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Absolute continuity of the periodic Schrödinger operator in transversal geometry

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Abstract. We show that the spectrum of a Schrödinger operator on $\mathbb{R}^n$, $n \geq 3$, with a periodic smooth Riemannian metric, whose conformal multiple has a product structure with one Euclidean direction, and with a periodic electric potential in $L^{n/2}_{\text{loc}}(\mathbb{R}^n)$, is purely absolutely continuous. Previous results in the case of a general metric are obtained in [12] (see also [9]) under the assumption that the metric, as well as the potential, are reflection symmetric.

Keywords. Periodic Schrödinger operator, spectrum, absolute continuity, Riemannian metric, spectral clusters, resolvent estimates

1. Introduction

Consider the Schrödinger operator

$$H = -\Delta_g + q \quad \text{ on } \mathbb{R}^n,\ n \geq 3.$$

Here $-\Delta_g$ is the Laplace–Beltrami operator associated to a C^∞-smooth Riemannian metric g, given by

$$-\Delta_g = |g|^{-1/2} D_{x_j}(|g|^{1/2} g^{jk} D_{x_k}),$$

where $D_{x_j} = i^{-1}\partial_{x_j}$, (g^{jk}) is the matrix inverse of (g_{jk}), and $|g| = \det(g_{jk})$. Throughout the paper we shall assume that the metric g and the electric potential q are 2π-periodic in all variables.

Let $q \in L^{n/2}_{\text{loc}}(\mathbb{R}^n)$ be real-valued. Then the operator H is the self-adjoint operator on $L^2(\mathbb{R}^n; |g|^{1/2}dx)$ given via the closed sesquilinear form, semibounded from below,

$$h[u, v] = \int_{\mathbb{R}^n} g^{jk} D_{x_k} u \overline{D_{x_j} v} |g|^{1/2}\, dx + \int_{\mathbb{R}^n} q u \overline{v} |g|^{1/2}\, dx, \tag{1.1}$$

with domain $\mathcal{D}(h) = H^1(\mathbb{R}^n)$, the standard L^2 based Sobolev space (see Appendix).

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Starting with the pioneering work [30], the structure of the spectrum of the periodic Schrödinger operator H in $\mathbb{R}^n$ has been intensively studied. We refer to [22], [2], [3], [24], and [25] for some of the works in this direction. In particular, in the case of the Euclidean metric, i.e. $g = I$, we know that the spectrum of H is purely absolutely continuous for a potential $q \in L^{n/2}_{\mathrm{loc}}(\mathbb{R}^n)$ when $n \geq 3$, thanks to [3] and [24], and for $q \in L^{1+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^2)$, $\varepsilon > 0$, thanks to [2]. These results can be extended to the case of a metric g conformal to the Euclidean, i.e. $g = cI$, where $c > 0$ is a smooth periodic function.

The absolute continuity of the spectrum for the magnetic Schrödinger operator with periodic electric and magnetic potentials, in the case of the Euclidean metric, was established in [1] in two dimensions and in [27] in higher dimensions. See also [15].

In two dimensions, the case of a general C^∞-smooth metric g was investigated completely in [21] and the absolute continuity of the spectrum of a periodic magnetic Schrödinger operator was established.

In higher dimensions, the case of a general metric is wide open and the only result concerning the absolute continuity of the spectrum of a periodic magnetic Schrödinger operator that we are aware of is due to [12], under the assumption that the operator is invariant under reflection $x_1 \mapsto -x_1$, in the case of smooth coefficients. The smoothness assumptions of [12] were relaxed in [9], and the absolute continuity of the spectrum was obtained for a Lipschitz continuous metric g, the magnetic potential $A \in L^{n+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^n)$, $\varepsilon > 0$, and the electric potential $q \in L^{n/2}_{\mathrm{loc}}(\mathbb{R}^n)$ (see also [11]).

The purpose of this paper is to consider the case $n \geq 3$ and to show the absolute continuity of the spectrum of the periodic Schrödinger operator H with a Riemannian metric g, whose conformal multiple has a product structure with one Euclidean direction, and $q \in L^{n/2}_{\mathrm{loc}}(\mathbb{R}^n)$. To be precise, we assume that

$$g(x_1, x') = c(x) \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix}, \tag{1.2}$$

where $c > 0$ is a positive smooth function, $x = (x_1, x') \in \mathbb{R}^n$, and g_0 is a Riemannian metric on $\mathbb{R}^{n-1}$.

Our main result is as follows.

Theorem 1.1. *Let g be a C^∞-smooth Riemannian metric on $\mathbb{R}^n$, $n \geq 3$, of the form* (1.2)*, and let $q \in L^{n/2}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{C})$. Assume that g and q are periodic with respect to the lattice $2\pi\mathbb{Z}^n$. Then the Schrödinger operator $H = -\Delta_g + q$ in $L^2(\mathbb{R}^n; |g|^{1/2}\, dx)$, defined by the sesquilinear form* (1.1)*, has no eigenvalues. In the case when q is real-valued, the spectrum of H is purely absolutely continuous.*

Remark 1.2. The metric $c^{-1}g$ is independent of x_1 and therefore satisfies the assumptions of [12]. On the other hand, no symmetry condition is imposed on the conformal factor c and the potential q.

Our inspiration for considering metrics of the form (1.2) came from recent works on inverse boundary value problems for Schrödinger operators on compact Riemannian manifolds with boundary, equipped with metrics of this form (see [6]).

We would like to mention that the problem of absolute continuity of the spectrum of the Schrödinger operator H on a smooth cylinder $M \times \mathbb{R}^m$ was treated in [8] (see also the references given there). Here M is a smooth compact Riemannian manifold, and the metric g on $M \times \mathbb{R}^m$ is a product of a Riemannian metric on M and the Euclidean metric on $\mathbb{R}^m$. Furthermore, the potential q is assumed to be periodic with respect to the Euclidean variables. In the case when the dimension of the cylinder is ≥ 3, the absolute continuity is established in [8] when $q \in L^{n/2+\varepsilon}_{\text{loc}}(M \times \mathbb{R}^m)$, $\varepsilon > 0$.

Let us finish the introduction by making some indications concerning the main steps in the proof of Theorem 1.1. First, as a consequence of general spectral theory, it will be seen that it is sufficient to treat the case when the conformal factor c in (1.2) satisfies $c = 1$, and therefore to work with the operator $D^2_{x_1} - \Delta_{g_0(x')} + q$. We would like to show the absence of eigenvalues for this operator, and replacing q by $q - \lambda$ we reduce the problem to establishing that zero is not an eigenvalue. An application of the Floquet theory combined with the Thomas approach (see Proposition A.1 in the Appendix) allows us next to conclude that it suffices to find $\theta \in \mathbb{C}^n$ such that

$$\operatorname{Ker}(H(\theta)) = \{0\}.$$

Here the operator $H(\theta)$, acting on the torus $\mathbb{T}^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n$, is given by

$$H(\theta) = |g|^{-1/2}(D_{x_j} + \theta_j)(|g|^{1/2} g^{jk}(D_{x_k} + \theta_k)) + q, \quad 1 \leq j, k \leq n.$$

See Subsection A.2 for the definition of this operator using the method of quadratic forms.

We shall make the following choice of the complex quasimomentum θ, for $\tau \in \mathbb{R}$:

$$\theta_\tau = (1/2 + i\tau, 0, \dots, 0) \in \mathbb{C}^n,$$

with the corresponding family of operators given by

$$H(\theta_\tau) = (D_{x_1} + 1/2)^2 + 2i\tau(D_{x_1} + 1/2) - \tau^2 - \Delta_{g_0(x')} + q.$$

In the case when $q \in L^\infty(\mathbb{R}^n)$, the fact that we know explicitly the eigenvalues of the one-dimensional normal operator $(D_{x_1} + 1/2)^2 + 2i\tau(D_{x_1} + 1/2) - \tau^2$ on $\mathbb{T}^1$ implies that for $\tau \in \mathbb{R}$ with $|\tau|$ sufficiently large, we have

$$\tfrac{1}{2}|\tau| \, \|u\|_{L^2(\mathbb{T}^n)} \leq \|H(\theta_\tau)u\|_{L^2(\mathbb{T}^n)}$$

for $u \in \mathcal{D}(H(\theta_\tau))$, and therefore $\operatorname{Ker}(H(\theta_\tau)) = \{0\}$.

In the case when $q \in L^{n/2}_{\text{loc}}(\mathbb{R}^n)$, we shall show that there exists a constant $C > 0$ such that for $\tau \in \mathbb{R}$ with $|\tau|$ sufficiently large,

$$\|u\|_{L^{2n/(n-2)}(\mathbb{T}^n)} \leq C \|H(\theta_\tau)u\|_{L^{2n/(n+2)}(\mathbb{T}^n)} \tag{1.3}$$

when $u \in \mathcal{D}(H(\theta_\tau))$, and thus $\operatorname{Ker}(H(\theta_\tau)) = \{0\}$.

When establishing (1.3), the crucial ingredients are spectral cluster estimates for the non-negative elliptic self-adjoint operator

$$(D_{x_1} + 1/2)^2 - \Delta_{g_0(x')}$$

acting on $L^2(\mathbb{T}^n)$ (see [28], [23], [29]), and uniform resolvent estimates for it, obtained recently in [5], [4], [18].

Let us point out that the idea of using the spectral cluster estimates of [28], and uniform L^p resolvent estimates for constant coefficient elliptic operators on the torus, in the study of absolute continuity of the spectrum of the Schrödinger operator goes back to [24], where the absolute continuity of the spectrum of a Schrödinger operator with a Euclidean metric and a potential in $L^{n/2}_{\mathrm{loc}}$ was established. The spectral cluster estimates of [28] were also used in [8]. In this paper we use the recently established uniform L^p resolvent estimates for elliptic self-adjoint operators with variable coefficients.

The paper is organized as follows. After explaining in Section 2 how to get rid of the conformal factor in the metric, Section 3 is devoted to the proof of Theorem 1.1 in the special case of a bounded potential. The proof in this case is quite straightforward and is presented here as a warm-up, before handling the general case in Section 4. The Appendix contains some standard material pertaining to the definition of our operators, review of Floquet theory, and a description of the Thomas approach to the absolute continuity problem. It is presented merely for the convenience of the reader.

2. Removing the conformal factor

In the case when $q \in L^{n/2}_{\mathrm{loc}}(\mathbb{R}^n)$ is real-valued, it follows from Remark A.2 in the Appendix that the singular continuous component of the spectrum of the Schrödinger operator H is empty, and the pure point spectrum is at most discrete, consisting only of isolated points without finite accumulation points. To establish the absolute continuity of H it suffices therefore to show the absence of eigenvalues. Hence, in what follows we shall concentrate on proving the absence of eigenvalues in the general non-self-adjoint case, i.e. when q is complex-valued.

We have the following conformal relation (see [6]):

$$c^{(n+2)/4}(-\Delta_g + q)(c^{-(n-2)/4}u) = (-\Delta_{c^{-1}g} + q_c)u$$

for $u \in \mathcal{D}$. Here

$$q_c = cq + c^{(n+2)/4}(-\Delta_g)(c^{-(n-2)/4}) \in L^{n/2}_{\mathrm{loc}}(\mathbb{R}^n)$$

is $2\pi\mathbb{Z}^n$-periodic, and

$$\begin{aligned}\mathcal{D} := \mathcal{D}(-\Delta_{c^{-1}g} + q_c) &= \{u \in H^1(\mathbb{R}^n) : c^{(n+2)/4}(-\Delta_g + q)c^{-(n-2)/4}u \in L^2(\mathbb{R}^n)\}\\ &= \{u \in H^1(\mathbb{R}^n) : (g^{jk}D_{x_j}D_{x_k} + q)u \in L^2(\mathbb{R}^n)\}.\end{aligned}$$

The last equality follows since $c \in C^\infty(\mathbb{R}^n)$ is a strictly positive periodic function.

Assume that $\lambda \in \mathbb{C}$ is an eigenvalue of $-\Delta_g + q$. Then for some $u \in \mathcal{D}$ not vanishing identically, we have

$$0 = c^{(n+2)/4}(-\Delta_g + q - \lambda)u = (-\Delta_{c^{-1}g} + q_c - c\lambda)(c^{(n-2)/4}u), \tag{2.1}$$

and therefore 0 is an eigenvalue of $-\Delta_{c^{-1}g} + q_c - c\lambda$.

To establish the absence of eigenvalues of the operator H it is thus sufficient to prove that zero is not an eigenvalue of the operator $-\Delta_{c^{-1}g} + q$ with an arbitrary periodic $q \in L^{n/2}_{\mathrm{loc}}(\mathbb{R}^n)$. In what follows we shall therefore assume that the metric g is of the form (1.2) with $c = 1$.

It follows from Proposition A.1 that to prove that zero is not an eigenvalue of $-\Delta_g + q$ it suffices to show that there exists $\theta \in \mathbb{C}^n$ such that the operator $H(\theta)$ defined in (A.5) is injective.

Let $\tau \in \mathbb{R}$, and set

$$\theta_\tau = (1/2 + i\tau, 0, \dots, 0) \in \mathbb{C}^n,$$
$$H_0(\theta_\tau) = (D_{x_1} + 1/2)^2 + 2i\tau(D_{x_1} + 1/2) - \tau^2 - \Delta_{g_0(x')},$$

so that

$$H(\theta_\tau) = H_0(\theta_\tau) + q.$$

Theorem 1.1 will follow once we prove that the operator $H(\theta_\tau)$ is injective for $\tau \in \mathbb{R}$ with $|\tau|$ sufficiently large.

3. Proof of Theorem 1.1 for q bounded

Proposition 3.1. *For all $\tau \in \mathbb{R}$ with $|\tau| \geq 1$,*

$$|\tau|\,\|u\|_{L^2(\mathbb{T}^n)} \leq \|H_0(\theta_\tau)u\|_{L^2(\mathbb{T}^n)} \tag{3.1}$$

for $u \in H^2(\mathbb{T}^n)$.

Proof. By a density argument it suffices to prove the estimate (3.1) for $u \in C^\infty(\mathbb{T}^n)$. Expanding u in the Fourier series with respect to x_1, we have

$$u(x_1, x') = \sum_{j \in \mathbb{Z}} e^{ijx_1} u_j(x'), \quad \text{where} \quad u_j(x') = \frac{1}{2\pi}\int_0^{2\pi} u(y_1, x') e^{-ijy_1}\, dy_1,$$

and therefore

$$\begin{aligned} H_0(\theta_\tau)u &= \big((D_{x_1} + 1/2)^2 + 2i\tau(D_{x_1} + 1/2) - \tau^2 - \Delta_{g_0(x')}\big)u \\ &= \sum_{j \in \mathbb{Z}} \big((j + 1/2)^2 + 2i\tau(j + 1/2) - \tau^2 - \Delta_{g_0(x')}\big) e^{ijx_1} u_j(x'). \end{aligned}$$

Since the operator $-\Delta_{g_0(x')}$ acting on $L^2(\mathbb{T}^{n-1})$ is self-adjoint, we have

$$\|(-\Delta_{g_0(x')} - z)^{-1}\|_{L^2(\mathbb{T}^{n-1}) \to L^2(\mathbb{T}^{n-1})} \leq 1/|\mathrm{Im}\, z|, \qquad \mathrm{Im}\, z \neq 0,$$

and hence, as $|j + 1/2| \geq 1/2$, $j \in \mathbb{Z}$, for $|\tau| \geq 1$ we get

$$\big\|\big((j + 1/2)^2 + 2i\tau(j + 1/2) - \tau^2 - \Delta_{g_0(x')}\big)^{-1}\big\|_{L^2(\mathbb{T}^{n-1}) \to L^2(\mathbb{T}^{n-1})} \leq 1/|\tau|.$$

By Parseval's identity, we obtain

$$\begin{aligned} \frac{1}{2\pi}\|H_0(\theta_\tau)u\|^2_{L^2(\mathbb{T}^n)} &= \sum_{j \in \mathbb{Z}} \big\|\big((j+1/2)^2 + 2i\tau(j+1/2) - \tau^2 - \Delta_{g_0(x')}\big)u_j(x')\big\|^2_{L^2(\mathbb{T}^{n-1})} \\ &\geq \frac{1}{2\pi}|\tau|^2 \|u\|^2_{L^2(\mathbb{T}^n)}. \end{aligned}$$

□

Let $q \in L^\infty(\mathbb{T}^n)$, so that $\mathcal{D}(H(\theta_\tau)) = H^2(\mathbb{T}^n)$. Thus, by Proposition 3.1 we conclude that for $|\tau| \geq 1$ sufficiently large,

$$\frac{1}{2}|\tau|\, \|u\|_{L^2(\mathbb{T}^n)} \leq \|(H_0(\theta_\tau)+q)u\|_{L^2(\mathbb{T}^n)}$$

for $u \in \mathcal{D}(H(\theta_\tau))$. The proof of Theorem 1.1 in the case $q \in L^\infty$ is therefore complete.

4. Proof of Theorem 1.1 for $q \in L^{n/2}(\mathbb{T}^n)$

Let us start by recalling the following chain of continuous inclusions, where the first and the last ones follow from the Sobolev embedding theorem:

$$H^1(\mathbb{T}^n) \hookrightarrow L^{2n/(n-2)}(\mathbb{T}^n) \hookrightarrow L^2(\mathbb{T}^n) \hookrightarrow L^{2n/(n+2)}(\mathbb{T}^n) \hookrightarrow H^{-1}(\mathbb{T}^n).$$

We shall need the following result.

Lemma 4.1. *Let $q \in L^{n/2}(\mathbb{T}^n)$. Then*

$$\mathcal{D}(H(\theta_\tau)) \subset W^{2,2n/(n+2)}(\mathbb{T}^n).$$

Proof. Let $u \in \mathcal{D}(H(\theta_\tau))$. Then $f := H(\theta_\tau)u \in L^2(\mathbb{T}^n)$. Using the fact that $\mathcal{D}(H(\theta_\tau)) \subset H^1(\mathbb{T}^n)$, Sobolev's embedding $H^1(\mathbb{T}^n) \hookrightarrow L^{2n/(n-2)}(\mathbb{T}^n)$, and Hölder's inequality

$$\|qu\|_{L^{2n/(n+2)}(\mathbb{T}^n)} \leq \|q\|_{L^{n/2}(\mathbb{T}^n)}\|u\|_{L^{2n/(n-2)}(\mathbb{T}^n)},$$

we get $qu \in L^{2n/(n+2)}(\mathbb{T}^n)$. Hence,

$$H_0(\theta_\tau)u = f - qu \in L^{2n/(n+2)}(\mathbb{T}^n).$$

As $u \in L^{2n/(n+2)}(\mathbb{T}^n)$ and the operator $H_0(\theta_\tau)$ is elliptic with smooth coefficients, by elliptic regularity we conclude that $u \in W^{2,2n/(n+2)}(\mathbb{T}^n)$. □

Proposition 4.2. *There exists a constant $C > 0$ such that for all $\tau \in \mathbb{R}$ with $|\tau|$ sufficiently large,*

$$\|u\|_{L^2(\mathbb{T}^n)} \leq \frac{C}{|\tau|^{1/2}}\|H_0(\theta_\tau)u\|_{L^{2n/(n+2)}(\mathbb{T}^n)}, \tag{4.1}$$

$$\|u\|_{L^{2n/(n-2)}(\mathbb{T}^n)} \leq C\|H_0(\theta_\tau)u\|_{L^{2n/(n+2)}(\mathbb{T}^n)}, \tag{4.2}$$

for $u \in W^{2,2n/(n+2)}(\mathbb{T}^n)$.

Proof. Here we use the notation and some ideas of [5]. We denote by $0 = \lambda_0 < \lambda_1 \leq \cdots$ the sequence of eigenvalues of $-\Delta_{g_0(x')}$ on $\mathbb{T}^{n-1}$, counted with their multiplicities, and by $(\psi_k)_{k\geq 0}$ the corresponding sequence of eigenfunctions forming an orthonormal basis of $L^2(\mathbb{T}^{n-1})$,

$$-\Delta_{g_0(x')}\psi_k = \lambda_k\psi_k.$$

The operator

$$H_0(\theta_0) = (D_{x_1} + 1/2)^2 - \Delta_{g_0(x')}$$

with domain $H^2(\mathbb{T}^n)$ is non-negative elliptic self-adjoint on $L^2(\mathbb{T}^n; |g|^{1/2}dx)$, with eigenvalues $(j + 1/2)^2 + \lambda_k$, $j \in \mathbb{Z}$, $k \in \mathbb{N}$, and the corresponding eigenfunctions $\widetilde{\psi}_{j,k}(x) = e^{ijx_1}\psi_k(x')$, i.e.

$$H_0(\theta_0)\widetilde{\psi}_{j,k} = ((j + 1/2)^2 + \lambda_k)\widetilde{\psi}_{j,k}.$$

We denote by $\pi_{j,k} : L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$ the orthogonal projection on the linear space spanned by the eigenfunction $\widetilde{\psi}_{j,k}$,

$$\pi_{j,k} f(x) = \frac{1}{2\pi}\left(\int_{\mathbb{T}^n} f(y)e^{-ijy_1}\overline{\psi_k(y')}\sqrt{|g_0|}\right)e^{ijx_1}\psi_k(x').$$

We have

$$\sum_{j\in\mathbb{Z},\, k\in\mathbb{N}} \pi_{j,k} = I.$$

Let us denote by χ_m the spectral projection operator on the space generated by the eigenfunctions corresponding to the mth spectral cluster of the operator $H_0(\theta_0)$,

$$\chi_m = \sum_{m\le\sqrt{(j+1/2)^2+\lambda_k}<m+1} \pi_{j,k}, \qquad m \in \mathbb{N}.$$

To establish the estimates (4.1) and (4.2) we shall need the spectral cluster estimates obtained in [28], [23] (see also [29]),

$$\|\chi_m f\|_{L^2(\mathbb{T}^n)} \le C(1+m)^{1/2}\|f\|_{L^{2n/(n+2)}(\mathbb{T}^n)}, \tag{4.3}$$

and the dual estimates

$$\|\chi_m f\|_{L^{2n/(n-2)}(\mathbb{T}^n)} \le C(1+m)^{1/2}\|f\|_{L^2(\mathbb{T}^n)}, \tag{4.4}$$

for $f \in C^\infty(\mathbb{T}^n)$.

By a density argument it suffices to establish (4.1) and (4.2) for $u \in C^\infty(\mathbb{T}^n)$. Consider the equation

$$H_0(\theta_\tau)u = f$$

with $u, f \in C^\infty(\mathbb{T}^n)$. Writing $u = \sum_{j,k} \pi_{j,k}u$ and $f = \sum_{j,k}\pi_{j,k}f$, we get

$$\big((j + 1/2)^2 + 2i\tau(j + 1/2) - \tau^2 + \lambda_k\big)\pi_{j,k}u = \pi_{j,k}f$$

for all $j \in \mathbb{Z}$ and $k \in \mathbb{N}$. As $|j + 1/2| \ge 1/2$ for $j \in \mathbb{Z}$, we have

$$|(j + 1/2)^2 + 2i\tau(j + 1/2) - \tau^2 + \lambda_k| \ge 2|\tau|\,|j + 1/2| \ge |\tau|,$$

and therefore, when $|\tau| \ge 1$,

$$u = G_\tau f := \sum_{j=-\infty}^{\infty}\sum_{k=0}^{\infty} \frac{\pi_{j,k} f}{(j + 1/2)^2 + 2i\tau(j + 1/2) - \tau^2 + \lambda_k}.$$

We now prove that there exists a constant $C > 0$ such that for $\tau \in \mathbb{R}$ with $|\tau| \geq 1$,

$$\|G_\tau f\|_{L^2(\mathbb{T}^n)} \leq \frac{C}{|\tau|^{1/2}} \|f\|_{L^{2n/(n+2)}(\mathbb{T}^n)} \tag{4.5}$$

for $f \in C^\infty(\mathbb{T}^n)$. Using (4.3), we get

$$\begin{aligned}
\|G_\tau f\|^2_{L^2(\mathbb{T}^n)} &= \sum_{j=-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{\|\pi_{j,k} f\|^2_{L^2(\mathbb{T}^n)}}{|(j+1/2)^2 + 2i\tau(j+1/2) - \tau^2 + \lambda_k|^2} \\
&\leq \sum_{m=0}^{\infty} \sup_{m \leq \sqrt{(j+1/2)^2+\lambda_k} < m+1} \frac{1}{|(j+1/2)^2 + 2i\tau(j+1/2) - \tau^2 + \lambda_k|^2} \|\chi_m f\|^2_{L^2(\mathbb{T}^n)} \\
&\leq CS \|f\|^2_{L^{2n/(n+2)}(\mathbb{T}^n)},
\end{aligned} \tag{4.6}$$

where

$$S := \sum_{m=0}^{\infty} (1+m) \sup_{m \leq \sqrt{(j+1/2)^2+\lambda_k} < m+1} \frac{1}{|(j+1/2)^2 + 2i\tau(j+1/2) - \tau^2 + \lambda_k|^2}. \tag{4.7}$$

Let us show that the series S converges and behaves as $1/|\tau|$ for $|\tau|$ large. To that end we observe that

$$2|(j+1/2)^2 + 2i\tau(j+1/2) - \tau^2 + \lambda_k| \geq |(j+1/2)^2 + \lambda_k - \tau^2| + |\tau|. \tag{4.8}$$

Assume now that $m \leq \sqrt{(j+1/2)^2 + \lambda_k} < m+1$ and let $m \leq |\tau|$. Then using the fact that $|\tau| \geq 1$, we obtain

$$\begin{aligned}
|m^2 - \tau^2| &\leq |(j+1/2)^2 + \lambda_k - \tau^2| + |m^2 - (j+1/2)^2 - \lambda_k| \\
&\leq |(j+1/2)^2 + \lambda_k - \tau^2| + 2m + 1 \leq |(j+1/2)^2 + \lambda_k - \tau^2| + 3|\tau|,
\end{aligned} \tag{4.9}$$

and therefore

$$4(|(j+1/2)^2 + \lambda_k - \tau^2| + |\tau|) \geq |m^2 - \tau^2| + |\tau|. \tag{4.10}$$

When $m > |\tau|$, we have

$$|(j+1/2)^2 + \lambda_k - \tau^2| \geq |m^2 - \tau^2|. \tag{4.11}$$

Now we are ready to estimate S given by (4.7). Using (4.8), (4.10) and (4.11), we get

$$\begin{aligned}
S &\lesssim \sum_{m=0}^{\infty} \frac{1+m}{(m^2-\tau^2)^2 + \tau^2} \lesssim \frac{1}{|\tau|^4} + \sum_{m=1}^{\infty} \frac{m}{(m^2-\tau^2)^2+\tau^2} \\
&\lesssim \frac{1}{|\tau|} + \int_0^\infty \frac{t\,dt}{(t^2-\tau^2)^2 + \tau^2} \lesssim \frac{1}{|\tau|} \int_{-\infty}^{\infty} \frac{ds}{s^2+1} \lesssim \frac{1}{|\tau|}.
\end{aligned} \tag{4.12}$$

Here we have used the fact that the function $t \mapsto t/((t^2-\tau^2)^2 + \tau^2)$ is increasing when $t \in [0, |\tau|/\sqrt{3})$ and decreasing when $t \in (|\tau|/\sqrt{3}, \infty)$, and we have performed the change of variables $s = t^2/|\tau| - |\tau|$ in the integral. Combining (4.6) and (4.12), we obtain (4.5), and therefore (4.1).

Let us now prove the estimate (4.2) for $u \in C^\infty(\mathbb{T}^n)$, which amounts to obtaining the uniform estimate

$$\|G_\tau f\|_{L^{2n/(n-2)}(\mathbb{T}^n)} \le C\|f\|_{L^{2n/(n+2)}(\mathbb{T}^n)} \tag{4.13}$$

for $f \in C^\infty(\mathbb{T}^n)$ and $\tau \in \mathbb{R}$ with $|\tau|$ sufficiently large.

To that end we shall need the following uniform resolvent estimate for the elliptic self-adjoint operator $H_0(\theta_0)$: for each $\delta \in (0, 1)$, there exists a constant $C > 0$ such that for all $u \in C^\infty(\mathbb{T}^n)$ and all $z \in \mathbb{C}$ with $\operatorname{Im} z \ge \delta$,

$$\|u\|_{L^{2n/(n-2)}(\mathbb{T}^n)} \le C\|(H_0(\theta_0) - z^2)u\|_{L^{2n/(n+2)}(\mathbb{T}^n)} \tag{4.14}$$

(see [5], [4] and [18]). When establishing (4.13), we shall follow [5] closely, and use a localization argument to deduce this estimate from (4.14) (see also [17] and [24]). We have

$$f(x_1, x') = \sum_{j\in\mathbb{Z}} e^{ijx_1} f_j(x'), \qquad f_j(x') = \frac{1}{2\pi}\int_0^{2\pi} f(y_1, x')e^{-ijy_1}\,dy_1.$$

Letting χ be the characteristic function of the interval $[1/2, 1)$, and further localizing f in frequency with respect to the variable x_1, we introduce

$$\widetilde{f_\nu}(x_1, x') = \chi(|D_{x_1}|/2^\nu) f(x_1, x') = \sum_{2^{\nu-1}\le|j|<2^\nu} e^{ijx_1} f_j(x'), \qquad \nu = 1, 2, \ldots,$$
$$\widetilde{f_0}(x_1, x') = f_0(x'),$$

so that $f = \sum_{\nu=0}^\infty \widetilde{f_\nu}$. Since the operators $\chi(|D_{x_1}|/2^\nu)$ and $H_0(\theta_\tau)$ commute, the localization of $G_\tau f$ in frequency with respect to the variable x_1 is given by

$$(\widetilde{G_\tau f})_\nu = G_\tau \widetilde{f_\nu}, \qquad \nu = 0, 1, 2, \ldots.$$

Standard arguments based on the one-dimensional Littlewood–Paley theory [7, Theorem 8.4] imply that in order to prove (4.13) it suffices to establish the uniform estimates

$$\|G_\tau \widetilde{f_\nu}\|_{L^{2n/(n-2)}(\mathbb{T}^n)} \le C\|\widetilde{f_\nu}\|_{L^{2n/(n+2)}(\mathbb{T}^n)}, \qquad \nu = 0, 1, 2, \ldots \tag{4.15}$$

(see also [18, Lemma 2.3]).

When proving (4.15), we introduce the resolvent of $H_0(\theta_0)$, given by

$$R(\zeta) := (H_0(\theta_0) - \zeta)^{-1} = \sum_{j=-\infty}^{\infty}\sum_{k=0}^{\infty} \frac{\pi_{j,k}}{(j + 1/2)^2 + \lambda_k - \zeta}, \qquad \zeta \notin \operatorname{Spec}(H_0(\theta_0)).$$

For future reference observe that for $\zeta = \tau^2 - i\rho\tau$ with $\rho \ge 1$, we have

$$\operatorname{Im}\sqrt{\zeta} = \sqrt{\frac{|\zeta| - \operatorname{Re}\zeta}{2}} = \sqrt{\frac{|\tau|\sqrt{\tau^2+\rho^2} - \tau^2}{2}} = \frac{\rho}{2} + \mathcal{O}\left(\frac{1}{\tau^2}\right) \ge \frac{1}{4}, \tag{4.16}$$

provided that $|\tau|$ is large.

In the case when $\nu = 0$, we have

$$G_\tau \widetilde{f_0} = R(\tau^2 - i\tau)\widetilde{f_0},$$

and thus (4.15) becomes the uniform resolvent estimate (4.14) with $z^2 = \tau^2 - i\tau$.

Let us show that in the case $\nu \ge 1$, (4.15) follows from the resolvent estimate

$$\|R(\tau^2 - i(2^\nu + 1)\tau)\widetilde{f_\nu}\|_{L^{2n/(n-2)}(\mathbb{T}^n)} \lesssim \|\widetilde{f_\nu}\|_{L^{2n/(n+2)}(\mathbb{T}^n)}, \tag{4.17}$$

where the implicit constant is independent of τ and ν. To that end, we write

$$(R(\tau^2 - i(2^\nu + 1)\tau) - G_\tau)\widetilde{f_\nu} = \sum_{j=-\infty}^{\infty}\sum_{k=0}^{\infty} a_{j,k,\nu}(\tau)\pi_{j,k}\widetilde{f_\nu},$$

where

$$a_{j,k,\nu}(\tau) = \frac{i\tau(2j - 2^\nu)1_{[2^{\nu-1},2^\nu)}(|j|)}{((j+1/2)^2 + 2i(j+1/2)\tau - \tau^2 + \lambda_k)((j+1/2)^2 + i(2^\nu+1)\tau - \tau^2 + \lambda_k)}.$$

Using the fact that $\sum_{m=0}^\infty \chi_m^2 = 1$, and spectral cluster estimates (4.3) and (4.4), we get

$$\begin{aligned}
&\|(R(\tau^2 - i(2^\nu + 1)\tau) - G_\tau)\widetilde{f_\nu}\|_{L^{2n/(n-2)}(\mathbb{T}^n)} \\
&\quad\lesssim \sum_{m=0}^{\infty}(1+m)^{1/2}\|\chi_m(R(\tau^2 - i(2^\nu+1)\tau) - G_\tau)\widetilde{f_\nu}\|_{L^2(\mathbb{T}^n)} \\
&\quad\lesssim \Big(\sum_{m=0}^{\infty}(1+m)\sup_{m\le\sqrt{(j+1/2)^2+\lambda_k}<m+1}|a_{j,k,\nu}(\tau)|\Big)\|\widetilde{f_\nu}\|_{L^{2n/(n+2)}(\mathbb{T}^n)}.
\end{aligned} \tag{4.18}$$

To see that the above series converges and is bounded uniformly with respect to τ with $|\tau| \ge 1$ and ν, we first observe using (4.9) and (4.11) that

$$\begin{aligned}
&\sup_{m\le\sqrt{(j+1/2)^2+\lambda_k}<m+1}|a_{j,k,\nu}(\tau)| \\
&\lesssim \sup_{m\le\sqrt{(j+1/2)^2+\lambda_k}<m+1}\frac{2^\nu|\tau|}{(|(j+1/2)^2+\lambda_k-\tau^2|+2^{\nu-1}|\tau|)^2} \lesssim \frac{2^\nu|\tau|}{(m^2-\tau^2)^2+4^{\nu-1}\tau^2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{m=0}^{\infty}(1+m)\sup_{m\le\sqrt{(j+1/2)^2+\lambda_k}<m+1}|a_{j,k,\nu}(\tau)| &\lesssim 2^\nu|\tau|\sum_{m=0}^{\infty}\frac{1+m}{(m^2-\tau^2)^2+4^{\nu-1}\tau^2} \\
&\lesssim \frac{1}{|\tau|} + 2^\nu|\tau|\sum_{m=1}^{\infty}\frac{m}{(m^2-\tau^2)^2+4^{\nu-1}\tau^2} \\
&\lesssim 1 + \int_0^\infty \frac{2^\nu|\tau|t}{(t^2-\tau^2)^2+4^{\nu-1}\tau^2}\,dt \lesssim \int_{-\infty}^{\infty}\frac{ds}{s^2+1} < \infty.
\end{aligned} \tag{4.19}$$

Here we have performed the change of variables $s = (t^2/|\tau| - |\tau|)/2^{\nu-1}$. Thus, it follows from (4.18) and (4.19) that

$$\|(R(\tau^2 - i(2^\nu + 1)\tau) - G_\tau)\widetilde{f}_\nu\|_{L^{2n/(n-2)}(\mathbb{T}^n)} \lesssim \|\widetilde{f}_\nu\|_{L^{2n/(n+2)}(\mathbb{T}^n)},$$

with a constant uniform in τ and ν. The estimate (4.15) is therefore a consequence of the uniform resolvent estimate (4.17), in view of (4.16). The proof of Proposition 4.2 is complete. □

It is now easy to finish the proof of Theorem 1.1. Let $q \in L^{n/2}(\mathbb{T}^n)$ and let us check that there exists a constant $C > 0$ such that for $\tau \in \mathbb{R}$ with $|\tau|$ sufficiently large,

$$\|u\|_{L^{2n/(n-2)}(\mathbb{T}^n)} \le C\|(H_0(\theta_\tau) + q)u\|_{L^{2n/(n+2)}(\mathbb{T}^n)} \tag{4.20}$$

for $u \in \mathcal{D}(H(\theta_\tau))$. Let $u \in \mathcal{D}(H(\theta_\tau))$. Then by Lemma 4.1, we know that $u \in W^{2,2n/(n+2)}(\mathbb{T}^n)$. Denoting by C_0 the uniform constant in the estimate (4.2), we write $q = q^\sharp + (q - q^\sharp)$ where $q^\sharp \in L^\infty(\mathbb{T}^n)$ and

$$\|q - q^\sharp\|_{L^{n/2}(\mathbb{T}^n)} \le \frac{1}{4C_0}. \tag{4.21}$$

By the embedding $L^2(\mathbb{T}^n) \hookrightarrow L^{2n/(n+2)}(\mathbb{T}^n)$ and the estimate (4.1), for $\tau \in \mathbb{R}$ with $|\tau| \ge 1$ sufficiently large we have

$$\|q^\sharp u\|_{L^{2n/(n+2)}(\mathbb{T}^n)} \le C\|q^\sharp\|_{L^\infty(\mathbb{T}^n)}\|u\|_{L^2(\mathbb{T}^n)} \le \tfrac{1}{2}\|H_0(\theta_\tau)u\|_{L^{2n/(n+2)}(\mathbb{T}^n)}. \tag{4.22}$$

Using Hölder's inequality and estimates (4.2), (4.21), and (4.22), for $\tau \in \mathbb{R}$ with $|\tau| \ge 1$ sufficiently large we have

$$\begin{aligned}
&\|(H_0(\theta_\tau) + q)u\|_{L^{2n/(n+2)}(\mathbb{T}^n)} \\
&\quad \ge \|H_0(\theta_\tau)u\|_{L^{2n/(n+2)}(\mathbb{T}^n)} - \|q^\sharp u\|_{L^{2n/(n+2)}(\mathbb{T}^n)} - \|(q - q^\sharp)u\|_{L^{2n/(n+2)}(\mathbb{T}^n)} \\
&\quad \ge \tfrac{1}{2}\|H_0(\theta_\tau)u\|_{L^{2n/(n+2)}(\mathbb{T}^n)} - \|q - q^\sharp\|_{L^{n/2}(\mathbb{T}^n)}\|u\|_{L^{2n/(n-2)}(\mathbb{T}^n)} \ge \frac{1}{4C_0}\|u\|_{L^{2n/(n-2)}(\mathbb{T}^n)},
\end{aligned}$$

which proves the estimate (4.20). This completes the proof of Theorem 1.1.

Appendix. Definition of operators using sesquilinear forms, Floquet theory and the Thomas approach

The material of this appendix is standard and is presented here for completeness and convenience of the reader (see [3], [22], [24], [26], [19], [20], and [31]).

A.1. Definition of the Schrödinger operator acting on $L^2(\mathbb{R}^n)$

Let us start by reviewing the definition of the Schrödinger operator $H = -\Delta_g + q$ on $\mathbb{R}^n$, $n \ge 3$, with a potential $q \in L^{n/2}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{C})$ and with a smooth Riemannian metric g, as a closed densely defined sectorial operator on $L^2(\mathbb{R}^n)$. We assume that g and q are periodic with respect to the lattice $2\pi\mathbb{Z}^n$.

In what follows all norms are defined using the Riemannian volume element $|g|^{1/2}\,dx$. Consider the sesquilinear form

$$h[u,v]=\int_{\mathbb{R}^n} g^{jk}D_{x_k}u\overline{D_{x_j}v}|g|^{1/2}\,dx+\int_{\mathbb{R}^n} qu\overline{v}|g|^{1/2}\,dx$$

for $u,v\in C_0^\infty(\mathbb{R}^n)$.

Let $(0,2\pi)^n$ be the interior of the fundamental domain of the lattice $2\pi\mathbb{Z}^n$. By Hölder's inequality and the Sobolev embedding $H^1((0,2\pi)^n)\subset L^{2n/(n-2)}((0,2\pi)^n)$, we get

$$\begin{aligned}\left|\int_{(0,2\pi)^n} qu\overline{v}|g|^{1/2}\,dx\right| &\le \|q\|_{L^{n/2}((0,2\pi)^n)}\|u\|_{L^{2n/(n-2)}((0,2\pi)^n)}\|v\|_{L^{2n/(n-2)}((0,2\pi)^n)}\\ &\le C\|u\|_{H^1((0,2\pi)^n)}\|v\|_{H^1((0,2\pi)^n)}.\end{aligned}$$

Replacing $(0,2\pi)^n$ by its translates $E_k=(0,2\pi)^n+k$, $k\in 2\pi\mathbb{Z}^n$, summing over all k, and using the Cauchy–Schwarz inequality, we get

$$\left|\int_{\mathbb{R}^n} qu\overline{v}|g|^{1/2}\,dx\right|\le C\sum_{k\in 2\pi\mathbb{Z}^n}\|u\|_{H^1(E_k)}\|v\|_{H^1(E_k)}\le C\|u\|_{H^1(\mathbb{R}^n)}\|v\|_{H^1(\mathbb{R}^n)}.$$

It follows that

$$|h[u,v]|\le C\|u\|_{H^1(\mathbb{R}^n)}\|v\|_{H^1(\mathbb{R}^n)} \tag{A.1}$$

for $u,v\in C_0^\infty(\mathbb{R}^n)$. Hence, the form h extends to a bounded sesquilinear form on $H^1(\mathbb{R}^n)$.

We shall next check the coercivity of h on $H^1(\mathbb{R}^n)$, i.e.,

$$\operatorname{Re} h[u,u]\ge c_0\|u\|^2_{H^1(\mathbb{R}^n)}-C_1\|u\|^2_{L^2(\mathbb{R}^n)},\qquad c_0>0,\ C_1\in\mathbb{R}. \tag{A.2}$$

Writing

$$q=q^\sharp+(q-q^\sharp), \tag{A.3}$$

where $q^\sharp\in L^\infty((0,2\pi)^n)$ and $\|q-q^\sharp\|_{L^{n/2}((0,2\pi)^n)}\le\varepsilon$ for some $\varepsilon>0$ small, we have

$$\begin{aligned}\int_{(0,2\pi)^n}|q|\,|u|^2|g|^{1/2}\,dx &\le \|q^\sharp\|_{L^\infty((0,2\pi)^n)}\|u\|^2_{L^2((0,2\pi)^n)}\\ &\quad+\|q-q^\sharp\|_{L^{n/2}((0,2\pi)^n)}\|u\|^2_{L^{2n/(n-2)}((0,2\pi)^n)}\\ &\le \mathcal{O}_\varepsilon(1)\|u\|^2_{L^2((0,2\pi)^n)}+\mathcal{O}(\varepsilon)\|u\|^2_{H^1((0,2\pi)^n)}.\end{aligned}$$

It follows that

$$\int_{\mathbb{R}^n}|q|\,|u|^2|g|^{1/2}\,dx\le\mathcal{O}_\varepsilon(1)\|u\|^2_{L^2(\mathbb{R}^n)}+\mathcal{O}(\varepsilon)\|u\|^2_{H^1(\mathbb{R}^n)}. \tag{A.4}$$

As g positive definite, using (A.4) and choosing ε sufficiently small, we get (A.2).

When equipped with the domain $\mathcal{D}(h) = H^1(\mathbb{R}^n)$, the form h is densely defined, closed and sectorial with the bound

$$|\mathrm{Im}\, h[u,u]| \le |h[u,u]| \le C c_0^{-1}(\mathrm{Re}\, h[u,u] + C_1 \|u\|^2_{L^2(\mathbb{R}^n)}).$$

Here we have used (A.1) and (A.2). By [14, Corollary 12.19], there exists a closed densely defined sectorial operator H on $L^2(\mathbb{R}^n)$, which we denote by $H = -\Delta_g + q$, with domain

$$\mathcal{D}(H) = \{u \in H^1(\mathbb{R}^n) : (-\Delta_g + q)u \in L^2(\mathbb{R}^n)\}$$

such that

$$h[u,v] = (Hu, v)_{L^2(\mathbb{R}^n)}, \quad u \in \mathcal{D}(H),\ v \in \mathcal{D}(h).$$

A.2. *Definition of a family of operators acting on $L^2(\mathbb{T}^n)$*

Let $\theta \in \mathbb{C}^n$. We shall next define a family of operators $H(\theta)$ acting on $L^2(\mathbb{T}^n)$, where $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$, formally given by

$$H(\theta) = e^{-ix\cdot\theta} H e^{ix\cdot\theta} = |g|^{-1/2}(D_{x_j} + \theta_j)(|g|^{1/2} g^{jk}(D_{x_k} + \theta_k)) + q. \tag{A.5}$$

Let $u, v \in C^\infty(\mathbb{T}^n)$ and consider the family of sesquilinear forms

$$h(\theta)[u,v] = \int_{\mathbb{T}^n} g^{jk} D_{x_k} u \overline{D_{x_j} v} |g|^{1/2}\, dx + w[\theta][u,v], \tag{A.6}$$

where

$$w[\theta][u,v] = \int_{\mathbb{T}^n} g^{jk}\big(\theta_k u \overline{D_{x_j} v} + (D_{x_k} u)\theta_j \overline{v} + \theta_k\theta_j u\overline{v}\big)|g|^{1/2}\, dx + \int_{\mathbb{T}^n} q u \overline{v} |g|^{1/2}\, dx.$$

By Hölder's inequality and the Sobolev embedding $H^1(\mathbb{T}^n) \subset L^{2n/(n-2)}(\mathbb{T}^n)$, we see that

$$|h(\theta)[u,v]| \le C\|u\|_{H^1(\mathbb{T}^n)} \|v\|_{H^1(\mathbb{T}^n)} \tag{A.7}$$

with $C = C(\theta) > 0$, and hence, for each $\theta \in \mathbb{C}^n$, the form $h(\theta)$ extends to a bounded sesquilinear form on $H^1(\mathbb{T}^n)$.

Using the Peter–Paul inequality, the decomposition (A.3), and the Sobolev embedding, we get

$$|w(\theta)[u,u]| \le \mathcal{O}(\varepsilon)\|u\|^2_{H^1(\mathbb{T}^n)} + \mathcal{O}_{\theta,\varepsilon}(1)\|u\|^2_{L^2(\mathbb{T}^n)}$$

for $\varepsilon > 0$. Hence, using the fact that g is positive definite, and choosing $\varepsilon > 0$ sufficiently small in the previous estimate, we obtain

$$\mathrm{Re}\, h[\theta][u,u] \ge c_0\|u\|^2_{H^1(\mathbb{T}^n)} - C_1\|u\|^2_{L^2(\mathbb{T}^n)}, \quad c_0 > 0,\ C_1 = C_1(\theta) \in \mathbb{R}. \tag{A.8}$$

When equipped with the domain $\mathcal{D}(h(\theta)) = H^1(\mathbb{T}^n)$, the form $h(\theta)$ is densely defined, closed and sectorial with the bound

$$|\mathrm{Im}\, h(\theta)[u,u]| \le |h(\theta)[u,u]| \le C c_0^{-1}(\mathrm{Re}\, h(\theta)[u,u] + C_1\|u\|^2_{L^2(\mathbb{T}^n)}). \tag{A.9}$$

Here we have used (A.7) and (A.8), and we may also notice that the constants C and C_1 are uniform in θ on compact subsets of $\mathbb{C}^n$.

By [14, Corollary 12.19], there exists a closed densely defined sectorial operator $H(\theta)$ on $L^2(\mathbb{T}^n)$, which we write as in (A.5), with domain

$$\mathcal{D}(H(\theta)) = \{u \in H^1(\mathbb{T}^n) : H(\theta)u \in L^2(\mathbb{T}^n)\}, \tag{A.10}$$

such that

$$h(\theta)[u, v] = (H(\theta)u, v)_{L^2(\mathbb{T}^n)}, \qquad u \in \mathcal{D}(H(\theta)),\ v \in \mathcal{D}(h(\theta)).$$

In view of (A.10) and (A.5), we have

$$\mathcal{D}(H(\theta)) = \{u \in H^1(\mathbb{T}^n) : (g^{jk} D_{x_j} D_{x_k} + q)u \in L^2(\mathbb{T}^n)\}, \tag{A.11}$$

and in particular we see that $\mathcal{D}(H(\theta))$ is independent of $\theta \in \mathbb{C}^n$.

Furthermore, by [14, Corollary 12.21] the spectrum of $H(\theta)$ is contained in the following angular set with opening $< \pi$:

$$\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq -C_1(\theta) + c_0,\ |\operatorname{Im} \lambda| \leq C(\theta) c_0^{-1} (\operatorname{Re} \lambda + C_1(\theta))\},$$

where the constants are taken from (A.8) and (A.9).

It follows that for each $\theta \in \mathbb{C}^n$, the resolvent of $H(\theta)$ is compact on $L^2(\mathbb{T}^n)$, and hence the spectrum of $H(\theta)$ is discrete, each eigenvalue having a finite algebraic multiplicity.

We conclude that the family of operators $H(\theta)$ is an entire holomorphic family of type (A) with respect to each of the complex variables $\theta_1, \dots, \theta_n$ (see [16, Section VII.2]).

A.3. The Floquet decomposition

The idea here is to pass from the operator H acting on functions on $\mathbb{R}^n$ to the family of operators $H(\theta)$ acting on functions on the torus $\mathbb{T}^n$.

To that end, for $u \in \mathcal{S}(\mathbb{R}^n)$, we define the Floquet–Bloch–Gelfand transform by

$$(Uu)(\theta, x) = e^{-ix\cdot\theta} \sum_{k \in \mathbb{Z}^n} e^{-2\pi k i \cdot \theta} u(x + 2\pi k), \qquad \theta \in \mathbb{R}^n,\ x \in [0, 2\pi]^n,$$

where the series converges in the C^∞-sense. We have

$$U : \mathcal{S}(\mathbb{R}^n) \to C^\infty_{\mathrm{Fl}}(\mathbb{R}^n_\theta \times \mathbb{T}^n_x),$$

where

$$C^\infty_{\mathrm{Fl}}(\mathbb{R}^n_\theta \times \mathbb{T}^n_x) = \{f \in C^\infty(\mathbb{R}^n_\theta \times \mathbb{T}^n_x) : f(\theta + l, x) = e^{-ix\cdot l} f(\theta, x),\ l \in \mathbb{Z}^n\}.$$

It follows that

$$\int_{(0,2\pi)^n} \int_{(0,1)^n} |(Uu)(\theta, x)|^2 \, d\theta \, |g|^{1/2} \, dx = \sum_{k \in \mathbb{Z}^n} \int_{(0,2\pi)^n} |u(x + 2\pi k)|^2 |g|^{1/2} \, dx,$$

and therefore

$$\|Uu\|_{L^2((0,1)^n_\theta \times (0,2\pi)^n_x)} = \|u\|_{L^2(\mathbb{R}^n)}.$$

Hence, U can be extended to an isometry

$$U : L^2(\mathbb{R}^n) \to L^2((0,1)^n_\theta \times \mathbb{T}^n_x).$$

Thus, $U^*U = I$, where U^* is the L^2-adjoint of U. A direct computation shows that for $v \in C^\infty_{\mathrm{Fl}}(\mathbb{R}^n_\theta \times \mathbb{T}^n_x)$, we have

$$(U^*v)(x) = \int_{(0,1)^n} e^{ix\cdot\theta} v(\theta, x)\, d\theta \in \mathcal{S}(\mathbb{R}^n). \tag{A.12}$$

To see that $UU^* = I$, we first let $v \in C^\infty_{\mathrm{Fl}}(\mathbb{R}^n_\theta \times \mathbb{T}^n_x)$. Then we have the Fourier series expansion with respect to θ,

$$v(\theta, x) = \sum_{k\in\mathbb{Z}^n} \left(\int_{(0,1)^n} v(\theta', x) e^{-i(2\pi k - x)\cdot\theta'}\, d\theta' \right) e^{i(2\pi k - x)\cdot\theta}.$$

We get

$$U(U^*v)(\theta, x) = \sum_{k\in\mathbb{Z}^n} e^{-i(x+2\pi k)\cdot\theta} \int_{(0,1)^n} e^{i(x+2\pi k)\cdot\theta'} v(\theta', x)\, d\theta' = v(\theta, x),$$

and therefore, by density, $UU^* = I$ on $L^2((0,1)^n_\theta \times \mathbb{T}^n_x)$.

Hence, the map

$$U : L^2(\mathbb{R}^n) \to L^2((0,1)^n_\theta \times \mathbb{T}^n_x) = L^2((0,1)^n_\theta; L^2(\mathbb{T}^n)) =: \int^{\oplus}_{(0,1)^n} L^2(\mathbb{T}^n)\, d\theta \tag{A.13}$$

is unitary.

We shall next show that

$$U : H^1(\mathbb{R}^n) \to L^2((0,1)^n; H^1(\mathbb{T}^n)) \tag{A.14}$$

is a linear homeomorphism. To that end, let $u \in H^1(\mathbb{R}^n)$, and let us check that $Uu \in L^2((0,1)^n; H^1(\mathbb{T}^n))$. Using the fact that for $u \in \mathcal{S}(\mathbb{R}^n)$,

$$D_{x_k}(Uu)(\theta, x) = (U(D_{x_k}u))(\theta, x) - \theta_k (Uu)(\theta, x), \quad k = 1, \ldots, n,$$

we get

$$\|D_{x_k}(Uu)\|_{L^2((0,1)^n; L^2(\mathbb{T}^n))} \le \|D_{x_k}u\|_{L^2(\mathbb{R}^n)} + \|u\|_{L^2(\mathbb{R}^n)}, \quad u \in \mathcal{S}(\mathbb{R}^n). \tag{A.15}$$

By a standard approximation argument, we get $Uu \in L^2((0,1)^n; H^1(\mathbb{T}^n))$, and the estimate (A.15) extends to $u \in H^1(\mathbb{R}^n)$. Hence, the map (A.14) is continuous.

It remains to show that the map (A.14) is surjective. Let $v \in L^2((0,1)^n; H^1(\mathbb{T}^n))$, and thus $U^{-1}v \in L^2(\mathbb{R}^n)$. As a consequence of (A.12), we get

$$\|D_{x_k}(U^{-1}v)\|_{L^2(\mathbb{R}^n)} \le \|D_{x_k}v\|_{L^2((0,1)^n; L^2(\mathbb{T}^n))} + \|v\|_{L^2((0,1)^n; L^2(\mathbb{T}^n))}$$

for $v \in C^\infty_{\mathrm{Fl}}(\mathbb{R}^n_\theta \times \mathbb{T}^n_x)$. An approximation argument yields $U^{-1}v \in H^1(\mathbb{R}^n)$, which shows the claim.

Now a direct computation shows that

$$h[u,v] = \int_{(0,1)^n} h(\theta)[(Uu)(\theta,\cdot),(Uv)(\theta,\cdot)]\,d\theta \tag{A.16}$$

for $u, v \in \mathcal{S}(\mathbb{R}^n)$. In view of the fact that the map (A.14) is a linear homeomorphism, and the forms h and $h(\theta)$ are continuous on $H^1(\mathbb{R}^n)$ and $H^1(\mathbb{T}^n)$, respectively, the decomposition (A.16) extends to $u, v \in H^1(\mathbb{R}^n)$.

Let us show that we have the following decomposition of the operator UHU^{-1} into a direct integral:

$$UHU^{-1} = \int^{\oplus}_{(0,1)^n} H(\theta)\,d\theta. \tag{A.17}$$

To that end let us recall the definition of the direct integral:

$$\mathcal{D}\Big(\int^{\oplus}_{(0,1)^n} H(\theta)\,d\theta\Big) = \Big\{\phi \in \int^{\oplus}_{(0,1)^n} L^2(\mathbb{T}^n)\,d\theta : \phi(\theta,\cdot) \in \mathcal{D}(H(\theta)) \text{ for a.a. } \theta \in (0,1)^n,$$
$$\int_{(0,1)^n} \|H(\theta)\phi(\theta,\cdot)\|^2_{L^2(\mathbb{T}^n)}\,d\theta < \infty\Big\},$$

and

$$\Big(\Big(\int^{\oplus}_{(0,1)^n} H(\theta)\,d\theta\Big)\phi\Big)(\theta,x) = (H(\theta)\phi(\theta,\cdot))(x)$$

for a.a. $\theta \in (0,1)^n$. Let us first prove that

$$\mathcal{D}(UHU^{-1}) = \mathcal{D}\Big(\int^{\oplus}_{(0,1)^n} H(\theta)\,d\theta\Big).$$

Using [14, Theorem 12.18], (A.13), (A.14) and (A.16), we get

$$\begin{aligned}
\mathcal{D}(UHU^{-1}) &= \{\phi \in L^2((0,1)^n; L^2(\mathbb{T}^n)) : U^{-1}\phi \in \mathcal{D}(H)\}\\
&= \{\phi \in L^2((0,1)^n; H^1(\mathbb{T}^n)) : \exists f \in L^2(\mathbb{R}^n) \text{ such that}\\
&\qquad h[U^{-1}\phi,\varphi] = (f,\varphi)_{L^2(\mathbb{R}^n)}, \forall \varphi \in H^1(\mathbb{R}^n)\}\\
&= \Big\{\phi \in L^2((0,1)^n; H^1(\mathbb{T}^n)) : \exists f \in L^2((0,1)^n; L^2(\mathbb{T}^n)) \text{ such that}\\
&\int_{(0,1)^n} h(\theta)[\phi,\psi]\,d\theta = (f,\psi)_{L^2((0,1)^n;L^2(\mathbb{T}^n))}, \forall \psi \in L^2((0,1)^n; H^1(\mathbb{T}^n))\Big\}.
\end{aligned}$$

Using (A.6), integrating by parts and modifying the function f by a suitable expression depending only on the first order partial derivatives of the metric g and the function ϕ in the variables $x \in \mathbb{T}^n$, we obtain

$$\mathcal{D}(UHU^{-1}) = \Big\{\phi \in L^2((0,1)^n; H^1(\mathbb{T}^n)) : \exists f \in L^2((0,1)^n; L^2(\mathbb{T}^n)) \text{ such that}$$

$$\int_{(0,1)^n}\int_{\mathbb{T}^n} \big(D_{x_k}\phi \overline{D_{x_j}(g^{jk}|g|^{1/2}\psi)} + q\phi\overline{\psi}|g|^{1/2}\big)\,dx\,d\theta = \int_{(0,1)^n}\int_{\mathbb{T}^n} f\overline{\psi}|g|^{1/2}\,dx\,d\theta,$$

$$\forall \psi \in L^2((0,1)^n; H^1(\mathbb{T}^n))\Big\}$$

$$= \{\phi \in L^2((0,1)^n; H^1(\mathbb{T}^n)) : (g^{jk}D_{x_j}D_{x_k} + q)\phi \in L^2((0,1)^n; L^2(\mathbb{T}^n))\}$$

$$= \mathcal{D}\Big(\int_{(0,1)^n}^{\oplus} H(\theta)\,d\theta\Big).$$

Here we have also used (A.11).

Let $\phi \in \mathcal{D}(UHU^{-1})$. For any $\psi \in C_0^\infty((0,1)^n; C^\infty(\mathbb{T}^n))$, we get

$$(UHU^{-1}\phi, \psi)_{L^2((0,1)^n; L^2(\mathbb{T}^n))} = h[U^{-1}\phi, U^{-1}\psi] = \int_{(0,1)^n} (H(\theta)\phi, \psi)_{L^2(\mathbb{T}^n)}\,d\theta$$

$$= \Big(\Big(\int_{(0,1)^n}^{\oplus} H(\theta)\,d\theta\Big)\phi, \psi\Big)_{L^2((0,1)^n; L^2(\mathbb{T}^n))},$$

which shows (A.17).

A.4. Thomas's approach

To show that the operator H has no eigenvalues, a fundamental idea of Thomas [30] is to complexify the quasimomentum θ and use analytic perturbation theory. As a consequence of this idea the following result is obtained. We have learned this result from [19], [20], and for the proof we follow [31].

Proposition A.1. *Let $\lambda \in \mathbb{C}$. If there exists $\theta = \theta(\lambda) \in \mathbb{C}^n$ such that the operator $H(\theta) - \lambda$ acting on $L^2(\mathbb{T}^n)$ has zero kernel, then λ is not an eigenvalue of the operator H acting on $L^2(\mathbb{R}^n)$.*

Proof. Seeking a contradiction, assume that λ is an eigenvalue of H acting on $L^2(\mathbb{R}^n)$, i.e. there exists $u \in \mathcal{D}(H)$ with $\|u\|_{L^2(\mathbb{R}^n)} = 1$ such that

$$(H - \lambda)u = 0. \tag{A.18}$$

We shall show that λ is an eigenvalue of $H(\theta)$ acting on $L^2(\mathbb{T}^n)$ for all $\theta \in \mathbb{C}^n$. Since the Floquet–Bloch–Gelfand transform U is an isometry, we have

$$\|u\|^2_{L^2(\mathbb{R}^n)} = \int_{\theta \in (0,1)^n} \|(Uu)(\theta, \cdot)\|^2_{L^2(\mathbb{T}^n)}\,d\theta = 1,$$

and therefore

$$\mu_n(\{\theta \in (0,1)^n : \|(Uu)(\theta, \cdot)\|_{L^2(\mathbb{T}^n)} \neq 0\}) > 0. \tag{A.19}$$

Here and in what follows, μ_n is the Lebesgue measure on $\mathbb{R}^n$.

It follows from (A.18) that

$$(UHU^{-1} - \lambda)Uu = 0.$$

This together with the decomposition (A.17) implies that

$$\left(\int_{(0,1)^n}^{\oplus} H(\theta)\, d\theta - \lambda \right) Uu = 0,$$

and hence

$$(H(\theta) - \lambda)(Uu)(\theta, \cdot) = 0 \tag{A.20}$$

for almost every $\theta \in (0, 1)^n$.

It follows from (A.19) and (A.20) that

$$\mu_n(\Theta) > 0, \qquad \Theta = \{\theta \in (0, 1)^n : \lambda \text{ is an eigenvalue of } H(\theta)\}.$$

When $1 \le j \le n$, let us consider the holomorphic family of operators

$$\mathcal{H}(\theta_j) = H(\theta_1^0, \dots, \theta_{j-1}^0, \theta_j, \theta_{j+1}^0, \dots, \theta_n^0), \qquad \theta_j \in \mathbb{C},$$

where the complex values θ_k^0, $k \ne j$, are kept fixed. The resolvent of the operator $\mathcal{H}(\theta_j)$ is compact for each θ_j, and an application of analytic Fredholm theory (see [16, Theorem VII.1.10]) allows us to conclude that either λ is an eigenvalue of the operator $\mathcal{H}(\theta_j)$ for each $\theta_j \in \mathbb{C}$, or the set of points $\theta_j \in \mathbb{C}$ for which λ is an eigenvalue of $\mathcal{H}(\theta_j)$ is discrete.

Let $\widetilde{\theta} \in \mathbb{C}^n$ be an arbitrary fixed vector and let us show that λ is an eigenvalue of $H(\widetilde{\theta})$. We shall show this by induction. First, consider the set

$$\Theta_2 = \left\{(\theta_2, \dots, \theta_n) \in (0, 1)^{n-1} : \mu_1(\{\theta_1 \in (0, 1) : (\theta_1, \theta_2, \dots, \theta_n) \in \Theta\}) > 0\right\}.$$

Thus, for any $(\theta_2, \dots, \theta_n) \in \Theta_2$, we have

$$\mu_1(\{\theta_1 \in (0, 1) : \lambda \text{ is an eigenvalue of } \mathcal{H}(\theta_1)\}) > 0.$$

Hence, by analytic Fredholm theory, we conclude that λ is an eigenvalue of $H(\theta)$ for all $\theta_1 \in \mathbb{C}$ and all $(\theta_2, \dots, \theta_n) \in \Theta_2$, and therefore λ is an eigenvalue of $H(\widetilde{\theta}_1, \theta_2, \dots, \theta_n)$ for all $(\theta_2, \dots, \theta_n) \in \Theta_2$.

As $\mu_n(\Theta) > 0$, by Fubini's theorem we have $\mu_{n-1}(\Theta_2) > 0$. Consider the set

$$\Theta_3 = \left\{(\theta_3, \dots, \theta_n) \in (0, 1)^{n-2} : \mu_1(\{\theta_2 \in (0, 1) : (\theta_2, \dots, \theta_n) \in \Theta_2\}) > 0\right\}.$$

Then for any $(\theta_3, \dots, \theta_n) \in \Theta_3$, since $\mu_1(\{\theta_2 \in (0, 1) : (\theta_2, \dots, \theta_n) \in \Theta_2\}) > 0$, by analytic Fredholm theory, we see that λ is an eigenvalue of the operator $H(\widetilde{\theta}_1, \theta_2, \theta_3, \dots, \theta_n)$ for all $\theta_2 \in \mathbb{C}$ and all $(\theta_3, \dots, \theta_n) \in \Theta_3$. In particular, λ is an eigenvalue of $H(\widetilde{\theta}_1, \widetilde{\theta}_2, \theta_3, \dots, \theta_n)$ for all $(\theta_3, \dots, \theta_n) \in \Theta_3$. Continuing in the same fashion, after $n - 2$ steps we find that λ is an eigenvalue of $H(\widetilde{\theta})$. This contradicts the assumption of the proposition. □

Remark A.2. In the case of a real valued periodic potential $q \in L^{n/2}_{loc}(\mathbb{R}^n)$, the sesquilinear form h is symmetric and bounded from below, and therefore the Schrödinger operator $H = -\Delta_g + q$ acting on $L^2(\mathbb{R}^n)$ is self-adjoint and bounded from below. For any $\theta \in (0,1)^n$, the sesquilinear form $h(\theta)$ is symmetric and bounded from below, and thus the operator $H(\theta)$ acting on $L^2(\mathbb{T}^n)$ is self-adjoint and bounded from below. Furthermore, the resolvent $(H(\theta)+i)^{-1}$ is a real-analytic function of $\theta \in (0,1)^n$, and $(H(\theta)+i)^{-1}$ is compact for every $\theta \in (0,1)^n$. Then using a general result of [10] and [13], concerning the spectrum of the analytic direct integral (A.17), we conclude that the singular continuous component of the spectrum of H is empty, and the pure point spectrum is at most discrete, consisting only of isolated points without finite accumulation points, and each eigenvalue λ of H is of infinite multiplicity. Hence, in the case of a real valued periodic potential q, the absence of eigenvalues implies that the spectrum of H is purely absolutely continuous.

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