J. Eur. Math. Soc. 19, 395–439

DOI 10.4171/JEMS/669

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Essential connectedness and the rigidity problem for Gaussian symmetrization

Received February 20, 2014

Abstract. We provide a geometric characterization of rigidity of equality cases in Ehrhard's symmetrization inequality for Gaussian perimeter. This condition is formulated in terms of a new measure-theoretic notion of connectedness for Borel sets, inspired by Federer's definition of indecomposable current, and of possible broader interest.

Keywords. Symmetrization, rigidity, equality cases, Gauss space

1. Introduction

1.1. Overview

Symmetrization inequalities are among the most basic tools used in the calculus of variations. The study of their equality cases plays a fundamental role in the explicit characterization of minimizers, thus in the computation of optimal constants in geometric and functional inequalities. Although it is usually easy to derive useful necessary conditions for equality cases, the analysis of *rigidity of equality cases* (that is, the situation when every set realizing equality in the given symmetrization inequality turns out to be symmetric) is a much subtler issue. Two deep results that provide *sufficient conditions* for the rigidity of equality cases are the Brothers–Ziemer theorem concerning Schwartz's symmetrization inequality for Dirichlet-type integral functionals [BZ88], and the Chlebík–Cianchi–Fusco theorem, concerning Steiner's symmetrization inequality for distributional perimeter [CCF05] (see [BCF13] for an extension of this last result to higher dimensional Steiner's symmetrization). In this paper we introduce a new point of view on

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Mathematics Subject Classification (2010): Primary 49K21

rigidity of equality cases, which will allow us to provide *characterizations* of rigidity (rather than merely sufficient conditions) in various situations.

We address the case of Ehrhard's symmetrization inequality for Gaussian perimeter. Ehrhard's symmetrization is a powerful device in the analysis of geometric variational problems in the Gauss space, the versatility of which is well-known in probability theory. Rigidity of equality cases for Ehrhard's inequality are an open problem, even at the level of finding sufficient conditions for rigidity. Theorem 1.3 below completely solves this problem, by providing a geometric characterization of rigidity of equality cases.

This characterization is obtained in terms of a measure-theoretic notion of connectedness, named *essential connectedness*, which is meaningful in the very general context of Borel sets, and is inspired by the notion of indecomposable current adopted in geometric measure theory [Fed69, 4.2.25]. We shall actually make precise the more general idea of having one (Borel) set disconnecting another (Borel) set. The rigidity results of this paper validate these new notions of connectedness as natural and usable concepts.

Constancy results for functions of bounded variation, based on the notion of indecomposability, have been exploited in nonlinear elasticity [DM95] and image reconstruction [ACMM01]. In the companion paper [CCDPM13] we address the rigidity problem in Steiner's inequality for Euclidean perimeter, and we exploit essential connectedness to formulate constancy results for measurable functions with no distributional gradient structure (namely, the functions defined by the barycenters of one-dimensional orthogonal sections of sets of finite perimeter). As similarly rough functions naturally arise in other problems (e.g., as minimizers of variational limits in singular perturbations models [DLO03]), we expect these ideas to be useful in obtaining rigidity results also outside the context of symmetrization inequalities.

The rest of this introduction is organized as follows. In Section 1.2 we introduce Gaussian perimeter, together with the Gaussian isoperimetric problem. This important variational problem motivates the notion of Ehrhard's symmetrization, presented in Section 1.3. In Sections 1.4 and 1.5 we introduce, respectively, the rigidity problem for Ehrhard's inequality, and the measure-theoretic notion of connectedness we shall exploit in its solution. In Section 1.6 we state our main result, Theorem 1.3, together with its proper reformulation in the planar setting.

1.2. Gaussian perimeter and the Gaussian isoperimetric problem

We introduce our setting. Given a Lebesgue measurable set $E \subset \mathbb{R}^n$, we define its *Gaussian volume* as

$$\gamma_n(E) = \frac{1}{(2\pi)^{n/2}} \int_E e^{-|x|^2/2}\, dx.$$

If $n \geq k \geq 1$, the k-dimensional *Gaussian-Hausdorff measure* of a Borel set $S \subset \mathbb{R}^n$ is

$$\mathcal{H}^k_\gamma(S) = \frac{1}{(2\pi)^{k/2}} \int_S e^{-|x|^2/2}\, d\mathcal{H}^k(x),$$

where $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure on $\mathbb{R}^n$. (In this way, $\gamma_n = \mathcal{H}^n_\gamma$ and $\mathcal{H}^k_\gamma(S) = 1$ whenever S is a k-dimensional plane containing the origin.) The *Gaussian* *perimeter* of an open set E with Lipschitz boundary is then defined as

$$P_\gamma(E) = \mathcal{H}^{n-1}_\gamma(\partial E) = \frac{1}{(2\pi)^{(n-1)/2}} \int_{\partial E} e^{-|x|^2/2}\, d\mathcal{H}^{n-1}(x). \tag{1.1}$$

The most basic geometric variational problem in the Gauss space is, of course, the *Gaussian isoperimetric problem*, which consists in the minimization of Gaussian perimeter at fixed Gaussian volume. As it turns out, (the only) isoperimetric sets are half-spaces. The Gaussian isoperimetric theorem can be translated into a geometric inequality, with a characterization of equality cases. Indeed, if we define $\Phi : \mathbb{R} \cup \{\pm\infty\} \to [0,1]$ and $\Psi = \Phi^{-1} : [0,1] \to \mathbb{R} \cup \{\pm\infty\}$ by setting

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-s^2/2}\, ds, \qquad t \in \mathbb{R} \cup \{\pm\infty\}, \tag{1.2}$$

then $\Phi(t)$ is the Gaussian volume of a half-space lying at "signed distance" t from the origin (more precisely, $\Phi(t) = \gamma_n(\{x_1 > t\})$ for every $t \in \mathbb{R}$). It is then clear that, given $\lambda \in (0,1)$, $e^{-\Psi(\lambda)^2/2}$ is the Gaussian perimeter of any half-space of Gaussian volume λ, and thus the *Gaussian isoperimetric inequality* takes the form

$$P_\gamma(E) \geq e^{-\Psi(\gamma_n(E))^2/2}, \tag{1.3}$$

with equality if and only if, up to rotations keeping the origin fixed, E is a half-space with the suitable Gaussian volume, that is,

$$E = \{x \in \mathbb{R}^n : x_n > \Psi(\gamma_n(E))\}.$$

Inequality (1.3) was first proved by Borell [Bor75] and by Sudakov and Cirel'son [SC74]. Alternative proofs, either of probabilistic [BL95, Bob97, Led98, BM00] or geometric [Ehr83, Ehr84, Ehr86] nature, have been proposed, although the characterization of equality cases has been obtained only recently, by probabilistic methods, by Carlen and Kerce [CK01]. Finally, a characterization of equality cases, and a stability inequality with sharp decay rate, were obtained in [CFMP11] building on the symmetrization methods introduced by Ehrhard [Ehr83]. Let us mention in passing that the study of stability issues for Gaussian isoperimetry still poses some difficult questions—see [MN12] for some recent progresses in this direction.

Let us notice that the natural domain of validity of the Gaussian isoperimetric inequality, and, in fact, of Ehrhard's symmetrization technique, is much broader than what we have explained so far. Indeed, Gaussian perimeter can be defined for every Lebesgue measurable set $E \subset \mathbb{R}^n$ by setting

$$P_\gamma(E) = \mathcal{H}^{n-1}_\gamma(\partial^{\mathrm{e}} E) \in [0, \infty].$$

We recall that the *essential boundary* $\partial^{\mathrm{e}} E$ of E is defined as

$$\partial^{\mathrm{e}} E = \mathbb{R}^n \setminus (E^{(0)} \cup E^{(1)}),$$

where, given $t \in [0, 1]$, $E^{(t)}$ denotes the set of points of density t of E,

$$E^{(t)} = \left\{ x \in \mathbb{R}^n : \lim_{r \to 0^+} \frac{\mathcal{H}^n(E \cap B(x, r))}{\omega_n r^n} = t \right\},$$

and ω_n is the volume of the Euclidean unit ball of $\mathbb{R}^n$. If E is an open set with Lipschitz boundary, then we trivially have $\partial^e E = \partial E$, and thus this new definition of $P_\gamma(E)$ provides a coherent extension of (1.1). In general, if $P_\gamma(E) < \infty$, then E is a set of locally finite perimeter, and in that case $P_\gamma(E) = \mathcal{H}^{n-1}_\gamma(\partial^* E)$, where $\partial^* E$ denotes the *reduced boundary* of E; see Section 2.5 for the terminology introduced here. (More generally, E is of locally finite perimeter if and only if E is of locally finite Gaussian perimeter, that is, $\mathcal{H}^{n-1}_\gamma(K \cap \partial^e E) < \infty$ for every compact set $K \subset \mathbb{R}^n$.) Finally, we notice that, with these definitions in force, inequality (1.3) holds true for every Lebesgue measurable set $E \subset \mathbb{R}^n$, and equality holds if and only if, up to rotations around the origin, E is $\mathcal{H}^n$-equivalent to the half-space $\{x \in \mathbb{R}^n : x_n > \Psi(\gamma_n(E))\}$.

1.3. *Ehrhard's symmetrization*

Ehrhard's approach [Ehr83, Ehr84, Ehr86] to the Gaussian isoperimetric inequality is based on a symmetrization procedure that is the natural analogue of Steiner's symmetrization in the Gaussian setting. The definition goes as follows. We decompose $\mathbb{R}^n$, $n \geq 2$, as the Cartesian product $\mathbb{R}^{n-1} \times \mathbb{R}$, denoting by $\mathbf{p} : \mathbb{R}^n \to \mathbb{R}^{n-1}$ and $\mathbf{q} : \mathbb{R}^n \to \mathbb{R}$ the horizontal and vertical projections, so that $x = (\mathbf{p}x, \mathbf{q}x)$, $\mathbf{p}x = (x_1, \dots, x_{n-1})$, and $\mathbf{q}x = x_n$ for every $x \in \mathbb{R}^n$. Given a set $E \subset \mathbb{R}^n$, we denote by E_z its vertical section with respect to $z \in \mathbb{R}^{n-1}$, that is,

$$E_z = \{t \in \mathbb{R} : (z, t) \in E\}, \qquad z \in \mathbb{R}^{n-1}. \tag{1.4}$$

Given a Lebesgue measurable function $v : \mathbb{R}^{n-1} \to [0, 1]$, we say that E is *v-distributed* provided $\mathcal{H}^1_\gamma(E_z) = v(z)$ for $\mathcal{H}^{n-1}$-a.e. $z \in \mathbb{R}^{n-1}$, and we write

$$F[v] = \{x \in \mathbb{R}^n : \mathbf{q}x > \Psi(v(\mathbf{p}x))\}, \tag{1.5}$$

for the v-distributed set whose vertical sections are positive half-lines in the x_n-direction. If E is a v-distributed set, then the *Ehrhard symmetral* E^s of E is defined as

$$E^s = F[v]$$

(see Figure 1.1). By Fubini's theorem, Gaussian volume is preserved under Ehrhard's symmetrization, that is, $\gamma_n(E) = \gamma_n(E^s)$. At the same time, Gaussian perimeter is decreased under Ehrhard's symmetrization. More precisely, if there exists a v-distributed set of finite Gaussian perimeter E, then $F[v]$ is of locally finite perimeter, and *Ehrhard's inequality*

$$P_\gamma(E) \geq P_\gamma(F[v]), \tag{1.6}$$

holds true. A proof of these facts based on the coarea formula is presented in [CFMP11, Section 4.1]. This approach also leads to the following theorem concerning equality cases, which will play an important role below. (Here, ν_E denotes the measure-theoretic outer unit normal to a set of locally finite perimeter E—see Section 2.5.)

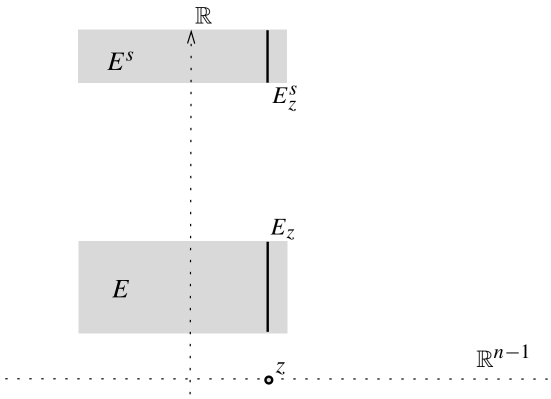

Fig. 1.1. Ehrhard's symmetrization consists in replacing the vertical sections of a set with vertical half-lines with the same Gaussian length and positive orientation. Note that, in this picture, the non-trivial vertical sections E_z of E are constantly equal to the same segment. The corresponding sections E^s_z of E^s are thus constantly equal to the half-line of Gaussian length $\mathcal{H}^1_\gamma(E_z)$.

Theorem A. *If $E \subset \mathbb{R}^n$ is a set of locally finite perimeter with $P_\gamma(E) = P_\gamma(E^s)$, then*

$$E_z \text{ is } \mathcal{H}^1\text{-equivalent to a half-line for } \mathcal{H}^{n-1}\text{-a.e. } z \in \mathbb{R}^{n-1}. \tag{1.7}$$

Moreover, if E satisfies (1.7), and $\partial^* E$ has no "vertical parts", that is,

$$\mathcal{H}^{n-1}(\{x \in \partial^* E : \mathbf{q}\nu_E(x) = 0\}) = 0, \tag{1.8}$$

then $P_\gamma(E) = P_\gamma(E^s)$.

1.4. The rigidity problem for Ehrhard's inequality

We now turn to the rigidity problem related to the Ehrhard inequality. Given $v : \mathbb{R}^{n-1} \to [0, 1]$ such that

$$\mathcal{M}(v) = \{E \subset \mathbb{R}^n : E \text{ is } v\text{-distributed and } P_\gamma(E) = P_\gamma(F[v]) < \infty\}$$

is non-empty, we ask about necessary and sufficient conditions for

$$E \in \mathcal{M}(v) \quad \text{if and only if} \quad \text{either } \mathcal{H}^n(E \,\triangle\, F[v]) = 0 \text{ or } \mathcal{H}^n(E \,\triangle\, g(F[v])) = 0, \tag{1.9}$$

where $g : \mathbb{R}^n \to \mathbb{R}^n$ denotes the reflection with respect to $\mathbb{R}^{n-1}$, that is,

$$g(x) = (\mathbf{p}x, -\mathbf{q}x), \qquad x \in \mathbb{R}^n.$$

Simple examples show that the rigidity condition (1.9) may fail if we allow v to take the values 0 or 1 (see Figure 1.2) and suggest that a reasonable sufficient condition for rigidity could amount to ruling out this possibility. At the same time, v may take the values 0 and/or 1 and still rigidity may hold: an example is depicted in Figure 1.3. Thus,

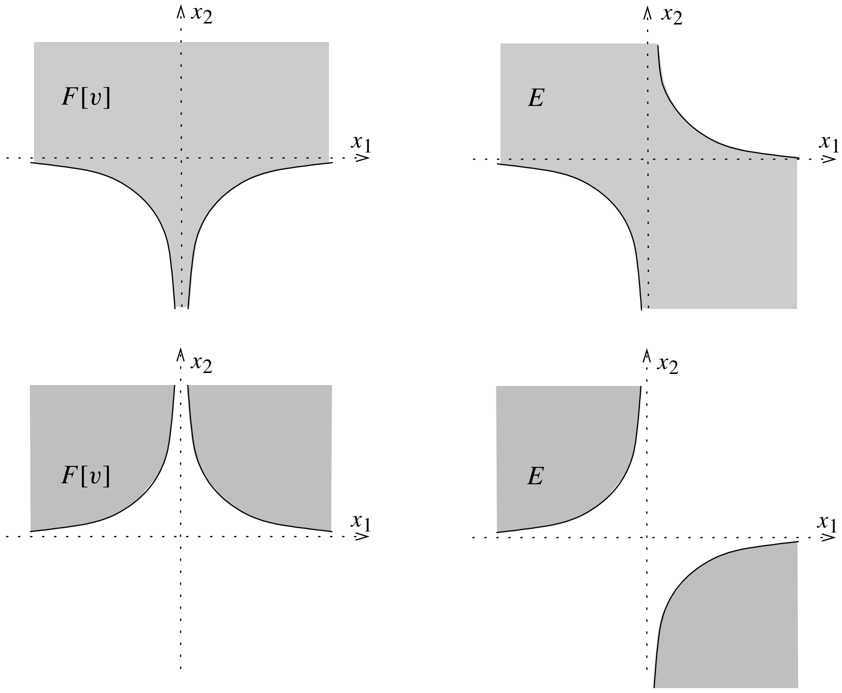

Fig. 1.2. In the first example (two top pictures), the function $v : \mathbb{R} \to [0, 1]$ takes the value 1 at the origin. The correspoding set $F[v]$ is connected and there exists $E \in \mathcal{M}(v)$ such that $\mathcal{H}^2(E \triangle F) = \mathcal{H}^2(E \triangle g(F)) = \infty$. In the second example (two bottom pictures), we observe the same features in the case of a function v that takes the value 0 at the origin.

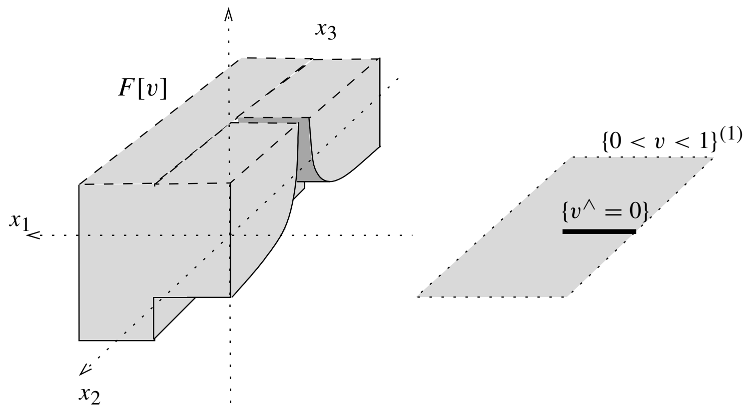

Fig. 1.3. In this example, $\{v^\wedge = 0\}$ is a segment lying inside $\{0 < v < 1\}^{(1)}$. Nevertheless, we have rigidity of equality, as a vertical reflection of $F[v]$ on any proper non-empty subset of $\{0 < v < 1\}$ will create extra Gaussian perimeter.

this plausible sufficient condition would be far from necessary. As it turns out, one needs to introduce some proper notions of connectedness in order to formulate conditions that effectively characterize rigidity.

Before entering into this, let us notice how the need for working in a measure-theoretic framework arises naturally here. Indeed, if $w = v$ $\mathcal{H}^{n-1}$-a.e. on $\mathbb{R}^{n-1}$, then $F[v]$ and $F[w]$ are $\mathcal{H}^n$-equivalent (thus $P_\gamma(F[v]) = P_\gamma(F[w]) \in [0,\infty]$), a set $E \subset \mathbb{R}^n$ is v-distributed if and only if it is w-distributed, and $\mathcal{M}(v) = \mathcal{M}(w)$. In particular, a condition like "v takes the value 0 or 1 on a given set S" has no meaning in our problem if $\mathcal{H}^{n-1}(S) = 0$. We shall rule out these ambiguities by exploiting the notions of approximate upper and lower limits of a Lebesgue measurable function $f : \mathbb{R}^m \to \mathbb{R}$. More precisely, the *approximate upper limit* $f^\vee(x)$ and the *approximate lower limit* $f^\wedge(x)$ of f at $x \in \mathbb{R}^m$ are defined by setting

$$f^\vee(x) = \inf\{t \in \mathbb{R} : x \in \{f > t\}^{(0)}\}, \tag{1.10}$$

$$f^\wedge(x) = \sup\{t \in \mathbb{R} : x \in \{f < t\}^{(0)}\}. \tag{1.11}$$

In this way, $f^\vee$ and $f^\wedge$ are defined at *every* point of $\mathbb{R}^m$, with values in $\mathbb{R}\cup\{\pm\infty\}$, in such a way that if $f_1 = f_2$ $\mathcal{H}^m$-a.e. on $\mathbb{R}^m$, then $f_1^\vee = f_2^\vee$ and $f_1^\wedge = f_2^\wedge$ *everywhere* on $\mathbb{R}^m$. Moreover, both $f^\vee$ and $f^\wedge$ turn out to be Borel functions on $\mathbb{R}^m$ (see Section 2.3).

1.5. A measure-theoretic notion of connectedness

Given an open set G and a hypersurface K in $\mathbb{R}^m$, the intuitive idea of what it means for K to disconnect G is pretty clear: one simply expects K to be the relative boundary inside G of two non-trivial, disjoint open sets G_+ and G_- such that $G_+ \cup G_- = G$. In this section, we precisely define what it means for a Borel set $K \subset \mathbb{R}^m$ to "essentially" disconnect a Borel set $G \subset \mathbb{R}^m$, in such a way that this definition is stable under modifications of K by $\mathcal{H}^{m-1}$-negligible sets, and of G by $\mathcal{H}^m$-negligible sets.

In order to introduce our definition, let us first recall the measure-theoretic notion of connectedness used in the theory of sets of finite perimeter. A set of finite perimeter $G \subset \mathbb{R}^m$ is *indecomposable* (see [DM95, Definition 2.11] or [ACMM01, Section 4]) if for every non-trivial partition $\{G_+, G_-\}$ of G into sets of finite perimeter modulo $\mathcal{H}^m$, i.e.

$$\mathcal{H}^m(G_+ \cap G_-) = 0, \quad \mathcal{H}^m(G \,\Delta\, (G_+ \cup G_-)) = 0, \quad \mathcal{H}^m(G_+)\mathcal{H}^m(G_-) > 0, \tag{1.12}$$

we have $P(G) < P(G_+) + P(G_-)$, where $P(G) = \mathcal{H}^{m-1}(\partial^* G) = \mathcal{H}^{m-1}(\partial^{\mathrm{e}} G)$. (The indecomposability of G in this sense is equivalent to the indecomposability in the sense of [Fed69, 4.2.25] of the m-dimensional integer current on $\mathbb{R}^m$ canonically associated to G.) More generally, we can say that a set of *locally* finite perimeter $G \subset \mathbb{R}^m$ is indecomposable if there exists $r_0 > 0$ such that $P(G; B_r) < P(G_+; B_r) + P(G_-; B_r)$ for every $r > r_0$ and for every non-trivial partition $\{G_+, G_-\}$ of G into sets of locally finite perimeter. For sets of finite perimeter, indecomposability plays the same role that connectedness plays for open sets: see, for example, the various results supporting this intuition collected in [ACMM01, Section 4]. In contrast to topological connectedness,

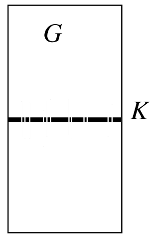

Fig. 1.4. If $G = [0,1] \times [-1,1] \subset \mathbb{R}^2$ and $K \subset \ell = [0,1] \times \{0\}$, then K essentially disconnects G if and only if $\mathcal{H}^1(\ell \setminus K) = 0$. Thus, the set of rational numbers in [0, 1] does not essentially disconnect G, while the set of irrational numbers in [0, 1] does.

indecomposability has however the following important stability property: if G_1 is an indecomposable set and G_2 is $\mathcal{H}^m$-equivalent to G_1, then G_2 is indecomposable too.

We now want to extend the notion of indecomposability to arbitrary Borel sets. Indeed, a pretty obvious necessary condition for rigidity in Ehrhard's inequality should be the "connectedness" of $\{0 < v < 1\}$. Of course, for the reasons explained so far, topological connectedness is not suitable here. Moreover, the Borel set $\{0 < v < 1\}$ defined by $v \in BV_{\rm loc}(\mathbb{R}^{n-1}; [0,1])$ may fail to be of locally finite perimeter (see Example 3.9), and in that case we cannot exploit indecomposability. Finally, we shall in fact need to give a precise meaning to the idea that a Borel set "disconnects" another Borel set. This is achieved as follows. Given two Borel sets K and G in $\mathbb{R}^m$, $m \geq 1$, we say that *K essentially disconnects G* if there exists a non-trivial *Borel* partition $\{G_+, G_-\}$ of G modulo $\mathcal{H}^m$ with

$$\mathcal{H}^{m-1}((G^{(1)} \cap \partial^e G_+ \cap \partial^e G_-) \setminus K) = 0. \tag{1.13}$$

Of course, we say that *K does not essentially disconnect G* if for every non-trivial Borel partition $\{G_+, G_-\}$ of G modulo $\mathcal{H}^m$ we have

$$\mathcal{H}^{m-1}((G^{(1)} \cap \partial^e G_+ \cap \partial^e G_-) \setminus K) > 0. \tag{1.14}$$

Finally, we say that G is *essentially connected* if $\emptyset$ does not essentially disconnect G. An example is depicted in Figure 1.4.

Remark 1.1. If $\mathcal{H}^m(G_1 \,\triangle\, G_2) = 0$, then $G_1^{(1)} = G_2^{(1)}$; thus, K essentially disconnects G_1 if and only if K essentially disconnects G_2. Similarly, if $\mathcal{H}^{m-1}(K_1 \,\triangle\, K_2) = 0$, then K_1 essentially disconnects G if and only if K_2 essentially disconnects G.

Remark 1.2. We shall prove in Remark 2.3 that a set $G \subset \mathbb{R}^m$ of locally finite perimeter is indecomposable if and only if $\mathcal{H}^{m-1}(G^{(1)} \cap \partial^e G_+ \cap \partial^e G_-) > 0$ for every non-trivial *Borel* partition $\{G_+, G_-\}$ of G modulo $\mathcal{H}^m$. Therefore, a set of locally finite perimeter is indecomposable if and only if it is essentially connected. At the same, the notion of essential connectedness makes sense on arbitrary Borel sets. Actually, by replacing $G^{(1)}$ with $(\mathbb{R}^m \setminus G)^{(0)}$ in the definition of $\partial^e G$, we define a notion of connectedness that should retain reasonable properties even when G is a not necessarily measurable set in $\mathbb{R}^m$.

1.6. Characterizations of rigidity for Ehrhard's inequality

We are finally in a position to state our characterization of rigidity of equality cases in Ehrhard's inequality.

Theorem 1.3. *If $v : \mathbb{R}^{n-1} \to [0, 1]$ is a Lebesgue measurable function with $P_\gamma(F[v]) < \infty$, then the following two statements are equivalent:*

(i) *if $E \in \mathcal{M}(v)$, then either $\mathcal{H}^n(E \,\triangle\, F[v]) = 0$, or $\mathcal{H}^n(E \,\triangle\, g(F[v])) = 0$;*
(ii) *the set $\{v^\wedge = 0\} \cup \{v^\vee = 1\}$ does not essentially disconnect $\{0 < v < 1\}$.*

Remark 1.4. If $v = w$ $\mathcal{H}^{n-1}$-a.e. on $\mathbb{R}^{n-1}$, then $v^\vee = w^\vee$ and $v^\wedge = w^\wedge$. In particular, the characterization (ii) of rigidity is independent of the representative of v.

Remark 1.5. The assumption $P_\gamma(F[v]) < \infty$ is of course the minimal hypothesis under which it makes sense to consider the rigidity problem. As we shall see in Proposition 3.1, it implies a minimal amount of regularity on v. More precisely, it implies that the Lebesgue measurable function $\Psi \circ v : \mathbb{R}^{n-1} \to \mathbb{R} \cup \{\pm\infty\}$ is an extended-real-valued function of generalized bounded variation (see Section 3.1).

Despite the geometric clarity of the characterization of rigidity presented in Theorem 1.3, its proof is actually quite delicate. We shall explain the reasons for this in the course of its proof, presented in Section 3. For the moment, let us just mention the following reformulation of Theorem 1.3 in the planar case $n = 2$.

Theorem 1.6. *If $v : \mathbb{R} \to [0, 1]$ is a Lebesgue measurable function with $P_\gamma(F[v]) < \infty$, then the following two statements are equivalent:*

(i) *if $E \in \mathcal{M}(v)$, then either $\mathcal{H}^2(E \,\triangle\, F[v]) = 0$, or $\mathcal{H}^2(E \,\triangle\, g(F[v])) = 0$;*
(ii) *$\{0 < v < 1\}$ is $\mathcal{H}^1$-equivalent to an open interval I, with $v^\wedge > 0$ and $v^\vee < 1$ on I.*

Remark 1.7. A natural problem is that of characterizing rigidity, or providing sufficient conditions for rigidity, in terms of indecomposability properties of $F[v]$. As shown by the examples in Figure 1.2, it is not enough to ask that either $F[v]$ or $\mathbb{R}^n \setminus F[v]$ be indecomposable sets. As it turns out, if we are in the planar case, and we ask that *both* $F[v]$ and $\mathbb{R}^2 \setminus F[v]$ be indecomposable, then rigidity holds (see Theorem 4.2). This last condition is not necessary for rigidity in the planar case (see Figure 1.5), and, in fact, it is not even sufficient for rigidity in $\mathbb{R}^n$ when $n \geq 3$ (see Figure 1.6). A sufficient condition for rigidity in $\mathbb{R}^n$, $n \geq 3$, is obtained by requiring the existence of $\varepsilon > 0$ such that

$$F[v] \cap (\{t < v < 1 - t\} \times \mathbb{R}) \text{ is indecomposable for a.e. } t < \varepsilon \tag{1.15}$$

(see Theorem 4.1). However, not even this last condition is necessary for rigidity in $\mathbb{R}^n$: for an example in the planar case, see Figure 1.7. In this case, (1.15) fails for every $t \in (0, 1)$, but, of course, rigidity holds true. In conclusion, it seems not possible to achieve a characterization of rigidity in terms of indecomposability properties of $F[v]$ and related

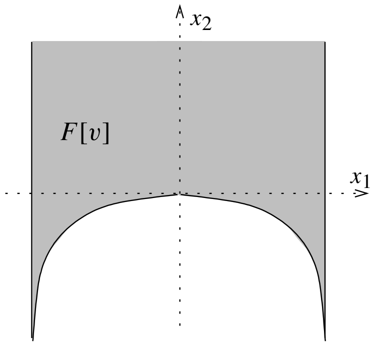

Fig. 1.5. Asking that both $F[v]$ and $\mathbb{R}^n \setminus F[v]$ be indecomposable is a sufficient condition for rigidity in $\mathbb{R}^n$ when $n = 2$, although it is not necessary, as this example shows.

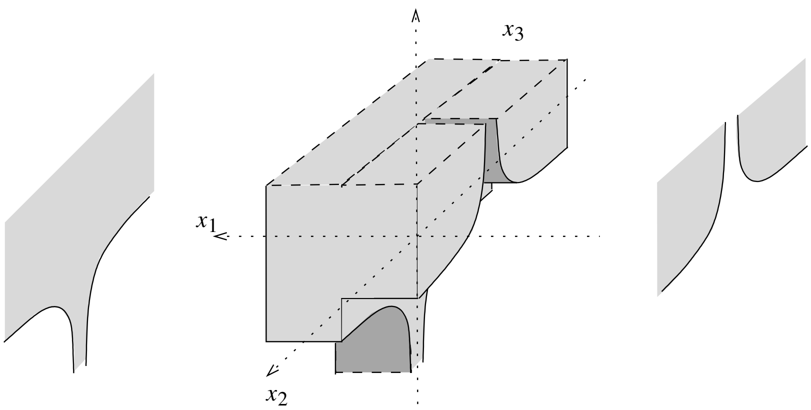

Fig. 1.6. It may happen that both $F[v]$ and $\mathbb{R}^3 \setminus F[v]$ are indecomposable, but rigidity fails. An example is obtained by setting

$$\begin{aligned} F[v] = &\{x \in \mathbb{R}^3 : 0 < x_1 < 1,\ |x_2| < 1,\ x_3 > -1/|x_2|\} \\ &\cup \{x \in \mathbb{R}^3 : -1 < x_1 < 0,\ |x_2| < 1,\ x_3 > 1/|x_2|\}. \end{aligned}$$

Notice that the section $F[v] \cap \{x \in \mathbb{R}^3 : x_1 = t\}$ for $t \in (0, 1)$ (depicted on the left) is an epigraph defined by two "negative" equilateral hyperbolas, while the section $F[v] \cap \{x \in \mathbb{R}^3 : x_1 = t\}$ for $t \in (-1, 0)$ (depicted on the right) is an epigraph defined by two "positive" equilateral hyperbolas. Also, $\{x \in \mathbb{R}^2 : -1 < x_1 < 0, x_2 = 0\} \subset \{v^\wedge = 0\}$ and $\{v^\vee = 1\} = \{x \in \mathbb{R}^2 : 0 < x_1 < 1, x_2 = 0\}$, so that $\{v^\wedge = 0\} \cup \{v^\vee = 1\}$ essentially disconnects $\{0 < v < 1\} = (-1, 1) \times (-1, 1)$, and by Theorem 1.3 regularity fails. Indeed, the set E defined by a vertical reflection of the part of $F[v]$ above $x_2 > 0$,

$$E = \{x \in F[v] : x_2 < 0\} \cup \{x \in \mathbb{R}^3 : g(x) \in F[v], x_2 > 0\},$$

is such that $\mathcal{H}^3(E \,\Delta\, F[v]) > 0$, $\mathcal{H}^3(E \,\Delta\, g(F[v])) > 0$, and $P_\gamma(E) = P_\gamma(F[v])$. We also notice that condition (1.15) does not hold true in this example.

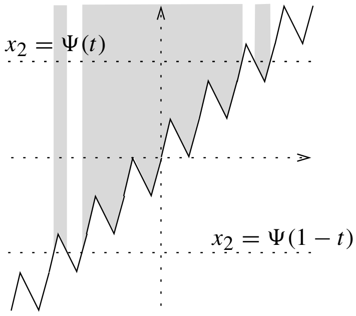

Fig. 1.7. A planar epigraph such that rigidity holds true but condition (1.15) fails. The grey shaded area corresponds, for a generic $t \in (0,1)$, to the set $F[v] \cap (\{t < v < 1-t\} \times \mathbb{R})$, which turns out to be disconnected.

sets. At the same time, it is natural to guess that a characterization of rigidity in terms of essential connectedness should be expressed by the requirement that

$$(\{v^\wedge = 0\} \cup \{v^\vee = 1\}) \times \mathbb{R} \text{ does not essentially disconnect } F[v].$$

Although we shall not pursue this last direction here, in Section 4 we shall provide proofs of the above stated sufficient conditions for rigidity (Theorems 4.1 and 4.2).

2. Notions from geometric measure theory

Here we gather some tools from geometric measure theory. The notions needed in this paper are treated in adequate generality in the monographs [GMS98, AFP00, Mag12].

2.1. General notation in $\mathbb{R}^n$

We denote by $B(x,r)$ and $\overline{B}(x,r)$ the open and closed Euclidean balls of radius $r > 0$ and center $x \in \mathbb{R}^n$. Given $x \in \mathbb{R}^n$ and $\nu \in S^{n-1}$ we denote by $H^+_{x,\nu}$ and $H^-_{x,\nu}$ the complementary half-spaces

$$H^+_{x,\nu} = \{y \in \mathbb{R}^n : (y-x)\cdot\nu \geq 0\}, \qquad H^-_{x,\nu} = \{y \in \mathbb{R}^n : (y-x)\cdot\nu \leq 0\}. \quad (2.1)$$

Finally, we decompose $\mathbb{R}^n$ as the product $\mathbb{R}^{n-1} \times \mathbb{R}$, and denote by $\mathbf{p} : \mathbb{R}^n \to \mathbb{R}^{n-1}$ and $\mathbf{q} : \mathbb{R}^n \to \mathbb{R}$ the corresponding horizontal and vertical projections, so that $x = (\mathbf{p}x, \mathbf{q}x) = (x', x_n)$ and $x' = (x_1, \dots, x_{n-1})$ for every $x \in \mathbb{R}^n$. We set

$$\mathbf{C}_{x,r} = \{y \in \mathbb{R}^n : |\mathbf{p}x - \mathbf{p}y| < r, |\mathbf{q}x - \mathbf{q}y| < r\},$$
$$\mathbf{D}_{z,r} = \{w \in \mathbb{R}^{n-1} : |w - z| < r\},$$

for the vertical cylinder of center $x \in \mathbb{R}^n$ and radius $r > 0$, and for the $(n-1)$-dimensional ball in $\mathbb{R}^{n-1}$ of center $z \in \mathbb{R}^{n-1}$ and radius $r > 0$, respectively. In this way, $\mathbf{C}_{x,r} = \mathbf{D}_{\mathbf{p}x,r} \times (\mathbf{q}x - r, \mathbf{q}x + r)$. We shall use the following two notions of convergence for

Lebesgue measurable subsets of $\mathbb{R}^n$. Given Lebesgue measurable sets $\{E_h\}_{h\in\mathbb{N}}$ and E in $\mathbb{R}^n$, we shall say that E_h *locally converges* to E, and write

$$E_h \xrightarrow{\text{loc}} E \quad \text{as } h \to \infty,$$

provided $\mathcal{H}^n((E_h \,\Delta\, E) \cap K) \to 0$ as $h \to \infty$ for every compact set $K \subset \mathbb{R}^n$; we say that E_h *converges* to E as $h \to \infty$, and write $E_h \to E$, provided $\mathcal{H}^n(E_h \,\Delta\, E) \to 0$ as $h \to \infty$.

2.2. *Density points*

If E is a Lebesgue measurable set in $\mathbb{R}^n$ and $x \in \mathbb{R}^n$, then we define the *upper* and *lower* n*-dimensional densities* of E at x as

$$\theta^*(E,x) = \limsup_{r\to 0^+} \frac{\mathcal{H}^n(E \cap \overline{B}(x,r))}{\omega_n r^n}, \qquad \theta_*(E,x) = \liminf_{r\to 0^+} \frac{\mathcal{H}^n(E \cap \overline{B}(x,r))}{\omega_n r^n},$$

respectively. In this way we define two Borel functions on $\mathbb{R}^n$, which agree a.e. on $\mathbb{R}^n$. In particular, the n*-dimensional density* of E at x,

$$\theta(E,x) = \lim_{r\to 0^+} \frac{\mathcal{H}^n(E \cap \overline{B}(x,r))}{\omega_n r^n} = \lim_{r\to 0} \frac{\mathcal{H}^n(E \cap B(x,r))}{\omega_n r^n},$$

is defined for a.e. $x \in \mathbb{R}^n$, and $\theta(E,\cdot)$ is a Borel function on $\mathbb{R}^n$ (after extending it by a constant value on some $\mathcal{H}^n$-negligible set). Furthermore, for $t \in [0,1]$, we set $E^{(t)} = \{x \in \mathbb{R}^n : \theta(E,x) = t\}$. By the Lebesgue differentiation theorem, $\{E^{(0)}, E^{(1)}\}$ is a partition of $\mathbb{R}^n$ up to an $\mathcal{H}^n$-negligible set. It is useful to keep in mind that

$$\begin{aligned} x \in E^{(1)} &\quad \text{if and only if} \quad E_{x,r} \xrightarrow{\text{loc}} \mathbb{R}^n \text{ as } r \to 0^+, \\ x \in E^{(0)} &\quad \text{if and only if} \quad E_{x,r} \xrightarrow{\text{loc}} \emptyset \text{ as } r \to 0^+, \end{aligned}$$

where $E_{x,r}$ denotes the *blow-up* of E at x at scale r, defined as

$$E_{x,r} = \frac{E - x}{r} = \left\{ \frac{y - x}{r} : y \in E \right\}, \qquad x \in \mathbb{R}^n,\ r > 0.$$

The set $\partial^{\mathrm{e}} E = \mathbb{R}^n \setminus (E^{(0)} \cup E^{(1)})$ is called the *essential boundary* of E. Thus, in general, we only have $\mathcal{H}^n(\partial^{\mathrm{e}} E) = 0$, and we do not claim $\partial^{\mathrm{e}} E$ to be "$(n-1)$-dimensional".

2.3. *Approximate limits*

Strictly related to the notion of density is that of approximate upper and lower limits of a measurable function. We shall stick to Federer's convention [Fed69, 2.9.12] in place of the one usually adopted in the study of functions of bounded variation [AFP00, Section 3.6] since we will mainly deal with functions of *generalized* bounded variation (see Section 2.5). Given a Lebesgue measurable function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ we define the (weak) *approximate upper* and *lower limits* of f at $x \in \mathbb{R}^n$ as

$$f^\vee(x) = \inf\big\{t \in \mathbb{R} : \theta(\{f > t\}, x) = 0\big\} = \inf\big\{t \in \mathbb{R} : \theta(\{f < t\}, x) = 1\big\},$$
$$f^\wedge(x) = \sup\big\{t \in \mathbb{R} : \theta(\{f < t\}, x) = 0\big\} = \sup\big\{t \in \mathbb{R} : \theta(\{f > t\}, x) = 1\big\}.$$

Note that $f^\vee$ and $f^\wedge$ are Borel functions with values in $\mathbb{R} \cup \{\pm\infty\}$, defined *at every point* x of $\mathbb{R}^n$, and they do not depend on the representative chosen for the function f. The *approximate jump* of f is the Borel function $[f] : \mathbb{R}^n \to [0, \infty]$ defined by

$$[f](x) = f^\vee(x) - f^\wedge(x), \qquad x \in \mathbb{R}^n.$$

We easily deduce the following properties, which hold true for every Lebesgue measurable $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ and for every $t \in \mathbb{R}$:

$$\{|f|^\vee < t\} = \{-t < f^\wedge\} \cap \{f^\vee < t\}, \tag{2.2}$$
$$\{f^\vee < t\} \subset \{f < t\}^{(1)} \subset \{f^\vee \le t\}, \tag{2.3}$$
$$\{f^\wedge > t\} \subset \{f > t\}^{(1)} \subset \{f^\wedge \ge t\}. \tag{2.4}$$

(Note that all the inclusions may be strict, that we also have $\{f < t\}^{(1)} = \{f^\vee < t\}^{(1)}$, and that all the other analogous relations hold true.) If f is non-negative and E is Lebesgue measurable, then for every $x \in E^{(1)}$, we have

$$(1_E f)^\vee(x) = f^\vee(x), \qquad (1_E f)^\wedge(x) = f^\wedge(x). \tag{2.5}$$

Finally, we notice that if I and J are intervals in $\mathbb{R} \cup \{\pm\infty\}$, $\varphi : I \to J$ is continuous and decreasing, and f takes values in I, then $v = \varphi \circ f$ is Lebesgue measurable on $\mathbb{R}^n$, with

$$v^\wedge = \varphi(f^\vee), \qquad v^\vee = \varphi(f^\wedge). \tag{2.6}$$

We now introduce the set S_f of *approximate discontinuity points* of a Lebesgue measurable function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$, which is defined as

$$S_f = \{x \in \mathbb{R}^n : f^\wedge(x) < f^\vee(x)\} = \{x \in \mathbb{R}^n : [f](x) > 0\}.$$

We have the following general fact, which is usually stated in the finite-valued case only. For this reason we have included a short proof.

Proposition 2.1. *If $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ is Lebesgue measurable, then $\{f^\wedge = f^\vee = f\}$ is $\mathcal{H}^n$-equivalent to $\mathbb{R}^n$. In particular, $f^\vee$ and $f^\wedge$ are representatives of f, and $\mathcal{H}^n(S_f) = 0$.*

Proof. Let us consider the function Φ defined in (1.2). Since $\Phi : \mathbb{R} \cup \{\pm\infty\} \to [0, 1]$ is continuous and decreasing, the function $v = \Phi \circ f : \mathbb{R}^n \to [0, 1]$ is Lebesgue measurable, with $v^\vee = \Phi \circ f^\wedge$ and $v^\wedge = \Phi \circ f^\vee$. Thus $S_v = S_f$, where, by [GMS98, Section 3.1.4, Proposition 3], $\mathcal{H}^n(S_v) = 0$. □

If $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ and $A \subset \mathbb{R}^n$ Lebesgue measurable, then we say that $t \in \mathbb{R} \cup \{\pm\infty\}$ is the *approximate* limit of f at x with respect to A, and write $t = \operatorname{ap\,lim}(f, A, x)$, if

$$\begin{aligned} &\theta(\{|f - t| > \varepsilon\} \cap A; x) = 0, && \forall \varepsilon > 0 && (t \in \mathbb{R}), \\ &\theta(\{f < M\} \cap A; x) = 0, && \forall M > 0 && (t = +\infty), \\ &\theta(\{f > -M\} \cap A; x) = 0, && \forall M > 0 && (t = -\infty). \end{aligned}$$

We say that $x \in S_f$ is a *jump point* of f if there exists $\nu \in S^{n-1}$ such that

$$f^\vee(x) = \operatorname{ap\,lim}(f, H^+_{x,\nu}, x), \qquad f^\wedge(x) = \operatorname{ap\,lim}(f, H^-_{x,\nu}, x).$$

If this is the case we set $\nu = \nu_f(x)$, the *approximate jump direction* of f at x. We denote by J_f the set of approximate jump points of f, so that $J_f \subset S_f$; moreover, $\nu_f : J_f \to S^{n-1}$ is a Borel function. It will be particularly useful to keep in mind the following proposition.

Proposition 2.2. *We have $x \in J_f$ if and only if for every $\varepsilon > 0$ such that $f^\wedge(x) + \varepsilon < f^\vee(x) - \varepsilon$ we have*

$$\{|f - f^\vee(x)| \le \varepsilon\}_{x,r} \xrightarrow{\text{loc}} H^+_{0,\nu}, \qquad \{|f - f^\wedge(x)| \le \varepsilon\}_{x,r} \xrightarrow{\text{loc}} H^-_{0,\nu}, \qquad \text{as } r \to 0^+.$$

Similarly, $x \in J_f$ if and only if for every $\tau \in (f^\wedge(x), f^\vee(x))$ we have

$$\{f > \tau\}_{x,r} \xrightarrow{\text{loc}} H^+_{0,\nu}, \qquad \{f < \tau\}_{x,r} \xrightarrow{\text{loc}} H^-_{0,\nu}, \qquad \text{as } r \to 0^+. \tag{2.7}$$

Proof. We prove the “only if” part of the first equivalence only, leaving the other implications to the reader. Set $t = f^\vee(x)$ and $s = f^\wedge(x)$. By assumption

$$(\{|f - t| > \varepsilon\} \cap H^+_{x,\nu})_{x,r} \xrightarrow{\text{loc}} \emptyset, \qquad (\{|f - s| > \varepsilon\} \cap H^-_{x,\nu})_{x,r} \xrightarrow{\text{loc}} \emptyset,$$

as $r \to 0^+$. As a consequence, as $r \to 0^+$,

$$(\{|f - t| \le \varepsilon\} \cup H^-_{x,\nu})_{x,r} \xrightarrow{\text{loc}} \mathbb{R}^n, \qquad (\{|f - s| \le \varepsilon\} \cup H^+_{x,\nu})_{x,r} \xrightarrow{\text{loc}} \mathbb{R}^n.$$

As $E^{(1)} \cap F^{(1)} = (E \cap F)^{(1)}$, we find

$$((\{|f - t| \le \varepsilon\} \cup H^-_{x,\nu}) \cap (\{|f - s| \le \varepsilon\} \cup H^+_{x,\nu}))_{x,r} \xrightarrow{\text{loc}} \mathbb{R}^n,$$

$$\text{that is,} \qquad (\{|f - t| \le \varepsilon\} \cap H^+_{x,\nu})_{x,r} \cup (\{|f - s| \le \varepsilon\} \cap H^-_{x,\nu})_{x,r} \xrightarrow{\text{loc}} \mathbb{R}^n,$$

Since the two sets are disjoint, the first one is contained in $H^+_{0,\nu}$, while the second is in $H^-_{0,\nu}$, we complete the proof. □

2.4. *Rectifiable sets*

Let $1 \le k \le n$, $k \in \mathbb{N}$. A Borel set $M \subset \mathbb{R}^n$ is *countably $\mathcal{H}^k$-rectifiable* if there exist Lipschitz functions $f_h : \mathbb{R}^k \to \mathbb{R}^n$ ($h \in \mathbb{N}$) such that

$$\mathcal{H}^k\Big(M \setminus \bigcup_{h\in\mathbb{N}} f_h(\mathbb{R}^k)\Big) = 0. \tag{2.8}$$

We further say that M is *locally $\mathcal{H}^k$-rectifiable* if $\mathcal{H}^k(M \cap K) < \infty$ for every compact set $K \subset \mathbb{R}^n$, or equivalently $\mathcal{H}^k \llcorner M$ is a Radon measure on $\mathbb{R}^n$. Hence, for a locally $\mathcal{H}^k$-rectifiable set M in $\mathbb{R}^n$ the following definition is well-posed: we say that M has a k-dimensional subspace L of $\mathbb{R}^n$ as its *approximate tangent* plane at $x \in \mathbb{R}^n$, $L = T_x M$, if

$$\lim_{r\to 0^+} \frac{1}{r^k} \int_{B(x,r)\cap M} \varphi\Big(\frac{y-x}{r}\Big)\, d\mathcal{H}^k(y) = \int_L \varphi \, d\mathcal{H}^k, \qquad \forall \varphi \in C_c^0(\mathbb{R}^n).$$

It turns out that $T_x M$ exists and is uniquely defined at $\mathcal{H}^k$-a.e. $x \in M$. Moreover, given two locally $\mathcal{H}^k$-rectifiable sets M_1 and M_2 in $\mathbb{R}^n$, we have $T_x M_1 = T_x M_2$ for $\mathcal{H}^k$-a.e. $x \in M_1 \cap M_2$. Since $f(\mathbb{R}^k)$ is locally $\mathcal{H}^k$-rectifiable whenever $f : \mathbb{R}^k \to \mathbb{R}^n$ is a Lipschitz function, if M is merely a countably $\mathcal{H}^k$-rectifiable set and $\{f_h\}_{h\in\mathbb{N}}$ is a sequence of Lipschitz functions satisfying (2.8), then we can find a partition $\{M_h\}_{h\in\mathbb{N}}$ of M modulo $\mathcal{H}^k$ into Borel sets such that $T_x f_h(\mathbb{R}^k)$ exists for every $x \in M_h$; we then set $T_x M = T_x f_h(\mathbb{R}^k)$ for $x \in M_h$. The definition is well-posed in the sense that the approximate tangent spaces defined by another family $\{g_h\}_{h\in\mathbb{N}}$ of Lipschitz functions satisfying (2.8) will just coincide at $\mathcal{H}^k$-a.e. $x \in M$ with the ones defined by $\{f_h\}_{h\in\mathbb{N}}$. In other words, $\{T_x M\}_{x\in M}$ is well-defined as an equivalence class modulo $\mathcal{H}^k$ of Borel functions from M to the set of k-planes in $\mathbb{R}^n$.

Finally, we mention the following consequence of [Fed69, 3.2.23]: if M is countably $\mathcal{H}^k$-rectifiable in $\mathbb{R}^n$, then $M \times \mathbb{R}^\ell$ is countably $\mathcal{H}^{k+\ell}$-rectifiable in $\mathbb{R}^{n+\ell}$, and

$$(\mathcal{H}^k \llcorner M) \times \mathcal{H}^\ell = \mathcal{H}^{k+\ell} \llcorner (M \times \mathbb{R}^\ell). \tag{2.9}$$

2.5. *Functions of bounded variation and sets of finite perimeter*

Given an open set $\Omega \subset \mathbb{R}^n$ and $f \in L^1(\Omega)$, we say that f has *bounded variation* in Ω, $f \in BV(\Omega)$, if the total variation of f in Ω, defined as

$$|Df|(\Omega) = \sup\Big\{ \int_\Omega f(x) \operatorname{div} T(x)\, dx : T \in C_c^1(\Omega; \mathbb{R}^n), |T| \le 1 \Big\},$$

is finite. We say that $f \in BV_{\mathrm{loc}}(\Omega)$ if $f : \Omega \to \mathbb{R}$ is Lebesgue measurable and $f \in BV(\Omega')$ for every open set $\Omega' \subset\subset \Omega$. If $f \in BV_{\mathrm{loc}}(\mathbb{R}^n)$ then the distributional derivative Df of f is an $\mathbb{R}^n$-valued Radon measure. The Radon–Nikodym decomposition of Df with respect to $\mathcal{H}^n$ is denoted by $Df = D^a f + D^s f$, where $D^s f$ and $\mathcal{H}^n$ are mutually singular, and where $D^a f \ll \mathcal{H}^n$. Moreover, S_f is countably $\mathcal{H}^{n-1}$-rectifiable, with $\mathcal{H}^{n-1}(S_f \setminus J_f) = 0$, $[f] \in L^1_{\mathrm{loc}}(\mathcal{H}^{n-1} \llcorner J_f)$, and the $\mathbb{R}^n$-valued Radon measure $D^j f$, defined as $D^j f = [f]\nu_f \, d\mathcal{H}^{n-1} \llcorner J_f$, is called the *jump part* of Df. Since $D^a f$ and $D^j f$ are mutually singular, by setting $D^c f = D^s f - D^j f$ we come to the canonical decomposition of Df into the sum $D^a f + D^j f + D^c f$, where $D^c f$ is called the *Cantorian part* of Df. It turns out that $|D^c f|(M) = 0$ whenever M is σ-finite with respect to $\mathcal{H}^{n-1}$.

A Lebesgue measurable set $E \subset \mathbb{R}^n$ is said to be *of locally finite perimeter* in $\mathbb{R}^n$ if $1_E \in BV_{\mathrm{loc}}(\mathbb{R}^n)$. In this case, we call $\mu_E = -D1_E$ the *Gauss–Green measure* of E, so that

$$\int_E \nabla\varphi(x)\, dx = \int_{\mathbb{R}^n} \varphi(x)\, d\mu_E(x), \qquad \forall \varphi \in C^1_c(\mathbb{R}^n).$$

The *reduced boundary* of E is the set $\partial^* E$ of those $x \in \mathbb{R}^n$ such that

$$\nu_E(x) = \lim_{r\to 0^+} \frac{\mu_E(B(x,r))}{|\mu_E|(B(x,r))} \quad \text{exists and belongs to } S^{n-1}.$$

The Borel function $\nu_E : \partial^* E \to S^{n-1}$ is called the *measure-theoretic outer unit normal* to E. It turns out that $\partial^* E$ is a locally $\mathcal{H}^{n-1}$-rectifiable set in $\mathbb{R}^n$ [Mag12, Corollary 16.1], and $\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial^* E$, so that

$$\int_E \nabla\varphi(x)\, dx = \int_{\partial^* E} \varphi(x)\nu_E(x)\, d\mathcal{H}^{n-1}(x), \qquad \forall \varphi \in C^1_c(\mathbb{R}^n).$$

We say that $x \in \mathbb{R}^n$ is a *jump point* of E if there exists $\nu \in S^{n-1}$ such that

$$E_{x,r} \xrightarrow{\text{loc}} H^+_{0,\nu} \quad \text{as } r \to 0^+, \tag{2.10}$$

and we denote by $\partial^J E$ the set of jump points of E. Notice that we always have $\partial^J E \subset E^{(1/2)} \subset \partial^{\mathrm{e}} E$. In fact, if E is a set of locally finite perimeter and $x \in \partial^* E$, then (2.10) holds true with $\nu = -\nu_E(x)$, so that $\partial^* E \subset \partial^J E$. Summarizing, if E is a set of locally finite perimeter, we have

$$\partial^* E \subset \partial^J E \subset E^{1/2} \subset \partial^{\mathrm{e}} E, \tag{2.11}$$

and moreover, by *Federer's theorem* [AFP00, Theorem 3.61], [Mag12, Theorem 16.2],

$$\mathcal{H}^{n-1}(\partial^{\mathrm{e}} E \setminus \partial^* E) = 0,$$

so that $\partial^{\mathrm{e}} E$ is locally $\mathcal{H}^{n-1}$-rectifiable in $\mathbb{R}^n$. We shall also need the following criterion for finite perimeter, known as *Federer's criterion* [Fed69, 4.5.11] (see also [EG92, Theorem 1, Section 5.11]): if E is a Lebesgue measurable set in $\mathbb{R}^n$ such that

$$\mathcal{H}^{n-1}(K \cap \partial^{\mathrm{e}} E) < \infty \qquad \text{for every compact set } K \subset \mathbb{R}^n,$$

then E is a set of locally finite perimeter. (Notice that Federer's criterion is actually more general than this.) We conclude this preliminary section with the following remark, which shows the equivalence for a set of locally finite perimeter between being indecomposable and being essentially connected (see Section 1.5 for the terminology).

Remark 2.3. If E is an indecomposable set in $\mathbb{R}^n$, then, whenever $\{F, G\}$ is a non-trivial partition of E *into Lebesgue measurable sets*, we have

$$\mathcal{H}^{n-1}(E^{(1)} \cap \partial^{\mathrm{e}} F \cap \partial^{\mathrm{e}} G) > 0. \tag{2.12}$$

Indeed, if $\{F, G\}$ is further assumed to be a partition into sets of locally finite perimeter, then, by definition of indecomposability, there exists r_0 such that $P(E; B_r) < P(F; B_r) + P(G; B_r)$ for every $r > r_0$. Thus, by Federer's theorem,

$$\begin{aligned}\mathcal{H}^{n-1}(B_r \cap \partial^{\mathrm{e}} E) &< \mathcal{H}^{n-1}(B_r \cap \partial^{\mathrm{e}} F) + \mathcal{H}^{n-1}(B_r \cap \partial^{\mathrm{e}} G)\\ &= \mathcal{H}^{n-1}(B_r \cap \partial^{\mathrm{e}} F \cap \partial^{\mathrm{e}} E) + \mathcal{H}^{n-1}(B_r \cap \partial^{\mathrm{e}} G \cap \partial^{\mathrm{e}} E)\\ &\quad + \mathcal{H}^{n-1}(B_r \cap \partial^{\mathrm{e}} F \cap E^{(1)}) + \mathcal{H}^{n-1}(B_r \cap \partial^{\mathrm{e}} G \cap E^{(1)}),\end{aligned} \tag{2.13}$$

where we have used the fact that since $F \subset E$, we have $\partial^{\mathrm{e}} F = (\partial^{\mathrm{e}} F \cap \partial^{\mathrm{e}} E) \cup (\partial^{\mathrm{e}} F \cap E^{(1)})$ (a similar remark applies to G). Since $(\partial^{\mathrm{e}} F \,\Delta\, \partial^{\mathrm{e}} G) \cap (E^{(1)} \cup E^{(0)}) = \emptyset$ and $\partial^J F \cap \partial^J G \subset E^{(1)}$, by Federer's theorem we find that $\partial^{\mathrm{e}} F \,\Delta\, \partial^{\mathrm{e}} G$ is $\mathcal{H}^{n-1}$-equivalent to $\partial^{\mathrm{e}} E$. Hence, (2.13) is equivalent to $0 < 2\mathcal{H}^{n-1}(\partial^{\mathrm{e}} F \cap \partial^{\mathrm{e}} G \cap E^{(1)} \cap B_r)$ for every $r > r_0$, that is, (2.12). To settle the general case, assume, towards a contradiction, the existence of a non-trivial Lebesgue measurable partition $\{F, G\}$ of E such that

$$0 = \mathcal{H}^{n-1}(E^{(1)} \cap \partial^{\mathrm{e}} F \cap \partial^{\mathrm{e}} G) = \mathcal{H}^{n-1}((\partial^{\mathrm{e}} F \cap \partial^{\mathrm{e}} G) \setminus \partial^{\mathrm{e}} E). \tag{2.14}$$

We are now going to show that, in this case, F and G are necessarily sets of locally finite perimeter, thus contradicting the fact that E is indecomposable. Indeed, since $F \subset E$, we have $E^{(0)} \subset F^{(0)}$, and thus $\partial^{\mathrm{e}} F \cap E^{(0)} = E^{(0)} \setminus (F^{(0)} \cup F^{(1)}) = \emptyset$; hence

$$\partial^{\mathrm{e}} F \subset \partial^{\mathrm{e}} E \cup (\partial^{\mathrm{e}} F \cap E^{(1)}). \tag{2.15}$$

At the same time, since $\partial^{\mathrm{e}} F \cap E^{(1)} \subset \partial^{\mathrm{e}} F \cap \partial^{\mathrm{e}} G$, we find

$$\partial^{\mathrm{e}} F \cap E^{(1)} \subset (\partial^{\mathrm{e}} F \cap \partial^{\mathrm{e}} G) \setminus \partial^{\mathrm{e}} E.$$

Therefore, by (2.14) and (2.15), for every compact set $K \subset \mathbb{R}^n$, and since E is of locally finite perimeter, $\mathcal{H}^{n-1}(K \cap \partial^{\mathrm{e}} F) \le \mathcal{H}^{n-1}(K \cap \partial^{\mathrm{e}} E) < \infty$. By Federer's criterion, F is a set of locally finite perimeter, and so is $G = E \setminus F$. We can thus repeat our initial argument to prove that $\mathcal{H}^{n-1}(E^{(1)} \cap \partial^{\mathrm{e}} F \cap \partial^{\mathrm{e}} G) > 0$ and obtain a contradiction.

3. Rigidity of equality cases in Ehrhard's inequality

This section contains the proofs of Theorems 1.3 and 1.6. In Section 3.1 we collect the basic results concerning epigraphs of locally finite perimeter. In Section 3.2 we show the implication (ii)⇒(i) of Theorem 1.3, while in Section 3.3 we prove (i)⇒(ii). In Section 3.4, we finally prove Theorem 1.6.

3.1. Epigraphs of locally finite perimeter and the space GBV_*

Write

$$\Sigma_f = \{x \in \mathbb{R}^n : \mathbf{q}x > f(\mathbf{p}x)\}$$

for the *epigraph* of $f : \mathbb{R}^{n-1} \to \mathbb{R} \cup \{\pm\infty\}$. In this section we analyze the situation when f defines an epigraph of locally finite perimeter. To this end, it is convenient to introduce the functions $\tau_M : \mathbb{R} \to \mathbb{R}$ $(M > 0)$ defined as

$$\tau_M(s) = \max\big\{-M, \min\{M, s\}\big\}, \qquad s \in \mathbb{R} \cup \{\pm\infty\},$$

and set the following definition: a Lebesgue measurable function $f : \mathbb{R}^{n-1} \to \mathbb{R} \cup \{\pm\infty\}$ is a function *of generalized bounded variation* with values in extended real numbers, $f \in GBV_*(\mathbb{R}^{n-1})$, if $\tau_M(f) \in BV_{\mathrm{loc}}(\mathbb{R}^{n-1})$ for every $M > 0$, or equivalently $\psi(f) \in BV_{\mathrm{loc}}(\mathbb{R}^{n-1})$ for every $\psi \in C^1(\mathbb{R})$ with $\psi' \in C^0_c(\mathbb{R})$. (Note that the composition makes sense since, for example, there are positive constants c and t_0 such that $\psi(t) = c$ for every $t > t_0$; therefore, we shall write $\psi(f) = c$ on $\{f = \infty\}$, and argue similarly on the set $\{f = -\infty\}$.) If we start from Lebesgue measurable functions $f : \mathbb{R}^{n-1} \to \mathbb{R}$, we shall set $GBV(\mathbb{R}^{n-1})$ for the corresponding space. The space $GBV_*(\mathbb{R}^{n-1})$ plays a particularly important role in our analysis because of the following proposition.

Proposition 3.1. *If $f : \mathbb{R}^{n-1} \to \mathbb{R} \cup \{\pm\infty\}$ is Lebesgue measurable, then $f \in GBV_*(\mathbb{R}^{n-1})$ if and only if Σ_f is of locally finite perimeter in $\mathbb{R}^n$; moreover, in this case, for a.e. $t \in \mathbb{R}$, the set $\{f < t\}$ is of locally finite perimeter in $\mathbb{R}^{n-1}$.*

Remark 3.2. If $\Omega \subset \mathbb{R}^{n-1}$ is an open set and $f \in L^1(\Omega)$, it is well-known that $f \in BV(\Omega)$ if and only if Σ_f is of finite perimeter in $\Omega \times \mathbb{R}$ (see e.g. [GMS98, Section 4.1.5]). This result, because of the artificial structures assumed in it (open set and summable function), will not suffice for our purposes. Moreover, it seems that the infinite-valued case is not covered in the literature. Therefore, we shall provide a proof of Proposition 3.1. Similar remarks apply to Proposition 3.4 and Lemma 3.6 below. We also notice that we shall need to refer to these proofs in some crucial steps of the proof of Theorem 1.3.

Remark 3.3. Note that if $v \in BV_{\mathrm{loc}}(\mathbb{R}^{n-1}; [0,1])$, then $f = \Psi \circ v \in GBV_*(\mathbb{R}^{n-1})$, where Ψ is defined as in (1.2). Indeed, if we pick any $\psi \in C^1(\mathbb{R})$ with $\psi' \in C^0_c(\mathbb{R})$, then $\psi \circ \Psi$ is real-valued on $[0,1]$, with $\psi \circ \Psi \in C^1([0,1])$ and $(\psi \circ \Psi)' \in C^0_c((0,1))$. Therefore, $\psi \circ f = (\psi \circ \Psi) \circ v \in BV_{\mathrm{loc}}(\mathbb{R}^{n-1})$ by the C^1 chain rule theorem on BV.

Proof of Proposition 3.1. Step 1. We show that if Σ_f is of locally finite perimeter then $f \in GBV_*(\mathbb{R}^{n-1})$. Let $\psi \in C^1(\mathbb{R})$ with $\psi' \in C^0_c(\mathbb{R})$, so that $\psi \circ f$ is defined on $\mathbb{R}^{n-1}$ with $\psi \circ f \in L^\infty(\mathbb{R}^{n-1}) \subset L^1_{\mathrm{loc}}(\mathbb{R}^{n-1})$. If $\psi \in C^2(\mathbb{R})$, then $\psi'(\mathbf{q}x)\varphi(\mathbf{p}x) \in C^1_c(\mathbb{R}^n)$ for every $\varphi \in C^1_c(\mathbb{R}^{n-1})$, and thus, setting $\nabla' = (\partial_1, \dots, \partial_{n-1})$, we have

$$\left|\int_{\Sigma_f} \nabla'(\psi'(\mathbf{q}x)\varphi(\mathbf{p}x))\,dx\right| = \left|\int_{\partial^*\Sigma_f} \psi'(\mathbf{q}x)\varphi(\mathbf{p}x)\mathbf{p}\nu_{\Sigma_f}(x)\,d\mathcal{H}^{n-1}(x)\right|$$
$$\leq \mathrm{Lip}(\psi)\sup|\varphi|\,P(\Sigma_f; \operatorname{spt}\varphi \times \operatorname{spt}\psi').$$

At the same time, by Fubini's theorem,

$$\int_{\Sigma_f} \nabla'(\psi'(\mathbf{q}x)\varphi(\mathbf{p}x))\,dx = \int_{\mathbb{R}^{n-1}} \nabla'\varphi(z)\,dz \int_{f(z)}^{\infty} \psi'(t)\,dt = -\int_{\mathbb{R}^{n-1}} \psi(f(z))\nabla'\varphi(z)\,dz.$$

Hence, for every $R > 0$,

$$\sup\left\{\left|\int_{\mathbb{R}^{n-1}} (\psi \circ f)\nabla'\varphi\right| : \varphi \in C_c^1(\mathbf{D}_R),\ |\varphi| \le 1\right\} \le \mathrm{Lip}(\psi)P(\Sigma_f; \mathbf{D}_R \times \operatorname{spt}\psi') < \infty,$$

that is, $\psi(f) \in BV_{\mathrm{loc}}(\mathbb{R}^{n-1})$ if $\psi \in C^2(\mathbb{R})$. By approximation, the same holds if we only have $\psi \in C^1(\mathbb{R})$, and thus $f \in GBV_*(\mathbb{R}^{n-1})$.

Step 2. If $f \in GBV_*(\mathbb{R}^{n-1})$, then $\tau_M \circ f \in BV_{\mathrm{loc}}(\mathbb{R}^{n-1})$, $\{\tau_M \circ f < t\} = \{f < t\}$ for every $|t| < M$, and $\{\tau_M \circ f < t\}$ is of locally finite perimeter for a.e. $t \in \mathbb{R}$. Hence, $\{f < t\}$ is of locally finite perimeter for a.e. $t \in \mathbb{R}$. Let now $\varphi \in C_c^1(\mathbb{R}^n)$ with $\operatorname{spt}\varphi \subset\subset \mathbf{D}_R \times (-R, R)$ for some $R > 0$. On the one hand, we have

$$\left|\int_{\Sigma_f} \partial_n\varphi\right| = \left|\int_{\mathbb{R}^{n-1}} dz \int_{f(z)}^{\infty} \partial_n\varphi\right| \le \sup_{\mathbb{R}^n} |\varphi|\,\mathcal{H}^{n-1}(\mathbf{D}_R); \tag{3.1}$$

on the other hand, since $\{f < t\}$ is of locally finite perimeter for a.e. $t \in \mathbb{R}$, we find

$$\begin{aligned}\left|\int_{\Sigma_f} \nabla'\varphi\right| &= \left|\int_{\mathbb{R}} dt \int_{\{f<t\}} \nabla'\varphi(z,t)\,dz\right| = \left|\int_{\mathbb{R}} dt \int_{\partial^*\{f<t\}} \varphi(z,t)\nu_{\{f<t\}}(z)\,d\mathcal{H}^{n-2}(z)\right| \\ &\le \sup_{\mathbb{R}^n} |\varphi| \int_{-R}^{R} P(\{f < t\}; \mathbf{D}_R)\,dt = \sup_{\mathbb{R}^n} |\varphi|\,|D(\tau_R \circ f)|(\mathbf{D}_R),\end{aligned} \tag{3.2}$$

by the coarea formula. By (3.1) and (3.2), Σ_f is a set of locally finite perimeter. □

Given a Lebesgue measurable function $f : \mathbb{R}^{n-1} \to \mathbb{R} \cup \{\pm\infty\}$, we set

$$\Gamma_f = \{x \in \mathbb{R}^n : f^\wedge(\mathbf{p}x) \le \mathbf{q}x \le f^\vee(\mathbf{p}x)\},$$
$$\Gamma_f^{\mathrm{v}} = \{x \in \mathbb{R}^n : f^\wedge(\mathbf{p}x) < \mathbf{q}x < f^\vee(\mathbf{p}x)\}.$$

We call Γ_f the *complete graph* of f, and Γ_f^{v} the *vertical graph* of f. Note that these objects are invariant in the $\mathcal{H}^{n-1}$-equivalence class of f.

Proposition 3.4. *If $f \in GBV_*(\mathbb{R}^{n-1})$, then*

$$\partial^*\Sigma_f \cap (S_f^c \times \mathbb{R}) =_{\mathcal{H}^{n-1}} \{x \in \mathbb{R}^n : \mathbf{q}x = f^\wedge(\mathbf{p}x) = f^\vee(\mathbf{p}x)\}, \tag{3.3}$$
$$\partial^*\Sigma_f \cap (S_f \times \mathbb{R}) =_{\mathcal{H}^{n-1}} \Gamma_f^{\mathrm{v}}, \tag{3.4}$$
$$\Sigma_f^{(1)} =_{\mathcal{H}^{n-1}} \{x \in \mathbb{R}^n : \mathbf{q}x > f^\vee(\mathbf{p}x)\}, \tag{3.5}$$
$$\Sigma_f^{(0)} =_{\mathcal{H}^{n-1}} \{x \in \mathbb{R}^n : \mathbf{q}x < f^\wedge(\mathbf{p}x)\}. \tag{3.6}$$

Moreover, S_f is countably $\mathcal{H}^{n-2}$-rectifiable with $\mathcal{H}^{n-2}(S_f \setminus J_f) = 0$. Finally, for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma_f^{\mathrm{v}}$, the outer unit normal $\nu_{\Sigma_f}(x)$ exists, S_f has an approximate tangent plane at $\mathbf{p}x$, and $\nu_{\Sigma_f}(x) = (\nu_{S_f}(\mathbf{p}x), 0)$, where $\nu_{S_f}(\mathbf{p}x)$ is a unit normal direction to $T_{\mathbf{p}x}S_f$ in $\mathbb{R}^{n-1}$.

Remark 3.5. Here and in the following, $A =_{\mathcal{H}^k} B$ stands for $\mathcal{H}^k(A \triangle B) = 0$. Moreover, $A^c = \mathbb{R}^m \setminus A$ whenever $A \subset \mathbb{R}^m$.

Proposition 3.4 is in turn based on the following lemma, which will play a crucial role also in the proof of Theorem 1.3.

Lemma 3.6. *If $f : \mathbb{R}^{n-1} \to \mathbb{R} \cup \{\pm\infty\}$ is a Lebesgue measurable function and I is a countable dense subset of $\mathbb{R}$ with the property that $\{f > t\}$ is of locally finite perimeter for every $t \in I$, and if we set*

$$N_f = \bigcup_{t \in I} \partial^{\mathrm{e}}\{f > t\} \setminus \partial^*\{f > t\},$$

then $\mathcal{H}^{n-2}(N_f) = 0$, and for every $z \in S_f \setminus N_f$ there exists $\nu(z) \in S^{n-2}$ such that

$$z \in \partial^J\{f > t\}, \qquad \forall t \in (f^\wedge(z), f^\vee(z)),$$

with jump direction $\nu(z)$. (In other words, the jump direction of $\{f > t\}$ at z is independent of t.) In particular,

$$S_f \setminus N_f \subset J_f, \qquad \mathcal{H}^{n-2}(S_f \setminus J_f) = 0.$$

Remark 3.7. Notice that the set N_f also depends on the choice of I.

Proof of Lemma 3.6. By Federer's theorem, $\mathcal{H}^{n-2}(N_f) = 0$. We now notice that

$$\begin{cases} z \in S_f, \\ f^\wedge(z) < t < s < f^\vee(z), \end{cases} \quad \Rightarrow \quad z \in \partial^{\mathrm{e}}\{f > t\} \cap \partial^{\mathrm{e}}\{f > s\}.$$

By taking into account that $z \in S_f \setminus N_f$ if and only if $z \in S_f$ and for every $t \in I$ either $z \notin \partial^{\mathrm{e}}\{f > t\}$ or $z \in \partial^*\{f > t\}$, we thus find

$$\begin{aligned} &\begin{cases} z \in S_f \setminus N_f, \\ f^\wedge(z) < t < s < f^\vee(z), \\ t, s \in I, \end{cases} \quad \Rightarrow \quad z \in \partial^*\{f > t\} \cap \partial^*\{f > s\} \\ &\qquad\qquad\qquad\qquad\quad \Rightarrow \quad \{f > t\}_{z,r} \xrightarrow{\text{loc}} H^+_{0,\nu(z)}, \quad \{f > s\}_{z,r} \xrightarrow{\text{loc}} H^+_{0,\nu(z)}, \\ &\qquad\qquad\qquad\qquad\qquad\quad \text{where } -\nu(z) = \nu_{\{f>t\}}(z) = \nu_{\{f>s\}}(z), \end{aligned} \tag{3.7}$$

as $E \subset F$ implies indeed that $\nu_E = \nu_F$ on $\partial^* E \cap \partial^* F$. In other words, for every $z \in S_f \setminus N_f$ there exists $\nu(z) \in S^{n-2}$ such that

$$\{f > t\}_{z,r} \xrightarrow{\text{loc}} H^+_{0,\nu(z)}, \qquad \forall t \in I \cap (f^\wedge(z), f^\vee(z)).$$

Finally, if $z \in S_f \setminus N_f$ with $f^\wedge(z) < t < f^\vee(z)$, then we may pick $s, s' \in I$ with $f^\wedge(z) < s < t < s' < f^\vee(z)$ and use

$$\{f > s\}_{z,r} \xrightarrow{\text{loc}} H^+_{0,\nu(z)}, \qquad \{f > s'\}_{z,r} \xrightarrow{\text{loc}} H^+_{0,\nu(z)},$$

to infer $\{f > t\}_{z,r} \stackrel{\text{loc}}{\longrightarrow} H^+_{0,\nu(z)}$. Indeed, as a general fact, if $E_h \subset F_h \subset G_h$ with $E_h \to E$ and $G_h \to E$ as $h \to \infty$, then $F_h \to E$ as $h \to \infty$. $\square$

Proof of Proposition 3.4. Step 1. We show that S_f is countably $\mathcal{H}^{n-2}$-rectifiable. Let $I \subset \mathbb{R}$ be a countable dense set in $\mathbb{R}$ such that for every $t \in I$ the set $\{f > t\}$ is of locally finite perimeter in $\mathbb{R}^{n-1}$. By Federer's theorem, if $t \in I$, then $\partial^*\{f > t\}$ is locally $\mathcal{H}^{n-2}$-rectifiable, with $\mathcal{H}^{n-2}(\partial^e\{f > t\} \setminus \partial^*\{f > t\}) = 0$. Since $t < f^\vee(z)$ gives $\theta^*(\{f > t\}, z) > 0$, while $t > f^\wedge(z)$ implies $\theta_*(\{f > t\}, z) < 1$, we find that for every $t \in \mathbb{R}$,

$$\{z \in \mathbb{R}^{n-1} : f^\vee(z) > t > f^\wedge(z)\} \subset \partial^e\{f > t\},$$

so that, as I is dense in $\mathbb{R}$,

$$S_f \subset \bigcup_{t\in I} \{z \in \mathbb{R}^{n-1} : f^\vee(z) > t > f^\wedge(z)\} \subset \bigcup_{t\in I} \partial^e\{f > t\}.$$

Thus S_f is countably $\mathcal{H}^{n-2}$-rectifiable, as, by Federer's theorem and since I is countable,

$$\mathcal{H}^{n-2}\Big(S_f \setminus \bigcup_{t\in I} \partial^*\{f > t\}\Big) = 0.$$

Step 2. We prove that

$$\partial^e\Sigma_f \cap (S_f \times \mathbb{R}) \subset_{\mathcal{H}^{n-1}} \Gamma^v_f, \tag{3.8}$$

$$\partial^e\Sigma_f \cap (S^c_f \times \mathbb{R}) \subset \{x \in \mathbb{R}^n : \mathbf{q}x = f^\wedge(\mathbf{p}x) = f^\vee(\mathbf{p}x)\}, \tag{3.9}$$

$$\{x \in \mathbb{R}^n : \mathbf{q}x < f^\wedge(\mathbf{p}x)\} \subset \Sigma^{(0)}_f, \tag{3.10}$$

$$\{x \in \mathbb{R}^n : \mathbf{q}x > f^\vee(\mathbf{p}x)\} \subset \Sigma^{(1)}_f. \tag{3.11}$$

We start by proving (3.10): if $x \in \mathbb{R}^n$ is such that $\mathbf{q}x < f^\wedge(\mathbf{p}x)$, then $f^\wedge(\mathbf{p}x) > -\infty$, and taking $t^* > \mathbf{q}x$ with $\theta(\{f < t^*\}, \mathbf{p}x) = 0$, for every $r < t^* - \mathbf{q}x$ we find

$$\begin{aligned}\mathcal{H}^n(\Sigma_f \cap \mathbf{C}_{x,r}) &= \int_{\mathbf{q}x-r}^{\mathbf{q}x+r} \mathcal{H}^{n-1}(\{f < s\} \cap \mathbf{D}_{\mathbf{p}x,r})\, ds \\ &\le 2r\mathcal{H}^{n-1}(\{f < t^*\} \cap \mathbf{D}_{\mathbf{p}x,r}) = o(r^n).\end{aligned}$$

This proves (3.10), and (3.11) follows similarly. As a consequence, $\partial^e\Sigma_f \subset \Gamma_f$, which yields (3.9), as well as $\partial^e\Sigma_f \cap (S_f \times \mathbb{R}) \subset \Gamma_f \cap (S_f \times \mathbb{R})$. This last inclusion implies (3.8), as

$$(\Gamma_f \cap (S_f \times \mathbb{R})) \setminus \Gamma^v_f = \{(z, f^\wedge(z)) : z \in S_f\} \cup \{(z, f^\vee(z)) : z \in S_f\},$$

is $\mathcal{H}^{n-1}$-negligible (indeed, it projects 2-to-1 over the countably $\mathcal{H}^{n-2}$-rectifiable set S_f).

Step 3. Let now N_f be as in Lemma 3.6. We claim that if $z \in S_f \setminus N_f$ and $f^\wedge(z) < t < f^\vee(z)$ (so that $z \in \partial^J\{f > t\}$ for every such t, with constant jump direction $\nu(z) \in S^{n-1} \cap \mathbb{R}^{n-1}$), then $(z, t) \in \partial^J\Sigma_f$ with jump direction given by $(-\nu(z), 0)$; in particular,

$$\Gamma^v_f \cap ((S_f \setminus N_f) \times \mathbb{R}) \subset \partial^J\Sigma_f. \tag{3.12}$$

Indeed, if $t_0, t_1 \in I$ are such that $f^\wedge(z) < t_0 < t < t_1 < f^\vee(z)$, then for r small enough,

$$\begin{aligned}
&\mathcal{H}^n\big((\Sigma_f \triangle H^+_{(z,t),(-\nu(z),0)}) \cap \mathbf{C}_{(z,t),r}\big) \\
&= \int_{t-r}^{t+r} \mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap H^-_{z,-\nu(z)} \cap \{f < s\}) + \mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap H^+_{z,-\nu(z)} \cap \{f \ge s\})\, ds \\
&\le 2r\mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap H^+_{z,\nu(z)} \cap \{f < t_1\}) + 2r\mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap H^-_{z,\nu(z)} \cap \{f \ge t_0\}) = o(r^n),
\end{aligned}$$

as $\{f < t_1\}_{z,r} \overset{\text{loc}}{\longrightarrow} H^-_{z,\nu(z)}$ and $\{f \ge t_0\}_{z,r} \overset{\text{loc}}{\longrightarrow} H^+_{z,\nu(z)}$. We conclude by Federer's theorem.

Step 4. By (3.9)–(3.11) and by Federer's theorem we deduce (3.3). By (3.8), (3.12), and by Federer's theorem, we get (3.4). Finally, a last application of Federer's theorem allows us to deduce (3.5) and (3.6) from (3.3), (3.4), (3.10), and (3.11). □

Recall that if $M \subset \mathbb{R}^n$ and $z \in \mathbb{R}^{n-1}$, then $M_z = \{t \in \mathbb{R} : (z,t) \in M\}$. As a corollary of Proposition 3.4 we find the following statement.

Corollary 3.8. *If $f \in GBV_*(\mathbb{R}^{n-1})$ and N_f is defined as in Lemma 3.6, then for every $z \in S_f \setminus N_f$ we have*

$$\begin{aligned}
(\Gamma^v_f)_z = (f^\wedge(z), f^\vee(z)) &\subset (\partial^J \Sigma_f \cap (S_f \times \mathbb{R}))_z \\
&\subset (\partial^e \Sigma_f \cap (S_f \times \mathbb{R}))_z \subset [f^\wedge(z), f^\vee(z)].
\end{aligned} \tag{3.13}$$

In particular, for every Borel set $A \subset S_f$,

$$P_\gamma(\Sigma_f; A \times \mathbb{R}) = \int_A \int_{f^\wedge(z)}^{f^\vee(z)} d\mathcal{H}^1_\gamma(t)\, d\mathcal{H}^{n-2}_\gamma(z).$$

Proof. The first inclusion in (3.13) follows immediately from (3.12), while the second is immediate from (2.11). The third inclusion follows of course from $\partial^e \Sigma_f \subset \Gamma_f$. Finally, since S_f is countably $\mathcal{H}^{n-2}$-rectifiable, (2.9) implies $\mathcal{H}^{n-1} \llcorner (S_f \times \mathbb{R}) = (\mathcal{H}^{n-2} \llcorner S_f) \times \mathcal{H}^1$. Thus, if A is a Borel set with $A \subset S_f$, then by (3.13) we find

$$\begin{aligned}
P_\gamma(\Sigma_f; A \times \mathbb{R}) = \mathcal{H}^{n-1}_\gamma(\partial^e \Sigma_f \cap (A \times \mathbb{R})) &= \int_A \mathcal{H}^1_\gamma((\partial^e \Sigma_f)_z)\, d\mathcal{H}^{n-2}_\gamma(z) \\
&= \int_A \int_{f^\wedge(z)}^{f^\vee(z)} d\mathcal{H}^1_\gamma(t)\, d\mathcal{H}^{n-2}_\gamma(z),
\end{aligned}$$

where the tensorization property of $e^{-|x|^2/2}$ was also taken into account. □

3.2. *Proof of Theorem 1.3, (ii) implies (i)*

In this section we present the proof of the implication (ii)⇒(i) in Theorem 1.3. At the end of the proof we collect some examples and remarks that should justify the rather involved technical argument we adopt.

Proof of Theorem 1.3, (ii) implies (i). Overview. We let $v : \mathbb{R}^{n-1} \to [0,1]$ be a Lebesgue measurable function such that $P_\gamma(F[v]) < \infty$ (and therefore $\{F[v], g(F[v])\} \subset \mathcal{M}(v)$). If we define $f : \mathbb{R}^{n-1} \to \mathbb{R} \cup \{\pm\infty\}$ as $f(z) = \Psi(v(z))$, $z \in \mathbb{R}^{n-1}$, then

$$F[v] = \Sigma_f = \text{epigraph of } f.$$

We shall set for brevity $F = F[v]$. Since F has finite Gaussian perimeter, it is of locally finite perimeter, and thus, by Proposition 3.1, $f \in GBV_*(\mathbb{R}^{n-1})$. Up to redefinition of v on an $\mathcal{H}^{n-1}$-negligible set, we can also assume that v is Borel measurable. (As noticed in the introduction, Theorem 1.3 is stable under modifications of v over $\mathcal{H}^{n-1}$-negligible sets.) We now consider the Borel set

$$G = \{z \in \mathbb{R}^{n-1} : 0 < v(z) < 1\} = \{z \in \mathbb{R}^{n-1} : f(z) \in \mathbb{R}\},$$

and assume that

$$\{v^\wedge = 0\} \cup \{v^\vee = 1\} \text{ does not essentially disconnect } G. \tag{3.14}$$

We want to prove that if E is a v-distributed set such that

$$P_\gamma(E) = P_\gamma(F), \tag{3.15}$$

then either $\mathcal{H}^n(E \,\Delta\, F) = 0$ or $\mathcal{H}^n(E \,\Delta\, g(F)) = 0$, where g denotes the reflection with respect to $\mathbb{R}^{n-1}$, $g(x) = (\mathbf{p}x, -\mathbf{q}x)$, $x \in \mathbb{R}^n$. To this end, write as usual $E_z = \{t \in \mathbb{R} : (z,t) \in E\}$ for $z \in \mathbb{R}^{n-1}$, and set

$$\begin{aligned}
G_+ &= \big\{z \in G : \mathcal{H}^1\big(E_z \,\Delta\, (f(z), \infty)\big) = 0\big\},\\
G_- &= \big\{z \in G : \mathcal{H}^1\big(E_z \,\Delta\, (-\infty, -f(z))\big) = 0\big\},\\
G_1 &= \{v = 1\} = \{z \in \mathbb{R}^{n-1} : \mathcal{H}^1(E_z \,\Delta\, \mathbb{R}) = 0\},\\
G_0 &= \{v = 0\} = \{z \in \mathbb{R}^{n-1} : \mathcal{H}^1(E_z) = 0\}.
\end{aligned}$$

By Theorem A we find that

$$E =_{\mathcal{H}^n} \big(F \cap ((G_+ \cup G_1) \times \mathbb{R})\big) \cup (g(F) \cap (G_- \times \mathbb{R})), \tag{3.16}$$

as well as that $\{G_+, G_-, G_1, G_0\}$ is a partition of $\mathbb{R}^{n-1}$ modulo $\mathcal{H}^{n-1}$, and $\{G_+, G_-\}$ is a partition of G modulo $\mathcal{H}^{n-1}$, where this last condition means

$$\mathcal{H}^{n-1}(G \,\Delta\, (G_+ \cup G_-)) = 0, \qquad \mathcal{H}^{n-1}(G_+ \cap G_-) = 0.$$

Clearly, $G = \{0 < v < 1\}$, $G_1 = \{v = 1\}$, and $G_0 = \{v = 0\}$ are Borel sets, as v is a Borel function. Notice that also G_+ and G_- are Lebesgue measurable sets. Indeed, if we define $\beta : \mathbb{R}^{n-1} \to \mathbb{R}$ as

$$\beta(z) = \begin{cases} \frac{1}{v(z)} \int_{E_z} t \, d\gamma_1(t), & z \in \{0 < v \le 1\},\\ 0, & z \in \{v = 0\},\end{cases}$$

(so that $\beta(z)$ is the Gaussian barycenter of E_z), then, by Fubini's theorem, β is a Lebesgue measurable function. At the same time, a simple computation shows that

$$\beta(z) = \frac{1}{\sqrt{2\pi}}\left(1_{G_+}(z)\frac{e^{-f(z)^2/2}}{v(z)} - 1_{G_-}(z)\frac{e^{-f(z)^2/2}}{v(z)}\right), \qquad \forall z \in G \cup G_1,$$

so that $G_+ = \{\beta > 0\}$ and $G_- = \{\beta < 0\}$. Thus, both G_+ and G_- are Lebesgue measurable sets. We now look back at (3.16), and notice that $\mathcal{H}^n(E \triangle F)\mathcal{H}^n(E \triangle g(F)) = 0$ if and only if $\mathcal{H}^{n-1}(G_+)\mathcal{H}^{n-1}(G_-) = 0$. To reach a contradiction, assume that rigidity fails because of E, which amounts to asking that

$$\mathcal{H}^{n-1}(G_+)\mathcal{H}^{n-1}(G_-) > 0. \tag{3.17}$$

In other words, $\{G_+, G_-\}$ is a non-trivial Lebesgue measurable partition of G. Hence, thanks to (3.14), by Borel regularity of the Lebesgue measure, and since $\partial^e A = \partial^e B$ if $A, B \subset \mathbb{R}^{n-1}$ with $\mathcal{H}^{n-1}(A \triangle B) = 0$, we find that

$$\mathcal{H}^{n-2}\big((G^{(1)} \cap \partial^e G_+ \cap \partial^e G_-) \setminus (\{v^\wedge = 0\} \cup \{v^\vee = 1\})\big) > 0. \tag{3.18}$$

Comparing (3.16) and (3.18) we see that E is obtained by reflecting F across a region of *non-trivial* $\mathcal{H}^{n-2}$ *measure* where the sections of F are *neither negligible nor equivalent to* $\mathbb{R}$; accordingly, we expect Gaussian perimeter to increase in this operation, that is, we expect (3.16) and (3.18) to imply $P_\gamma(E) > P_\gamma(F)$, thus contradicting (3.15). The main difficulty in proving that this actually happens relies on the fact that the set $G^{(1)} \cap \partial^e G_+ \cap \partial^e G_-$ may not have a reasonable metric structure, that is, it may fail to be countably $\mathcal{H}^{n-2}$-rectifiable. (Examples 3.9 and 3.10 below show that, respectively, G may fail to be of locally finite perimeter, and $G^{(1)} \cap \partial^e G_+ \cap \partial^e G_-$ may fail to be countably $\mathcal{H}^{n-2}$-rectifiable even if $v \in \mathrm{Lip}(\mathbb{R}^{n-1}; [0, 1])$.) We shall avoid this difficulty by showing the existence of a countably $\mathcal{H}^{n-2}$-rectifiable set Σ such that

$$\Sigma \subset (G^{(1)} \cap \partial^e G_+ \cap \partial^e G_-) \setminus (\{v^\wedge = 0\} \cup \{v^\vee = 1\}), \qquad \mathcal{H}^{n-2}(\Sigma) > 0.$$

We shall then deduce that, as simple drawings suggest, $P_\gamma(E; \Sigma \times \mathbb{R}) > P_\gamma(F; \Sigma \times \mathbb{R})$. Finally, by taking into account that $P_\gamma(E; A \times \mathbb{R}) \geq P_\gamma(F; A \times \mathbb{R})$ for every Borel set $A \subset \mathbb{R}^{n-1}$, we shall find $P_\gamma(E) > P_\gamma(F)$. We divide this argument into nine steps.

Step 1. We use the information that E is of locally finite perimeter to deduce that for every $k \in \mathbb{N}$ the function $u_k : \mathbb{R}^{n-1} \to \mathbb{R}$ defined as

$$u_k = (k - |f|)1_{\{|f|<k\}}(1_{G_+} - 1_{G_-}) \text{ is in } BV_{\mathrm{loc}}(\mathbb{R}^{n-1}).$$

Indeed, if we take into account (3.16) and repeat the argument in the proof of Proposition 3.1 with E in place of $F = \Sigma_f$, then we find

$$\begin{aligned} P(E; K \times I) \geq &\int_{G_+} \nabla'\varphi(z)\,dz \int_{f(z)}^{\infty} \psi'(t)\,dt + \int_{G_-} \nabla'\varphi(z)\,dz \int_{-\infty}^{-f(z)} \psi'(t)\,dt \\ &+ \int_{G_1} \nabla'\varphi(z)\,dz \int_{-\infty}^{\infty} \psi'(t)\,dt \end{aligned} \tag{3.19}$$

whenever $\varphi \in C_c^1(\mathbb{R}^{n-1})$ with $\operatorname{spt}\varphi \subset K \subset\subset \mathbb{R}^{n-1}$ and $|\varphi| \le 1$, and $\psi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function with $\operatorname{spt}\psi' \subset I \subset\subset \mathbb{R}$ and $\operatorname{Lip}(\psi) \le 1$. If we apply (3.19) with ψ defined by $\psi(t) = k$ for $|t| > k$ and $\psi(t) = |t|$ for $|t| \le k$, then we deduce our assertion by exploiting the relations (valid for every $a \in \mathbb{R}$)

$$\int_a^\infty \psi' = (k - |a|)1_{(-k,k)}(a), \qquad \int_{-\infty}^{-a} \psi' = -(k - |a|)1_{(-k,k)}(a), \qquad \int_{-\infty}^{\infty} \psi' = 0.$$

Step 2. We show that, for every $k \in \mathbb{N}$,

$$\{|f|^\vee < k/2\} \cap G_+^{(1)} \subset \{u_k^\wedge > k/2\} \cap G_+^{(1)}.$$

It suffices to prove that if $z \in \{|f|^\vee < k/2\} \cap G_+^{(1)}$ and $\varepsilon < (k/2) - |f|^\vee(z)$, then

$$\theta(\{u_k < s\}, z) = 0, \qquad \forall s < k/2 + \varepsilon.$$

Indeed, thanks to (2.3), we have $\{|f|^\vee < k/2\} \subset \{|f| < k/2\}^{(1)}$. Thus, for every such s,

$$\begin{aligned}
&\mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap \{u_k < s\}) \\
&= \mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap \{u_k < s\} \cap G_+) + o(r^{n-1}) \qquad (\text{since } z \in G_+^{(1)}) \\
&= \mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap \{u_k < s\} \cap \{|f| < k/2\} \cap G_+) + o(r^{n-1}) \qquad (\text{since } z \in \{|f| < k/2\}^{(1)}) \\
&= \mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap \{k - |f| < s\} \cap \{|f| < k/2\} \cap G_+) + o(r^{n-1}) \\
&\le \mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap \{k - s < |f|\}) + o(r^{n-1}) = o(r^{n-1}),
\end{aligned}$$

where the last identity follows by definition of $|f|^\vee$ since $k - s > k/2 - \varepsilon > |f|^\vee(z)$.

Step 3. We set

$$\Sigma_k = \partial^{\mathrm{e}} G_+ \cap \partial^{\mathrm{e}} G_- \cap \{-k/2 < f^\wedge \le f^\vee < k/2\}^{(1)}, \qquad k \in \mathbb{N},$$

and prove that

$$\Sigma_k \subset \{u_k^\vee \ge k/2\} \cap \{u_k^\wedge \le -k/2\}, \qquad \forall k \in \mathbb{N}. \tag{3.20}$$

To show this, we start by noticing that for every $z \in \Sigma_k$ we have

$$\begin{aligned}
\mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap \{u_k > k/2\}) &= \mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap \{u_k > k/2\}^{(1)}) \\
&\ge \mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap \{u_k > k/2\}^{(1)} \cap G_+^{(1)}) \\
&\ge \mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap \{u_k^\wedge > k/2\} \cap G_+^{(1)}),
\end{aligned} \tag{3.21}$$

where the last inequality follows from (2.4). Now, by Step 2 and (2.2),

$$\{u_k^\wedge > k/2\} \cap G_+^{(1)} \supset \{|f|^\vee < k/2\} \cap G_+^{(1)} = \{-k/2 < f^\wedge \le f^\vee < k/2\} \cap G_+^{(1)},$$

so that, by (3.21),

$$\begin{aligned}
\mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap \{u_k > k/2\}) &\ge \mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap \{-k/2 < f^\wedge \le f^\vee < k/2\} \cap G_+^{(1)}) \\
&= \mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap G_+) + o(r^{n-1}),
\end{aligned}$$

where in the last identity we have used the fact that $z \in \{k/2 > f^\vee \geq f^\wedge > -k/2\}^{(1)}$. Since, by assumption, $z \in \partial^e G_+$, we conclude that

$$0 < \theta^*(G_+, z) \leq \theta^*(\{u_k > k/2\}, z),$$

which in turn gives $u_k^\vee(z) \geq k/2$. One can prove analogously that $u_k^\wedge(z) \leq -k/2$.

Step 4. We show that, for every $k \in \mathbb{N}$,

$$\Sigma_k \text{ is locally } \mathcal{H}^{n-2}\text{-rectifiable.}$$

From Step 3 we know that $\Sigma_k \subset S_{u_k}$. Since $u_k \in BV_{\mathrm{loc}}(\mathbb{R}^{n-1})$, this implies that Σ_k is countably $\mathcal{H}^{n-2}$-rectifiable, and it remains to show that Σ_k is locally $\mathcal{H}^{n-2}$-finite. To this end, let $K \subset \mathbb{R}^{n-1}$ be a compact set; since

$$\Sigma_k = [\Sigma_k \cap (S_{u_k} \setminus J_{u_k})] \cup (\Sigma_k \cap J_{u_k})$$

and $\mathcal{H}^{n-2}(S_{u_k} \setminus J_{u_k}) = 0$, we have

$$\mathcal{H}^{n-2}(\Sigma_k \cap K) = \mathcal{H}^{n-2}(\Sigma_k \cap J_{u_k} \cap K) \in [0, \infty].$$

By Step 3 and since $u_k \in BV_{\mathrm{loc}}(\mathbb{R}^{n-1})$,

$$k\mathcal{H}^{n-2}(\Sigma_k \cap J_{u_k} \cap K) \leq \int_{\Sigma_k \cap J_{u_k} \cap K} (u_k^\vee - u_k^\wedge)\, d\mathcal{H}^{n-2} \leq |D^j u_k|(K).$$

Thus, if $K \subset \mathbb{R}^{n-1}$ is compact and $k \in \mathbb{N}$, then $\mathcal{H}^{n-2}(K \cap \Sigma_k) \leq k^{-1}|D^j u_k|(K) < \infty$. This proves Σ_k is locally $\mathcal{H}^{n-2}$-finite.

Step 5. We are now going to deduce from (3.18) that, for k sufficiently large,

$$\mathcal{H}^{n-2}(\Sigma_k) > 0. \tag{3.22}$$

We start by proving the following identity:

$$\bigcup_{k \in \mathbb{N}} \Sigma_k = (\partial^e G_+ \cap \partial^e G_-) \setminus (\{f^\vee = \infty\} \cup \{f^\wedge = -\infty\}). \tag{3.23}$$

Indeed, by definition of Σ_k, and by repeatedly applying (2.3) and (2.4),

$$\begin{aligned}
\Sigma_k &= \partial^e G_+ \cap \partial^e G_- \cap \{-k/2 < f^\wedge \leq f^\vee < k/2\}^{(1)} \\
&\subset \partial^e G_+ \cap \partial^e G_- \cap \{-k/2 \leq f^\wedge \leq f^\vee \leq k/2\} \\
&\subset \partial^e G_+ \cap \partial^e G_- \cap \{-(k+1)/2 < f^\wedge \leq f^\vee < (k+1)/2\} \\
&\subset \partial^e G_+ \cap \partial^e G_- \cap \{-(k+1)/2 < f^\wedge \leq f^\vee < (k+1)/2\}^{(1)} = \Sigma_{k+1},
\end{aligned} \tag{3.24}$$

from which (3.23) immediately follows. Since $f = \Psi(v)$ with Ψ continuous and decreasing, and thanks to (2.6), we have $\{f^\vee = \infty\} = \{v^\wedge = 0\}$ and $\{f^\wedge = -\infty\} = \{v^\vee = 1\}$, so that (3.23) is equivalent to

$$\bigcup_{k \in \mathbb{N}} \Sigma_k = (\partial^e G_+ \cap \partial^e G_-) \setminus (\{v^\wedge = 0\} \cup \{v^\vee = 1\}). \tag{3.25}$$

Finally, by (3.25), (3.24), and (3.18), we find

$$\lim_{k\to\infty} \mathcal{H}^{n-2}(\Sigma_k) = \mathcal{H}^{n-2}\big((\partial^e G_+ \cap \partial^e G_-) \setminus (\{v^\wedge = 0\} \cup \{v^\vee = 1\})\big) > 0.$$

Step 6. We show that if $W \subset \Sigma_k$ is a Borel set, then

$$P_\gamma(F; W \times \mathbb{R}) = \int_W d\mathcal{H}^{n-2}_\gamma(z) \int_{f^\wedge(z)}^{f^\vee(z)} d\mathcal{H}^1_\gamma. \tag{3.26}$$

Indeed, (3.26) follows immediately from Corollary 3.8 provided $W \subset S_f$. Since the right-hand side of (3.26) is trivially equal to zero if $W \subset S_f^c$, it remains to prove that

$$P_\gamma(F; (\Sigma_k \cap S_f^c) \times \mathbb{R}) = 0.$$

To this end, we notice that, by Proposition 3.4,

$$\partial^e F \cap (S_f^c \times \mathbb{R}) \subset_{\mathcal{H}^{n-1}} \{x \in \mathbb{R}^n : \mathbf{p}x \in S_f^c,\ \mathbf{q}x = f^\wedge(\mathbf{p}x) = f^\vee(\mathbf{p}x)\}.$$

If L denotes the set on the right-hand side of this inclusion, then $\mathcal{H}^0(L_z) = 1$ for every $z \in S_f^c$. As Σ_k is countably $\mathcal{H}^{n-2}$-rectifiable, by (2.9) we find that

$$\begin{aligned} P_\gamma(F; (S_f^c \cap \Sigma_k) \times \mathbb{R}) &= \mathcal{H}^{n-1}_\gamma\big(\partial^e F \cap ((S_f^c \cap \Sigma_k) \times \mathbb{R})\big) \\ &\le \mathcal{H}^{n-1}_\gamma\big(L \cap ((S_f^c \cap \Sigma_k) \times \mathbb{R})\big) = \int_{S_f^c \cap \Sigma_k} \mathcal{H}^1_\gamma(L_z)\, d\mathcal{H}^{n-2}_\gamma(z) = 0. \end{aligned}$$

We have thus completed the proof of (3.26).

Step 7. We show that if $z \in \Sigma_k \setminus N_{u_k}$ (with N_{u_k} defined as in Lemma 3.6), then there exists $\nu \in S^{n-1} \cap \mathbb{R}^{n-1}$ such that

$$(G_+)_{z,r} \xrightarrow{\text{loc}} H^+_{0,\nu}, \qquad (G_-)_{z,r} \xrightarrow{\text{loc}} H^-_{0,\nu}, \tag{3.27}$$

$$\{u_k > t\}_{z,r} \xrightarrow{\text{loc}} H^+_{0,\nu}, \quad \forall t \in (u_k^\wedge(z), u_k^\vee(z)). \tag{3.28}$$

By (3.23) and since $\mathcal{H}^{n-2}(N_{u_k}) = 0$, this will imply in particular that

$$\Sigma_k \subset_{\mathcal{H}^{n-2}} \partial^J G_+ \cap \partial^J G_- \cap \{|f|^\vee < \infty\}. \tag{3.29}$$

We first recall that, by Lemma 3.6, if $z \in S_{u_k} \setminus N_{u_k}$, then there exists $\nu = \nu(z) \in S^{n-2}$ such that (3.28) holds true. Now, we easily find that

$$\{u_k > t\} = G_+ \cap \{|f| < k - t\}, \qquad \forall t > 0,$$

which in particular gives

$$z \in \bigcap_{0<t<u_k^\vee(z)} \partial^J(G_+ \cap \{|f| < k - t\}).$$

Since, by (3.20), $u_k^\vee(z) \geq k/2$ for every $z \in \Sigma_k$, for ε small enough we find that

$$\Sigma_k \setminus N_{u_k} \subset \partial^J(G_+ \cap \{|f| < k - (k/2 - \varepsilon)\}) = \partial^J(G_+ \cap \{|f| < k/2 + \varepsilon\}).$$

Taking now into account that $\partial^J(A \cap B) \cap B^{(1)} \subset (\partial^J A) \cap B^{(1)}$, we thus find

$$(\Sigma_k \setminus N_{u_k}) \cap \{|f| < k/2 + \varepsilon\}^{(1)} \subset \partial^J G_+.$$

Finally, since $\Sigma_k \subset \{|f| < (k/2) + \varepsilon\}^{(1)}$, we conclude that $\Sigma_k \setminus N_{u_k} \subset \partial^J G_+$. One proves analogously the inclusion in $\partial^J G^-$.

Step 8. We have proved so far that if k is large enough, then Σ_k is a locally $\mathcal{H}^{n-2}$-rectifiable set in $\mathbb{R}^{n-1}$ with $\mathcal{H}^{n-2}(\Sigma_k) > 0$, and $\Sigma_k \subset \partial^J G_+ \cap \partial^J G_- \cap \{|f|^\vee < \infty\}$ (modulo $\mathcal{H}^{n-2}$). Moreover, we have computed the Gaussian perimeter of F above Σ_k. We now want to compute $P_\gamma(E; \Sigma_k \times \mathbb{R})$, in order to show that this last quantity is strictly larger than $P_\gamma(F; \Sigma_k \times \mathbb{R})$. To this end, it is convenient to divide Σ_k into two parts, defined by the sets Π_+ and Π_- introduced in this and in the following step. More precisely, we start this conclusive part of our argument by considering the set Π_+ of those

$$z \in \partial^J G_+ \cap \partial^J G_- \cap \{|f|^\vee < \infty\} \cap (S_f^c \cup J_f)$$

such that, for some $\nu \in S^{n-1} \cap \mathbb{R}^{n-1}$,

$$(G_+)_{z,r} \xrightarrow{\text{loc}} H_{0,\nu}^+, \qquad (G_-)_{z,r} \xrightarrow{\text{loc}} H_{0,\nu}^-, \tag{3.30}$$

$$\{f > s\} \xrightarrow{\text{loc}} H_{0,\nu}^+ \quad \text{if } z \in J_f \text{ and } s \in (f^\wedge(z), f^\vee(z)). \tag{3.31}$$

We want to characterize $(\partial^J E)_z$ for $z \in \Pi_+$ by showing that

$$(\partial^J E)_z =_{\mathcal{H}^1} (-\infty, -f^\wedge(z)) \cup (f^\vee(z), \infty), \quad \forall z \in \Pi_+ \cap \{f^\vee \geq -f^\wedge\}, \tag{3.32}$$

$$(\partial^J E)_z =_{\mathcal{H}^1} (-\infty, f^\vee(z)) \cup (-f^\wedge(z), \infty), \quad \forall z \in \Pi_+ \cap \{f^\vee \leq -f^\wedge\}. \tag{3.33}$$

In particular, we shall prove that if $z \in \Pi_+$ and $f^\vee(z) \geq -f^\wedge(z)$, then

$$(z,t) \in \partial^J E, \quad \forall t \in (-\infty, -f^\wedge(z)) \cup (f^\vee(z), \infty), \tag{3.34}$$

$$(z,t) \in E^{(0)} \subset \mathbb{R}^n \setminus \partial^e E, \quad \forall t \in (-f^\wedge(z), f^\vee(z)) \tag{3.35}$$

(so that (3.32) holds true, see Figure 3.1), while if $z \in \Pi_+$ and $f^\vee(z) \leq -f^\wedge(z)$, then

$$(z,t) \in \partial^J E, \quad \forall t \in (-\infty, f^\vee(z)) \cup (-f^\wedge(z), \infty), \tag{3.36}$$

$$(z,t) \in E^{(1)} \subset \mathbb{R}^n \setminus \partial^e E, \quad \forall t \in (f^\vee(z), -f^\wedge(z)) \tag{3.37}$$

(thus proving (3.33)—see, once again, Figure 3.1). Before entering into the proof of (3.34)–(3.37), let us notice that (3.32) and (3.33) imply that

$$(\partial^J E)_z =_{\mathcal{H}^1} (-\infty, a(z)) \cup (b(z), \infty), \quad \forall z \in \Pi_+, \tag{3.38}$$

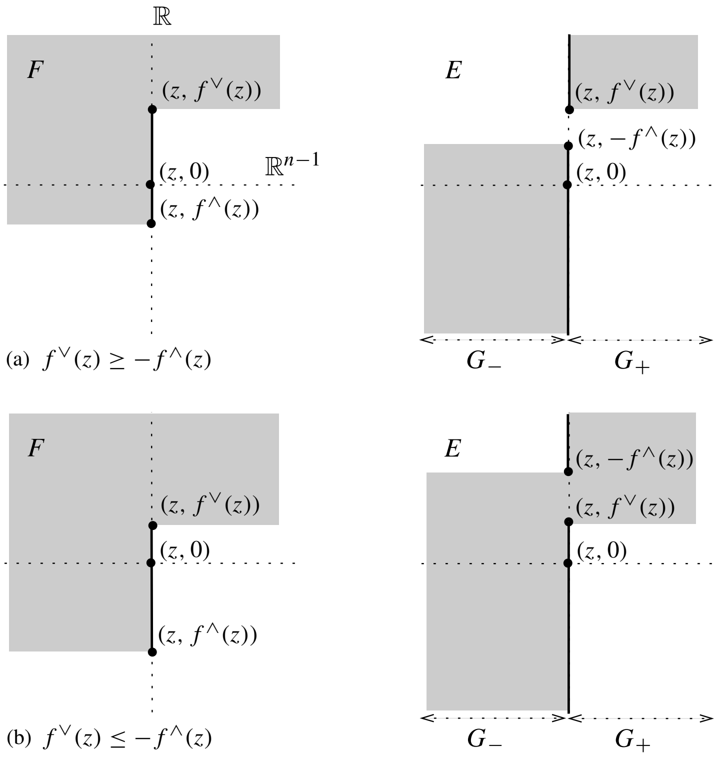

Fig. 3.1. In panel (a) we consider the case when $z \in \Pi_+$ and $f^\vee(z) \geq -f^\wedge(z)$. In this case we must have $f^\vee(z) \geq 0$, while, of course, $f^\wedge(z)$ has arbitrary sign. Moreover, $(\partial^e E)_z$ is $\mathcal{H}^1$-equivalent to $(-\infty, -f^\wedge(z)) \cup (f^\vee(z), \infty)$ (see (3.34)), and $(-f^\wedge(z), f^\vee(z))$ is $\mathcal{H}^1$-equivalent to $(E^{(0)})_z$ (see (3.35)). In panel (b) we consider the complementary case when $z \in \Pi_+$ and $f^\vee(z) \leq -f^\wedge(z)$. In this case $(\partial^e E)_z$ is $\mathcal{H}^1$-equivalent to $(-\infty, f^\vee(z)) \cup (-f^\wedge(z), \infty)$ (see (3.36)), while $(f^\vee(z), -f^\wedge(z))$ is $\mathcal{H}^1$-equivalent to $(E^{(1)})_z$ (see (3.37)). In both cases, of course, $(\partial^e F)_z$ is $\mathcal{H}^1$-equivalent to $(f^\wedge(z), f^\vee(z))$.

where we have set

$$a(z) = \min\{-f^\wedge(z), f^\vee(z)\}, \qquad b(z) = \max\{-f^\wedge(z), f^\vee(z)\}. \tag{3.39}$$

We shall now provide the details of the proof of (3.34), noticing that (3.35)–(3.37) can be proved by entirely analogous arguments. Let $z \in \Pi_+$ with $f^\vee(z) \geq -f^\wedge(z)$, and notice that necessarily $f^\vee(z) \geq (f^\vee(z) + f^\wedge(z))/2 \geq 0$. We now consider two separate cases.

Proof of (3.34) *when* $t > f^\vee(z)$. Let $r_* > 0$ be such that $t - r_* > f^\vee(z)$, so that

$$\{f < s\}_{z,r} \xrightarrow{\text{loc}} \mathbb{R}^{n-1}, \qquad \{f < -s\}_{z,r} \xrightarrow{\text{loc}} \emptyset, \qquad \forall s \in [t - r_*, t + r_*], \tag{3.40}$$

thanks to the fact that $f^\vee(z) \geq 0$. Since $z \in G_+^{(1/2)} \cap G_-^{(1/2)} \subset G_1^{(0)} \cap G_0^{(0)}$, we have

$$\mathcal{H}^n\big(\mathbf{C}_{(z,t),r} \cap ((G_1 \cup G_0) \times \mathbb{R})\big) = o(r^n); \tag{3.41}$$

moreover, if $r < r_*$, then by (3.40) and (3.30) we have

$$\mathcal{H}^n(E \cap \mathbf{C}_{(z,t),r} \cap (G_- \times \mathbb{R})) = \int_{t-r}^{t+r} \mathcal{H}^{n-1}(G_- \cap \{f < -s\} \cap \mathbf{D}_{z,r})\, ds$$
$$\leq 2r\mathcal{H}^{n-1}(\{f < -(t-r_*)\} \cap \mathbf{D}_{z,r}) = o(r^n), \tag{3.42}$$

as well as

$$\mathcal{H}^n(H^+_{(z,t),(\nu,0)} \cap \mathbf{C}_{(z,t),r} \cap (G_- \times \mathbb{R})) = 2r\mathcal{H}^{n-1}(H^+_{z,\nu} \cap G_- \cap \mathbf{D}_{z,r}) = o(r^n). \tag{3.43}$$

Hence, by (3.41)–(3.43), and taking again into account (3.40) and (3.30), we obtain

$$\begin{aligned}\mathcal{H}^n((E \,\triangle\, H^+_{(z,t),(\nu,0)}) \cap \mathbf{C}_{(z,t),r}) &= o(r^n) + \int_{t-r}^{t+r} \mathcal{H}^{n-1}\big((H^+_{z,\nu} \,\triangle\, (G_+ \cap \{f < s\})) \cap \mathbf{D}_{z,r}\big)\, ds \\ &= o(r^n) + \int_{t-r}^{t+r} \mathcal{H}^{n-1}((G_+ \,\triangle\, H^+_{z,\nu}) \cap \mathbf{D}_{z,r})\, ds = o(r^n).\end{aligned}$$

This proves that if $t > f^\vee(z)$, then $(z,t) \in \partial^J E$ with $E_{(z,t),r} \xrightarrow{\text{loc}} H^+_{(0,0),(\nu,0)}$.

Proof of (3.34) *when* $t < -f^\wedge(z)$. In the subcase that $t < -f^\vee(z)$, we immediately see (by symmetry) that $(z,t) \in \partial^J E$ with

$$E_{(z,t),r} \xrightarrow{\text{loc}} H^-_{(0,0),(\nu,0)}. \tag{3.44}$$

In particular, if $z \in S^c_f$, this concludes the proof of (3.34). It remains to consider the case that $z \in J_f$ and $-f^\vee(z) < t < -f^\wedge(z)$. In this case, we still record the validity of (3.44), but this time, in the proof, we also have to take (3.31) into account: indeed, by (3.31) we have

$$\{f < s\}_{z,r} \xrightarrow{\text{loc}} H^-_{0,\nu}, \qquad \forall s \in (f^\wedge(z), f^\vee(z)),$$

therefore, if $-f^\vee(z) < t < -f^\wedge(z)$ then there exists $r_* > 0$ such that

$$\{f < -s\}_{z,r} \xrightarrow{\text{loc}} H^-_{0,\nu}, \qquad \forall s \in [t - r_*, t + r_*]. \tag{3.45}$$

We now notice that since $t + r^* < -f^\wedge(z) \leq f^\wedge(z)$, we have $\{f < t + r_*\}_{z,r} \xrightarrow{\text{loc}} \emptyset$, and so

$$\mathcal{H}^n(E \cap \mathbf{C}_{(z,t),r} \cap (G_+ \times \mathbb{R})) \leq \int_{t-r}^{t+r} \mathcal{H}^{n-1}(G_+ \cap \{f < s\} \cap \mathbf{D}_{z,r})\, ds$$
$$\leq 2r\mathcal{H}^{n-1}(G_+ \cap \{f < t + r_*\} \cap \mathbf{D}_{z,r}) = o(r^n). \tag{3.46}$$

By (3.30), we similarly have

$$\mathcal{H}^n(H^-_{(z,t),(\nu,0)} \cap \mathbf{C}_{(z,t),r} \cap (G_+ \times \mathbb{R})) = 2r\mathcal{H}^{n-1}(H^-_{z,\nu} \cap G_+ \cap \mathbf{D}_{z,r}) = o(r^n). \tag{3.47}$$

By combining (3.46) and (3.47) with (3.41) (which holds true simply because $z \in G_+^{(1/2)} \cap G_-^{(1/2)}$), we thus find

$$\begin{aligned}
&\mathcal{H}^n((E \triangle H^-_{(z,t),(\nu,0)}) \cap \mathbf{C}_{(z,t),r}) \\
&\qquad = o(r^n) + \int_{t-r}^{t+r} \mathcal{H}^{n-1}\big((H^-_{z,\nu} \triangle (G_- \cap \{f < -s\})) \cap \mathbf{D}_{z,r}\big)\, ds \\
&\qquad = o(r^n) + \int_{t-r}^{t+r} \mathcal{H}^{n-1}(G_- \cap \{f < -s\} \cap H^+_{z,\nu} \cap \mathbf{D}_{z,r})\, ds \\
&\qquad\quad + \int_{t-r}^{t+r} \mathcal{H}^{n-1}\big((H^-_{z,\nu} \setminus (G_- \cap \{f < -s\})) \cap \mathbf{D}_{z,r}\big)\, ds \\
&\qquad \le o(r^n) + 2r\mathcal{H}^{n-1}(G_- \cap \{f < -(t - r_*)\} \cap H^+_{z,\nu} \cap \mathbf{D}_{z,r}) \\
&\qquad\quad + 2r\mathcal{H}^{n-1}\big((H^-_{z,\nu} \setminus (G_- \cap \{f < -(t + r_*)\})) \cap \mathbf{D}_{z,r}\big) = o(r^n),
\end{aligned}$$

where in the last step we have also used (3.30) and (3.45). This concludes the proof of (3.34).

Step 9. We finally find a contradiction. To this end, define Π_- as the set of those

$$z \in \partial^J G_+ \cap \partial^J G_- \cap \{|f|^\vee < \infty\} \cap (S_f^c \cup J_f)$$

such that, for some $\nu \in S^{n-1}$,

$$(G_+)_{z,r} \xrightarrow{\text{loc}} H^+_{0,\nu}, \quad (G_-)_{z,r} \xrightarrow{\text{loc}} H^-_{0,\nu}, \tag{3.48}$$

$$\{f > s\} \xrightarrow{\text{loc}} H^-_{0,\nu} \quad \text{if } z \in J_f \text{ and } s \in (f^\wedge(z), f^\vee(z)). \tag{3.49}$$

Now notice the following two facts. First, trivially,

$$(\Pi_+ \cup \Pi_-) \cap S_f^c = \partial^J G_+ \cap \partial^J G_- \cap \{|f|^\vee < \infty\} \cap S_f^c. \tag{3.50}$$

Second, since J_f and S_{u_k} are both countably $\mathcal{H}^{n-2}$-rectifiable, we have $\nu_f = \pm\nu_{u_k}$ $\mathcal{H}^{n-2}$-a.e. on $J_f \cap S_{u_k}$, and thus by (3.27) and (3.28),

$$(\Pi_+ \cup \Pi_-) \cap J_f \cap S_{u_k} =_{\mathcal{H}^{n-2}} \partial^J G_+ \cap \partial^J G_- \cap \{|f|^\vee < \infty\} \cap J_f \cap S_{u_k}. \tag{3.51}$$

Since $\mathcal{H}^{n-2}(S_f \setminus J_f) = 0$, we finally conclude that

$$(\Pi_+ \cup \Pi_-) \cap S_{u_k} =_{\mathcal{H}^{n-2}} \partial^J G_+ \cap \partial^J G_- \cap \{|f|^\vee < \infty\} \cap S_{u_k}.$$

In particular, by (3.22) and (3.29), we may assume (up to replacing E with $g(E)$) that

$$\mathcal{H}^{n-2}(\Sigma_k \cap \Pi_+) > 0$$

for sufficiently large values of k. Since Σ_k is countably $\mathcal{H}^{n-2}$-rectifiable, by (2.9) and (3.38) we find

$$P_\gamma(E; (\Sigma_k \cap \Pi_+) \times \mathbb{R}) = \int_{\Sigma_k \cap \Pi_+} d\mathcal{H}^{n-2}_\gamma(z) \int_{(\partial^J E)_z} d\mathcal{H}^1_\gamma$$
$$= \int_{\Sigma_k \cap \Pi_+} d\mathcal{H}^{n-2}_\gamma(z) \left(\int_{-\infty}^{a(z)} d\mathcal{H}^1_\gamma + \int_{b(z)}^{\infty} d\mathcal{H}^1_\gamma \right),$$

where a and b have been defined as in (3.39). Since $\mathcal{H}^1_\gamma(\mathbb{R}) = 1$, we thus have

$$P_\gamma(E; (\Sigma_k \cap \Pi_+) \times \mathbb{R}) = \int_{\Sigma_k \cap \Pi_+} (1 - \gamma_1(a(z), b(z)))\, d\mathcal{H}^{n-2}_\gamma(z),$$

while, by (3.26),

$$P_\gamma(F; (\Sigma_k \cap \Pi_+) \times \mathbb{R}) = \int_{\Sigma_k \cap \Pi_+} \gamma_1(f^\wedge(z), f^\vee(z))\, d\mathcal{H}^{n-2}_\gamma(z).$$

Since $P_\gamma(E; W \times \mathbb{R}) \geq P_\gamma(F; W \times \mathbb{R})$ for every Borel set $W \subset \mathbb{R}^{n-1}$, by using $P_\gamma(E) = P_\gamma(F)$ we find that

$$P_\gamma(E; (\Sigma_k \cap \Pi_+) \times \mathbb{R}) = P_\gamma(F; (\Sigma_k \cap \Pi_+) \times \mathbb{R}).$$

This contradicts $\mathcal{H}^{n-2}(\Sigma_k \cap \Pi_+) > 0$ and the fact that the function

$$\delta(\alpha, \beta) = 1 - \gamma_1(\min\{-\alpha, \beta\}, \max\{-\alpha, \beta\}) - \gamma_1(\alpha, \beta), \qquad \forall \beta \geq \alpha,$$

is strictly positive on $\{(\alpha, \beta) \in \mathbb{R}^2 : \beta \geq \alpha\}$. Indeed, if $-\alpha \leq \beta$, then

$$\delta(\alpha, \beta) = 1 - \gamma_1(-\alpha, \beta) - \gamma_1(\alpha, \beta) = 1 - \gamma_1(-\alpha, \beta) - \gamma_1(-\beta, -\alpha)$$
$$= 1 - \gamma_1(-\beta, \beta) > 0;$$

if instead $-\alpha > \beta$, then

$$\delta(\alpha, \beta) = 1 - \gamma_1(\beta, -\alpha) - \gamma_1(\alpha, \beta) = 1 - \gamma_1(\alpha, -\alpha) > 0.$$

This completes the proof of the implication (ii)⇒(i). □

Example 3.9. It may happen that $v \in BV(\mathbb{R}^{n-1}; [0, 1])$ but $G = \{0 < v < 1\}$ is not of locally finite perimeter in $\mathbb{R}^{n-1}$. For example, if $n \geq 3$, take

$$v(z) = \frac{|z|^2}{2} \sum_{h=1}^{\infty} 1_{[1/(2h+1)^{1/(n-2)}, 1/(2h)^{1/(n-2)}]}(|z|), \qquad z \in \mathbb{R}^{n-1}.$$

In this case $G = \{0 < v < 1\}$ is not of locally finite perimeter, as

$$\mathcal{H}^{n-2}(\mathbf{D}_r \cap \partial^e G) = \mathcal{H}^{n-2}(\mathbf{D}_r \cap \partial G) = (n-1)\omega_{n-1} \sum_{h=h(r)}^{\infty} \left(\frac{1}{2h} + \frac{1}{2h+1} \right) = \infty, \quad \forall r > 0.$$

At the same time $v \in BV(\mathbb{R}^{n-1}; [0, 1])$, as

$$|Dv|(\mathbb{R}^{n-1}) \leq \sqrt{2}\, \mathcal{H}^{n-1}(G) + 2(n-1)\omega_{n-1} \sum_{h=1}^{\infty} \frac{1}{(2h)^{2/(n-2)}} \frac{1}{2h} < \infty.$$

Example 3.10. The set $G^{(1)} \cap \partial^e G_+ \cap \partial^e G_-$ may fail to be countably $\mathcal{H}^{n-2}$-rectifiable even if $v \in \mathrm{Lip}(\mathbb{R}^{n-1}; [0, 1])$. To construct an example, consider an open equilateral triangle T in $\mathbb{R}^2$, and define an increasing sequence $\{T_h\}_{h=0}^{\infty}$ of open sets by setting $T_0 = T$; T_1 is obtained from T_0 by adding a copy of T rescaled by a factor of $1/3$ with respect to the center of each side of T_0; and so on. In this way, the open set $A = \bigcup_{h=0}^{\infty} T_h$ has the well-known von Koch curve K as its topological boundary. If we set

$$v(z) = \min\{1/2, \mathrm{dist}(z, K)\}, \qquad z \in \mathbb{R}^2,$$

then v is a Lipschitz function on $\mathbb{R}^2$ with $G = \{0 < v < 1\} = \mathbb{R}^2 \setminus K$. Notice that

$$K = \{v^\wedge = 0\} = \{v = 0\} \subset G^{(1)}, \qquad \{v^\vee = 1\} = \emptyset,$$

that is, $G^{(1)} \cap \{v^\wedge = 0\} \cap \{v^\vee = 1\} = K$, and thus it is not countably $\mathcal{H}^1$-rectifiable. (Indeed, the Hausdorff dimension of K is equal to $\log(4)/\log(3)$.) In particular, given a Borel partition $\{G_+, G_-\}$ of G we cannot expect the set

$$G^{(1)} \cap \partial^e G_+ \cap \partial^e G_- \cap (\{v^\wedge = 0\} \cup \{v^\vee = 1\}) \subset K$$

to possess any rectifiability property. Notice also that, in this example, $K = \{v^\wedge = 0\}$ essentially disconnects $\{0 < v < 1\}$, as is seen by considering the non-trivial Borel partition $\{G_+, G_-\}$ of G defined by $G_+ = A$ and $G_- = \mathbb{R}^2 \setminus \overline{A}$. (Indeed, we easily find that $\partial^e G_+ = \partial^e G_- \subset K$.) Also, by Theorem 1.3, we expect rigidity to fail. A counterexample to rigidity is obtained by setting

$$E = (F \cap (G_+ \times \mathbb{R})) \cup (g(F) \cap (G_- \times \mathbb{R})).$$

The fact that $P_\gamma(E) = P_\gamma(F)$ comes from the proof of (i)$\Rightarrow$(ii) in Section 3.3.

3.3. *Proof of Theorem 1.3, (i) implies (ii)*

In this section we present the proof of the implication (i)$\Rightarrow$(ii) in Theorem 1.3. Recall the following general relation for essential boundaries:

$$\partial^e(A \cap B) \cap B^{(1)} = (\partial^e A) \cap B^{(1)}, \tag{3.52}$$

which holds true for every pair of Lebesgue measurable sets $A, B \subset \mathbb{R}^n$.

Proof of Theorem 1.3, (i) implies (ii). Overview. We shall prove that if (ii) fails then (i) fails. More precisely, assume there exists a non-trivial Borel partition $\{G_+, G_-\}$ of $G = \{0 < v < 1\}$ such that

$$\mathcal{H}^{n-2}\big((G^{(1)} \cap \partial^e G_+ \cap \partial^e G_-) \setminus (\{v^\wedge = 0\} \cup \{v^\vee = 1\})\big) = 0. \tag{3.53}$$

We set $G_1 = \{v = 1\}$, $G_0 = \{v = 0\}$, and then consider the Borel set

$$E = \big(F \cap ((G_+ \cup G_1) \times \mathbb{R})\big) \cup (g(F) \cap (G_- \times \mathbb{R})).$$

The idea here is that since E is obtained by reflecting F across a region where the sections of F are *either negligible or equivalent to* $\mathbb{R}$, then we should have $P_\gamma(E) = P_\gamma(F)$;

however, since $\mathcal{H}^{n-1}(G_+)\mathcal{H}^{n-1}(G_-) > 0$ by assumption, this would imply that both $\mathcal{H}^n(E \triangle F) > 0$ and $\mathcal{H}^n(E \triangle g(F)) > 0$, and thus that (i) fails. In order to prove that $P_\gamma(E) = P_\gamma(F)$ we shall first need to prove that E is of locally finite perimeter, and then use the information that its reduced boundary is $\mathcal{H}^{n-1}$-equivalent to its essential boundary in order to be able to check that no additional Gaussian perimeter is created in passing from F to E.

Step 1. In this step we gather some preliminary remarks. We start by noticing that if we set for brevity

$$G_{10+} = G_1 \cup G_0 \cup G_+, \qquad G_{10-} = G_1 \cup G_0 \cup G_-,$$

then by (3.52) and since $F \cap (G_{10+} \times \mathbb{R}) = E \cap (G_{10+} \times \mathbb{R})$ we find that

$$\partial^e F \cap (G^{(1)}_{10+} \times \mathbb{R}) = \partial^e E \cap (G^{(1)}_{10+} \times \mathbb{R}). \tag{3.54}$$

Similarly, starting from $g(F) \cap (G_{10-} \times \mathbb{R}) = E \cap (G_{10-} \times \mathbb{R})$, we deduce that

$$\partial^e (g(F)) \cap (G^{(1)}_{10-} \times \mathbb{R}) = \partial^e E \cap (G^{(1)}_{10-} \times \mathbb{R}). \tag{3.55}$$

By (3.54) and (3.55), we thus find

$$\mathcal{H}^{n-1}_\gamma(\partial^e E \cap (G^{(1)}_{10+} \times \mathbb{R})) = \mathcal{H}^{n-1}_\gamma(\partial^e F \cap (G^{(1)}_{10+} \times \mathbb{R})), \tag{3.56}$$

$$\begin{aligned}\mathcal{H}^{n-1}_\gamma(\partial^e E \cap (G^{(1)}_{10-} \times \mathbb{R})) &= \mathcal{H}^{n-1}_\gamma(\partial^e g(F) \cap (G^{(1)}_{10-} \times \mathbb{R})) \\ &= \mathcal{H}^{n-1}_\gamma(\partial^e F \cap (G^{(1)}_{10-} \times \mathbb{R})).\end{aligned} \tag{3.57}$$

By (3.56) and (3.57), it remains to understand the situation outside the cylinder of basis $\mathbb{R}^{n-1} \setminus (G^{(1)}_{10+} \cup G^{(1)}_{10-})$. To this end, notice that

$$G^{(0)}_{10+} = G^{(1)}_-, \qquad G^{(0)}_{10-} = G^{(1)}_+, \qquad \partial^e G_{10+} = \partial^e G_-, \qquad \partial^e G_{10-} = \partial^e G_+,$$

so that

$$\begin{aligned}\mathbb{R}^{n-1} \setminus (G^{(1)}_{10+} \cup G^{(1)}_{10-}) &= (G^{(0)}_{10+} \cup \partial^e G_{10+}) \cap (G^{(0)}_{10-} \cup \partial^e G_{10-}) \\ &= \partial^e G_+ \cap \partial^e G_-.\end{aligned} \tag{3.58}$$

Also notice that, by (3.53) and [Fed69, 2.10.45],

$$\mathcal{H}^{n-1}\big([(G^{(1)} \cap \partial^e G_+ \cap \partial^e G_-) \setminus (\{v^\wedge = 0\} \cup \{v^\vee = 1\})] \times \mathbb{R}\big) = 0. \tag{3.59}$$

(We cannot apply (2.9) here, since $\partial^e G_+ \cap \partial^e G_-$ may fail to be countably $\mathcal{H}^{n-2}$-rectifiable —see Example 3.10.) By taking into account that $\partial^e G_\sigma = (\partial^e G_\sigma \cap \partial^e G) \cup (\partial^e G_\sigma \cap G^{(1)})$ for $\sigma \in \{+, -\}$, it remains to understand the situation inside the cylinder $(W_1 \cup W_2) \times \mathbb{R}$, where we have set

$$\begin{aligned} W_1 &= G^{(1)} \cap \partial^e G_+ \cap \partial^e G_- \cap (\{v^\wedge = 0\} \cup \{v^\vee = 1\}), \\ W_2 &= \partial^e G \cap \partial^e G_+ \cap \partial^e G_-. \end{aligned}$$

In fact, by taking into account that

$$\begin{aligned}\partial^{\mathrm{e}} G &\subset \big\{z \in \mathbb{R}^{n-1} : \theta^*(\{v=0\}, z) > 0\big\} \cup \big\{z \in \mathbb{R}^{n-1} : \theta^*(\{v=1\}, z) > 0\big\}\\ &\subset \{v^\wedge = 0\} \cup \{v^\vee = 1\},\end{aligned}$$

we find

$$W_2 = \partial^{\mathrm{e}} G \cap \partial^{\mathrm{e}} G_+ \cap \partial^{\mathrm{e}} G_- \cap (\{v^\wedge = 0\} \cup \{v^\vee = 1\}),$$

so that

$$W_1 \cup W_2 = \partial^{\mathrm{e}} G_+ \cap \partial^{\mathrm{e}} G_- \cap (\{v^\wedge = 0\} \cup \{v^\vee = 1\}). \tag{3.60}$$

Step 2. We show that E and F have no essential boundary above $\{v^\vee = 0\} \cup \{v^\wedge = 1\}$. Indeed, we are going to prove

$$\{v^\vee = 0\} \times \mathbb{R} \subset E^{(0)} \cap F^{(0)}, \tag{3.61}$$

$$\{v^\wedge = 1\} \times \mathbb{R} \subset E^{(1)} \cap F^{(1)}, \tag{3.62}$$

thus deducing that

$$\mathcal{H}^{n-1}_\gamma(\partial^{\mathrm{e}} F \cap (\{v^\vee = 0\} \times \mathbb{R})) = \mathcal{H}^{n-1}_\gamma(\partial^{\mathrm{e}} E \cap (\{v^\vee = 0\} \times \mathbb{R})) = 0, \tag{3.63}$$

$$\mathcal{H}^{n-1}_\gamma(\partial^{\mathrm{e}} F \cap (\{v^\wedge = 1\} \times \mathbb{R})) = \mathcal{H}^{n-1}_\gamma(\partial^{\mathrm{e}} E \cap (\{v^\wedge = 1\} \times \mathbb{R})) = 0. \tag{3.64}$$

Let us show for example that if $z \in \{v^\vee = 0\}$, then $(z,s) \in E^{(0)}$ for every $s \in \mathbb{R}$. Indeed, if $s \in \mathbb{R}$ and $r < 1$, then

$$\begin{aligned}\mathcal{H}^n(E \cap \mathbf{C}_{(z,s),r}) &= 2r\mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap G_1) + \int_{s-r}^{s+r} \mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap G_+ \cap \{f < t\})\, dt\\ &\quad + \int_{s-r}^{s+r} \mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap G_- \cap \{f < -t\})\, dt\\ &\le 2r\mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap G_1) + 2r\mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap \{f < |s|+1\}) = o(r^n),\end{aligned}$$

where in the last identity we have used the assumption that $v^\vee(z) = 0$ (and thus $f^\wedge(z) = +\infty$) to deduce that $\theta(\{f < |s|+1\}, z) = 0$. This proves (3.61), and (3.62) follows analogously.

Step 3. We show that E is of locally finite perimeter. To this end, by taking into account Steps 1 and 2, it suffices to prove that

$$\mathcal{H}^{n-1}_\gamma(\partial^{\mathrm{e}} E \cap (\Sigma_1 \times \mathbb{R})) < \infty, \tag{3.65}$$

where we have set

$$\Sigma_1 = \partial^{\mathrm{e}} G_+ \cap \partial^{\mathrm{e}} G_- \cap (\{0 = v^\wedge < v^\vee\} \cup \{v^\wedge < v^\vee = 1\}).$$

We now claim that if $z \in \{0 = v^\wedge < v^\vee\} \cup \{v^\wedge < v^\vee = 1\}$, then

$$(\partial^{\mathrm{e}} E)_z \subset_{\mathcal{H}^1} (\partial^{\mathrm{e}} F)_z \cup (\partial^{\mathrm{e}} g(F))_z. \tag{3.66}$$

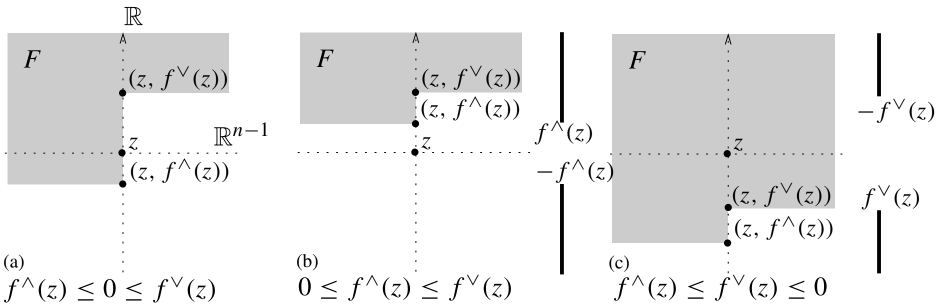

Fig. 3.2. The three cases one has to consider in describing $(\partial^e E)_z$. Notice that, in case (a), both inclusions (3.69) and (3.70) are trivial; in case (b), (3.70) is trivial, and (3.69) carries all the useful information; finally, in case (c), (3.69) is trivial and (3.70) is not.

Indeed, on the one hand, by (3.13) we have

$$(\partial^e F)_z =_{\mathcal{H}^1} [f^\wedge(z), \infty), \qquad \forall z \in \{0 = v^\wedge < v^\vee\}, \tag{3.67}$$

$$(\partial^e F)_z =_{\mathcal{H}^1} (-\infty, f^\vee(z)], \qquad \forall z \in \{v^\wedge < v^\vee = 1\}; \tag{3.68}$$

on the other hand, we also have, *for every* $z \in \mathbb{R}^{n-1}$,

$$(\partial^e E)_z \subset (-\infty, -f^\wedge(z)] \cup [f^\wedge(z), \infty), \tag{3.69}$$

$$(\partial^e E)_z \subset (-\infty, f^\vee(z)] \cup [-f^\vee(z), \infty) \tag{3.70}$$

(see Figure 3.2). Let us show, for example, the validity of (3.69): if $f^\wedge(z) \le 0$, then the inclusion is trivial; if $f^\wedge(z) > 0$, then $v^\vee(z) < 1/2$, thus

$$0 = \theta(\{v > 2/3\}, z) \ge \theta(G_1, z),$$

that is, $z \in G_1^{(0)}$. Hence,

$$\begin{aligned}\mathcal{H}^n(E \cap \mathbf{C}_{(z,t),r}) &= 2r\mathcal{H}^{n-1}(G_1 \cap \mathbf{D}_{z,r}) + \int_{t-r}^{t+r} \mathcal{H}^{n-1}(G_+ \cap \{f < s\} \cap \mathbf{D}_{z,r})\, ds \\ &\quad + \int_{t-r}^{t+r} \mathcal{H}^{n-1}(G_- \cap \{f < -s\} \cap \mathbf{D}_{z,r})\, ds \\ &\le o(r^n) + 2r\mathcal{H}^{n-1}(\{f < |t| + r\} \cap \mathbf{D}_{z,r});\end{aligned}$$

therefore, if $t \in (-f^\wedge(z), f^\wedge(z))$ and $r < r_*$ for a suitable r_*, then

$$\mathcal{H}^n(E \cap \mathbf{C}_{(z,t),r}) \le o(r^n) + 2r\mathcal{H}^{n-1}(\{f < |t| + r_*\} \cap \mathbf{D}_{z,r}) = o(r^n),$$

that is, $(z,t) \in E^{(0)}$; in other words,

$$(-f^\wedge(z), f^\wedge(z)) \subset (E^{(0)})_z \subset \mathbb{R} \setminus (\partial^e E)_z,$$

that is, (3.69) holds. The proof of (3.70) is analogous; by taking into account (3.67)–(3.70), we thus obtain (3.66), which in particular gives

$$\mathcal{H}^{n-1}_\gamma(\partial^e E \cap (\Sigma_1 \times \mathbb{R})) \leq 2\mathcal{H}^{n-1}_\gamma(\partial^e F \cap (\Sigma_1 \times \mathbb{R})), \tag{3.71}$$

and this proves (3.65). By (3.56)–(3.60) and (3.63)–(3.65), we find $\mathcal{H}^{n-1}_\gamma(\partial^e E) < \infty$. Hence, by Federer's criterion, E is of locally finite perimeter.

Step 4. We have proved so far that E is of locally finite perimeter with

$$P_\gamma(E; (\mathbb{R}^{n-1} \setminus \Sigma_1) \times \mathbb{R}) = P_\gamma(F; (\mathbb{R}^{n-1} \setminus \Sigma_1) \times \mathbb{R}).$$

Since $P_\gamma(E; W \times \mathbb{R}) \geq P_\gamma(F; W \times \mathbb{R})$ for every Borel set $W \subset \mathbb{R}^{n-1}$, we only need to show

$$P_\gamma(E; \Sigma_1 \times \mathbb{R}) \leq P_\gamma(F; \Sigma_1 \times \mathbb{R}). \tag{3.72}$$

By Federer's theorem, $\mathcal{H}^{n-1}(\partial^e E \setminus \partial^J E) = 0$, and moreover by Proposition 3.4 we have $\mathcal{H}^{n-2}(S_f \setminus J_f) = 0$ (so that $\mathcal{H}^{n-1}((S_f \setminus J_f) \times \mathbb{R}) = 0$). Since $\{v^\wedge = 0\} = \{f^\vee = \infty\}$ and $\{v^\vee = 1\} = \{f^\wedge = -\infty\}$, we conclude that (3.72) follows from

$$\mathcal{H}^{n-1}_\gamma(\partial^J E \cap (\Sigma_2 \times \mathbb{R})) \leq \mathcal{H}^{n-1}_\gamma(\partial^e F \cap (\Sigma_2 \times \mathbb{R})), \tag{3.73}$$

where

$$\Sigma_2 = \partial^e G_+ \cap \partial^e G_- \cap J_f \cap (\{-\infty < f^\wedge < f^\vee = \infty\} \cup \{-\infty = f^\wedge < f^\vee < \infty\}).$$

We now turn to the proof of (3.73), and thus complete the proof of (ii)⇒(i). To this end, we pick

$$z \in J_f \cap (\{-\infty < f^\wedge < f^\vee = \infty\} \cup \{-\infty = f^\wedge < f^\vee < \infty\})$$

and show that either $(\partial^J E)_z \subset_{\mathcal{H}^1} (\partial^J F)_z$ or $(\partial^J E)_z \subset_{\mathcal{H}^1} g((\partial^J F)_z)$. In fact, by symmetry, we only have to consider the case

$$z \in J_f \cap \{-\infty < f^\wedge < f^\vee = \infty\}. \tag{3.74}$$

Under assumption (3.74), we thus want to show that

$$\text{either} \quad (\partial^J E)_z \subset_{\mathcal{H}^1} (\partial^J F)_z =_{\mathcal{H}^1} (f^\wedge(z), \infty), \tag{3.75}$$

$$\text{or} \quad (\partial^J E)_z \subset_{\mathcal{H}^1} g((\partial^J F)_z) =_{\mathcal{H}^1} (-\infty, -f^\wedge(z)). \tag{3.76}$$

We first notice that, by Lemma 3.6, there exists $\nu \in S^{n-1} \cap \mathbb{R}^{n-1}$ such that

$$\{f < s\}_{z,r} \xrightarrow{\text{loc}} H^+_{z,\nu}, \qquad \forall s > f^\wedge(z), \tag{3.77}$$

$$\{f < s\}_{z,r} \xrightarrow{\text{loc}} \emptyset, \qquad \forall s < f^\wedge(z). \tag{3.78}$$

Moreover, we have the inclusions

$$(\partial^J E)_z \cap (f^\wedge(z), -f^\wedge(z)) \subset_{\mathcal{H}^1} (\partial^J F)_z \qquad \text{if } f^\wedge(z) \le 0, \tag{3.79}$$

$$(\partial^J E)_z \cap (-f^\wedge(z), f^\wedge(z)) \subset_{\mathcal{H}^1} \emptyset \qquad \text{if } f^\wedge(z) > 0, \tag{3.80}$$

which follow from (3.67) and (3.69). We now divide our argument into two cases.

Case 1. Assuming that there exists $t_0 \in (\partial^J E)_z$ with $t_0 > |f^\wedge(z)|$ we show that

$$\{t \in (\partial^J E)_z : |t| > |f^\wedge(z)|\} = (|f^\wedge(z)|, \infty). \tag{3.81}$$

Case 2. Assuming that there exists $t_0 \in (\partial^J E)_z$ with $t_0 < -|f^\wedge(z)|$ we show that

$$\{t \in (\partial^J E)_z : |t| > |f^\wedge(z)|\} = (-\infty, -|f^\wedge(z)|). \tag{3.82}$$

Before entering into the proof of the two cases, notice how they allow us to complete the proof of the theorem (see also Figure 3.3). Indeed, if none of the two cases holds true, then $(\partial^J E)_z \subset_{\mathcal{H}^1} (-|f^\wedge(z)|, |f^\wedge(z)|)$, and the validity of either (3.75) or (3.76) follows from (3.79) and (3.80). (Just notice that if $f^\wedge(z) \le 0$, then $(-|f^\wedge(z)|, |f^\wedge(z)|) \subset (\partial^J F)_z \cap g(\partial^J F)_z$.) Similarly, if we are in the first case, and $f^\wedge(z) > 0$, then (3.75) follows by combining (3.80) with (3.81); if we are in the first case and $f^\wedge(z) \le 0$, then (3.75) follows from (3.79) and (3.81); finally, if we are in the second case then (3.76) holds true by combining (3.79), (3.80), and (3.82). We prove (3.81) and (3.82) in the next step.

Step 5. We assume to be in the first case, and prove (3.81). Let us first show that

$$\mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap G_{1+} \cap H^+_{z,\nu}) = \omega_{n-1} r^{n-1}/2 + o(r^{n-1}), \tag{3.83}$$

and thus clearly

$$\mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap G_{0-} \cap H^+_{z,\nu}) = o(r^{n-1}), \tag{3.84}$$

where we have set $G_{1+} = G_+ \cup G_1$ and $G_{0-} = G_- \cup G_0$. To prove (3.83), we pick t_1 and t_2 such that $|f^\wedge(z)| < t_1 < t_0 < t_2$. Since $(z, t_0) \in \partial^J E$, and since every half-space H with $x \in \partial H$ cuts $\mathbf{C}_{x,r}$ into two halves of equal volume, we find that

$$\begin{aligned}
&\mathcal{H}^n(\mathbf{C}_{(z,t_0),r})/2 + o(r^n) = \mathcal{H}^n(E \cap \mathbf{C}_{(z,t_0),r}) \\
&= \int_{t_0-r}^{t_0+r} [\mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap G_{1+} \cap \{f < s\}) + \mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap G_- \cap \{f < -s\})]\, ds \\
&\le 2r\mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap G_{1+} \cap \{f < t_2\})\, ds + o(r^n),
\end{aligned}$$

where in the last identity we have used (3.78) with $s = -t_1 < -|f^\wedge(z)| \le f^\wedge(z)$. By applying (3.77) with $s = t_2 > |f^\wedge(z)| \ge f^\wedge(z)$, and since $\mathcal{H}^{n-1}(\mathbf{C}_r) = 2\omega_{n-1} r^n$, we find

$$\omega_{n-1} r^n + o(r^n) \le 2r\mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap G_{1+} \cap H^+_{z,\nu}) + o(r^n),$$

that is,

$$\omega_{n-1} r^{n-1}/2 + o(r^{n-1}) \le \mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap G_{1+} \cap H^+_{z,\nu}) \le \omega_{n-1} r^{n-1}/2.$$

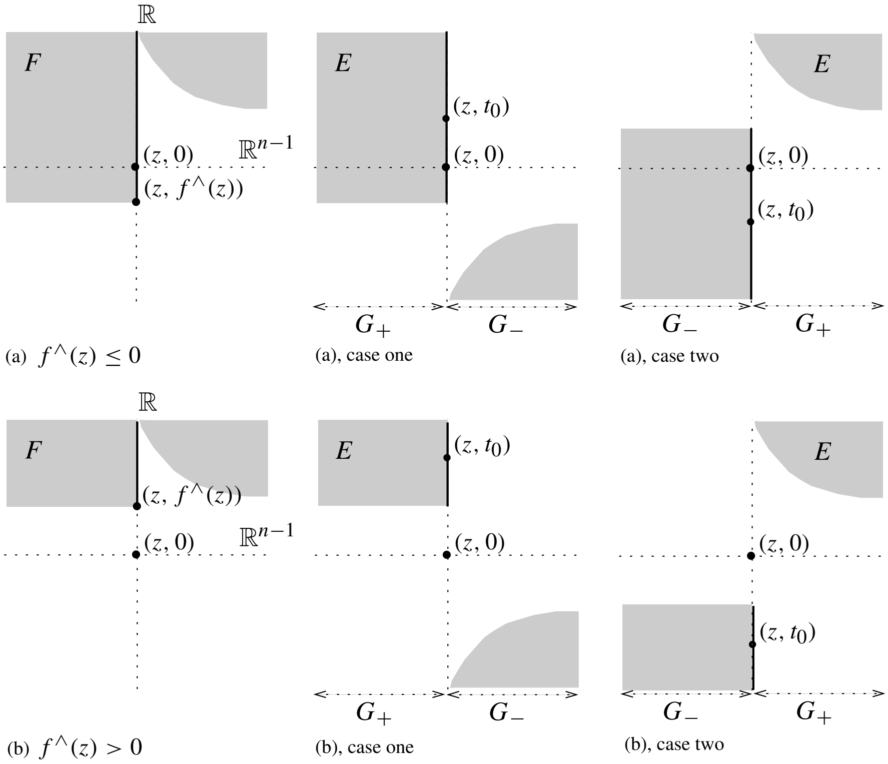

Fig. 3.3. The situation in the proof of (3.75) and (3.76). If $f^\wedge(z) \le 0$, then (3.79) shows that $(f^\wedge(z), -f^\wedge(z))$ is contained in both $(\partial^e F)_z$ and $(\partial^e E)_z$. Moreover, if $f^\wedge(z) \le 0$ and we are in case one, then (see (3.81)) there exists $t_0 > -f^\wedge(z)$ such that $(z, t_0) \in \partial^J E$, $(\partial^e E)_z$ and $(\partial^e F)_z$ are both $\mathcal{H}^1$-equivalent to $(f^\wedge(z), \infty)$, and (3.75) holds true. Finally, if $f^\wedge(z) \le 0$ and we are in case two, then (see (3.82)) there exists $t_0 < f^\wedge(z)$ such that $(z, t_0) \in \partial^J E$, $(\partial^e E)_z$ and $g((\partial^e F)_z)$ are both $\mathcal{H}^1$ equivalent to $(-\infty, f^\wedge(z))$, and thus (3.76) holds true. Similar remarks apply when $f^\wedge(z) > 0$.

This proves (3.83), and thus (3.84). We now pick $t > |f^\wedge(z)|$, choose t_1 and t_2 such that $|f^\wedge(z)| < t_1 < t < t_2$, and notice that

$$\begin{aligned}\mathcal{H}^n((E \,\Delta\, H^+_{(z,t),(\nu,0)}) \cap \mathbf{C}_{(z,t),r}) &= \int_{t-r}^{t+r} \mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap G_{1+} \cap \{f < s\} \cap H^-_{z,\nu})\, ds \\ &+ \int_{t-r}^{t+r} \mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap G_{1+} \cap \{f \ge s\} \cap H^+_{z,\nu})\, ds \\ &+ \int_{t-r}^{t+r} \mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap G_- \cap \{f < -s\} \cap H^-_{z,\nu})\, ds \\ &+ \int_{t-r}^{t+r} \mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap G_- \cap \{f \ge -s\} \cap H^+_{z,\nu})\, ds,\end{aligned}$$

so that $\mathcal{H}^n((E \triangle H^+_{(z,t),(\nu,0)}) \cap \mathbf{C}_{(z,t),r}) \le 2r(I_1 + I_2 + I_3 + I_4)$ where

$$\begin{aligned} I_1 &= \mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap G_{1+} \cap \{f < t_2\} \cap H^-_{z,\nu}), \\ I_2 &= \mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap G_{1+} \cap \{f \ge t_1\} \cap H^+_{z,\nu}), \\ I_3 &= \mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap G_- \cap \{f < -t_1\} \cap H^-_{z,\nu}), \\ I_4 &= \mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap G_- \cap \{f \ge -t_2\} \cap H^+_{z,\nu}). \end{aligned}$$

We see that $I_1 = I_2 = o(r^{n-1})$ by (3.77), while $I_3 = o(r^{n-1})$ by (3.78), and $I_4 = o(r^{n-1})$ by (3.84). We have thus proved that

$$(|f^\wedge(z)|, \infty) \subset (\partial^J E)_z.$$

In order to conclude the proof of (3.81) we will now prove that

$$(-\infty, -|f^\wedge(z)|) \subset E^{(0)}.$$

Indeed, pick $t < -|f^\wedge(z)|$. This time we take t_1 and t_2 such that $t_1 < t < t_2 < -|f^\wedge(z)|$. In this way, by arguing as above, and by also recalling that $z \in G_1^{(0)}$, we find

$$\begin{aligned} \mathcal{H}^n(E \cap \mathbf{C}_{(z,t),r}) \le 2r\mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap G_+ \cap \{f < t_2\}) \\ + 2r\mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap G_- \cap \{f < -t_1\}), \end{aligned}$$

where the first term is $o(r^n)$ by (3.78). By (3.77) we thus find

$$\mathcal{H}^n(E \cap \mathbf{C}_{(z,t),r}) = o(r^n) + 2r\mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap G_- \cap H^+_{z,\nu}) = o(r^n),$$

where the last identity follows from (3.84). Hence $(z,t) \in E^{(0)}$, as claimed, and the proof of (3.81) is completed. In order to prove (3.82), we notice that the existence of $t_0 < -|f^\wedge(z)|$ such that $(z, t_0) \in \partial^J E$ implies

$$\mathcal{H}^{n-1}(\mathbf{D}_{z,r} \cap G_{1+} \cap H^+_{z,\nu}) = o(r^{n-1}). \tag{3.85}$$

The proof of (3.82) is then analogous to that of (3.81), with (3.85) in place of (3.84). □

Proof of Theorem 1.3. The equivalence of (i) and (ii) is proved in Sections 3.2 and 3.3. □

3.4. Proof of Theorem 1.6

Step 1. We show that if a Borel set $G \subset \mathbb{R}$ is essentially connected, then $G^{(1)}$ is an interval. Indeed, let us prove that if $a, b \in G^{(1)}$ with $a < b$ and $c \in (a, b)$, then $c \in G^{(1)}$. To see this, we set $G_+ = G \cap (c, \infty)$, $G_- = G \cap (-\infty, c)$, so that $\{G_+, G_-\}$ is a Borel partition of G modulo $\mathcal{H}^1$. In fact, $\mathcal{H}^1(G_+)\mathcal{H}^1(G_-) > 0$. Indeed, should $\mathcal{H}^1(G_+)$ be 0, we would have $(G_+)^{(1)} = \emptyset$, and thus

$$b \in G^{(1)} \cap (c, \infty)^{(1)} \subset (G \cap (c, \infty))^{(1)} = (G_+)^{(1)} = \emptyset,$$

a contradiction. Since G is essentially connected, we find

$$\mathcal{H}^0(G^{(1)} \cap \partial^e G_+ \cap \partial^e G_-) > 0. \tag{3.86}$$

Since $G^{(1)} \cap \partial^e G_+ = G^{(1)} \cap \{c\}$ and $G^{(1)} \cap \partial^e G_- = G^{(1)} \cap \{c\}$, (3.86) gives $c \in G^{(1)}$.

Step 2. If $\{v^\wedge = 0\} \cup \{v^\vee = 1\}$ does not essentially disconnect $\{0 < v < 1\}$, then in particular $\{0 < v < 1\}$ is essentially connected, and thus $\mathcal{H}^1$-equivalent to an open interval I by Step 1. Let now $c \in I$, and assume that $v^\wedge(c) = 0$. Since $\{c\}$ (thus $\{v^\wedge = 0\}$) essentially disconnects I, by Remark 1.1 we find that $\{v^\wedge = 0\}$ essentially disconnects $\{0 < v < 1\}$, a contradiction. Therefore, $v^\wedge > 0$ on I. We similarly see that $v^\vee < 1$ on I. This shows that assumption (ii) in Theorem 1.3 implies assumption (ii) in Theorem 1.6. Since the reverse implication is trivial, we are done. □

4. Some further conditions for rigidity

As noticed in Remark 1.7, a natural question is whether it is possible to formulate sufficient conditions for rigidity in terms of suitable connectedness properties of $F[v]$. Referring the readers to the remark for a list of examples and possible conditions, we prove two results that provide simple sufficient conditions for rigidity.

Theorem 4.1. *If $v : \mathbb{R}^{n-1} \to [0, 1]$ is Lebesgue measurable and $P_\gamma(F[v]) < \infty$, and if there exists a sequence $t_h \to 0$ as $h \to \infty$ such that, for every $h \in \mathbb{N}$,*

$$F[v] \cap (\{t_h < v < 1 - t_h\} \times \mathbb{R}) \text{ is essentially connected in } \mathbb{R}^n, \tag{4.1}$$

then $E \in \mathcal{M}(v)$ if and only if $\mathcal{H}^n(E \,\triangle\, F[v]) = 0$ or $\mathcal{H}^n(E \,\triangle\, g(F[v])) = 0$.

Proof. We notice that in the proof of (ii)⇒ (i) in Theorem 1.3, assumption (ii) was used only to guarantee the validity of (3.18), which in turn was used in Step 5 of that proof to deduce that $\mathcal{H}^{n-2}(\Sigma_k) > 0$. Thus, in order to prove that (4.1) implies rigidity, it will suffice to show that it implies $\mathcal{H}^{n-2}(\Sigma_k) > 0$ for k large enough. Now set

$$G_h = \{t_h < v < 1 - t_h\}, \qquad F_h = F \cap (G_h \times \mathbb{R}), \qquad h \in \mathbb{N}.$$

If we write $G_{h,+} = G_+ \cap G_h$ and $G_{h,-} = G_- \cap G_h$, then $\mathcal{H}^{n-1}(G_{h,\pm}) \to \mathcal{H}^{n-1}(G_\pm)$ as $h \to \infty$. Hence, $\mathcal{H}^{n-1}(G_{h,+})\mathcal{H}^{n-1}(G_{h,-}) > 0$ for h large enough, and so the sets

$$F_{h,+} = F \cap (G_{h,+} \times \mathbb{R}), \qquad F_{h,-} = F \cap (G_{h,-} \times \mathbb{R})$$

define a non-trivial Lebesgue measurable partition of F_h. By (4.1),

$$\mathcal{H}^{n-1}(\partial^e F_{h,+} \cap \partial^e F_{h,-} \cap F_h^{(1)}) > 0 \tag{4.2}$$

for h large enough. Now set

$$\Lambda_h = \mathbf{p}(\partial^e F_{h,+} \cap \partial^e F_{h,-} \cap F_h^{(1)}), \qquad \forall h \in \mathbb{N}.$$

If $\mathcal{H}^{n-2}(\Lambda_h) < \infty$, then, by [Fed69, 2.10.45], for every $R > 0$ we have

$$\begin{aligned}\mathcal{H}^{n-2}(\Lambda_h)\mathcal{L}^1((-R,R)) &\geq c(n)\mathcal{H}^{n-1}(\Lambda_h \times (-R,R))\\ &\geq c(n)\mathcal{H}^{n-1}(\partial^e F_{h,+} \cap \partial^e F_{h,-} \cap F_h^{(1)} \cap \{|\mathbf{q}x| < R\}),\end{aligned}$$

so that, by (4.2), $\mathcal{H}^{n-2}(\Lambda_h) > 0$ for every h large enough.

We now claim that given $h \in \mathbb{N}$ there exists $k_h \in \mathbb{N}$ such that

$$\Lambda_h \subset \Sigma_k, \qquad \forall k \geq k_h; \tag{4.3}$$

this will conclude the proof. To show (4.3), we start by noticing that

$$\begin{aligned} z \in G_+^{(0)} \quad &\Rightarrow \quad z \in G_{h,+}^{(0)}\\ &\Rightarrow \quad (z,s) \in (G_{h,+} \times \mathbb{R})^{(0)}, \qquad \forall s \in \mathbb{R},\\ &\Rightarrow \quad (z,s) \in [F \cap (G_{h,+} \times \mathbb{R})]^{(0)}, \quad \forall s \in \mathbb{R},\\ &\Rightarrow \quad z \notin \mathbf{p}(\partial^e F_{h,+}); \end{aligned}$$

similarly, since G_+ and G_- are disjoint, $z \in G_+^{(1)}$ implies $z \in G_-^{(0)}$, and consequently $z \notin \mathbf{p}(\partial^e F_{h,-})$. We have thus proved so far that

$$\Lambda_h \subset \mathbf{p}(\partial^e F_{h,+} \cap \partial^e F_{h,-}) \subset \partial^e G_+ \cap \partial^e G_-, \qquad \forall h \in \mathbb{N}. \tag{4.4}$$

We now notice that

$$\begin{aligned} G_h^{(1)} &\subset \{v > t_h\}^{(1)} \cap \{v < 1 - t_h\}^{(1)}\\ &\subset \{v^\wedge \geq t_h\} \cap \{v^\vee \leq 1 - t_h\} \qquad \text{(by (2.3) and (2.4))}\\ &\subset \{f^\vee \leq \Psi(t_h)\} \cap \{f^\wedge \geq \Psi(1 - t_h)\} \qquad \text{(by (2.6))}. \end{aligned} \tag{4.5}$$

Hence, if $x \in F_h^{(1)}$, then $x \in (G_h \times \mathbb{R})^{(1)}$, and thus $\mathbf{p}x = z \in G_h^{(1)}$, so that, by (4.5),

$$\Lambda_h \subset G_h^{(1)} \subset \{f^\vee \leq \Psi(t_h)\} \cap \{f^\wedge \geq \Psi(1 - t_h)\}, \qquad \forall h \in \mathbb{N}. \tag{4.6}$$

By combining (4.4), (4.6), and the definition of Σ_k, we thus obtain (4.3) provided we choose k_h such that $k_h > \Psi(t_h)$ and $-k_h < \Psi(1 - t_h)$. □

Theorem 4.2. *If $v : \mathbb{R} \to [0,1]$ is Lebesgue measurable with $P_\gamma(F[v]) < \infty$, and the sets $F[v]$ and $\mathbb{R}^2 \setminus F[v]$ are both indecomposable, then $E \in \mathcal{M}(v)$ if and only if $\mathcal{H}^2(E \,\triangle\, F[v]) = 0$ or $\mathcal{H}^2(E \,\triangle\, g(F[v])) = 0$.*

Proof. Step 1. We show that if $F = F[v]$ is indecomposable in $\mathbb{R}^2$ and $v^\wedge(c) = 0$, then

$$\mathcal{H}^2(F \cap ((c,\infty) \times \mathbb{R}))\mathcal{H}^2(F \cap ((-\infty,c) \times \mathbb{R})) = 0. \tag{4.7}$$

Indeed, assume this is not the case, and set $F_+ = F \cap ((c,\infty) \times \mathbb{R})$ and $F_- = F \cap ((-\infty,c) \times \mathbb{R})$. We claim that $\{F_+, F_-\}$ is a non-trivial partition of F into sets of locally finite perimeter with

$$F^{(1)} \cap \partial^e F_+ \cap \partial^e F_- = \emptyset, \tag{4.8}$$

contrary to the indecomposability of F. To show that (4.8) holds true, notice that since F_+ and F_- are disjoint subsets of F whose union is F, we have

$$F^{(1)} \cap \partial^e F_+ \cap \partial^e F_- = F^{(1)} \cap \partial^e F_+ = F^{(1)} \cap (\{c\} \times \mathbb{R}).$$

However, if $(c,t) \in F^{(1)}$ for some $t \in \mathbb{R}$, then for every $r < 1$,

$$\begin{aligned} 4r^2 + o(r^2) = \mathcal{H}^2(F \cap \mathbf{C}_{(c,t),r}) &= \int_{t-r}^{t+r} \mathcal{H}^1(\mathbf{D}_{c,r} \cap \{f < s\})\,ds \\ &\le 2r\mathcal{H}^1(\mathbf{D}_{c,r} \cap \{f < t+1\}), \end{aligned}$$

which leads to a contradiction, because $v^\wedge(c) = 0$ (that is, $f^\vee(c) = +\infty$) implies

$$\liminf_{r \to 0^+} \frac{\mathcal{H}^1(\mathbf{D}_{c,r} \cap \{f < t+1\})}{2r} < 1.$$

This proves (4.8), and thus our claim.

Step 2. By arguing as in Step 1, we notice that if $\mathbb{R}^2 \setminus F$ is indecomposable in $\mathbb{R}^2$ and $v^\vee(c) = 1$, then

$$\mathcal{H}^2\big(((c,\infty) \times \mathbb{R}) \setminus F\big)\mathcal{H}^2\big(((-\infty,c) \times \mathbb{R}) \setminus F\big) = 0. \tag{4.9}$$

Step 3. We show that if both F and $\mathbb{R}^2 \setminus F$ are indecomposable, then $\{0 < v < 1\}$ is $\mathcal{H}^1$-equivalent to an open interval. Indeed, let I be the least closed interval that contains $\{0 < v < 1\}$ modulo $\mathcal{H}^1$. If $\{0 < v < 1\}$ is not $\mathcal{H}^1$-equivalent to I, then there exists $J \subset I \cap (\{v = 0\} \cup \{v = 1\})$ with $\mathcal{H}^1(J) > 0$. In particular, if $\varepsilon = \mathcal{H}^1(J)/3$, then there exists $c \in J^{(1)}$ with

$$c > \inf I + \varepsilon, \qquad c < \sup I - \varepsilon. \tag{4.10}$$

By (4.10), and by minimality of I, we see that

$$\mathcal{H}^2\big(F \cap ((c,\infty) \times \mathbb{R})\big)\mathcal{H}^2\big(F \cap ((-\infty,c) \times \mathbb{R})\big) > 0, \tag{4.11}$$

$$\mathcal{H}^2\big(((c,\infty) \times \mathbb{R}) \setminus F\big)\mathcal{H}^2\big(((-\infty,c) \times \mathbb{R}) \setminus F\big) > 0. \tag{4.12}$$

Since $c \in J^{(1)}$ we find that $c \in (\{v = 0\} \cup \{v = 1\})^{(1)}$, and thus either $\theta^*(\{v = 0\}, c) > 0$ or $\theta^*(\{v = 1\}, c) > 0$; therefore, either $v^\wedge(c) = 0$ (but then (4.11) contradicts (4.7)), or $v^\vee(c) = 1$ (but then (4.12) contradicts (4.9)). Hence, $\{0 < v < 1\}$ is $\mathcal{H}^1$-equivalent to the interval I.

Step 4. We prove the validity of condition (ii) in Theorem 1.6 by a simple combination of the first three steps. Hence, rigidity holds true by Theorem 1.6. □

Acknowledgments. This work was carried out while FC, MC, and GDP were visiting the University of Texas at Austin. The work of FC was partially supported by the UT Austin-Portugal partnership through the FCT post-doctoral fellowship SFRH/BPD/51349/2011. The work of GDP was partially supported by ERC under FP7, Advanced Grant no. 246923. The work of FM was partially supported by ERC under FP7, Starting Grant no. 258685 and Advanced Grant no. 226234, by the Institute for Computational Engineering and Sciences and by the Mathematics Department of the University of Texas at Austin during the time he was visiting these institutions, and by NSF Grant DMS-1265910.

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