J. Eur. Math. Soc. 19, 983–1011 © European Mathematical Society 2017
DOI 10.4171/JEMS/686

Moritz Kassmann · Ante Mimica†

Intrinsic scaling properties for nonlocal operators

Received August 2, 2014 and in revised form May 7, 2015

Abstract. We study integrodifferential operators and regularity estimates for solutions to integrodifferential equations. Our emphasis is on kernels with a critically low singularity which does not allow for standard scaling. For example, we treat operators that have a logarithmic order of differentiability. For corresponding equations we prove a growth lemma and derive a priori estimates. We derive these estimates by classical methods developed for partial differential operators. Since the integrodifferential operators under consideration generate Markov jump processes, we are able to offer an alternative approach using probabilistic techniques.

Keywords. Integrodifferential operators, regularity, jump processes, intrinsic scaling

1. Introduction

In recent years, regularity results for linear and nonlinear integrodifferential operators have been addressed by many research articles. Scaling properties are crucially used in these approaches. We reconsider these cases and, at the same time, include limit cases where standard scaling properties do not hold anymore. We study linear operators of the form

$$Au(x) = \int_{\mathbb{R}^d} \big(u(x+h) - u(x) - \langle \nabla u(x), h\rangle \mathbb{1}_{B_1}(h)\big) K(x,h)\, dh,$$

which, provided certain assumptions on $K(x, h)$ are satisfied, are well defined for smooth and bounded functions $u : \mathbb{R}^d \to \mathbb{R}$. The quantity $K(x, h)$ equals the jump intensity of jumps from $x \in \mathbb{R}^d$ to $x+h \in \mathbb{R}^d$ for the Markov process X that is generated by the linear operator A. If K is independent of the first variable, then X is a Lévy process. If $K(x, h) = |h|^{-\alpha-d} m(h/|h|)$ for all $h \neq 0$ and some appropriate function $m : \mathbb{S}^{d-1} \to [0, \infty]$, then the increments of X have stable distributions. Looking at the operator A as an integrodifferential operator, this property is important because it allows one to use scaling techniques.

M. Kassmann: Fakultät für Mathematik, Universität Bielefeld,
Postfach 100131, D-33501 Bielefeld, Germany; e-mail: moritz.kassmann@uni-bielefeld.de

A. Mimica: Department of Mathematics, University of Zagreb,
Bijenička c. 30, 10000 Zagreb, Croatia; url: https://web.math.pmf.unizg.hr/˜amimica/

†Unfortunately, the second author was not able to witness the publication of this work. Ante Mimica passed away on June 9, 2016, after a courageous two-year battle against a brain tumor.

Mathematics Subject Classification (2010): Primary 35B65; Secondary 35R09, 47G20, 60J75, 31B05

Scaling techniques themselves are crucial when studying regularity properties of functions $u : \mathbb{R}^d \to \mathbb{R}$ satisfying the equation $Au = f$ in a domain $\Omega \subset \mathbb{R}^d$ for some function $f : \Omega \to \mathbb{R}$. In this work we study such properties with the emphasis on two features. We do not assume any regularity of the kernel function K with respect to the first variable except for boundedness. Moreover, and this is the main new contribution, we systematically study classes of kernels that do not possess the aforementioned scaling property.

Our results include a growth lemma (expansion of positivity) and Hölder-type regularity estimates. Moreover, we provide several estimates on the corresponding Markov jump process. Recall that, in the case of an elliptic operator of second order $Au = a_{ij}(\cdot)\partial_i\partial_j u$, the standard growth lemma reads as follows:

Lemma 1. *There is a constant $\theta \in (0,1)$ such that if $R > 0$ and $u : \mathbb{R}^d \to \mathbb{R}$ satisfy*

$$-Au \le 0 \text{ in } B_{2R}, \qquad u \le 1 \text{ in } B_{2R}, \qquad |(B_{2R} \setminus B_R) \cap \{u \le 0\}| \ge \tfrac{1}{2}|B_{2R} \setminus B_R|,$$

then $u \le 1 - \theta$ in B_R.

The above lemma also holds true for several nonlinear operators. Such lemmas are systematically studied and applied in [Lan71]. Their importance is underlined in [KS79], where the authors establish a priori bounds for elliptic equations of second order with bounded measurable coefficients. Nowadays they form a standard tool for the study of various questions for nonlinear partial differential equations of second order (see [CC95] and [DGV12]). Note that the property formulated in Lemma 1 is also referred to as expansion of positivity which describes the corresponding property for $1 - u$.

In this work we prove a similar growth lemma for integrodifferential operators (see Lemma 2 below). An important instance of an operator A that we have in mind is

$$Au(x) = \int_{\mathbb{R}^d} [u(x+h) - u(x)] a(x,h) |h|^{-d} \mathbb{1}_{B_1}(h)\, dh \tag{1}$$

for some measurable function $a : \mathbb{R}^d \times \mathbb{R}^d \to [1,2]$. Note that, in the last years, similar results have been studied for kernels of the form $a(x,h) \asymp |h|^{-\alpha}$ for some $\alpha \in (0,2)$ and we refer the reader to the short discussion below. The case $\alpha = 0$ is of particular interest because in this case the corresponding growth lemma fails. Our results apply to more general kernels than the one appearing in (1).

1.1. Main assumptions and results

Let $K : \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) \to [0,\infty)$ be a measurable function such that $K(x,h) = K(x,-h)$ for all x, h and

$$\kappa^{-1}|h|^{-d}\ell(|h|) \le K(x,h) \le \kappa |h|^{-d}\ell(|h|) \quad \text{for } 0 < |h| < R_0, \tag{A$_1$}$$

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d\setminus\{0\}} (1 \wedge |h|^2) K(x,h)\, dh \le K_0, \tag{K$_0$}$$

where $\kappa, K_0 \ge 1$, $R_0 \in (0,\infty]$ are fixed constants and $\ell : (0,R_0) \to (0,\infty)$ is a function satisfying $\int_0^{R_0} (\ell(s)/s)\, ds = \infty$ and, for some $c_L \in (0,1)$, $c_U \ge 1$, and $\gamma \in (0,2)$, the following:

$$\int_r^{R_0} \ell(s)\,\frac{ds}{s} < \infty \qquad \text{for } 0 < r < R_0, \tag{ℓ_1}$$

$$\ell(r\lambda)/\ell(r) \geq c_L \lambda^{-\gamma} \qquad \text{for } r > 0 \text{ and } 1 \leq \lambda < R_0/r, \tag{ℓ_2}$$

$$\ell(r\lambda)/\ell(r) \leq c_U \lambda^{d} \qquad \text{for } r > 0 \text{ and } 1 \leq \lambda < R_0/r. \tag{ℓ_3}$$

The last two conditions are often referred to as *weak scaling conditions*.

Examples. The standard example is given by $\ell(s) = s^{-\alpha}$ for some $\alpha \in (0,2)$. Other examples include $\ell(s) = s^{-\alpha} g(s)$ for $\alpha \in (0,2)$ and g a function that varies slowly at 0. More generally, the conditions (ℓ_2) and (ℓ_3) are satisfied if the function ℓ is regularly varying at zero of order $-\alpha \in (-2, 0]$ and satisfies some weak bounds for large values of s. The case $\alpha = 0$ is very interesting. The choice $\ell(s) = 1$ is possible if $R_0 < \infty$. In the case $R_0 = \infty$ an interesting example is provided by $\ell(s) = \mathbb{1}_{(0,1)}(s) + s^{-\gamma}\mathbb{1}_{[1,\infty)}(s)$ for some $\gamma > 0$.

We define an auxiliary function $L\colon (0, R_0) \to (0,\infty)$ by

$$L(r) = \int_r^{R_0} \frac{\ell(s)}{s}\,ds,$$

which is strictly decreasing. Note that under our assumptions, $L(0+) = \infty$ and $L(R_0-) = 0$. Furthermore, we define a measure μ on $\mathcal{B}(\mathbb{R}^d)$ by

$$\mu(dy) = \frac{\ell(|y|)}{L(|y|)}\,\frac{dy}{|y|^d}$$

and, for $a > 1$, a scale function $\varphi_a = \varphi\colon (0, R_0) \to (0,\infty)$ by $\varphi(r) = L^{-1}(a^{-1}L(r))$.

Define an operator $A\colon C_b^2(\mathbb{R}^d) \to C(\mathbb{R}^d)$ by

$$Au(x) = \int_{\mathbb{R}^d\setminus\{0\}} \big(u(x+h) - u(x) - \langle \nabla u(x), h\rangle \mathbb{1}_{B_1}(h)\big) K(x,h)\,dh \tag{2}$$

where $K : \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) \to [0,\infty)$ satisfies (A_1) and (K_0).

Now we can formulate our first main result, a growth lemma for nonlocal operators. We state the result for functions which, together with their first and second derivatives, are continuous and bounded. It is an important feature of this result that none of the constants depends on the regularity of the function under consideration. Thus, the result is tailored for later applications to viscosity solutions of fully nonlinear partial differential equations.

Lemma 2. *Assume* (K_0) *and* (A_1) *hold true with* $R_0 = \infty$. *Let* $\eta, \delta \in (0,1)$ *and* $C_0 > 0$. *There exist constants* $a > 2$ *and* $\theta \in (0,1)$ *such that if* $r > 0$ *and* $v \in C_b^2(\mathbb{R}^d)$ *satisfy*

$$\begin{aligned}
&-Av(x) \leq L(\varphi(r)) \quad (= a^{-1}L(r)) && \textit{for } x \in B_{\varphi(r)},\\
&v(x) \leq 1 && \textit{for } x \in B_{\varphi(r)},\\
&v(x) \leq C_0 \left(\frac{L(\varphi(r))}{L(|x|)}\right)^{\eta} && \textit{for } x \in \mathbb{R}^d \setminus B_{\varphi(r)},\\
&\mu\big((B_{\varphi(r)} \setminus B_r) \cap \{v \leq 0\}\big) \geq \delta\mu(B_{\varphi(r)} \setminus B_r),
\end{aligned}$$

then

$$v(x) \leq 1 - \theta \qquad \textit{for all } x \in B_r. \tag{3}$$

Remark. As the proof shows, the value of θ is a multiple of a^{-1}.

Note that, in the by now well-known case where $\ell(r) = r^{-\alpha}$ and $L(r) \asymp r^{-\alpha}$ for $\alpha \in (0, 2)$, this result reduces to a growth lemma which is very similar to those given in [Sil06] and [CS09]. Let us now formulate our second main result.

Theorem 3. *Assume* (K_0) *and* (A_1) *hold true with* $R_0 \in (0, \infty]$. *There exist constants* $c > 0$ *and* $\beta \in (0, 1)$ *such that if* $0 < r \le R_0/2$, $f \in L^\infty(B_r)$, *and* $u \in C_b^2(\mathbb{R}^d)$ *satisfy* $Au = f$ *in* B_r, *then*

$$\sup_{x,y\in B_{r/4}} \frac{|u(x) - u(y)|}{L(|x - y|)^{-\beta}} \le cL(r)^{\beta}\|u\|_\infty + cL(r)^{\beta-1}\|f\|_{L^\infty(B_r)}. \tag{4}$$

If $R_0 = \infty$, *then* (4) *holds true for every* $r > 0$.

In the case $\ell(r) = r^{-\alpha}$, we set $\nu = \alpha\beta$, and the estimate (4) reduces to

$$\sup_{x,y\in B_{r/4}} \frac{|u(x) - u(y)|}{|x - y|^{\nu}} \le cr^{-\nu}\|u\|_\infty + cr^{\alpha-\nu}\|f\|_{L^\infty(B_r)},$$

which one would expect from standard scaling behavior of the integrodifferential operator.

In the case $R_0 = \infty$ we obtain a Liouville theorem.

Corollary 4. *If* (A_1) *holds for* $R_0 = \infty$, *then every function* $u \in C_b^2(\mathbb{R}^d)$ *satisfying* $Au = 0$ *on* $\mathbb{R}^d$ *is a constant function.*

Proof. Since u is harmonic in every ball B_r we can consider $r \to \infty$ in Theorem 3 and use $\lim_{r\to\infty} L(r) = 0$ in order to prove that u is a constant function. □

Our method to prove Lemma 2 and Theorem 3 is based on a purely analytic technique introduced in [Sil06]. As mentioned above, a second aim of this work is to explain a probabilistic approach to results like Theorem 3. The starting point for these observations is that, for several linear differential or integrodifferential operators A, variants of Lemma 1 can be established with the help of the corresponding Markov processes. Let X be the strong Markov process associated with the operator A, i.e. we assume that the martingale problem has a unique solution. Denote by T_A, τ_A the hitting resp. exit time for a measurable set $A \subset \mathbb{R}^d$ and by $\mathbb{P}_x$ the measure on the path space with $\mathbb{P}_x(X_0 = x) = 1$. The following property then implies Lemma 1.

Proposition 5. *There is a constant* $c \in (0, 1)$ *such that for every* $R > 0$, *every measurable set* $A \subset B_{2R} \setminus B_R$ *with* $|(B_{2R} \setminus B_R) \cap A| \ge \frac{1}{2}|B_{2R} \setminus B_R|$ *and every* $x \in B_R$,

$$\mathbb{P}_x(T_A < \tau_{B_{2R}}) \ge c. \tag{5}$$

This result is established for nondegenerate diffusions in [KS79], thus leading to a result like Theorem 3 for elliptic differential operators of second order. The case of integrodifferential operators with fractional order of differentiability $\alpha \in (0, 2)$ is treated in [BL02]. Therein it is shown that Proposition 5 holds true for jump processes X generated by integral operators A of the form (2) under the assumptions $K(x,h) = K(x,-h)$ and $K(x,h) \asymp |h|^{-d-\alpha}$ for all x and h where $\alpha \in (0,2)$ is fixed. Note that this class includes the case $Au = -(-\Delta)^{\alpha/2}u$ and versions with bounded measurable coefficients.

Proposition 5 fails to hold for several cases we are interested in. One example is given by A as in (2) with $K(x,h) = k(h) \asymp |h|^{-d}$ for $|h| \leq 1$ and some appropriate condition for $|h| > 1$. For example, the geometric stable process with generator $-\ln(1+(-\Delta)^{\alpha/2})$, $0 < \alpha \leq 2$, can be represented by (2) with a kernel $K(x,h) = k(h)$ with such behavior for $|h|$ close to zero. The operator resp. the corresponding stochastic process can be shown not to satisfy a uniformly hitting estimate like (5) (see [Mim14]).

This leads to the question whether a priori estimates can be obtained by the approach from [BL02] at all. In the second part of this work we address this question. It turns out that our main idea, i.e., to determine a new intrinsic scale, can also be used to establish a modification of (5). As we did in the proof of Lemma 2, we choose a measure different from the Lebesgue measure for the assumption $|(B_{2R} \setminus B_R) \cap A| \geq \frac{1}{2}|B_{2R} \setminus B_R|$. We refer the reader to Section 6 for further details.

Since we employ methods from two different fields: partial differential operators as in [Sil06] and stochastic analysis as in [BL02], it is interesting to compare both approaches. In both, we need to make several assumptions, e.g., solvability of the equation and existence of the corresponding Markov jump process. The conditions in the analytic approach are slightly less restrictive than those imposed when using stochastic analysis. Note that although we assume the solutions u are twice differentiable in the first part, the assertions resp. the constants in our results do not depend on the regularity of the functions u. Thus, the techniques and assertions presented here can be applied to nonlinear problems.

1.2. Examples

Let us look at different choices for the function ℓ used in condition (A_1). Note that (ℓ_2) does not allow $\ell(h)$ to be zero (unlike $K(x,h)$). Since the behavior of ℓ at zero is most important and characteristic, we provide examples of functions $\ell : (0,1) \to (0,\infty)$. For $a > 1$, $r \in (0,1)$, set $L(r) = \int_r^1 (\ell(s)/s)\,ds$ and $\varphi_a(r) = L^{-1}(a^{-1}L(r))$.

Table 1. Different choices for a function ℓ when $\beta \in (0,2)$, $a > 1$.

$\ell(s)$	$L(s)$	$\varphi_a(s)$
$s^{-\beta}\ln(2/s)^2$	$\asymp s^{-\beta}\ln(2/s)^2$	$\asymp s$
$s^{-\beta}$	$\beta^{-1}(s^{-\beta}-1)$	$\asymp s$
$\ln(2/s)$	$\asymp \ln(2/s)^2$	$\asymp s^{1/\sqrt{a}}$
1	$\ln(1/s)$	$s^{1/a}$
$\ln(2/s)^{-1}$	$\asymp \ln(\ln(2/s))$	$\asymp \exp(-\ln(2/s)^{1/a})$

1.3. *Related results in the literature*

Let us comment on related results in the literature. The probabilistic approach, which we explain in Section 5, is based on the approach of [BL02]. The analytic method, which we employ in the first part of the article, is based on [Sil06]. Both approaches have been refined in many articles, allowing for more general kernels and treating fully nonlinear integrodifferential equations, but all these articles assume standard scaling properties, i.e., something like $K(x,h) \asymp |h|^{-d-\alpha}$ for some $\alpha \in (0,2)$. We refer to [KS14, SS14] for further references. Note that our regularity result, Theorem 3, is stronger than Theorem 12 because we can allow for right-hand sides f in the integrodifferential equation and for more general kernels.

The current work comprises the two preprints [KM13] and [KM14] where the approaches by analytic and probabilistic methods are explained separately. After [KM13] had appeared, several articles have made use of the ideas therein. In [Bae15] nonlocal problems are studied where the kernels are supposed to satisfy certain upper and lower scaling conditions. These assumptions do not include limit cases like (1) since some comparability with kernels like $|h|^{-d-\alpha}$ for $\alpha \in (0,2)$ is still assumed. In [KKL16] the authors study fully nonlinear problems with assumptions on the kernels similar to [Bae15]. In [CZ15] the authors extend the regularity estimates of [KM13] to time-dependent equations with drifts. The article [JW16] is not directly related to [KM13] but mentions the need to consider $f \neq 0$. We solve this problem.

1.4. *Organization of the article*

In Section 2 we review the relation between translation invariant nonlocal operators and semigroups/Lévy processes. Presumably, Proposition 6 is of some interest to many readers since it establishes a one-to-one relation between the behavior of a Lévy measure at zero and the behavior of the multiplier of the corresponding generator for large values of $|\xi|$. Sections 3 and 4 contain the proof of Lemma 2 and Theorem 3 respectively. In Section 5 we explain the probabilistic approach to Theorem 3, which leads to Theorem 12. Note that we are slightly changing the assumptions there. The probabilistic approach is based on a Krylov–Safonov type hitting lemma, which is Proposition 13. Section 6 contains the proof of this result and of Theorem 12. The last section is an Appendix in which we collect important properties of regularly resp. slowly varying functions.

2. Multipliers and Lévy measure: analysis meets probability

The aim of this section is to provide some background about translation invariant integrodifferential operators and related stochastic processes. The results explained here motivated the search for a new scale function which is a key element of the whole project. However, the material of this section is not needed for the proofs of the main results.

In this paper we provide two approaches to Theorem 3. One approach uses techniques from analysis, the other uses stochastic processes. Note that the quantity $K(x,h)\,dh$ in (2) has a clear interpretation in terms of probability. For fixed x, the quantity $\int_M K(x,h)\,dh$

describes the intensity with which the corresponding process performs jumps from some point $x \in \mathbb{R}^d$ to a point from the set $x + M$. In this sense, the conditions (ℓ_1)–(ℓ_3) say something about the behavior of the process. On the other hand, the conditions say something about mapping properties of the operator A. In this section we explain the link between these two viewpoints. We restrict ourselves to the cases of translation invariant operators, i.e., we assume $K(x, h)$ to be independent of the first variable. This allows us to give a focused presentation. Note that the results of this section are not used in the rest of the article.

In the translation invariant case, i.e. when $K(x, h)$ does not depend on x, there is a one-to-one correspondence between A and multipliers, semigroups and stochastic processes. One aim is to prove how the behavior of $\ell(|h|)$ for small values of $|h|$ translates into properties of the multiplier or characteristic exponent $\psi(|\xi|)$ for large values of $|\xi|$. This is achieved in Proposition 6. We add a subsection where we discuss which regularity results are known in critical cases of the (much simpler) translation invariant case. Note that our set-up, although allowing for an irregular dependence of $K(x, h)$ on $x \in \mathbb{R}^d$, leads to new results in these critical cases.

2.1. Semigroups, generators and Lévy processes

A stochastic process $X = (X_t)_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *Lévy process* if it has stationary and independent increments, $\mathbb{P}(X_0 = 0) = 1$ and its paths are $\mathbb{P}$-a.s. right continuous with left limits. For $x \in \mathbb{R}^d$ we define $\mathbb{P}_x$ to be the law of the process $X + x$. In particular, $\mathbb{P}_x(X_t \in B) = \mathbb{P}(X_t \in B - x)$ for $t \geq 0$ and measurable sets $B \subset \mathbb{R}^d$.

Due to stationarity and independence of increments, the characteristic exponent of X_t is given by

$$\mathbb{E}[e^{i\langle \xi, X_t\rangle}] = e^{-t\psi(\xi)},$$

where ψ is called the *characteristic exponent* of X. It has the following Lévy–Khintchine representation:

$$\psi(\xi) = \frac{1}{2}\langle A\xi, \xi\rangle + \langle b, \xi\rangle + \int_{\mathbb{R}^d\setminus\{0\}} \big(1 - e^{i\langle \xi, h\rangle} + i\langle \xi, h\rangle \mathbb{1}_{B_1}(h)\big)\, \nu(dh), \tag{6}$$

where A is a symmetric nonnegative definite matrix, $b \in \mathbb{R}^d$ and ν is a measure on $\mathbb{R}^d\setminus\{0\}$ satisfying $\int_{\mathbb{R}^d\setminus\{0\}} (1 \wedge |y|^2)\, \nu(dy) < \infty$, called the *Lévy measure* of X.

The converse also holds; that is, given ψ as in the Lévy–Khintchine representation (6), there exists a Lévy process $X = (X_t)_{t\geq 0}$ with the characteristic exponent ψ. The equality (6) provides a link to an analytic viewpoint on Lévy processes. If ν is a Lévy measure, i.e., a Borel measure on $\mathcal{B}(\mathbb{R}^d \setminus \{0\})$, then one can construct a convolution semigroup $(\nu_t)_{t>0}$ of probability measures such that the Fourier transform of ν_t equals $e^{-t\psi}$ with ψ as in (6). This approach can be found in [BF75].

Let $X = (X_t)_{t\geq 0}$ be a Lévy process corresponding to the characteristic exponent ψ as in (6) with $A = 0$, $b = 0$ and a Lévy measure $\nu(dh)$. Then $P_t f(x) := \mathbb{E}_x[f(X_t)]$ defines a strongly continuous contraction semigroup $(P_t)_{t\geq 0}$ of operators on the space of bounded uniformly continuous functions on $\mathbb{R}^d$ equipped with the supremum norm. Moreover, it is a convolution semigroup, since

$$\mathbb{P}_t f(x) = \mathbb{E}_0[f(x + X_t)] = \int_{\mathbb{R}^d} f(x + y)\, \nu_t(dy)$$

with $\nu_t(B) := \mathbb{P}(X_t \in B)$. The infinitesimal generator A of the semigroup $(P_t)_{t\geq 0}$ is given by

$$Au(x) = \int_{\mathbb{R}^d\setminus\{0\}} \big(u(x + h) - u(x) - \langle \nabla u(x), h\rangle \mathbb{1}_{B_1}(h)\big)\, \nu(dh) \tag{7}$$

if u is sufficiently regular (see [Sat13, proof of Theorem 31.5]. Note that the process $(u(X_t) - u(X_0) - \int_0^t Au(X_s)\, ds)_{t\geq 0}$ is a martingale (with respect to the natural filtration) for every $u \in C_b^2(\mathbb{R}^d)$ (see [RY99, proof of Proposition VII.1.6]. In this sense the process X corresponds to the given Lévy measure ν and, in our set-up, to the kernel $K(x, h) = k(h)$. For details about Lévy processes we refer to [Ber96, Sat13].

Let us now explain the connection between the characteristic exponent ψ and the symbol of the operator A. To be more precise, if $\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{i\langle\xi,x\rangle} f(x)\, dx$ denotes the Fourier transform of a function $f \in L^1(\mathbb{R}^d)$, then

$$\widehat{Af}(\xi) = -\psi(-\xi)\hat{f}(\xi)$$

for any $f \in \mathcal{S}(\mathbb{R}^d)$ with $Af \in L^1(\mathbb{R}^d)$, where $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz space (see [Ber96, Proposition I.2.9]). Hence, $-\psi(-\xi)$ is the symbol (multiplier) of the operator A. The following result explains how, in the case $\nu(dh) = k(h)dh = K(x, h)dh$, the kernel $K(x, h) = k(h)$ is related to the characteristic exponent resp. the multiplier.

Proposition 6. *Assume that the operator A defined on $\mathcal{S}$ is given by (7). Assume $\nu(dh) = k(h)dh = K(x, h)dh$ where $K: \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) \to [0, \infty)$ is a measurable function with $K(x, h) = K(x, -h)$ for almost all x, h. Assume that K satisfies (A_1) and (K_0) with $R_0 \in (0, \infty]$. Set $L(r) = \int_r^{R_0} (\ell(s)/s)\, ds$. Then there are constants $c, r_0 > 0$ such that*

$$c^{-1}L(|\xi|^{-1}) \leq \psi(\xi) \leq cL(|\xi|^{-1}) \quad \textit{for } \xi \in \mathbb{R}^d,\ |\xi| \geq r_0.$$

The assumptions of Proposition 6 allow one to treat sophisticated examples. However, it is instructive to think about the simple examples

$$\begin{aligned} K(x, h) &= |h|^{-d-\alpha} \quad \text{for some } \alpha \in (0, 2),\\ K(x, h) &= |h|^{-d}\mathbb{1}_{B_1}(h),\\ K(x, h) &= |h|^{-d} \ln(2/|h|)^{\pm 1}\mathbb{1}_{B_1}(h). \end{aligned}$$

Proof of Proposition 6. Note first that, by (A_1),

$$\kappa^{-1} j(|h|) \leq k(h) \leq \kappa j(|h|) \quad \text{for } 0 < |h| < R_0$$

for $j(s) = s^{-d}\ell(s)$. Since $1 - \cos x \leq \frac{1}{2}x^2$, it follows from ($\ell_2$) and Lemma 7 below that

$$\begin{aligned}\psi(\xi) &\le \tfrac{1}{2}|\xi|^2 \int_{|h|\le|\xi|^{-1}} |h|^2 j(|h|)\,dh + 2\int_{|\xi|^{-1}<|h|<R_0} j(|h|)\,dh + 2\int_{|h|\ge R_0} k(h)\,dh \\ &\le c_1\Big[|\xi|^2 \int_0^{|\xi|^{-1}} s\ell(s)\,ds + L(|\xi|^{-1}) + 1\Big] \\ &\le c_2\Big[|\xi|^2\ell(|\xi|^{-1}) \int_0^{|\xi|^{-1}} s c_L^{-1}(s/|\xi|^{-1})^{-\gamma}\,ds + L(|\xi|^{-1})\Big] \\ &\le c_3\big(\ell(|\xi|^{-1}) + L(|\xi|^{-1})\big) \le c_4 L(|\xi|^{-1}).\end{aligned}$$

In order to prove the lower bound, we employ an idea of [Grz14]. Let us first consider the case $R_0 = \infty$. We choose an orthogonal transformation of the form $Oe_1 = |\xi|^{-1}\xi$, where $e_1 := (1, 0, \dots, 0) \in \mathbb{R}^d$. Then a change of variable yields

$$\psi(\xi) = \int_{\mathbb{R}^d\setminus\{0\}} (1-\cos(\langle \xi, h\rangle)) j(|h|)\,dh = \int_{\mathbb{R}^d\setminus\{0\}} (1-\cos(|\xi|h_1)) j(|h|)\,dh.$$

By the Fubini theorem, we obtain

$$\psi(\xi) \ge 2\int_0^\infty (1-\cos(|\xi|r))F(r)\,dr,$$

where $F(r) := \int_{\mathbb{R}^{d-1}} j(\sqrt{|z|^2+r^2})\,dz$ for $r > 0$. Using (ℓ_3) we deduce that for every $0 < r \le s$,

$$F(r) = \int_{\mathbb{R}^{d-1}} \left(\frac{|z|^2+s^2}{|z|^2+r^2}\right)^{d/2} \frac{\ell(\sqrt{|z|^2+r^2})}{\ell(\sqrt{|z|^2+s^2})} j(\sqrt{|z|^2+s^2})\,dz \ge c_U^{-1}F(s).$$

Now,

$$\begin{aligned}\psi(\xi) &\ge 2\sum_{k=0}^\infty \int_{|\xi|^{-1}(\pi/2+2k\pi)}^{|\xi|^{-1}(3\pi/2+2k\pi)} (1-\cos(|\xi|r))F(r)\,dr \\ &\ge \frac{c_U^{-1}\pi}{|\xi|}\sum_{k=0}^\infty F(|\xi|^{-1}(3\pi/2+2k\pi)) \ge c_U^{-2}\sum_{k=0}^\infty \int_{|\xi|^{-1}(3\pi/2+2k\pi)}^{|\xi|^{-1}(3\pi/2+(2k+1)\pi)} F(r)\,dr \\ &\ge c_U^{-2}\int_{\frac{3\pi}{2}|\xi|^{-1}}^\infty F(r)\,dr \ge c_5 \int_{|h|\ge\frac{3\pi}{2}|\xi|^{-1}} j(|h|)\,dh = c_6 L\big(\tfrac{3\pi}{2}|\xi|^{-1}\big) \ge c_7 L(|\xi|^{-1}),\end{aligned}$$

where in the last inequality we have used Lemma 7. The case $R_0 < \infty$ can be proved similarly. □

2.2. *Some related results from potential theory*

Let us explain which results, related to Theorem 3, have been obtained in the case where $K(x,h)$ is independent of $x \in \mathbb{R}^d$. In this case, methods from potential theory can be used.

Hölder estimates of harmonic functions are obtained for the Lévy process with the characteristic exponent $\psi(\xi) = |\xi|^2/\ln(1+|\xi|^2) - 1$ in [Mim13] by establishing a Krylov–Safonov type estimate replacing the Lebesgue measure with the capacity of the sets involved. Recently, regularity estimates have been obtained in [Grz14] for a class of isotropic unimodal Lévy processes which is quite general but does not include Lévy processes with slowly varying Lévy exponents such as geometric stable processes. Regularity of harmonic functions for such processes is investigated in [Mim14], where it is shown that a result like Proposition 5 fails. Using the Green function, logarithmic bounds for the modulus of continuity are obtained. At this point it is worth mentioning that the transition density $p_t(x,y)$ of the geometric stable process satisfies $p_1(x,x) = \infty$ (see [ŠSV06]). This illustrates that regularity results like Theorem 3 in the case $K(x,h) = |h|^{-d}\mathbb{1}_{B_1}(h)$ and in similar cases are quite delicate.

3. Proof of Lemma 2

Before we proceed to the main proof, let us provide two auxiliary statements which indicate the link of the scale function φ with the kernels K.

Lemma 7. *The following properties of the function L hold true:*

$$L(r) \geq \gamma^{-1} c_L \ell(r) \qquad \text{for } r > 0, \tag{i}$$

In the case $R_0 = \infty$:
$$\frac{L(r\lambda)}{L(r)} \geq c_L \lambda^{-\gamma} \qquad \text{for } \lambda \geq 1,\ r > 0, \tag{ii-a}$$

In the case $R_0 < \infty$:
$$\frac{L(r\lambda)}{L(r)} \geq \frac{c_L}{2}\lambda^{-\gamma} \qquad \text{for } 1 \leq \lambda < \lambda_1,\ 0 < r < r_1, \tag{ii-b}$$

where $\lambda_1 > 1$ can be chosen arbitrarily and $r_1 > 0$ depends on λ_1 and R_0.

Proof. From (ℓ_2) we deduce

$$L(r) = \ell(r)\int_r^{R_0} \frac{\ell(s)}{\ell(r)}\frac{ds}{s} \geq c_L \ell(r) r^{\gamma} \int_r^{R_0} s^{-1-\gamma}\, ds = c_L \gamma^{-1} \ell(r),$$

which proves part (i). In the case $R_0 = \infty$ we obtain

$$\begin{aligned} L(r\lambda) &= \int_{r\lambda}^{\infty} \ell(s)\frac{ds}{s} = \int_r^{\infty} \ell(s\lambda)\frac{ds}{s} = \int_r^{\infty} \frac{\ell(s\lambda)}{\ell(s)}\ell(s)\frac{ds}{s} \geq c_L\lambda^{-\gamma}\int_r^{\infty}\ell(s)\frac{ds}{s} \\ &= c_L \lambda^{-\gamma} L(r), \end{aligned}$$

which is one part of claim (ii). In the case $R_0 < \infty$ we deduce

$$\begin{aligned} L(r\lambda) &= \int_{r\lambda}^{R_0} \ell(s)\frac{ds}{s} = \int_r^{R_0/\lambda} \ell(s\lambda)\frac{ds}{s} \geq c_L\lambda^{-\gamma}\int_r^{R_0/\lambda}\ell(s)\frac{ds}{s} \\ &\geq c_L\lambda^{-\gamma}(L(r) - L(R_0/\lambda)) \geq c_L\lambda^{-\gamma}(L(r) - L(R_0/\lambda_1)) \\ &= \frac{c_L}{2}\lambda^{-\gamma}L(r) + \frac{c_L}{2}\lambda^{-\gamma}(L(r) - 2L(R_0/\lambda_1)) \geq \frac{c_L}{2}\lambda^{-\gamma}L(r), \end{aligned}$$

which proves the remaining case and completes the proof. □

Lemma 8. *Assume $a > 1$. Then $\mu(B_{\varphi_a(r)} \setminus B_r) = |\partial B_1| \ln a$, where $|\partial B_1|$ denotes the surface area of the unit sphere in $\mathbb{R}^d$.*

Proof. This follows by introducing polar coordinates:

$$\mu(B_{\varphi_a(r)} \setminus B_r) = |\partial B_1| \int_r^{\varphi_a(r)} \frac{1}{L(s)} \frac{\ell(s)\, ds}{s} = |\partial B_1| \ln \frac{L(r)}{L(\varphi_a(r))} = |\partial B_1| \ln a. \qquad \square$$

Lemma 9. *Set $j(s) = s^{-d}\ell(s)$ for $0 < s < R_0$. Let $M \geq 1$. Then*

$$s \leq Mt \leq R_0 \quad \textit{and} \quad t \leq R_0 \quad \textit{imply} \quad j(t) \leq c j(s)$$

with $c = \max\{c_U, M^{\gamma+d} c_L^{-1}\}$.

Proof. Assume $s, t > 0$ with $s \leq Mt \leq R_0$ for some $M \geq 1$. We consider two cases. If $s < t$, then

$$j(t) = t^{-d}\ell(t) = (s/t)^d s^{-d} \ell(s(t/s)) \leq c_U s^{-d} \ell(s),$$

where we have applied (ℓ_3). If $t \leq s \leq Mt$, then

$$j(t) = t^{-d}\ell(t) \leq (s/t)^{\gamma} c_L^{-1} (M^{-1}s)^{-d} \ell(s) \leq M^{\gamma+d} c_L^{-1} j(s),$$

where we have applied (ℓ_2). The proof is complete. $\square$

We are now able to provide the proof of our first main result. Recall that this result is proved under the assumption $R_0 = \infty$.

Proof of Lemma 2. Define $\beta\colon [0, \infty) \to [0, \infty)$, $\beta(r) = \exp(-r^2)$, and further

$$b(x) := \beta(|x|) \quad \text{and} \quad b_r(x) := \beta_r(|x|) := \beta(r^{-1}|x|) \quad \text{for } x \in \mathbb{R}^d,\ r > 0.$$

First we estimate $-Ab_r$. For $r > 0$ we can deduce from (ℓ_1), (ℓ_2) and Lemma 7 that

$$\begin{aligned}
-Ab_r(x) &= \int_{\mathbb{R}^d} \left(b\left(\frac{x}{r}\right) - b\left(\frac{x+y}{r}\right) + \left\langle \nabla b\left(\frac{x}{r}\right), \frac{y}{r} \right\rangle \mathbb{1}_{B_1}(y) \right) K(x, y)\, dy \\
&\leq c_1 \int_{\mathbb{R}^d} \left(\left(\frac{|y|}{r} \right)^2 \mathbb{1}_{B_r}(y) + \mathbb{1}_{B_r^c}(y) \right) \frac{\ell(|y|)}{|y|^d}\, dy \\
&= c_2 \left(r^{-2} \int_0^r s\ell(s)\, ds + \int_r^\infty \ell(s) \frac{ds}{s} \right) = c_2 \left(r^{-2}\ell(r) \int_0^r s \frac{\ell(s)}{\ell(r)}\, ds + L(r) \right) \\
&\leq c_3 \left(r^{-2}\ell(r) \int_0^r s \left(\frac{r}{s}\right)^{\gamma} ds + L(r) \right) \leq c_4(\ell(r) + L(r)) \leq c_5 L(r).
\end{aligned}$$

Hence,

$$\sup_{x \in \mathbb{R}^d} -Ab_r(x) \leq c_6 L(r). \tag{8}$$

Set

$$\theta := \frac{1}{a}\left(\beta(1) - \beta\left(\frac{3}{2}\right) \right) = \frac{1}{a}\left(\beta_r(r) - \beta_r\left(\frac{3r}{2}\right) \right),$$

where $a > 2$ will be chosen later independently of v and r.

We claim that one can choose $a > 2$ so large that $v(x) \leq 1 - \theta$ for $x \in B_r$. Assume that this is not true. Then for $a > 2$ there is $x_0 \in B_r$ satisfying

$$v(x_0) \geq 1 - \theta = 1 - a^{-1}\beta_r(r) + a^{-1}\beta_r(3r/2).$$

Since $|x_0| < r$,

$$\begin{aligned} v(x_0) + a^{-1}b_r(x_0) &\geq 1 + a^{-1}\beta_r(|x_0|) - a^{-1}\beta_r(r) + a^{-1}\beta_r(3r/2) \\ &> 1 + a^{-1}\beta_r(3r/2) \\ &\geq v(y) + a^{-1}\beta_r(|y|) \quad \text{for all } y \in B_{\varphi(r)} \setminus B_{3r/2}, \end{aligned} \tag{9}$$

where the last inequality follows from the assumption $v(x) \leq 1$ for $x \in B_{\varphi(r)}$ and $\beta_r(|y|) \leq \beta_r(3r/2)$. By choosing a sufficiently large, we will ensure that $\varphi(r) > 3r/2$. It follows from (9) that $v+a^{-1}b_r$ attains its maximum at $x_1 \in B_{3r/2}$ and $(v+a^{-1}b_r)(x_1) \geq 1 + a^{-1}\beta_r(3r/2) > 1$.

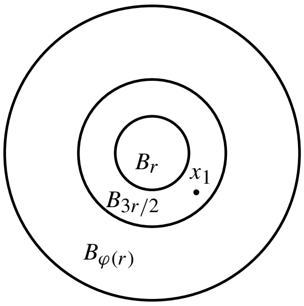

Fig. 1. $B_r \subset B_{3r/2} \subset B_{\varphi(r)}$.

The idea now is to establish a contradiction by evaluating $-A(v + a^{-1}b_r)(x_1)$ in two different ways. First, by (8),

$$-A(v + a^{-1}b_r)(x_1) \leq a^{-1}L(r) + c_6a^{-1}L(r) = (1 + c_6)a^{-1}L(r).$$

On the other hand, since $v + a^{-1}b_r$ attains its maximum at x_1, $\nabla(v + a^{-1}b_r)(x_1) = 0$ and hence

$$\begin{aligned} -A(v + a^{-1}b_r)(x_1) &= \int_{\mathbb{R}^d\setminus\{0\}} \big((v + a^{-1}b_r)(x_1) - (v + a^{-1}b_r)(x_1 + y)\big)K(x_1, y)\,dy \\ &= \int_{\{y\in\mathbb{R}^d\setminus\{0\}:\ x_1+y\in B_{\varphi(r)}\}} \big((v + a^{-1}b_r)(x_1) - (v + a^{-1}b_r)(x_1 + y)\big)K(x_1, y)\,dy \\ &\quad + \int_{\{y\in\mathbb{R}^d\setminus\{0\}:\ x_1+y\notin B_{\varphi(r)}\}} \big((v + a^{-1}b_r)(x_1) - (v + a^{-1}b_r)(x_1 + y)\big)K(x_1, y)\,dy \\ &=: I_1 + I_2. \end{aligned}$$

Since $v + a^{-1}b_r$ attains its maximum on $B_{\varphi(r)}$ at x_1 with $v + a^{-1}b_r > 1$, by (A_1) we obtain

$$I_1 \geq \int_{\{y\in\mathbb{R}^d\setminus\{0\}:\, x_1+y\in B_{\varphi(r)}\setminus B_r,\, v(x_1+y)\leq 0\}} \big((v+a^{-1}b_r)(x_1)-(v+a^{-1}b_r)(x_1+y)\big)K(x_1,y)\,dy$$
$$\geq c_7(1-a^{-1}\|b\|_\infty)\int_{\{y\in\mathbb{R}^d\setminus\{0\}:\, x_1+y\in B_{\varphi(r)}\setminus B_r,\, v(x_1+y)\leq 0\}} j(|y|)\,dy$$

with $j(s) := s^{-d}\ell(s)$.

Using $|y| \leq |x_1+y|+|x_1| \leq |x_1+y|+3r/2 \leq \frac{5}{2}|x_1+y|$ for $x_1+y \in B_{\varphi(r)}\setminus B_r$, we deduce from ($\ell_2$) that $j(|y|) \geq c_8 j(|x_1+y|)$. Here we have applied Lemma 9. The assumptions of the lemma imply

$$\begin{aligned}
I_1 &\geq c_9(1-a^{-1}\|b\|_\infty)\int_{\{y\in\mathbb{R}^d\setminus\{0\}:\, x_1+y\in B_{\varphi(r)}\setminus B_r,\, v(x_1+y)\leq 0\}} j(|x_1+y|)\,dy\\
&= c_9(1-a^{-1})\int_{\{y\in\mathbb{R}^d\setminus\{0\}:\, y\in B_{\varphi(r)}\setminus B_r,\, v(y)\leq 0\}} j(|y|)\,dy\\
&= c_9(1-a^{-1})\int_{\{y\in\mathbb{R}^d\setminus\{0\}:\, y\in B_{\varphi(r)}\setminus B_r,\, v(y)\leq 0\}} L(|y|)\,\mu(dy)\\
&\geq c_9(1-a^{-1})L(\varphi(r))\mu((B_{\varphi(r)}\setminus B_r)\cap\{v\leq 0\})\\
&\geq c_9(1-a^{-1})a^{-1}L(r)\delta\mu(B_{\varphi(r)}\setminus B_r) \geq c_{10}(1-a^{-1})\delta L(r)(\ln a)/a,
\end{aligned}$$

where in the last inequality we have used Lemma 8.

Lemma 7 implies that, if we consider $a > c_L^{-1}(5/2)^\gamma$, then $L(r)/L(5r/2) \leq c_L^{-1}(5/2)^\gamma$. Hence

$$L(\varphi(r)) = a^{-1}\frac{L(r)}{L(5r/2)}L(5r/2) \leq L(5r/2),$$

and since L is decreasing, we obtain $\varphi(r) \geq 5r/2$. To estimate I_2 we note that for $x_1+y \notin B_{\varphi(r)}$ it follows from (9) that

$$(v+a^{-1}b_r)(x_1)-a^{-1}b_r(x_1+y) \geq 1+a^{-1}\beta_r(3r/2)-a^{-1}\beta_r(\varphi(r)) \geq 1.$$

Together with the growth assumption on v, this yields

$$I_2 \geq -c_{11}\int_{\{y:\, x_1+y\in\mathbb{R}^d\setminus B_{\varphi(r)}\}} \left(\frac{L(\varphi(r))}{L(|x_1+y|)}\right)^\eta j(|y|)\,dy.$$

Note that $x_1+y \notin B_{\varphi(r)}$ implies $|y| \geq |x_1+y|-|x_1| \geq 5r/2-3r/2 = r$ and

$$|x_1+y| \leq 3r/2+|y| \leq \tfrac{3}{2}|y|+|y| = \tfrac{5}{2}|y| \quad \text{and} \quad y \notin B_{2\varphi(r)/5}.$$

In this case $L(|x_1+y|) \geq L\big(\frac{5}{2}|y|\big)$ and

$$\{y\in\mathbb{R}^d\setminus\{0\}:\, x_1+y\notin B_{\varphi(r)}\} \subset \big\{y\in\mathbb{R}^d\setminus\{0\}:\, |y| \geq \tfrac{2}{5}\varphi(r)\big\}.$$

Thus we obtain

$$
\begin{aligned}
I_2 &\geq -c_{12} \int_{\{y:: x_1+y \in \mathbb{R}^d \setminus B_{\varphi(r)}\}} \left(\frac{L(\varphi(r))}{L(5|y|)/2} \right)^{\eta} j(|y|)\, dy \\
&\geq -c_{13} \int_{\frac{2}{5}\varphi(r)}^{\infty} \left(\frac{L(\varphi(r))}{L(5s/2)} \right)^{\eta} \frac{\ell(s)}{s}\, ds \geq -c_{14} \int_{\frac{2}{5}\varphi(r)}^{\infty} \left(\frac{L(\varphi(r))}{L(s)} \right)^{\eta} (-L'(s))\, ds \\
&= -c_{14} L(\varphi(r))^{\eta} (1-\eta)^{-1} L\big(\tfrac{2}{5}\varphi(r)\big)^{1-\eta} \geq -c_{15} L(\varphi(r)) = -c_{16} \frac{L(r)}{a},
\end{aligned}
$$

where in the third inequality Lemma 7 and in the last inequality monotonicity of L and Lemma 7 again have been used.

Finally, we obtain

$$
(1+c_6)\frac{L(r)}{a} \geq c_{10}(1-a^{-1})\delta L(r) \frac{\ln a}{a} - c_{16} \frac{L(r)}{a},
$$

or $1+c_6+c_{16} \geq c_{10}(1-a^{-1})\delta \ln a$. Choosing $a > 2$ large enough leads to a contradiction.

This means that we have proved that there exists $a > 2$ such that

$$
v(x) \leq 1 - a^{-1}(\beta(1) - \beta(3/2)) = 1 - \theta \quad \text{ for all } x \in B_r.
$$

Note that our choice of a does not depend on r; hence the assertion of the theorem holds for every $r > 0$ with the same choice of a and θ. □

4. Proof of Theorem 3

First of all, we restrict the general case and assume that conditions (A_1) and (K_0) hold true with $R_0 = \infty$. At the end of the proof we explain how to reduce the general case to this case.

Let $r > 0$. Assume $u \in C_b^2(\mathbb{R}^d)$ satisfies $Au = f$ in B_r where f is essentially bounded. We assume $u \not\equiv 0$ and prove assertion (4) in the simplified case $\|u\|_\infty \leq 1/2$ and $\|f\|_{L^\infty(B_r)} \leq \frac{1}{2}L(r/2)$. Let us briefly explain why this is sufficient. In the general case we would set

$$
\widetilde{u} = \frac{u}{2\|u\|_\infty + 2L(r/2)^{-1}\|f\|_{L^\infty(B_r)}}.
$$

If u solved $Au = f$ in B_r, then $\widetilde{u}$ would solve $A\widetilde{u} = \widetilde{f}$ in B_r with $\|\widetilde{u}\|_\infty \leq 1/2$ and $\|\widetilde{f}\|_{L^\infty(B_r)} \leq \frac{1}{2}L(r/2)$. Thus we could apply the result in the simplified case and obtain

$$
\begin{aligned}
\sup_{x,y \in B_{r/4}} \frac{|u(x)-u(y)|}{L(|x-y|)^{-\beta}} &\leq \left(\frac{c}{2}L(r)^{\beta} + \frac{c}{2}L(r/2)L(r)^{\beta-1} \right) \big(2\|u\|_\infty + 2L(r/2)^{-1}\|f\|_{L^\infty(B_r)}\big) \\
&\leq \widetilde{c}L(r)^{\beta}\|u\|_\infty + \widetilde{c}L(r)^{\beta-1}\|f\|_{L^\infty(B_r)},
\end{aligned}
$$

where $\widetilde{c}$ is another constant, depending on c_L and γ because of Lemma 7.

Hence we can restrict ourselves to $\|u\|_\infty \leq 1/2$ and $\|f\|_{L^\infty(B_r)} \leq \frac{1}{2}L(r/2)$. Let $x_0 \in B_{r/4}$. Without loss of generality we may assume $u(x_0) > 0$.

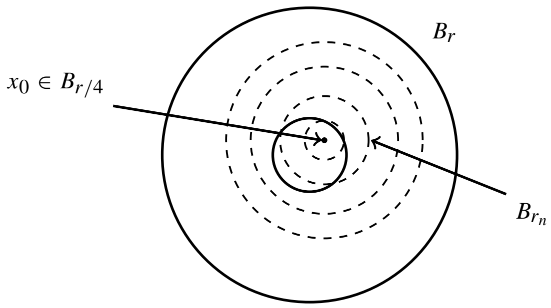

Fig. 2. Reduction of oscillation at x_0.

It is sufficient to show that

$$|u(x) - u(x_0)| \le c\frac{L(|x-x_0|)^{-\beta}}{L(r)^{-\beta}} \quad \text{for any } x \in B_r.$$

Define $r_n = L^{-1}(a^{n-1}L(r/2))$ for $n \in \mathbb{N}$, where $a > 2$ will be chosen in the course of the proof independently of r, u and f. We will construct a nondecreasing sequence $(c_n)_{n\in\mathbb{N}}$ and a nonincreasing sequence $(d_n)_{n\in\mathbb{N}}$ of positive numbers so that

$$c_n \le u(x) \le d_n \text{ for all } x \in B_{r_n} := B_{r_n}(x_0) \quad \text{and} \quad d_n - c_n \le b^{-n+1}, \tag{10}$$

where $b = 2/(2-\theta) \in (1,2)$ and $\theta \in (0,1)$ will be chosen later independently of r, u and f. This will be enough, since for $r_{n+1} \le |x - x_0| < r_n$ we will then have

$$\begin{aligned}
|u(x) - u(x_0)| \le b^{-n+1} &= b\left(\frac{1}{a^n}\right)^{\ln b/\ln a} = b\left(\frac{L(r/2)}{a^n L(r/2)}\right)^{\ln b/\ln a} \\
&= b\left(\frac{L(r/2)}{L(r_{n+1})}\right)^{\ln b/\ln a} \le b\left(\frac{L(r/2)}{L(|x-x_0|)}\right)^{\ln b/\ln a} \\
&\le b\left(\frac{2^\gamma}{c_L}\right)^{\ln b/\ln a}\left(\frac{L(|x-x_0|)}{L(r)}\right)^{-\ln b/\ln a},
\end{aligned}$$

where in the last inequality we have used Lemma 7.

We prove (10) and construct sequences (c_n) and (d_n) inductively. We set

$$c_1 := \inf_{\mathbb{R}^d} u \quad \text{and} \quad d_1 := c_1 + 1.$$

Let $n \in \mathbb{N}$, $n \ge 2$. Assume that c_k and d_k have been constructed for $k \le n$ and that (10) holds for $k \in \mathbb{N}$, $k \le n$. We are now going to construct c_{n+1} and d_{n+1}.

Set $m := (c_n + d_n)/2$. By (10) it follows that for $x \in B_{r_n}$,

$$u(x) - m \le \tfrac{1}{2}(d_n - c_n) \le \tfrac{1}{2}b^{-n+1}.$$

Define a function $v\colon \mathbb{R}^d \to \mathbb{R}$ by $v(x) := 2b^{n-1}(u(x_0+x)-m)$. Then $v(x) \le 1$ for $x \in B_{r_n}$ and $Av = 2b^{n-1} f$ in B_{r_n}.

Assume that $\mu(\{x \in B_{r_n} \setminus B_{r_{n+1}} : v(x) \le 0\}) \ge \frac{1}{2}\mu(B_{r_n} \setminus B_{r_{n+1}})$. We recall that the ball $B_{r/2}$ has center 0 and the balls B_{r_n} have center x_0. For $x \in B_{r/2} \setminus B_{r_n}$ there exists $k \in \mathbb{N}$ with $k \le n-1$ such that $r_{k+1} \le |x| < r_k$. Then by (10) we have

$$\begin{aligned} v(x) &= 2b^{n-1}(u(x_0+x)-m) \le 2b^{n-1}(d_k - m) \le 2b^{n-1}(d_k - c_k) \\ &= 2b^{n-k} = 2b\left(\frac{a^{n-1}L(r/2)}{a^k L(r/2)}\right)^{\ln b/\ln a} = 2b\left(\frac{L(r_n)}{L(r_{k+1})}\right)^{\ln b/\ln a} \\ &\le 2b\left(\frac{L(r_n)}{L(|x|)}\right)^{\ln b/\ln a}. \end{aligned}$$

If $x \in B^c_{r/2}$, then $L(|x|) \le L(r/2)$ and $u(x_0+x) - m \le d_1 - c_1 = 1$; hence

$$v(x) \le 2b^{n-1} = 2\left(\frac{a^{n-1}L(r/2)}{L(r/2)}\right)^{\ln b/\ln a} = 2\left(\frac{L(r_n)}{L(r/2)}\right)^{\ln b/\ln a} \le 2\left(\frac{L(r_n)}{L(|x|)}\right)^{\ln b/\ln a}.$$

We want to apply Lemma 2 with $r = r_{n+1}$. Note that $r_n = \varphi_a(r_{n+1})$. In order to apply Lemma 2 we need to verify that $2b^{n-1}|f| \le L(\varphi(r_{n+1})) = a^{n-1}L(r/2)$. But this holds true because $|f| \le \frac{1}{2}L(r/2)$ and $(b/a)^{n-1} \le b/a \le 2a^{-1} \le 1$. Thus we find that for some $a > 2$ and $\theta \in (0,1)$, not depending on v or r, we have $v(x) \le 1-\theta$ on $B_{r_{n+1}}$. Going back to u, we deduce

$$u(x) \le \frac{1-\theta}{2} b^{1-n} + \frac{c_n + d_n}{2} \qquad \text{for } x \in B_{r_{n+1}}.$$

We take $c_{n+1} = c_n$ and $d_{n+1} = \min\{d_n, \frac{1-\theta}{2}b^{1-n} + \frac{c_n+d_n}{2}\}$. This choice implies that $d_{n+1} - c_{n+1} \le b^{-n}$.

In the case $\mu(\{x \in B_{r_n} \setminus B_{r_{n+1}} : v(x) \le 0\}) < \frac{1}{2}\mu(B_{r_n} \setminus B_{r_{n+1}})$ we repeat the previous argument with $-v$ instead of v and deduce

$$u(x) \ge -\frac{1-\theta}{2} b^{1-n} + \frac{c_n + d_n}{2} \qquad \text{for } x \in B_{r_{n+1}}.$$

This time we choose $c_{n+1} = \max\{c_n, -\frac{1-\theta}{2}b^{1-n} + \frac{c_n+d_n}{2}\}$ and $d_{n+1} = d_n$. Finally, we set $\beta := \ln b/\ln a$.

We have completed the proof in the special case where conditions (A_1) and (K_0) hold true with $R_0 = \infty$. Now, let us assume that conditions (A_1) and (K_0) hold true for some $R_0 < \infty$, i.e., there is $\ell\colon (0,R_0) \to (0,\infty)$ satisfying $\int_0^{R_0} (\ell(s)/s)\,ds = \infty$ and conditions (ℓ_1)–(ℓ_3) for some $c_L \in (0,1)$, $c_U \ge 1$, and $\gamma \in (0,2)$. We define $\widetilde{\ell}\colon (0,\infty) \to (0,\infty)$ by

$$\widetilde{\ell}(s) = \begin{cases} \ell(s) & \text{for } 0 < s < R_0, \\ (s - R_0/2)^{-\gamma} & \text{for } s \ge R_0. \end{cases}$$

Lemma 10. *The function* $\widetilde{\ell}$ *satisfies* (ℓ_1)–(ℓ_3) *with* R_0 *being replaced by* ∞ *and for some* $\widetilde{c}_L \in (0,1)$, $\widetilde{c}_U \geq 1$, *and* $\gamma \in (0,2)$.

Proof. Assume $R_0 > 0$. Condition (ℓ_1) obviously holds true. When checking (ℓ_2) and (ℓ_3) we need to consider several cases. Both conditions hold true for $\lambda r < R_0$ because the functions $\widetilde{\ell}$ and ℓ coincide in this range. For $\lambda r \geq R_0$, $r \geq R_0$ the conditions can easily be verified. The only challenging case is $\lambda r \geq R_0$, $r < R_0$. Let us verify (ℓ_2) in this case, i.e., show that

$$\widetilde{\ell}(r\lambda) \geq \widetilde{c}_L \lambda^{-\gamma} \widetilde{\ell}(r) \;\Leftrightarrow\; (r\lambda - R_0/2)^{-\gamma} \geq \widetilde{c}_L \lambda^{-\gamma} \ell(r), \tag{11}$$

which is equivalent to

$$\frac{(r\lambda - R_0/2)^{-\gamma}}{(r\lambda)^{-\gamma}} r^{-\gamma} \geq \widetilde{c}_L \ell(r) \quad \text{and} \quad \left(\frac{r\lambda}{r\lambda - R_0/2}\right)^{\gamma} r^{-\gamma} \geq \widetilde{c}_L \ell(r).$$

Since the fraction on the left-hand side is bounded in $[1,2]$ for $r\lambda \geq R_0$, it is sufficient to show $r^{-\gamma} > \widetilde{c}_L \ell(r)$ in the case $r < R_0 \leq \lambda r$ for some $\widetilde{c}_L \in (0,1)$. Let us prove this assertion in two cases separately. In the case $R_0/2 < r < R_0 \leq \lambda r$ we conclude that

$$\ell(r) = \ell(R_0/2)\frac{\ell(r)}{\ell(R_0/2)} \overset{(\ell_3)}{\leq} \ell(R_0/2) c_U 2^d R_0^{-d} r^d \leq \ell(R_0/2) c_U 2^d \leq \ell(R_0/2) c_U 2^d \frac{r^{-\gamma}}{R_0^{-\gamma}},$$

which proves the assertion. In the case $r < R_0/2 < R_0 \leq \lambda r$ we proceed as follows:

$$\ell(r) = \frac{\ell(R_0/2)}{\frac{\ell(R_0/2)}{\ell(r)}} \overset{(\ell_2)}{\leq} \frac{\ell(R_0/2)}{c_L (R_0/2r)^{-\gamma}} = c_L (R_0/2)^{\gamma} \ell(R_0/2) r^{-\gamma}.$$

Thus we have shown

$$\widetilde{\ell}(r\lambda) \geq \widetilde{c}_L \lambda^{-\gamma} \widetilde{\ell}(r) \qquad \text{for all } r > 0 \text{ and } \lambda > 1$$

for an appropriate choice of $\widetilde{c}_L$. In other words, the function $\widetilde{\ell}$ satisfies condition (ℓ_2) with R_0 being replaced by ∞. It remains to show

$$\widetilde{\ell}(r\lambda) \leq \widetilde{c}_U \lambda^d \widetilde{\ell}(r) \qquad \text{for all } r > 0 \text{ and } \lambda > 1$$

for an appropriate choice of $\widetilde{c}_U$. As explained above, it remains to show this estimate in the case $r < R_0 \leq \lambda r$. For $r \leq R_0/2 < R_0 \leq \lambda r$ we obtain

$$\begin{aligned}\frac{\ell(\lambda r)}{\ell(r)} &= \frac{(\lambda r - R_0/2)^{-\gamma}}{\ell(R_0/2)} \frac{\ell(R_0/2)}{\ell(r)} \overset{(\ell_3)}{\leq} 2^{\gamma} R_0^{-\gamma} \ell(R_0/2)^{-1} c_U (R_0/2r)^d \\ &\leq 2^{\gamma} R_0^{-\gamma} \ell(R_0/2)^{-1} c_U \lambda^d.\end{aligned}$$

If $R_0/2 < r < R_0 \leq \lambda r$,

$$\begin{aligned}\frac{\ell(\lambda r)}{\ell(r)} &= \frac{(\lambda r - R_0/2)^{-\gamma}}{\ell(R_0/2)} \frac{\ell(R_0/2)}{\ell(r)} \overset{(\ell_2)}{\leq} 2^{\gamma} R_0^{-\gamma} \ell(R_0/2)^{-1} c_L^{-1} (R_0/2r)^{\gamma} \\ &\leq 2^{\gamma} R_0^{-\gamma} \ell(R_0/2)^{-1} c_L^{-1} \leq 2^{\gamma} R_0^{-\gamma} \ell(R_0/2)^{-1} c_L^{-1} \lambda^d.\end{aligned}$$

□

Next, define a modified kernel function $\widetilde{K}(x,h)$ by

$$\widetilde{K}(x,h) = \begin{cases} K(x,h) & \text{for } 0 < |h| < R_0, \\ |h|^{-d}(|h| - R_0/2)^{-\gamma} & \text{for } |h| \geq R_0, \end{cases}$$

with γ as in (ℓ_2). Let us denote the integrodifferential operator corresponding to $\widetilde{K}$ by $\widetilde{A}$. Since u solves $Au = f$, the function u also solves

$$\widetilde{A}u = f + (\widetilde{A} - A)u =: \widetilde{f}.$$

Due to the definition of $\widetilde{K}$, the image of u under the operator $\widetilde{A} - A$ is a bounded function, hence $\widetilde{f}$ is a bounded function. Now we apply the previous proof ($R_0 = \infty$) and obtain, for every $r > 0$,

$$\sup_{x,y\in B_{r/4}} \frac{|u(x)-u(y)|}{L(|x-y|)^{-\beta}} \leq cL(r)^{\beta}\|u\|_\infty + cL(r)^{\beta-1}\|f\|_{L^\infty(B_r)} + cL(r)^{\beta-1}\|(\widetilde{A}-A)u\|_{L^\infty(B_r)} \tag{12}$$

for some positive constant $c > 0$. Note that

$$\|(\widetilde{A}-A)u\|_{L^\infty(B_r)} \leq 2\|u\|_\infty \sup_{x\in B_r} \int_{\mathbb{R}^d\setminus B_{R_0}} (K(x,h) + \widetilde{K}(x,h))\,dh \leq c'\|u\|_\infty$$

for some constant $c' > 0$. Since $L(R_0/2) \leq L(r)$ for $0 < r \leq R_0/2$ we conclude that

$$\|(\widetilde{A}-A)u\|_{L^\infty(B_r)} \leq c'L(R_0/2)^{-1}L(r)\|u\|_\infty$$

and finally

$$\sup_{x,y\in B_{r/4}} \frac{|u(x)-u(y)|}{L(|x-y|)^{-\beta}} \leq (c + c'L(R_0/2)^{-1})L(r)^{\beta}\|u\|_\infty + cL(r)^{\beta-1}\|f\|_{L^\infty(B_r)}. \qquad \square$$

5. An approach to regularity via stochastic processes

As explained in the introduction, the aim of this section is to provide an alternative approach to Theorem 3 using stochastic processes. First, let us formulate our assumptions and results. As in the first part, we assume that $0 \leq \alpha < 2$ and $K\colon \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) \to [0,\infty)$ is a measurable function satisfying (K_0) and the symmetry condition $K(x,h) = K(x,-h)$ for all x, h. Instead of condition (A_1) we assume the following:

$$\kappa^{-1}\frac{\ell(|h|)}{|h|^d} \leq K(x,h) \leq \kappa\frac{\ell(|h|)}{|h|^d} \quad \text{for } 0 < |h| \leq 1, \tag{K}$$

where $\kappa > 1$ and $\ell\colon (0,1) \to (0,\infty)$ is locally bounded and varies regularly at zero with index $-\alpha \in (-2, 0]$. Possible examples could be $\ell(s) = 1$, $\ell(s) = s^{-3/2}$ and $\ell(s) = s^{-\beta}\ln(2/s)^2$ for some $\beta \in (0,2)$ (see Appendix for a more detailed discussion).

These assumptions on $K(x,h)$ differ slightly from the ones in Section 1. Concerning the behavior of $K(x,h)$ for small values of $|h|$, these assumptions are slightly more restrictive. We suppose that there exists a strong Markov process $X = (X_t, \mathbb{P}_x)$ with trajectories that are right continuous with left limits associated with A in the sense that for every $x \in \mathbb{R}^d$,

(i) $\mathbb{P}_x(X_0 = x) = 1$;
(ii) for any $f \in C_b^2(\mathbb{R}^d)$ the process $(f(X_t) - f(X_0) - \int_0^t Af(X_s)\,ds)_{t\geq 0}$ is a martingale under $\mathbb{P}_x$.

Note that the existence of such a Markov process comes for free in the case when $K(x,h)$ is independent of x (see Section 2). In the general case it has been established by many authors in different general contexts (see the discussion in [AK09]). Denote by $\tau_A = \inf\{t > 0 \colon X_t \notin A\}$ and $T_A = \inf\{t > 0 \colon X_t \in A\}$ the first exit time resp. hitting time of the process X for a measurable set $A \subset \mathbb{R}^d$.

Definition 11. A bounded function $u \colon \mathbb{R}^d \to \mathbb{R}$ is said to be *harmonic* in an open subset $D \subset \mathbb{R}^d$ *with respect to* X (and A) if for any bounded open set $B \subset \overline{B} \subset D$ the stochastic process $(u(X_{\tau_B \wedge t}))_{t\geq 0}$ is a $\mathbb{P}_x$-martingale for every $x \in \mathbb{R}^d$.

Before we can formulate our results we need to introduce an additional quantity. Note that (K_0) and (K) imply that $\int_0^1 s\ell(s)\,ds \leq c$ for some constant $c > 0$. Let $L \colon (0,1) \to (0,\infty)$ be defined by $L(r) = \int_r^1 (\ell(s)/s)\,ds$. The function L is well defined because $L(r) \leq r^{-2}\int_r^1 s^2(\ell(s)/s)\,ds \leq cr^{-2}$. See the table in Section 1.2 for several examples. We note that the function L is always decreasing.

Remark. The definition of L here is different from the definition in Section 1. The reason is that here we are able to work without specific assumptions on $K(x,h)$ for large values of $|h|$.

The result analogous to Theorem 3, which we prove with probabilistic techniques, is the following theorem.

Theorem 12. *There exist constants $c > 0$ and $\gamma \in (0,1)$ such that for all $r \in (0,1/2)$ and $x_0 \in \mathbb{R}^d$,*

$$|u(x) - u(y)| \leq c\|u\|_\infty \frac{L(|x-y|)^{-\gamma}}{L(r)^{-\gamma}}, \qquad x, y \in B_{r/4}(x_0), \tag{13}$$

for all bounded functions $u \colon \mathbb{R}^d \to \mathbb{R}$ that are harmonic in $B_r(x_0)$ with respect to A.

Let us comment on this result. It is important to note that the result trivially holds if the function $L : (0,1) \to (0,\infty)$ satisfies $\lim_{r\to 0+} L(r) < \infty$. This is equivalent to

$$\int_{B_1} \frac{\ell(|h|)}{|h|^d}\,dh < \infty, \tag{14}$$

which, in the case $K(x,h) = k(h)$, means that the Lévy measure is finite. Thus, for the proof, we can concentrate on cases where (14) does not hold. One could say that our result holds true up to and across the phase boundary determined by whether the kernel $K(x,\cdot)$ is integrable (finite Lévy measure) or not.

Furthermore, note that the main result of [BL02] is implied by Theorem 12 since the choice $\ell(s) = s^{-\alpha}$, $\alpha \in (0,2)$, leads to $L(r) \asymp r^{-\alpha}$. Given the whole spectrum of possible operators covered by our approach, this choice is a very specific one. It allows one to use scaling methods which are not at our disposal here. Table 1 in Section 1 contains several admissible examples. Note that (13) becomes trivial if $L(0) < \infty$.

The main ingredient in the proof of Theorem 12 is a new version of Proposition 5 which we provide now. Define a measure μ by

$$\mu(dx) = \frac{\ell(|x|)}{L(|x|)|x|^d} \mathbb{1}_{B_1}(x)\,dx. \tag{15}$$

Moreover, for $a > 1$, we define a function $\varphi_a : (0,1) \to (0,1)$ by $\varphi_a(r) = L^{-1}(a^{-1}L(r))$. The following result is our modification of Proposition 5.

Proposition 13. *There exists a constant $c > 0$ such that for all $a > 1$, $r \in (0,1/2)$ and measurable sets $A \subset B_{\varphi_a(r)} \setminus B_r$ with $\mu(A) \geq \frac{1}{2}\mu(B_{\varphi_a(r)} \setminus B_r)$, the inequality*

$$\mathbb{P}_x(T_A < \tau_{B_{\varphi_a(r)}}) \geq \mathbb{P}_x(X_{\tau_{B_r}} \in A) \geq c\,\frac{\ln a}{a}$$

holds true for all $x \in B_{r/2}$.

The novelty of Proposition 13 is the definition and use of the measure μ. It allows us to deal with the classical cases as well as with critical cases, e.g. given by $K(x,h) \asymp |h|^{-d}\mathbb{1}_{B_1}(h)$.

Note that we use the notation $f(r) \asymp g(r)$ to denote that the ratio $f(r)/g(r)$ stays between two positive constants as r converges to some value of interest.

6. Probabilistic estimates

Proposition 14. *There exists a constant $C_1 > 0$ such that for $x_0 \in \mathbb{R}^d$, $r \in (0,1/2)$ and $t > 0$,*

$$\mathbb{P}_{x_0}(\tau_{B_r(x_0)} \leq t) \leq C_1 t L(r).$$

Proof. Let $x_0 \in \mathbb{R}^d$, $0 < r < 1$ and let $f \in C^2(\mathbb{R}^d)$ be a positive function such that

$$f(x) = \begin{cases} |x-x_0|^2, & |x-x_0| \leq r/2, \\ r^2, & |x-x_0| \geq r, \end{cases}$$

and for some $c_1 > 0$,

$$|f(x)| \leq c_1 r^2, \qquad \left|\frac{\partial f}{\partial x_i}(x)\right| \leq c_1 r \quad \text{and} \quad \left|\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right| \leq c_1.$$

By the optional stopping theorem we get

$$\mathbb{E}_x f(X_{t\wedge\tau_{B_r(x_0)}}) - f(x_0) = \mathbb{E}^x \int_0^{t\wedge\tau_{B_r(x_0)}} Af(X_s)\,ds, \qquad t > 0. \tag{16}$$

Let $x \in B_r(x_0)$. We estimate $Af(x)$ by splitting the integral in (2) into three parts. First

$$\int_{B_r} (f(x+h) - f(x) - \nabla f(x)\cdot h\mathbb{1}_{\{|h|\le 1\}})K(x,h)\,dh$$
$$\le c_2 \int_{B_r} |h|^2 K(x,h)\,dh \le c_2\kappa \int_{B_r} |h|^{2-d}\ell(|h|)\,dh \le c_3 r^2\ell(r),$$

where in the last line we have used Karamata's theorem (see property (2) in Appendix). On the other hand, on B_r^c we have

$$\int_{B_r^c} (f(x+h) - f(x))K(x,h)\,dh \le 2\|f\|_\infty \int_{B_r^c} K(x,h)\,dh$$
$$\le 2\|f\|_\infty \left(\kappa \int_{B_1\setminus B_r} |h|^{-d}\ell(|h|)\,dh + \int_{B_1^c} K(x,h)\,dh\right) \le c_4 r^2 L(r),$$

where we have applied property (5) from Appendix. Note that

$$\left|\int_{B_1\setminus B_r} h\cdot\nabla f(x)K(x,h)\,dh\right| = 0.$$

Therefore, by property (1) from Appendix we conclude that there is a constant $c_6 > 0$ such that for all $x \in B_r(x_0)$ and $r \in (0,1)$ we have

$$Af(x) \le c_6 r^2 L(r). \tag{17}$$

Let us look again at (16). On $\{\tau_{B_r(x_0)} \le t\}$ we have $X_{t\wedge\tau_{B_r(x_0)}} \in B_r(x_0)^c$ and so $f(X_{t\wedge\tau_{B_r(x_0)}}) \ge r^2$. Thus, by (17) and (16) we get

$$\mathbb{P}_{x_0}(\tau_{B_r(x_0)} \le t) \le c_6 L(r)t. \qquad \square$$

Proposition 15. *There are constants $C_2, C_3 > 0$ such that for $x_0 \in \mathbb{R}^d$,*

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}_x \tau_{B_r(x_0)} \le \frac{C_2}{L(r)}, \qquad r \in (0,1/2),$$
$$\inf_{x\in B_{r/2}(x_0)} \mathbb{E}_x \tau_{B_r(x_0)} \ge \frac{C_3}{L(r)}, \qquad r \in (0,1/2).$$

Proof. The proof is similar to the proof of the exit time estimates in [BL02] (see also the proof of Proposition 17):

(a) First we prove the upper estimate for the exit time. Let $x \in \mathbb{R}^d$, $r \in (0,1/2)$ and let

$$S = \inf\{t > 0 \colon |X_t - X_{t-}| > 2r\}$$

be the first time of a jump larger than $2r$. With the help of the Lévy system formula (see [BL02, Proposition 2.3]) and (K) we can deduce

$$\mathbb{P}_x(S \le L(r)^{-1}) = \mathbb{E}_x \sum_{t \le L(r)^{-1} \wedge S} \mathbb{1}_{\{|X_t - X_{t-}| > 2r\}} = \mathbb{E}_x \int_0^{L(r)^{-1} \wedge S} \int_{B_{2r}^c} K(X_s, h)\, dh\, ds$$
$$\ge c_1 \mathbb{E}_x[L(r)^{-1} \wedge S] \int_{2r}^1 \frac{\ell(t)}{t}\, dt. \tag{18}$$

Since L is regularly varying at zero,

$$\mathbb{E}_x[L(r)^{-1} \wedge S] \ge L(r)^{-1}\mathbb{P}_x(S > L(r)^{-1}) \ge c_2 L(2r)^{-1}\big(1 - \mathbb{P}_x(S \le L(r)^{-1})\big)$$

and so it follows from (18) that

$$\mathbb{P}_x(S \le L(r)^{-1}) \ge c_3 \tag{19}$$

with $c_3 = \frac{c_1 c_2}{c_1 c_2 + 1} \in (0, 1)$. The strong Markov property and (18) lead to

$$\mathbb{P}_x(S > mL(r)^{-1}) \le (1 - c_3)^m, \qquad m \in \mathbb{N}.$$

Since $\tau_{B_r(x_0)} \le S$,

$$\mathbb{E}_x \tau_{B_r(x_0)} \le \mathbb{E}_x S \le L(r)^{-1} \sum_{m=0}^{\infty} (m+1)\mathbb{P}_x(S > L(r)^{-1} m)$$
$$\le L(r)^{-1} \sum_{m=0}^{\infty} (m+1)(1 - c_3)^m.$$

(b) Now we prove the lower estimate of the exit time. Let $r \in (0, 1/2)$ and $y \in B_{r/2}(x_0)$. By Proposition 14,

$$\mathbb{P}_y(\tau_{B_r(x_0)} \le t) \le \mathbb{P}_y(\tau_{B_{r/2}(y)} \le t) \le C_1 t L(r/2), \qquad t > 0,$$

since $B_{r/2}(y) \subset B_r(x_0)$. Choose $t = \frac{1}{2C_1 L(r/2)}$. Then

$$\mathbb{E}_y \tau_{B_r(x_0)} \ge \mathbb{E}_y[\tau_{B_r(x_0)};\ \tau_{B_r(x_0)} > t] \ge t\mathbb{P}_y(\tau_{B_r(x_0)} > t)$$
$$\ge t(1 - C_1 L(r/2) t) = \frac{1}{4C_1 L(r/2)}.$$

By (3) from Appendix we know that L is regularly varying at zero. Hence there is a constant $c_1 > 0$ such that $L(r/2) \le c_1 L(r)$ for all $r \in (0, 1/2)$. Therefore

$$\mathbb{E}_y \tau_{B_r(x_0)} \ge \frac{1}{4C_1 c_1 L(r)}.$$

□

Proposition 16. *There is a constant $C_4 > 0$ such that for all $x_0 \in \mathbb{R}^d$ and $r, s \in (0, 1/2)$ satisfying $2r < s$,*

$$\sup_{x \in B_r(x_0)} \mathbb{P}_x(X_{\tau_{B_r(x_0)}} \notin B_s(x_0)) \le C_4 \frac{L(s)}{L(r)}.$$

Proof. Let $x_0 \in \mathbb{R}^d$, $r, s \in (0, 1/2)$ and $x \in B_r(x_0)$. Set $B_r := B_r(x_0)$. By the Lévy system formula, for $t > 0$,

$$\mathbb{P}_x(X_{\tau_{B_r} \wedge t} \notin B_s) = \mathbb{E}_x \sum_{v \le \tau_{B_r} \wedge t} \mathbb{1}_{\{X_{v-} \in B_r,\, X_v \in B_s^c\}} = \mathbb{E}_x \int_0^{\tau_{B_r} \wedge t} \int_{B_s^c} K(X_v, z - X_v)\, dz\, dv.$$

Let $y \in B_r$. Since $s \ge 2r$, it follows that $B_{s/2}(y) \subset B_s$ and hence

$$\int_{B_s^c} K(y, z - y)\, dz \le \int_{B_{s/2}(y)^c} K(y, z - y)\, dz \le c_1 \int_{s/2}^1 \frac{\ell(u)}{u}\, du + c_2 \le c_3 L(s),$$

where in the last inequality we have used that L varies regularly at zero and that $\lim_{r \to 0+} L(r) > 0$ (see (5) in Appendix).

The above considerations together with Proposition 15 imply

$$\mathbb{P}_x(X_{\tau_{B_r} \wedge t} \notin B_s) \le c_3 L(s) \mathbb{E}_x \tau_{B_r} \le c_4 \frac{L(s)}{L(r)}.$$

Letting $t \to \infty$ we obtain the desired estimate. □

For $x_0 \in \mathbb{R}^d$ and $r \in (0, 1)$ we define the measure

$$\mu_{x_0}(dx) = \frac{\ell(|x - x_0|)}{L(|x - x_0|)} |x - x_0|^{-d} \mathbb{1}_{\{|x - x_0| < 1\}}\, dx. \tag{20}$$

Define $\varphi_a(r) = L^{-1}(a^{-1} L(r))$ for $r \in (0, 1)$ and $a > 1$. The following property is important for the construction below:

$$r = L^{-1}(L(r)) \le L^{-1}(a^{-1} L(r)) = \varphi_a(r). \tag{21}$$

Now we can prove a Krylov–Safonov type hitting estimate which includes Proposition 13 as a special case.

Proposition 17. *There exists a constant $C_5 > 0$ such that for all $x_0 \in \mathbb{R}^d$, $a > 1$, $r \in (0, 1/2)$ and $A \subset B_{\varphi_a(r)}(x_0) \setminus B_r(x_0)$ satisfying $\mu_{x_0}(A) \ge \frac{1}{2} \mu_{x_0}(B_{\varphi_a(r)}(x_0) \setminus B_r(x_0))$,*

$$\mathbb{P}_y(T_A < \tau_{B_{\varphi_a(r)}(x_0)}) \ge \mathbb{P}_y(X_{\tau_{B_r(x_0)}} \in A) \ge C_5 \frac{\ln a}{a}, \qquad y \in B_{r/2}(x_0).$$

Proof. Consider $x_0 \in \mathbb{R}^d$, $a > 1$, $r \in (0, 1/2)$ and a set $A \subset B_{\varphi_a(r)}(x_0) \setminus B_r(x_0)$ satisfying $\mu_{x_0}(A) \ge \frac{1}{2} \mu_{x_0}(B_{\varphi_a(r)}(x_0) \setminus B_r(x_0))$. Set $\mu := \mu_{x_0}$, $\varphi := \varphi_a$, $B_s := B_s(x_0)$ and let $y \in B_{r/2}$. The first inequality follows from $\{X_{\tau_{B_r}} \in A\} \subset \{T_A < \tau_{B_{\varphi(r)}}\}$ since $A \subset B_{\varphi(r)} \setminus B_r$.

By the Lévy system formula, for $t > 0$,

$$\mathbb{P}_y(X_{\tau_{B_r}\wedge t} \in A) = \mathbb{E}_y \sum_{s \le \tau_{B_r}\wedge t} \mathbb{1}_{\{X_{s-}\in B_r,\, X_s\in A\}} = \mathbb{E}_y \int_0^{\tau_{B_r}\wedge t} \int_A K(X_s, z - X_s)\, dz\, ds. \tag{22}$$

Since $|z-x| \le |z-x_0| + |x_0 - x| \le |z-x_0| + r \le 2|z-x_0|$ for $x \in B_r$ and $z \in B_r^c$,

$$\mathbb{E}_y \int_0^{\tau_{B_r}\wedge t} \int_A K(X_s, z - X_s)\, dz\, ds \ge c_1 \mathbb{E}_y[\tau_{B_r} \wedge t] \int_A \frac{\ell(|z-x_0|)}{|z-x_0|^d}\, dz, \tag{23}$$

where we have used property (4) of Appendix.

Since L is decreasing,

$$\int_A \frac{\ell(|z-x_0|)}{|z-x_0|^d}\, dz = \int_A L(|z-x_0|)\, \mu(dz) \ge L(\varphi(r))\mu(A) \ge \frac{L(r)}{2a} \mu(B_{\varphi(r)} \setminus B_r). \tag{24}$$

Noting that

$$\mu(B_{\varphi(r)} \setminus B_r) = c_2 \int_r^{\varphi(r)} \frac{1}{L(s)} \frac{\ell(s)\, ds}{s} = -c_2 \ln L(s)|_r^{\varphi(r)} = c_2 \ln a,$$

we conclude from (22)–(24) that

$$\mathbb{P}_y(T_A < \tau_{B_{\varphi_a(r)}(x_0)}) \ge c_3 L(r) \frac{\ln a}{a} \mathbb{E}_y[\tau_{B_r} \wedge t].$$

Note that the above estimate provides an alternative proof of the first assertion of Proposition 15. Letting $t \to \infty$ and using the lower bound in Proposition 15 we get

$$\mathbb{P}_y(T_A < \tau_{B_{\varphi_a(r)}(x_0)}) \ge c_3 L(r) \frac{\ln a}{a} \mathbb{E}_y \tau_{B_r} \ge c_3 L(r) \frac{\ln a}{a} C_3 L(r)^{-1} = c_3 C_3 \frac{\ln a}{a}. \qquad \square$$

7. Proof of Theorem 12

Let $x_0 \in \mathbb{R}^d$, $r \in (0, 1/2)$ and $x \in B_{r/4}(x_0)$. Using (4) from Appendix with $\delta = 1$, we see that there is a constant $c_0 \ge 1$ such that

$$\frac{L(s)}{L(s')} \le c_0 \left(\frac{s}{s'}\right)^{-\alpha-1}, \quad 0 < s < s' < 1/2. \tag{25}$$

For $n \in \mathbb{N}$ define

$$r_n := L^{-1}(L(r/2)a^{n-1}) \quad \text{and} \quad s_n := 3\|u\|_\infty b^{-(n-1)}$$

for some constants $b \in (1, 3/2)$ and $a > c_0 2^{\alpha+1}$ that will be chosen in the proof independently of n, r and u. As we explained in the introduction, Theorem 12 trivially holds true if $\lim_{r\to 0+} L(r)$ is finite. Thus, we can assume $\lim_{r\to 0+} L(r)$ is infinite. This implies that $r_n \to 0$ as $n \to \infty$ as it should be.

We will use the following abbreviations:

$$B_n := B_{r_n}(x), \quad \tau_n := \tau_{B_n}, \quad m_n := \inf_{B_n} u, \quad M_n := \sup_{B_n} u.$$

We are going to prove

$$M_k - m_k \le s_k \tag{26}$$

for all $k \ge 1$.

Assume for a moment that (26) is proved. Then, for any $r \in (0, 1/2)$ and $y \in B_{r/4}(x_0) \subset B_{r/2}(x)$ we can find $n \in \mathbb{N}$ such that

$$r_{n+1} \le |y - x| < r_n.$$

Furthermore, since L is decreasing, we obtain with $\gamma = \ln b/\ln a \in (0, 1)$

$$\begin{aligned} |u(y) - u(x)| \le s_n = 3b\|u\|_\infty a^{-n\ln b/\ln a} &= 3b\|u\|_\infty \left[\frac{L(r_{n+1})}{L(r/2)}\right]^{-\ln b/\ln a} \\ &\le 3b\|u\|_\infty \left[\frac{L(|x-y|)}{L(r/2)}\right]^{-\gamma}, \end{aligned}$$

which proves our assertion. Thus it remains to prove (26).

We use an inductive argument. Obviously, $M_1 - m_1 \le 2\|u\|_\infty \le s_1$. Since $1 < b < 3/2$, it follows that

$$M_2 - m_2 \le 2\|u\|_\infty \le 3\|u\|_\infty b^{-1} = s_2.$$

Assume now that (26) is true for all $k \in \{1, \dots, n\}$ for some $n \ge 2$.

Let $\varepsilon > 0$ and take $z_1, z_2 \in B_{n+1}$ so that

$$u(z_1) \le m_{n+1} + \varepsilon/2, \qquad u(z_2) \ge M_{n+1} - \varepsilon/2.$$

It is enough to show that

$$u(z_2) - u(z_1) \le s_{n+1}, \tag{27}$$

since then $M_{n+1} - m_{m+1} - \varepsilon \le s_{n+1}$, which implies (26) for $k = n + 1$, since $\varepsilon > 0$ was arbitrary.

By the optional stopping theorem,

$$\begin{aligned} u(z_2) - u(z_1) &= \mathbb{E}_{z_2}[u(X_{\tau_n}) - u(z_1)] \\ &= \mathbb{E}_{z_2}[u(X_{\tau_n}) - u(z_1);\ X_{\tau_n} \in B_{n-1}] \\ &\quad + \sum_{i=1}^{n-2} \mathbb{E}_{z_2}[u(X_{\tau_n}) - u(z_1);\ X_{\tau_n} \in B_{n-i-1} \setminus B_{n-i}] \\ &\quad + \mathbb{E}_{z_2}[u(X_{\tau_n}) - u(z_1);\ X_{\tau_n} \in B_1^c] = I_1 + I_2 + I_3. \end{aligned}$$

Let $A = \{z \in B_{n-1} \setminus B_n : u(z) \le (m_n + M_n)/2\}$. It is sufficient to consider the case $\mu_x(A) \ge \frac{1}{2}\mu_x(B_{n-1} \setminus B_n)$, where μ_x is the measure defined by (20). In the other case we would use $\mu_x((B_{n-1} \setminus B_n) \setminus A) \geq \frac{1}{2}\mu_x(B_{n-1} \setminus B_n)$ and could continue the proof with $\|u\|_\infty - u$ and

$$(B_{n-1} \setminus B_n) \setminus A = \left\{ z \in B_{n-1} \setminus B_n \colon \|u\|_\infty - u(z) \leq \frac{\|u\|_\infty - m_n + \|u\|_\infty - M_n}{2} \right\}$$

instead of u and A.

The estimate (25) implies $a = L(r_{n+1})/L(r_n) \leq c_0(r_{n+1}/r_n)^{-\alpha-1}$, from which we deduce $r_{n+1} \leq r_n(c_0 a^{-1})^{1/(\alpha+1)} \leq r_n/2$ because of $a > c_0 2^{\alpha+1}$. Next, we make use of the following property:

$$r_{n-1} = L^{-1}(L(r/2)a^{n-2}) = L^{-1}(a^{-1}L(r/2)a^{n-1}) = L^{-1}(a^{-1}L(r_n)) = \varphi_a(r_n). \tag{28}$$

Then by Proposition 17 (with $r = r_n$ and $x_0 = x$) we get

$$p_n := \mathbb{P}_{z_2}(X_{\tau_n} \in A) \geq C_5(\ln a)/a.$$

Hence,

$$\begin{aligned} I_1 &= \mathbb{E}_{z_2}[u(X_{\tau_n}) - u(z_1);\ X_{\tau_n} \in B_{n-1}] \\ &= \mathbb{E}_{z_2}[u(X_{\tau_n}) - u(z_1);\ X_{\tau_n} \in A] + \mathbb{E}_{z_2}[u(X_{\tau_n}) - u(z_1);\ X_{\tau_n} \in B_{n-1} \setminus A] \\ &\leq \left(\frac{m_n + M_n}{2} - m_n \right) p_n + s_{n-1}(1 - p_n) \\ &\leq \tfrac{1}{2} s_n p_n + s_{n-1}(1 - p_n) \leq s_{n-1}\left(1 - \tfrac{1}{2} p_n\right) \leq s_{n-1}\left(1 - \frac{C_5 \ln a}{2a}\right). \end{aligned}$$

By Proposition 16,

$$\begin{aligned} I_2 &\leq \sum_{i=1}^{n-2} s_{n-i-1} \mathbb{P}_{z_2}(X_{\tau_n} \notin B_{n-i}) \leq C_4 \sum_{i=1}^{n-2} s_{n-i-1} \frac{L(r_{n-i})}{L(r_n)} \\ &\leq 3C_4 \|u\|_\infty \sum_{i=1}^{n-2} b^{-(n-i-2)} \frac{a^{n-i-1}}{a^{n-1}} \leq 3C_4 \|u\|_\infty \frac{b^{-n+3}}{a-b} \leq C_4 \frac{b^3}{a-b} s_{n+1}. \end{aligned}$$

Similarly, by Proposition 16,

$$I_3 \leq 2\|u\|_\infty \mathbb{P}_{z_2}(X_{\tau_n} \notin B_1) \leq 2C_4 \|u\|_\infty \frac{L(r_1)}{L(r_n)} = \frac{2C_4}{3} b \left(\frac{b}{a}\right)^{n-1} s_{n+1} \leq C_4 \frac{b^2}{a} s_{n+1}.$$

Hence,

$$u(z_2) - u(z_1) \leq s_{n+1} b^2 \left[1 - \frac{C_5 \ln a}{2a} + \frac{C_4 b}{a-b} + \frac{C_4}{a} \right].$$

Since $a - b \geq a/4$ for $b \in (1, 3/2)$ and $a > c_0 2^{\alpha+1} \geq 2$, it follows that

$$q := 1 - \frac{C_5 \ln a}{2a} + \frac{C_4 b}{a-b} + \frac{C_4}{a} \leq 1 - \frac{C_5 \ln a}{2a} + \frac{7C_4}{a} = 1 - \frac{C_5 \ln a - 14C_4}{2a}.$$

Next, we choose $a > c_0 2^{\alpha+1}$ so large that $C_5 \ln a - 14C_4 > 0$. Thus $q < 1$. Finally, we choose $b \in (1, 3/2)$ sufficiently small so that $b^2 q < 1$.

Hence, (27) holds, which finishes the proof. □

Appendix. Slow and regular variation

In this section we collect some properties of slowly resp. regularly varying functions that are used in Sections 6 and 7.

Definition 18. A measurable function $\ell\colon (0,1) \to (0,\infty)$ is said to *vary regularly at zero with index* $\rho \in \mathbb{R}$ if for every $\lambda > 0$,

$$\lim_{r\to 0+} \frac{\ell(\lambda r)}{\ell(r)} = \lambda^{\rho}.$$

If a function varies regularly at zero with index 0, it is said to *vary slowly at zero*. For simplicity, we call such functions *regularly varying* resp. *slowly varying*.

Note that slowly resp. regularly varying functions include functions which are neither increasing nor decreasing. By [BGT87, Theorem 1.4.1(iii)], any function ℓ that varies regularly with index $\rho \in \mathbb{R}$ is of the form $\ell(r) = r^{\rho}\ell_0(r)$ for some function ℓ_0 that varies slowly.

Assume $\int_0^1 s\ell(s)\,ds \le c$ for some $c > 0$. Let $L\colon (0,1) \to (0,\infty)$ be defined by

$$L(r) = \int_r^1 \frac{\ell(s)}{s}\,ds\ .$$

The function L is well defined because $L(r) = r^{-2}\int_r^1 r^2(\ell(s)/s)\,ds \le r^{-2}\int_r^1 s\ell(s)\,ds \le cr^{-2}$. Note that ($K_0$) and (K) imply that $\int_0^1 s\ell(s)\,ds \le c$ does hold in our setting. We note that the function L is always decreasing.

Let us list further properties which are made use of in our proofs. Note that they are established in [BGT87] for functions which are slowly resp. regularly varying at ∞. By a simple inversion we adapt the results to functions which are slowly resp. regularly varying at 0.

(1) If ℓ is slowly varying, then by [BGT87, Proposition 1.5.9a], L is slowly varying with

$$\lim_{r\to 0+} L(r) = \infty \quad \text{and} \quad \lim_{r\to 0+} \frac{\ell(r)}{L(r)} = 0.$$

(2) If ℓ is slowly varying and $\rho > -1$, then Karamata's theorem [BGT87, Proposition 1.5.8] ensures

$$\lim_{r\to 0+} \frac{\int_0^r s^{\rho}\ell(s)\,ds}{r^{\rho+1}\ell(r)} = (\rho+1)^{-1}.$$

(3) If ℓ is regularly varying of order $-\alpha < 0$ (in our case $0 < \alpha < 2$), then [BGT87, Theorem 1.5.11]

$$\lim_{r\to 0+} \frac{L(r)}{\ell(r)} = \alpha^{-1}.$$

In particular, if ℓ is regularly varying of order $-\alpha < 0$, then so is L.

(4) Assume ℓ is regularly varying of order $-\alpha \leq 0$ and stays bounded away from 0 and ∞ on every compact subset of $(0, 1)$. Then Potter's theorem [BGT87, Theorem 1.5.6(ii)] implies that for every $\delta > 0$ there is a constant $C = C(\delta) \geq 1$ such that for $r, s \in (0, 1)$,

$$\frac{\ell(r)}{\ell(s)} \leq C \max\left\{\left(\frac{r}{s}\right)^{-\alpha-\delta}, \left(\frac{r}{s}\right)^{-\alpha+\delta}\right\}.$$

(5) Since L is nonincreasing, we observe $\lim_{r\to 0+} L(r) \in (0, \infty]$.

*Acknowledgments.*The authors thank Tomasz Grzywny and Jongchun Bae for discussions about the assumptions used in this article. Further thanks are due to an anonymous referee for comments which helped us to improve the presentation.

This research was supported by German Science Foundation (DFG) via SFB 701, and by Croatian Science Foundation under the project 3526.

References

[AK09] Abels, H., Kassmann, M.: The Cauchy problem and the martingale problem for integro-differential operators with non-smooth kernels. Osaka J. Math. **46**, 661–683 (2009) Zbl 1196.47037 MR 2583323

[Bae15] Bae, J.: Regularity for fully nonlinear equations driven by spatial-inhomogeneous nonlocal operators. Potential Anal. **43**, 611–624 (2015) Zbl 06535137 MR 3432451

[BL02] Bass, R. F., Levin, D. A.: Harnack inequalities for jump processes. Potential Anal. **17**, 375–388 (2002) Zbl 0997.60089 MR 1918242

[BF75] Berg, C., Forst, G.: Potential Theory on Locally Compact Abelian Groups. Ergeb. Math. Grenzgeb. 87, Springer, New York (1975) Zbl 0308.31001 MR 0481057

[Ber96] Bertoin, J.: Lévy Processes. Cambridge Tracts in Math. 121, Cambridge Univ. Press, Cambridge (1996) Zbl 0938.60005 MR 1465812

[BGT87] Bingham, N. H., Goldie, C. M., Teugels, J. L.: Regular Variation. Encyclopedia Math. Appl. 27, Cambridge Univ. Press, Cambridge (1987) Zbl 0667.26003 MR 1015093

[CC95] Caffarelli, L. A., Cabré, X.: Fully Nonlinear Elliptic Equations. Amer. Math. Soc. Colloq. Publ. 43, Amer. Math. Soc., Providence, RI (1995) Zbl 0834.35002 MR 1351007

[CS09] Caffarelli, L., Silvestre, L.: Regularity theory for fully nonlinear integro-differential equations. Comm. Pure Appl. Math. **62**, 597–638 (2009) Zbl 1170.45006 MR 2494809

[CZ15] Chen, Z.-Q., Zhang, X.: Hölder estimates for nonlocal-diffusion equations with drifts. Comm. Math. Statist. **2**, 331–348 (2015) Zbl 1310.60103 MR 3326235

[DGV12] DiBenedetto, E., Gianazza, U., Vespri, V.: Harnack's Inequality for Degenerate and Singular Parabolic Equations. Springer Monogr. Math., Springer, New York (2012) Zbl 1237.35004 MR 2865434

[Grz14] Grzywny, T.: On Harnack inequality and Hölder regularity for isotropic unimodal Lévy processes. Potential Anal. **41**, 1–29 (2014) Zbl 1302.60109 MR 3225805

[JW16] Jarohs, S., Weth, T.: Symmetry via antisymmetric maximum principles in nonlocal problems of variable order. Ann. Mat. Pura Appl. (4) **195**, 273–291 (2016) Zbl 06548059 MR 3453602

[KM13] Kassmann, M., Mimica, A.: Intrinsic scaling properties for nonlocal operators. arXiv:1310.5371v2 (2013)

[KM14] Kassmann, M., Mimica, A.: Intrinsic scaling properties for nonlocal operators II. arXiv:1412.7566v2 (2014)

[KS14] Kassmann, M., Schwab, R. W.: Regularity results for nonlocal parabolic equations. Riv. Mat. Univ. Parma (N.S.) **5**, 183–212 (2014) Zbl 1329.35095 MR 3289601

[KKL16] Kim, S., Kim, Y.-C., Lee, K.-A.: Regularity for fully nonlinear integro-differential operators with regularly varying kernels. Potential Anal. **44**, 673–705 (2016) Zbl 06580626 MR 3490545

[KS79] Krylov, N. V., Safonov, M. V.: An estimate for the probability of a diffusion process hitting a set of positive measure. Dokl. Akad. Nauk SSSR **245**, 18–20 (1979) (in Russian) Zbl 0459.60067 MR 0525227

[Lan71] Landis, E. M.: Second Order Equations of Elliptic and Parabolic Type. Izdat. “Nauka”, Moscow (1971) (in Russian) Zbl 0226.35001 MR 0320507

[Mim13] Mimica, A.: Harnack inequality and Hölder regularity estimates for a Lévy process with small jumps of high intensity. J. Theor. Probab. **26**, 329–348 (2013) Zbl 1279.60100 MR 3055806

[Mim14] Mimica, A.: On harmonic functions of symmetric Lévy processes. Ann. Inst. H. Poincaré Probab. Statist. **50**, 214–235 (2014) Zbl 1298.60054 MR 3161529

[RY99] Revuz, D., Yor, M.: Continuous Martingales and Brownian Motion. 3rd ed., Grundlehren Math. Wiss. 293, Springer, Berlin (1999) Zbl 1087.60040 MR 1725357

[Sat13] Sato, K.-I.: Lévy Processes and Infinitely Divisible Distributions. Cambridge Stud. Adv. Math. 68, Cambridge Univ. Press, Cambridge (2013)

[ŠSV06] Šikić, H., Song, R., Vondraček, Z.: Potential theory of geometric stable processes. Probab. Theory Related Fields **135**, 547–575 (2006) Zbl 1099.60051 MR 2240700

[Sil06] Silvestre, L.: Hölder estimates for solutions of integro-differential equations like the fractional Laplace. Indiana Univ. Math. J. **55**, 1155–1174 (2006) Zbl 1101.45004 MR 2244602

[SS14] Silvestre, L., Schwab, R.: Regularity for parabolic integro-differential equations with very irregular kernels. arXiv:1412.3790v1 (2014)