_j - 1)$-dimensional measure $O(t_j^{r_j - 1})$. Hence

$$\begin{aligned} |\mathfrak{J} - \mathfrak{J}(H)| &\ll \int_{\substack{\mathbf{t} \in \mathbb{R}^N_{>0} \\ |\mathbf{t}| > H}} |\mathbf{t}|^{\varepsilon/2} \min_{j \in \Delta} \{t_j^{-1/s_j}\} \Big(\prod_{j \in \Delta} t_j^{r_j - 1} \Big) \, d\mathbf{t} \\ &\leq \int_{\substack{\mathbf{t} \in \mathbb{R}^N_{>0} \\ |\mathbf{t}| > H}} |\mathbf{t}|^{\varepsilon} \min_{j \in \Delta} \{t_j^{-1/s_j}\} \Big(\prod_{j \in \Delta} t_j^{r_j - 1 - \varepsilon/(2N)} \Big) \, d\mathbf{t}. \end{aligned}$$

We will consider the contribution to the right hand side from $\mathbf{t}$ for which $|\mathbf{t}| = t_{j_0}$ for some $j_0 \in \Delta$. If $H \geq 1$ we have

$$\min_{j \in \Delta} \{t_j^{-1/s_j}\} \ll \min_{j \in \Delta} \{(1 + t_j)^{-1/s_j}\} \leq \prod_{j \in \Delta} (1 + t_j)^{-\lambda_j / s_j},$$

with λ_j given by (8.8). This therefore leads to the overall contribution

$$\ll \int_{\substack{\mathbf{t} \in \mathbb{R}^N_{>0} \\ |\mathbf{t}| = t_{j_0} > H}} \frac{t_{j_0}^{\varepsilon - \theta/s_{j_0}}}{\prod_{j \in \Delta} (1 + t_j)^{1 + \varepsilon/(2N)}} \, d\mathbf{t} \ll \int_H^\infty t_{j_0}^{\varepsilon - \theta/s_{j_0} - 1} \, dt_{j_0}.$$

This establishes (8.6) for a suitable $\delta > 0$, as required, provided only that $\theta > 0$. Recalling that $\theta = 1 - (s_1r_1 + \cdots + s_Dr_D)$, this condition is ensured by (8.2), which thereby completes the proof of Lemma 8.1.

9. Proof of Theorem 1.6

We begin by disposing of the case in which D is the only degree present, so that $r_D = R$ and $\mathcal{D} = RD$. In this situation

$$n_0 = R(R+1)(D-1)2^{D-1}$$

as in Birch's result [4]. Thus Theorem 1.6 is trivial in the case $D = 1$, and for $D \geq 2$ we have to show that $n_0 + R - 1 \leq R^2D^22^{D-1}$ and $n_0 + R - 1 \leq (RD-1)2^{RD}$. However,

$$\begin{aligned} R(R+1)(D-1)2^{D-1} + R - 1 &\leq \{R(R+1)(D-1) + R\}2^{D-1} \\ &\leq (2R^2(D-1) + R^2)2^{D-1} \leq R^2D^22^{D-1} \end{aligned}$$

since $2D - 1 \leq D^2$. The first estimate then follows. For the second, we observe that

$$\begin{aligned} R(R+1)(D-1)2^{D-1} + R - 1 &\leq \{R(R+1)(D-1) + R - 1\}2^{D-1} \\ &\leq \{R(R+1) + R - 1\}(RD-1)2^{D-1}, \end{aligned}$$

since we are now supposing that $D \geq 2$. However, $R(R+1) + R - 1 \leq 2^{2R-1}$ for any $R \geq 1$ and $2^{D-1+2R-1} \leq 2^{RD}$ for $D \geq 2$. This establishes the second estimate.

We may assume henceforth that not all the forms have the same degree, whence $R \geq 2$ and $D \geq 2$. We also note that $D + R - 1 \leq \mathcal{D} \leq DR - 1$. We now proceed to dispose of the case in which $n_0 = n_0(D)$. We have $n_0(D) = \mathcal{D}2^{D-1}$, so that we need to show that

$$\mathcal{D}2^{D-1} + R - 1 \leq \mathcal{D}^22^{D-1} \quad \text{and} \quad \mathcal{D}2^{D-1} + R - 1 \leq (\mathcal{D}-1)2^{\mathcal{D}}.$$

We begin by observing that

$$\mathcal{D}2^{D-1} + R - 1 \leq (\mathcal{D} + R - 1)2^{D-1} \leq 2\mathcal{D}2^{D-1}.$$

The first bound then follows since $2\mathcal{D} \leq \mathcal{D}^2$. Moreover $2\mathcal{D} \leq 4(\mathcal{D}-1)$ and $D + 1 \leq \mathcal{D}$, whence

$$2\mathcal{D}2^{D-1} \leq (\mathcal{D}-1)2^{D+1} \leq (\mathcal{D}-1)2^{\mathcal{D}},$$

as required for the second bound.

For the rest of our argument we examine $n_0(d)$ for $d < D$, and we shall assume that $\#\Delta \geq 2$. This allows us to set $E := \max\{d \in \Delta : d < D\}$. We begin by observing that

$$t_d = \sum_{k=d}^{D} 2^{k-1}(k-1)r_k \leq 2^{D-1}\sum_{k=1}^{D}(k-1)r_k = 2^{D-1}(\mathcal{D} - R)$$

for every $d \geq 1$, whence

$$t_{d+1} + \sum_{j=d+1}^{D} t_j r_j \leq 2^{D-1}(\mathcal{D} - R)\{1 + \sum_{j=1}^{D} r_j\} = 2^{D-1}(\mathcal{D} - R)(1 + R).$$

We also have

$$\mathcal{D}_d \leq \mathcal{D} - D \leq \mathcal{D} - 2 \quad \text{and} \quad \mathcal{D}_d \leq E(R-1)$$

for $0 \leq d \leq D-1$. Thus

$$n_0(d) \leq 2^{D-1}\{(\mathcal{D}-2)(\mathcal{D}-R+1) + (\mathcal{D}-R)(1+R)\}$$

and

$$n_0(d) \leq 2^{D-1}\{E(R-1)(\mathcal{D}-R+1) + (\mathcal{D}-R)(1+R)\},$$

for $0 \leq d \leq D-1$.

It will therefore suffice to show that

$$2^{D-1}\{(\mathcal{D}-2)(\mathcal{D}-R+1) + (\mathcal{D}-R)(1+R)\} + R - 1 \leq \mathcal{D}^2 2^{D-1}$$

and

$$2^{D-1}\{E(R-1)(\mathcal{D}-R+1) + (\mathcal{D}-R)(1+R)\} + R - 1 \leq (\mathcal{D}-1)2^{\mathcal{D}}.$$

For the first inequality we note that the left hand side is at most

$$\begin{aligned} &2^{D-1}\{(\mathcal{D}-2)(\mathcal{D}-R+1) + (\mathcal{D}-R)(1+R) + R - 1\} \\ &\qquad = 2^{D-1}\{\mathcal{D}^2 - R^2 + 2R - 3\} \leq 2^{D-1}\mathcal{D}^2. \end{aligned}$$

For the second inequality one sees that the left hand side is at most

$$2^{D-1}\{E(R-1)(\mathcal{D}-R+1) + (\mathcal{D}-R)(1+R) + R\},$$

and

$$\begin{aligned} E(R-1)&(\mathcal{D}-R+1) + (\mathcal{D}-R)(1+R) + R \\ &\leq E(R-1)(\mathcal{D}-1) + (\mathcal{D}-1)(R+1) \\ &= \{ER - E + R + 1\}(\mathcal{D}-1) \leq 2RE(\mathcal{D}-1). \end{aligned}$$

To complete the argument we observe that $R \leq 2^{R-1}$ and $E \leq 2^{E-1}$. Moreover, since $D + R + E - 2 \leq \mathcal{D}$, we also have $2^{D-1+R-1+E-1} \leq 2^{\mathcal{D}-1}$.

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