J. Eur. Math. Soc. 19, 957–966 

DOI 10.4171/JEMS/684

Serena Dipierro · Ovidiu Savin · Enrico Valdinoci

# All functions are locally $s$-harmonic up to a small error

Received June 11, 2014

**Abstract.** We show that we can approximate every function $f \in C^k(\overline{B_1})$ by an $s$-harmonic function in $B_1$ that vanishes outside a compact set.

That is, $s$-harmonic functions are dense in $C^k_{\mathrm{loc}}$. This result is clearly in contrast with the rigidity of harmonic functions in the classical case and can be viewed as a purely nonlocal feature.

**Keywords.** Density properties, approximation, $s$-harmonic functions

## 1. Introduction

It is a well-known fact that harmonic functions are very rigid. For instance, in dimension 1, they reduce to linear functions and, in any dimension, they never possess local extrema.

The goal of this paper is to show that the situation for fractional harmonic functions is completely different, namely one can fix any function in a given domain and find an $s$-harmonic function[1] arbitrarily close to it.

Heuristically speaking, the reason for this phenomenon is that while classical harmonic functions are determined once their trace on the boundary is fixed, in the fractional setting the operator sees all the data outside the domain. Hence, a careful choice of these data allows an $s$-harmonic function to "bend up and down" basically without any restriction.

S. Dipierro: School of Mathematics and Statistics, University of Melbourne, Peter Hall Building, Parkville, VIC 3010, Australia; e-mail: s.dipierro@unimelb.edu.au

O. Savin: Department of Mathematics, Columbia University,
2990 Broadway, New York, NY 10027, USA; e-mail: savin@math.columbia.edu

E. Valdinoci: School of Mathematics and Statistics, University of Melbourne,
Peter Hall Building, Parkville, VIC 3010, Australia, and Dipartimento di Matematica,
Università degli studi di Milano, Via Saldini 50, 20133 Milano, Italy, and
Istituto di Matematica Applicata e Tecnologie Informatiche, Consiglio Nazionale delle Ricerche,
Via Ferrata 1, 27100 Pavia, Italy, and Weierstraß Institut für Angewandte Analysis und Stochastik,
Mohrenstraße 39, 10117 Berlin, Germany; e-mail: enrico@mat.uniroma3.it

*Mathematics Subject Classification (2010):* 35R11, 60G22, 35A35, 34A08

[1] Of course, we are saying here that a (say, bounded and smooth) function $u$ is *$s$-harmonic* in a domain $\Omega$ if $(-\Delta)^s u(x) = 0$ for any $x \in \Omega$.

The rigorous statement of this fact is in the following Theorem 1.1. For this, we recall that, given $s \in (0, 1)$, the fractional Laplace operator of a function $u$ is defined (up to a normalizing constant) as

$$(-\Delta)^s u(x) := \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}}\, dy.$$

We refer to [5, 9, 11, 12] for other equivalent definitions, motivations and applications.

**Theorem 1.1.** *Fix $k \in \mathbb{N}$. Then, given any function $f \in C^k(\overline{B_1})$ and any $\epsilon > 0$, there exist $R > 1$ and $u \in H^s(\mathbb{R}^n) \cap C^s(\mathbb{R}^n)$ such that*

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } B_1, \\ u = 0 & \text{in } \mathbb{R}^n \setminus B_R \end{cases}$$

*and*

$$\|f - u\|_{C^k(B_1)} \le \epsilon.$$

As usual, in Theorem 1.1, we have denoted by $C^k(\overline{B_1})$ the space of all the functions $f : \overline{B_1} \to \mathbb{R}$ that possess an extension $\tilde{f} \in C^k(B_{1+\mu})$ (i.e. $\tilde{f} = f$ in $B_1$) for some $\mu > 0$.

We also mention that an important rigidity feature for classical harmonic functions is imposed by the Harnack inequality: if $u$ is harmonic and nonnegative in $B_1$ then $u(x)$ and $u(y)$ are comparable for any $x, y \in B_{1/2}$. A striking difference with the nonlocal case is that this type of Harnack inequality fails for the fractional Laplacian (namely it is necessary to require that $u$ is nonnegative in the whole of $\mathbb{R}^n$ and not only in $B_1$, see e.g. [8, Theorem 2.2]). As an application of Theorem 1.1, we point out that one can construct examples of $s$-harmonic functions with a "wild" behavior, that oscillate as many times as we want, and reach interior extrema basically at any prescribed point. In particular, one can construct $s$-harmonic functions to be used as barriers basically without any geometric restriction.

As a further observation, we would like to stress that, while Theorem 1.1 reveals a purely nonlocal phenomenon, a similar result does not hold for any nonlocal operator. For instance, it is not possible to replace the sentence "any fuction is locally $s$-harmonic up to a small error" with "any surface is locally a nonlocal minimal surface up to a small error", that is, it is not true that any surface may be locally approximated by nonlocal minimal surfaces. Indeed, the uniform density estimates satisfied by nonlocal minimal surfaces impose severe geometric restrictions that prevent the formation of sharp edges and thin spikes.

We refer to [3] for the definition of nonlocal minimal surfaces and for their density properties; as a matter of fact, one of the consequences of Theorem 1.1 is that density properties do not hold true for nonlocal minimal surfaces, so $s$-harmonic functions and nonlocal minimal surfaces may have very different behaviors.

Finally, we would like to point out that, while Theorem 1.1 states that "*up to a small error*, all functions are $s$-harmonic", it is not true that "all functions are $s$-harmonic" (or, more precisely, that any given function, say in $B_1$, may be conveniently extended outside $B_1$ to make it $s$-harmonic near the origin). For instance, a function that vanishes on an open subset of $B_1$ cannot be extended to a function that is $s$-harmonic in $B_1$, unless it vanishes identically, in view of the Unique Continuation Principle (see [7]). This provides an example of a function which is not $s$-harmonic in $B_1$ (but, by Theorem 1.1, may be arbitrarily well approximated by $s$-harmonic functions).

We think that it is an interesting problem to determine whether a density result as in Theorem 1.1 holds true under additional assumptions on the function $u$: for instance, whether one can also require that $u$ is supported in a ball of universal radius (i.e. independent of $\epsilon$) or whether one can have meaningful bounds on its global norms. Moreover, it would be interesting to find constructive and efficient algorithms to explicitly determine $u$.

The proof of Theorem 1.1 can be summarized in three steps:

- One may reduce to the case in which $f$ is a polynomial, by density in $C^k(B_1)$, and so to the case in which $f$ is a monomial, by the linearity of the operator. Therefore, it is enough to find an $s$-harmonic function that approximates $x^\beta/\beta!$ in $C^k(B_1)$.
- One can construct an $s$-harmonic function $v$ with an arbitrarily large number of derivatives prescribed at a given point; in particular, one obtains an $s$-harmonic function that has the same derivatives as $x^\beta/\beta!$ up to order $|\beta|$ at the origin (this is indeed the main step needed for the proof).
- One can rescale the function $v$ above by preserving the derivatives of order $|\beta|$ at the origin. By this rescaling, the higher order derivatives (of order between $|\beta|+1$ and $k$) go to zero and so they become a better and better approximation of the higher derivatives of $x^\beta/\beta!$, which establishes Theorem 1.1.

The rest of the paper is organized as follows. In Section 2 we collect some preliminary results, such as a (probably well-known) generalization of the Stone–Weierstrass Theorem and the construction of an $s$-harmonic function in $B_1$ that has a well-defined growth from the boundary. Then, in Section 3, we construct an $s$-harmonic function with an arbitrarily large number of derivatives prescribed. This is, in a sense, already the core of our argument, since these types of properties are typical for the fractional case and do not hold for classical harmonic functions. Also, from this result, the proof of Theorem 1.1 will follow via scaling and approximation.

We refer to [2] for recent results in the spirit of this paper dealing with processes with memory. See also [6] for recent results concerning approximation of arbitrary functions by solutions of very general linear nonlocal equations.

## 2. Preliminary observations

In this section we collect some auxiliary results that will be needed in the rest of the paper.

First of all, we recall a version of the Stone–Weierstrass Theorem for smooth functions. We give a quick proof since in general this result is presented only in the continuous setting.

**Lemma 2.1.** *For any* $f \in C^k(\overline{B_1})$ *and any* $\epsilon > 0$ *there exists a polynomial* $P$ *such that* $\|f - P\|_{C^k(B_1)} \leq \epsilon$.

*Proof.* Without loss of generality we may suppose that $f \in C_0^k(B_2)$. Also, given $\epsilon > 0$, we fix $R > 0$ such that

$$\int_{\mathbb{R}^n \setminus B_R} e^{-|x|^2}\, dx \le \epsilon. \tag{1}$$

Then we fix $\eta > 0$, to be taken arbitrarily small (possibly in dependence on $\epsilon$ and $R$, which are fixed once and for all), and we take $J_\eta \in \mathbb{N}$ large enough such that

$$\sum_{j > J_\eta} \frac{(-1)^j}{j!\eta^j} \le e^{-1/\eta}. \tag{2}$$

Let also

$$\begin{aligned} Q(x) &:= (\pi\eta)^{-n/2} \sum_{j=0}^{J_\eta} \frac{(-1)^j |x|^{2j}}{j!\eta^j}, \\ P(x) &:= \int_{\mathbb{R}^n} f(y) Q(x-y)\, dy, \\ G(x) &:= (\pi\eta)^{-n/2} e^{-|x|^2/\eta}. \end{aligned}$$

We remark that $Q$ is a polynomial in $x$, hence so is $P$. Moreover, by Taylor expansion,

$$G(x) = Q(x) + (\pi\eta)^{-n/2} \sum_{j > J_\eta} \frac{(-1)^j |x|^{2j}}{j!\eta^j},$$

and so, using (2), we conclude that, for any $x \in B_3$,

$$|G(x) - Q(x)| \le e^{-1/\sqrt{\eta}}, \tag{3}$$

provided that $\eta$ is sufficiently small.

Now we recall (1) and we observe that, for any $\alpha \in \mathbb{N}^n$ with $|\alpha| \le k$ and any $x \in B_1$,

$$\begin{aligned} |D^\alpha (G * f)(x) - D^\alpha f(x)| &= \left| \int_{\mathbb{R}^n} G(y) \big( D^\alpha f(x-y) - D^\alpha f(x) \big)\, dy \right| \\ &\le \pi^{-n/2} \int_{\mathbb{R}^n} e^{-|z|^2} |D^\alpha f(x - \sqrt{\eta}\, z) - D^\alpha f(x)|\, dz \\ &\le 2\pi^{-n/2} \epsilon \|f\|_{C^k(\mathbb{R}^n)} + \pi^{-n/2} \int_{B_R} e^{-|z|^2} |D^\alpha f(x - \sqrt{\eta}\, z) - D^\alpha f(x)|\, dz \\ &\le C \Big( \epsilon + R^n \sup_{z \in B_R} |D^\alpha f(x - \sqrt{\eta}\, z) - D^\alpha f(x)| \Big) \end{aligned}$$

for some $C > 0$. Now, if $\eta$ is sufficiently small, we have

$$\sup_{|x-y| \le \sqrt{\eta}\, R} |D^\alpha f(x) - D^\alpha f(y)| \le R^{-n} \epsilon,$$

and we conclude that

$$|D^\alpha(G*f)(x) - D^\alpha f(x)| \leq C\epsilon \tag{4}$$

for any $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq k$ and any $x \in B_1$, for a suitable $C > 0$.

Furthermore, using (3) we see that, for any $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq k$ and any $x \in B_1$,

$$\begin{aligned}
|D^\alpha(G*f)(x) - D^\alpha P(x)| &= |D^\alpha(G*f)(x) - D^\alpha(Q*f)(x)| \\
&= \left| \int_{B_3} (G(y) - Q(y)) D^\alpha f(x-y)\, dy \right| \\
&\leq C \|f\|_{C^k(\mathbb{R}^n)} e^{-1/\sqrt{\eta}} \leq \epsilon,
\end{aligned}$$

as long as $\eta$ is small enough. From this and (4) we obtain

$$\|f - P\|_{C^k(\mathbb{R}^n)} \leq \|f - (G*f)\|_{C^k(\mathbb{R}^n)} + \|(G*f) - P\|_{C^k(\mathbb{R}^n)} \leq C\epsilon$$

for some $C > 0$, which is the desired result, up to renaming $\epsilon$. □

Now, we construct an $s$-harmonic function in $B_1$ that has a well-defined growth close to the boundary:

**Lemma 2.2.** *Let $\bar\psi \in C^\infty(\mathbb{R}, [0,1])$ be such that $\bar\psi(t) = 0$ for any $t \in \mathbb{R} \setminus (2,3)$ and $\bar\psi(t) > 0$ for any $t \in (2,3)$. Let $\psi_0(x) := \bar\psi(|x|)$ and $\psi \in H^s(\mathbb{R}^n) \cap C^s(\mathbb{R}^n)$ be the solution of*

$$\begin{cases} (-\Delta)^s \psi = 0 & \text{in } B_1, \\ \psi = \psi_0 & \text{in } \mathbb{R}^n \setminus B_1. \end{cases}$$

*Then, if $x \in \partial B_{1-\epsilon}$, we have*

$$\psi(x) = \kappa \epsilon^s + o(\epsilon^s) \tag{5}$$

*as $\epsilon \to 0^+$, for some $\kappa > 0$.*

*Proof.* We notice that the function $\psi \in H^s(\mathbb{R}^n)$ may be constructed by the direct method of the calculus of variations, and also $\psi \in C^s(\mathbb{R}^n)$ (see e.g. [10]).

Also, we use the Poisson kernel representation (see e.g. [1,9]) to write, for any $x \in B_1$,

$$\begin{aligned}
\psi(x) &= c \int_{\mathbb{R}^n \setminus B_1} \frac{\psi_0(y)(1-|x|^2)^s}{(|y|^2-1)^s |x-y|^n}\, dy \\
&= c(1-|x|^2)^s \int_2^3 \left[ \int_{S^{n-1}} \frac{\rho^{n-1}\bar\psi(\rho)}{(\rho^2-1)^s |x-\rho\omega|^n}\, d\omega \right] d\rho
\end{aligned}$$

for some $c > 0$. Now we take $x \in B_1$ with $|x| = 1 - \epsilon$, and we obtain

$$\begin{aligned}
\psi(x) &= c(2\epsilon - \epsilon^2)^s \int_2^3 \left[ \int_{S^{n-1}} \frac{\rho^{n-1}\bar\psi(\rho)}{(\rho^2-1)^s |(1-\epsilon)e_1 - \rho\omega|^n}\, d\omega \right] d\rho \\
&= 2^s c \epsilon^s \int_2^3 \left[ \int_{S^{n-1}} \frac{\rho^{n-1}\bar\psi(\rho)}{(\rho^2-1)^s |e_1 - \rho\omega|^n}\, d\omega \right] d\rho + o(\epsilon^s) = \kappa\epsilon^s + o(\epsilon^s)
\end{aligned}$$

for some $\kappa > 0$, as desired. □

We observe that alternative proofs of Lemma 2.2 may be obtained from a boundary Harnack inequality in the extended problem and from explicit barriers (see [4, 10]).

By blowing up the functions constructed in Lemma 2.2 we obtain the existence of a sequence of $s$-harmonic functions approaching $(x\cdot e)_+^s$ for a fixed unit vector $e$, as stated below:

**Corollary 2.3.** *For a fixed $e\in\partial B_1$, there exists a sequence of functions $v_{e,j}\in H^s(\mathbb{R}^n)\cap C^s(\mathbb{R}^n)$ such that $(-\Delta)^s v_{e,j}=0$ in $B_1(e)$, $v_{e,j}=0$ in $\mathbb{R}^n\setminus B_{4j}(e)$, and*

$$v_{e,j}(x)\to\kappa(x\cdot e)_+^s \qquad \text{in } L^1(B_1(e))$$

*as $j\to\infty$, for some $\kappa>0$.*

*Proof.* Let $\psi$ be as in Lemma 2.2 and

$$v_{e,j}(x):=j^s\psi(j^{-1}x-e).$$

The $s$-harmonicity of $v_{e,j}$ and the property of its support can be derived from those of $\psi$. We now prove the convergence. For this, given $x\in B_1(e)$ we write $p_j:=j^{-1}x-e$ and $\epsilon_j:=1-|p_j|=1-|j^{-1}x-e|$. We remark that

$$1>|x-e|^2=|x|^2-2x\cdot e+1,$$

which implies that

$$|x|^2<2x\cdot e \quad \text{and} \quad x\cdot e>0 \quad \text{for all } x\in B_1(e). \tag{6}$$

As a consequence,

$$|p_j|^2=|j^{-1}x-e|^2=j^{-2}|x|^2+1-2j^{-1}x\cdot e=1-2j^{-1}(x\cdot e)_++o(j^{-1})(x\cdot e)_+^2,$$

and so

$$\epsilon_j=j^{-1}(1+o(1))(x\cdot e)_+.$$

Therefore, using (5), we have

$$\begin{aligned} v_{e,j}(x)&=j^s\psi(p_j)=j^s(\kappa\epsilon_j^s+o(\epsilon_j^s))\\ &=j^s\big(\kappa j^{-s}(x\cdot e)_+^s+o(j^{-s})\big)\\ &=\kappa(x\cdot e)_+^s+o(1). \end{aligned}$$

Integrating over $B_1(e)$ we obtain the desired convergence. □

## 3. Spanning the derivative of a function and proof of Theorem 1.1

The main result of this section is that we can find an $s$-harmonic function with an arbitrarily large number of derivatives prescribed. That is, we can arbitrarily prescribe the Taylor polynomial of $s$-harmonic functions (in sharp contrast to the case of classical harmonic functions, in which the Hessian must be of trace zero). This feature will be accomplished by a "vector space" procedure on a suitable "germ space": hence, in light of the method exploited in this linear space, we say, with some abuse of terminology, that we "span" the derivatives with $s$-harmonic functions.

To this end, we use the standard norm notation $|\alpha| := \alpha_1 + \cdots + \alpha_n$ for a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$.

**Theorem 3.1.** *For any $\beta \in \mathbb{N}^n$ there exist $R > r > 0$, $p \in \mathbb{R}^n$, and $v \in H^s(\mathbb{R}^n) \cap C^s(\mathbb{R}^n)$ such that*

$$\begin{cases} (-\Delta)^s v = 0 & \text{in } B_r(p), \\ v = 0 & \text{in } \mathbb{R}^n \setminus B_R(p), \end{cases} \tag{7}$$

$$D^\alpha v(p) = 0 \quad \text{for any } \alpha \in \mathbb{N}^n \text{ with } |\alpha| \le |\beta| - 1, \tag{8}$$

$$D^\alpha v(p) = 0 \quad \text{for any } \alpha \in \mathbb{N}^n \text{ with } |\alpha| = |\beta| \text{ and } \alpha \neq \beta, \tag{9}$$

$$D^\beta v(p) = 1. \tag{10}$$

*Proof.* We denote by $\mathcal{Z}$ the set containing the couples $(v, x)$ of all functions $v \in H^s(\mathbb{R}^n) \cap C^s(\mathbb{R}^n)$ and points $x \in B_r(p)$ that satisfy (7) for some $R > r > 0$ and $p \in \mathbb{R}^n$.

We let

$$N := \sum_{j=0}^{|\beta|} n^j.$$

To any $(v, x) \in \mathcal{Z}$ we can associate a vector in $\mathbb{R}^N$ by listing all the derivatives of $v$ up to order $|\beta|$ evaluated at $x$, that is,

$$(D^\alpha v(x))_{|\alpha| \le |\beta|} \in \mathbb{R}^N.$$

We claim that the vector space spanned by this construction exhausts $\mathbb{R}^N$ (if we prove this, then we obtain (8)–(10) by writing the vector with entry 1 when $\alpha = \beta$ and 0 otherwise as a linear combination of the above functions).

To reach a contradiction, assume that the vector space above does not exhaust $\mathbb{R}^N$, that is, there exists $c = (c_\alpha)_{|\alpha| \le |\beta|} \in \mathbb{R}^N \setminus \{0\}$ such that

$$\sum_{|\alpha| \le |\beta|} c_\alpha D^\alpha v(x) = 0 \tag{11}$$

for any $(v, x) \in \mathcal{Z}$.

We observe that the couple $(v_{e,j}, x)$ with $v_{e,j}$ given by Corollary 2.3 and $x \in B_1(e)$ belongs to $\mathcal{Z}$. Therefore, for any $\xi \in \mathbb{R}^n \setminus B_{1/2}$ and $e := \xi/|\xi|$, (11) holds when $v := v_{e,j}$ and $x \in B_1(e)$.

Accordingly, for every $\varphi \in C_0^\infty(B_1(e))$, we use integration by parts and the convergence result in Corollary 2.3 to obtain

$$\begin{aligned} 0 &= \lim_{j\to\infty} \int_{\mathbb{R}^n} \sum_{|\alpha|\le|\beta|} c_\alpha D^\alpha v_{e,j}(x)\,\varphi(x)\,dx \\ &= \lim_{j\to\infty} \int_{\mathbb{R}^n} \sum_{|\alpha|\le|\beta|} (-1)^{|\alpha|} c_\alpha v_{e,j}(x) D^\alpha \varphi(x)\,dx \\ &= \kappa \int_{\mathbb{R}^n} \sum_{|\alpha|\le|\beta|} (-1)^{|\alpha|} c_\alpha (x\cdot e)_+^s D^\alpha \varphi(x)\,dx \\ &= \kappa \int_{\mathbb{R}^n} \sum_{|\alpha|\le|\beta|} c_\alpha D^\alpha (x\cdot e)_+^s \varphi(x)\,dx. \end{aligned}$$

Consequently, for any $x \in B_1(e)$,

$$\sum_{|\alpha|\le|\beta|} c_\alpha D^\alpha (x\cdot e)_+^s = 0. \tag{12}$$

Recalling (6), we observe that, for any $x \in B_1(e)$,

$$D^\alpha (x\cdot e)_+^s = s(s-1)\dots(s-|\alpha|+1)(x\cdot e)_+^{s-|\alpha|} e_1^{\alpha_1}\dots e_n^{\alpha_n}.$$

So we take $x := e/|\xi| \in B_1(e)$, and we obtain

$$D^\alpha (x\cdot e)_+^s|_{x=e/|\xi|} = s(s-1)\dots(s-|\alpha|+1)|\xi|^{-s}\xi_1^{\alpha_1}\dots\xi_n^{\alpha_n}.$$

Hence we can write (12) as

$$\sum_{|\alpha|\le|\beta|} c_\alpha s(s-1)\dots(s-|\alpha|+1)\xi^\alpha = 0 \tag{13}$$

for any $\xi \in \mathbb{R}^n \setminus B_{1/2}$. Now, (13) says that a polynomial in the variable $\xi$ is identically equal to 0 in an open subset of $\mathbb{R}^n$, so all of its coefficients must vanish:

$$s(s-1)\dots(s-|\alpha|+1)c_\alpha = 0 \tag{14}$$

for any $|\alpha| \le |\beta|$. Notice that none of the factors $s$, $(s-1)$, $\dots$, $(s-|\alpha|+1)$ vanishes since $s$ is not an integer. Hence $c_\alpha = 0$ for any $|\alpha| \le |\beta|$, that is, $c = 0$, contrary to our assumption. □

We stress that Theorem 3.1 reflects a purely nonlocal feature. Indeed, in the local case (i.e. when $s = 1$) the statement of Theorem 3.1 would be clearly false when $|m| \ge 2$, since the sum of the pure second derivatives of any harmonic function must vanish and cannot sum up to 1.

With the aid of Theorem 3.1, we can now complete the proof of Theorem 1.1:

*Proof of Theorem 1.1.* By Lemma 2.1, we can limit ourselves to the case in which $f$ is a polynomial, and the linearity of the fractional Laplace operator allows us to reduce the argument to the case where $f$ is a monomial, say

$$f(x) = x^\beta/\beta!$$

for some $\beta \in \mathbb{N}^n$. Then we take $v$ as in Theorem 3.1 and we define

$$u_\eta(x) := \eta^{-|\beta|} v(\eta x + p),$$

with $\eta \in (0, 1/2)$ to be taken conveniently small (in dependence on $\epsilon$ that is fixed in the statement of Theorem 1.1).

For simplicity, the function $u_\eta$ will be called just $u$; it will give, for a suitable choice of $\eta$, the function seeked in the statement of Theorem 1.1.

Let also $g(x) := u(x) - f(x) = u(x) - (\beta!)^{-1}x^\beta$. By Theorem 3.1 we know that $D^\alpha g(0) = 0$ for any $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq |\beta|$. Furthermore, if $|\alpha| \geq |\beta| + 1$, then

$$|D^\alpha g(x)| = \eta^{|\alpha|-|\beta|}|D^\alpha v(\eta x + p)| \leq C_{|\alpha|}\eta \|v\|_{C^{|\alpha|}(B_{1/2}(p))}$$

for any $x \in B_1$, with some $C_{|\alpha|} > 0$. As a consequence, defining $k' := k + |\beta| + 1$ and fixing any $\gamma \in \mathbb{N}^n$ with $|\gamma| \leq k' - 1$ and any $x \in B_1$, we deduce by Taylor expansion that

$$D^\gamma g(x) = \sum_{|\beta|+1\leq|\gamma|+|\alpha|\leq k'-1} \frac{D^{\gamma+\alpha}g(0)}{\alpha!}x^\alpha + \sum_{|\gamma|+|\alpha|=k'} \frac{k'}{\alpha!}\int_0^1 (1-t)^{k'-1} D^{\gamma+\alpha}g(tx)\,dt\, x^\alpha,$$

and so $|D^\gamma g(x)| \leq C\eta$, with $C > 0$ possibly depending also on $v$.

Since this is valid for any $x \in B_1$, we obtain

$$\|u - f\|_{C^k(B_1)} = \|g\|_{C^k(B_1)} \leq \|g\|_{C^{k'-1}(B_1)} \leq C\eta$$

for some $C > 0$, which implies the statement of Theorem 1.1 as long as $\eta \in (0, C^{-1}\epsilon)$. □

*Acknowledgments.* The first author has been supported by EPSRC grant EP/K024566/1 "Monotonicity formula methods for nonlinear PDEs". The second author has been supported by NSF grant DMS-1200701. The third author has been supported by ERC grant 277749 "EPSILON Elliptic Pde's and Symmetry of Interfaces and Layers for Odd Nonlinearities" and PRIN grant 201274FYK7 "Aspetti variazionali e perturbativi nei problemi differenziali nonlineari".

## References

[1] Bucur, C.: Some observations on the Green function for the ball in the fractional Laplace framework. Comm. Pure Appl. Anal. **15**, 657–699 (2016) Zbl 1334.35383 MR 3461641
[2] Bucur, C.: Local density of Caputo-stationary functions in the space of smooth functions. ESAIM Control Optim. Calc. Var. (online) (2017)
[3] Caffarelli, L., Roquejoffre, J.-M., Savin, O.: Nonlocal minimal surfaces. Comm. Pure Appl. Math. **63**, 1111–1144 (2010) Zbl 1248.53009 MR 2675483

[4] Caffarelli, L., Silvestre, L.: An extension problem related to the fractional Laplacian. Comm. Partial Differential Equations **32**, 1245–1260 (2007) Zbl 1143.26002 MR 2354493
[5] Di Nezza, E., Palatucci, G., Valdinoci, E.: Hitchhiker's guide to the fractional Sobolev spaces. Bull. Sci. Math. **136**, 521–573 (2012) Zbl 1252.46023 MR 2944369
[6] Dipierro, S., Savin, O., Valdinoci, E.: Local approximation of arbitrary functions by solutions of nonlocal equations. arXiv:1609.04438 (2016)
[7] Fall, M. M., Felli, V.: Unique continuation property and local asymptotics of solutions to fractional elliptic equations. Comm. Partial Differential Equations **39**, 354–397 (2014) Zbl 1286.35250 MR 3169789
[8] Kassmann, M.: A new formulation of Harnack's inequality for nonlocal operators. C. R. Math. Acad. Sci. Paris **349**, 637–640 (2011) Zbl 1236.31003 MR 2817382
[9] Landkof, N. S.: Foundations of Modern Potential Theory. Grundlehren Math. Wiss. 180, Springer, New York (1972) Zbl 0253.31001 MR 0350027
[10] Ros-Oton, X., Serra, J.: The Dirichlet problem for the fractional Laplacian: Regularity up to the boundary. J. Math. Pures Appl. (9) **101**, 275–302 (2014) Zbl 1285.35020 MR 3168912
[11] Silvestre, L. E.: Regularity of the obstacle problem for a fractional power of the Laplace operator. PhD thesis, Univ. of Texas at Austin (2005) MR 2707618
[12] Stein, E. M.: Singular Integrals and Differentiability Properties of Functions. Princeton Math. Ser. 30, Princeton Univ. Press, Princeton, NJ (1970) Zbl 0207.13501 MR 0290095