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Alessandro Chiodo · Gavril Farkas

Singularities of the moduli space of level curves

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Abstract. We describe the singular locus of the compactification of the moduli space $\mathcal{R}_{g,\ell}$ of curves of genus g paired with an ℓ-torsion point in their Jacobian. Generalising previous work for $\ell \leq 2$, we also describe the sublocus of noncanonical singularities for any positive integer ℓ. For $g \geq 4$ and $\ell = 3, 4, 6$, this allows us to provide a lifting result on pluricanonical forms playing an essential role in the computation of the Kodaira dimension of $\mathcal{R}_{g,\ell}$: for those values of ℓ, every pluricanonical form on the smooth locus of the moduli space extends to a desingularisation of the compactified moduli space.

Keywords. Moduli of curves

The modular curve $X_1(\ell) := \mathcal{H}/\Gamma_1(\ell)$ classifying elliptic curves together with an ℓ-torsion point in their Jacobian is among the most studied objects in arithmetic geometry. In a series of recent papers, the birational geometry of its higher genus generalisations and their variants (e.g. theta characteristics) has been systematically studied and proved to be, in many cases such as $\ell = 2$, better understandable than that of the underlying moduli space of curves $\mathcal{M}_g$. As an example, we refer to the complete computation of the Kodaira dimension of all components of the moduli of theta characteristics ($L^{\otimes 2} \cong \omega$)—see [23, 14, 16, 17].

In this paper, for $g \geq 2$ and for all positive levels ℓ, we consider the moduli space $\mathcal{R}_{g,\ell}$ parametrising level-ℓ curves, i.e. triples (C, L, ϕ) where C is a smooth curve equipped with a line bundle L and a trivialisation $\phi\colon L^{\otimes \ell} \xrightarrow{\sim} \mathcal{O}$. The Kodaira dimension of $\mathcal{R}_{g,\ell}$ is defined as the Kodaira dimension of an arbitrary resolution of singularities of a completion; therefore, as a first step toward the birational classification of $\mathcal{R}_{g,\ell}$, we consider a natural compactification $\overline{\mathcal{R}}_{g,\ell}$ and study the singular locus $\mathrm{Sing}(\overline{\mathcal{R}}_{g,\ell})$. More precisely one needs to determine the sublocus $\mathrm{Sing}_{\mathrm{nc}}(\overline{\mathcal{R}}_{g,\ell}) \subseteq \mathrm{Sing}(\overline{\mathcal{R}}_{g,\ell})$ of noncanonical singularities.

For $\ell = 2$, this analysis has been carried out by the second author and Ludwig in [15] using Cornalba's compactification in terms of quasistable curves [11] of $\overline{\mathcal{R}}_{g,2}$.

A. Chiodo: Université Pierre et Marie Curie, Institut de mathématiques de Jussieu PRG, 75252 Paris, France; e-mail: alessandro.chiodo@imj-prg.fr

G. Farkas: Institut für Mathematik, Humboldt Universität zu Berlin, 10099 Berlin, Germany; e-mail: farkas@math.hu-berlin.de

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Clearly, we can leave out the case $\ell = 1$, which coincides with Deligne and Mumford's functor of stable curves $\overline{\mathcal{M}}_g = \overline{\mathcal{R}}_{g,1}$. The passage to all higher levels presents a new feature from Abramovich and Vistoli's theory of stable maps to stacks: the points of the compactification cannot be interpreted in terms of ℓ-torsion line bundles on a scheme-theoretic curve, but rather on a stack-theoretic curve. Instead of the above triples $(C, L \in \mathrm{Pic}(C), \phi\colon L^{\otimes \ell} \xrightarrow{\sim} \mathcal{O})$, we simply consider their stack-theoretic analogues

$$(\mathsf{C}, \mathsf{L} \in \mathrm{Pic}(\mathsf{C}), \phi\colon \mathsf{L}^{\otimes \ell} \xrightarrow{\sim} \mathcal{O}) \in \overline{\mathcal{R}}_{g,\ell},$$

where C is a one-dimensional stack, whose nodes may have nontrivial stabilisers $\boldsymbol{\mu}_r \subseteq \boldsymbol{\mu}_\ell$, and where $\mathsf{L} \to \mathsf{C}$ is a line bundle whose fibres are faithful representations (see Definition 1.5). This yields a compactification which is represented by a smooth Deligne–Mumford stack.

In analogy with the moduli space of stable curves $\overline{\mathcal{M}}_g$, the boundary locus $\overline{\mathcal{R}}_{g,\ell} \setminus \mathcal{R}_{g,\ell}$ can be described in terms of the combinatorics of the standard dual graph Γ whose vertices correspond to the irreducible components of the curve and whose edges correspond to the nodes of the curve. In §1.4.5, we revisit this well known description by emphasising the natural role of an extra *multiplicity* datum enriching the graph. Indeed, the stack-theoretic structure of the underlying curve C and the line bundle $\mathsf{L} \to \mathsf{C}$ are determined, locally at a node, by assigning to each oriented edge e a character $\chi_e \in \mathrm{Hom}(\boldsymbol{\mu}_r, \mathbb{G}_\mathrm{m}) = \mathbb{Z}/r \subseteq \mathbb{Z}/\ell$ of the stabiliser. Hence, to each point of the boundary we attach a dual graph Γ and a $\mathbb{Z}/\ell$-valued 1-cochain $M\colon e \mapsto \chi_e$ in $C^1(\Gamma; \mathbb{Z}/\ell)$ which we refer to as the *multiplicity of the level curve*. (Proposition 1.11 recalls that a multiplicity cochain arises at the boundary if and only if it lies in the kernel of the homomorphism $\partial\colon C^1(\Gamma; \mathbb{Z}/\ell) \to C^0(\Gamma; \mathbb{Z}/\ell)$.)

In order to describe the singular locus of $\overline{\mathcal{R}}_{g,\ell}$, we lift to the moduli of level curves a result of Harris and Mumford [18]. Theorem 2 in [18] implies that, for $g \geq 4$, the local structure $\mathrm{Def}(C)/\mathrm{Aut}(C)$ of $\overline{\mathcal{M}}_g$ is singular if and only if C is equipped with an automorphism which is *not* the product of "elliptic tail involutions" (ETI for short):

$$\mathrm{Sing}(\overline{\mathcal{M}}_g) = N_1 := \{C \mid \mathrm{Aut}(C) \ni \alpha \textit{ not } \text{a product of ETI}\}.$$

By definition, an ETI operates nontrivially on the curve C only at a genus-1 component E which meets the rest of the curve in exactly one node n; its restriction to the "tail" (E, n) is the canonical involution. These automorphisms are the only nontrivial automorphisms of curves (and also of level curves) which do not yield singularities: their action on moduli is simply a quasireflection. An example of a point of N_1 is given by choosing a tail (E, n) with $\mathrm{Aut}(E, n) \cong \boldsymbol{\mu}_6$. This type of curves fill up a sublocus $T_1 \subset N_1$, of codimension 2 within $\overline{\mathcal{M}}_g$, which plays a remarkable role in this paper. Indeed, the order-6 automorphism α spanning $\mathrm{Aut}(E, n)$ and fixing $C \setminus E$ is clearly not a product of ETI and, most important, yields a noncanonical singularity. This can be checked by the Reid–Shepherd-Barron–Tai criterion: α operates on the regular space $\mathrm{Def}(C)/\langle \mathrm{ETI} \rangle$ as $(1/3, 1/3, 0, \dots, 0) := \mathrm{Diag}(\xi_3, \xi_3, 1, \dots, 1)$ and modding out α yields a noncanonical singularity, since the *age* $1/3 + 1/3 + 0 + \cdots + 0$ of α is less than 1 (see Definition 2.35). Harris and Mumford show that these special tailed curves are the only possible curves carrying a *junior* (i.e. aged less than 1) automorphism; this amounts to the following statement:

$$\operatorname{Sing}_{\mathrm{nc}}(\overline{\mathcal{M}}_g) = T_1 := \{C \mid C \supset E,\ C \cap \overline{C \setminus E} = \{n\},\ \operatorname{Aut}(E,n) \cong \boldsymbol{\mu}_6\}.$$

The generalisation of this statement to level-ℓ curves poses no problems on the interior: the variety $\mathcal{R}_{g,\ell}$ has only canonical singularities; furthermore, the singular locus is contained in the inverse image of the singular locus of $\mathcal{M}_g$, but may be smaller in general: an automorphism α of a smooth curve C does not necessarily give rise to an automorphism of (C, L) if $\alpha^* L \ncong L$.

When we consider the boundary locus $\overline{\mathcal{R}}_{g,\ell} \setminus \mathcal{R}_{g,\ell}$ the analysis becomes subtle due to a new phenomenon: stack-theoretic curves C may be equipped with *ghost automorphisms* $\mathsf{a} \in \operatorname{Aut}_C(\mathsf{C})$ which fix all geometric points of C and yet operate nontrivially on the stack C. The group $\operatorname{Aut}_C(\mathsf{C})$ has been completely determined by Abramovich, Corti, and Vistoli [1]; here, we describe the ghosts of level structures $(\mathsf{C}, \mathsf{L}, \phi)$,

$$\underline{\operatorname{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi) = \{\mathsf{a} \in \operatorname{Aut}_C(\mathsf{C}) \mid \mathsf{a}^* \mathsf{L} \cong \mathsf{L}\}.$$

The loci N_1 and T_1 naturally lift to N_ℓ and T_ℓ within $\overline{\mathcal{R}}_{g,\ell}$. For the definition of N_ℓ, no modification is needed (we require that $\underline{\operatorname{Aut}}(\mathsf{C}, \mathsf{L}, \phi)$ contains at least one automorphism which is not the product of ghost automorphisms or of ETI, using the obvious generalisation of ETI to stack-theoretic curves, Definition 2.12). The locus T_ℓ is defined as we did for T_1 by requiring the presence of an elliptic tail (E, n) with $\operatorname{Aut}(E, n) \cong \boldsymbol{\mu}_6$, but also by imposing the extra condition that the line bundle be trivial on the genus-1 tail (see Definition 2.51). For general values of ℓ, we have proper inclusions $N_\ell \subsetneq \operatorname{Sing}(\overline{\mathcal{R}}_{g,\ell})$ and $T_\ell \subsetneq \operatorname{Sing}_{\mathrm{nc}}(\overline{\mathcal{R}}_{g,\ell})$. In order to obtain $\operatorname{Sing}(\overline{\mathcal{R}}_{g,\ell})$ one needs to include also the entire locus of level curves with a nontrivial ghost (*haunted* level curves)

$$H_\ell = \{(\mathsf{C}, \mathsf{L}, \phi) \mid \underline{\operatorname{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi) \neq 1\}.$$

Similarly, in order to obtain $\operatorname{Sing}_{\mathrm{nc}}(\overline{\mathcal{R}}_{g,\ell})$ one needs to take the union of T_ℓ and of the locus of level curves haunted by a *junior ghost*,

$$J_\ell = \{(\mathsf{C}, \mathsf{L}, \phi) \mid \underline{\operatorname{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi) \ni \mathsf{a},\ \operatorname{age}(\mathsf{a}) < 1\},$$

where, as above, the age refers to the action on the regular space $\operatorname{Def}(C)/\langle \mathrm{ETI} \rangle$. This locus turns out to be entirely contained in the inverse image of the locus of curves with at least three nonseparating nodes (see Remark 2.43). In this way, J_ℓ has codimension at least three within $\overline{\mathcal{R}}_{g,\ell}$ and is a closed subvariety, reducible in general, lying in the inverse image of the boundary divisor $\delta_0^{\mathrm{stable}}$, the closure of the locus of irreducible one-nodal curves. We deduce that T_ℓ is the only irreducible component of $\operatorname{Sing}_{\mathrm{nc}}(\overline{\mathcal{R}}_{g,\ell})$ having codimension 2 within $\overline{\mathcal{R}}_{g,\ell}$.

Summarising the above discussion and taking advantage of the study of H_ℓ and J_ℓ carried out in Theorems 2.28, 2.44 and 2.52, we provide the desired extension of pluricanonical forms for $\ell = 3, 4, 6$.

Theorem. *Let $g \geq 4$. We have*

$$\mathrm{Sing}(\overline{\mathcal{R}}_{g,\ell}) = N_\ell \cup H_\ell \quad \textit{and} \quad \mathrm{Sing}_{\mathrm{nc}}(\overline{\mathcal{R}}_{g,\ell}) = T_\ell \cup J_\ell.$$

Furthermore, the locus J_ℓ is empty if and only if $5 \neq \ell \leq 6$; therefore, for $g \geq 4$ and $5 \neq \ell \leq 6$, we have

$$\Gamma\big((\overline{\mathcal{R}}_{g,\ell})^{\mathrm{reg}}, K^{\otimes q}_{\overline{\mathcal{R}}_{g,\ell}}\big) \cong \Gamma(\widehat{\mathcal{R}}_{g,\ell}, K^{\otimes q}_{\widehat{\mathcal{R}}_{g,\ell}}) \tag{1}$$

for any desingularisation $\widehat{\mathcal{R}}_{g,\ell} \to \overline{\mathcal{R}}_{g,\ell}$ and for all integers $q \geq 0$.

The case $\ell = 1$ is proven by Harris and Mumford [18]. The case $\ell = 2$ is proven by the second author in collaboration with Ludwig [15] (following work of Ludwig [23]). The above formulation presents the isomorphism (1) as a consequence of $J_\ell = \emptyset$ (and Harris and Mumford's work on the locus T_1). However, the question of whether (1) holds in the remaining cases (for $\ell = 5$ or $\ell > 6$) remains open. In §2.4, we provide a complete computation of the group $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$, which is interesting in its own right. This shows in particular that the existence of a point in J_ℓ over a stable curve C is a combinatorial condition depending only on the dual graph of C and on ℓ. The computation of $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$ will certainly allow one to further detail the geometry of J_ℓ (e.g. the irreducible components) and of $\overline{\mathcal{R}}_{g,\ell}$. We show for instance a simple combinatorial device (ghost camera) detecting the presence of ghosts and counting their number.

Write ℓ as $\prod_{p|\ell} p^{e_p}$, where p denotes a prime divisor of ℓ and e_p the p-adic valuation of ℓ. Fix a level curve $(\mathsf{C}, \mathsf{L}, \phi)$, its dual graph Γ and the multiplicity $M\colon e \mapsto \chi_e$. Consider the sequence of subgraphs

$$\emptyset \subseteq \Delta_p^{e_p} \subseteq \cdots \subseteq \Delta_p^k := \{e \mid \chi_e \in (p^k) \text{ in } \mathbb{Z}/(p^{e_p})\} \subseteq \cdots \subseteq \Delta_p^1 \subseteq \Delta_p^0 = \Gamma, \tag{2}$$

where $\chi_e \in \mathbb{Z}/\ell$ is regarded as an element of $\mathbb{Z}/(p^{e_p})$. The contraction to points of the respective subsets of edges yields

$$\Gamma \to \Gamma_p^{e_p} \to \cdots \to \Gamma_p^k \to \cdots \to \Gamma_p^1 \to \bullet. \tag{3}$$

Then all the ghost automorphisms are trivial, i.e. $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi) = 1$, if and only if $\Gamma_p^{e_p}$ are bouquets (connected graphs with a single vertex) for all p. Lemma 2.22 provides an explicit description of the group structure of $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$. In particular, we get the number of ghosts.

Corollary. *We have $\#\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi) = \frac{1}{\ell}\prod_{p|\ell} p^{V_p}$, where V_p is the total number of vertices appearing in the graphs Γ_p^j for $1 \leq j \leq e_p$.*

Note that if Γ_p^j is a bouquet for all p and j, then $\#\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi) = \frac{1}{\ell}\prod_{p|\ell} p^{e_p} = 1$. See Example 2.24 for a simple demonstration. In §2.4.6 the above formula is used to match Caporaso, Casagrande, and Cornalba's computation [7] of the length of the fibre of the moduli of level curves over the moduli of stable curves.

The above description leads to the claim that junior ghosts (hence noncanonical singularities of the form $\mathrm{Def}/\mathrm{Aut}_C(\mathsf{C}, \mathsf{L}, \phi)$) can be completely ruled out for $5 \neq \ell \leq 6$ and are relatively rare in general: their appearance is due to the presence of age-delay edges which we describe in the proof of the No-Ghost Lemma 2.44.

The computation of the Kodaira dimension of $\overline{\mathcal{R}}_{g,\ell}$ for $\ell \leq 6$ and $\ell \neq 5$ can be carried out without further study of resolutions of noncanonical singularities; for instance, in [10], in collaboration with Eisenbud and Schreyer, we show the following statement.

Theorem ([10, Thm. 0.2]). *$\mathcal{R}_{g,3}$ is a variety of general type for $g \geq 12$. Furthermore, the Kodaira dimension of $\mathcal{R}_{11,3}$ is at least 19.*

Structure of the paper. In Section 1 we introduce moduli of smooth level curves, their compactification, the relevant combinatorics and the boundary locus of the compactified moduli space. In Section 2 we study the local structure of the moduli space, we develop the suitable machinery for the computation of the ghost automorphism group and we deduce the theorem stated above.

1. Level curves

We work over an algebraically closed field k and we always denote by ℓ a positive integer prime to $\mathrm{char}(k)$.

1.1. Preliminary conventions on coarse spaces and local pictures

The interplay between stacks and their coarse spaces is crucial in this paper. Any stack X of Deligne–Mumford (DM) type admits an algebraic space X and a morphism $\epsilon_{\mathsf{X}}\colon \mathsf{X} \to X$ universal with respect to morphisms from X to algebraic spaces [22]. We regard this operation as a functor. The *coarsening* of any DM stack X is the algebraic space X (also called coarse space). The *coarsening of a morphism* $\mathsf{f}\colon \mathsf{X} \to \mathsf{Y}$ between DM stacks is the corresponding morphism $f\colon X \to Y$,

$$\mathsf{X} \rightsquigarrow X \quad \text{and} \quad \mathsf{f} \rightsquigarrow f \qquad \text{(coarsening)}.$$

We will use this notion both for curves, possibly stack-theoretic ones and equipped with level structures, and for their moduli, which are represented by stacks. For clarity let us provide two simple examples. (1) Consider the quotient DM stack $\mathsf{C} = [\mathbb{P}^1_z/\boldsymbol{\mu}_k]$ with $\zeta \in \boldsymbol{\mu}_k$ acting as $z \mapsto \zeta z$ ($k \geq 2$); the coarsening C of C is the (smooth) quotient scheme $C = \mathbb{P}^1_z/\boldsymbol{\mu}_k \cong \mathbb{P}^1_{z^k}$. (2) The coarsening of the proper, smooth, $3g-3$-dimensional DM stack $\overline{\mathsf{M}}_g$ of stable curves of genus $g \geq 2$ is the $3g-3$-dimensional projective scheme $\overline{\mathcal{M}}_g$.

When we refer to the local picture of X at the geometric point p, we mean the strict Henselisation of X at p. Hence, the local pictures of $\overline{\mathsf{M}}_g$ and of $\overline{\mathcal{M}}_g$ at the points representing C are the quotient stack $[\mathrm{Def}(C)/\mathrm{Aut}(C)]$ and the quotient scheme $\mathrm{Def}(C)/\mathrm{Aut}(C)$, respectively.

1.2. Smooth level curves

We set up $\mathsf{R}_{g,\ell}$, the space $\mathcal{R}_{g,\ell}$, and the compactification problem.

1.2.1. The moduli stack of level smooth curves. The integers $g \geq 2$ and $\ell \geq 1$ denote the *genus* and the *level*. In this way, we do not consider smooth curves with infinite automorphism groups. We further assume that the level is prime to the characteristic of the base field.

Definition 1.1. The stack $\mathsf{R}_{g,\ell}$ is the category of level-ℓ curves (C, L, ϕ) where C is a smooth genus-g curve (over a base scheme B), L is a line bundle on C, and ϕ is an isomorphism $\phi\colon L^{\otimes \ell} \to \mathcal{O}_C$. We additionally require that the order of the isomorphism class of L in $\operatorname{Pic}(C)$ is exactly ℓ. A morphism from a family $(C \to B, L, \phi)$ to a family $(C' \to B', L', \phi')$ is given by a pair (s, ρ) where $s\colon (C'/B') \to (C''/B'')$ is a morphism of curves and ρ is an isomorphism $s^*L'' \to L'$ of line bundles satisfying $\phi' \circ \rho^{\otimes \ell} = s^*\phi''$.

The category $\mathsf{R}_{g,\ell}$ is a DM stack. Its points have finite stabilisers and we have a coarsening $\mathcal{R}_{g,\ell}$ and a morphism

$$\mathsf{R}_{g,\ell} \to \mathcal{R}_{g,\ell}.$$

The forgetful functor $\mathsf{f}\colon \mathsf{R}_{g,\ell} \to \mathsf{M}_g$ to the category of smooth genus-g curves is an étale, connected cover, and indeed a finite morphism of stacks. Finiteness can be regarded as a consequence of the fact that every fibre (pullback of f via a geometric point) consists of $\Phi_{2g}(\ell)$ geometric points, with

$$\Phi_n(\ell) = \ell^n \prod_{p|\ell}\left(1 - \frac{1}{p^n}\right) \quad (\Phi_n(\ell) = \ell^n - 1 \text{ if } \ell \text{ is prime}).$$

Each of such points of the fibre is isomorphic to the stack $\mathsf{B}\boldsymbol{\mu}_\ell = [\operatorname{Spec}\mathbb{C}/\boldsymbol{\mu}_\ell]$. This happens because each point has *quasitrivial* automorphisms acting on C as the identity (i.e., s equals id_C), and scaling the fibres of L by multiplication by $\zeta \in \boldsymbol{\mu}_\ell$. Since $\mathsf{B}\boldsymbol{\mu}_\ell$ has degree $1/\ell$ over $\operatorname{Spec}\mathbb{C}$, we get

$$\deg(\mathsf{f}\colon \mathsf{R}_{g,\ell} \to \mathsf{M}_g) = \Phi_{2g}(\ell)/\ell \quad (= (\ell^{2g} - 1)/\ell \text{ if } \ell \text{ is prime}).$$

When we pass to the coarsening $f\colon \mathcal{R}_{g,\ell} \to \mathcal{M}_g$, the automorphisms are forgotten. The morphism f is still a finite connected cover, but it may well be ramified.

The stack $\mathsf{R}_{g,\ell}$ is not compact. If we allow triples $(C_{\mathrm{st}}, L, \phi)$, where C_{st} is a stable genus-g curve, and keep the rest of the definition unchanged, we obtain an étale cover of $\overline{\mathsf{M}}_g$. Properness fails: the cardinality of the fibre is not constant, as can be easily checked when C is a one-nodal irreducible curve: $\#\operatorname{Pic}(C)[\ell] = \ell^{2g-1}$.

1.3. Twisted level curves

The compactification becomes straightforward once we use the analogue of nodal curves in the context of DM stacks (for a scheme-theoretic translation see Remark 1.6).

1.3.1. Twisted curves. We point out that a less restrictive definition of twisted curve occurs in the literature, where no stability condition on C is preimposed (see for instance [25]).

Definition 1.2. A *twisted curve* C is a DM stack whose coarse space is a stable curve, whose smooth locus is represented by a scheme, and whose singularities are nodes whose local picture is given by $[\{xy = 0\}/\boldsymbol{\mu}_r]$ with $\zeta \in \boldsymbol{\mu}_r$ acting as $\zeta \cdot (x, y) = (\zeta x, \zeta^{-1} y)$.

1.3.2. Faithful line bundles. A line bundle L on a twisted curve C may be pulled back from the coarse space C or from an intermediate twisted curve fitting in a sequence of morphisms $\mathsf{C} \to \mathsf{C}' \to C$ (with $\mathsf{C}' \neq \mathsf{C}$ and $C' = C$). The following condition rules out this possibility.

Definition 1.3. A *faithful* line bundle on a twisted curve is a line bundle $\mathsf{L} \to \mathsf{C}$ for which the associated morphism $\mathsf{C} \to \mathsf{B}\mathbb{G}_{\mathrm{m}}$ is representable.

Remark 1.4. Let us phrase the condition explicitly in terms of the local picture of the fibre bundle mapping from the total space of L to C. The local picture of $\mathsf{L} \to \mathsf{C}$ at a node n of C is the projection $\mathbb{A}^1 \times \{xy = 0\} \to \{xy = 0\}$, with $\zeta \in \boldsymbol{\mu}_r$ acting as $\zeta \cdot (x, y) = (\zeta x, \zeta^{-1} y)$ on $\{xy = 0\}$ and as $\zeta \cdot (t, x, y) = (\zeta^m t, \zeta x, \zeta^{-1} y)$ on $\mathbb{C} \times \{xy = 0\}$ for a suitable index m (modulo r). Notice that the index $m \in \mathbb{Z}/r$ is uniquely determined as soon as we assign a privileged choice of a branch of the node on which $\boldsymbol{\mu}_r$ acts by the character $1 \in \operatorname{Hom}(\boldsymbol{\mu}_r, \mathbb{G}_{\mathrm{m}})$ (the action on the remaining branch is opposite). In this setting, we may restate faithfulness as follows:

$$\mathsf{L} \text{ is faithful at } \mathsf{n} \iff \text{the representation } \mathsf{L}|_{\mathsf{n}} \text{ is faithful} \iff \gcd(m, r) = 1.$$

Notice that if we switch the roles of the two branches, then m changes sign modulo r. Faithfulness does not depend on the sign of m or the choice of the branch.

1.3.3. Twisted curves and their level structures. Once the notion of twisted curve and the notion of faithful line bundle are given, level-ℓ structures are defined as for smooth curves. This is the main advantage of the twisted curve approach.

Definition 1.5. A *level-ℓ twisted curve* $(\mathsf{C} \to B, \mathsf{L}, \phi)$ consists of a twisted curve C of genus g over a base scheme B, a faithful line bundle L, and an isomorphism $\phi \colon \mathsf{L}^{\otimes \ell} \to \mathcal{O}_{\mathsf{C}}$. We additionally require that the order of the isomorphism class of L in $\operatorname{Pic}(\mathsf{C})$ is exactly ℓ.

The category of level-ℓ twisted curves forms a smooth DM stack $\overline{\mathsf{R}}_{g,\ell}$ of dimension $3g - 3$, with a finite forgetful morphism over the stack of stable curves $\mathsf{f} \colon \overline{\mathsf{R}}_{g,\ell} \to \overline{\mathsf{M}}_g$ of degree $\deg(\mathsf{f}) = \Phi_{2g}(\ell)/\ell$ (or, simply, $(\ell^{2g} - 1)/\ell$ when ℓ is prime). This definition is given implicitly in [5] by Abramovich and Vistoli (level-ℓ curves correspond to a connected component of the moduli stack of stable maps to $\mathsf{B}\boldsymbol{\mu}_\ell$). The forgetful morphism f is ramified as we illustrate in §1.5. See also work of the first author [8] for a slightly modified version, which preserves the étaleness of the forgetful morphism from level-ℓ smooth curves.

Remark 1.6. We can regard the data $\mathsf{L} \colon \mathsf{C} \to \mathsf{B}\mathbb{G}_{\mathrm{m}}$ alongside with $\phi \colon \mathsf{L}^{\otimes \ell} \to \mathcal{O}_{\mathsf{C}}$ as a representable map $\mathsf{f} \colon \mathsf{C} \to \mathsf{B}\boldsymbol{\mu}_\ell$. Then, by exploiting the representability of the map f, one can pull back the universal $\boldsymbol{\mu}_\ell$-cover $\operatorname{Spec} k \to \mathsf{B}\boldsymbol{\mu}_\ell$ to C and obtain a scheme-theoretic curve P equipped with a $\boldsymbol{\mu}_\ell$-action. In this way we can equivalently interpret the data of a level curve $(\mathsf{C}, \mathsf{L}, \phi)$, or more simply the data of a map $\mathsf{f} \colon \mathsf{C} \to \mathsf{B}\boldsymbol{\mu}_\ell$, as a $\boldsymbol{\mu}_\ell$-action on a scheme-theoretic curve P, with $\zeta \in \boldsymbol{\mu}_r$ acting as $\zeta \cdot (x, y) = (\zeta x, \zeta^{-1} y)$ at each node $\{xy = 0\}$. We refer the reader to [1] and [4, p. 506, (i)–(iii)] for this interpretation. We notice that P, equipped with its $\boldsymbol{\mu}_\ell$-action, is a $\boldsymbol{\mu}_\ell$-torsor on C (notice that all fibres over geometric points $\operatorname{Spec} k \to \mathsf{C}$ consist of ℓ distinct points which constitute a $\boldsymbol{\mu}_\ell$-orbit). On the other hand, when we regard P as a cover of C (after composition with $\mathsf{C} \to C$), we get an admissible $\boldsymbol{\mu}_\ell$-cover of the coarsening C in the sense of [4, p. 506, (i)–(iii)] (some orbits may consist of $\ell/r < \ell$ points, and in this case all points in the orbit are nodes).

1.3.4. Local indices. Consider the local picture from Remark 1.4 of a level-ℓ curve at a node:

$$\zeta \cdot (t, x, y) = (\zeta^m t, \zeta x, \zeta^{-1} y), \qquad \zeta \in \boldsymbol{\mu}_r.$$

Notice that $\mathsf{L}^{\otimes \ell} \cong \mathcal{O}$ implies $(\zeta^m)^\ell = 1$, that is, $\ell m \in r\mathbb{Z}$ with $r \geq 1$ and m in $\{0, \dots, r-1\}$. Faithfulness implies $\gcd(r, m) = 1$; hence $r \mid \ell$. In the rest of the paper, we often use a single *multiplicity index* $M = m\ell/r$ to encode the local indices r and m:

$$\begin{aligned} &r(M) = \frac{\ell}{\gcd(M, \ell)}, \quad m(M) = \frac{M}{\gcd(M, \ell)} \quad (M \in \{0, \dots, \ell - 1\}), \\ &M(r, m) = m\ell/r \quad (rx \mid \ell, m \in \{0, \dots, r-1\},\ \gcd(r, m) = 1). \end{aligned} \tag{4}$$

The first interesting example is $\ell = 3$. In this case, M equals m, and once we choose a privileged branch at a node, there are three possible local pictures:

- $M = 0$ (i.e. $(m, r) = (0, 1)$), trivial stabiliser;
- $M = 1$ (i.e. $(m, r) = (1, 3)$), nontrivial $\boldsymbol{\mu}_3$-action: the restriction of L to the privileged branch parametrised by x is $\zeta \cdot (t, x) = (\zeta t, \zeta x)$ (with $\zeta \in \boldsymbol{\mu}_3$);
- $M = 2$ (i.e. $(m, r) = (2, 3)$), nontrivial $\boldsymbol{\mu}_3$-action: the restriction of L to the privileged branch parametrised by x is $\zeta \cdot (t, x) = (\zeta^2 t, \zeta x)$ (with $\zeta \in \boldsymbol{\mu}_3$).

Let us fix a node with a given choice of a branch falling under case $M = 1$ (resp. $M = 2$); note that if we change the choice of the branch, this case falls under case $M = 2$ (resp. $M = 1$). Therefore, we can summarise this analysis by saying that the nodes of level-3 twisted curves are either trivial ($M = 0$) or nontrivial ($M \neq 0$), and in the latter case equipped with a distinguished choice of a branch so that M equals 1.

1.4. Dual graphs of twisted curves and multiplicity of level curves

The dual graph of a twisted curve is simply the dual graph of the coarse curve.

1.4.1. Dual graphs. Dual graphs arising from the standard construction recalled below are connected nonoriented graphs, possibly containing multiple edges (edges linking the same two vertices) and loops (edges starting and ending at the same vertex). Consider a twisted curve C and its normalisation $\mathsf{nor}\colon \mathsf{C}' \to \mathsf{C}$. Locally at a node of C the normalisation is given by $[\operatorname{Spec} \mathbb{C}[x]/\boldsymbol{\mu}_r] \sqcup [\operatorname{Spec} \mathbb{C}[y]/\boldsymbol{\mu}_r] \to [\{xy = 0\}/\boldsymbol{\mu}_r]$ with $\zeta \in \boldsymbol{\mu}_r$ operating on x as $\zeta \cdot x = \zeta x$ and on y as $\zeta \cdot y = \zeta^{-1} y$.

Definition 1.7. The vertex set V of the dual graph is the set of connected components of C'. The edge set E of the dual graph is the set of nodes of C. The two sets V and E determine a graph as follows: a node identifies the connected components of C' where its preimages lie, in this way an edge links two (possibly equal) vertices.

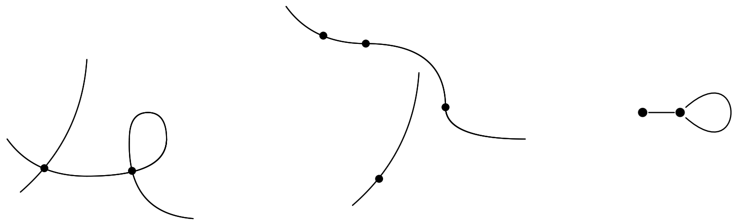


Fig. 1. Normalisation and dual graph of a twisted curve

1.4.2. Cochains. Each node of C has two branches. Let $\mathbb{E}$ be the set of branches of each node of C. The cardinality of $\mathbb{E}$ is twice that of E; there is a 2-to-1 projection $\mathbb{E} \to E$ and an involution $e \mapsto \overline{e}$ of $\mathbb{E}$. On $\mathbb{E}$ we can define a function $\mathbb{E} \to V$, denoted $e \mapsto e_+ \in V$, assigning to each oriented edge the vertex $v = e_+$ corresponding to the connected component C'_v of C' where the chosen branch lies. We get $e \mapsto e_-$ by applying $(\ )_+$ after the involution. If $e_+ = e_-$ we have a loop (Figure 1): $\overline{e} \neq e$ in $\mathbb{E}$ map to the same vertex via $e \mapsto e_+$.

We define the groups of 1-cochains and 0-cochains of the dual graph with coefficients in $\mathbb{Z}$. We define $C^0(\Gamma)$ as the set of $\mathbb{Z}$-valued functions on V

$$C^0(\Gamma) = \{a \colon V \to \mathbb{Z}\} = \bigoplus_{v \in V} \mathbb{Z}.$$

We define 1-cochains as antisymmetric $\mathbb{Z}$-valued functions on $\mathbb{E}$,

$$C^1(\Gamma) = \{b \colon \mathbb{E} \to \mathbb{Z} \mid b(\overline{e}) = -b(e)\},$$

where $\overline{e}$ and e are oriented edges with opposite orientations. After assigning an orientation to each edge $e \in E$, we may identify $C^1(\Gamma)$ to $\bigoplus_{e \in E} \mathbb{Z}$, but we prefer working with $\mathbb{E}$.

The spaces of $\mathbb{Z}$-valued 0-cochains and 1-cochains, $C^0(\Gamma)$ and $C^1(\Gamma)$, are equipped with nondegenerate bilinear $\mathbb{Z}$-valued forms

$$\langle a_1, a_2 \rangle = \sum_{v \in V} a_1(v) a_2(v), \qquad \langle b_1, b_2 \rangle = \frac{1}{2} \sum_{e \in \mathbb{E}} b_1(e) b_2(e), \tag{5}$$

with $a_1, a_2 \in C^0$ and $b_1, b_2 \in C^1$. The exterior differential is

$$\delta \colon C^0(\Gamma) \to C^1(\Gamma), \qquad a \mapsto \delta a, \qquad \text{with} \qquad \delta a(e) = a(e_+) - a(e_-).$$

The adjoint operator with respect to $\langle\ ,\ \rangle$ is given by

$$\partial \colon C^1(\Gamma) \to C^0(\Gamma), \qquad b \mapsto \partial b, \qquad \text{with} \qquad \partial b(v) = \sum_{\substack{e \in \mathbb{E} \\ e_+ = v}} b(e).$$

Remark 1.8 (cuts and circuits). The image $\mathrm{im}(\delta)$ is freely generated by $\#V-1$ cuts (see [6, Ch. 4]),

$$\mathrm{im}(\delta)\cong\mathbb{Z}^{\oplus(\#V-1)}. \tag{6}$$

We recall that a cut is determined by a proper nonempty subset W of the vertex set V of Γ: the sets W and $V\setminus W$ form a partition of V. Cuts are 1-cochains $b\colon\mathbb{E}\to\mathbb{Z}$ in $C^1(\Gamma)$ equal to 1 on the (nonempty) set H_W of edges having only one end on W and oriented from W to $V\setminus W$, equal to -1 on $\overline{H}_W=\{\overline{e}\mid e\in H_W\}$, and vanishing elsewhere. By construction, H_W and $\overline{H}_W$ contain no loops. For this reason, in graph theory literature the image of δ is often referred to as the *cut space*.

The kernel $\ker\partial$ is freely generated by $b_1=1-\chi(\Gamma)=1-\#V+\#E$ circuits,

$$\ker\partial\cong\mathbb{Z}^{\oplus(1-\#V+\#E)}.$$

We recall that a *circuit* within a graph is a sequence of n oriented edges $e_0,\dots,e_{n-1}\in\mathbb{E}$ labelled by $i\in\mathbb{Z}/n$, overlying n distinct nonoriented edges in E, so that the head $(e_i)_+$ is also the tail $(e_{i+1})_-$ for all $i\in\mathbb{Z}/n$ and the n vertices $v_i=(e_i)_-$ are distinct. If we remove the condition $(e_0)_-=(e_{n-1})_+$ we obtain a *path* of edges joining $v_0=(e_0)_-$ to $v=(e_{n-1})_+$. Here, we treat circuits and paths as 1-cochains, regarding their characteristic function (given by 1 on e_i, -1 on $\overline{e}_i$ and 0 elsewhere) as an element of $C^1(\Gamma)$. Circuits formed by a single oriented edge will be called *loops*.

Since δ is the adjoint of ∂, for any $s\in\ker\partial$ and $\delta t\in\mathrm{im}\,\delta$ we have $\langle s,\delta t\rangle=\langle\partial s,t\rangle=0$. Conversely, the condition $\langle s,b\rangle=0$ for all $s\in\ker\partial$ implies $b\in\mathrm{im}\,\delta$. In order to see this, (for every connected component) fix a vertex $v_0\in V$ and define $a\in C^0(\Gamma)$ as $a(v)=\sum_{i=0}^{n-1}b(e_i)$ for a path joining v_0 to v. The definition of a does not depend on the chosen path because the difference between two paths lies in $\ker\partial$ and we have $\langle s,b\rangle=0$ for all $s\in\ker\partial$. By construction, we have $\delta a=b$. In this way, we get a simple criterion for $b\in C^1(\Gamma)$ to lie in $\mathrm{im}\,\delta$:

$$b\in C^1(\Gamma)\text{ is in }\mathrm{im}\,\delta\ \Leftrightarrow\ b(K)=\sum_{0\le i<n}b(e_i)=0\text{ for all circuits }K=\sum_{0\le i<n}e_i\text{ of }\Gamma. \tag{7}$$

Remark 1.9. For any abelian group A, by taking $\partial\otimes_{\mathbb{Z}}A$ and $\delta\otimes_{\mathbb{Z}}A$, we recover the simplicial cohomology and homology complexes with coefficients in A,

$$\delta_A\colon C^0(\Gamma;A)\to C^1(\Gamma;A),\qquad \partial_A\colon C^1(\Gamma;A)\to C^0(\Gamma;A)\qquad (C^i(\Gamma;A)=C^i\otimes_{\mathbb{Z}}A).$$

The forms (5) extend to pairings

$$\langle\ ,\ \rangle\colon C^0(\Gamma)\otimes_{\mathbb{Z}}C^0(\Gamma;A)\to A\quad\text{and}\quad\langle\ ,\ \rangle\colon C^1(\Gamma)\otimes_{\mathbb{Z}}C^1(\Gamma;A)\to A.$$

with the same definition, where $a_1(v)a_2(v)$ is in A for $a_1(v)\in\mathbb{Z}$ and $a_2(v)\in A$, and similarly $b_1(e)b_2(e)$ is in A for $b_1(e)\in\mathbb{Z}$ and $b_2(e)\in A$. Notice that we still have the equalities $\langle\delta s_0,t_1\rangle=\langle s_0,\partial_A t_1\rangle$ and $\langle s_1,\delta_A t_0\rangle=\langle\partial s_1,t_0\rangle$ for any $s_i\in C^i(\Gamma)$ and $t_j\in C^j(\Gamma;A)$.

Then, any elements $\delta_A t \in \operatorname{im}\delta_A \subset C^1(\Gamma; A)$ and $s \in \ker\partial \in C^1(\Gamma)$ satisfy the condition $\langle s, \delta_A t\rangle = \langle \partial s, t\rangle = 0$. As above, the condition $\langle s, b\rangle = 0$ for all $s \in \ker\partial$ implies $b \in \operatorname{im}\delta_A$. We conclude that (7) still holds. More precisely, for a circuit $K = \sum_{0\le i<n} e_i$, we set $b(K) := \sum_{0\le i<n} b(e_i)$ so that the claim (7) generalises to $C^1(\Gamma; A)$ and $\operatorname{im}\delta_A$ verbatim. Notice that if A is multiplicative (e.g. $\boldsymbol{\mu}_\ell$ and $\mathbb{G}_{\mathrm{m}}$ below) all notation should be read accordingly; for instance, the condition $b(\overline{e}) = -b(e)$ defining $C^1(\Gamma; A)$ should be read as $b(\overline{e}) = b(e)^{-1}$, and similarly the sum $\sum_{0\le i<n} b(e_i)$ defining $b(K)$ above should be read as $\prod_{0\le i<n} b(e_i)$.

1.4.3. The group of line bundles with trivial normalisation and the $\mathbb{G}_{\mathrm{m}}$-valued cut space. The cohomology of the short exact sequence of sheaves $1 \to \mathbb{G}_{\mathrm{m}} \to \mathsf{nor}_*\mathsf{nor}^*\mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}|_{\mathsf{Sing}\mathsf{C}} \to 1$ and the analogous sequence for $\boldsymbol{\mu}_\ell$ yields the exact sequences

$$C^0(\Gamma; \mathbb{G}_{\mathrm{m}}) \xrightarrow{\delta} C^1(\Gamma; \mathbb{G}_{\mathrm{m}}) \xrightarrow{\tau} \operatorname{Pic}(\mathsf{C}) \xrightarrow{\mathsf{nor}^*} \operatorname{Pic}(\mathsf{C}'), \tag{8}$$

$$C^0(\Gamma; \boldsymbol{\mu}_\ell) \xrightarrow{\delta} C^1(\Gamma; \boldsymbol{\mu}_\ell) \xrightarrow{\tau} \operatorname{Pic}(\mathsf{C})[\ell] \xrightarrow{\mathsf{nor}^*} \operatorname{Pic}(\mathsf{C}')[\ell]. \tag{9}$$

Let us state explicitly the definition of the homomorphism τ. It is enough to consider a 1-cochain b vanishing on all edges except e_0 and $\overline{e}_0$ where it equals ζ and ζ^{-1} respectively (the cochain is $\mathbb{G}_{\mathrm{m}}$-valued if ζ lies in $\mathbb{G}_{\mathrm{m}}$ and $\boldsymbol{\mu}_\ell$-valued if ζ^ℓ equals 1). The line bundle $\tau(b)$ is the locally free sheaf of regular functions f on the normalisation of C at the node n satisfying $f(\mathsf{x}) = \zeta f(\mathsf{y})$ for x and y preimages of n, with x lying on the branch corresponding to e_0 and y lying on the remaining branch.

1.4.4. Line bundles on an ℓ-twisted curve C up to pullbacks from C are $\boldsymbol{\mu}_\ell$-valued circuits. For any stable curve C, up to isomorphism, we can define a unique twisted curve $\widetilde{\mathsf{C}}$ with order-ℓ stabilisers at all nodes. We may call the curve C the *ℓ-twisted curve* attached to C (in another context [8] it is called ℓ-stable, because imposing that all stabilisers have the same cardinality amounts to a stability condition). We consider the line bundles of $\operatorname{Pic}(\widetilde{\mathsf{C}})$ up to pullbacks from $\operatorname{Pic}(C)$, or—what is the same—$\operatorname{Pic}(\widetilde{\mathsf{C}})[\ell]$ modulo $\operatorname{Pic}(\widetilde{C})[\ell]$. By [8, Cor. 3.1], the long exact sequence of cohomology of the Kummer sequence $1 \to \boldsymbol{\mu}_\ell \to \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}} \to 1$, combined with that of $1 \to A \to \mathsf{nor}_*\mathsf{nor}^*A \to A|_{\mathsf{Sing}\mathsf{C}} \to 1$ for $A = \mathbb{G}_{\mathrm{m}}$ and $\boldsymbol{\mu}_\ell$, yields the exact sequence

$$1 \to \operatorname{Pic}(C)[\ell] \to \operatorname{Pic}(\widetilde{\mathsf{C}})[\ell] \to C^1(\Gamma; \mathbb{Z}/\ell) \xrightarrow{\partial} C^0(\Gamma; \mathbb{Z}/\ell).$$

Here, it should be noticed that the cohomology with coefficients in $\boldsymbol{\mu}_\ell$ naturally produces $\mathbb{Z}/\ell$-valued cochains. For instance, the $\boldsymbol{\mu}_\ell$-valued second cohomology group of a curve is canonically identified with $\mathbb{Z}/\ell$ (see [24, §14]). On the other hand, $C^1(\Gamma; \mathbb{Z}/\ell)$ equals $\bigoplus_{e\in E} H^1(B\boldsymbol{\mu}_\ell, \boldsymbol{\mu}_\ell)$, where each summand is the ℓ-torsion subgroup of the group of characters $\operatorname{Hom}(\boldsymbol{\mu}_\ell, \mathbb{G}_{\mathrm{m}})$, which—by definition—equals $\mathbb{Z}/\ell$.

1.4.5. Multiplicity and $\ker\partial$. Since oriented edges are in one-to-one correspondence with branches of nodes of C, using §1.3.4 we define the multiplicity cochain.

Definition 1.10 ($\mathbb{Z}/\ell$-valued multiplicity 1-cochain of level-ℓ curves). Consider a level-ℓ curve $(\mathsf{C}, \mathsf{L}, \phi)$. To each oriented edge e, we can attach the multiplicity $M(e)$ of $(\mathsf{C}, \mathsf{L}, \phi)$ at the node (with its prescribed branch). The function $M\colon e \mapsto M(e) \in \mathbb{Z}/\ell$ satisfies $M(\overline{e}) = -M(e)$ for all $e \in \mathbb{E}$; in this way we have

$$M \in C^1(\Gamma; \mathbb{Z}/\ell).$$

Proposition 1.11. *Let C be a stable curve and consider the set of level-ℓ curves $(\mathsf{C}, \mathsf{L}, \phi)$ with coarsening C. Consider the dual graph of C and the differential ∂. Then associating to $(\mathsf{C}, \mathsf{L}, \phi)$ its multiplicity 1-cochain M yields a surjective map from the set of level-ℓ curves with coarsening C to* $\ker \partial \subseteq C^1(\Gamma; \mathbb{Z}/\ell)$.

Proof. All level-ℓ structures overlying C may be regarded as elements in $\mathrm{Pic}(\widetilde{\mathsf{C}})[\ell]$, where $\widetilde{\mathsf{C}}$ is the twisted curve $\widetilde{\mathsf{C}}$ with $\boldsymbol{\mu}_\ell$-stabilisers at all nodes (note that in $\mathrm{Pic}(\widetilde{\mathsf{C}})[\ell]$ we do not impose faithfulness). The multiplicity cochain lifts to a homomorphism $M\colon \mathrm{Pic}(\widetilde{\mathsf{C}})[\ell] \to C^1(\Gamma; \mathbb{Z}/\ell)$. The above claim follows from the exact sequence $1 \to \mathrm{Pic}(C)[\ell] \to \mathrm{Pic}(\widetilde{\mathsf{C}})[\ell] \to C^1(\Gamma; \mathbb{Z}/\ell) \to C^0(\Gamma; \mathbb{Z}/\ell)$ (see [8, Cor. 3.1]) and from the existence of an element in $\mathrm{Pic}(C)[\ell]$ (for $g \geq 2$) whose order equals ℓ. ◻

Remark 1.12. The above $\mathbb{Z}/\ell$-valued 1-cochain is (in another context) the same as the 1-cochain associated to a weighted subgraph of Γ in [7, Rem. 2.2.1], including the fact that it takes values in $\ker \partial$. Similarly, the description of ghost automorphisms for ℓ prime in §2.2 is related to [20, Proposition 4.1.11] and [7, Lemma 2.3.2].

Remark 1.13. The fact that M takes values in $\ker \partial$ may be regarded as saying that the multiplicities $M_1, \dots, M_N$ at all special points $\mathsf{p}_1, \dots, \mathsf{p}_N$ of a connected component of the normalisation must add up to zero modulo ℓ. This is easy to see also directly once we express the line bundle induced by L on a connected component X of the normalisation of C as the line bundle $\mathcal{O}_{\mathsf{X}}(D)$, where D is a Cartier divisor on a smooth stack-theoretic curve X. The divisor D is the sum of a divisor I with integer coefficients plus the rational coefficients divisor $S = \sum_{i=1}^N \frac{M_i}{\ell}[\mathsf{p}_i]$ supported on the special points. The condition $\mathsf{L}^{\otimes \ell} \cong \mathcal{O}$ implies that $\ell I + \ell S$ is a principal divisor; in particular, it has degree zero and we have

$$\sum_{i=1}^N M_i = \ell \deg S = -\ell \deg I \in \ell\mathbb{Z}.$$

Example 1.14. Consider a two-component twisted curve obtained as the union of two smooth one-dimensional stacks X and Y meeting transversely at two nodes. For each node, let us measure the multiplicities with respect to the branch lying in X. Proposition 1.11 says that the multiplicities M_1 and M_2 should add up to 0 (modulo ℓ). Let us examine in greater detail the case $\ell = 3$, $M_1 = 1$ and $M_2 = 2$. Over X the third root L of $\mathcal{O}$ is given by a divisor D' of degree 0 (a root of $\mathcal{O}_{\mathsf{X}}$) with rational coefficients of the form $D' = \lfloor D' \rfloor + [\mathsf{x}_1]/3 + 2[\mathsf{x}_2]/3$, where $\mathsf{x}_1\colon \operatorname{Spec}\mathbb{C} \to \mathsf{X}$ and $\mathsf{x}_2\colon \operatorname{Spec}\mathbb{C} \to \mathsf{X}$ are the geometric points lifting n_1 and n_2 to X. Conversely, $\mathsf{L}|_{\mathsf{Y}}$ can be expressed as the degree-0 line bundle $\mathcal{O}(D'')$ with $D'' = \lfloor D'' \rfloor + 2[\mathsf{y}_1]/3 + [\mathsf{y}_2]/3$, where again y_1 and y_2 lift n_1 and n_2 to Y.

The multiplicity 1-cochain encodes much of the relevant topological information characterising a level curve. In what follows, we describe some natural invariants of $\mathbb{Z}/\ell$-valued 1-cochains.

1.4.6. The support and its characteristic function. For any 1-cochain $c\colon \mathbb{E} \to \mathbb{Z}/\ell$ we consider the characteristic function of the support of c taking values in the extended set $\mathbb{Z} \cup \{\infty\}$ (we use the standard conventions $a < \infty$ and $a + \infty = \infty$ for $a \in \mathbb{Z}$):

$$\nu_c(e) = \begin{cases} \infty & \text{if } c(e) = 0 \in \mathbb{Z}/\ell, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Proposition 1.11 implies $\nu_c(e) = \infty$ for any separating edge.

For any abelian group A, we present a natural subcomplex $C^\bullet_\nu(\Gamma; A)$ of $C^\bullet(\Gamma; A)$ attached to a given symmetric characteristic function $\nu\colon \mathbb{E} \to \{0, \infty\}$, i.e. to any subset of E. In §2.4 we generalise this construction by allowing, instead of characteristic functions, more general functions arising as truncated valuations of M (see (31)). When ℓ is prime we recover the above defined function ν_c.

1.4.7. The contracted graph $\Gamma(\nu)$. We define precisely the graphs obtained by iterated edge-contractions of Γ mentioned in the introduction. Let us consider any symmetric characteristic function $\nu\colon \mathbb{E} \to \{0, \infty\}$ (since ν is symmetric it descends to E and we sometimes abuse the notation by regarding it as a function on E). We attach to Γ a new graph $\Gamma(\nu)$ whose sets of vertices and edges $(V(\nu), E(\nu))$ are obtained from (V, E)

(i) by setting $E(\nu) = \{e \mid \nu(e) = 0\}$,
(ii) by modding out V by the relations $e_+ \sim_\nu e_-$ if $\nu(e) = \infty$, i.e. $V(\nu) = V/{\sim_\nu}$.

In the new graph, the set of vertices of the edge $e \in E(\nu)$ is the set of vertices of $e \in E$ in V modulo the relation $\sim_\nu$. In simple terms, $\Gamma(\nu)$ is the contraction of all edges where $\upsilon > 0$. We refer to $\Gamma(\nu)$ as a *contraction* of Γ, and to Γ as a *blowup* of $\Gamma(\nu)$ (often in graph theory literature, the graph obtained from an iterated edge-contraction is a "minor" of the initial graph, but we do not use this terminology here).

1.4.8. The complex $C^\bullet_\nu(\Gamma; A)$. The inclusion $i\colon E(\nu) \hookrightarrow E$ and the projection $p\colon V \twoheadrightarrow V(\nu)$ yield homomorphisms $p_*\colon C^0(\Gamma; A) \twoheadrightarrow C^0(\Gamma(\nu); A)$ and $i^*\colon C^1(\Gamma; A) \twoheadrightarrow C^1(\Gamma(\nu); A)$ and the *contraction homomorphism* between complexes with differentials given by ∂,

$$\mathcal{C}\colon (C^\bullet(\Gamma; A), \partial) \twoheadrightarrow (C^\bullet(\Gamma(\nu); A), \partial). \tag{11}$$

Conversely, the homomorphisms $p^*\colon C^0(\Gamma(\nu); A) \hookrightarrow C^0(\Gamma; A)$ and $i_*\colon C^1(\Gamma(\nu); A) \hookrightarrow C^1(\Gamma; A)$ yield the *blowup homomorphism* between complexes with differential δ,

$$\mathcal{B}\colon (C^\bullet(\Gamma(\nu); A), \delta) \hookrightarrow (C^\bullet(\Gamma; A), \delta). \tag{12}$$

The subcomplex $\mathcal{B}(C^\bullet(\Gamma(\nu); A), \delta)$ consists of the 0-cochains $a \in C^0(\Gamma; A)$ and the 1-cochains $b \in C^1(\Gamma; A)$ satisfying $a(e_+) = a(e_-)$ and $b(e) = 0$ if $\nu(e) = \infty$. Within $(C^\bullet(\Gamma; A), \delta)$ we denote such a subcomplex by

$$C^\bullet_\nu(\Gamma; A) \subseteq C^\bullet(\Gamma; A).$$

In fact, we have

$$\mathcal{B}(\operatorname{im}\delta) = \operatorname{im}\delta \cap C^1_\nu(\Gamma; A). \tag{13}$$

The inclusion from left to right follows from (12). Conversely, $b = \delta(a)$ is in $C^1_\nu(\Gamma; A)$ only if, for any contracted edge e, we have $a(e_+) = a(e_-)$, that is, only if a lies in $C^0_\nu(\Gamma; A)$). Passing to the adjoint operator we also get

$$\mathcal{C}(\ker\partial) = \ker\partial. \tag{14}$$

Summarising, the contraction of a circuit is a circuit and the blowup of a cut is a cut.

1.5. The boundary locus

We describe $\overline{\mathsf{R}}_{g,\ell} \setminus \mathsf{R}_{g,\ell}$ by classifying one-nodal level curves.

1.5.1. Reducible one-nodal curves. Consider the union $\mathsf{C} = \mathsf{C}_1 \cup \mathsf{C}_2$ of two smooth stack-theoretic curves C_1 and C_2 of genus i and $g-i$ meeting transversely at a point. Proposition 1.11 implies that the node has multiplicity zero or, in other words, trivial stabiliser. Hence, we have $\mathsf{C} = C$, i.e. C is an ordinary stable curve of compact type, $C = C_1 \cup C_2$. The line bundle $\mathsf{L} = L$ on C is determined by the choice of two line bundles L_1 and L_2 satisfying $L_1^{\otimes\ell} \cong \mathcal{O}_{C_1}$ and $L_2^{\otimes\ell} \cong \mathcal{O}_{C_1}$. There are three possibilities:

$$\text{(i) } L_1 \cong \mathcal{O}, L_2 \ncong \mathcal{O}; \quad \text{(ii) } L_1 \ncong \mathcal{O}, L_2 \cong \mathcal{O}; \quad \text{(iii) } L_1, L_2 \ncong \mathcal{O}$$

(since we have $\mathsf{L} \ncong \mathcal{O}$, the possibility that both line bundles are trivial is excluded). If $0 < i < g/2$, these three cases characterise three loci in the moduli space whose closures are the divisors Δ_{g-i}, Δ_i and $\Delta_{i:g-i}$ respectively. We write δ_{g-i}, δ_i and $\delta_{i:g-i}$ for the corresponding $\mathbb{Q}$-divisors defined by the same conditions in the moduli stack. The morphism f is not ramified along these divisors. We have

$$\mathsf{f}^*(\delta_i^{\text{stable}}) = \delta_{g-i} + \delta_i + \delta_{i:g-i}, \tag{15}$$

where δ_i^{stable} is the $\mathbb{Q}$-divisor class in $\overline{\mathsf{M}}_g$ defined by stable curves with at least one node separating the curve into two components of genus i and $g-i$.

If $i = g/2$ the same classification reduces to two divisors: the closure of the locus of one-nodal level curves for which only one line bundle among L_1 and L_2 is trivial yields $\Delta_{g/2}$, and the closure of the locus classifying curve where both L_1 and L_2 are nontrivial yields $\Delta_{g/2:g/2}$.

1.5.2. Irreducible one-nodal curves. If C is irreducible and has one node, then the node is of nonseparating type: the normalisation $\mathsf{nor}\colon \mathsf{C}' \to \mathsf{C}$ is given by a connected curve. There are three possibilities:

$$\text{(i) } M = 0 \text{ and } \mathsf{nor}^*\mathsf{L} \ncong \mathcal{O}; \quad \text{(ii) } M = 0 \text{ and } \mathsf{nor}^*\mathsf{L} \cong \mathcal{O}; \quad \text{(iii) } M \neq 0.$$

The closures of the loci of level curves satisfying the three conditions above determine three divisors denoted by Δ_0', Δ_0'', Δ_0^{ram} in the moduli space. We write again δ_0', δ_0'', δ_0^{ram} for the corresponding classes of divisors defined by the same conditions in the moduli stack. The morphism f is not ramified along δ_0' and δ_0''. When ℓ is prime, f is ramified with order ℓ along δ_0^{ram}. More precisely, we have (cf. [10])

$$\mathsf{f}^*(\delta_0^{\mathrm{stable}}) = \delta_0' + \delta_0'' + \ell\delta_0^{\mathrm{ram}} \quad (\ell \text{ prime}). \tag{16}$$

In general, δ_0^{ram} can be decomposed into several components depending on the value of the multiplicity index M; we refer to §1.5.4 for the study of the order of the ramification.

This calls for an analysis of the irreducible components of the boundary divisors δ_0', δ_0'', δ_0^{ram} as well as for the previous divisors δ_i, $\delta_{i:g-i}$ for $1 \le i \le g/2$. We carry it out in the last part of this section (§1.5.3 and §1.5.4) as a nice application of the combinatorial invariants of level curves illustrated above. On the other hand, the present description of the boundary locus is sufficient for the entire Section 2 and may already be already regarded as a decomposition into irreducible components of the boundary for $\ell = 3$ (see Examples 1.16 and 1.18). Therefore, it is worthwhile to illustrate it further by an example, which will play an important role in the rest of the paper: the case of level structures on elliptic-tailed curves.

Example 1.15 (two level-ℓ structures on the elliptic-tailed curve). We provide examples of two distinct twisted level curves, one representing a point of $\Delta_1 \cap \Delta_0^{\mathrm{ram}}$, and the other representing a point in $\Delta_1 \cap \Delta_0''$. Consider the stack-theoretic quotient E of $\widetilde{E} = \mathbb{P}^1/(\Omega_{P'})$ by $\boldsymbol{\mu}_\ell$, with $\zeta \in \boldsymbol{\mu}_\ell$ operating by multiplication on the local parameter of $\mathbb{P}^1$ at 0. Now let C be a twisted curve containing, as a subcurve, a copy of such a genus-1 stack-theoretic curve E. We assume $\mathsf{E} \cap \overline{\mathsf{C} \setminus \mathsf{E}} = \{n\}$, where n is a separating node with trivial stabiliser (see Proposition 1.11).

Level-ℓ structures in Δ_1 can be defined on C by extending trivially on $\overline{\mathsf{C} \setminus \mathsf{E}}$ nontrivial ℓth roots of $\mathcal{O}$ on E. To this end, we can exploit $\mathsf{p}\colon \widetilde{E} \to \mathsf{E}$, which is an étale $\boldsymbol{\mu}_\ell$-cyclic cover of E. The rank-ℓ locally free sheaf $\mathsf{p}_*\mathcal{O}$ carries a $\boldsymbol{\mu}_\ell$-representation and admits an isotypical decomposition $\mathsf{p}_*\mathcal{O} = \bigoplus_{\chi \in \mathbb{Z}/\ell = \mathrm{Hom}(\boldsymbol{\mu}_\ell, \mathbb{G}_{\mathrm{m}})} \mathsf{L}_\chi$. We set $\mathsf{L}_{\mathrm{ram}} := \mathsf{L}_1$, where $\chi = 1$ is the character $(1\colon \boldsymbol{\mu}_\ell \subset \mathbb{G}_{\mathrm{m}}) \in \mathbb{Z}/\ell$. In this way $\mathsf{L}_{\mathrm{ram}}$ is is equipped with an isomorphism $\phi_{\mathrm{ram}}\colon \mathsf{L}_{\mathrm{ram}}^{\otimes \ell} \cong \mathcal{O}$. Then $\mathsf{L}_{\mathrm{ram}} \to \mathsf{C}$ yields an object $(\mathsf{C}, \mathsf{L}_{\mathrm{ram}}, \phi_{\mathrm{ram}})$ in $\Delta_1 \cap \Delta_0^{\mathrm{ram}}$ because the multiplicity of $\mathsf{L}_{\mathrm{ram}}$ at the nonseparating node is $\neq 0$ (1 or $l-1$ depending on the chosen branch).

The projection to the coarse space $\epsilon_{\mathsf{E}}\colon \mathsf{E} \to E$ allows us to define another nontrivial line bundle in $\mathrm{Pic}(\mathsf{C})[\ell]$ as follows. On E, simply consider the pullback of the line bundle of regular functions f on the normalisation $E' \cong \mathbb{P}^1$ satisfying $f(\infty) = \zeta f(0)$ for any $\zeta \in \boldsymbol{\mu}_\ell$. This is $\tau(\zeta)$ in the notation of §1.4.3. If ζ is a primitive root of unity, then we get a line bundle $\mathsf{L}_{\text{ét}} \to \mathsf{C}$ yielding a point in $\Delta_1 \cap \Delta_0''$ (the multiplicity at the nonseparating node is 0 and $\mathsf{L}_{\text{ét}}$ is trivial on the normalisation by construction).

1.5.3. The closure of the locus of reducible one-nodal curves: irreducible components. We provide a decomposition into irreducible components of the divisor defined above as the closure of the substack of reducible level-ℓ one-nodal curves. It is convenient to reformulate the problem in $\overline{\mathsf{M}}_g$: we study the divisor

$$\mathsf{D}_{\text{red}}^{\text{stable}} = \sum_{1 \le i \le g/2} \delta_i^{\text{stable}} \tag{17}$$

of stable curves with at least one separating node. We do so, by analysing the degree-2 map $\widetilde{\mathsf{D}}_{\text{red}}^{\text{stable}} \to \mathsf{D}_{\text{red}}^{\text{stable}}$ classifying stable curves alongside with a separating node and a branch of the node. We have the natural decomposition $\widetilde{\mathsf{D}}_{\text{red}}^{\text{stable}} = \bigsqcup_{i=1}^{g-1} \widetilde{\mathsf{D}}_i^{\text{stable}}$ where $\widetilde{\mathsf{D}}_i^{\text{stable}}$ classifies objects where the chosen branch lies in the genus-i connected component Z of the normalisation of the separating node. Then, for $i = 1, \dots, g-1$, we write $\mathsf{D}_i^{\text{stable}}$ for the pushforward in $\overline{\mathsf{M}}_g$ of the cycle $\widetilde{\mathsf{D}}_i^{\text{stable}}$ via the map forgetting the branch; for $i \neq g/2$, the forgetful map from $\widetilde{\mathsf{D}}_i^{\text{stable}}$ has degree 1 and we have $\mathsf{D}_i^{\text{stable}} = \mathsf{D}_{g-i}^{\text{stable}}$, while for $i = g/2$ the forgetful map $\widetilde{\mathsf{D}}_{g/2}^{\text{stable}}$ is a degree-2 morphism. In this way, we reformulate (17) as follows:

$$\mathsf{D}_{\text{red}}^{\text{stable}} = \frac{1}{2} \sum_{i=1}^{g-1} \mathsf{D}_i^{\text{stable}}.$$

For level curves, consider the stack $\widetilde{\mathsf{D}}_{\text{red}}$ classifying level-ℓ curves alongside with a separating node and a branch of the node. Hence, we get the decomposition of $\widetilde{\mathsf{D}}_{\text{red}}$ into connected components and the corresponding decomposition of D_{red} into irreducible components,

$$\widetilde{\mathsf{D}}_{\text{red}} = \bigsqcup_{d_1, d_2, i} \widetilde{\mathsf{D}}_i^{d_1, d_2} \quad \text{and} \quad \mathsf{D}_{\text{red}} = \frac{1}{2} \sum_{d_1, d_2, i} \mathsf{D}_i^{d_1, d_2},$$

where d_1 and d_2 are divisors of ℓ whose least common multiple equals ℓ, i ranges between 1 and $g-1$, and the loci $\widetilde{\mathsf{D}}_i^{d_1,d_2}$ and $\mathsf{D}_i^{d_1,d_2}$ are defined as follows. The stack $\widetilde{\mathsf{D}}_i^{d_1,d_2}$ is the full subcategory of objects where the data of the chosen branch and of the genus-i connected component Z of the normalisation of the separating node satisfy

(i) the branch lies in Z and $g(\mathsf{Z}) = i$,
(ii) the order of L on Z equals d_1,
(iii) the order of L on $\overline{\mathsf{C} \setminus \mathsf{Z}}$ equals d_2.

The divisor $\mathsf{D}_i^{d_1,d_2}$ is the pushforward of the cycle $\widetilde{\mathsf{D}}_i^{d_1,d_2}$ via the forgetful functor forgetting the choice of the branch and of the node. Since the stack-theoretic structure of one-nodal level-ℓ curves of compact type is trivial, there is no ramification of f along D_{red}: we have $\mathsf{D}_{\text{red}} = \mathsf{f}^* \mathsf{D}_{\text{red}}^{\text{stable}}$. The factor $1/2$ in the above expression D_{red} eliminates the factor 2 due to $\mathsf{D}_i^{d_1,d_2} = \mathsf{D}_{g-i}^{d_2,d_1}$ for any $(i, d_1, d_2) \neq (g/2, \ell, \ell)$, and to the degree 2 of the map $\widetilde{\mathsf{D}}_{\ell/2}^{\ell,\ell} \to \mathsf{D}_{\ell/2}^{\ell,\ell}$ when g is even.

Example 1.16. For ℓ prime, we notice that δ_i, δ_{g-i} and $\delta_{i:g-i}$ are precisely the divisors $\mathsf{D}_i^{\ell,1}$, $\mathsf{D}_i^{1,\ell}$ and $\mathsf{D}_i^{\ell,\ell}$ for $i \neq g/2$ and $\frac{1}{2}\mathsf{D}_{g/2}^{\ell,\ell}$ otherwise. For $i \neq g/2$, they are the three irreducible components of $\mathsf{f}^*\delta_i^{\text{stable}}$ of degrees $(\ell^{2i}-1)/\ell$, $(\ell^{2g-2i}-1)/\ell$ and $(\ell^{2g-2i}-1)(\ell^{2i}-1)/\ell$ over δ_i^{stable} (we check that they add up to $\deg(\mathsf{f}) = (\ell^{2g}-1)/\ell$).

1.5.4. The closure of the locus of irreducible one-nodal curves: irreducible components. We study the divisor δ_0^{stable} of stable curves with at least one separating node. As in §1.5.3, we write $\mathsf{D}_{\text{irr}}^{\text{stable}} = \delta_0^{\text{stable}}$ and we analyze the degree-2 morphism $\widetilde{\mathsf{D}}_{\text{irr}}^{\text{stable}} \to \mathsf{D}_{\text{irr}}^{\text{stable}}$ classifying stable curves alongside with a nonseparating node and a branch of the node. Consider the stack $\widetilde{\mathsf{D}}_{\text{irr}}$ classifying level-ℓ curves $(\mathsf{C}, \mathsf{L}, \phi)$ equipped with a prescribed choice of a nonseparating node and of a branch of that node; this yields a notation x and y for the points lifting the node to the normalisation $\text{nor}\colon \mathsf{C}' \to \mathsf{C}$ of the nonseparating node. On $\widetilde{\mathsf{D}}_{\text{irr}}$, we can define the data M, d, h:

- the multiplicity $M \in \mathbb{Z}/\ell$,
- the order d (dividing ℓ and multiple of $\ell/\gcd(M,\ell)$) of $\text{nor}^*\mathsf{L}$ on C',
- the gluing datum of a root of unity $h \in \boldsymbol{\mu}_{\ell/d}$ satisfying $f(\mathsf{x}) = hf(\mathsf{y})$ for the sections f of $(\text{nor}^*\mathsf{L})^{\otimes d} \cong \mathcal{O}$.

Within $\widetilde{\mathsf{D}}_{\text{irr}}$, we write $\widetilde{\mathsf{D}}_{\text{irr}}^{M,d,h}$ for the locus where the multiplicity, the order and the gluing datum are respectively M, d and h. Since L has order ℓ, within $\widetilde{\mathsf{D}}_{\text{irr}}$, the gluing datum is always a primitive ℓ/dth root of unity; however, the same definition, without any condition on the order of L, yields a stack for any $h \in \boldsymbol{\mu}_{\ell/d}$ and we have $\widetilde{\mathsf{D}}_{\text{irr}}^{M,d,1} \cong \widetilde{\mathsf{D}}_{\text{irr}}^{M,d,h}$, via $\mathsf{L} \mapsto \mathsf{L} \otimes \tau(\zeta)$ for $\zeta \in \boldsymbol{\mu}_\ell$ with $\zeta^d = h$ (see §1.4.3). The moduli stack $\widetilde{\mathsf{D}}_{\text{irr}}^{M,d,h}$ is connected because $\widetilde{\mathsf{D}}_{\text{irr}}^{M,d,1}$ is. Indeed, following Remark 1.6, $\widetilde{\mathsf{D}}_{\text{irr}}^{M,d,1}$ classifies $\boldsymbol{\mu}_d$-covers $\pi\colon P \to C$ of genus-g curves with a specified nonseparating node n in C corresponding to an orbit of d/r nodes for $r = \ell/\gcd(M,\ell)$. We further require the following properties: (1) there is a privileged branch at n and the action of $\boldsymbol{\mu}_r \subseteq \boldsymbol{\mu}_d$ is of the form $\zeta \cdot (z,w) = (\zeta z, \zeta^{-1}w)$ on P and is given at the privileged branch by the character $m = M/\gcd(M,\ell) \in \mathbb{Z}/r$, (2) the normalisation of C at n and of P at the d/r points of $\pi^{-1}(n)$ is a connected $\boldsymbol{\mu}_d$-cover $\pi\colon P' \to C'$. The connectedness of $\widetilde{\mathsf{D}}_{\text{irr}}^{M,d,1}$ follows precisely from the connectedness of P' and it may be interesting to see it explicitly. We do it hereafter.

Lemma 1.17. *The moduli stack* $\widetilde{\mathsf{D}}_{\text{irr}}^{M,d,1}$ *is connected.*

Proof. For simplicity, let us first consider the case $r = 1$ (i.e. $M \in \ell\mathbb{Z}$). The connected $\boldsymbol{\mu}_d$-cover $\pi'\colon P' \to C'$ contains two distinguished orbits D_x and $D_y \subset P'$ lying above the preimages x and $y \in C'$ of the node n. The claim follows from the existence of a family ranging through all possible ways to glue back this normalised $\boldsymbol{\mu}_d$-cover of C' to form a $\boldsymbol{\mu}_d$-cover of C (in general there are d/r distinguished possibilities; here we have d choices). By deformation, it is enough to show the claim when C' is $\mathbb{P}^1/(0 \sim \infty)$ marked at x and y and P' is the connected étale $\boldsymbol{\mu}_d$-cover attached to $\tau(\xi_d)$. We take P' itself as a base scheme and we define a family of $\boldsymbol{\mu}_d$-covers over it. Above any point p of $\Omega_{P'} = P' \setminus (\text{Sing} \cup D_x)$ we can consider the cover $P' \to C'$ and two distinguished orbits:

D_x and the orbit of p. By taking the limit $\boldsymbol{\mu}_d$-cover using the properness of $\overline{\mathsf{R}}_{g,d}$ (or simply by blowing up conveniently within $\mathbb{P}^1 \times \mathbb{P}^1$), the family extends uniquely across the nodes of P and the points of D_x. We obtain in this way a family $\mathcal{P}'$ of $\boldsymbol{\mu}_d$-covers over the base scheme P' with a section δ extending the diagonal of $(\Omega_{P'})^2$ and disjoint from the closure of the orbit $D_x \times \Omega_{P'}$. Fix a point $t \in D_x$; then, for any $g \in \boldsymbol{\mu}_d$, we glue the image of $g\delta$ with the (closure of) $gt \times \Omega_{P'}$ and we get a family of $\boldsymbol{\mu}_d$-covers which embraces all the possible ways to glue back $P' \to C'$ to a $\boldsymbol{\mu}_d$-cover of the initial curve C. These are precisely the d fibres above the d points of D_y.

If we drop the condition $r = 1$, we regard the initial data $\pi' \colon P' \to C'$ as the composite of the étale $\boldsymbol{\mu}_d/\boldsymbol{\mu}_r$-cover $\varepsilon' \colon E' \to C'$ given by $\tau(\xi_{d/r})$, and a branched $\boldsymbol{\mu}_r$-cover with action $m \in \mathbb{Z}/r$ at the points of $D_x = (\varepsilon')^{-1}(x)$ and $r - m \in \mathbb{Z}/r$ at the points of $D_y = (\varepsilon')^{-1}(x)$. By construction, this branched cover is unique up to isomorphism and amounts to extracting an rth root of $\mathcal{O}_{E'}(-mD_x - (r-m)D_y)$. We proceed as above by defining a family of $\boldsymbol{\mu}_d$-covers over the base scheme E'. To this end, if p lies in $\Omega_{E'} = E' \setminus (\mathrm{Sing} \cup D_x)$ we consider the $\boldsymbol{\mu}_d$-cover of C' given by the $\boldsymbol{\mu}_r$-cover of E' itself induced by an rth root of $\mathcal{O}_{E'}(-mD_x - (r-m)\Delta)$, where Δ is the orbit of p. Again, this family of $\boldsymbol{\mu}_d$-covers extends uniquely across $\mathrm{Sing} \cup D_x$ and admits two sections with disjoint orbits (lifting the closure of the diagonal of $(\Omega_{E'})^2$ and the closure of a section of $D_x \times \Omega_{E'} \to \Omega_{E'}$). By gluing along these sections as above, we get a family of $\boldsymbol{\mu}_d$-covers embracing all the possible ways to glue back $P' \to C'$ to a $\boldsymbol{\mu}_d$-cover of the initial curve C. These are precisely the d/r fibres above the d/r points of D_y. □

Hence, we have decomposed $\widetilde{\mathsf{D}}_{\mathrm{irr}}$ into a disjoint union of $\sum_{M=0}^{\ell-1} \gcd(M, \ell)$ connected loci $\widetilde{\mathsf{D}}_{\mathrm{irr}}^{M,d,h}$, where $M \in \{0, \dots, \ell-1\}$, h is a $\gcd(M, \ell)$th root of unity and d is determined by h: we set $d = \ell/\mathrm{ord}(h)$ so that h is a primitive root in $\boldsymbol{\mu}_{\ell/d}$. We may remove d from the notation and we get the desired decomposition into irreducible components of $\mathsf{D}_{\mathrm{irr}}$:

$$\widetilde{\mathsf{D}}_{\mathrm{irr}} = \bigsqcup_{\substack{M \in \mathbb{Z}/\ell \\ h \in \boldsymbol{\mu}_{\gcd(M,\ell)}}} \widetilde{\mathsf{D}}_{\mathrm{irr}}^{M,h} \quad \text{and} \quad \mathsf{D}_{\mathrm{irr}} = \frac{1}{2} \sum_{\substack{M \in \mathbb{Z}/\ell \\ h \in \boldsymbol{\mu}_{\gcd(M,\ell)}}} \mathsf{D}_{\mathrm{irr}}^{M,h}.$$

Here $\mathsf{D}_{\mathrm{irr}}^{M,h}$ are the pushforwards in $\overline{\mathsf{R}}_{g,\ell}$ of the cycles $\widetilde{\mathsf{D}}_{\mathrm{irr}}^{M,h}$ via the morphism forgetting the prescribed branch.

Note that if $M \in \{0, \ell/2\}$ and $h \in \{1, -1\}$, this forgetful morphism is a degree-2 morphism and the factor $1/2$ removes the degree factor appearing in the direct image of $\widetilde{\mathsf{D}}_{\mathrm{irr}}^{M,h}$. Actually, not all combinations with $M = 0, \ell/2$ and $h = 1, -1$ occur: if ℓ is odd, only $(M, h) = (0, 1)$ occurs; if $\ell \in 2\mathbb{Z} \setminus 4\mathbb{Z}$, all combinations except $(\ell/2, -1)$ occur ($\boldsymbol{\mu}_{\ell/2}$ does not contain -1); if ℓ is in $4\mathbb{Z}$, any of the four combinations occurs.

In all the remaining cases $\mathsf{D}_{\mathrm{irr}}^{M,h}$ equals $\mathsf{D}_{\mathrm{irr}}^{\ell-M,h^{-1}}$. For these terms, the sum is redundant and the factor $1/2$ removes the factor 2 arising from summing twice the same divisor.

Notice also that the order of the ramification of the morphism f along $\mathsf{D}_{\mathrm{irr}}^{M,d,h}$ equals the order of M in $\mathbb{Z}/\ell$; that is precisely $r = \ell/\gcd(M, \ell)$.

Example 1.18. If $\ell = 3$, the stack $\widetilde{\mathsf{D}}_{\mathrm{irr}}$ has five connected components, as many as $\sum_{M=0}^{2} \gcd(M, 3) = 3 + 1 + 1$. Since ℓ is odd, only one of these yields a connected degree-2 cover: $\widetilde{\mathsf{D}}_{\mathrm{irr}}^{0,1} \to \mathsf{D}_{\mathrm{irr}}^{0,1}$. The remaining cases are paired as follows: we have $\widetilde{\mathsf{D}}_{\mathrm{irr}}^{0,\xi_3} = \widetilde{\mathsf{D}}_{\mathrm{irr}}^{0,\xi_3^2}$ and $\widetilde{\mathsf{D}}_{\mathrm{irr}}^{1,1} = \widetilde{\mathsf{D}}_{\mathrm{irr}}^{2,1}$.

The divisors δ_0', δ_0'' and δ_0^{ram} over $\delta_0^{\mathrm{stable}}$ can be recovered as follows:

- the divisor δ_0' is the image in $\overline{\mathsf{R}}_{g,\ell}$ of $\widetilde{\mathsf{D}}_{\mathrm{irr}}^{0,1}$;
- the divisor δ_0'' is the image of $\widetilde{\mathsf{D}}_{\mathrm{irr}}^{0,\xi_3}$ and it can also be written as $\widetilde{\mathsf{D}}_{\mathrm{irr}}^{0,\xi_3^2}$;
- finally, δ_0^{ram} is the image of $\widetilde{\mathsf{D}}_{\mathrm{irr}}^{1,1}$ and it can also be written as $\widetilde{\mathsf{D}}_{\mathrm{irr}}^{2,1}$.

These divisors coincide with the irreducible components of $\mathsf{D}_{\mathrm{irr}}$. As substacks over $\delta_0^{\mathrm{stable}} = \mathsf{D}_{\mathrm{irr}}^{\mathrm{stable}}$ they have respectively degree $1/3$ times $3(3^{2g-2} - 1)$, 2, and $2(3^{2g-2})$; using (16), we count the degree of δ_0^{ram} over δ_0 with multiplicity 3 and we obtain again $\deg(\mathsf{f}) = (3^{2g} - 1)/3$.

In view of the next example and further generalisations, we can perform this degree check more systematically via pushforward via $\widetilde{\mathsf{f}} \colon \widetilde{\mathsf{D}}_{\mathrm{irr}} \to \widetilde{\mathsf{D}}_{\mathrm{irr}}^{\mathrm{stable}}$, $\mathsf{i} \colon \widetilde{\mathsf{D}}_{\mathrm{irr}}^{\mathrm{stable}} \to \overline{\mathsf{M}}_g^{\mathrm{stable}}$, $\mathsf{j} \colon \widetilde{\mathsf{D}}_{\mathrm{irr}} \to \overline{\mathsf{R}}_{g,\ell}^{\mathrm{stable}}$ and $\mathsf{f} \colon \overline{\mathsf{R}}_{g,\ell} \to \overline{\mathsf{M}}_g$ (the composite of the first two maps equals the composite of the last two). Write $\mathsf{f}^*\delta_0^{\mathrm{stable}} = \delta_0' + \delta_0'' + 3\delta_0^{\mathrm{ram}}$ as

$$\mathsf{f}^*\delta_0^{\mathrm{stable}} = \tfrac{1}{2}(\mathsf{D}_{\mathrm{irr}}^{0,1} + \mathsf{D}_{\mathrm{irr}}^{0,\xi_3} + \mathsf{D}_{\mathrm{irr}}^{0,\xi_3^2} + 3\mathsf{D}_{\mathrm{irr}}^{1,1} + 3\mathsf{D}_{\mathrm{irr}}^{2,1})$$

and take the pushforward

$$\mathsf{f}_*\mathsf{f}^*\delta_0^{\mathrm{stable}} = \tfrac{1}{2}(\mathsf{f}_*\mathsf{D}_{\mathrm{irr}}^{0,1} + \mathsf{f}_*\mathsf{D}_{\mathrm{irr}}^{0,\xi_3} + \mathsf{f}_*\mathsf{D}_{\mathrm{irr}}^{0,\xi_3^2} + 3\mathsf{f}_*\mathsf{D}_{\mathrm{irr}}^{1,1} + 3\mathsf{f}_*\mathsf{D}_{\mathrm{irr}}^{2,1}).$$

We write $\mathsf{D}_{\mathrm{irr}}^{M,h} = \mathsf{j}_*\widetilde{\mathsf{D}}_{\mathrm{irr}}^{M,h}$ and we replace each $\mathsf{f}_*\mathsf{j}_*\widetilde{\mathsf{D}}_{\mathrm{irr}}^{M,h}$ by $\mathsf{i}_*\widetilde{\mathsf{f}}_*\widetilde{\mathsf{D}}_{\mathrm{irr}}^{M,h} = d^{M,h}(\mathsf{i}_*\widetilde{\mathsf{D}}_{\mathrm{irr}}^{\mathrm{stable}})$ where $d^{M,h}$ is the degree of the forgetful morphism $\widetilde{\mathsf{D}}_{\mathrm{irr}}^{M,h} \to \widetilde{\mathsf{D}}_{\mathrm{irr}}^{\mathrm{stable}}$ onto its image. We obtain

$$\begin{aligned}
\mathsf{f}_*\mathsf{f}^*\delta_0^{\mathrm{stable}} &= (d^{0,1} + d^{0,\xi_3} + d^{0,\xi_3^2} + 3d^{1,1} + 3d^{2,1})\tfrac{1}{2}\mathsf{i}_*\widetilde{\mathsf{D}}_{\mathrm{irr}}^{\mathrm{stable}} \\
&= \left(\frac{3(3^{2g-2} - 1)}{3} + \frac{1}{3} + \frac{1}{3} + 3\frac{3^{2g-2}}{3} + 3\frac{3^{2g-2}}{3}\right)\delta_0^{\mathrm{stable}} = \frac{3^{2g} - 1}{3}\delta_0^{\mathrm{stable}}.
\end{aligned}$$

Example 1.19. If $\ell = 4$, the stack $\widetilde{\mathsf{D}}_{\mathrm{irr}}$ has eight connected components, as many as $\sum_{M=0}^{3} \gcd(M, 4) = 4 + 1 + 2 + 1$. Four of them are paired and yield the same boundary divisor: $\mathsf{D}_{\mathrm{irr}}^{0,\xi_4} = \mathsf{D}_{\mathrm{irr}}^{0,\xi_4^3}$ and $\mathsf{D}_{\mathrm{irr}}^{1,1} = \mathsf{D}_{\mathrm{irr}}^{3,1}$. The remaining four, $\mathsf{D}_{\mathrm{irr}}^{0,1}$, $\mathsf{D}_{\mathrm{irr}}^{0,\xi_2}$, $\mathsf{D}_{\mathrm{irr}}^{2,1}$ and $\mathsf{D}_{\mathrm{irr}}^{2,\xi_2}$, yield boundary divisors with multiplicity 2. We can write the fundamental class of the boundary as

$$\mathsf{D}_{\mathrm{irr}} = \tfrac{1}{2}(\mathsf{D}_{\mathrm{irr}}^{0,1} + \mathsf{D}_{\mathrm{irr}}^{0,\xi_2} + \mathsf{D}_{\mathrm{irr}}^{0,\xi_4} + \mathsf{D}_{\mathrm{irr}}^{0,\xi_4^3} + \mathsf{D}_{\mathrm{irr}}^{1,1} + \mathsf{D}_{\mathrm{irr}}^{2,1} + \mathsf{D}_{\mathrm{irr}}^{2,\xi_2} + \mathsf{D}_{\mathrm{irr}}^{3,1}),$$

or equivalently, highlighting its five irreducible components, as

$$\mathsf{D}_{\mathrm{irr}} = \tfrac{1}{2}\mathsf{D}_{\mathrm{irr}}^{0,1} + \tfrac{1}{2}\mathsf{D}_{\mathrm{irr}}^{0,\xi_4^2} + \mathsf{D}_{\mathrm{irr}}^{0,\xi_4} + \mathsf{D}_{\mathrm{irr}}^{1,1} + \tfrac{1}{2}\mathsf{D}_{\mathrm{irr}}^{2,1} + \tfrac{1}{2}\mathsf{D}_{\mathrm{irr}}^{2,\xi_2}.$$

By pulling back $\delta_0^{\text{stable}} = \mathsf{D}_{\text{irr}}^{\text{stable}}$ we get the same locus with multiplicities; following the last computation given for $\ell = 3$ we check that this decomposition is compatible with $\deg(\mathsf{f}) = \Phi_{2g}(4)/4 = (4^{2g} - 2^{2g})/4$. We have

$$\mathsf{f}^*\delta_0^{\text{stable}} = \tfrac{1}{2}(\mathsf{D}_{\text{irr}}^{0,1} + \mathsf{D}_{\text{irr}}^{0,\xi_4} + \mathsf{D}_{\text{irr}}^{0,\xi_4^2} + \mathsf{D}_{\text{irr}}^{0,\xi_4^3} + 4\mathsf{D}_{\text{irr}}^{1,1} + 2\mathsf{D}_{\text{irr}}^{2,1} + 2\mathsf{D}_{\text{irr}}^{2,\xi_2} + 4\mathsf{D}_{\text{irr}}^{3,1}),$$

hence

$$\begin{aligned}\mathsf{f}_*\mathsf{f}^*\delta_0^{\text{stable}} &= (d^{0,1} + d^{0,\xi_4} + d^{0,\xi_4^2} + d^{0,\xi_4^3} + 4d^{1,1} + 2d^{2,1} + 2d^{2,\xi_2} + 4d^{3,1})\tfrac{1}{2}\mathsf{i}_*\widetilde{\mathsf{D}}_{\text{irr}}^{\text{stable}} \\ &= \left(\frac{4\Phi_{2g-2}(4)}{4} + \frac{1}{4} + \frac{2\Phi_{2g-2}(2)}{4} + \frac{1}{4} + 4\frac{4^{2g-2}}{4} + 2\frac{2\Phi_{2g-2}(4)}{4} + 2\frac{2^{2g-2}}{4} + 4\frac{4^{2g-2}}{4}\right)\delta_0^{\text{stable}} \\ &= \frac{\Phi_{2g}(4)}{4}\delta_0^{\text{stable}},\end{aligned}$$

where we have used $\Phi_n(4) = 4^n - 2^n$ and $\Phi_n(2) = 2^n - 1$.

The combinatorics involved in the previous examples is subsumed under the general treatment of §2.4.6, where we provide the computation of the length of any fibre of moduli of level curves.

Remark 1.20 (Compatibility with the terminology of [10]). We have the following relations between the coarse decomposition in terms of δ-divisors and the finer analysis in terms of D-divisors. We have

$$\delta_i = \mathsf{D}_i^{1,\ell} = \mathsf{D}_{g-i}^{\ell,1}, \quad \delta_{g-i} = \mathsf{D}_i^{\ell,1} = \mathsf{D}_i^{\ell,1}, \quad \delta_{i:g-i} = \sum_{\substack{d_1,d_2|\ell \\ \text{lcm}(d_1,d_2)=\ell}} \mathsf{D}_i^{d_1,d_2} \quad (i \neq g/2),$$

$$\delta_0' = \frac{1}{2}\sum_{h\in\boldsymbol{\mu}_\ell} \mathsf{D}_{\text{irr}}^{0,h}, \qquad \delta_0'' = \tfrac{1}{2}\mathsf{D}_{\text{irr}}^{0,1}, \qquad \delta_0^{\text{ram}} = \frac{1}{2}\sum_{\substack{0\neq M\in\mathbb{Z}/\ell \\ h\in\boldsymbol{\mu}_{\gcd(M,\ell)}}} \mathsf{D}_{\text{irr}}^{M,h}$$

(for $g \in 2\mathbb{Z}$ we have $\delta_{g/2:g/2} = \frac{1}{2}\sum_{d_1,d_2|\ell} \mathsf{D}_{g/2}^{d_1,d_2}$ where again we impose $\text{lcm}(d_1, d_2) = \ell$).

Since the ramification index at $\mathsf{D}_{\text{irr}}^{M,h}$ equals $r(M)$, it follows that the equation $\mathsf{f}^*(\delta_0^{\text{stable}}) = \delta_0' + \delta_0'' + \ell\delta_0^{\text{ram}}$ only holds for ℓ prime. However, we point out that the same equation holds if we replace δ_0^{ram} by $\sum_{a=1}^{\lfloor \ell/2\rfloor} \delta_0^{(a)}$ and we set, for any $a = 1, \ldots, \lfloor \ell/2\rfloor$,

$$\delta_0^{(a)} = \frac{1}{2}\sum_{\substack{M=a,-a \\ h\in\boldsymbol{\mu}_{\gcd(M,\ell)}}} \frac{1}{\gcd(a,\ell)}\mathsf{D}_{\text{irr}}^{M,h}.$$

When ℓ is prime, which is the main focus in [10], the divisor above arises naturally as a substack within $\overline{\mathsf{R}}_{g,\ell}$. For composite values of ℓ, the above divisor can still be obtained as a codimension-1 substack of a suitable compactification of $\mathsf{R}_{g,\ell}$; indeed in [10, §1.3] we illustrate how in $\delta_0^{(a)}$ the multiplicities $1/\gcd(a, \ell)$ arise naturally when working with the compactification of [8] which simply imposes stabilisers of order ℓ at all nonseparating nodes instead of imposing the faithfulness condition on L. These compactifications have the same coarse space $\overline{\mathcal{R}}_{g,\ell}$, bur they are not very convenient for the study of the singularities of $\overline{\mathcal{R}}_{g,\ell}$ because their stabilisers are extensions by quasireflections of the stabilisers of $\overline{\mathsf{R}}_{g,\ell}$.

2. The singularities of the moduli space of level curves

In this section we assume $g \geq 4$; this is a standard condition in the study of the singularity locus of the coarse moduli space of curves essentially motivated by Harris and Mumford's work [18] (see Remark 2.11 and Proposition 2.13 and also the role played by this condition in the proof of Theorem 2.44).

At the point represented by $(\mathsf{C}, \mathsf{L}, \phi)$, the local pictures of $\overline{\mathsf{R}}_{g,\ell}$ and of $\overline{\mathcal{R}}_{g,\ell}$ are given by $[\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)/\mathrm{Aut}(\mathsf{C}, \mathsf{L}, \phi)]$ and $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)/\mathrm{Aut}(\mathsf{C}, \mathsf{L}, \phi)$. We relate these local pictures to $[\mathrm{Def}(C)/\mathrm{Aut}(C)]$ and $\mathrm{Def}(C)/\mathrm{Aut}(C)$, the local pictures of $\overline{\mathsf{M}}_g$ and $\overline{M}_g$ at C.

2.1. *Deformation spaces and automorphism groups*

The space $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)$ can be expressed in terms of $\mathrm{Def}(C)$.

2.1.1. Deformations of C. We only consider the stable curve C. We denote by $\mathrm{Def}(C, \mathrm{Sing}(C))$ the space of deformations of the curve C alongside its set of nodes $\mathrm{Sing}(C)$. It may be decomposed canonically as

$$\mathrm{Def}(C, \mathrm{Sing}(C)) = \bigoplus_{v \in V} H^1(C'_v, T(-D_v)),$$

where we denote by $C'_v \subseteq C$ the connected component of the normalisation of C attached to v, and by D_v the divisor formed by the inverse images of the nodes of C under the normalisation map. Indeed, the group $H^1(C'_v, T(-D_v))$ parametrises deformations of the pair (C'_v, D_v).

Note that $\mathrm{Def}(C, \mathrm{Sing}(C))$ is a subspace of $\mathrm{Def}(C)$; by modding it out we obtain

$$\mathrm{Def}(C)/\mathrm{Def}(C, \mathrm{Sing}(C)) = \bigoplus_{e \in E} N_e,$$

where the decomposition is canonical and the term N_e denotes the fibre over $[C]$ of the normal bundle to the locus of deformations preserving the node attached to e. In fact N_e is one-dimensional; giving a (noncanonical) parametrisation

$$N_e \cong \mathrm{Spec}(\mathbb{C}[t_e]) =: \mathbb{A}^1_{t_e}$$

is equivalent to choosing a smoothing[1] of the node attached to e along t_e.

[1] A *smoothing* of a node $n \in C$ is an infinitesimal deformation $\mathcal{C} \to \mathrm{Spec}\,\mathbb{C}[t_e]/(t_e^2)$ of the curve C, where n is a regular point within the scheme $\mathcal{C}$.

2.1.2. Deformations of $(\mathsf{C},\mathsf{L},\phi)$. The deformation space $\mathrm{Def}(\mathsf{C},\mathsf{L},\phi)$ is canonically identified with $\mathrm{Def}(\mathsf{C})$ via the étale forgetful functor $(\mathsf{C},\mathsf{L},\phi)\mapsto\mathsf{C}$. The picture of $\mathrm{Def}(\mathsf{C})$ is analogue to the above picture for $\mathrm{Def}(C)$.

Within $\mathrm{Def}(\mathsf{C},\mathsf{L},\phi)=\mathrm{Def}(\mathsf{C})$, we consider $\mathrm{Def}(\mathsf{C},\mathsf{L},\phi,\mathrm{Sing}(\mathsf{C}))=\mathrm{Def}(\mathsf{C},\mathrm{Sing}(\mathsf{C}))$, the subspace of deformations where $\mathrm{Sing}(\mathsf{C})$ deforms alongside C (the topological type of the curve is preserved). In fact, via the natural forgetful map $\mathrm{Def}(\mathsf{C},\mathsf{L},\phi)=\mathrm{Def}(\mathsf{C})\to\mathrm{Def}(C)$, this space is canonically identified to $\mathrm{Def}(C,\mathrm{Sing}(C))$ (this happens because $H^1(C'_v,T(-D_v))$ and the stack-theoretic counterpart are canonically isomorphic [5, Lem. 2.3.4]). Therefore, we have

$$\mathrm{Def}(\mathsf{C},\mathsf{L},\phi,\mathrm{Sing}(\mathsf{C}))=\mathrm{Def}(\mathsf{C},\mathrm{Sing}(\mathsf{C}))=\bigoplus_{v\in V}H^1(C'_v,T(-D_v)).\tag{18}$$

The corresponding quotient space canonically decomposes as

$$\mathrm{Def}(\mathsf{C},\mathsf{L},\phi)/\mathrm{Def}(\mathsf{C},\mathsf{L},\phi,\mathrm{Sing}(\mathsf{C}))=\mathrm{Def}(\mathsf{C})/\mathrm{Def}(\mathsf{C},\mathrm{Sing}(\mathsf{C}))=\bigoplus_{e\in E}K_e.\tag{19}$$

As in §2.1.1, K_e is one-dimensional; indeed, it can be parametrised by τ_e, the $r(e)$th root of the above mentioned parameter t_e ($r(e)$ is the local index from §1.3.4). In this way τ_e may be geometrically interpreted as the parameter smoothing the node of C corresponding to e and the map between quotients $\mathrm{Def}(\mathsf{C},\mathsf{L},\phi)/\mathrm{Def}(\mathsf{C},\mathsf{L},\phi,\mathrm{Sing}(\mathsf{C}))\to\mathrm{Def}(C)/\mathrm{Def}(C,\mathrm{Sing}(C))$ is the direct sum, for e in E, of

$$\mathbb{A}^1_{t_e}\to\mathbb{A}^1_{\tau_e},\qquad t_e\mapsto t_e^{r(e)}.$$

2.1.3. Automorphisms of $(\mathsf{C},\mathsf{L},\phi)$. An automorphism of a level curve $(\mathsf{C},\mathsf{L},\phi)$ is given by (s,ρ) where s is an isomorphism of C, and ρ is an isomorphism $\mathsf{s}^*\mathsf{L}\to\mathsf{L}$ of line bundles satisfying $\phi\circ\rho^{\otimes\ell}=\mathsf{s}^*\phi$:

$$\begin{tikzcd}
\mathsf{s}^*(\mathsf{L}^{\otimes\ell}) \arrow[r, "="] \arrow[d, "\mathsf{s}^*\phi"] & (\mathsf{s}^*\mathsf{L})^{\otimes\ell} \arrow[r, "\rho^{\otimes r}"] & \mathsf{L}^{\otimes\ell} \arrow[d, "\phi"] \\
\mathsf{s}^*\mathcal{O} \arrow[rr, "="] & & \mathcal{O}
\end{tikzcd}$$

We write

$$\mathrm{Aut}(\mathsf{C},\mathsf{L},\phi)=\{(\mathsf{s},\rho)\mid \mathsf{s}\in\mathrm{Aut}(\mathsf{C}),\ \rho\colon\mathsf{s}^*\mathsf{L}\xrightarrow{\cong}\mathsf{L},\ \phi\circ\rho^{\otimes\ell}=\mathsf{s}^*\rho\}.$$

On the other hand, we consider

$$\underline{\mathrm{Aut}}(\mathsf{C},\mathsf{L},\phi)=\{\mathsf{s}\in\mathrm{Aut}(\mathsf{C})\mid\mathsf{s}^*\mathsf{L}\cong\mathsf{L}\}.$$

It is easy to see that for each element $\mathsf{s}\in\underline{\mathrm{Aut}}(\mathsf{C},\mathsf{L},\phi)$ there exists $(\mathsf{s},\rho)\in\mathrm{Aut}(\mathsf{C},\mathsf{L},\phi)$. Two pairs of this form differ by a power of a quasitrivial automorphism of the form $(\mathrm{id}_{\mathsf{C}},\xi_\ell)$ operating by scaling the fibres. We have the exact sequence

$$1\to\boldsymbol{\mu}_\ell\to\mathrm{Aut}(\mathsf{C},\mathsf{L},\phi)\to\underline{\mathrm{Aut}}(\mathsf{C},\mathsf{L},\phi)\to 1.$$

As already mentioned, quasitrivial isomorphisms act trivially on $\mathrm{Def}(\mathsf{C},\mathsf{L},\phi)$. Therefore, it is natural to study the action of $\mathrm{Aut}(\mathsf{C},\mathsf{L},\phi)$ on $\mathrm{Def}(\mathsf{C},\mathsf{L},\phi)$ by focusing on $\underline{\mathrm{Aut}}(\mathsf{C},\mathsf{L},\phi)=\mathrm{Aut}(\mathsf{C},\mathsf{L},\phi)/\boldsymbol{\mu}_\ell$.

The coarsening $\mathsf{s} \mapsto s$ induces a group homomorphism

$$\mathfrak{coarse}\colon \underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi) \to \mathrm{Aut}(C).$$

Its kernel and image are natural geometric objects of independent interest. We denote them by $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$ and $\mathrm{Aut}'(C)$ and we refer to them as the group of *ghost automorphisms* and the group of *automorphisms of C lifting to* $(\mathsf{C}, \mathsf{L}, \phi)$:

$$1 \to \underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi) \to \underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi) \to \mathrm{Aut}'(C) \to 1. \tag{20}$$

2.1.4. Ghosts automorphisms. The kernel of $\mathfrak{coarse}$ is the group of ghosts automorphisms: automorphisms s of C fixing at the same time the underlying curve C and the isomorphism class of the overlying line bundle L; we write

$$\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi) := \ker(\mathfrak{coarse}).$$

It is worth pointing out that an automorphism of a stack X may well be nontrivial and, at the same time, operate as the identity on the coarse space X. In our case, stabilisers are isolated and we may treat this issue locally. Consider $\mathsf{U} = [\{xy = 0\}/\boldsymbol{\mu}_r]$ the quotient stack where ξ_r acts on (x, y) as $(\xi_r x, \xi_r^{-1} y)$. All automorphisms $(x, y) \mapsto (\xi_r^b x, \xi_r^a y)$ induce the identity on the quotient space. The automorphisms fixing the coarsening U up to natural transformations (the 2-isomorphisms $(x, y) \mapsto (\xi_r^i x, \xi_r^{-i} y)$) form a group $\mathrm{Aut}_U(\mathsf{U}) \cong \boldsymbol{\mu}_r$ generated by $(x, y) \mapsto (\xi_r x, y)$. In this way, the automorphisms of a twisted curve C with order-r stabilisers at k nodes which fix C are freely generated by k automorphisms, each one operating as $(x, y) \mapsto (\xi_r x, y)$ at a node [1, §7]. Note that no branch has been privileged: via the natural transformation $(x, y) \mapsto (\xi_r x, \xi_r^{-1} y)$, the automorphism $(x, y) \mapsto (\xi_r x, y)$ is 2-isomorphic to $(x, y) \mapsto (x, \xi_r y)$.

This explains the canonical identification from [1, §7, Prop. 7.1.1],

$$\mathrm{Aut}_C(\mathsf{C}) = \bigoplus_{e \in E} \boldsymbol{\mu}_{r(e)}. \tag{21}$$

We notice that, throughout the paper, we adopt for clarity the additive notation for sums and direct sums (e.g. we write $\bigoplus_{e\in E} \boldsymbol{\mu}_{r(e)}$, $(\boldsymbol{\mu}_\ell)^{\oplus \# V}$, and, where sums over a set I of indices are not direct, we use the symbol $\sum_{i \in I}$)

The summand labelled by e on the right hand side corresponds to $\mathrm{Aut}_{\mathsf{C}\setminus\{\mathsf{n}_e\}}(\mathsf{C})$, the subgroup of automorphisms of C operating as the identity off the node n_e attached to e. The action of $\mathrm{Aut}_{\mathsf{C}\setminus\{\mathsf{n}_e\}}(\mathsf{C})$ on $\mathrm{Def}(\mathsf{C})/\mathrm{Def}(\mathsf{C}, \mathrm{Sing}(\mathsf{C})) = \bigoplus_{e \in E} K_e$ (see in (2.1.2)) coincides with the natural action of $\boldsymbol{\mu}_{r(e)}$ on the one-dimensional term K_e: the character 1 in $\mathrm{Hom}(\boldsymbol{\mu}_{r(e)}, \mathbb{G}_{\mathrm{m}}) = \mathbb{Z}/r(e)$.

2.1.5. Automorphisms of C lifting to $(\mathsf{C}, \mathsf{L}, \phi)$. The image of $\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi)$ via $\mathfrak{coarse}$ is the group of automorphisms s of C which can be obtained as the coarsening of a morphism s of C satisfying $\mathsf{s}^*\mathsf{L} \cong \mathsf{L}$. Clearly, this group differs in general from $\mathrm{Aut}(C)$; notice for instance that automorphisms of the coarse curve C that do not preserve the order of the overlying stabiliser of C cannot be lifted to C. More precisely, we have the obvious inclusion

$$\mathrm{Aut}'(C) := \mathrm{im}(\mathfrak{coarse}) \subseteq \{s \in \mathrm{Aut}(C) \mid s_\Gamma^* M = M\}$$

where s_Γ is the dual graph automorphism induced by s. The condition $s_\Gamma^* M = M$ is restrictive in general (it is not, of course, when M vanishes), but it does not guarantee the existence of an automorphism s lifting s. For a simple counterexample, consider a point of the divisor $\Delta_{g/2}$ from §1.5 lying over the isomorphism class in $\Delta_{g/2}^{\mathrm{stable}}$ of two isomorphic 1-pointed genus-$g/2$ curves meeting transversely at their marked point; here the involution of the underlying stable curve respects the multiplicity cochain, but does not lift to the level structure. We also point out that in general, even when a lift s exists, there may well be no canonical choice for s. Lifting a morphism that maps a $\mathsf{B}\boldsymbol{\mu}_k$-node to another $\mathsf{B}\boldsymbol{\mu}_k$-node amounts to extracting a kth root of the identifications between local parameters on both branches (there may be no distinguished choice, although all choices can be identified via a ghost isomorphism, up to natural transformation).

Example 2.1. We conclude this subsection with the study of automorphisms of the genus-1 curve $\mathsf{E} = [\widetilde{E}/\boldsymbol{\mu}_\ell]$, a stack quotient of a nodal cubic $\mathbb{P}^1/(0 \sim \infty)$, from Example 1.15. Although the group of automorphisms of E and of $E = \widetilde{E}/\boldsymbol{\mu}_\ell$ is not finite (E is not stable), the study of this case is relevant to the study of level curves over a stable curve containing, as a subcurve, a copy of E meeting the rest of the curve in one separating node n (the orbit $\boldsymbol{\mu}_\ell \cdot 1$) with trivial stabiliser by Proposition 1.11. To this end, it is crucial to study the finite group of automorphisms of E that fix n,

$$\mathrm{Aut}(\mathsf{E}, n) = \{\mathsf{s} \in \mathrm{Aut}(E) \mid \mathsf{s}(n) = n\}.$$

The exact sequence $1 \to \mathrm{Aut}_E(\mathsf{E}, n) \to \mathrm{Aut}(\mathsf{E}, n) \to \mathrm{Aut}(E, n)$ reads

$$1 \to \boldsymbol{\mu}_\ell \to \mathrm{Aut}(\mathsf{E}, n) \xrightarrow{\mathfrak{coarse}} \boldsymbol{\mu}_2.$$

After choosing ξ_ℓ, $\boldsymbol{\mu}_\ell$ is generated by the automorphism g with coarsening $g = \mathrm{id}$ and local picture $(x, y) \mapsto (\xi_\ell x, y)$ at the node. On the other hand, $\boldsymbol{\mu}_2$ is generated by the unique involution i fixing n and the node, and interchanging the branches at the node. In this special case, $\mathfrak{coarse}$ is surjective and the involution i admits a distinguished lift $\mathsf{i} \in \mathrm{Aut}[\widetilde{E}/\boldsymbol{\mu}_\ell]$ as follows. At the level of $\widetilde{E}$, consider the unique involution of $\widetilde{E}$ fixing the node of $\widetilde{E}$ and the point 1 and exchanging the branches of the node. At the level of the group $\boldsymbol{\mu}_\ell$, consider the passage to the inverse. We obtain $\mathsf{i}\colon [\widetilde{E}/\boldsymbol{\mu}_\ell] \to [\widetilde{E}/\boldsymbol{\mu}_\ell]$ and we have the short exact sequence[2]

$$0 \to \boldsymbol{\mu}_\ell \to \mathrm{Aut}(\mathsf{E}, n) \to \boldsymbol{\mu}_2 \to 0.$$

[2] One can observe explicitly that $\mathrm{Aut}(\mathsf{E}, n)$ is the direct product $\boldsymbol{\mu}_\ell \times \boldsymbol{\mu}_2$, i.e. the involution i commutes with the ghost g defined locally at the node as $(x, y) \mapsto (\xi_\ell x, y)$. We only need to check $\mathsf{g} \circ \mathsf{i} = \mathsf{i} \circ \mathsf{g}$ in the local picture $[\{xy = 0\}/\boldsymbol{\mu}_\ell]$ at the node of $[\widetilde{E}/\boldsymbol{\mu}_\ell]$. There, the morphism i may be described as the map interchanging the branches $(x, y) \mapsto (y, x)$ and $\mathsf{i} \circ \mathsf{g}\colon (x, y) \mapsto (\xi_\ell y, x)$ equals $\mathsf{g} \circ \mathsf{i}\colon (x, y) \mapsto (y, \xi_\ell x)$ up to the natural transformation $(x, y) \mapsto (\xi_\ell x, \xi_\ell^{-1} y)$.

We now set $\ell = 2$ and consider the automorphisms of an explicitly defined level-2 curve. Let C be a twisted curve, the union of $\mathsf{E} = [\widetilde{E}/\boldsymbol{\mu}_2]$ with a smooth $(g-1)$-curve X with $\operatorname{Aut}(X) = \{\mathrm{id}_X\}$. The curves E and X meet transversely at n and the coarse spaces form a genus-g stable curve C. Therefore, by construction, the above short exact sequence reads $0 \to \operatorname{Aut}_C(\mathsf{C}) \to \operatorname{Aut}(\mathsf{C}) \to \operatorname{Aut}(C) \to 0$. Let $(\mathsf{C} = X \cup \mathsf{E}, \mathsf{L} = \mathcal{O} \cup (\mathsf{L}_{\mathrm{ram}} \otimes \mathsf{L}_{\mathrm{\acute{e}t}}), \mathsf{s})$ be the unique level-2 curve obtained by glueing over n the fibre of $\mathcal{O}_X$ and that of $\mathsf{L}_{\mathrm{ram}} \otimes \mathsf{L}_{\mathrm{\acute{e}t}}$ from Example 1.15. By construction i^* operates trivially on both $\mathsf{L}_{\mathrm{ram}}$ and $\mathsf{L}_{\mathrm{\acute{e}t}}$; therefore in this example $\underline{\operatorname{Aut}}(\mathsf{C}, \mathsf{L}, \mathsf{s}) \to \operatorname{Aut}(C)$ is surjective and $\operatorname{Aut}'(C) = \operatorname{Aut}(C)$. On the other hand, g^* acts trivially on $\mathsf{L}_{\mathrm{\acute{e}t}}$ but nontrivially on $\mathsf{L}_{\mathrm{ram}}$:

$$\mathsf{g}^*\mathsf{L}_{\mathrm{ram}} = \mathsf{L}_{\mathrm{ram}} \otimes \mathsf{L}_{\mathrm{\acute{e}t}}$$

(this relation can be shown directly, but we refer to (29) for a general rule). Notice that, in fact, there is a second level-2 curve $(\mathsf{C}, \mathsf{L}_0 = \mathcal{O} \cup \mathsf{L}_{\mathrm{ram}}, \mathsf{s}_0)$ which is isomorphic to $(\mathsf{C}, \mathsf{L}, \mathsf{s})$ via g^*, but $\mathsf{L}_0 \ncong \mathsf{L}$.

We deduce that $\underline{\operatorname{Aut}}_C(\mathsf{C}, \mathsf{L}, \mathsf{s})$, in the example $(\mathsf{C}, \mathsf{L}, \mathsf{s})$ given above, is trivial: there are no ghost automorphisms. This is a consequence of the more general No-Ghosts Lemma 2.10. The sequence (20) reads $0 \to 0 \to \boldsymbol{\mu}_2 \to \boldsymbol{\mu}_2 \to 0$ and $\underline{\operatorname{Aut}}(\mathsf{C}, \mathsf{L}, \mathsf{s}) = \boldsymbol{\mu}_2$ operates nontrivially only on the parameter $\tau_n = t_n$ appearing in (18) and corresponding to the family smoothing the node n (the local picture is $\tau_n \mapsto -\tau_n$ because i operates trivially on the y-branch lying on X and operates by a change of sign on the x-branch lying on the component E, and τ_n equals xy). In other words, i fixes a hyperplane of $\operatorname{Def}(\mathsf{C}, \mathsf{L}, \mathsf{s})$, i.e. i is a quasireflection.

2.2. *Dual graph and ghost automorphisms when the level is prime*

Only for this section the index ℓ is assumed to be prime. Ghost automorphisms of the level curve $(\mathsf{C}, \mathsf{L}, \phi)$ can be described in terms of the dual graph Γ of C.

2.2.1. Setup. Consider the characteristic function $\nu = \nu_M$ of the support of the multiplicity M of $(\mathsf{C}, \mathsf{L}, \phi)$ and the corresponding contraction $\Gamma \to \Gamma(\nu)$ (the condition $\nu > 0$, or $\nu = \infty$, holds if and only if $M = 0$ and singles out contracted edges, see (10)). Recall $(C^\bullet_\nu(\Gamma; \boldsymbol{\mu}_\ell), \delta)$:

$$C^0_\nu(\Gamma; \boldsymbol{\mu}_\ell) = \{a \colon V \to \boldsymbol{\mu}_\ell \mid a(e_+) = a(e_-) \text{ if } \nu(e) > 0\}, \tag{22}$$

$$C^1_\nu(\Gamma; \boldsymbol{\mu}_\ell) = \{b \colon \mathbb{E} \to \boldsymbol{\mu}_\ell \mid b(\overline{e}) = b(e)^{-1}, \text{ and } b(e) = 1 \text{ if } \nu(e) > 0\}. \tag{23}$$

By (13) we have the following identification via $\mathcal{B}$:

$$\operatorname{im}\bigl(\delta \colon C^0(\Gamma(\nu); \boldsymbol{\mu}_\ell) \to C^1(\Gamma(\nu); \boldsymbol{\mu}_\ell)\bigr) \cong C^1_\nu(\Gamma; \boldsymbol{\mu}_\ell) \cap \operatorname{im}\delta.$$

2.2.2. Automorphisms of C *via* Γ *and* ν*.* It is natural to define the group of symmetric $\boldsymbol{\mu}_\ell$-valued functions vanishing on the set of edges with zero multiplicity,

$$S_\nu(\Gamma; \boldsymbol{\mu}_\ell) = \{b \colon \mathbb{E} \to \boldsymbol{\mu}_\ell \mid b(\overline{e}) = b(e), \text{ and } b(e) = 1 \text{ for } \nu(e) > 0\},$$

canonically isomorphic to $\bigoplus_{e|\nu(e)>0}\boldsymbol{\mu}_\ell$. As mentioned in (21), the group $\mathrm{Aut}_C(\mathsf{C})$ is easy to describe by [1, §7]. For ℓ prime, there is a canonical isomorphism

$$\mathrm{Aut}_C(\mathsf{C}) = S_\nu(\Gamma;\boldsymbol{\mu}_\ell), \tag{24}$$

where $\mathsf{a}\colon e \mapsto \mathsf{a}(e) \in \boldsymbol{\mu}_\ell$ corresponds to $\mathsf{a} \in \mathrm{Aut}_C(\mathsf{C})$ acting at the node attached to $e \in E$ as

$$(x, y) \mapsto (\mathsf{a}(e)x, y) \equiv (x, \mathsf{a}(e)y). \tag{25}$$

2.2.3. Ghost automorphisms via Γ *and* ν. We characterise ghost automorphisms of the level structure $(\mathsf{C},\mathsf{L},\phi)$. To begin, recall that ℓ is prime, M is the multiplicity of $(\mathsf{C},\mathsf{L},\phi)$, and ν is equal to ∞ where M vanishes and to 0 elsewhere (see (10)). Via $\mathbb{Z}/\ell = \mathrm{Hom}(\boldsymbol{\mu}_\ell,\mathbb{G}_{\mathrm{m}})$, we have the product

$$S_\nu(\Gamma;\boldsymbol{\mu}_\ell)\times C^1_\nu(\Gamma;\mathbb{Z}/\ell)\to C^1_\nu(\Gamma;\boldsymbol{\mu}_\ell) \qquad (\mathsf{a}, f)\mapsto \mathsf{a}\odot f := f(\mathsf{a}). \tag{26}$$

Indeed, since a takes values in $\boldsymbol{\mu}_\ell$, and f in $\mathbb{Z}/\ell = \mathrm{Hom}(\boldsymbol{\mu}_\ell,\mathbb{G}_{\mathrm{m}})$, we express the result of the action of the automorphism a on f by $\mathsf{a}\odot f := f(\mathsf{a})$, i.e. by the evaluation at each edge e of the homomorphism $f(e)$ at $\mathsf{a}(e)$. This could be stated more explicitly: within $\boldsymbol{\mu}_\ell$ we have $(\mathsf{a}\odot f)(e) = \mathsf{a}(e)^{f(e)}$. The notation $\mathsf{a}\odot f$ emphasises that a operates on f and becomes convenient once we fix isomorphisms $\boldsymbol{\mu}_r \cong \mathbb{Z}/r$ in the last part of the paper (see Assumption 2.34); then $\mathsf{a}\odot f$ is actually a product in $\mathbb{Z}/\ell$ (see (44)).

Since M lies in $C^1_\nu(\Gamma;\mathbb{Z}/\ell)$ we get the isomorphisms

$$M\colon S_\nu(\Gamma;\boldsymbol{\mu}_\ell)\to C^1_\nu(\Gamma;\boldsymbol{\mu}_\ell) \quad\text{and}\quad M^{-1}\colon C^1_\nu(\Gamma;\boldsymbol{\mu}_\ell)\to S_\nu(\Gamma;\boldsymbol{\mu}_\ell) \tag{27}$$

mapping $\mathsf{a} \in S_\nu(\Gamma;\boldsymbol{\mu}_\ell)$ to $\mathsf{a}\odot M$, and, conversely, the 1-cochain $b\colon e \mapsto b(e)$ of $C_\nu(\Gamma;\boldsymbol{\mu}_\ell)$ to the symmetric function $\mathsf{a} = M^{-1}b$,

$$\mathsf{a}\colon e\mapsto \begin{cases}[M(e)^{-1}]_\ell(b(e)) = \mathsf{a}(e) & \text{if } M(e)\neq 0,\\ 1 & \text{if } M(e)=0,\end{cases} \tag{28}$$

(where $[M(e)^{-1}]_\ell$ is the inverse of $M(e)$ in $\mathbb{Z}/\ell$ and is regarded as an invertible homomorphism applied to $b(e)\in\boldsymbol{\mu}_\ell$; again, this turns into a product under Assumption 2.34).

Now, for any $\mathsf{a}\in \mathrm{Aut}_C(\mathsf{C}) = S_\nu(\Gamma;\mathbb{Z}/\ell)$, we have (see [8, Prop. 2.18])

$$\mathsf{a}^*\mathsf{L}\cong \mathsf{L}\otimes\tau(\mathsf{a}\odot M), \tag{29}$$

where τ is the homomorphism defined in §1.4.3 associating to a $\boldsymbol{\mu}_\ell$-valued 1-cochain the line bundle with the corresponding descent data. For completeness, we recall here the argument proving the above identity. Let us write $\{xy=0\}$ for the local picture at a chosen node attached to the oriented edge e (as already observed, the choice of the notation (x, y) yields $e\in\mathbb{E}$). Then, consider the pullback via the automorphism $\mathsf{a}\colon (x, y)\mapsto(\xi_\ell x, y)$ of the line bundle L defined by the action $\xi_\ell\cdot(x,y,t) = (\xi_\ell x, \xi_\ell^{-1}y, \xi_\ell t)$ on $\{xy=0\}\times\mathbb{A}^1$ locally at the chosen node and trivial elsewhere. This definition of L makes sense because the quotient is canonically trivialised off the node by the invariant sections xt^{-1} on one branch and by yt on the other branch. Pulling back via a changes the trivialisation only at one branch; in other words, by (9), it is equivalent to tensoring by $\tau(\mathsf{a}\odot M)$.

The above statement implies (via (9)) that a is a ghost if and only if $\mathsf{a}\odot M$ lies in $\ker\tau = \operatorname{im}\delta$. This completely justifies the following notation.

Definition 2.2. Set $G_\nu(\Gamma;\boldsymbol{\mu}_\ell) = C^1_\nu(\Gamma;\boldsymbol{\mu}_\ell)\cap \operatorname{im}\delta$.

Remark 2.3. Via the contraction $\Gamma\to\Gamma(\nu)$ and (13), we get the alternative presentation

$$G_\nu(\Gamma;\boldsymbol{\mu}_\ell) = \operatorname{im}\big(\delta\colon C^0(\Gamma(\nu);\boldsymbol{\mu}_\ell)\to C^1(\Gamma(\nu);\boldsymbol{\mu}_\ell)\big) \tag{30}$$

yielding the isomorphism $G_\nu(\Gamma;\mathbb{Z}/\ell) = (\boldsymbol{\mu}_\ell)^{\oplus(\#V(\nu)-1)}$.

Proposition 2.4. *For ℓ prime, let $(\mathsf{C},\mathsf{L},\phi)$ be a level-ℓ curve. We have a canonical identification*

$$\underline{\operatorname{Aut}}_C(\mathsf{C},\mathsf{L},\phi)\cong G_\nu(\Gamma;\boldsymbol{\mu}_\ell).$$

A 1-cochain $b\colon e\mapsto b(e)$ of $G_\nu(\Gamma;\mathbb{Z}/\ell)$ corresponds to the symmetric function $M^{-1}b$. □

Remark 2.5. As an easy consequence of the above analysis, a ghost automorphism $\mathsf{a}\in\underline{\operatorname{Aut}}_C(\mathsf{C},\mathsf{L},\phi)$ fixes every irreducible component $\mathsf{Z}\subseteq\mathsf{C}$. Indeed, the restriction of a may operate nontrivially only at the nodes of Z. These are represented by loops in the dual graph. Indeed $G_\nu(\Gamma;\boldsymbol{\mu}_\ell)$ is not supported on the loops (cuts are supported off the loops).

Example 2.6. Assume $\ell = 3$. Consider the case where the dual graph is formed by a single circuit K consisting of n edges. Then $\ker\partial\cong\mathbb{Z}/3=\langle K\rangle$. There are two possibilities: $M=0$, where $\underline{\operatorname{Aut}}_C(\mathsf{C},\mathsf{L},\phi)=1$, and $M\neq 0$, where $C^1_\nu(\Gamma;\boldsymbol{\mu}_3)=C^1(\Gamma;\boldsymbol{\mu}_3)$ and the group of ghosts $G_\nu(\Gamma;\boldsymbol{\mu}_\ell)$ is isomorphic to $\operatorname{im}\delta\cong(\boldsymbol{\mu}_\ell)^{\oplus(\#V-1)}=(\boldsymbol{\mu}_3)^{\oplus(n-1)}$. The elements of $\underline{\operatorname{Aut}}_C(\mathsf{C},\mathsf{L},\phi)$ are the functions $\mathsf{a}\in S_\nu(\Gamma;\boldsymbol{\mu}_3)$ such that $\mathsf{a}\odot M(K)=1$ (see (7)).

(i) Assume $n=3$. In this case M lies in $\operatorname{im}\delta$ and we get an element of $G_\nu(\Gamma;\boldsymbol{\mu}_3)$ by taking $\mathsf{a}\colon\mathbb{E}\to\boldsymbol{\mu}_3$ constant. To fix ideas, fix a primitive third root of unity ξ_3 and set a constant and equal to ξ_3. Then a is a ghost operating as $(x,y)\mapsto(\xi_3x,y)$ at all nodes and acting on $\operatorname{Def}(\mathsf{C},\mathsf{L},\phi)$ as $(\xi_3\mathbb{I}_3)\oplus\operatorname{id}$ (see (18)). This argument holds in general whenever M is in $\operatorname{im}\delta_{\mathbb{Z}/\ell}$; then, for $a(e)=\zeta$ for all e, we find that $\mathsf{a}\odot M(e)=\zeta^{M(e)}$ for all e, and by (7) the $\mathbb{Z}/\ell$-valued 1-cochain $M\in\operatorname{im}\delta_{\mathbb{Z}/\ell}$ yields a $\boldsymbol{\mu}_\ell$-valued 1-cochain $\mathsf{a}\odot M\in\operatorname{im}\delta_{\boldsymbol{\mu}_\ell}$.

(ii) Assume $n=2$, and let e_1 and e_2 be the two edges. Here $M=0$ or $M\notin\operatorname{im}\delta$. Again by choosing ξ_3, we define a symmetric function $\mathsf{a}\colon\mathbb{E}\to\mathbb{Z}/\ell$ mapping one edge to ξ_3 ($e_1,\overline{e}_1\mapsto\xi_3$) and the other to its inverse ξ_3^2 ($e_2,\overline{e}_2\mapsto\xi_3^2$); then $\mathsf{a}\odot M$ is a cut, lies in $\operatorname{im}\delta$ and acts on (18) as $\operatorname{Diag}(\xi_3,\xi_3^2)\oplus\operatorname{id}$.

(iii) If the circuit has a single edge, then $\operatorname{im}\delta=(0)$. There are no nontrivial ghosts.

Example 2.7. The argument at point (iii) shows that the level structures $\mathcal{O}\cup(\mathsf{L}_{\mathrm{ram}}\otimes\mathsf{L}_{\mathrm{\acute{e}t}})$ and $\mathcal{O}\cup\mathsf{L}_{\mathrm{ram}}$ introduced in Example 2.1 have no nontrivial ghosts. Indeed, the dual graph in that case has two vertices v_X and v_{E} corresponding to X and E, one edge e_n connecting them and corresponding to the node n, and a second edge e_{loop} with both ends on v_{E}. The multiplicity is supported on this last vertex, and the vertex set $V(\nu)$ of the graph $\Gamma(\nu)$ obtained by contracting all edges with vanishing multiplicity reduces to a single vertex. We have $\operatorname{im}(\delta\colon C^0(\Gamma(\nu);\boldsymbol{\mu}_\ell) \to C^1(\Gamma(\nu);\boldsymbol{\mu}_\ell)) \cong (\boldsymbol{\mu}_\ell)^{\oplus(\#V(\nu)-1)} = 1$. Notice that this argument holds for any tree-like graph (that is, a graph that becomes a tree once the loops are removed)—see Corollary 2.14.

Example 2.8. Consider a dual graph with two vertices v_1, v_2 and three edges, each of them linking the two vertices to each other. As a multiplicity cochain we choose $e \mapsto M(e)$ equal to 1 on the oriented edges of $\mathbb{E}$ oriented from v_1 to v_2. For $\ell = 3$, it is easy to check that M belongs to the kernel and is indeed the sum of two different two-edge circuits. The cochain M lies also in $\operatorname{im}\delta$ (it is the $\boldsymbol{\mu}_3$-valued cut attached to the proper nonempty subset $H = \{v_1\}$). Therefore a constant $\mathsf{a} \equiv \zeta \in \boldsymbol{\mu}_3 \in S_\nu(\Gamma;\boldsymbol{\mu}_3)$ satisfies $\mathsf{a} \odot M \in \operatorname{im}\delta$ and acts on $\operatorname{Def}(\mathsf{C},\mathsf{L},\phi)$ as $\zeta\mathbb{I}_3 \oplus \operatorname{id}$. (See also Example 2.6(i).)

2.3. The singular points of the moduli space

Notice that in all the above examples of ghost automorphisms $\mathsf{g} \in \underline{\operatorname{Aut}}(\mathsf{C},\mathsf{L},\phi)$, the fixed space $\{v \in \operatorname{Def}(\mathsf{C},\mathsf{L},\phi) \mid \mathsf{g}\cdot v = v\}$ is never a hyperplane. An automorphism of an affine space whose fixed space coincides with a hyperplane is called a *quasireflection*. A general property of nontrivial ghosts is that they never act as quasireflections. Let us recall that this is crucial for classifying singularities.

Fact 2.9. *The scheme-theoretic quotient* $\operatorname{Def}(\mathsf{C},\mathsf{L},\phi)/\operatorname{Aut}(\mathsf{C},\mathsf{L},\phi)$ *is smooth if and only if* $\operatorname{Aut}(\mathsf{C},\mathsf{L},\phi)$ *is spanned by elements acting as the identity or as quasireflections* (*see* [26]).

2.3.1. Nontrivial ghosts are not quasireflections. Here is a consequence of Proposition 2.4.

Lemma 2.10. *If* $\mathsf{a} \in \underline{\operatorname{Aut}}_C(\mathsf{C},\mathsf{L},\phi)$ *fixes a hyperplane of* $\operatorname{Def}(\mathsf{C},\mathsf{L},\phi)$, *then* $\mathsf{a} = \operatorname{id}_{\mathsf{C}}$.

As we argue in Remark 2.27, this lemma generalises word for word and straightforwardly to the case when ℓ is composite; so, we did not impose the condition on ℓ to be prime in the statement.

Proof of Lemma 2.10. Let b be a nontrivial ghost $G_\nu(\Gamma;\boldsymbol{\mu}_\ell)$, i.e. a 1-cochain b in $C^1_\nu(\Gamma;\boldsymbol{\mu}_\ell)$ lying in $\operatorname{im}\delta$. Then there exists a (nonseparating) edge e with $\nu(e) \neq \infty$ and $b(e) \neq 1$. In this case, there is a circuit K passing through e. Now, K satisfies $b(K) = 1$ by (7). Hence, the support of b contains an oriented edge e' which differs from e regardless of its orientation. Proposition 2.4 claims that the unique automorphism a such that $\mathsf{a} \odot M = b$ acts nontrivially on $\mathbb{A}^1_{\tau_e}$ and $\mathbb{A}^1_{\tau_{e'}}$. □

Remark 2.11 ($\underline{\operatorname{Aut}}(\mathsf{C},\mathsf{L},\phi)$ operates faithfully on $\operatorname{Def}(\mathsf{C},\mathsf{L},\phi)$). Under the assumption $g \geq 4$, any nontrivial automorphism $a \in \operatorname{Aut}(C)$ acts nontrivially on $\operatorname{Def}(C)$ (see [18]). Then the faithfulness of $\underline{\operatorname{Aut}}(\mathsf{C},\mathsf{L},\phi)$ follows from that of $\underline{\operatorname{Aut}}_C(\mathsf{C},\mathsf{L},\phi)$ and from the above lemma.

2.3.2. Elliptic tail involutions. In [18, Thm. 2, §2], Harris and Mumford prove that an automorphism $a \in \mathrm{Aut}(C)$ is a quasireflection of $\mathrm{Def}(C)$ if and only if a is an *elliptic tail involution* (ETI): the curve C contains a genus-1 subcurve E meeting the rest of the curve in a single point n, and a is the identity on $\overline{C \setminus E}$ and is the nontrivial canonical involution i of $\mathrm{Aut}(E, n)$. This involution is canonically identified both if E is elliptic or rational: it is the hyperelliptic involution in the first case, whereas in the second case it is the unique involution fixing the point n and the node of $E \cong \mathbb{P}^1/(0 \equiv \infty)$ and interchanging the branches of that node. We need to generalise the notion of ETI to twisted curves. Because all separating nodes of level-ℓ curves have trivial stabilisers, a genus-1 subcurve E meeting the rest of the curve in a single point n is either a scheme (E, n) or is isomorphic to the pointed stack-theoretic curve (E, n) of Example 1.15. In both cases, these tails are equipped with a canonical involution i.

Definition 2.12 (elliptic tail and ETI). Let $(\mathsf{C}, \mathsf{L}, \phi)$ be a level-ℓ twisted curve. An *elliptic tail* is a genus-1 subcurve E meeting the rest of the curve C in a single point. An *elliptic tail involution* of $(\mathsf{C}, \mathsf{L}, \phi)$ is an automorphism of C such that the restriction to $\overline{\mathsf{C} \setminus \mathsf{E}}$ is the identity and the restriction to E equals the canonical involution i and satisfies $\mathsf{i}^*(\mathsf{L}|_{\mathsf{E}}) = \mathsf{L}|_{\mathsf{E}}$.

Proposition 2.13. *Consider a stable genus-g level-ℓ curve with $g \geq 4$. An automorphism $\mathsf{s} \in \underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi)$ acts as a quasireflection on $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)$ if and only if it is an ETI.*

As we argue in Remark 2.27, this proposition also generalises immediately to the case of ℓ composite.

Proof of Proposition 2.13. Let s be an automorphism of $(\mathsf{C}, \mathsf{L}, \phi)$ acting as a quasireflection on $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)$. Then its coarsening s acts either as the identity or as a quasireflection on $\mathrm{Def}(C)$. We rule out $s = \mathrm{id}_C$: in this case s would be a ghost, and by Lemma 2.10 there is no ghost acting as quasireflection. Then, by [18], s operates as an ETI on C. If the elliptic tail is represented by a scheme, then s is an ETI (using Lemma 2.10 on $\overline{C \setminus \mathsf{E}}$). Otherwise, the elliptic tail is the curve E of Example 1.15 and we need to check that i is the only automorphism lifting the ETI i and operating as a quasireflection on $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)$. By (20) the remaining automorphisms are of the form $\mathsf{i} \circ \mathsf{g}^n$ with $\mathsf{g}^n \neq \mathrm{id}$ (using the notation of Example 1.15); due to Proposition 2.4, the automorphism g^n acts nontrivially on $\mathrm{Def}(\mathsf{C})$ and $\mathsf{i} \circ \mathsf{g}^n$ is not a quasireflection. □

2.3.3. No-Ghosts. By Remark 1.8 and Proposition 2.4, $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$ is trivial if and only if the multiplicity graph $\Gamma(\nu)$ has only one vertex. We call such graphs *bouquets*.

Corollary 2.14. *Let ℓ be prime. The group of ghost automorphisms $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$ is trivial if and only if $\Gamma(\nu)$ is a bouquet.* □

Combining Corollary 2.14 and Proposition 2.13 we get the following result.

Theorem 2.15. *Let ℓ be prime and assume $g \geq 4$. The following conditions are equivalent:*

(i) *The point of $\overline{\mathcal{R}}_{g,\ell}$ representing $(\mathsf{C}, \mathsf{L}, \phi)$ is smooth.*
(ii) *The group $\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi)$ is spanned by ETIs of C.*
(iii) *The graph $\Gamma(\nu)$ is a bouquet and $\mathrm{Aut}'(C)$ is spanned by ETIs of C.*

Proof. The point representing $(\mathsf{C}, \mathsf{L}, \phi)$ is smooth if and only $\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi)$ is generated by elements operating on $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)$ as the identity or as quasireflections. Nontrivial elements of $\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi)$ never operate as the identity (see Remark 2.11). By Proposition 2.13 elements operating as quasireflections are precisely the ETIs of C; hence (i)$\Leftrightarrow$(ii). Now (iii) implies $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi) = 1$ and $\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi) = \mathrm{Aut}'(C)$ generated by ETIs of C; we deduce (ii) because the ETIs generating $\mathrm{Aut}'(C)$ lift canonically to ETIs generating $\mathrm{Aut}(\mathsf{C}, \mathsf{L}, \phi)$. Conversely, (ii) holds only if there are no nontrivial ghosts ($\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi) = 1$) because any nontrivial composition of ETIs has a nontrivial coarsening. Hence, $\Gamma(\nu)$ is a bouquet, $\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi) = \mathrm{Aut}'(C)$, and the coarsening of the ETIs spanning $\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi)$ are the ETIs spanning $\mathrm{Aut}'(C)$. □

2.4. *Generalisation to the case of level curves of composite level*

The generalisation of the above statement requires a modification of the condition "$\Gamma(\nu)$ is a bouquet" in part (iii); we introduce a new set of contractions. We are grateful to Roland Bacher for several ideas that helped us a great deal in finding the correct setup for this section.

2.4.1. The truncated valuation of $\mathbb{Z}/p^n$. For any prime p we recall that the ring $\mathbb{Z}/p^n$ is a truncated valuation ring in the sense of [13, §1.1]. We recall the definition, which applies to any local ring R whose maximal ideal $\mathfrak{m}$ is generated by a nilpotent element. We set the valuation $\mathrm{val}_{\mathfrak{m}} \colon R \to \mathbb{Z} \cup \{\infty\}$, $x \mapsto \sup\{i \mid x \in \mathfrak{m}^i\}$, taking values in $\mathbb{Z} \cap [0, \mathrm{length}(R) - 1]$ on $R \setminus \{0\}$ and satisfying $\mathrm{val}_{\mathfrak{m}}(0) = \infty$ (if $R = \mathbb{Z}/p^n$, then $\mathfrak{m} = (p)$ and $\mathrm{length}(R) = n$).

2.4.2. The vector-valued function $\boldsymbol{\nu}$. Consider the prime factorisation of ℓ,

$$\ell = \prod_{p \mid \ell} p^{e_p},$$

where e_p is the p-adic valuation of ℓ. Then the following vector-valued function $\boldsymbol{\nu}_M$, or simply $\boldsymbol{\nu}$, encodes the truncated valuations of $M(e) \bmod p^{e_p}$ in $\mathbb{Z}/p^{e_p}$ for all $p \mid \ell$:

$$e \mapsto \boldsymbol{\nu}(e) = (\nu_p(e))_{p \mid \ell} \quad \text{where} \quad \nu_p(e) := \mathrm{val}_{(p)}(M(e) \bmod p^{e_p}). \tag{31}$$

Notice that when ℓ is prime, we recover the characteristic function ν_M of the support of M,

$$\mathrm{val}_{(p)}(M(e)) = \nu_M(e).$$

2.4.3. Contractions. For each $p \mid \ell$, the coordinate ν_p of $\boldsymbol{\nu} = (\nu_p)_{p|\ell}$ yields a filtration

$$\emptyset \subseteq \{\nu_p \geq e_p\}_E \subseteq \{\nu_p \geq e_p - 1\}_E \subseteq \cdots \subseteq \{\nu_p \geq k\}_E \\ \subseteq \cdots \subseteq \{\nu_p \geq 1\}_E \subseteq \{\nu_p \geq 0\}_E = E.$$

To each of the above edge subsets we can naturally associate a subgraph (the vertex set is formed by the heads and the tails of the chosen edges):

$$\emptyset \subseteq \Delta(\nu_p^{e_p}) \subseteq \Delta(\nu_p^{e_p-1}) \subseteq \cdots \subseteq \Delta(\nu_p^k) \subseteq \cdots \subseteq \Delta(\nu_p^1) \subseteq \Delta(\nu_p^0) = \Gamma. \tag{32}$$

The respective contractions $\Gamma(\nu_p^k)$ of $\{\nu_p \geq k\}_E$ fit in the sequence of contractions

$$\Gamma \to \Gamma(\nu_p^{e_p}) \to \Gamma(\nu_p^{e_p-1}) \to \cdots \to \Gamma(\nu_p^k) \to \cdots \to \Gamma(\nu_p^1) \to \Gamma(\nu_p^0), \tag{33}$$

where $\Gamma(\nu_p^0)$ is the null graph (Γ is connected). The sets of vertices $V(\nu_p^k)$ fit in

$$V \twoheadrightarrow V(\nu_p^{e_p}) \twoheadrightarrow V(\nu_p^{e_p-1}) \twoheadrightarrow \cdots \twoheadrightarrow V(\nu_p^k) \twoheadrightarrow \cdots \twoheadrightarrow V(\nu_p^1) \twoheadrightarrow V(\nu_p^0) = \{\bullet\}. \tag{34}$$

The sets of edges $E(\nu_p^k)$ are related by the reverse inclusions

$$E \supseteq E(\nu_p^{e_p}) \supseteq E(\nu_p^{e_p-1}) \supseteq \cdots \supseteq E(\nu_p^k) \supseteq \cdots \supseteq E(\nu_p^1) \supseteq E(\nu_p^0) = \emptyset. \tag{35}$$

In the introduction, for brevity, we used the notation Δ_p^k and Γ_p^k for the graphs $\Delta(\nu_p^k)$ and $\Gamma(\nu_p^k)$. Contracting $\{\nu_p \geq k\}_E$ makes sense for any k in $\mathbb{Z} \cup \{\infty\}$; for $k \geq e_p$ we get $\Gamma(\nu_p^k) = \Gamma(\nu_p^{e_p})$, for $k \leq 0$ we get the null graph $\Gamma(\nu_p^k) = \Gamma(\nu_p^0)$. For $k \in \{0, \ldots, e_p\}$, the following holds.

Definition 2.16 (the graph $\Gamma(\nu_p^k)$). For p prime dividing ℓ, and $k \in \{0, \ldots, e_p\}$, the map $\Gamma \to \Gamma(\nu_p^k)$ is given by contracting the edges e for which p^k divides $M(e) \in \mathbb{Z}/p^{e_p}$.

2.4.4. The subcomplex $C^\bullet_{\boldsymbol{\nu}}(\Gamma; \boldsymbol{\mu}_\ell)$. Let us point out that for

$$\overline{\nu}_p(e) = \min(e_p, \nu_p(e)),$$

we have $\gcd(M(e), \ell) = \prod_{p|\ell} p^{\overline{\nu}_p(e)} (= \ell/r(e))$. We systematically use the canonical morphisms

$$\bigoplus_{p|\ell} \boldsymbol{\mu}_{p^{e_p - \overline{\nu}_p(e)}} \underset{\subseteq}{\to} \bigoplus_{p|\ell} \boldsymbol{\mu}_{p^{e_p}} = \boldsymbol{\mu}_\ell, \tag{36}$$

where the term on the left hand side may be regarded, via a canonical identification, as $\boldsymbol{\mu}_{r(e)}$.

Now, we generalise the above mentioned subcomplex $C^\bullet_\nu(\Gamma; \boldsymbol{\mu}_\ell)$ (see (22) and (23)). Set

$$C^0_{\boldsymbol{\nu}}(\Gamma; \boldsymbol{\mu}_\ell) = \left\{ a \colon V \to \boldsymbol{\mu}_\ell \;\middle|\; a(e_+)(a(e_-))^{-1} \in \bigoplus_{p|\ell} \boldsymbol{\mu}_{p^{e_p - \overline{\nu}_p(e)}} = \boldsymbol{\mu}_{r(e)} \right\},$$

$$C^1_{\boldsymbol{\nu}}(\Gamma; \boldsymbol{\mu}_\ell) = \left\{ b \colon \mathbb{E} \to \boldsymbol{\mu}_\ell \;\middle|\; b(\overline{e}) = b(e)^{-1} \text{ and } b(e) \in \bigoplus_{p|\ell} \boldsymbol{\mu}_{p^{e_p - \overline{\nu}_p(e)}} = \boldsymbol{\mu}_{r(e)} \right\}.$$

By restricting δ we get the differential

$$C^0_{\boldsymbol{\nu}}(\Gamma;\boldsymbol{\mu}_\ell)\xrightarrow{\delta(\boldsymbol{\nu})}C^1_{\boldsymbol{\nu}}(\Gamma;\boldsymbol{\mu}_\ell);$$

and $(C^\bullet_{\boldsymbol{\nu}}(\Gamma;\boldsymbol{\mu}_\ell),\delta)$ is a subcomplex of $(C^\bullet(\Gamma;\boldsymbol{\mu}_\ell),\delta)$. Definition 2.2 extends word for word.

Definition 2.17. Set $G_{\boldsymbol{\nu}}(\Gamma;\boldsymbol{\mu}_\ell)=C^1_{\boldsymbol{\nu}}(\Gamma;\boldsymbol{\mu}_\ell)\cap\operatorname{im}\delta$.

By construction $G_{\boldsymbol{\nu}}(\Gamma;\boldsymbol{\mu}_\ell)$ equals $\operatorname{im}\delta(\boldsymbol{\nu})$ via the inclusion $C^1_{\boldsymbol{\nu}}(\Gamma;\boldsymbol{\mu}_\ell)\subseteq C^1(\Gamma;\boldsymbol{\mu}_\ell)$. The following theorem proves that, with this setup, $G_{\boldsymbol{\nu}}(\Gamma;\boldsymbol{\mu}_\ell)$ is again isomorphic to the group of ghost automorphisms. First, we introduce the generalised group of symmetric functions

$$S_{\boldsymbol{\nu}}(\Gamma;\boldsymbol{\mu}_\ell)=\left\{b\colon\mathbb{E}\to\boldsymbol{\mu}_\ell\;\middle|\;b(\overline{e})=b(e)\ \text{ and }\ b(e)\in\bigoplus_{p\mid\ell}\boldsymbol{\mu}_{p^{e_p-\overline{v}_p(e)}}=\boldsymbol{\mu}_{r(e)}\right\}.$$

Via $\mathbb{Z}/r(e)=\operatorname{Hom}(\boldsymbol{\mu}_{r(e)},\mathbb{G}_{\mathrm{m}})$, we have the product

$$S_{\boldsymbol{\nu}}(\Gamma;\boldsymbol{\mu}_\ell)\times C^1_{\boldsymbol{\nu}}(\Gamma;\mathbb{Z}/\ell)\to C^1_{\boldsymbol{\nu}}(\Gamma;\boldsymbol{\mu}_\ell),\qquad(\mathsf{a},f)\mapsto\mathsf{a}\odot f:=f(\mathsf{a}).$$

Again, since M lies by construction in $C^1_{\boldsymbol{\nu}}(\Gamma;\mathbb{Z}/\ell)$, we get the isomorphisms

$$M\colon S_{\boldsymbol{\nu}}(\Gamma;\boldsymbol{\mu}_\ell)\to C^1_{\boldsymbol{\nu}}(\Gamma;\boldsymbol{\mu}_\ell)\quad\text{and}\quad M^{-1}\colon C^1_{\boldsymbol{\nu}}(\Gamma;\boldsymbol{\mu}_\ell)\to S_{\boldsymbol{\nu}}(\Gamma;\boldsymbol{\mu}_\ell).\tag{37}$$

Here M maps the symmetric function $\mathsf{a}\colon e\mapsto\mathsf{a}(e)$ to the 1-cochain $\mathsf{a}\odot M$ given by applying at each edge e the homomorphism $m(e)\in\mathbb{Z}/r(e)=\operatorname{Hom}(\boldsymbol{\mu}_{r(e)},\mathbb{G}_{\mathrm{m}})$ to the $r(e)$th root $\mathsf{a}(e)$. Conversely, M^{-1} maps the 1-cochain $b\colon e\mapsto b(e)$ of $C_{\boldsymbol{\nu}}(\Gamma;\boldsymbol{\mu}_\ell)$ to the symmetric function $M^{-1}b=\mathsf{a}$, defined as

$$M^{-1}b\colon e\mapsto\begin{cases}[m(e)^{-1}]_{r(e)}(b(e)) & \text{if } M(e)\neq 0,\\ 1 & \text{if } M(e)=0,\end{cases}\tag{38}$$

where $[m(e)^{-1}]_{r(e)}$ is the inverse of $m(e)$ in $\mathbb{Z}/r(e)$.

Theorem 2.18. *Let $(\mathsf{C},\mathsf{L},\phi)$ be a level curve of level $\ell\in\mathbb{N}^\times$; write M for its multiplicity and $\boldsymbol{\nu}$ for the corresponding vector-valued function (31).*

(i) *There is a canonical isomorphism* $\operatorname{Aut}(\mathsf{C},\mathsf{L},\phi)=S_{\boldsymbol{\nu}}(\Gamma;\boldsymbol{\mu}_\ell)$. *The above local description of $\mathsf{a}\in S_{\boldsymbol{\nu}}(\Gamma;\boldsymbol{\mu}_\ell)$ holds without changes if we write a as a $\boldsymbol{\mu}_\ell$-valued function.*

(ii) *Let $\mathsf{a}\in S_{\boldsymbol{\nu}}(\Gamma;\boldsymbol{\mu}_\ell)$. Then $\mathsf{a}^*\mathsf{L}=\mathsf{L}\otimes\tau(\mathsf{a}\odot M)$ (using (29)).*

(iii) *We have*

$$\underline{\operatorname{Aut}}_C(\mathsf{C},\mathsf{L},\phi)\cong G_{\boldsymbol{\nu}}(\Gamma;\boldsymbol{\mu}_\ell).$$

The 1-cochain $b\in G_{\boldsymbol{\nu}}(\Gamma;\boldsymbol{\mu}_\ell)\subset C^1_{\boldsymbol{\nu}}(\Gamma;\boldsymbol{\mu}_\ell)$ identifies the ghost automorphism a corresponding to the symmetric function $M^{-1}b$ explicitly defined above.

Proof. Since $S_{\boldsymbol{\nu}}(\Gamma;\mathbb{Z}/\ell) = \bigoplus_{e\in E}\boldsymbol{\mu}_{r(e)}$, we recover the group $\mathrm{Aut}_C(\mathsf{C})$ of [1, §7, Prop. 7.1.1]. Point (ii) yields (iii) immediately and is a direct consequence of [8, Prop. 2.18] as before. □

Remark 2.19. If we work with a fixed primitive ℓth root of unity, then giving $\mathsf{a}(e)\in\boldsymbol{\mu}_{r(e)}$ amounts to specifying a multiple $\widetilde{\mathsf{a}}(e)$ of $\ell/r(e)$ modulo ℓ and the ghost a operates on $\mathrm{Def}(\mathsf{C},\mathsf{L},\phi)$ as

$$\Big\{\bigoplus_e(\tau_e\mapsto\xi_\ell^{\widetilde{\mathsf{a}}(e)}\tau_e)\Big\}\oplus\mathrm{id}.$$

This follows from the analysis of the quotient $\mathrm{Def}(\mathsf{C},\mathsf{L},\phi)/\mathrm{Def}(\mathsf{C},\mathsf{L},\phi,\mathrm{Sing}(\mathsf{C}))$ carried out in §2.1.2 and of the action of the ghosts on it (see §2.1.4).

Remark 2.20. Every ghost restricts to the identity on the irreducible components of C.

2.4.5. Computing the group $G_{\boldsymbol{\nu}}(\Gamma;\boldsymbol{\mu}_\ell)$. When ℓ is a prime integer, the group of ghosts is a free $\boldsymbol{\mu}_\ell$-module and Remark 2.3 allowed us to compute its rank over $\boldsymbol{\mu}_\ell$: the number of vertices of the contracted graph minus 1. In general, when ℓ is composite, the group of ghosts is not free over $\boldsymbol{\mu}_\ell$. By generalising Remark 2.3, we provide an explicit formula for its elementary divisors.

Remark 2.21. Once an orientation $E\to\mathbb{E}$ is specified, $C^1_{\boldsymbol{\nu}}(\Gamma;\boldsymbol{\mu}_\ell)$ may be written as

$$\bigoplus_{e\in E}\bigoplus_{p\mid\ell}\boldsymbol{\mu}_{p^{e_p-\overline{\nu}_p(e)}}.$$

We may invert the order of the direct sums and rewrite the summands as usual,

$$\bigoplus_{p\mid\ell}\Big(\bigoplus_{e\in E}\boldsymbol{\mu}_{p^{e_p}-p^{\overline{\nu}_p(e)}}\Big).$$

The summands of the first direct sum (over the prime divisors p) equal the (nondirect) sums of subgroups

$$\sum_{1\le k\le e_p}\mathcal{B}C^1(\Gamma(\nu_p^k);\boldsymbol{\mu}_{p^{e_p-k+1}})\subseteq C^1(\Gamma;\boldsymbol{\mu}_\ell).$$

We deduce from this characterisation the following identity which does not involve any fixed orientation $E\to\mathbb{E}$ and holds for both 1-cochains and 0-cochains:

$$C^i_{\boldsymbol{\nu}}(\Gamma;\boldsymbol{\mu}_\ell)=\bigoplus_{p\mid\ell}\sum_{1\le k\le e_p}\mathcal{B}C^i(\Gamma(\nu_p^k);\boldsymbol{\mu}_{p^{e_p-k+1}})\qquad i=0,1. \tag{39}$$

Moreover, we immediately get an explicit computation of the groups $C^1_{\boldsymbol{\nu}}(\Gamma;\boldsymbol{\mu}_\ell)$: because $\mathcal{B}C^0(\Gamma(\nu);\boldsymbol{\mu}_{p^h})\cong(\boldsymbol{\mu}_{p^h})^{\oplus\#V(\nu)}$ and $\mathcal{B}C^1(\Gamma(\nu);\boldsymbol{\mu}_{p^h})\cong(\boldsymbol{\mu}_{p^h})^{\oplus\#E(\nu)}$, we have

$$C^i_{\boldsymbol{\nu}}(\Gamma;\boldsymbol{\mu}_\ell)\cong\bigoplus_{p\mid\ell}\bigoplus_{k=1}^{e_p}(\boldsymbol{\mu}_{p^k})^{\oplus\eta^i(\nu_p^k)},$$

where, using the Kronecker delta, we can compute $\eta^i(\nu_p^k)$ from (34) and (35):

$$\eta^i(\nu_p^k) := \begin{cases} \#V(\nu_p^{e_p-k+1}) - \delta_{k,e_p}\#V(\nu_p^{e_p-k}), & i=0, \\ \#E(\nu_p^{e_p-k+1}) - \delta_{k,e_p}\#E(\nu_p^{e_p-k}), & i=1. \end{cases}$$

The following lemma, embodying the corollary stated in the introduction, follows.

Lemma 2.22. *We have*

$$G_{\boldsymbol{\nu}}(\Gamma; \boldsymbol{\mu}_\ell) = \bigoplus_{p|\ell} \sum_{1\le k\le e_p} \operatorname{im}\delta(\nu_p^k),$$

where $\delta(\nu_p^k)$ *is* $C^0(\Gamma(\nu_p^k); \boldsymbol{\mu}_{p^{e_p-k+1}}) \to C^1(\Gamma(\nu_p^k); \boldsymbol{\mu}_{p^{e_p-k+1}})$. *More explicitly, set*

$$\alpha_p^k := \#V(\nu_p^{e_p-k+1}) - \#V(\nu_p^{e_p-k});$$

then the group $G_{\boldsymbol{\nu}}(\Gamma; \boldsymbol{\mu}_\ell)$ *decomposes as*

$$G_{\boldsymbol{\nu}}(\Gamma; \mathbb{Z}/\ell) \cong \bigoplus_{p|\ell} \bigoplus_{k=1}^{e_p} (\boldsymbol{\mu}_{p^k})^{\oplus \alpha_p^k}$$

and has order $\frac{1}{\ell}\prod_{p|\ell} p^{\#V_p}$ *for* $V_p = \bigsqcup_{k=1}^{e_p} V(\nu_p^k)$. □

Remark 2.23. For ℓ prime, we recover (6): (34) reads $V(\nu) \to \{\bullet\}$, and we find that $\alpha_p^1 = \#V(\nu) - 1$. (Note that the Kronecker delta does not occur in the formula for the elementary divisors α_p^k.)

Example 2.24. We consider the dual graph Γ in Figure 2 of a level-8 curve. The multiplicities assigned to each oriented edge define a cocycle $M \in \ker\partial$. Here, 2 is the only prime divisor of ℓ. In Figure 3, we write, next to each edge e, the value of $\nu_2(e)$. Then, in Figure 4, we show the corresponding contractions. We observe that, in this case, at each step the number of vertices decreases by 1. Therefore, by Lemma 2.22, we compute $\alpha_2^k = 1$ for $k = 1, 2, 3$. We finally obtain 64 ghosts

$$G_{\boldsymbol{\nu}}(\Gamma; \boldsymbol{\mu}_8) \cong \boldsymbol{\mu}_2 \oplus \boldsymbol{\mu}_4 \oplus \boldsymbol{\mu}_8$$

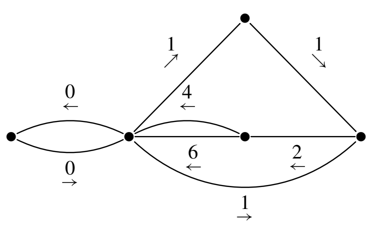


Fig. 2. A dual graph Γ of a level-8 curve with multiplicities.

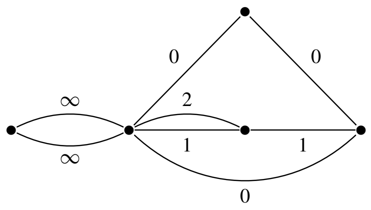


Fig. 3. The truncated valuation associated to (2) in $\mathbb{Z}/8$.

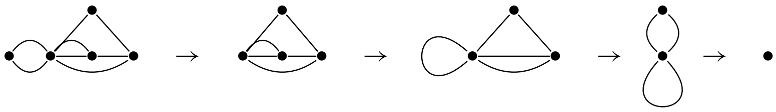


Fig. 4. The contractions $\Gamma \to \Gamma(\nu_2^3) \to \Gamma(\nu_2^2) \to \Gamma(\nu_2^1) \to \bullet$; we have $\#V_3 = 9$.

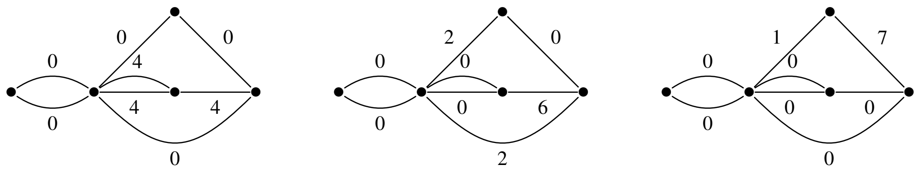


Fig. 5. Three generators of order 2, 4 and 8 of the group of ghosts.

that can be spanned by ghosts of order 2, 4 and 8 corresponding to the $\boldsymbol{\mu}_8$-valued symmetric functions in $S_{\boldsymbol{\nu}}(\Gamma, \boldsymbol{\mu}_8)$ displayed in Fig. 5 (in order to simplify the notation we specify 8th roots of unity with respect to a chosen primitive root ξ_8 of unity: we write integers mod 8 next to each edge). We check the corollary stated in the introduction: there are nine vertices in $\Gamma(\nu_2^3)$, $\Gamma(\nu_2^2)$, and $\Gamma(\nu_2^1)$ and there are $2^9/8$ (i.e. 64) ghosts.

2.4.6. The length of the fibre of the morphism forgetting level structures. In [7], Caporaso, Casagrande and Cornalba check that the length of the fibre of the forgetful morphism from their compactified moduli of ℓth roots to the moduli of stable curves equals ℓ^{2g} (see [7, §4.1, after Lem. 4.1.1]). We show that in the twisted curve approach used in this paper the length of the fibre is still ℓ^{2g}. The computation here is more involved because our moduli functor yields more geometric points, which reflects the fact that the compactified moduli spaces of this paper are smooth and actually provide the normalisations of the possibly singular spaces of [7] (see [10, §1.2-3]). The contractions $\Gamma \to \Gamma(\nu_\ell^k)$ allow us to organise our computation efficiently.

The authors of [7] consider ℓth roots of a line bundle N and the respective moduli functor which can be naturally regarded as a fibred category (over the category of schemes). The authors are not considering the stack representing such a category and are mainly interested in the scheme *coarsely* representing this moduli functor. For any family of curves $\pi : C \to B$, there exists a scheme $S^\ell(N, \pi)$ representing ℓth roots of N. In general $S^\ell(N, \pi)$ is smooth over B, but not proper over B. In the same spirit of the present paper, the authors introduce a new, less restrictive, notion of root: the "limit ℓth root of N". The corresponding moduli functor is shown in [7] to be coarsely represented by a proper, but possibly singular, scheme $\overline{S}^\ell(N,\pi)$ over B (singularities occur when ℓ is not prime, see discussion after [7, Thm. 4.2.3]).

We assume $N = \mathcal{O}$; then for any family $\pi\colon C \to B$ we have a possibly singular scheme $\overline{S}^\ell(\mathcal{O},\pi)$ and, as a byproduct, a moduli space $\overline{S}^\ell_g$, which is a finite ℓ^{2g}-cover of the proper moduli space $\overline{\mathcal{M}}_g$. We consider, for any $g \geq 2$ and $\ell \geq 1$, the moduli stacks $\mathsf{T}_{g,\ell} = \bigsqcup_{d\mid\ell} \mathsf{R}_{g,d}$ and the compactifications $\overline{\mathsf{T}}_{g,\ell} = \bigsqcup_{d\mid\ell} \overline{\mathsf{R}}_{g,d}$ yielding the finite cover

$$\mathsf{p}\colon \overline{\mathsf{T}}_{g,\ell} \to \overline{\mathsf{M}}_g.$$

We can now state precisely what we mean (both here and in [7]) by "the fibre over a point of" $\overline{\mathsf{M}}_g$. In [7], for a stable curve C over k, the main focus is $\overline{S}^\ell(\mathcal{O}, C \to \operatorname{Spec} k)$, which is the zero-dimensional scheme coarsely representing the fibred product of categories obtained by pulling back the fibred category of limit ℓth roots of $\mathcal{O}$ over $\overline{\mathsf{M}}_g$ via the map $b\colon \operatorname{Spec} k \to \overline{\mathsf{M}}_g$ induced by C. In complete analogy, our focus here is the coarsening of the fibred product $\operatorname{Spec} k \ {}_b{\times}_{\mathsf{p}} \overline{\mathsf{T}}_{g,\ell}$, which we denote by F_b. Notice that, by definition, this does not involve the automorphism group of C (as it would have been the case if we had considered $\operatorname{Spec} k \ {}_b{\times}_{\mathsf{p}} \overline{T}_{g,\ell}$ instead). Of course the reader may also read this computation under the additional assumption that the curve C has trivial automorphism group (in this case we are actually computing $\operatorname{Spec} k \ {}_b{\times}_{\mathsf{p}} \overline{T}_{g,\ell}$).

Fix a stable curve C with dual graph Γ, that is, a geometric point $b \to \overline{\mathsf{M}}_g$. We check $\operatorname{length}(F_b) = \ell^{2g}$ for the scheme-theoretic fibre F_b, coarsening of the base change of $\overline{\mathsf{T}}_{g,\ell} \to \overline{\mathsf{M}}_g$ to b. Each connected component of F_b is a, possibly nonreduced, zero-dimensional scheme corresponding to an isomorphism class of a triple $(\mathsf{C},\mathsf{L},\phi\colon \mathsf{L}^{\otimes\ell} \to \mathcal{O})$ with a multiplicity M and corresponding characteristic functions $\boldsymbol{\nu} = (\nu_p)$. By Theorem 2.18, the length of such a zero-dimensional scheme is

$$\#\operatorname{Aut}_C(\mathsf{C})/\#\underline{\operatorname{Aut}}(\mathsf{C},\mathsf{L},\phi) = \Big(\prod_{p\mid\ell}\prod_{k=1}^{e_p} p^{\#E(\nu_p^k)}/p^{\#V(\nu_p^k)-1}\Big) = \prod_{p\mid\ell}\prod_{k=1}^{e_p} p^{b_1(\Gamma(\nu_p^k))}.$$

(By the definition of F_b, we are not considering the action of $\operatorname{Aut}(C)$.)

The number of connected components is

$$\sum_{\substack{M\in\ker\partial\\ \boldsymbol{\nu}_M=(\nu_p)}} \ell^{2p_g(C)} \prod_{p\mid\ell}\prod_{k=1}^{e_p} p^{b_1(\Delta(\nu_p^k))}.$$

This happens because the multiplicities range over the elements of $\ker\partial$ by Proposition 1.11. Furthermore, once the multiplicity is specified, the numbers of ℓth roots equal the summands appearing above. Indeed, we can count by taking a product over the prime factors of ℓ and reduce to showing the claim for $\ell = p^e$. Then, we need to show that the number of ℓth roots sharing the same multiplicity M is

$$\prod_{k=1}^{e} p^{2p_g(C)} p^{b_1(\Delta(\nu_p^k))}.$$

This amounts to showing that the factors above are the numbers of p^kth roots up to p^{k-1}st roots for any $k = 1, \dots, e$. The factor $p^{2p_g(C)}$ counts p^kth roots up to p^{k-1}st roots on the normalisation. The last factor involves $\Delta(\nu_p^k)$, the subgraph of Γ formed by the edges e where $p^k \mid M(e)$. By (29), if p^k does not divide $M(e)$, iterated pull-backs via $(x, y) \mapsto (\xi_{r(e)}x, y)$ at the node n corresponding to e identify to each other all gluing data in $p^{k-1}\mathbb{Z}/p^k\mathbb{Z}$ along n. Therefore the gluings, up to automorphisms, are determined by the subgraphs $\Delta(\nu_p^k)$ and their number is the number of elements of $H^1(\Delta(\nu_p^k)), \boldsymbol{\mu}_{p^k}/\boldsymbol{\mu}_{p^{k-1}})$. We get exactly the power of p appearing in the last factor of the displayed formula above.

Finally, since $\Gamma(\nu_p^k)$ is given by collapsing the subgraph $\Delta(\nu_p^k)$, the Betti numbers $b_1(\Delta(\nu_p^k))$ and $b_1(\Gamma(\nu_p^k))$ add up to $b_1(\Gamma)$; we get

$$\operatorname{length}(F_b) = \sum_{\substack{M \in \ker \partial \\ \boldsymbol{\nu}_M = (\nu_p)}} \ell^{2p_g(C)} \prod_{p \mid \ell} \prod_{k=1}^{e_p} p^{b_1(\Delta(\nu_p^k)) + b_1(\Gamma(\nu_p^k))} = \sum_{\substack{M \in \ker \partial \\ \boldsymbol{\nu}_M = (\nu_p)}} \ell^{2p_g(C)} \ell^{b_1(\Gamma)} = \ell^{2g}.$$

2.4.7. No-Ghosts. Theorem 2.18 and Lemma 2.22 imply a no (nontrivial) ghost criterion.

Corollary 2.25. *Let ℓ be any positive integer. The group $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$ is trivial if and only if for any prime factor p of ℓ the graph $\Gamma(\nu_p^{e_p})$ is a bouquet.* □

Remark 2.26. In analogy with the case where ℓ is prime, one may consider the condition “the contraction Γ' of $\{e \mid \ell \text{ divides } M(e)\}$ is a bouquet”, which clearly implies the above no-ghosts condition. The converse is false: for $\ell = 6$, consider Γ with vertices v_1, v_2, edges e_1, e_2, e_3 going from v_1 to v_2, set $M(e_i) = i$.

Remark 2.27. Lemma 2.10 and Proposition 2.13 generalise verbatim, and, by Corollary 2.25, the same holds for Theorem 2.15 once we replace “$\Gamma(\nu)$ is a bouquet” by “$\Gamma(\nu_p^{e_p})$ is a bouquet for any prime $p \mid \ell$”.

We can also state the generalisation as follows.

Theorem 2.28. *Let $g \geq 4$ and let $\ell \geq 1$. The point representing $(\mathsf{C}, \mathsf{L}, \phi)$ in $\overline{\mathcal{R}}_{g,\ell}$ is smooth if and only if the group $\mathrm{Aut}'(C)$ is generated by ETIs of C and the graphs $\Gamma(\nu_p^{e_p})$ (obtained by contracting the edges e for which $M(e) \in \mathbb{Z}/p^{e_p}$ vanishes) are bouquets for any prime $p \mid \ell$.*

2.4.8. Automorphism group of level structures over stack-theoretic elliptic tails. We describe the action of the automorphism group of the stack-theoretic elliptic tail of Example 1.15 on Pic. We are interested in level structures on a curve with an elliptic tail E; it is natural to fix a divisor l of ℓ, which should be thought of as the the order of the restriction of the level-ℓ structure to E. We refer the reader to Example 2.1 for the study of level structures on E when E is an irreducible twisted curve with a single node.

The 1-pointed 1-nodal twisted genus-1 curve (E, n) is given by the stack-theoretic quotient of $\widetilde{E} = \mathbb{P}^1/(0 \sim \infty)$ by $\boldsymbol{\mu}_r$ where r divides l and $\boldsymbol{\mu}_r$ acts by multiplication as usual. Consider $\mathsf{p}\colon \widetilde{E} \to \mathsf{E}$, the isotypical decomposition $\mathsf{p}_*\mathcal{O} = \bigoplus_{\chi\in\mathbb{Z}/r} \mathsf{L}_\chi$, and the l-torsion line bundle $\mathsf{L}_{\mathrm{ram}} := \mathsf{L}_{\chi=1}$ on E with $\phi_{\mathrm{ram}}\colon \mathsf{L}_{\mathrm{ram}}^{\otimes l} \to \mathcal{O}$ obtained by taking the (l/r)th tensor power of the isomorphism $\mathsf{L}_{\mathrm{ram}}^{\otimes r} \cong \mathcal{O}$. We also consider the l-torsion line bundle $\mathsf{L}_{\text{ét}}$, the pullback via $\epsilon_{\mathsf{E}}\colon \mathsf{E} \to E$ of the sheaf of regular functions f on the normalisation satisfying $f(\infty) = \xi_l f(0)$ for an lth primitive root of unity ξ_l.

We have

$$\mathrm{Pic}(\mathsf{E})[l] \cong \boldsymbol{\mu}_l \oplus \mathbb{Z}/r. \tag{40}$$

The second summand has the distinguished generator $\mathsf{L}_{\mathrm{ram}} := \mathsf{L}_1$. The first summand is generated by $\mathsf{L}_{\text{ét}}$, defined after choosing a primitive root of unity ξ_l.

We have

$$\mathrm{Aut}(\mathsf{E}, n) = \{\mathsf{a} \in \mathrm{Aut}(\mathsf{E}) \mid \mathsf{a}(n) = n\} \cong \boldsymbol{\mu}_2 \oplus \boldsymbol{\mu}_r,$$

where the first summand is generated by the distinguished involution i, whereas the second summand is generated by g, defined after choosing an rth root of unity ξ_r by the local picture $\mathsf{g}\colon (x, y) \mapsto (\xi_r x, y)$ at n, and the condition $\mathsf{g}|_{\mathsf{E}\setminus\{n\}} = \mathrm{id}$.

Then i operates on $\mathrm{Pic}(\mathsf{E})[l]$ as the passage to the inverse

$$\mathsf{i}\colon (\alpha \in \boldsymbol{\mu}_l, k \in \mathbb{Z}/r) \mapsto (\alpha^{-1}, -k).$$

On the other hand, any given root of unity $\zeta \in \boldsymbol{\mu}_r$ operates on $\mathrm{Pic}(\mathsf{E})[l]$ as

$$\zeta\colon (\alpha \in \boldsymbol{\mu}_l, k \in \mathbb{Z}/r) \mapsto (\alpha k(\zeta), k),$$

where the product between $\alpha \in \boldsymbol{\mu}_l$ and $k(\zeta) \in \mathbb{G}_{\mathrm{m}}$ is obviously taken within $\mathbb{G}_{\mathrm{m}}$.

More explicitly, in terms of the explicit bases mentioned above, we have the additive groups $\mathrm{Pic}(\mathsf{E})[l] \cong \langle \mathsf{L}_{\text{ét}}, \mathsf{L}_{\mathrm{ram}}\rangle = \mathbb{Z}/l \oplus \mathbb{Z}/r$ and $\mathrm{Aut}(\mathsf{E}, n) \cong \langle \mathsf{i}, \mathsf{g}\rangle = \mathbb{Z}/2 \oplus \mathbb{Z}/r$ and the action of $(a_1, a_2) = \mathsf{i}^{a_1} \circ \mathsf{g}^{a_2} \in \mathrm{Aut}(\mathsf{E}, n)$ on the line bundle $(k_1, k_2) = (\mathsf{L}_{\text{ét}})^{\otimes k_1} \otimes \mathsf{L}_{\mathrm{ram}}^{\otimes k_2}$ in $\mathrm{Pic}(\mathsf{E})[l]$ yields

$$(a_1, a_2) \cdot (k_1, k_2) = ((-1)^{a_1} k_1 + (l/r) a_2 k_2, (-1)^{a_1} k_2), \tag{41}$$

where $a_2 k_2$ is the product in $\mathbb{Z}/r$.

In view of the study of ghost automorphisms of level-l curves we consider a faithful order-l line bundle L on E; in other words, we consider an order-l element $(\alpha, k) \in \boldsymbol{\mu}_l \oplus \mathbb{Z}/r \cong \mathrm{Pic}(\mathsf{E})[l]$ where k is prime to r (faithfulness).

Proposition 2.29. *The complete list of nontrivial automorphisms* $(\sigma \in \boldsymbol{\mu}_2, \zeta \in \boldsymbol{\mu}_r)$ *in* $\mathrm{Aut}(\mathsf{E}, n)$ *fixing the isomorphism class of the order-l line bundle* L *is as follows:*

(i) $l = 1$, $r = 1$, $\mathsf{L} = \mathcal{O}$, *and* $(\sigma, \zeta) = (-1, 1)$*;*
(ii) $l = 2$, $r = 1$, $\mathsf{L} \in \mathrm{Pic}[2] \setminus \{\mathcal{O}\}$, *and* $(\sigma, \zeta) = (-1, 1)$*;*
(iii) $l = 2$, $r = 2$, $\mathsf{L} = (1, \mathsf{L}_{\mathrm{ram}})$ *or* $(-1, \mathsf{L}_{\mathrm{ram}}) \in \mathrm{Pic}[2] = \boldsymbol{\mu}_2 \oplus \mathbb{Z}/2$, *and* $(\sigma, \zeta) = (-1, 1)$*;*
(iv) $l = 4$, $r = 2$, $\mathsf{L} = (\alpha, \mathsf{L}_{\mathrm{ram}}) \in \mathrm{Pic}[4] = \boldsymbol{\mu}_4 \oplus \mathbb{Z}/2$ *(α primitive), and* $(\sigma, \zeta) = (-1, -1)$.

Proof. There are no nontrivial solutions (σ, ζ) of the form $(1, \zeta)$, because this yields $\alpha k(\zeta) = \alpha$, which implies $\zeta = 1$ ($\ker k = 1$). Then, we look for solutions (σ, ζ) of the form $(-1, \zeta)$; hence we solve the equations $\alpha^{-1}k(\zeta) = \alpha$ and $-k = k \mod r$ (with k prime to r). Then $k = 0$ (and $r = 1$) or $k = r/2$ (and $r = 2$). Cases (i) and (ii) arise from $k = 0$, which yields $\zeta = 1$ and $\alpha = 1$ (case (i)) or $\alpha = -1$ (case (ii)). Cases (iii) and (iv) arise from $k = 1$, which yields $\zeta = 1$ and $\alpha^2 = 1$ (case (iii)) or $\zeta = -1$ and $\alpha^2 = -1$ (case (iv)). □

Remark 2.30. Notice that in cases (i)–(iii), the automorphism is the canonical involution i. This may be thought of as the restriction on an elliptic tail (E, n) of the automorphism of a level-ℓ curve $(\mathsf{C}, \mathsf{L}, \phi)$; then the ETI fixing $\overline{\mathsf{C} \setminus \mathsf{E}}$ and yielding i on E operates on $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)$ as the quasireflection $(-\mathbb{I}_1) \oplus \mathrm{id}$.

Again, if we choose explicit bases $\mathrm{Pic}(\mathsf{E})[l] \cong \langle \mathsf{L}_{\text{ét}}, \mathsf{L}_{\text{ram}} \rangle = \mathbb{Z}/l \oplus \mathbb{Z}/r$ and $\mathrm{Aut}(\mathsf{E}, n) \cong \langle \mathsf{i}, \mathsf{g} \rangle = \mathbb{Z}/2 \oplus \mathbb{Z}/r$ we can explicitly realise the fixed line bundle: $\mathcal{O}$ in case (i), $\mathsf{L}_{\text{ét}}$ in case (ii), and L_{ram} and $\mathsf{L}_{\text{ram}} \otimes \mathsf{L}_{\text{ét}}$ in case (iii).

Remark 2.31. In case (iv), the automorphism is the involution obtained as the composition of i with the order-2 ghost g operating locally at the node as $(x, y) \mapsto (-x, -y)$. Again (E, n), with this automorphism and its fixed 4-torsion bundle, may be thought of as the elliptic tail of a level-ℓ curve $(\mathsf{C}, \mathsf{L}, \phi)$. The involution fixing $\overline{\mathsf{C} \setminus \mathsf{E}}$ and yielding $\mathsf{i} \circ \mathsf{g}$ on E does not act as a quasireflection (see Prop. 2.13). Indeed, the action on $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)/\mathrm{Def}(\mathsf{C}, \mathsf{L}, \mathsf{L}, \mathrm{Sing}(\mathsf{C}))$ is nontrivial only on the parameter τ_1 smoothing n: we have $\tau_1 \mapsto -\tau_1$. On the other hand, on $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \mathsf{L}, \mathrm{Sing}(\mathsf{C}))$ the action is trivial except on the parameter τ_2 deforming only the tail: we have $\tau_2 \mapsto -\tau_2$. Therefore the involution fixes a codimension-2 subspace of $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)$ and operates as $-\mathbb{I}_2 \oplus \mathrm{id}$. Finally, when we choose the above explicit bases of Pic and Aut, we may realise the level-4 structure on (E, n) as $\mathsf{L}_{(\mathsf{E},n)} := \mathsf{L}_{\text{ram}} \otimes \mathsf{L}_{\text{ét}}$. Indeed, we have

$$(\mathsf{i} \circ \mathsf{g})^*(\mathsf{L}_{(\mathsf{E},n)}) = \mathsf{i}^*(\mathsf{g}^*\mathsf{L}_{\text{ram}} \otimes \mathsf{g}^*\mathsf{L}_{\text{ét}}) \overset{(41)}{=} \mathsf{i}^*(\mathsf{L}_{\text{ram}} \otimes \mathsf{L}_{\text{ét}}^{\otimes 2} \otimes \mathsf{L}_{\text{ét}}) = \mathsf{i}^*(\mathsf{L}_{\text{ram}}^\vee \otimes \mathsf{L}_{\text{ét}}^\vee) = \mathsf{L}_{(\mathsf{E},n)}.$$

2.5. *Noncanonical singularities*

The problem of describing the locus of noncanonical singularities within the moduli space of level-ℓ curves is treated locally: we systematically study the action of $\mathrm{Aut}(\mathsf{C}, \mathsf{L}, \phi)$ on $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)$. By the Reid–Shepherd-Barron–Tai criterion, the age invariant introduced below detects in terms of rational numbers the cases where noncanonical singularities occur.

Throughout the rest of the paper we use the notation $\{x\}$, which stands for the fractional part of a real number x; in other words, we set $\{x\} := x - \lfloor x \rfloor$.

Although we do not use this point of view in this paper, we mention in passing that Abramovich, Graber, and Vistoli [2] have introduced a global age grading function defined on the cyclotomic inertia stack

$$\mathsf{AGE}\colon I_{\boldsymbol{\mu}}(\overline{\mathsf{R}}_{g,\ell}) \to \mathbb{Q}_{\geq 0}.$$

One could state our description of the noncanonical singularities locus as a description of the locus $\mathsf{AGE}^{-1}(]0,1[)$ within the cyclotomic inertia stack. We are indebted to the authors of [2] for this point of view; nevertheless, the following introduction of the age grading is elementary and can be read without referring to [2].

2.5.1. The age of representations of $\boldsymbol{\mu}_r$. We consider the group $\boldsymbol{\mu}_r$ for any positive integer r and we define an additive age grading over the representation ring $R\boldsymbol{\mu}_r$. Since $\mathrm{Hom}(\boldsymbol{\mu}_r, \mathbb{G}_\mathrm{m})$ is canonically identified with $\mathbb{Z}/r$, we can define the age grading of the character $k \in \mathbb{Z}/r$ as $k/r \in \mathbb{Q}$. Since the characters in $\mathrm{Hom}(\boldsymbol{\mu}_r, \mathbb{G}_\mathrm{m})$ form a basis for the representation ring $R\boldsymbol{\mu}_r$, this yields an additive homomorphism $\mathrm{age}\colon R\boldsymbol{\mu}_r \to \mathbb{Q}$.

2.5.2. Cyclotomic injections and group elements. Let G be a finite group. When working over the complex numbers there is a canonical identification between the set of group elements and the set of cyclotomic injections,

$$\{g \mid g \in G\} \stackrel{1:1}{\longleftrightarrow} \bigsqcup_{r\geq 1}\{\gamma \mid \gamma\colon \boldsymbol{\mu}_r \hookrightarrow G\}. \tag{42}$$

The identification is the obvious one: to an element $g \in G$ of order r we attach the homomorphism $\gamma\colon \boldsymbol{\mu}_r \hookrightarrow G$ mapping $\exp(2\pi i/r)$ to g; in the other direction, we set $g = \gamma(\exp(2\pi i/r))$.

Over any base field this identification depends on the choice of a primitive root of unity $\xi_r \in \boldsymbol{\mu}_r$ for any positive integer r. Below, we define—without the need of any such choice—the age grading of cyclotomic injections within a group G operating on $V = \mathbb{A}^m$.

2.5.3. The age grading for a G-representation. Let $\rho\colon G \to \mathrm{GL}(V)$ be a G-representation, where $V = \mathbb{A}^m$. Any injective homomorphism $\gamma\colon \boldsymbol{\mu}_r \hookrightarrow G$ yields, by composing with ρ, a $\boldsymbol{\mu}_r$-representation. We get an invariant of the G-representation

$$\mathrm{age}_V\colon \bigsqcup_{r\geq 1}\{\gamma \mid \gamma\colon \boldsymbol{\mu}_r \hookrightarrow G\} \to \mathbb{Q}, \qquad \gamma \mapsto \mathrm{age}(\rho\circ\gamma). \tag{43}$$

Explicitly, $\mathrm{age}_V(\gamma)$ is defined as follows: for any primitive root of unity ζ in $\boldsymbol{\mu}_r$ the matrix corresponding to the action of $\gamma(\zeta)$ on V is conjugate to $\mathrm{Diag}((\zeta)^{a_1}, \dots, (\zeta)^{a_m})$ and we have

$$\mathrm{age}_V(\gamma) = a_1/r + \dots + a_m/r \in \mathbb{Q}.$$

The coefficients $a_1, \dots, a_m$ are uniquely determined by imposing $0 \leq a_i < r$ and do not depend on the choice of the primitive root of unity ξ_r.

Over the complex numbers, the notion of group element and that of cyclotomic injection are interchangeable and age_V can be defined directly on G. Moreover, the explicit definition above can be given by fixing $\zeta := \exp(2\pi i/r)$.

2.5.4. The Reid–Shepherd-Barron–Tai criterion. Assume that the point at the origin of V modulo $G \in \mathrm{GL}(V)$ is singular. Such a singularity is *canonical* if and only if any pluricanonical form on the smooth locus extends to any desingularisation of V/G. In other words, for all $q \in \mathbb{Z}$ sufficiently high and divisible, we have

$$\Gamma((V/G)^{\mathrm{reg}}, \omega^{\otimes q}) = \Gamma(\widehat{V/G}, \omega^{\otimes q}) \quad \text{for any desingularisation } \widehat{V/G} \to V/G.$$

Theorem 2.32 (Reid–Shepherd-Barron–Tai criterion [27, 30, 28]). *Assume that the finite group G operates on V without quasireflections. The scheme-theoretic quotient V/G has a noncanonical singularity at the origin if and only if the image of* age_V (*see* (43)) *intersects* $]0, 1[$.

Remark 2.33. The above definition does not depend on any choice of primitive roots of unity. If we fix a root of unity ξ_r for every order $r \geq 1$, the age grading age_V can be defined directly on G via (42): we get

$$G \xrightarrow{1:1} \bigsqcup_{r\geq 1} \{\gamma \mid \gamma : \boldsymbol{\mu}_r \hookrightarrow G\} \longrightarrow \mathbb{Q}.$$

It is important to stress that the image of the above map only depends on the second morphism. More explicitly, for a fixed element $g \in G$ of order r, we have the following relation between the gradings age' and age'' attached to two choices ζ' and ζ'' of primitive rth roots of unity in $\boldsymbol{\mu}_r$. If $\zeta' = (\zeta'')^a$ for a suitable a prime to r, then $\mathrm{age}''(g) = \mathrm{age}'(g^a)$.

Therefore, we fix, once and for all, a system of roots of unity in order to simplify the combinatorial analysis. In particular, this will allow us to specify ghosts simply by writing $\mathbb{Z}/\ell$-valued symmetric functions (as we already did in Fig. 5). Furthermore, this will allow us to define the age of a ghost acting on the deformation space.

Assumption 2.34 (choice of rth primitive roots of unity for all r). We now fix, for any positive integer r, a primitive rth root of unity $\xi_r \in \boldsymbol{\mu}_r$. This is the same as fixing isomorphisms $\mathbb{Z}/r \to \boldsymbol{\mu}_r$, $k \mapsto (\xi_r)^k$, for all $r \in \mathbb{Z}_{\geq 1}$. In particular, we work with a fixed identification (42), and to a given representation $G \to \mathrm{GL}(V)$ we attach $\mathrm{age}_V : G \to \mathbb{Q}$, the nonnegative grading directly defined on G. Note that, under any chosen identification $\mathbb{Z}/r \cong \boldsymbol{\mu}_r$, the pairing $\mathbb{Z}/r \times \boldsymbol{\mu}_r = \mathrm{Hom}(\boldsymbol{\mu}_r, \mathbb{G}_{\mathrm{m}}) \to \boldsymbol{\mu}_r$ matches the product of the ring $\mathbb{Z}/r$.

In this way for $M \in \mathbb{Z}/\ell$ and $\mathsf{a} \in \boldsymbol{\mu}_r = \boldsymbol{\mu}_\ell^M$ we can express $\mathsf{a} \odot M$ as a product. Via $\boldsymbol{\mu}_r \cong \mathbb{Z}/r \subseteq \gcd(M, \ell)\mathbb{Z}/\ell$, we write a as a multiple of $\gcd(M, \ell)$ modulo ℓ; then we have

$$\mathsf{a} \odot M = \frac{aM}{\gcd(M, \ell)} \in \boldsymbol{\mu}_r \cong \gcd(M, \ell)\mathbb{Z}/\ell. \tag{44}$$

When ℓ is prime, the product $\odot$ is simply the product within the ring $\mathbb{Z}/\ell$.

Definition 2.35 (junior and senior group elements). An element $g \in G$ operating nontrivially on V is *senior on V* if $\mathrm{age}_V(g) \geq 1$, and is *junior on V* if $0 < \mathrm{age}_V(g) < 1$ (Ito and Reid's [19] terminology).

Now, Theorem 2.32 may be regarded as saying: V/G has a noncanonical singularity at the origin if and only if there exists an element $g \in G$ which is junior on V.

2.5.5. The computation of the age of an automorphism on $\mathrm{Def}(\mathsf{C},\mathsf{L},\phi)/\langle\mathrm{QR}\rangle$. We mod out $\mathrm{Def}(\mathsf{C},\mathsf{L},\phi)$ by the group $\langle\mathrm{QR}\rangle$ of automorphisms spanned by quasireflections; by Proposition 2.13 this amounts to modding out the ETIs restricting to the identity on the entire curve except for an elliptic tail component E where the canonical involution i fixes $\mathsf{L}|_{\mathsf{E}}$. These involutions operate simply by changing the sign of the parameter τ_e smoothing the node where E meets the rest of the curve. We refer to E as a quasireflection elliptic tail component, or simply *quasireflection tail* (QR tail). We refer to the node joining the quasireflection tail to the rest of the curve as a *quasireflection node* (QR node) and we identify in this way a partition $\mathrm{Sing}(\mathsf{C}) \cong \mathrm{Sing}_{\mathrm{QR}}(\mathsf{C}) \sqcup \mathrm{Sing}_{\mathrm{nonQR}}(\mathsf{C})$ and a partition $E = E_{\mathrm{QR}} \sqcup E_{\mathrm{nonQR}}$. Equations (18) and (19) yield

$$\mathrm{Def}(\mathsf{C},\mathsf{L},\phi)/\langle\mathrm{QR}\rangle \cong \Big(\bigoplus_{e\in E}\mathbb{A}^1_{\overline{\tau}_e}\Big)\oplus\Big(\bigoplus_{v\in V}H^1(C'_v, T(-D_v))\Big)$$

with

$$\overline{\tau}_e = \begin{cases}\tau_e^2 & \text{for } e\in E_{\mathrm{QR}},\\ \tau_e & \text{for } e\in E\setminus E_{\mathrm{QR}}.\end{cases}$$

The action of $\mathrm{Aut}(\mathsf{C},\mathsf{L},\phi)$ on $(\mathsf{C},\mathsf{L},\phi)$ descends to an action without quasireflections on the above space.

Corollary 2.36. *The point at the origin of* $\mathrm{Def}(\mathsf{C},\mathsf{L},\phi)/\mathrm{Aut}(\mathsf{C},\mathsf{L},\phi)$ *is a noncanonical singularity if and only if there exists an automorphism which is junior on* $\mathrm{Def}/\langle\mathrm{QR}\rangle$.

Example 2.37. The stack-theoretic ETI of Remark 2.30 acts trivially on $\mathrm{Def}/\langle\mathrm{QR}\rangle$.

Example 2.38. The automorphism a extending $\mathsf{i}\circ\mathsf{g}$ in Remark 2.31 operates on $\mathbb{A}^1_{\tau_1}\oplus\mathbb{A}^1_{\tau_2}\oplus\mathbb{A}^{3g-1}$ as $(-\mathbb{I}_2)\oplus\mathrm{id}$. Furthermore, by Proposition 2.29, i is not an automorphism of $(\mathsf{C},\mathsf{L},\phi)$. We have $\mathrm{Def}/\langle\mathrm{QR}\rangle = \mathbb{A}^1_{\tau_1}\oplus\mathbb{A}^1_{\tau_2}\oplus\mathbb{A}^{3g-1}$ and a is senior: $\mathrm{age} = 1/2+1/2 = 1$.

2.5.6. The computation of the age of a ghost. Using Proposition 2.4 and Theorem 2.18 we can easily compute the age of a ghost automorphism $\mathsf{a}\in\underline{\mathrm{Aut}}_C(\mathsf{C},\mathsf{L},\phi)$ attached to $b\in G_{\boldsymbol{\nu}}(\Gamma;\boldsymbol{\mu}_\ell)$. Assumption 2.34 allows us to regard b as a $\mathbb{Z}/\ell$-valued 1-chain $b\in G_{\boldsymbol{\nu}}(\Gamma;\mathbb{Z}/\ell)$. We point out that the explicit expressions $[M(e)^{-1}]_\ell b(e)$ in (28) and $[m(e)^{-1}]_{r(e)}b(e)$ in (38) may be interpreted as multiplications in the ring $\mathbb{Z}/\ell$.

When ℓ is prime we have

$$\mathrm{age}(\mathsf{a}) = \sum_{e\in E}\left\{\frac{\mathsf{a}(e)}{\ell}\right\} = \sum_{e\in E}\left\{\frac{[M(e)^{-1}]_\ell b(e)}{\ell}\right\} \quad (\ell \text{ prime}), \tag{45}$$

where $\{\ \}$ denotes fractional part and the terms in the numerators are integer representatives of $\mathsf{a}(e)$, $b(e)$ and $[M(e)^{-1}]_\ell$ in $\mathbb{Z}/\ell$ (each summand in the above expression is clearly independent of the choice of the representatives modulo ℓ). For composite ℓ, Theorem 2.18(3) yields

$$\mathrm{age}(\mathsf{a}) = \sum_{e\in E}\left\{\frac{\mathsf{a}(e)}{\ell}\right\} = \sum_{e\in E}\left\{\frac{[m(e)^{-1}]_{r(e)}b(e)}{r(e)}\right\} = \sum_{e\in E}\left\{\frac{[m(e)^{-1}]_{r(e)}\widetilde{b}(e)}{\ell}\right\}, \tag{46}$$

where $\widetilde{b}(e) \in \mathbb{Z}/\ell$ is the image of $b(e) \in \mathbb{Z}/r(e)$ via the identification $\widetilde{b}(e) = (\ell/r(e))b(e) = \gcd(\ell, M(e))b(e)$. Again the above definition does not depend on the choices of the integer representatives of $\mathsf{a}(e) \in \mathbb{Z}/\ell$ and of $[m(e)^{-1}]_{r(e)}, b(e) \in \mathbb{Z}/r(e)$.

In Example 2.24, we presented three ghost automorphisms; the corresponding symmetric functions $e \mapsto \mathsf{a}(e)$ on the set of oriented edges are given in Fig. 5. Equation (46) allows us to compute their age. According to (46), the order-2 automorphism has age $3/2$, the order-4 automorphism has age $5/4$, whereas the order-8 automorphism has age 1. Hence, in all these three cases the ghosts are senior. However, ghost automorphisms operating as junior ghosts actually occur; we provide some examples, which will play a role in the proof of Theorem 2.44.

Example 2.39. Let $\ell = 5$. Consider a level curve whose dual graph has multiplicity M, pictured in the first diagram of Fig. 6; write $\boldsymbol{\nu}$ for the characteristic function of the support of M. Here we have $\boldsymbol{\nu} = \mathbf{0}$. In the second and third diagrams we specify the symmetric function $\mathsf{a} \in S_{\boldsymbol{\nu}}(\Gamma; \mathbb{Z}/5)$ and the corresponding 1-cochain $b \in G_{\boldsymbol{\nu}}(\Gamma; \mathbb{Z}/5)$. Using (45) we get $\mathrm{age}(\mathsf{a}) = 1/5 + 1/5 + 1/5 + 1/5 = 4/5$.

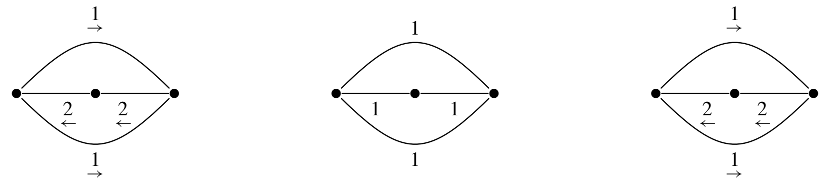


Fig. 6. The multiplicity cochain M, the symmetric function $e \mapsto \mathsf{a}(e)$ and the cochain $e \mapsto b(e)$.

Example 2.40. We consider a level-5 curve again, but this time we only need three nodes and two components. The dual graph has multiplicity M pictured in the first diagram of Fig. 7. Again, we have $\boldsymbol{\nu} = \mathbf{0}$ and in the second and third diagrams we specify the symmetric function $\mathsf{a} \in S_{\boldsymbol{\nu}}(\Gamma; \mathbb{Z}/5)$ and the corresponding 1-cochain $b \in G_{\boldsymbol{\nu}}(\Gamma; \mathbb{Z}/5)$. Using (45) we get $\mathrm{age}(\mathsf{a}) = 2/5 + 1/5 + 1/5 = 4/5$.

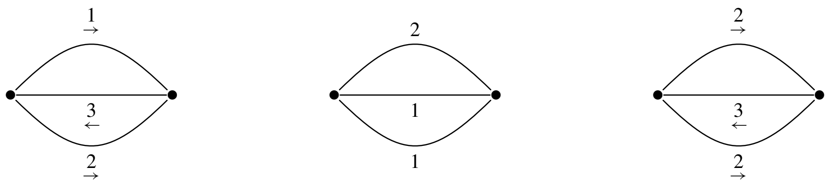


Fig. 7. The multiplicity cochain M, the symmetric function $e \mapsto \mathsf{a}(e)$ and the cochain $e \mapsto b(e)$.

Example 2.41. Let $\ell = 8$. We adopt the notation M and $\boldsymbol{\nu}$ as above. This time $\boldsymbol{\nu}$ is the vector-valued function attached to M. Again, the second and third diagrams specify the symmetric function $\mathsf{a} \in S_{\nu}(\Gamma; \mathbb{Z}/8)$ and the corresponding 1-cochain $b \in G_{\boldsymbol{\nu}}(\Gamma; \mathbb{Z}/8)$. More precisely, next to each edge we have written the values of $\widetilde{\mathsf{a}}$ and $\widetilde{b}$ in $\mathbb{Z}/8$; e.g., “2” appearing in the second diagram represents the order-4 element $2 \mod 8$ in $\mathbb{Z}/8$. Using (46) we get $\mathrm{age}(\mathsf{a}) = 1/8 + 1/8 + 1/8 + 1/8 + 2/8 = 3/4$.

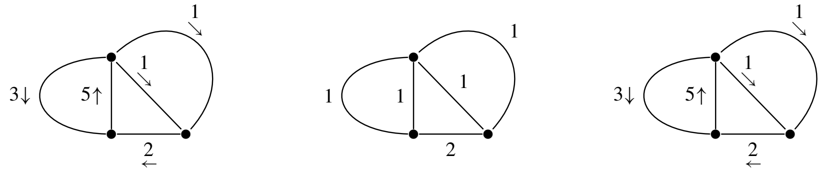


Fig. 8. The multiplicity cochain M, the $\mathbb{Z}/8$-valued symmetric function $e \mapsto \widetilde{\mathsf{a}}(e) = \mathsf{a}(e)\gcd(8, M(e))$ and the cochain $e \mapsto \widetilde{b}(e) = b(e)\gcd(8, M(e))$.

Example 2.42. Let $\ell = 12$. In view of the proof of Theorem 2.44 we slightly generalise Example 2.41. We refer to Figure 9, where we adopt the established conventions. Using (46) we get $\operatorname{age}(\mathsf{a}) = 1/12 + 1/12 + 1/12 + 1/12 + 2/12 + 2/12 = 2/3$.

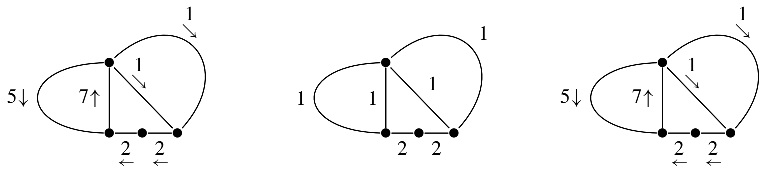


Fig. 9. The multiplicity cochain M, the $\mathbb{Z}/12$-valued symmetric function $e \mapsto \widetilde{\mathsf{a}}(e) = \mathsf{a}(e)\gcd(12, M(e))$ and the cochain $e \mapsto \widetilde{b}(e) = b(e)\gcd(12, M(e))$.

2.5.7. The locus of noncanonical singularities of $\overline{\mathcal{R}}_{g,\ell}$. We may apply the above criterion as follows. Within Deligne and Mumford's moduli stack $\overline{\mathsf{M}}_g$ of stable curves, consider the locus

$$\overline{\mathsf{M}}^{\circ}_g = \{C \mid \operatorname{Aut}(C) = 0\}_{/\cong}$$

of stable curves with trivial automorphism group. This is a stack that can be represented by a smooth scheme. We study the overlying stack

$$\overline{\mathsf{R}}^{\circ}_{g,\ell} = \{(\mathsf{C}, \mathsf{L}, \phi) \mid \operatorname{Aut}(C) = 0\}_{/\cong}$$

of level curves $(\mathsf{C}, \mathsf{L}, \phi)$ such that the coarsening C of C has trivial automorphism group $\operatorname{Aut}(C)$. The scheme $\overline{\mathcal{R}}^{\circ}_{g,\ell}$ coarsely representing $\overline{\mathsf{R}}^{\circ}_{g,\ell}$ may well have singularities; this happens as soon as $\underline{\operatorname{Aut}}(\mathsf{C}, \mathsf{L}, \phi) = \underline{\operatorname{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$ is nontrivial (by Lemma 2.10, nontrivial ghosts cannot operate as the identity or as quasireflections; hence singular points in $\overline{\mathcal{R}}^{\circ}_{g,\ell}$ are characterised by the presence of nontrivial ghosts). Furthermore, since the action of $\operatorname{Aut}(\mathsf{C}, \mathsf{L}, \phi)$ on $\operatorname{Def}(\mathsf{C}, \mathsf{L}, \phi)$ satisfies the hypotheses of the Reid–Shepherd-Barron–Tai criterion (Theorem 2.32), noncanonical singular points are characterised by the presence of junior nontrivial ghosts in the sense of Definition 2.35. Examples 2.39–2.42 already allow a few remarks on the codimension of the locus of noncanonical singularities. By Example 2.40, within $\overline{\mathcal{R}}^{\circ}_{g,5h}$, the locus of noncanonical singularities has codimension 3. Furthermore, the theorem that follows may be regarded as saying: for levels 2, 3, 4, and 6, all singularities of the scheme $\overline{\mathcal{R}}^{\circ}_{g,\ell}$ are canonical. We spell out the statements in the following remark and theorem in terms of the entire space $\overline{\mathcal{R}}_{g,\ell}$.

Remark 2.43. In Examples 2.39–2.42 the edges are all nonseparating and are more than 2. These are general features: the edges are nonseparating because $r(e)$ vanishes on separating edges. In other words, J_ℓ lies over the divisor δ_0^{stable} of curves having at least one nonseparating node. Furthermore, a graph Γ with only two separating edges can only be a graph whose nonseparating edges are two loops, or a graph with a single circuit of length 2. In any case, two circuits never overlap in Γ. As a consequence, ghosts are always senior if the graph has only two nonseparating edges. We conclude that the codimension of J_ℓ is higher than 2. This means that within the locus of noncanonical singularities there is only one irreducible component which has codimension 2 in $\overline{\mathcal{R}}_{g,\ell}$: the locus T_ℓ.

Theorem 2.44 (No-Junior-Ghosts Theorem). *For $g \geq 4$ and $\ell \geq 1$ consider the stack of level-ℓ genus-g curves $\overline{\mathsf{R}}_{g,\ell}$. In $\overline{\mathsf{R}}_{g,\ell}$, every nontrivial ghost automorphism is senior if and only if $\ell \leq 6$ and $\ell \neq 5$.*

Proof. Proving the "only if" part of the statement for a given level ℓ and genus g amounts to exhibiting a dual graph Γ attached to an object of $\overline{\mathsf{R}}_{g,\ell}$ with a multiplicity $M \in \ker \partial$ and a symmetric function $\mathsf{a}\colon E \to \mathbb{Z}/\ell$ defining a junior ghost. Notice that if there exists such a triple (Γ, M, a) for $\overline{\mathsf{R}}_{g,\ell}$, then we can exhibit a triple $(\Gamma, (\ell'/\ell) \times M, (\ell'/\ell) \times \mathsf{a})$ for $\overline{\mathsf{R}}_{g,\ell'}$ for any multiple ℓ' of ℓ (here, Proposition 1.11 has been implicitly used). Examples 2.39, 2.41, 2.42 actually occur for $g \geq 4$ and exhibit junior ghosts for positive levels $\ell \in 5\mathbb{Z} \cup 8\mathbb{Z} \cup 12\mathbb{Z}$. By halving a single straight edge in Fig. 6 (•—• → •—•—•), we can immediately generalise Example 2.39 from $\ell = 5$ to $\ell = 7$; iterating this procedure, for all odd levels $\ell \geq 5$ and for their multiples, we exhibit junior ghosts. The "only if" part is proven: in order for junior ghost not to occur, ℓ should be a positive integer of the form $2^a 3^b$ with $a \in \mathbb{N}$ and $b = 0, 1$ (i.e. not a multiple of an odd integer ≥ 5), with $a < 3$ (i.e. not a multiple of 8) and with $a < 2$ if $b = 1$ (i.e. not a multiple of 12).

The "if" part of the statement means that there is no junior ghost a in $G_{\boldsymbol{\nu}}(M; \mathbb{Z}/\ell)$ for any stable graph Γ with $\boldsymbol{\nu} = \boldsymbol{\nu}_M$ attached to $M \in \ker \partial$. Throughout the entire proof, we will use the following necessary conditions for the existence of a junior ghost a:

(i) $\operatorname{age}(\mathsf{a}) < 1$ (i.e. a is junior);
(ii) $M = \sum_{i \in I} K_i$, where I is a finite set of circuits (i.e. $M \in \ker \partial$);
(iii) $\mathsf{a} \odot M(K) \equiv 0$ for any circuit K (i.e. $\mathsf{a} \odot M \in \operatorname{im} \delta$).

In order to prove that such conditions are incompatible, we provide tables showing all possible values of M and $\mathsf{a} \in \mathbb{Z}/\ell$ and the corresponding value of $\mathsf{a} \odot M$ for $\ell = 2, 3, 4, 6$. In the first line we list all values $M = 0, 1, \dots, \ell - 1$. In the first column, we list the possible values $i = 0, 1, \dots, \ell - 1$ that a may take at an edge e of multiplicity M. We have $\mathsf{a}(e) = i$ only if i satisfies the compatibility condition $\gcd(M, \ell) \mid i$. Then, we fill the ith slot of the jth column in the table with the corresponding value of $\mathsf{a} \odot M$ if and only if $\mathsf{a} = i$ is compatible with $M = j$. We draw a box around the configurations where $\mathsf{a} = i$ is strictly less than $\mathsf{a} \odot M \in \{0, \dots, \ell - 1\}$. Indeed, the presence of oriented edges e with the corresponding values $\mathsf{a}(e)$, $M(e)$ is a necessary condition for a to be junior. If no such oriented edges occur, as it happens for $\ell = 2$, then, for a nontrivial element a,

$\ell=2$	0	1
0	0	0
1		1

$\ell=3$	0	1	2
0	0	0	0
1		1	$\boxed{2}$
2		2	1

$\ell=4$	0	1	2	3
0	0	0	0	0
1		1		$\boxed{3}$
2		2	2	2
3		3		1

$\ell=6$	0	1	2	3	4	5
0	0	0	0	0	0	0
1		1				$\boxed{5}$
2		2	2		$\boxed{4}$	$\boxed{4}$
3		3		3		3
4		4	4		2	2
5		5				1

Fig. 10. Multiplication tables for $\odot$ and $\ell = 2, 3, 4$ and 6.

condition (iii) is incompatible with condition (i). Indeed, we have

$$\operatorname{age}(\mathsf{a}) = \sum_E \frac{\mathsf{a}(e)}{\ell} \geq \sum_K \frac{(\mathsf{a} \odot M)(e)}{\ell} \geq 1 \ \text{ (by (iii))},$$

where K is a circuit passing through an edge where $\mathsf{a}(e)$ is nontrivial. This settles the case $\ell = 2$ and motivates the following definitions.

Definition 2.45. An edge e is *active* with respect to an automorphism a if $\mathsf{a}(e)$ is nontrivial. For an oriented edge e, we refer to the values $(M(e), \mathsf{a}(e))$ as the *type* of e. An oriented edge e is an *age-delay* edge if after reduction modulo ℓ within $\{0, \dots, \ell-1\}$ we have

$$a(e) < \mathsf{a} \odot M(e). \tag{47}$$

An age-delay edge is automatically active, otherwise both sides of the inequality vanish. We say that a circuit K is active if it passes through an active edge and we say that it is age-delay if it passes through an age-delay edge. An age-delay circuit is automatically active.

With this terminology, the previous argument may be rephrased.

Lemma 2.46. *Let* a *be a junior automorphism. Then an active circuit is necessarily age-delay.* $\square$

We can also prove that the type of the active edges of an active circuit cannot be constant.

Lemma 2.47. *Let* a *be a junior automorphism. Consider an active circuit* $K = \sum_{i=0}^{n-1} e_i$ *where the head of* e_i *is the tail of* e_{i+1} *for all* $i \in \mathbb{Z}/n$*. Then the active edges* e_i *of* K *cannot be all of the same type.*

Proof. By way of contradiction, assume there is an active circuit K whose active edges are all of type $(M(e), \mathsf{a}(e)) = (J, I)$ for some $J, I \in \{1, \dots, \ell-1\}$ with $\gcd(J, \ell) \mid I$.

Then condition (iii) may be expressed via (44) as $\sum_K IJ/\gcd(J,\ell)^2 \in \ell/\gcd(J,\ell)\mathbb{Z}$. In particular, $\ell/\gcd(J,\ell)$ divides $k^\# IJ/\gcd(J,\ell)^2$ where $k^\#$ is the number of active edges in K. We conclude that $\ell/\gcd(J,\ell)$, which is prime to $J/\gcd(J,\ell)$, divides $k^\# I/\gcd(J,\ell)$, and ℓ divides $k^\# I > 0$. This contradicts $\operatorname{age}(\mathsf{a}) < 1$, because $\operatorname{age}(\mathsf{a}) \geq k^\# I/\ell > 1$. □

In the case $\ell = 3$ (resp. $\ell = 4$) the only age-delay edges are of type $(\ell - 1, 1)$. A non-trivial junior automorphism should contain an age-delay circuit $K = \sum_{i=0}^{n-1} e_i$ with $(M(e_0), \mathsf{a}(e_0)) = (\ell - 1, 1)$. For $i \neq 0$ the total value of a should be strictly less than 2 (resp. 3) by condition (i). Furthermore the total value of $\mathsf{a} \odot M$ reduced within $\{0, \dots, \ell - 1\}$ modulo ℓ should be 1 (resp. 1) by condition (iii). Then only one of the edges e_i for $i \neq 0$ is active, and its type is $(1, 1)$ (resp. is $(1, 1)$). For a junior automorphism, any active circuit contains exactly two active edges of type $(1, 1)$ and $(\ell - 1, 1)$. Then the automorphism cannot be junior; the claim follows from this slightly more general statement (which we make in view of $\ell = 6$).

Lemma 2.48. *Let* a *be an automorphism for which all active circuits have only an even number* $2k$ *of active edges equally divided into* k *edges of type* $(1, 1)$ *and* k *edges of type* $(\ell - 1, 1)$. *Then* a *is either trivial or senior.*

Proof. We consider the set of active edges, which by the hypothesis, can only be of multiplicity $M = 1$ or $\ell - 1$ depending on their orientation. We pick an orientation for all edges in the edge set E in such a way that $M = 1$ on all active edges. Notice that a circuit, which is by definition a sequence of oriented edges $e_0, \dots, e_{n-1} \in \mathbb{E}$, can now be regarded, with respect to the chosen orientation, as a characteristic function $\chi_K : E \to \mathbb{Z}/\ell$, which equals 1 (resp. $-1 \in \mathbb{Z}/\ell$) on e if $e = e_i$ (resp $e = \overline{e}_i$ for some i) and vanishes otherwise. Condition (ii) may be regarded as saying that the multiplicity is a $\mathbb{Z}/\ell$-valued sum of these characteristic functions of circuits. If we add up the values of the multiplicities M of the active circuits, we obtain $0 \in \mathbb{Z}/\ell$ because each function χ_K restricted to the active circuits has total value $k - k = 0$ by the hypothesis of the lemma. Since $M = 1$ on all active edges, the number of active edges is a multiple of ℓ. Then a is senior or trivial, since it equals 1 on all active edges. □

For $\ell = 6$ the value of a on an age-delay edge is 1 or 2. By excluding the second case the claim will be deduced below as for $\ell = 3$ and 4.

Lemma 2.49. *Let* $\ell = 6$ *and let* a *be junior. Then* $\mathsf{a}(e) \neq 2$ *on all edges.*

Proof. By way of contradiction, let e be an oriented edge with $\mathsf{a}(e) = 2$ and consider a circuit $K = \sum_{i=0}^{n-1} e_i$ through it with $e = e_0$. The value of M can be $1, 2, 4, 5$ because $\gcd(M(e_0), \ell)$ should divide 2. By conveniently choosing the orientation of e_0 and of the circuit $K = \sum_{i=0}^{n-1} e_i$ we assume $M(e_0) = 4$ or 5, which implies $(\mathsf{a} \odot M)(e_0) = 4$ (e_0 is age-delay).

The age contribution of e_0 is $a(e_0)/\ell = 1/3$. Furthermore, the function $\mathsf{a} \odot M$ should add up to $2 \in \mathbb{Z}/6$ on the remaining active edges of K (condition (iii)). Condition (i),

$\mathrm{age}(\mathsf{a}) < 1$, requires edges with $\mathsf{a} < 4$. We have two possibilities for the set of active edges of K:

(a) it is formed by e_0 and two active edges e'', e''' of type $(1, 1)$ where $\mathsf{a} \odot M$ equals 1;
(b) it is of the form $\{e_0, e''\}$ with e'' of type $(1, 2)$ or $(2, 2)$, and $\mathsf{a} \odot M(e'') = 2$.

In any of these cases K contributes 2/3 to $\mathrm{age}(\mathsf{a})$ and there is exactly one more active edge e' of Γ outside K (by Lemma 2.47 and $\mathrm{age}(\mathsf{a}) < 1$): we have $\mathsf{a}(e') = 1$ and $M(e')$ odd. We argue by parity, that is, we compose the functions M and $\mathsf{a} \odot M$ with $P: \mathbb{Z}/6 \to \mathbb{Z}/2$. Note that $P(M)$ does not depend on the choice of the orientation. The value of $P(\mathsf{a} \odot M)$ on the only edge of H that lies off K is odd; therefore, $P(\mathsf{a} \odot M)$ should add up to $1 \in \mathbb{Z}/2$ also on the set of edges shared by K and H. Thus we exclude case (b), where $P(\mathsf{a} \odot M)$ is zero identically.

The set of active edges of Γ is formed by e_0, e', e'', e'''. The function $P(\mathsf{a} \odot M)$ is even on e_0 and odd on e', e'' and e'''; hence, any active circuit should go through $\{e', e'', e''\}$ an even number of times; by (ii), this implies $P(M(e')) + P(M(e'')) + P(M(e''')) = 0$. This is impossible, because $P(M)$ equals 1 identically on $\{e', e'', e'''\}$. □

For $\ell = 6$, any age-delay edge is of type $(5, 1)$; therefore, in order to be compatible with (i) and (iii) and the above lemma, an age-delay circuit has either exactly two active edges of type $(1, 1)$ and $(5, 1)$, or four active edges, divided into two edges of type $(1, 1)$ and two edges of type $(5, 1)$. Lemma 2.48 implies the claim. □

Definition 2.50. A level-ℓ curve $(\mathsf{C}, \mathsf{L}, \phi)$ is a *J-curve* if $\mathrm{Aut}(\mathsf{C}, \mathsf{L}, \phi)$ contains a junior ghost.

The points representing J-curves are noncanonical singularities by definition. Noncanonical singularities may occur even if the level curve has no junior ghost automorphisms and regardless of the level ℓ. Indeed, this is the case of level curves of type T (or simply T-curves) which we now illustrate. T-curves represent a codimension-1 locus within the divisor Δ_{g-1}, i.e. a codimension-2 locus in $\overline{\mathsf{R}}_{g,\ell}$.

Definition 2.51. A level-ℓ curve $(\mathsf{C}, \mathsf{L}, \phi)$ is a *T-curve* if

- C contains an elliptic tail (that is, $E \subset \mathsf{C}$ with $\mathsf{C} \cap \overline{\mathsf{C} \setminus E} = \{n\}$) (*Tail-condition*);
- E admits an order-3 automorphism (that is, $\mathrm{Aut}(E, n) \cong \boldsymbol{\mu}_6$) (*Three-condition*);
- L is trivial on the elliptic tail, i.e. $(\mathsf{C}, \mathsf{L}, \phi) \in \Delta_{g-1}$ (*Triviality-condition*).

Theorem 2.52. *The point representing* $(\mathsf{C}, \mathsf{L}, \phi)$ *in* $\overline{\mathsf{R}}_{g,\ell}$ *is a noncanonical singularity if and only if* $(\mathsf{C}, \mathsf{L}, \phi)$ *is a T-curve or a J-curve.*

Proof. For the “if” part, we only need to show that a T-curve $(\mathsf{C}, \mathsf{L}, \phi)$ has a junior automorphism. Let us define the automorphism $\mathsf{a}_{1/6} \in \mathrm{Aut}(\mathsf{C}, \mathsf{L}, \phi)$, whose restriction to $\overline{\mathsf{C} \setminus E}$ is the identity and whose restriction to E generates $\mathrm{Aut}(E, n) \cong \boldsymbol{\mu}_6$ and operates on the local parameter of E at n as $z \mapsto \xi_6 z$. The coordinates τ_1 and τ_2 correspond to the direction smoothing the node n and to the direction preserving the node and varying along Δ_{g-1}. The action of $\mathsf{a}_{1/6}$ on $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)$ is given by $\mathrm{Diag}(\xi_6, \xi_6^2, 1, \dots, 1)$, where the first coordinates are τ_1 and τ_2. The action of $\mathsf{a}_{1/6}$ on $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)/\langle \mathrm{QR} \rangle$ is given by

$\mathrm{Diag}(\xi_6^2, \xi_6^2, 1, \dots, 1)$, where the first coordinates are $\overline{\tau}_1 = \tau_1^2$ and $\overline{\tau}_2 = \tau_2$ (see §2.5.5). The age of $\mathsf{a}_{1/6}$ on $\mathrm{Def}/\langle \mathrm{QR}\rangle$ is $1/3 + 1/3 = 2/3 < 1$, and Def /Aut has a noncanonical singularity. Notice that all the junior automorphisms operating as $z \mapsto \lambda z$ on the tail (and fixing the rest of the curve) are of the form $\mathsf{a}_{1/6}$ up to ETI.

The "only if" part reduces to the following proposition.

Proposition 2.53. *Let* $(\mathsf{C}, \mathsf{L}, \phi)$ *be a level-*ℓ *curve which is not a J-curve and has a junior automorphism* a. *Then it is a T-curve and the isomorphism* a *coincides, up to ETIs, with the above isomorphism* $\mathsf{a}_{1/6}$.

Preliminary 1. As in [18, p. 33] we begin by slightly simplifying the problem by adding a further condition to the hypotheses. A level curve $(\mathsf{C}, \mathsf{L}, \phi)$ representing a noncanonical singularity in $\overline{\mathcal{R}}_{g,\ell}$ is $(\star)$-*smoothable* if:

(a) There is a junior automorphism $\mathsf{a} \in \mathrm{Aut}(\mathsf{C}, \mathsf{L}, \phi)$ and m nodes $\mathsf{n}_0, \dots, \mathsf{n}_{m-1}$ lying in $\mathrm{Sing}(\mathsf{C}) \setminus \{\mathrm{QR\ nodes}\}$ labelled by $j \in \mathbb{Z}/m$ so that $\mathsf{a}(\mathsf{n}_j) = \mathsf{n}_{j+1}$.

(b) $\prod_{i=0}^{m-1} c_j = 1$, where c_j are the complex nonvanishing constants satisfying the condition $\mathsf{a}^*\tau_{j+1} = c_j\tau_j$ for all $j \in \mathbb{Z}/m$, and τ_j is the parameter smoothing n_j.

By [18, p. 33], if $(\mathsf{C}, \mathsf{L}, \phi)$ is $(\star)$-smoothable, then the data $\mathsf{a} \in \mathrm{Aut}(\mathsf{C}, \mathsf{L}, \phi)$ can be deformed to $\mathsf{a}' \in \mathrm{Aut}(\mathsf{C}', \mathsf{L}', \phi')$ in such a way that the m nodes above are smoothed and the age of the action on $\mathrm{Def}/\langle \mathrm{QR}\rangle$ is preserved. In [23, Prop. 3.6], Ludwig proves a generalisation applying to moduli of roots of any line bundle; in particular we can use this fact for level-ℓ curves. Hence, by iterating such deformations, within the locus of noncanonical singular points in $\overline{\mathcal{R}}_{g,\ell}$, we can smooth any $(\star)$-smoothable curve to a curve which is no more $(\star)$-smoothable; we refer to this condition as $(\star)$-*rigidity*. The loci of T-curves and of J-curves are closed: in the above deformation, if $(\mathsf{C}', \mathsf{L}', \phi')$ is a J-curve [a T-curve], then $(\mathsf{C}, \mathsf{L}, \phi)$ is a J-curve [a T-curve]. Proposition 2.53 can be shown under the following assumption.

Assumption 2.54. In the proof of Proposition 2.53 we assume that $(\mathsf{C}, \mathsf{L}, \phi)$ is $(\star)$-rigid.

Preliminary 2. Set $\mathfrak{ord}(\mathsf{a}) = \mathrm{ord}(a)$; this is also the least integer for which a^m is a ghost and is a divisor of $\mathrm{ord}(\mathsf{a})$. We can provide lower bounds for the age of a.

Lemma 2.55. *Consider a level-*ℓ *curve* $(\mathsf{C}, \mathsf{L}, \phi)$.

0. *For any automorphism* $\mathsf{a} \in \mathrm{Aut}(\mathsf{C}, \mathsf{L}, \phi)$, *we have* $\mathrm{age}(\mathsf{a}) \geq (\#E - N)/2$, *where* N *is the number of cycles of the permutation of* E *induced by* a.

We can improve the lower bound in the following situations:

1. *Assume that* $(\mathsf{C}, \mathsf{L}, \phi)$ *is a noncanonical* $(\star)$-*rigid singularity in* $\overline{\mathcal{R}}_{g,\ell}$. *Then, for any subcurve* Z *such that* $\mathsf{a}(\mathsf{Z}) = \mathsf{Z}$ *and for any length-*k *cycle of the induced permutation of* $\mathrm{Sing}_{\mathrm{nonQR}}(\mathsf{C}) \cap \mathrm{Sing}(\mathsf{Z})$, *we have* $\mathrm{age}(\mathsf{a}) \geq k/\mathrm{ord}(\mathsf{a}|_{\mathsf{Z}}) + (\#E - N)/2$.
2. *If* $\mathsf{a}^{\mathfrak{ord}(\mathsf{a})}$ *is a senior ghost, then* $\mathrm{age}(\mathsf{a}) \geq 1/\mathfrak{ord}(\mathsf{a}) + (\#E - N)/2$.

Proof. We can express the action of a on $\bigoplus_{e\in E}\mathbb{A}^1_{\overline{\tau}_e}$ (see §2.5.5 for the notation $\overline{\tau}_e$) in terms of a block-diagonal matrix H whose blocks $H_1,\dots,H_N$ are of the form

$$H_i = D_i P_i = \mathrm{Diag}((\xi_R)^{q_0^{(i)}},\dots,(\xi_R)^{q_{n_i-1}^{(i)}})P_i,$$

where P_i is the permutation matrix attached to the cycle permutation operating on $\mathbb{Z}/n_i$ as $\sigma(j)=j+1$, R is a suitable positive integer and the exponents $q_j^{(i)}$ are contained in $[0,R-1]$. Note that n_i divides the order of the permutation of E induced by a. Then (see also [23, Prop. 3.7]), since the characteristic polynomial of H_i is $x^{n_i}-\det D_i$, we have

$$\mathrm{age}(\mathsf{a}) \ge \sum_{i=1}^{N}\left(\left\{\sum_{j=0}^{n_i-1}\frac{q_j^{(i)}}{R}\right\}+\frac{n_i-1}{2}\right)=\sum_{i=1}^{N}\left\{\sum_{j=0}^{n_i-1}\frac{q_j^{(i)}}{R}\right\}+\frac{\#E-N}{2},$$

where the right hand side is of the form $A+(\#E-N)/2$ with $A\ge 0$.

In case 1, there is a $k\times k$-block $H=H_{i_0}$ with $D=D_{i_0}=\mathrm{Diag}((\xi_R)^{q_0},\dots,(\xi_R)^{q_{k-1}})$, $H^k=(\xi_R)^q\mathbb{I}$, and $q/R=\{\sum_{j=0}^{k-1}q_j/R\}\neq 0$ (see condition (b) defining ($\star$)-smoothability). Since, for $w=\mathrm{ord}(\mathsf{a}|_Z)$ we have $H^w=\mathrm{id}$, it follows that $\frac{w}{k}\frac{q}{R}\in\mathbb{Z}$; hence, $A\ge q/R\ge k/w$ as required.

In case 2 we are assuming that a^m is senior for $m=\mathfrak{ord}(\mathsf{a})$. Notice that m/n_i is an integer for all i. We want to show $A\ge 1/m$. Assume $A<1/m$. Then, for all i, we notice

$$\left\{\frac{m}{n_i}\sum_{j=0}^{n_i-1}\frac{q_j^{(i)}}{R}\right\}=\frac{m}{n_i}\left\{\sum_{j=0}^{n_i-1}\frac{q_j^{(i)}}{R}\right\}. \tag{48}$$

This happens because

$$\frac{m}{n_i}\left\{\sum_{j=0}^{n_i-1}\frac{q_j^{(i)}}{R}\right\}\le\sum_{i=1}^{N}m\left\{\sum_{j=0}^{n_i-1}\frac{q_j^{(i)}}{R}\right\}=mA<1.$$

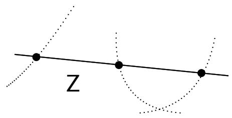
On the other hand, a^m is a ghost automorphism and operates on $\bigoplus_{e\in E}\mathbb{A}^1_{\overline{\tau}_e}$ as the diagonal matrix H^m with n_i eigenvalues equal to $(\det D_i)^{m/n_i}$ for $i=1,\dots,N$; hence using (48), we get

$$\mathrm{age}(\mathsf{a}^m)=\mathrm{age}(H^m)=\sum_{i=1}^{N}n_i\left\{\sum_{j=0}^{n_i-1}\frac{m}{n_i}\frac{q_j^{(i)}}{R}\right\}=\sum_{i=1}^{N}m\left\{\sum_{j=0}^{n_i-1}\frac{q_j^{(i)}}{R}\right\}<1,$$

contradicting the assumption that a^m is senior. □

Step 1: the automorphism a *fixes all nodes except possibly a single transposition of two nodes.* Indeed, by [18, p. 34] (embodied in the first part of Lemma 2.55), each node transposition contributes $1/2$ to $\mathrm{age}(\mathsf{a})$.

Step 2: for each irreducible component Z *we have* $\mathsf{a}(\mathsf{Z}) = \mathsf{Z}$. Harris and Mumford's argument excludes[3] the condition $\mathsf{a}(\mathsf{Z}) \neq \mathsf{Z}$ apart from one situation which we now state more precisely. Case (a) of p. 35 in [18] concerns a smooth, rational, irreducible component Z,



meeting the rest of the curve in three special points; in the present setup we should of course allow nontrivial stabilisers at these three points. Then [18] relies on the following claim in the special case $\mathsf{a} = a \in \mathrm{Aut}(C)$. We state a generalised version, which is due to Ludwig [23, end of proof of Prop. 3.8].

Lemma 2.56. *Assume the coarsening of* $\mathsf{a} \in \mathrm{Aut}(\mathsf{C}, \mathsf{L}, \phi)$ *operates locally at a scheme-theoretic node* $\mathrm{Spec}\,\mathbb{C}[x, y]/(xy)$ *of* C *as* $(x, y) \mapsto (y, x)$. *Then* a *fixes the parameter smoothing the node in* $\mathrm{Def}(C)$ *and operates on the parameter* τ *smoothing the node in* $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)$ *as either* $\tau \mapsto \tau$ *or* $\tau \mapsto -\tau$. *In the first case the curve is not* $(\star)$*-rigid.* □

It is worthwhile to sketch the proof since Ludwig uses the different setup of quasistable curves (which is equivalent in this case). If $\mathsf{a} = a$ we have $xy = t = \tau = \tau$, hence $\tau \mapsto \tau$. Otherwise the multiplicity at the oriented edge e corresponding to the above node satisfies $M(e) = -M(\overline{e}) = -M(e)$; hence $M(e) = \ell/2$ and the action on τ_e is $\tau_e \mapsto \tau_e$ or $\tau_e \mapsto -\tau_e$.

Now, assume $\mathsf{a}(\mathsf{Z}) \neq \mathsf{Z}$ and apply the fact that a is junior and $(\mathsf{C}, \mathsf{L}, \phi)$ is $(\star)$-rigid. The three special points of Z are nodes of C. If they are fixed they have two branches, one in Z and one in $\mathsf{a}(\mathsf{Z})$. Since the coarsening Z of Z is a projective line, these fixed nodes satisfy the condition of the above lemma and, by $(\star)$-rigidity, yield age contribution $1/2$. Recall that each nonfixed node also contributes $1/2$. The age is at least 1 (with one pair of nodes exchanged and the remaining node fixed). So, the argument of [18] holds true: $\mathsf{a}(\mathsf{Z}) \neq \mathsf{Z}$ is ruled out.

Step 3: classification of the irreducible components. For any irreducible component Z of C we set up the notation for the rest of the proof. We write $\mathsf{N} \to \mathsf{Z}$ for its normalisation, $\mathsf{D} \subset \mathsf{N}$ for the divisor representing special points lifting the nodes of C, r for the restriction $\mathsf{a}|_{\mathsf{Z}}$, and $\mathsf{r}_{\mathsf{N}} \in \mathrm{Aut}(\mathsf{N})$ for the lift to N. Coarsening yields $N \to Z$, $D \subset N$, $r \in \mathrm{Aut}(Z)$ and $r_N \in \mathrm{Aut}(N)$. Since all components are fixed, we establish a list of possible cases by recalling the classification [18, Prop., p. 28] of nontrivial automorphisms r_N of a smooth

[3] The space parametrising the deformations of a hypothetical component Z for which $\mathsf{a}(\mathsf{Z}) \neq \mathsf{Z}$ (alongside its special points) should have dimension $d = 0$ or 1 and in this second case we must have $\mathsf{a}(\mathsf{a}(\mathsf{Z})) = \mathsf{Z}$, i.e. the cycle of irreducible components obtained by applying the automorphism a iteratively starting from Z must have length $m = 2$ (indeed, via an age estimate analogue to Lemma 2.55, one can prove age $\geq d(m-1)/2$). The case $d = 1$ corresponding to (c), (d) and (e) in [18, p. 35] is ruled out by the authors, as also is the case named (b) where Z is a singular elliptic tail, because it yields $g(\mathsf{C}) \leq 3$.

scheme-theoretic curve N paired with a divisor $D \subset N$ operating on $H^1(N, T(-D))$ with age less than 1:

(I) N rational with $r_N : z \mapsto \xi_n z$ for $n = 2, 4$,
(II) N elliptic with r_N of order 2, 3, 4, or 6,
(III) N hyperelliptic of genus 2 or 3 with r_N the hyperelliptic involution,
(IV) N of genus-2 with an involution r_N such that $N/\langle r_N \rangle$ is an elliptic curve.

Step 4: classification of the irreducible components Z *such that* a *fixes all nodes of* $\mathsf{Z} \cap \mathrm{Sing}(\mathsf{C})$. We keep the above notation $\mathsf{N}, \mathsf{D}, \mathsf{r}, \mathsf{r}_{\mathsf{N}}, N, D, r, r_N$.

First, case (I) does not occur. Indeed, we argue as in [18, case (b), p. 37]: Since the nodes of $\mathsf{Z} \cap \mathrm{Sing}(\mathsf{C})$ are fixed, the special points of N are either fixed or form orbits of two points with respect to r_{N}. We deduce that r_N necessarily operates on the coarse space as $z \mapsto \xi_2 z$. Then, using the stability condition, it is easy to show that there is at least one pair of points with opposite coordinate on N mapping to a node n of Z. By Lemma 2.56 and ($\star$)-rigidity this yields age contribution $1/2$ and the nontriviality of the stabiliser over n; we deduce (using $\mathrm{age}(a) < 1$) that there is exactly one node of Z whose preimages in N are interchanged by r_N. The remaining nodes lying in Z are contained in the images of the two fixed points of $z \mapsto \xi_2 z$. Note that there cannot be two such nodes, otherwise the action on $H^1(N, T(-D))$ gives an extra contribution of at least $1/2$, because the order-2 automorphism does not deform to the general four-pointed rational curve. Therefore, the only possibility is that Z is a stack-theoretic genus-1 tail as in Definition 2.12. Since a operates by changing the sign of the parameter deforming the elliptic tail, we are necessarily in the situation (iv) of Proposition 2.29, and by Remark 2.31 we have $\mathrm{age}(\mathsf{a}) \geq 1$, a contradiction.

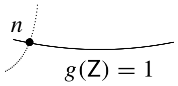
By a simple age computation,[4] [18, p. 39, case (e)] rules out, without changes, the genus-2 curve of case (IV).

Second, case (II) occurs only if r_N fixes at least one point. Assume, by way of contradiction, that r_N is a nontrivial translation $z \mapsto z + t_0$. Since the translation does not allow fixed points, it should allow two-point orbits. In this way r_N is a translation of order 2. This implies that $\mathsf{C} = \mathsf{Z}$, i.e. C is irreducible. Since $g(\mathsf{C}) \geq 4$, there are at least three nodes satisfying the conditions of Lemma 2.56. Applying ($\star$)-rigidity we get an age contribution of $3/2$ and we can conclude as in [18] that r_N fixes at least one point; then we can use Harris and Mumford's list of cases "(c2)–(c5)" at [18, pp. 37–39] specifying the configuration of the elliptic components and their age contribution. We summarise this in (i) and (ii) below.

We can reproduce Harris and Mumford's list of possible irreducible components Z for which the restriction $\mathsf{r} = \mathsf{a}|_{\mathsf{Z}}$ does not satisfy $r = \mathrm{id}_Z$ and fixes all points of $\mathsf{Z} \cap \mathrm{Sing}(\mathsf{C})$.

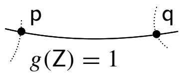
(i) Z is a scheme-theoretic elliptic tail; r is the ETI (age contribution 0) or an automorphism of order 3, 4 or 6 of a smooth elliptic tail $\mathsf{Z} = Z$ meeting the rest of the curve at n acting on $H^1(Z, T(n))$ with age $1/3$, $1/2$ or $1/3$ (see figures on pages 38 and 39 in [18]).

[4] The dimension of the (-1)-eigenspace of r_N on $H^1(N, T(P))$ (where P is a fixed point of r_N) is 2.



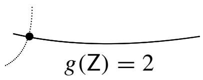
ord $= 3, 4, 6,$ age contribution $= 1/3, 1/2, 1/3.$

(ii) Z is a smooth genus-1 component ($Z = Z$) meeting the rest of the curve in two points p and q; the action on $H^1(Z, T(p+q))$ has order 2 or 4 and age $1/2$ or $3/4$ (see figures on pages 38 and 39 in [18]).



ord $= 2, 4,$ age contribution $= 1/2, 3/4.$

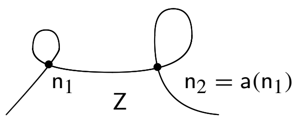
(iii) Z is a hyperelliptic tail of genus $g = 2$; the restriction r is the hyperelliptic involution contributing $1/2$ to age(a) (see case (d) of [18, p. 39]).



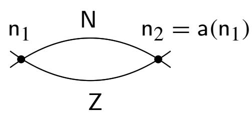
ord $= 2,$ age contribution $= 1/2.$

Step 5: $a \in \operatorname{Aut}(C)$ *fixes all nodes.* To see this, by Step 1, we need to rule out the cases where a transposes a pair (n_1,n_2) of nodes. Since a node transposition contributes $1/2$ to age(a), we can exclude the presence in the curve of components of the form (ii) and (iii). We can assume that a operates as the identity on the elliptic tails (i). If this is not the case, we can simply modify a by restricting to $B = \overline{C \setminus \{\text{elliptic tails}\}}$ and by trivially extending to C; the resulting automorphism has lower age but it is still nontrivial because it exchanges two nodes; hence it is a nontrivial junior automorphism, which we will refer to as a in this step.

We now see that $n_2 = a(n_1)$ yields a contradiction; since all irreducible components are (globally) fixed by Step 2, we reduce to the following cases.



case (a)



case (b)

(a) All the branches of n_1 and of $n_2 = a(n_1)$ lie in the same irreducible component Z. Then Lemma 2.55(1) yields age contribution $2/n + 1/2$, where $n = \operatorname{ord}(r)$ and fits in the conditions required by 2.55. We observe that $\operatorname{ord}(r) = \operatorname{ord}(r)$ because every ghost is the identity on the irreducible components of C (see Remark 2.20). The age contribution coincides with that used in [18, pp. 36–37] in order to rule out this case.
(b) There is a component Z containing exactly one branch for each of the nodes n_1 and n_2. Let H be the second component through n_1 and n_2. Notice that $a^{\operatorname{ord}(a)}$ is either the identity or a senior ghost because (C, L, ϕ) is not a J-curve. Then, by Lemma 2.55(1)–(2), the age of a is at least $1/n+1/2$ where n is the order of the coarsening of $a|_{Z \cup H}$. According to the list of cases (I)–(IV), n can be 2, 4, 6, or 12. Since the lower bound $1/n+1/2$ is smaller than the lower bound $2/n+1/2$ found in [18, pp. 36–37] we can only conclude for $n = 2$. In particular we should study more carefully the

case $n = 4$ where no extra argument was needed in [18]. The same issue arises in [23, proof of Prop. 3.10], where Ludwig notices that when n equals 4, there is an extra age contribution of $1/4$. Indeed, either Z or H is an elliptic curve on which the coarsening a operates, locally at a point $p \neq n_1, n_2$, as $z \mapsto \xi_4 z$. This yields an extra age contribution of $1/4$. (Ludwig also checks that the arguments of Harris and Mumford allow one to consider $n = 6$ and 12 because of the respective extra age contributions $1/3$ and $1/2$ that they find in these two cases. The argument fits equally well here.)

Step 6. We are left with the problem of patching together the few curves of genus 1 and 2 listed in (i)–(iii) with lots of identity components, i.e. components where the coarsening of a restricts to the identity. We do it by following [18, p. 39] (see also [23, Propositions 3.12–15]). In case there is a component of type (iii), the second component H through the node separating Z from the rest of the curve cannot be of type (iii) (each component of type (iii) adds $1/2$ to $\operatorname{age}(\mathsf{a})$). By the same argument we rule out H of type (ii). On the other hand, H cannot be an elliptic tail because $g(\mathsf{C}) \geq 4$. Finally, H cannot be an identity component, because this yields a $1/2$-age contribution due to the parameter smoothing the node $\mathsf{H} \cap \mathsf{Z}$. As a consequence, case (iii) is impossible.

Assume that there is a component Z of type (ii), that is, a so-called elliptic ladder. Since such components contribute at least $1/2$ to $\operatorname{age}(\mathsf{a})$, we assume there is exactly one such case. We argue as in Step 5 where we have replaced a by another junior automorphism operating as the identity on the elliptic tails. In this way, we have $n = \mathfrak{ord}(\mathsf{r}) = \mathfrak{ord}(\mathsf{a})$. If $\mathsf{a}^{\mathfrak{ord}(\mathsf{a})}$ is a nontrivial ghost, then it is senior, because $(\mathsf{C}, \mathsf{L}, \phi)$ is not a J-curve; by Lemma 2.55(2) we have $\operatorname{age}(\mathsf{a}) \geq 1/n$. The same inequality holds, by Lemma 2.55(1), if $\mathsf{a}^{\mathfrak{ord}(\mathsf{a})}$ is trivial, i.e. if $\mathfrak{ord}(\mathsf{a}) = \operatorname{ord}(\mathsf{a})$ (since $g \geq 4$, there is at least one fixed node in $\mathsf{C} \setminus \{\text{elliptic tails}\}$). Now, for $n = 2$, the total age contribution is $1/2 + 1/2$, and for $n = 4$, the total age contribution is $3/4 + 1/4$. We may rule out this case.

Now the coarsening of a is the identity on all components that are not elliptic tails. In fact, a is actually the identity on all such components; if this were not the case, we could replace a by a junior ghost automorphism of $(\mathsf{C}, \mathsf{L}, \phi)$, contradicting the assumption that $(\mathsf{C}, \mathsf{L}, \phi)$ is not a J-curve. So, a is the identity everywhere except for some scheme-theoretic elliptic tails. We can now go through the study of elliptic tails (i) and add the age contribution from the parameter smoothing the QR node where the tail meets the rest of the curve. As in [18] and [23], we conclude that a has order 6 and operates on the elliptic tail precisely as prescribed by the statement of Proposition 2.53. ◻

By definition, noncanonical singularities are local obstructions to the extension of pluricanonical forms. On the other hand, Harris and Mumford show that noncanonical singularities at T-curves do not pose a global obstruction: pluricanonical forms extend across the locus T of level curves of type T as soon as they are globally defined off T. Their statement can immediately be adapted to level curves (the argument is spelled out in [15, Thm. 6.1] and [23, Thm 4.1] and relies on the fact that the morphism forgetting the level structure is not ramified along δ_{g-1}). The precise statement is as follows.

Corollary 2.57. *Fix $g \geq 4$ and $5 \neq \ell \leq 6$. Let $\widehat{\mathcal{R}}_{g,\ell} \to \overline{\mathcal{R}}_{g,\ell}$ be any desingularisation. Then every pluricanonical form defined on the smooth locus $(\overline{\mathcal{R}}_{g,\ell})^{\mathrm{reg}}$ of $\overline{\mathcal{R}}_{g,\ell}$ extends holomorphically to $\widehat{\mathcal{R}}_{g,\ell}$, that is, for all integers $q \geq 0$ we have isomorphisms*

$$\Gamma\big((\overline{\mathcal{R}}_{g,\ell})^{\mathrm{reg}}, K^{\otimes q}_{\overline{\mathcal{R}}_{g,\ell}}\big) \cong \Gamma\big(\widehat{\mathcal{R}}_{g,\ell}, K^{\otimes q}_{\widehat{\mathcal{R}}_{g,\ell}}\big). \qquad \square$$

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