J. Eur. Math. Soc. 19, 1071–1126

DOI 10.4171/JEMS/688

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A reduction theorem for Dade's projective conjecture

Received May 20, 2014 and in revised form March 20, 2015

Abstract. In this paper, we propose a strengthening of Dade's Conjecture. This version, called the Character Triple Conjecture, once assumed for quasisimple groups, is shown to imply Dade's Projective Conjecture for all finite groups. In particular Dade's Projective Conjecture holds for a group whose nonabelian simple sections have only covering groups satisfying the Character Triple Conjecture. We verify the new conjecture for some classes of quasisimple groups.

Keywords. Dade's (projective) conjecture, block theory, representations of finite groups

1. Introduction

In [Da92], [Da94] and [Da97], E. C. Dade proposed a series of conjectures on linear representations of finite groups. Those conjectures generalize earlier conjectural statements due to Alperin–McKay [Al76] and to Alperin [Al87], and an important reformulation of the latter due to Knörr–Robinson [KR89]. We propose here a strategy to reduce Dade's conjectures to statements on quasisimple groups. Namely we introduce a strengthening of one of Dade's conjectures, called below the Character Triple Conjecture 1.2, and prove that Dade's Projective Conjecture (see [Da94, 15.5]) holds for all finite groups if the Character Triple Conjecture holds for all quasisimple groups (see Theorem 1.3 below).

We hope that our reduction theorem can give further motivation to the checking of Dade's conjectures for various quasisimple groups (see for instance [Br06]). For simple groups with cyclic outer automorphism group, our inductive condition is equivalent to Dade's Invariant Projective Conjecture (see Proposition 6.6 below). Note that our work gives evidence for the project brought forth by Dade [Da97, Section 5] although we do not take into account the various refinements of Dade's conjecture given by Glesser [Gl07] or Uno [Uno04].

Roughly speaking, our Character Triple Conjecture can be considered as an adaptation to Dade's conjecture of the inductive conditions given by Isaacs–Malle–Navarro, Navarro–Tiep and the author in the context of other local/global conjectures (see [IMN07], [NT11], [Spä13a] and [Spä13b]).

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Mathematics Subject Classification (2010): Primary 20C20; Secondary 20C33, 20C34

Concerning the methods used to prove Theorem 1.3, it is understood that any reduction statement has to give Clifford theory a central rôle (see [IMN07], [NT11], [Spä13a] and [Spä13b]). Namely, studying triples (H, M, χ), where M is a normal subgroup of the finite group H and $\chi \in \mathrm{Irr}(M)$ is an irreducible character of M, is of crucial importance. In particular, characterizing how two such character triples are to be considered equivalent with regard to our particular purpose seems to be a good way to start any study. This method allows a remarkable flexibility as to what can be required in terms of blocks or character degrees, while remaining very elementary in the questions raised.

Continuing our investigations developed in [Spä13a], [NS14] and [KS15], we propose here another variant of equivalence relation on character triples (see Definition 3.6), and show several properties useful for our study of Dade's conjectures. This leads us to formulate our reduction theorem in those new terms.

Before stating the conjecture and the main result we recall some relevant notation. Fix a prime p. For a finite group G we denote by $\mathfrak{P}(G|\mathrm{O}_p(G))$ the set of p-chains of G starting with $\mathrm{O}_p(G)$ (see Definition 6.1). For $\mathbb{D} \in \mathfrak{P}(G|\mathrm{O}_p(G))$ with

$$\mathbb{D} = (D_0 = \mathrm{O}_p(G) \lneq D_1 \lneq \cdots \lneq D_l)$$

we denote by $|\mathbb{D}|$ the integer l, called the *length* of $\mathbb{D}$. This notion partitions $\mathfrak{P}(G|\mathrm{O}_p(G))$ into the set of p-chains of even length, denoted by $\mathfrak{P}(G|\mathrm{O}_p(G))_+$, and the set of p-chains of odd length, denoted by $\mathfrak{P}(G|\mathrm{O}_p(G))_-$. For a p-block B of G and an element $\mathbb{D} \in \mathfrak{P}(G)$ we denote by $G_{\mathbb{D}}$ the normalizer of $\mathbb{D}$ in G and by $B_{\mathbb{D}}$ the set of all p-blocks b of $G_{\mathbb{D}}$ with $b^G = B$, where b^G is the block obtained via Brauer induction (of blocks).

For $\chi \in \mathrm{Irr}(G)$ we denote by $d(\chi)$ the *defect* of χ, which is the integer i such that $p^i \chi(1)_p = |G|_p$, where as usual for an integer j we denote by j_p the maximal p-power dividing j. For a nonnegative integer d, and any set C of p-blocks of G, the set $\mathrm{Irr}^d(C)$ consists of all characters with defect d belonging to a block in C. Now for a p-block B of G, $\mathcal{C}^d(B)_+$ is defined to be the set of all pairs $(\mathbb{D}, \theta)$ with $\mathbb{D} \in \mathfrak{P}(G|\mathrm{O}_p(G))_+$ and $\theta \in \mathrm{Irr}^d(B_{\mathbb{D}})$, and one obtains $\mathcal{C}^d(B)_-$ analogously. The group G acts naturally on $\mathcal{C}^d(B)_+$ and $\mathcal{C}^d(B)_-$ by conjugation. For $(\mathbb{D}, \theta) \in \mathcal{C}^d(B)_+$ we denote by $\overline{(\mathbb{D}, \theta)}$ its G-orbit and by $\overline{\mathcal{C}^d(B)_+}$ and $\overline{\mathcal{C}^d(B)_-}$ the corresponding sets of G-orbits.

With this notation, Dade's Projective Conjecture 15.5 from [Da94] can be rephrased in the following way (see Proposition 6.2 for more details).

Conjecture 1.1. *Let p be a prime, d an integer, G a finite group with $\mathrm{O}_p(G) \leq \mathrm{Z}(G)$, and B a p-block of G with a noncentral defect group. Then there exists a bijection*

$$\Omega : \overline{\mathcal{C}^d(B)_+} \to \overline{\mathcal{C}^d(B)_-}$$

such that $\theta_{\mathrm{O}_p(G)}$ and $\theta'_{\mathrm{O}_p(G)}$ are multiples of the same irreducible character whenever $(\mathbb{D}, \theta) \in \mathcal{C}^d(B)_+$ and $(\mathbb{D}', \theta') \in \Omega(\overline{(\mathbb{D}, \theta)})$.

According to [Da94, Theorem (18.14)] the above conjecture for all groups implies the Alperin–McKay conjecture for all groups, and according to [Da92, Theorem 8.3] it also implies the Alperin weight conjecture for all groups.

As above, one can interpret Dade's Projective Conjecture as the existence of a bijection. Then one can strengthen this conjecture by requiring additionally that characters associated to each other via this bijection give character triples that satisfy the new equivalence relation introduced in Definition 3.6.

Conjecture 1.2 (Character Triple Conjecture). *Let p be a prime, d an integer, G a finite group with $\mathrm{O}_p(G) \leq \mathrm{Z}(G)$, and B a p-block of G with a noncentral defect group. Suppose that $G \lhd A$. Then there exists an A_B-equivariant bijection*

$$\Omega : \overline{\mathcal{C}^d(B)_+} \to \overline{\mathcal{C}^d(B)_-}$$

such that for every $(\mathbb{D}, \theta) \in \mathcal{C}^d(B)_+$, some $(\mathbb{D}, \theta') \in \Omega(\overline{(\mathbb{D}, \theta)})$ satisfies

$$(A_{\mathbb{D},\theta}, G_{\mathbb{D}}, \theta) \sim_G (A_{\mathbb{D}',\theta'}, G_{\mathbb{D}'}, \theta')$$

in the sense of Definition 3.6.

This conjecture for a given group G implies Dade's Extended Projective Conjecture [Da97, 4.10] for G (see Proposition 6.4). The author suspects that it is in fact stronger than Dade's Final Conjecture proposed in [Da97, 5.8], but this should be the subject of future investigations. In addition our new conjecture holds for p-solvable groups (see [Spä14]).

The main aim of the paper is to prove the following statement.

Theorem 1.3. *Let $\mathcal{S}$ be a set of simple nonabelian groups such that every covering group X of some $S \in \mathcal{S}$ satisfies Conjecture 1.2 with respect to $X \lhd X \rtimes \mathrm{Aut}(X)$. Then Dade's Projective Conjecture (Conjecture 1.1) holds for every finite group G if every nonabelian simple group involved in G is contained in $\mathcal{S}$.*

This gives another approach to verifying the Alperin weight and the Alperin–McKay Conjectures via making use of the classification of finite simple groups.

The proof of Theorem 1.3 is mainly built on the study and properties of the equivalence relation from Definition 3.6 on character triples. This equivalence relation allows us to control the Clifford theory of characters, especially with respect to blocks. In Section 8 this is applied to verify the reduction statement via the following steps:

(i) A minimal counterexample G has been described in [ER02] and has a normal perfect subgroup K such that $K/\mathrm{Z}(K)$ is the direct product S^r of groups isomorphic to a nonabelian simple group S. This group generates together with the centre of G the generalized Fitting subgroup.

(ii) According to [Ro02] it is sufficient to consider the cardinality of character sets associated with the chains of K.

(iii) The universal covering of K is a group of the form $\widehat{S}^r$, where $\widehat{S}$ is the universal covering group of S. The p-chains of K and the cardinalities of the character sets from (ii) can be studied via characters of subgroups of $\widehat{S}^r \rtimes \mathrm{Aut}(\widehat{S}^r)$ thanks to Theorem 5.3.

(iv) Chains of $\widehat{S}$ and associated characters in the interplay with the wreath product structure of $\widehat{S}^r \rtimes \mathrm{Aut}(\widehat{S}^r)$ are studied in Section 7. (Slightly similar considerations already appear in [EH02].)

Later on we give examples where Conjecture 1.2 holds for covering groups of nonabelian simple groups. Based on earlier work on Dade's Invariant Conjecture we prove that the Character Triple Conjecture holds for most sporadic groups. Blocks with cyclic defect also satisfy this conjecture. A further example is given by the quasisimple groups $\mathrm{SL}_2(q)$.

The paper is structured in the following way: After setting the notation and some basic lemmas in Section 2 we introduce in Section 3 our equivalence relation between character triples. Basic properties of this relation are then investigated in Section 4. In Section 5 we prove that two major constructions of irreducible characters give a way to obtain new pairs of equivalent character triples and investigate their nature (see Theorem 5.3). Using this new equivalence relation we propose in Section 6 a new version of Dade's Conjecture and prove that it is a strengthening of Dade's Extended Projective Conjecture. In Section 7 we investigate p-chains of direct products and study how wreath products act on them. This proves some implications of the Character Triple Conjecture for quasisimple groups. In Section 8 we construct an equivariant bijection between certain pairs of characters and p-chains using combinatorial arguments and apply it to prove our main result, namely that a minimal counterexample of Dade's Projective Conjecture cannot exist if all quasisimple groups satisfy the Character Triple Conjecture. We conclude by giving some examples of quasisimple groups satisfying the latter conjecture.

2. Notation and basic observations

This section introduces some notation and gathers basic results about induced blocks. The notation is based on that introduced in [NT89] and [Na98]. All groups in this paper are finite. We use R. Brauer's definition of induced blocks (see [Na98, p. 87]).

Notation 2.1. We denote by $\mathrm{tr}(M)$ the trace of a matrix M. Let p be a prime. Let R be the ring of algebraic integers, and let $\mathcal{O}$ be a localization of R at some maximal ideal containing pR. Let $\mathbb{F}$ be the residue field of $\mathcal{O}$. Accordingly $\mathrm{char}\,\mathbb{F} = p$. See Chapter 2 of [Na98] for details and exact definitions. Let $()^* : \mathcal{O} \to \mathbb{F}$ be the associated canonical epimorphism.

For a finite group G we denote by $\mathrm{Irr}(G)$ the set of ordinary irreducible characters of G. For a character $\phi_1 \in \mathrm{Irr}(G)$ and $H \leq G$ we denote by $\phi_{1,H}$ the restriction of ϕ_1 to H. The set of p-blocks of G form $\mathrm{Bl}(G)$, and $\mathrm{Bl}_{\mathrm{nc}}(G)$ is defined to be the set of p-blocks with noncentral defect groups. Recall that for $\chi \in \mathrm{Irr}(G)$ we denote by $d(\chi)$ the *defect* of χ, that is, the integer d with $p^d = (|G|/\chi(1))_p$. This defines the set $\mathrm{Irr}^d(G) := \{\chi \in \mathrm{Irr}(G) \mid d(\chi) = d\}$ for any integer d.

For a character $\chi \in \mathbb{Z}_{\geq 0}\mathrm{Irr}(G)$ we denote by $\mathrm{Irr}(\chi)$ the set of irreducible constituents of χ and by $\mathrm{Irr}(G \mid \nu)$ the set of irreducible constituents of ν^G where $N \leq G$ and $\nu \in \mathrm{Irr}(N)$. As always let $\mathrm{Char}(G \mid \nu)$ be the set of characters of G whose irreducible constituents are contained in $\mathrm{Irr}(G \mid \nu)$. For $\chi \in \mathrm{Irr}(G)$ we denote by $\lambda_\chi : \mathrm{Z}(\mathbb{F}G) \to \mathbb{F}$ the associated central function. For $b \in \mathrm{Bl}(G)$ the central function $\lambda_b : \mathrm{Z}(\mathbb{F}G) \to \mathbb{F}$ is defined as $\lambda_b = \lambda_\chi$ for any $\chi \in \mathrm{Irr}(b)$. Also, if $\chi \in \mathrm{Irr}(G)$ we denote by $\mathrm{bl}(\chi)$ the p-block of G containing χ. Moreover $\mathfrak{Cl}_G(x)$ is the G-conjugacy class containing $x \in G$. For any subset $C \subseteq G$, C^+ denotes the sum $\sum_{x \in C} x$ seen as an element in $\mathbb{F}G$ or $\mathbb{Z}G$.

For $H \leq G$ and $b \in \mathrm{Bl}(H)$ we denote by b^G the induced block, when it is defined, and if $H \lhd G$, then $\mathrm{Bl}(G \mid b)$ is the set of p-blocks of G covering b. For B a p-block or a set (sum) of p-blocks of G we denote by $\mathrm{Irr}(B)$ the ordinary characters belonging to (a block in) B. For nonnegative integers d we set $\mathrm{Irr}^d(B) := \mathrm{Irr}^d(G) \cap \mathrm{Irr}(B)$ and $\mathrm{Irr}^d(B \mid \nu) := \mathrm{Irr}^d(B) \cap \mathrm{Irr}(G \mid \nu)$ where $N \lhd G$ and $\nu \in \mathrm{Irr}(N)$.

If $N \lhd G$ and $\tau \in \mathrm{Irr}(N)$ we denote by G_τ the stabilizer of τ in G, also sometimes called the inertia group and denoted by $\mathrm{I}_G(\tau)$. For $B \in \mathrm{Bl}(G)$ we define $\mathrm{Irr}(B \mid \tau) := \mathrm{Irr}(B) \cap \mathrm{Irr}(G \mid \tau)$.

Lemma 2.2. *Let $N \lhd G$ and let $\epsilon : G \to G/N = \overline{G}$ be the canonical epimorphism. Let $\theta \in \mathrm{Irr}(N)$ and suppose that there is an extension $\widetilde{\theta} \in \mathrm{Irr}(G)$ of θ. Let $\overline{\eta} \in \mathrm{Irr}(\overline{G})$ and $\eta := \overline{\eta} \circ \epsilon \in \mathrm{Irr}(G)$. If $x \in G$, then*

$$\lambda_{\widetilde{\theta}\eta}(\mathfrak{Cl}_G(x)^+) = \lambda_{\widetilde{\theta}_{\langle N,x\rangle}}((\mathfrak{Cl}_G(x) \cap xN)^+)\lambda_{\overline{\eta}}(\mathfrak{Cl}_{\overline{G}}(\overline{x})^+).$$

(Note that $\widetilde{\theta}\eta \in \mathrm{Irr}(G)$ according to [Is76, *Corollary* (6.15)].)

Proof. From [NS14, Lemma 2.2] we know

$$\lambda_{\widetilde{\theta}\eta}(\mathfrak{Cl}_G(x)^+) = \lambda_{\widetilde{\theta}_{\langle N,x\rangle}}(\mathfrak{Cl}_L(x)^+)\lambda_{\overline{\eta}}(\mathfrak{Cl}_{\overline{G}}(\overline{x})^+),$$

where $L/N = \mathrm{C}_{\overline{G}}(\overline{x})$ and $\overline{x} := \epsilon(x)$. By the definition of L we observe that $\mathfrak{Cl}_L(x)$ is contained in xN and coincides with $\mathfrak{Cl}_G(x) \cap xN$, hence $\lambda_{\widetilde{\theta}_L}(\mathfrak{Cl}_L(x)^+) = \lambda_{\widetilde{\theta}_{\langle N,x\rangle}}((\mathfrak{Cl}_G(x) \cap xN)^+)$. □

The above formula is used for the proof of an adaptation of [NS14, Proposition 2.3] to the situation we consider in the later sections.

Proposition 2.3. *Let $N \lhd G$ and $H_1, H_2 \leq G$ such that $NH_1 = NH_2 = G$. Write $M_1 := N \cap H_1$ and $M_2 := N \cap H_2$. For $i = 1, 2$ let $\widetilde{\theta}_i \in \mathrm{Irr}(H_i)$ with $\widetilde{\theta}_{i,M_i} \in \mathrm{Irr}(M_i)$. Assume moreover that:*

(i) *for every $N \leq J \leq G$ with J/N cyclic, the blocks $\mathrm{bl}(\widetilde{\theta}_{1,J\cap H_1})^J$ and $\mathrm{bl}(\widetilde{\theta}_{2,J\cap H_2})^J$ are defined and equal:*

$$\mathrm{bl}(\widetilde{\theta}_{1,J\cap H_1})^J = \mathrm{bl}(\widetilde{\theta}_{2,J\cap H_2})^J,$$

(ii) *for every $c \in \mathrm{Bl}(H_1 \mid \mathrm{bl}(\theta_1))$ the block c^G is defined.*

Then for every $\eta \in \mathrm{Irr}(G)$ with $N \leq \ker(\eta)$, the block $\mathrm{bl}(\widetilde{\theta}_2\eta_{H_2})^G$ is defined and $\mathrm{bl}(\widetilde{\theta}_1\eta_{H_1})^G = \mathrm{bl}(\widetilde{\theta}_2\eta_{H_2})^G$. (Note that $\mathrm{bl}(\widetilde{\theta}_1\eta_{H_1})$ and $\mathrm{bl}(\widetilde{\theta}_2\eta_{H_2})$ are well-defined since $H_1/M_1 \cong H_2/M_2 \cong G/N$ implies $\widetilde{\theta}_1\eta_{H_1} \in \mathrm{Irr}(H_1)$ and $\widetilde{\theta}_2\eta_{H_2} \in \mathrm{Irr}(H_2)$ according to [Is76, *Corollary* (6.15)].)

Proof. Let $\eta_1 := \eta_{H_1}$ and $\eta_2 := \eta_{H_2}$. Note that because $\mathrm{bl}(\widetilde{\theta}_1\eta_1) \in \mathrm{Bl}(H_1 \mid \mathrm{bl}(\theta_1))$, the block $\mathrm{bl}(\widetilde{\theta}_1\eta_1)^G$ is defined by (ii). Hence, by definition, it is sufficient to show

$$\lambda_{\widetilde{\theta}_1\eta_1}((\mathfrak{Cl}_G(x) \cap H_1)^+) = \lambda_{\widetilde{\theta}_2\eta_2}((\mathfrak{Cl}_G(x) \cap H_2)^+) \quad \text{for every } x \in G,$$

since this implies that $\mathrm{bl}(\widetilde{\theta}_2\eta_2)^G$ is defined and $\mathrm{bl}(\widetilde{\theta}_1\eta_1)^G = \mathrm{bl}(\widetilde{\theta}_2\eta_2)^G$. Let $\overline{x} := xN$, let $\overline{x}_1$ be the image of $\overline{x}$ via the isomorphism $G/N \cong H_1/M_1$ and let $x_1 \in H_1$ be such that $x_1M_1 = \overline{x}_1$. According to Lemma 2.2, we have

$$\lambda_{\widetilde{\theta}_1\eta_1}((\mathfrak{Cl}_G(x) \cap H_1)^+) = \lambda_{\widetilde{\theta}_{1,\langle M_1,x_1\rangle}}((\mathfrak{Cl}_G(x) \cap x_1M_1)^+)\lambda_{\overline{\eta}_1}(\mathfrak{Cl}_{H_1/M_1}(\overline{x}_1))$$

where $\overline{\eta}_1 \in \mathrm{Irr}(H_1/M_1)$ is the character that lifts to η_1. Let $\overline{x}_2$ be the image of $\overline{x}$ via $G/N \cong H_2/M_2$, and let $x_2 \in H_2$ be such that $x_2M_2 = \overline{x}_2$. Analogously

$$\lambda_{\widetilde{\theta}_2\eta_2}((\mathfrak{Cl}_G(x) \cap H_2)^+) = \lambda_{\widetilde{\theta}_{2,\langle M_2,x_2\rangle}}((\mathfrak{Cl}_G(x) \cap x_2M_2)^+)\lambda_{\overline{\eta}_2}(\mathfrak{Cl}_{H_2/M_2}(\overline{x}_2)),$$

where $\overline{\eta}_2 \in \mathrm{Irr}(H_2/M_2)$ lifts to η_2. By the definition of $\overline{\eta}_1$ and $\overline{\eta}_2$ we see that

$$\lambda_{\overline{\eta}_1}(\mathfrak{Cl}_{H_1/M_1}(\overline{x}_1)) = \lambda_{\overline{\eta}_2}(\mathfrak{Cl}_{H_2/M_2}(\overline{x}_2)).$$

Since $\mathrm{bl}(\widetilde{\theta}_{1,\langle M_1,x_1\rangle})^{\langle N,x\rangle} = \mathrm{bl}(\widetilde{\theta}_{2,\langle M_2,x_2\rangle})^{\langle N,x\rangle}$ by assumption (i), we have

$$\lambda_{\widetilde{\theta}_{\langle M_1,x_1\rangle}}((\mathfrak{Cl}_G(x) \cap x_1M_1)^+) = \lambda_{\widetilde{\theta}_{\langle M_2,x_2\rangle}}((\mathfrak{Cl}_G(x) \cap x_2M_2)^+).$$

This implies

$$\lambda_{\widetilde{\theta}_1\eta_1}((\mathfrak{Cl}_G(x) \cap H_1)^+) = \lambda_{\widetilde{\theta}_2\eta_2}((\mathfrak{Cl}_G(x) \cap H_2)^+),$$

and hence $\mathrm{bl}(\widetilde{\theta}_1\eta_1)^G = \mathrm{bl}(\widetilde{\theta}_2\eta_2)^G$. □

2.4. Dade's ramification group $G[b]$

Later we compare induced blocks. This task is significantly simplified by using Dade's ramification group $G[b]$, introduced in [Da73]. If $N \lhd G$ and $b \in \mathrm{Bl}(N)$, we denote by $G[b]$ the group generated by N and all elements $x \in G$ with $\lambda_{\widetilde{b}^{(x)}}(\mathfrak{Cl}_{\langle N,x\rangle}(x)^+) \neq 0$ for some block $\widetilde{b}^{(x)} \in \mathrm{Bl}(\langle N, x\rangle)$ covering b. (This is an equivalent definition of $G[b]$ following ideas of [Mu13]; see [KS15, Proposition 3.1] for more details.)

Proposition 2.5 (Properties of Dade's ramification group). *Let $N \lhd G$ and $b \in \mathrm{Bl}(N)$ be G-invariant. Then*

(a) *$G[b] \leq N\mathrm{C}_G(D)$, where D is a defect group of b,*
(b) *$\lambda_{\widetilde{b}}(\mathfrak{Cl}_J(x)^+) = 0$ for every $N \leq J \leq G$, $x \in J \setminus G[b]$ and $\widetilde{b} \in \mathrm{Bl}(J \mid b)$.*

Proof. Part (a) is a consequence of Dade's description of $G[b]$ given in [Da73, Corollary 12.6] (see also [Mu13, Theorem 3.13]). Part (b) follows from the definition of $G[b]$ given above. □

3. A new equivalence relation between character triples

The goal of this section is to establish the announced new equivalence relation between character triples and describe its first properties. This equivalence relation helps to prove Theorem 1.3, since it enables one to control and compare the Clifford theory of different characters.

Recall that a triple (G, N, χ) is called a *character triple* if $N \lhd G$ and χ is a G-invariant irreducible character of N (see [Is76, p. 186]).

Many properties of a character triple are reflected by a projective representation that can be deduced from the character of the (character) triple. In our setting we are mainly interested in projective representations obtained that way satisfying some additional properties (see also [NT89, 3.5.7] and [NS14, Section 3]).

Definition 3.1 (Projective representations associated with a character). Let (G, N, χ) be a character triple and $\mathcal{D}$ a (linear) representation of N affording χ. Then a projective representation $\mathcal{P}$ of G is called a *projective representation of G associated with χ* if

(i) $\mathcal{P}(gn) = \mathcal{P}(g)\mathcal{D}(n)$ and $\mathcal{P}(ng) = \mathcal{D}(n)\mathcal{P}(g)$ for every $n \in N$ and $g \in G$, i.e., $\mathcal{P}_N = \mathcal{D}$ and the factor set α of $\mathcal{P}$ is trivial on $(G \times N) \cup (N \times G)$,

(ii) the values of the factor set α are roots of unity.

For a given character triple (G, N, χ) and a linear representation $\mathcal{D}$ of N affording χ there exists a projective representation $\mathcal{P}$ of G associated with χ (see [NS14, Theorem 3.1(a)] based on [Is76, Theorem (11.2)]). Then the factor set α of $\mathcal{P}$ determines a map $\overline{\alpha} : G/N \times G/N \to \mathbb{C}$ (see also [NS14, remark before Theorem 3.2]).

Recall the definition of an isomorphism of character triples. Let (H_1, M_1, θ_1) and (H_2, M_2, θ_2) be character triples and let $\iota : H_1/M_1 \to H_2/M_2$ be an isomorphism. For every group J with $M_1 \leq J \leq H_1$ we denote by J^ι the group with $M_2 \leq J^\iota \leq H_2$ and $J^\iota/M_2 = \iota(J/M_1)$, and by η^ι the character of J^ι/M_2 associated to $\eta \in \mathrm{Irr}(J/M_1)$ via ι. Suppose that for every subgroup J with $M_1 \leq J \leq H_1$ there exists an additive bijection

$$\sigma_J : \mathrm{Char}(J \mid \theta_1) \to \mathrm{Char}(J^\iota \mid \theta_2)$$

satisfying $\sigma_J(\mathrm{Irr}(J \mid \theta_1)) = \mathrm{Irr}(J^\iota \mid \theta_2)$ such that for I, J with $M_1 \leq I \leq J \leq H_1$ and $\psi, \psi' \in \mathrm{Char}(J \mid \theta_1)$ the following conditions hold:

(i) $\sigma_I(\psi_I) = \sigma_J(\psi)_{I^\iota}$,

(ii) $\sigma_J(\psi\eta) = \sigma(\psi)\eta^\iota$ for every $\eta \in \mathrm{Irr}(J/M_1)$.

Let σ denote the union of the maps σ_J for $M_1 \leq J_1 \leq H_1$. Then the map $(\iota, \sigma) : (H_1, M_1, \theta_1) \to (H_2, M_2, \theta_2)$ is called an *isomorphism of character triples* (see also [Is76, Definition (11.23)]).

Further the isomorphism (ι, σ) is *strong* if

$$\sigma_J(\psi)^{h_2} = \sigma_{J^{h_1}}(\psi^{h_1})$$

for every J with $M_1 \leq J \leq H_1$, $h_1 \in H_1$ and $\psi \in \mathrm{Char}(J \mid \theta_1)$, where $h_2M_2 = \iota(h_1M_1)$. (See also [Is76, Exercise (11.13)].)

We use the following generalization of [NS14, Theorem 3.2] and assume thereby the following setting.

Hypothesis 3.2. Let $N \lhd G$, and let (H_1, M_1, θ_1) and (H_2, M_2, θ_2) be character triples with $G = NH_1 = NH_2$, $M_1 = N \cap H_1$ and $M_2 = N \cap H_2$. Let $\iota : H_1/M_1 \to H_2/M_2$ be the canonical isomorphism. Assume that there exist projective representations $\mathcal{P}_1$ and $\mathcal{P}_2$ of H_1 and H_2 associated with θ_1 and θ_2 respectively, with factor sets α_1 and α_2 such that $\overline{\alpha}_1(x, y) = \overline{\alpha}_2(\iota(x), \iota(y))$ for all $x, y \in H_1/M_1$.

Theorem 3.3. *Assume Hypothesis 3.2. Then there exists a strong isomorphism of character triples*

$$(\iota, \sigma) : (H_1, M_1, \theta_1) \to (H_2, M_2, \theta_2),$$

where for every $N \leq J \leq G$ the map

$$\sigma_{J\cap H_1} : \mathrm{Char}(J \cap H_1 \mid \theta_1) \to \mathrm{Char}(J \cap H_2 \mid \theta_2)$$

is given by

$$\sigma_{J\cap H_1}(\mathrm{tr}(\mathcal{Q}_{J\cap H_1} \otimes \mathcal{P}_{1,J\cap H_1})) = \mathrm{tr}(\mathcal{Q}_{J\cap H_2} \otimes \mathcal{P}_{2,J\cap H_2})$$

for any projective representation $\mathcal{Q}$ of J whose factor set is the inverse of that of $\mathcal{P}_{1,J\cap H_1}$ and which is the lift of a projective representation of J/N.

Proof. The arguments from [NS14, proof of Theorem 3.2] apply here: By [NS14, Theorem 3.1(c)], every $\gamma \in \mathrm{Irr}(J \cap H_1 \mid \theta_1)$ is the trace of some representation of the form $\mathcal{Q}_1 \otimes \mathcal{P}_{1,J\cap H_1}$, where $\mathcal{Q}_1$ is a projective representation of $J \cap H_1$ that is the lift of a projective representation of $(J \cap H_1)/M_1$ whose factor set is the inverse of that of $\mathcal{P}_{1,J\cap H_1}$. Also, $\mathcal{Q}_1$ is uniquely determined up to similarity and defines a projective representation of J and $J \cap H_2$. It is straightforward to check that this defines a strong isomorphism of character triples. □

In a slight generalization of the terminology of [NS14] we call $(\iota, \sigma) : (H_1, M_1, \theta_1) \to (H_2, M_2, \theta_2)$ as above an *isomorphism of character triples given by* (*the projective representations*) $\mathcal{P}_1$ *and* $\mathcal{P}_2$.

If $\mathrm{C}_G(N) \leq H_1 \cap H_2$, by Schur's Lemma the matrices $\mathcal{P}_1(x)$ and $\mathcal{P}_2(x)$ are scalar for all $x \in \mathrm{C}_G(N)$. Recall that for J with $J \leq K$ and $\psi \in \mathrm{Irr}(K)$ we denote by $\mathrm{Irr}(\psi_J)$ the set of irreducible constituents of ψ_J.

Lemma 3.4. *Assume Hypothesis 3.2. Let $(\iota, \sigma) : (H_1, M_1, \theta_1) \to (H_2, M_2, \theta_2)$ be an isomorphism of character triples given by the projective representations $\mathcal{P}_1$ and $\mathcal{P}_2$. Assume $\mathrm{C}_G(N) \leq H_1 \cap H_2$. Then the following are equivalent:*

(i) *For every $x \in \mathrm{C}_G(N)$ the matrices $\mathcal{P}_1(x)$ and $\mathcal{P}_2(x)$ are scalar matrices associated with the same $\zeta \in \mathbb{C}$.*
(ii) $\mathrm{Irr}(\psi_{\mathrm{C}_J(N)}) = \mathrm{Irr}(\sigma_{J\cap H_1}(\psi)_{\mathrm{C}_J(N)})$ *for every J with $N \leq J \leq G$ and $\psi \in \mathrm{Irr}(J \mid \theta_1)$.*

Proof. The arguments of [NS14, proof of Lemma 3.3] apply with mild modifications. □

In the situation of Lemma 3.4, we say that (H_1, M_1, θ_1) and (H_2, M_2, θ_2) are *N-central isomorphic character triples* and that (ι, σ) is an *N-central isomorphism of character triples*.

The following lemma is often used and is basic for the definition of N-block isomorphic character triples in Definition 3.6.

Lemma 3.5. *Assume Hypothesis 3.2. Additionally assume that for $i = 1, 2$ some defect group D_i of $\mathrm{bl}(\theta_i)$ satisfies $\mathrm{C}_G(D_i) \leq H_i$. Then*

(a) $\mathrm{C}_G(N) \leq H_1 \cap H_2$,
(b) *for every $c_i \in \mathrm{Bl}(H_i \mid \theta_i)$ the block $\mathrm{bl}(c_i)^G$ is defined.*

Proof. This follows from [NT89, Lemma 5.5.14 and Theorem 5.5.16(ii)]. □

Definition 3.6 (N-block isomorphism of character triples). Let $(\iota, \sigma) : (H_1, M_1, \theta_1) \to (H_2, M_2, \theta_2)$ be an N-central isomorphism of character triples. Assume that for $i = 1, 2$ there exists some defect group D_i of $\mathrm{bl}(\theta_i)$ with $\mathrm{C}_G(D_i) \leq H_i$. If for every $N \leq J \leq G$ and $\psi \in \mathrm{Irr}(J \mid \theta)$ the equality

$$\mathrm{bl}(\sigma_{J \cap H_1}(\psi))^J = \mathrm{bl}(\psi)^J$$

holds then we say that (ι, σ) is an *N-block isomorphism of character triples*. (Recall that according to Lemma 3.5(b) the blocks $\mathrm{bl}(\sigma_{J \cap H_1}(\psi))^J$ and $\mathrm{bl}(\psi)^J$ are defined.) In this situation we write

$$(H_1, M_1, \theta_1) \sim_N (H_2, M_2, \theta_2) \quad \text{via } (\iota, \sigma)$$

and call (H_1, M_1, θ_1) and (H_2, M_2, θ_2) (a pair of) *N-block isomorphic character triples*.

In Lemma 3.8(b) we will see that $\sim_N$ is an equivalence relation. The following remark lists the properties that have to be checked in order to verify that two character triples are N-block isomorphic.

Remark 3.7. Let $N \lhd G$, and let (H_1, M_1, θ_1) and (H_2, M_2, θ_2) be two character triples. Then

$$(H_1, M_1, \theta_1) \sim_N (H_2, M_2, \theta_2)$$

if

(i) $G = NH_1 = NH_2$, $M_1 = N \cap H_1$ and $M_2 = N \cap H_2$ (let $\iota : H_1/M_1 \to H_2/M_2$ be the canonical epimorphism),
(ii) for $i = 1, 2$ some defect group D_i of $\mathrm{bl}(\theta_i)$ satisfies $\mathrm{C}_G(D_i) \leq H_i$,
(iii) there exist projective representations $\mathcal{P}_1$ and $\mathcal{P}_2$ associated with θ_1 and θ_2 with factor sets α_1 and α_2 such that $\overline{\alpha}_1(h, h') = \overline{\alpha}_2(\iota(h), \iota(h'))$ for all $h, h' \in H_1/M_1$, and the scalar matrices $\mathcal{P}_1(x)$ and $\mathcal{P}_2(x)$ are associated with the same scalars for every $x \in \mathrm{C}_G(N)$,
(iv) for every $N \leq J \leq G$ and $\psi \in \mathrm{Irr}(J \cap H_1 \mid \theta_1)$ the blocks satisfy

$$\mathrm{bl}(\sigma_{J \cap H_1}(\psi))^J = \mathrm{bl}(\psi)^J,$$

where $(\iota, \sigma) : (H_1, N_1, \theta_1) \to (H_2, N_2, \theta_2)$ is the N-central isomorphism of character triples given by $\mathcal{P}_1$ and $\mathcal{P}_2$.

Then (ι, σ) is an N-block isomorphism of character triples.

Note that by Lemma 3.5(a) the condition in (ii) implies that the matrices in (iii) are well-defined.

Proof. Assumptions (i) and (ii) imply that $C_G(N) \leq C_G(D_1) \cap C_G(D_2) \leq H_1 \cap H_2$. Then Lemma 3.4 implies that $\mathcal{P}_1$ and $\mathcal{P}_2$ determine an N-central isomorphism (ι, σ) of character triples. For every $N \leq J \leq G$ and $\psi \in \mathrm{Irr}(J \cap H_1 \mid \theta_1)$ the blocks $\mathrm{bl}(\psi)^J$ and $\mathrm{bl}(\sigma_{J\cap H_1}(\psi))^J$ are defined according to Lemma 3.5, and those blocks satisfy $\mathrm{bl}(\psi)^J = \mathrm{bl}(\sigma_{J\cap H_1}(\psi))^J$ according to (iv). □

The next statement proves that $\sim_N$ is an equivalence relation on the set of character triples.

Lemma 3.8. (a) *If* $(H_1, M_1, \theta_1) \sim_N (H_2, M_2, \theta_2)$ *and* $(H_2, M_2, \theta_2) \sim_N (H_3, M_3, \theta_3)$, *then*

$$(H_1, M_1, \theta_1) \sim_N (H_3, M_3, \theta_3).$$

(b) *If* $(H_1, M_1, \theta_1) \sim_N (H_2, M_2, \theta_2)$, *then*

$$(H_1 \cap J, M_1, \theta_1) \sim_N (H_2 \cap J, M_2, \theta_2) \quad \text{for every } N \leq J \leq G.$$

(c) *If* $(H_1, M_1, \theta_1) \sim_N (H_2, M_2, \theta_2)$ *and* $n \in N$, *then* $(H_2^n, M_2^n, \theta_2^n) \sim_N (H_1, M_1, \theta_1)$.

Proof. This follows from straightforward calculations. □

While the relation $\sim_b$ introduced in [NS14, Definition 3.6] was in fact an order relation and enabled us to control the relative height of characters and defect groups, an N-block isomorphism of character triples affords less control since the characters involved have to satisfy fewer requirements with respect to the defect groups of the associated blocks or their heights.

Proposition 3.9. *Let* $(\iota, \sigma) : (H_1, M_1, \theta_1) \to (H_2, M_2, \theta_2)$ *be an* N*-block isomorphism of character triples (see Definition 3.6). Let* J *be a group with* $N \leq J \leq G$, $\psi \in \mathrm{Irr}(J \mid \theta)$ *and* $\psi' := \sigma_{J\cap H_1}(\psi)$. *Then:*

(a) $\mathrm{Irr}(\psi_{C_J(G)}) = \mathrm{Irr}(\psi'_{C_J(G)})$.
(b) $d(\psi) - d(\theta_1) = d(\psi') - d(\theta_2)$.
(c) *Assume* $d(\theta_1) = d(\theta_2)$ *and* $J \lhd G$. *Let* $B \in \mathrm{Bl}(G)$ *and* B_i *the sum of blocks* b *in* $\mathrm{Bl}(H_i)$ *with* $b^G = B$ *for* $i = 1, 2$. *Then the restriction of* σ_{H_1} *to* $\mathrm{Irr}(B_1 \mid \psi)$ *gives a defect preserving bijection between* $\mathrm{Irr}(B_1 \mid \psi)$ *and* $\mathrm{Irr}(B_2 \mid \psi')$.

Proof. Part (a) follows directly from Lemma 3.4. According to Theorem 3.3 the maps (ι, σ) define an isomorphism of character triples, hence $\psi(1)/\theta_1(1) = \psi'(1)/\theta_2(1)$ according to [Is76, Lemma (11.24)]. This implies (b).

If $d(\theta_1) = d(\theta_2)$ then the bijection σ_{H_1} is a defect preserving bijection thanks to (b). According to the properties of isomorphisms between character triples, $\sigma_{H_1}(\psi^{H_1})$ is mapped to $(\sigma_{H_1\cap J}(\psi))^{H_2} = (\psi')^{H_2}$. This implies

$$\sigma_{H_1}(\mathrm{Irr}(H_1 \mid \psi)) = \mathrm{Irr}(H_2 \mid \psi').$$

According to Definition 3.6 every character $\tau \in \mathrm{Irr}(H_1 \mid \psi)$ satisfies $\mathrm{bl}(\tau)^G = \mathrm{bl}(\sigma_{H_1}(\tau))^G$. This implies $\sigma_{H_1}(\mathrm{Irr}(B_1 \mid \psi)) = \mathrm{Irr}(B_2 \mid \psi')$. □

Lemma 3.10. *Assume Hypothesis 3.2. In addition assume that for $i = 1, 2$ there exists some defect group D_i of $\mathrm{bl}(\theta_i)$ with $\mathrm{C}_G(D_i) \leq H_i$. Suppose that for $i = 1, 2$ the projective representation $\mathcal{P}_i$ associated with θ_i is a linear representation affording a character $\widetilde{\theta}_i$ with*

(i) $\mathrm{Irr}(\widetilde{\theta}_{1,\mathrm{C}_G(N)}) = \mathrm{Irr}(\widetilde{\theta}_{2,\mathrm{C}_G(N)})$,
(ii) $\mathrm{bl}(\widetilde{\theta}_{2,J\cap H_2})^J = \mathrm{bl}(\widetilde{\theta}_{1,J\cap H_1})^J$ *for every $N \leq J \leq G$ with J/N cyclic.*

Then

$$(H_1, M_1, \theta_1) \sim_N (H_2, M_2, \theta_2)$$

via the isomorphism of character triples given by $\mathcal{P}_1$ and $\mathcal{P}_2$. (Note that according to Lemma 3.5 the characters $\widetilde{\theta}_{1,\mathrm{C}_G(N)}$ and $\widetilde{\theta}_{2,\mathrm{C}_G(N)}$ from (i) *and the blocks $\mathrm{bl}(\widetilde{\theta}_{2,J\cap H_2})^J$ and $\mathrm{bl}(\widetilde{\theta}_{1,J\cap H_1})^J$ are well-defined.)*

Proof. We want to apply Remark 3.7 and check the assumptions made there. The assumptions 3.7(i) and 3.7(ii) are satisfied by the assumptions of our lemma.

The projective representations $\mathcal{P}_1$ and $\mathcal{P}_2$ are linear and hence their factor sets coincide as required in 3.7(iii). Furthermore for $x \in \mathrm{C}_G(N)$ the matrices $\mathcal{P}_1(x)$ and $\mathcal{P}_2(x)$ are scalar matrices associated to the same $\zeta \in \mathbb{C}$ because of assumption (i) on $\widetilde{\theta}_1$ and $\widetilde{\theta}_2$. The isomorphism (ι, σ) of character triples given by $\mathcal{P}_1$ and $\mathcal{P}_2$, as defined in Theorem 3.3, is the following: if $N \leq J \leq G$, then

$$\sigma_J : \mathrm{Char}(J \cap H_1 \mid \theta_1) \to \mathrm{Char}(J \cap H_2 \mid \theta_2)$$

satisfies

$$\widetilde{\theta}_{1,J\cap H_1}\eta_{J\cap H_1} \mapsto \widetilde{\theta}_{2,J\cap H_2}\eta_{J\cap H_2}$$

for every $\eta \in \mathrm{Irr}(J)$ with $N \leq \ker(\eta)$. Then by hypothesis and Proposition 2.3 we have

$$\mathrm{bl}(\psi)^J = \mathrm{bl}(\sigma_{J\cap H_1}(\psi))^J \quad \text{for every } \psi \in \mathrm{Irr}(J \mid \theta_1).$$

This proves that condition 3.7(iv) is satisfied. □

The above statement helps to shorten the assumptions made in Remark 3.7 in the case where H_1/M_1 is cyclic. In this situation the existence of an N-block isomorphism of character triples requires mainly group-theoretic properties.

Proposition 3.11. *Let $N \lhd G$ with cyclic G/N, and let (H_1, M_1, θ_1) and (H_2, M_2, θ_2) be character triples with $H_1, H_2 \leq G$ and $\mathrm{Irr}(\theta_{1,Z(N)\cap H_1}) = \mathrm{Irr}(\theta_{2,Z(N)\cap H_2})$. Assume*

(i) *$G = NH_1 = NH_2$, $M_1 = N \cap H_1$ and $M_2 = N \cap H_2$, in particular H_1/M_1 and H_2/M_2 are then cyclic,*
(ii) *for $i = 1, 2$ some defect group D_i of $\mathrm{bl}(\theta_i)$ satisfies $\mathrm{C}_G(D_i) \leq H_i$.*

Then there is an N-block isomorphism $(\iota, \sigma) : (H_1, M_1, \theta_1) \to (H_2, M_2, \theta_2)$, where $\iota : H_1/M_1 \to H_2/M_2$ is the canonical epimorphism.

Proof. In this situation we know that θ_1 extends to some $\widetilde{\theta}_1 \in \mathrm{Irr}(H_1)$ according to [Is76, Corollary (11.22)]. We use Dade's ramification group introduced in [Da73] (see also Proposition 2.5). Recall that we denote by $G[c]$ Dade's ramification group of a block $c \in \mathrm{Bl}(N)$ (see 2.4). According to [KS15, Theorem C(a)(2)] there exists an extension $\widetilde{\psi}$ of θ_2 to $G[\mathrm{bl}(\theta_1)^G] \cap H_2$ such that $\mathrm{bl}(\widetilde{\theta}_{1,J\cap H1})^J = \mathrm{bl}(\widetilde{\psi}_{J\cap H_2})^J$ for every group J with $N \leq J \leq G[\mathrm{bl}(\theta_1)^G]$. The character $\widetilde{\psi}$ extends to some $\widetilde{\theta}_2 \in \mathrm{Irr}(H_2)$ since H_2/M_2 is cyclic.

Let $G_{p'}$ be the group with $N \leq G_{p'}$ such that $G/G_{p'}$ is isomorphic to a Sylow p-subgroup of G/N. For $J := N\mathrm{C}_{G_{p'}}(N)$ we have $\mathrm{bl}(\widetilde{\theta}_{1,H_1\cap J})^J = \mathrm{bl}(\widetilde{\theta}_{2,H_2\cap J})^J$ and we see that the same unique block of $\mathrm{C}_{G_{p'}}(N)$ is covered by $\mathrm{bl}(\widetilde{\theta}_{1,H_1\cap J})$ and $\mathrm{bl}(\widetilde{\theta}_{2,H_2\cap J})$. Since $\mathrm{Irr}(\theta_{1,Z(N)\cap H_1}) = \mathrm{Irr}(\theta_{2,Z(N)\cap H_2})$, this implies $\mathrm{Irr}(\widetilde{\theta}_{1,\mathrm{C}_{G_{p'}}(N)}) = \mathrm{Irr}(\widetilde{\theta}_{2,\mathrm{C}_{G_{p'}}(N)})$.

Since $G/G_{p'}$ is cyclic, there exists some $\eta \in \mathrm{Irr}(G)$ with $G_{p'} \leq \ker(\eta)$ such that $\widetilde{\theta}_2\eta_{H_2} \in \mathrm{Irr}(H_2 \mid \nu)$ where $\nu \in \mathrm{Irr}(\widetilde{\theta}_{1,\mathrm{C}_G(N)})$. Then straightforward arguments using [NT89, Corollary 5.1.12] prove that $\mathrm{bl}(\widetilde{\theta}_{2,H_2\cap J}\,\eta_{H_2\cap J}) = \mathrm{bl}(\widetilde{\theta}_{2,H_2\cap J})$ for every J with $N \leq J \leq G$.

The characters $\widetilde{\theta}_1$ and $\widetilde{\theta}_2\eta_{H_2}$ have all of the properties required in Lemma 3.10, and hence $(H_1, M_1, \theta_1) \sim_N (H_2, M_2, \theta_2)$. □

Notation 3.12. Let $Z \leq \mathrm{C}_G(N)$ with $Z \lhd G$ and $Z \cap N = 1$. Let $(\iota, \sigma) : (H_1, M_1, \theta_1) \to (H_2, M_2, \theta_2)$ be an N-central isomorphism of character triples (see Lemma 3.4). For $\overline{G} = G/Z$, $\overline{N} := NZ/Z \cong N$ and $\overline{M}_i := M_iZ/Z \cong M_i$, and the characters $\overline{\theta}_i \in \mathrm{Irr}(\overline{M}_i)$ corresponding to θ_i there is an isomorphism of character triples $(\overline{\iota}, \overline{\sigma}) : (\overline{H}_1, \overline{M}_1, \overline{\theta}_1) \to (\overline{H}_2, \overline{M}_2, \overline{\theta}_2)$ given by projective representations such that

- the isomorphism $\iota : H_1/M_1 \to H_2/M_2$ induces an isomorphism $\overline{\iota} : \overline{H}_1/\overline{M}_1 \to \overline{H}_2/\overline{M}_2$,
- for every $NZ \leq J \leq G$ and any character $\overline{\chi} \in \mathrm{Irr}(J/Z \mid \overline{\theta}_1)$ the character $\overline{\sigma}_{J/Z}(\overline{\chi})$ lifts to $\sigma_J(\chi)$, where $\chi \in \mathrm{Irr}(J \mid \theta_1)$ is the lift of $\overline{\chi}$.

We call $(\overline{\iota}, \overline{\sigma})$ *induced by* (ι, σ) and (ι, σ) a *lift* of $(\overline{\iota}, \overline{\sigma})$.

Recall that every block of a quotient of a given group H is contained (or dominated by) a unique block of H (see [Na98, p. 198]).

Proposition 3.13. *Assume Hypothesis 3.2 with $(H_1, M_1, \theta_1) \sim_N (H_2, M_2, \theta_2)$ via (ι, σ). Let $Z \leq H_1 \cap H_2$ with $Z \lhd G$ and $N \cap Z = 1$. Let $\overline{G} = G/Z$, $\overline{N} := NZ/Z \cong N$ and $\overline{M}_i := M_iZ/Z \cong M_i$. Let $\overline{\theta}_i \in \mathrm{Irr}(\overline{M}_i)$ be the character corresponding to θ_i.*

(a) *Assume that either Z is a p'-group or $Z \leq \mathrm{Z}(G)$ is a p-group. Then $(\overline{H}_1, \overline{M}_1, \overline{\theta}_1) \sim_{\overline{N}} (\overline{H}_2, \overline{M}_2, \overline{\theta}_2)$.*
(b) *Assume that $Z \leq \mathrm{Z}(G)$ is central. Then $(\overline{H}_1, \overline{M}_1, \overline{\theta}_1) \sim_{\overline{N}} (\overline{H}_2, \overline{M}_2, \overline{\theta}_2)$ via $(\overline{\iota}, \overline{\sigma})$, where $(\overline{\iota}, \overline{\sigma})$ is induced by (ι, σ).*

Proof. For the proof of (a) let $\mathcal{P}_1$ and $\mathcal{P}_2$ be projective representations associated with θ_1 and θ_2 that give the N-block isomorphism (ι, σ) between the character triples. Now consider the linear representation $\overline{\mathcal{X}}_i$ of $\overline{M}_i$ defined by $\overline{\mathcal{X}}_i(mZ) = \mathcal{P}_i(m)$ for $m \in M_i$ that affords $\overline{\theta}_i$. In addition by [NS14, Theorem 3.1] there exists a projective representation $\overline{\mathcal{P}}_1$ of $\overline{H}_1$ associated with $\overline{\theta}_1$ such that

$$\overline{\mathcal{P}}_1(mZ) = \overline{\mathcal{X}}_1(mZ) = \mathcal{P}_1(m) \quad \text{for every } m \in M_1.$$

The map $\mathcal{D}_1$ on H_1 defined by $\mathcal{D}_1(h) := \overline{\mathcal{P}}_1(hZ)$ for every $h \in H_1$ is a projective representation of H_1 with $\mathcal{D}_{1,M_1} = \mathcal{P}_{1,M_1}$. According to [NS14, Theorem 3.1(b)] there exists a map $\xi_1 : H_1/M_1 \to \mathbb{C}^\times$ such that

$$\overline{\mathcal{P}}_1(gZ) = \xi_1(gM_1)\mathcal{P}_1(g) \quad \text{for every } g \in H_1.$$

The map ξ_1 defines a map $\xi_2 : H_2/M_2 \to \mathbb{C}^\times$ by means of the canonical isomorphism $\iota : H_1/M_1 \to H_2/M_2$. Then $\xi_2\mathcal{P}_2$ is a projective representation of H_2 associated with θ_2 as well. Since the factor sets of $\mathcal{P}_1$ and $\mathcal{P}_2$ correspond via ι, the factor sets of $\xi_1\mathcal{P}_1$ and $\xi_2\mathcal{P}_2$ coincide as well. Straightforward calculations show that $\xi_1\mathcal{P}_1$ and $\xi_2\mathcal{P}_2$ give the same isomorphism (ι, σ) of character triples.

Now notice that $\mathrm{C}_{G/Z}(NZ/Z) = \mathrm{C}_G(N)/Z$ since $Z \lhd G$ and $N \cap Z = 1$. As $\mathcal{P}_1(x)$ and $\mathcal{P}_2(x)$ are associated with the same scalar for $x \in \mathrm{C}_G(N)$, the projective representations $\xi_1\mathcal{P}_1$ and $\xi_2\mathcal{P}_2$ have the same property. Since the factor sets of $\xi_1\mathcal{P}_1$ and $\xi_2\mathcal{P}_2$ coincide via ι, we see that $\xi_2\mathcal{P}_2$ uniquely determines a projective representation $\overline{\mathcal{P}}_2$ of H_2/Z associated with $\overline{\theta}_2$.

The isomorphism of character triples $(\overline{\iota}, \overline{\sigma}) : (\overline{H}_1, \overline{M}_1, \overline{\theta}_1) \to (\overline{H}_2, \overline{M}_2, \overline{\theta}_2)$ given by $\overline{\mathcal{P}}_1$ and $\overline{\mathcal{P}}_2$ is the one induced by (ι, σ) in the sense of 3.12.

By the above arguments it is clear that $(\overline{\iota}, \overline{\sigma})$ is an $\overline{N}$-central isomorphism of character triples. Note that some defect group D_i of $\mathrm{bl}(\theta_i)$ satisfies $\mathrm{C}_G(D_i) \leq H_i$, since $(H_1, M_1, \theta_1) \sim_N (H_2, M_2, \theta_2)$. Moreover $\overline{D_i} := D_iZ/Z$ is a defect group of $\mathrm{bl}(\overline{\theta}_i)$ according to [NT89, Theorems 5.8.8 and 5.8.10]. Now $\mathrm{C}_{G/Z}(D_iZ/Z) = \mathrm{C}_G(D_i)Z/Z$ since $N \cap Z = 1$ and $Z \lhd G$. This implies $\mathrm{C}_{\overline{G}}(\overline{D}_i) \leq \overline{H}_i$.

Let $NZ \leq J \leq G$ and $\overline{\psi} \in \mathrm{Irr}(\overline{J} \cap \overline{H}_1 \mid \overline{\theta}_1)$ with $\overline{J} := J/Z$. By its definition the character $\overline{\sigma}_{\overline{J}\cap\overline{H}_1}(\overline{\psi})$ lifts to $\sigma_{J\cap H_1}(\psi)$ whenever $\psi \in \mathrm{Irr}(J \cap H_1 \mid \theta_1)$ is the lift of $\overline{\psi}$. By assumption $\mathrm{bl}(\psi)^J = \mathrm{bl}(\sigma_{J\cap H_1}(\psi))^J$. Let $B := \mathrm{bl}(\psi)^J$, $b_1 := \mathrm{bl}(\psi) \in \mathrm{Bl}(J \cap H_1)$ and $b_2 := \mathrm{bl}(\sigma_{J\cap H_1}(\psi)) \in \mathrm{Bl}(J \cap H_2)$. Moreover let $\overline{B} \in \mathrm{Bl}(\overline{J})$ be the block contained in B in the sense of [Na98, p. 198], $\overline{b}_1 \in \mathrm{Bl}(\overline{J} \cap \overline{H}_1)$ the one contained in b_1 and $\overline{b}_2 \in \mathrm{Bl}(\overline{J} \cap \overline{H}_2)$ the one contained in b_2. (The existence and uniqueness of $\overline{B}$, $\overline{b}_1$ and $\overline{b}_2$ follows from the fact that $p \nmid |Z|$ or $Z \leq \mathrm{Z}(J)$ according to [NT89, Theorems 5.8.8 and 5.8.11].) The blocks $\overline{b}_1^{\overline{J}}$ and $\overline{b}_2^{\overline{J}}$ are defined according to [NT89, Lemma 5.5.14 and Theorem 5.5.16(ii)]. Then according to [NS14, Proposition 2.4(b) and (c)] we have

$$\overline{b}_1^{\overline{J}} = \overline{B} = \overline{b}_2^{\overline{J}}.$$

Since $\overline{b}_1 = \mathrm{bl}(\overline{\psi})$ and $\overline{b}_2 = \mathrm{bl}(\sigma_{J\cap H_1}(\overline{\psi}))$, this implies $(\overline{H}_1, \overline{M}_1, \overline{\theta}_1) \sim_{\overline{N}} (\overline{H}_2, \overline{M}_2, \overline{\theta}_2)$ in the situation of (a). In order to prove (b), one applies part (a) with $\mathrm{O}_p(Z)$ and then again with $Z/\mathrm{O}_p(Z)$ (in place of Z). □

4. N-block isomorphic character triples via projective representations

In this section we study N-block isomorphisms of character triples in more detail and derive a criterion in terms of projective representations. This is done in the following steps: Before considering the general case one analyses when two character triples (H_1, M_1, θ_1) and (H_2, M_2, θ_2) are N-block isomorphic character triples under the assumption that θ_1 and θ_2 extend. The general case is later studied using the theory of projective representations. Therefore we gather some essentially well-known statements that allow us to relate the case with an extending character to the general case. This can be seen as an adaptation of [NS14, Theorem 4.1] to this new equivalence relation of character triples.

If we have an epimorphism $\epsilon : \widehat{G} \to G$ with kernel Z, a *Z-section* of ϵ is any map $\mathrm{rep} : G \to \widehat{G}$ with $\epsilon \circ \mathrm{rep} = \mathrm{Id}_G$ and $\mathrm{rep}(1_G) = 1_{\widehat{G}}$.

Theorem 4.1. *Let $N \lhd G$ and (H_1, M_1, θ_1) be a character triple such that $H_1N = G$ and $H_1 \cap N = M_1$. Let $\mathcal{P}_1$ be a projective representation of H_1 associated with θ_1. Then there is a group $\widehat{G}$, a surjective homomorphism $\epsilon : \widehat{G} \to G$ with finite cyclic central kernel Z and a Z-section* $\mathrm{rep} : G \to \widehat{G}$ *of ϵ satisfying the following properties:*

(a) *$\widehat{N} = N_0 \times Z$ where $\widehat{N} := \epsilon^{-1}(N)$, N_0 is isomorphic to N via $\epsilon_{N_0} : N_0 \to N$ and $N_0 \lhd \widehat{G}$. Also the action of $\widehat{G}$ on N_0 coincides with the action of G on N via ϵ.*

(b) *Let $M_{1,0} := \epsilon^{-1}(M_1) \cap N_0$. The character $\theta_{1,0} := \theta_1 \circ \epsilon_{M_{1,0}} \in \mathrm{Irr}(M_{1,0})$ extends to $\widehat{H}_1 := \epsilon^{-1}(H_1)$. There exists a linear representation $\mathcal{D}_1$ of $\widehat{H}_1$ with*

$$\mathcal{D}_1(\mathrm{rep}(g)) = \mathcal{P}_1(g) \quad \textit{for every } g \in H_1,$$

and this representation affords an extension $\widetilde{\theta}_{1,0} \in \mathrm{Irr}(\widehat{H}_1)$ of $\theta_{1,0}$. The Z-section rep *satisfies*

$$\mathrm{rep}(n) \in N_0 \quad \textit{and} \quad \mathrm{rep}(ng) = \mathrm{rep}(n)\,\mathrm{rep}(g) \quad \textit{for every } n \in N \textit{ and } g \in G.$$

(c) *The unique irreducible constituent ν of $(\widetilde{\theta}_{1,0})_Z$ is faithful.*

(d) *If $N \leq J \leq G$ with $\mathrm{C}_G(J) \leq H_1$ and $\widehat{J} := \epsilon^{-1}(J)$, then $\epsilon(\mathrm{C}_{\widehat{G}}(\widehat{J})) = \mathrm{C}_G(J)$.*

Assume there exists some projective representation $\mathcal{P}_2$ of H_2 associated with θ_2 such that the isomorphism

$$(\iota, \sigma) : (H_1, M_1, \theta_1) \to (H_2, M_2, \theta_2)$$

given by $\mathcal{P}_1$ and $\mathcal{P}_2$ is N-central. Then:

(e) *$\widehat{M}_2 = M_{2,0} \times Z$, where $\widehat{M}_2 := \epsilon^{-1}(M_2)$ and $M_{2,0} := N_0 \cap \epsilon^{-1}(M_2)$.*

(f) *The character $\theta_{2,0} := \theta_2 \circ \epsilon_{M_{2,0}} \in \mathrm{Irr}(M_{2,0})$ extends to $\widehat{H}_2 := \epsilon^{-1}(H_2)$. There exists a linear representation $\mathcal{D}_2$ of $\widehat{H}_2$ with*

$$\mathcal{D}_2(\mathrm{rep}(g)) = \mathcal{P}_2(g) \quad \textit{for every } g \in H_2,$$

and this representation affords an extension $\widetilde{\theta}_{2,0} \in \mathrm{Irr}(\widehat{H}_2)$ of $\theta_{2,0}$.

(g) $\{\nu\} = \mathrm{Irr}((\widetilde{\theta}_{2,0})_Z) = \mathrm{Irr}((\widetilde{\theta}_{1,0})_Z)$.

(h) *The isomorphism of character triples*

$$(\widehat{\iota}, \widehat{\sigma}) : (\widehat{H}_1, \widehat{M}_{1,0}, \theta_{1,0}) \to (\widehat{H}_2, \widehat{M}_{2,0}, \theta_{2,0})$$

given by $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *is* N_0*-central and is the lift of* (ι, σ) *in the sense of* 3.12.

(i) *If* $(\widehat{H}_1, M_{1,0}, \theta_{1,0}) \sim_{N_0} (\widehat{H}_2, M_{2,0}, \theta_{2,0})$ *via* $(\widehat{\iota}, \widehat{\sigma})$, *then* $(H_1, M_1, \theta_1) \sim_N (H_2, M_2, \theta_2)$ *via* (ι, σ).

Proof. The construction of $\widehat{G}$ can be found in [Na98, proof of Theorem (8.28)] and in [Is76, proof of Lemma (11.28)]. We fix Z to be the finite subgroup of $\mathbb{C}^\times$ that is generated by all values of α, where α denotes the factor set of $\mathcal{P}_1$. The elements of $\widehat{G}$ are defined to be $\{(g, z) \mid g \in G,\ z \in Z\}$ and multiplication is given by

$$(g_1, z_1)(g_2, z_2) = (g_1 g_2, z_1 z_2 \alpha(g_1, g_2)).$$

Let $\epsilon : \widehat{G} \to G$ be the epimorphism given by $(g, z) \mapsto g$, with kernel $1 \times Z \leq \mathrm{Z}(\widehat{G})$, which we identify with Z. We define a Z-section rep $: G \to \widehat{G}$ by $g \mapsto (g, 1)$. Then $N_0 := \{(n, 1) \mid n \in N\} \lhd \widehat{G}$ and N_0 is isomorphic to N via ϵ_{N_0}.

Then $M_{1,0} := \{(m, 1) \mid m \in M_1\}$ is isomorphic to M_1. Let $\theta_{1,0} := \theta_1 \circ \epsilon_{M_{1,0}} \in \mathrm{Irr}(M_{1,0})$. The map $\mathcal{D}_1$ defined on $\widehat{H}_1 := \epsilon^{-1}(H_1)$ by

$$\mathcal{D}_1(h, z) = z\mathcal{P}_1(h) \quad \text{ for every } z \in Z \text{ and } h \in H_1,$$

is an irreducible linear representation of $\widehat{H}_1$. Let $\widetilde{\theta}_{1,0} \in \mathrm{Irr}(\widehat{H}_1)$ be the character afforded by $\mathcal{D}_1$. With these definitions, (a)–(c) are satisfied.

In the following we denote by $\widehat{U}$ the group $\epsilon^{-1}(U) \leq \widehat{G}$ for $U \leq G$. Let $N \leq J \leq G$ and $\widehat{J} := \epsilon^{-1}(J)$. We assume that $\mathrm{C}_G(J) \leq H_1$. If $c \in \mathrm{C}_G(J) \leq \mathrm{C}_{H_1}(M_1)$ the matrix $\mathcal{P}_1(c)$ is scalar by Schur's Lemma. Using the fact that $\mathcal{P}_1(c) = \mathcal{P}_1(ncn^{-1})$ for $n \in J$ we now see that $\epsilon(\mathrm{C}_{\widehat{G}}(\widehat{J})) = \mathrm{C}_G(J)$, as claimed in (d).

By the definition of character triple isomorphisms given by projective representations the factor sets of $\mathcal{P}_1$ and $\mathcal{P}_2$ coincide via the canonical isomorphism $\iota : H_1/M_1 \to H_2/M_2$. In particular the values of the factor set α_2 of $\mathcal{P}_2$ are contained in Z. Then the map $\mathcal{D}_2$ defined on $\widehat{H}_2$ by

$$\mathcal{D}_2(h, z) = z\mathcal{P}_1(h) \quad \text{ for every } z \in Z \text{ and } h \in H_2$$

is an irreducible linear representation of $\widehat{H}_2$. If $\widetilde{\theta}_{2,0} \in \mathrm{Irr}(\widehat{H}_2 \mid \theta_{2,0})$ is the character afforded by $\mathcal{D}_2$, then it is an extension of $\theta_{2,0}$. Also by definition $\{\nu\} = \mathrm{Irr}((\widetilde{\theta}_{2,0})_Z) = \mathrm{Irr}((\widetilde{\theta}_{1,0})_Z)$. We see that $\epsilon^{-1}(M_2) = M_{2,0} \times Z$.

According to (d) we have $\mathrm{C}_{\widehat{G}}(N_0) = \mathrm{C}_{\widehat{G}}(\widehat{N}) = \widehat{\mathrm{C}_G(N)} \leq \widehat{H}_1 \cap \widehat{H}_2$, as (H_1, M_1, θ_1) and (H_2, M_2, θ_2) are N-central isomorphic character triples, and hence $\mathrm{C}_G(N) \leq H_1 \cap H_2$. Since $\mathcal{P}_1$ and $\mathcal{P}_2$ define an N-central isomorphism of character triples, it is clear that $\mathcal{D}_1(x, z)$ and $\mathcal{D}_2(x, z)$ for $(x, z) \in \mathrm{C}_{\widehat{G}}(N_0)$ are associated with the same scalar. This proves that $(\widehat{H}_1, \widehat{M}_{1,0}, \theta_{1,0})$ and $(\widehat{H}_2, \widehat{M}_{2,0}, \theta_{2,0})$ are N_0-central isomorphic character triples via $(\widehat{\iota}, \widehat{\sigma})$.

Finally we consider part (h) and assume that $(\widehat{H}_1, M_{1,0}, \theta_{1,0}) \sim_{N_0} (\widehat{H}_2, M_{2,0}, \theta_{2,0})$. Then $(H_1, M_1, \theta_1) \sim_N (H_2, M_2, \theta_2)$ according to Proposition 3.13(b). □

When checking if two given character triples are N-block isomorphic, usually the most complicated task is to verify the equality of induced blocks coming from corresponding characters. We facilitate that by using Dade's ramification group, mentioned in 2.4.

Lemma 4.2. *Let $N \lhd G$ and $H_1, H_2 \leq G$ with $NH_1 = NH_2 = G$. For $i = 1, 2$ write $M_i := N \cap H_i$ and let $\widetilde{\theta}_i \in \mathrm{Irr}(H_i)$ with $\theta_i := \widetilde{\theta}_{i,M_i} \in \mathrm{Irr}(M_i)$. Suppose that $\mathrm{bl}(\theta_1)$ and $\mathrm{bl}(\theta_2)$ have defect groups D_1 and D_2 with $\mathrm{C}_G(D_1) \leq H_1$ and $\mathrm{C}_G(D_2) \leq H_2$. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be linear representations of H_1 and H_2 affording $\widetilde{\theta}_1$ and $\widetilde{\theta}_2$, respectively.*

(a) *For $x \in G$ and $J := \langle N, x\rangle$, $\mathcal{P}_1((\mathfrak{Cl}_J(x) \cap H_1)^+)$ is a scalar matrix associated to an element of $\mathcal{O}$. So there is some $\xi \in \mathbb{F}$ associated with $\mathcal{P}_1((\mathfrak{Cl}_J(x) \cap H_1)^+)^*$.*
(b) *If $x \in G \setminus G[\mathrm{bl}(\theta_i)^N]$ and $J := \langle N, x\rangle$, then $\mathcal{P}_i((\mathfrak{Cl}_J(x) \cap H_i)^+)^*$ is the zero matrix.*
(c) *If $x \in G[\mathrm{bl}(\theta_1)^N]$ and $\mathrm{bl}(\theta_1)^N = \mathrm{bl}(\theta_2)^N$, then for $J := \langle N, x\rangle$ the following two statements are equivalent:*
   (i) $\mathrm{bl}(\widetilde{\theta}_{1,J\cap H_1})^J = \mathrm{bl}(\widetilde{\theta}_{2,J\cap H_2})^J$.
   (ii) *The matrices $\mathcal{P}_1((\mathfrak{Cl}_J(y) \cap H_1)^+)^*$ and $\mathcal{P}_2((\mathfrak{Cl}_J(y) \cap H_2)^+)^*$ are associated with the same $\xi \in \mathbb{F}$ for every $y \in xN$.*

Proof. Note that $\mathfrak{Cl}_J(x) \cap H_1$ is either empty or contained in $x'M_1$ for some $x' \in H_1$. In the first case we have $\mathcal{P}_1((\mathfrak{Cl}_J(x) \cap H_1)^+) = 0$. Otherwise $\mathfrak{Cl}_J(x) \cap H_1$ is a disjoint union of J_1-conjugacy classes for $J_1 := J \cap H_1$. Accordingly $\mathcal{P}_1((\mathfrak{Cl}_J(x) \cap H_1)^+)$ is a scalar matrix associated to an algebraic integer by [Is76, Section 3]. This proves (a).

Note that $\widetilde{\theta}_{1,J_1}$ is afforded by $\mathcal{P}_{1,J_1}$. Let $b := \mathrm{bl}(\theta_1)^N$. By definition the element $\lambda_{\widetilde{\theta}_{1,J_1}}((\mathfrak{Cl}_J(x) \cap H_1)^+)$ is the scalar associated with $\mathcal{P}_1((\mathfrak{Cl}_J(x) \cap H_1)^+)^*$. If $x \in G \setminus G[b]$, then $\lambda_{\widetilde{\theta}_{1,J_1}}((\mathfrak{Cl}_J(x) \cap H_1)^+) = 0$ by the definition of $G[b]$ (see Proposition 2.5(b)).

Assume now $x \in G[b]$ and $b = \mathrm{bl}(\theta_2)^N$. Since $\mathcal{P}_{1,J_1}$ affords $\widetilde{\theta}_{1,J_1}$ and $\mathcal{P}_{2,J_2}$ affords $\widetilde{\theta}_{2,J_2}$ for $J_2 := J \cap H_2$, the equality $\mathrm{bl}(\widetilde{\theta}_{1,J\cap H_1})^J = \mathrm{bl}(\widetilde{\theta}_{2,J\cap H_2})^J$ implies

$$\lambda_{\widetilde{\theta}_{1,J_1}}((\mathfrak{Cl}_J(y) \cap H_1)^+) = \lambda_{\widetilde{\theta}_{2,J_2}}((\mathfrak{Cl}_J(y) \cap H_2)^+) \quad \text{for every } y \in J.$$

This shows that (i) implies (ii) in part (c).

On the other hand, following [KS15, Theorem B] together with [Na98, Theorem (9.2)] there exists $\zeta \in \mathrm{Irr}(J_2)$ with $M_2 \leq \ker(\zeta)$ such that $\mathrm{bl}(\widetilde{\theta}_{1,J\cap H_1})^J = \mathrm{bl}(\zeta\widetilde{\theta}_{2,J\cap H_2})^J$. Note that ζ is a linear character since J_2/M_2 is cyclic by definition. This implies

$$\lambda_{\widetilde{\theta}_{1,J_1}}((\mathfrak{Cl}_J(y) \cap H_1)^+) = \zeta(y_2)^*\lambda_{\widetilde{\theta}_{2,J_2}}((\mathfrak{Cl}_J(y) \cap H_2)^+) \quad \text{for every } y \in J,$$

where $y_2 \in yN \cap H_2$. Now by straightforward calculations we see that for some $y \in xN$ the matrices $\mathcal{P}_1((\mathfrak{Cl}_J(y) \cap H_1)^+)^*$ and $\mathcal{P}_2((\mathfrak{Cl}_J(y) \cap H_2)^+)^*$ are nonzero. Hence the values of $\lambda_{\widetilde{\theta}_{1,J_1}}((\mathfrak{Cl}_J(y) \cap H_1)^+)$ and $\lambda_{\widetilde{\theta}_{2,J_2}}((\mathfrak{Cl}_J(y) \cap H_2)^+)$ are nonzero. By the assumption in (ii) this implies $\zeta(y_2)^* = 1$, and hence all p-regular elements of J_2 are contained in the kernel of ζ since J_2/M_2 is cyclic. Accordingly $\mathrm{bl}(\zeta\widetilde{\theta}_{2,J\cap H_2}) = \mathrm{bl}(\widetilde{\theta}_{2,J\cap H_2})$ by [NT89, 5.1.12]. This completes the proof of (c). □

The following statement gives a general criterion for the existence of an N-block isomorphism of character triples. It takes into account the projective representations associated with the characters.

Theorem 4.3. *Let $N \lhd G$, and let (H_1, M_1, θ_1) and (H_2, M_2, θ_2) be character triples with $NH_1 = NH_2 = G$, $M_1 = N \cap H_1$ and $M_2 = N \cap H_2$. Suppose there are defect groups D_1 and D_2 of $\mathrm{bl}(\theta_1)$ and $\mathrm{bl}(\theta_2)$ with $\mathrm{C}_G(D_1) \leq H_1$ and $\mathrm{C}_G(D_2) \leq H_2$. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be projective representations of H_1 and H_2 associated with θ_1 and θ_2, respectively, in the sense of Definition 3.1. Then the following two statements are equivalent:*

(i) *The projective representations $\mathcal{P}_1$ and $\mathcal{P}_2$ of H_1 and H_2 satisfy:*

- (i.a) *The factor sets of $\mathcal{P}_1$ and $\mathcal{P}_2$ coincide via the canonical isomorphism ι : $H_1/M_1 \to H_2/M_2$.*
- (i.b) *$\mathcal{P}_1(x)$ and $\mathcal{P}_2(x)$ are scalar matrices associated with the same scalar for every $x \in \mathrm{C}_G(N)$.*
- (i.c) *$(\mathcal{P}_1((\mathfrak{Cl}_J(x) \cap H_1)^+))^*$ and $(\mathcal{P}_2((\mathfrak{Cl}_J(x) \cap H_2)^+))^*$ are scalar matrices associated with the same scalar for every $x \in G[\mathrm{bl}(\theta_1)^N] \leq N\mathrm{C}_G(D_1) \cap N\mathrm{C}_G(D_2)$, where $J := \langle N, x\rangle$.*

(ii) *$(H_1, M_1, \theta_1) \sim_N (H_2, M_2, \theta_2)$ via the isomorphism of character triples given by $\mathcal{P}_1$ and $\mathcal{P}_2$.*

(Note that $\mathrm{C}_G(D_1) \leq H_1$ and $\mathrm{C}_G(D_2) \leq H_2$ imply $\mathrm{C}_G(N) \leq H_1 \cap H_2$ by Lemma 3.5(a), and hence $\mathcal{P}_1(x)$ and $\mathcal{P}_2(x)$ in (i.b) *are well-defined. Moreover, by Lemma 4.2(a) the elements of $\mathbb{F}$ occurring in* (i.c) *are well-defined.)*

Proof. We first prove that (i) implies (ii). By Lemma 3.4 the projective representations determine an N-central isomorphism $(\iota, \sigma) : (H_1, M_1, \theta_1) \to (H_2, M_2, \theta_2)$ of character triples. By assumption, θ_1 and θ_2 have defect groups D_1 and D_2 with $\mathrm{C}_G(D_1) \leq H_1$ and $\mathrm{C}_G(D_2) \leq H_2$. By Lemma 3.5(b) this implies $\mathrm{bl}(\psi_i)^G$ is defined for all $\psi_i \in \mathrm{Irr}(H_i \mid \theta_i)$.

Following Theorem 4.1, $\mathcal{P}_1$ with its factor set α_1 determines a group $\widehat{G} = \{(g, z) \mid g \in G,\ z \in Z\}$, a surjective homomorphism $\epsilon : \widehat{G} \to G$ and a cyclic group $Z \lhd \widehat{G}$. By 4.1(a), N is naturally isomorphic to $N_0 := \{(n, 1) \mid n \in N\}$ via ϵ_{N_0}. Furthermore by 4.1(a) the action of $\widehat{G}$ on $\widehat{N} := \epsilon^{-1}(N)$ satisfies

$$(n, z)^{(g,z')} = (n^g, z) \qquad \text{for every } g \in G,\ z, z' \in Z \text{ and } n \in N. \tag{4.1}$$

Let $M_{1,0}$ and $\theta_{1,0} \in \mathrm{Irr}(M_{1,0})$ and $\widehat{H}_1$ be defined as in 4.1(b). Moreover let $\widetilde{\theta}_{1,0} \in \mathrm{Irr}(\widehat{H}_1)$ be the extension of $\theta_{1,0}$ afforded by a representation $\mathcal{D}_1$ defined as in 4.1(b).

The assumptions (i.a) and (i.b) on $\mathcal{P}_1$ and $\mathcal{P}_2$ imply that (H_1, M_1, θ_1) and (H_2, M_2, θ_2) are N-central isomorphic character triples via the isomorphism given by $\mathcal{P}_1$ and $\mathcal{P}_2$ (see Theorem 3.3 and Lemma 3.4).

Let $\widehat{H}_2 \leq \widehat{G}$, $M_{2,0}$, $\theta_{2,0} \in \mathrm{Irr}(M_{2,0})$, $\mathcal{D}_2$ and $\widetilde{\theta}_{2,0}$ be defined as in Theorem 4.1(f). Then the isomorphism $(\widehat{\iota}, \widehat{\sigma})$ of character triples given by $\mathcal{D}_1$ and $\mathcal{D}_2$ is N_0-central and is the lift of (ι, σ) in the sense of 3.12 (see Theorem 4.1(h)).

By Theorem 4.1(i) it is sufficient to prove that $(\widehat{\iota}, \widehat{\sigma})$ is an N_0-block isomorphism of character triples, as this implies $(H_1, M_1, \theta_1) \sim_N (H_2, M_2, \theta_2)$ via (ι, σ). We prove in the following that $(\widehat{\iota}, \widehat{\sigma})$ is an N_0-block isomorphism of character triples by checking the conditions given in Lemma 3.10:

Straightforward arguments prove that Hypothesis 3.2 is satisfied. Recall that D_1 is a defect group of $\mathrm{bl}(\theta_1)$ with $\mathrm{C}_G(D_1) \leq H_1$. Then $D_{1,0} := \{(d,1) \mid d \in D_1\}$ is a defect group of $\mathrm{bl}(\theta_{1,0})$ and by Theorem 4.1(a) it satisfies $\mathrm{C}_{\widehat{G}}(D_{1,0}) \leq \widehat{H}_1$. Analogously for a defect group D_2 of $\mathrm{bl}(\theta_2)$ with $\mathrm{C}_G(D_2) \leq H_2$, the group $D_{2,0} := \{(d,1) \mid d \in D_2\}$ is a defect group of $\mathrm{bl}(\theta_{2,0})$ and $\mathrm{C}_{\widehat{G}}(D_{2,0}) \leq \widehat{H}_2$. Hence the group-theoretic assumptions of Lemma 3.10 are satisfied.

Assume that there exist projective representations $\mathcal{P}_1$ and $\mathcal{P}_2$ of H_1 and H_2 associated with θ_1 and θ_2 respectively, with factor sets α_1 and α_2 such that

$$\overline{\alpha}_1(x,y) = \overline{\alpha}_2(\iota(x),\iota(y)) \quad \text{for all } x,y \in H_1/M_1.$$

Now $\mathrm{C}_{\widehat{G}}(N_0)/Z = \mathrm{C}_G(N)$ according to (4.1) (see also Theorem 4.1(a)). Hence for $x \in \mathrm{C}_G(N)$, $\mathcal{D}_1(x)$ and $\mathcal{D}_2(x)$ are associated with the same scalar by assumption (i.b) and the construction of $\mathcal{D}_1$ and $\mathcal{D}_2$. This implies $\mathrm{Irr}((\widetilde{\theta}_{1,0})_{\mathrm{C}_{\widehat{G}}(N_0)}) = \mathrm{Irr}((\widetilde{\theta}_{2,0})_{\mathrm{C}_{\widehat{G}}(N_0)})$, assumption (i) of Lemma 3.10.

In the next step we verify assumption (ii) of Lemma 3.10, i.e., $\mathrm{bl}((\widetilde{\theta}_{2,0})_{J\cap\widehat{H}_2})^{J_0} = \mathrm{bl}((\widetilde{\theta}_{1,0})_{J_0\cap\widehat{H}_1})^{J_0}$ for every $N_0 \leq J_0 \leq \widehat{G}$ with cyclic J_0/N_0. According to Lemma 4.2(c) it is sufficient to check that $\mathcal{D}_1((\mathfrak{Cl}_{J_0}(x_0) \cap \widehat{H}_1)^+)^*$ and $\mathcal{D}_2((\mathfrak{Cl}_{J_0}(x_0) \cap \widehat{H}_2)^+)^*$ are associated with the same scalar in $\mathbb{F}$ for every $x_0 \in \widehat{G}[\mathrm{bl}(\theta_{1,0})^{N_0}]$, where $J_0 = \langle N_0, x_0\rangle$.

Note that by (i.c) for elements of N we have $\mathrm{bl}(\theta_1)^N = \mathrm{bl}(\theta_2)^N$. Proposition 2.5(a) together with [Na98, Lemma (4.13)] implies $G[\mathrm{bl}(\theta_1)^N] \leq N\mathrm{C}_G(D_1) \cap N\mathrm{C}_G(D_2)$. Let $x \in G[\mathrm{bl}(\theta_1)^N]$ and $J := \langle N, x\rangle$. Then $\mathfrak{Cl}_J(x)$ coincides with the N-orbit containing x, and hence $\mathfrak{Cl}_J(x) = \{nx \mid n \in \mathcal{L}_x(N)\}$, where $\mathcal{L}_x : N \to N$ is the map given by $n \mapsto n^{-1}xnx^{-1}$. Let $y := (x,1)$, $J_0 := \langle N_0, y\rangle$ and define $\mathcal{L}_y : N_0 \to N_0$ by $n \mapsto n^{-1}yny^{-1}$. Then the set $\mathfrak{Cl}_{J_0}(y)$ coincides with $\mathfrak{Cl}_{J_0}(y) = \{ly \mid l \in \mathcal{L}_y(N_0)\}$, since the action of y on N_0 coincides with the one of x on N according to 4.1(a).

Assume first that $\mathfrak{Cl}_J(x) \cap H_1 \neq \emptyset$, so there exists some $x_0 \in H_1 \cap xN \cap \mathfrak{Cl}_J(x)$. Straightforward computations show that $y_0 := (x_0, 1) \in \mathfrak{Cl}_{J_0}(y)$. The definition of $\mathcal{D}_1$ implies

$$\begin{aligned}\mathcal{D}_1((\mathfrak{Cl}_{J_0}(y) \cap \widehat{H}_1)^+) &= \sum_{\substack{l \in \mathcal{L}_y(N_0)\\ ly \in \widehat{H}_1}} \mathcal{D}_1(ly) = \sum_{\substack{l \in \mathcal{L}_y(N_0)\\ ly \in \widehat{H}_1}} \mathcal{D}_1(lyy_0^{-1})\mathcal{D}_1(y_0)\\ &= \sum_{\substack{l \in \mathcal{L}_x(N)\\ lx \in H_1}} \mathcal{P}_1(lxx_0^{-1})\mathcal{P}_1(x_0) = \sum_{\substack{k \in \mathfrak{Cl}_J(x_0)\\ k \in H_1}} \mathcal{P}_1(k)\\ &= \mathcal{P}_1((\mathfrak{Cl}_J(x) \cap H_1)^+).\end{aligned}$$

If $\mathfrak{Cl}_J(x) \cap H_1 = \emptyset$ then for $y = (x,1)$ we also have $\mathcal{D}_1((\mathfrak{Cl}_{J_0}(y) \cap \widehat{H}_1)^+) = 0 = \mathcal{P}_1((\mathfrak{Cl}_J(x) \cap \widehat{H}_1)^+)$. Analogously we see

$$\mathcal{D}_2((\mathfrak{Cl}_{J_0}(y) \cap \widehat{H}_2)^+) = \mathcal{P}_2((\mathfrak{Cl}_J(x) \cap H_2)^+).$$

Let $y' := (x,z)$. Moreover $\mathfrak{Cl}_{\langle N_0, y'\rangle}(y') = \{k(1,z) \mid k \in \mathfrak{Cl}_{\langle N_0,(x,1)\rangle}(x,1)\}$. This implies the equalities $\mathcal{D}_1((\mathfrak{Cl}_{J_0}(y') \cap \widehat{H}_2)^+) = z\mathcal{P}_1((\mathfrak{Cl}_J(x) \cap H_2)^+)$ and $\mathcal{D}_2((\mathfrak{Cl}_{J_0}(y') \cap \widehat{H}_2)^+) =$

$z\mathcal{P}_2((\mathfrak{Cl}_J(x) \cap H_2)^+)$. Together with assumption (i.c) and Lemma 4.2(b) this implies that the scalars associated with $\mathcal{D}_1((\mathfrak{Cl}_{J_0}(y') \cap \widehat{H}_1)^+)^*$ and $\mathcal{D}_2((\mathfrak{Cl}_{J_0}(y') \cap \widehat{H}_2)^+)^*$ coincide whenever $y' \in N_0 \mathrm{C}_{\widehat{G}}(D_{1,0}) \cap N_0 \mathrm{C}_{\widehat{G}}(D_{2,0})$ and $J_0 := \langle N_0, y' \rangle$. Note that $\widehat{G}[\mathrm{bl}(\theta_{1,0})^{N_0}] \leq N_0 \mathrm{C}_{\widehat{G}}(D_{1,0})$ according to Proposition 2.5(a). Hence by Lemma 4.2(c) we have $\mathrm{bl}((\widetilde{\theta}_{1,0})_{J \cap \widehat{H}_1})^J = \mathrm{bl}((\widetilde{\theta}_{2,0})_{J \cap \widehat{H}_2})^J$ for every $N_0 \leq J \leq \widehat{G}$ with J/N_0 cyclic. This shows by Lemma 3.10 that

$$(\widehat{H}_1, M_{1,0}, \theta_{1,0}) \sim_{N_0} (\widehat{H}_2, M_{2,0}, \theta_{2,0}).$$

We now prove that (ii) implies (i). Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be projective representations of H_1 and H_2 associated with θ_1 and θ_2 that give the N-block isomorphism $(\iota, \sigma) : (H_1, M_1, \theta_1) \to (H_2, M_2, \theta_2)$ of character triples.

According to Lemma 3.4 the projective representations $\mathcal{P}_1$ and $\mathcal{P}_2$ have the properties described in (i.a) and (i.b). It remains to verify the property described in (i.c) for elements $x \in G$ and $J := \langle N, x \rangle$. In addition let $\widetilde{\theta}_1 \in \mathrm{Irr}(J \cap H_1 \mid \theta_1)$ be an extension of θ_1. Then $\widetilde{\theta}_2 := \sigma_{J \cap H_1}(\widetilde{\theta}_1) \in \mathrm{Irr}(J \cap H_2)$ is an extension of θ_2. Let $\mathcal{Q}$ be the projective representation of J/N such that $\mathcal{Q}_{J \cap H_1} \otimes \mathcal{P}_{1, J \cap H_1}$ affords $\widetilde{\theta}_1$. Then $\widetilde{\theta}_2$ is afforded by $\mathcal{Q}_{J \cap H_2} \otimes \mathcal{P}_{2, J \cap H_2}$. Note that since J/N is cyclic, $\mathcal{Q}$ is a one-dimensional projective representation with $\mathcal{Q}(x)^* \neq 0$. Now $\mathrm{bl}(\widetilde{\theta}_1)^J = \mathrm{bl}(\widetilde{\theta}_2)^J$ implies according to Lemma 4.2(c) that the matrices $(\mathcal{Q}_{J \cap H_1} \otimes \mathcal{P}_1((\mathfrak{Cl}_J(x) \cap H_1)^+))^* = \mathcal{Q}(x)^* \mathcal{P}_1((\mathfrak{Cl}_J(x) \cap H_1)^+)^*$ and $\mathcal{Q}(x)^* \mathcal{P}_2((\mathfrak{Cl}_J(x) \cap H_2)^+)^*$ are scalar matrices associated with the same scalar in $\mathbb{F}$. This implies that $\mathcal{P}_1$ and $\mathcal{P}_2$ satisfy the condition from (i.c). □

Using the above criterion we can prove that N-block isomorphic character triples can be obtained by passing to quotients or to (central) extensions.

Corollary 4.4. *Suppose that $N \lhd G$, $Z \lhd G$ with $Z \leq \mathrm{Z}(N)$ and $H_1, H_2 \leq G$ with $NH_1 = NH_2 = G$. Assume $(H_1/Z, M_1/Z, \overline{\theta}_1) \sim_{N/Z} (H_2/Z, M_2/Z, \overline{\theta}_2)$. Let $\theta_1 \in \mathrm{Irr}(H_1)$ and $\theta_2 \in \mathrm{Irr}(H_2)$ be the lifts of $\overline{\theta}_1$ and $\overline{\theta}_2$. Then $(H_1, M_1, \theta_1) \sim_N (H_2, M_2, \theta_2)$.*

Proof. In order to prove the statement we will check that the assumptions made in Remark 3.7 are satisfied.

Let $\overline{H}_i := H_i/Z$, $\overline{M}_i := M_i/Z$, $\overline{N} := N/Z$ and $\overline{D}_i$ be a defect group of $\mathrm{bl}(\overline{\theta}_i)$. According to [NT89, Theorems 5.8.8, 5.8.10 and 5.8.11] some defect group D_i of $\mathrm{bl}(\theta_i)$ satisfies $D_i Z/Z = \overline{D}_i$. Hence $\mathrm{C}_{\overline{G}}(\overline{D}_i) \leq \overline{H}_i$ implies $\mathrm{C}_G(D_i) \leq H_i$. This proves that the general assumptions made in Theorem 4.3 are satisfied.

Let $\overline{\mathcal{P}}_1$ and $\overline{\mathcal{P}}_2$ be projective representations of $\overline{H}_1$ and $\overline{H}_2$ associated with $\overline{\theta}_1$ and $\overline{\theta}_2$ that give an $\overline{N}$-block isomorphism of character triples. Then they satisfy the assumptions in Theorem 4.3(i).

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the projective representations of H_1 and H_2 respectively given by

$$\mathcal{P}_i(h_i) = \overline{\mathcal{P}}_i(h_i Z) \quad \text{for every } h_i \in H_i \text{ and } i \in \{1, 2\}.$$

We now check that those projective representations have the properties described in 4.3(i).

Since $\mathrm{C}_{\overline{G}}(\overline{N}) \geq \mathrm{C}_G(N)/Z$ for $\overline{G} := G/Z$, the projective representations $\mathcal{P}_1$ and $\mathcal{P}_2$ define an N-central isomorphism of character triples according to Lemma 3.4. Hence $\mathcal{P}_1$ and $\mathcal{P}_2$ satisfy assumptions (i.a) and (i.b) of Theorem 4.3.

Let $x \in G$, $J := \langle N, x\rangle$, $\overline{x} := xZ$ and $\overline{J} := J/Z$. Let $\mathbb{T} \subset Z$ be defined by $\{xz \mid z \in \mathbb{T}\} = \mathfrak{Cl}_J(x) \cap \{xz \mid z \in Z\}$. Then we obtain

$$\mathcal{P}_i((\mathfrak{Cl}_J(x) \cap H_i)^+) = |\mathbb{T}|\overline{\mathcal{P}}_i((\mathfrak{Cl}_{\overline{J}}(\overline{x}) \cap \overline{H}_i)^+).$$

By Theorem 4.3, the matrices $\overline{\mathcal{P}}_1((\mathfrak{Cl}_{\overline{J}}(\overline{x}) \cap \overline{H}_1)^+)^*$ and $\overline{\mathcal{P}}_2((\mathfrak{Cl}_{\overline{J}}(\overline{x}) \cap \overline{H}_2)^+)^*$ are associated with the same scalar. So $\mathcal{P}_1$ and $\mathcal{P}_2$ satisfy 4.3(i.c) in view of the above equality.

Since the conditions of Theorem 4.3(i) are satisfied, (ι, σ) is an N-block isomorphism of character triples. □

In the opposite direction only a weak version of the above statement can be proven. Although it is not essential in the later proofs, we give this statement and its proof for completeness and possible future use.

Corollary 4.5. *Suppose that $N \lhd G$ and $(H_1, M_1, \theta_1) \sim_N (H_2, M_2, \theta_2)$ where $NH_1 = NH_2 = G$. Let $Z \leq \mathrm{Z}(N) \cap \ker(\theta_1) \cap \ker(\theta_2)$ with $\mathrm{C}_G(N)/Z = \mathrm{C}_{G/Z}(N/Z)$. Assume that either Z is a p'-group or $Z \leq \mathrm{Z}(G)$. Let $\overline{\theta}_1 \in \mathrm{Irr}(M_1/Z)$ and $\overline{\theta}_2 \in \mathrm{Irr}(M_2/Z)$ be the associated characters of the quotients. If $\mathrm{C}_{G/Z}(\overline{D}_1) \leq H_1/Z$ and $\mathrm{C}_{G/Z}(\overline{D}_2) \leq H_2/Z$ for some defect groups $\overline{D}_1$ of $\mathrm{bl}(\overline{\theta}_1)$ and $\overline{D}_2$ of $\mathrm{bl}(\overline{\theta}_2)$, then $(H_1/Z, M_1/Z, \overline{\theta}_1) \sim_{N/Z} (H_2/Z, M_2/Z, \overline{\theta}_2)$.*

Proof. Let $\overline{H}_i := H_i/Z$, $\overline{M}_i := M_i/Z$ and $\overline{N} := N/Z$. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be projective representations of H_1 and H_2 respectively that give the N-block isomorphism $(\iota, \sigma) : (H_1, M_1, \theta_1) \to (H_2, M_2, \theta_2)$ of character triples. Because $Z \leq \mathrm{Z}(N) \cap \ker(\theta_i)$, the projective representation $\mathcal{P}_i$ is constant on Z-cosets for $i = 1, 2$, and hence defines a projective representation $\overline{\mathcal{P}}_i$ of $\overline{H}_i$ associated with $\overline{\theta}_i$.

We check successively that $\overline{\mathcal{P}}_1$ and $\overline{\mathcal{P}}_2$ satisfy the requirements from Remark 3.7. The defect groups $\overline{D}_i$ satisfy $\mathrm{C}_{\overline{G}}(\overline{D}_i) \leq \overline{H}_i$, the requirement from 3.7(ii), by the given assumptions.

According to Theorem 4.3 the projective representations $\mathcal{P}_1$ and $\mathcal{P}_2$ have factor sets coinciding via the canonical isomorphism $\iota : H_1/M_1 \to H_2/M_2$. Hence via the canonical isomorphism $\overline{\iota} : \overline{H}_1/\overline{M}_1 \to \overline{H}_2/\overline{M}_2$ the factor sets of $\overline{\mathcal{P}}_1$ and $\overline{\mathcal{P}}_2$ coincide as well. This is assumption 3.7(i).

For $x \in \mathrm{C}_G(N)$, $\mathcal{P}_1(x)$ and $\mathcal{P}_2(x)$ are scalar matrices associated with the same scalar. Now since $\mathrm{C}_G(N)/Z = \mathrm{C}_{G/Z}(N/Z)$, the scalars associated with $\overline{\mathcal{P}}_1(x)$ and $\overline{\mathcal{P}}_2(x)$ coincide as well. This is assumption 3.7(iii).

Let $(\overline{\iota}, \overline{\sigma})$ be the character triple isomorphism given by $\overline{\mathcal{P}}_1$ and $\overline{\mathcal{P}}_2$. Let $N \leq J \leq G$ and $\psi \in \mathrm{Irr}(H_1 \cap J \mid \theta_1)$. Then $\sigma_{H_1 \cap J}(\psi)$ is the lift of $\overline{\sigma}_{\overline{H}_1 \cap \overline{J}}(\overline{\psi})$, where $\overline{J} := J/Z$ and $\overline{\psi} \in \mathrm{Irr}(\overline{H}_1 \cap \overline{J})$ lifts to ψ. Moreover $\mathrm{bl}(\psi)^J = \mathrm{bl}(\sigma_{H_1 \cap J}(\psi))^J$. According to [NS14, Proposition 2.4(b)] (for $p \nmid |Z|$) and [NS14, Proposition 2.4(c)] (if $Z \leq \mathrm{Z}(G)$) this implies $\mathrm{bl}(\overline{\psi})^{\overline{J}} = \mathrm{bl}(\overline{\sigma}_{\overline{H}_1 \cap \overline{J}}(\overline{\psi}))^{\overline{J}}$ since $\mathrm{bl}(\psi) \supseteq \mathrm{bl}(\overline{\psi})$ and $\mathrm{bl}(\sigma_{H_1 \cap J}(\psi)) \supseteq \mathrm{bl}(\overline{\sigma}_{\overline{H}_1 \cap \overline{J}}(\overline{\psi}))$. By Remark 3.7 this proves the statement. □

5. Construction of N-block isomorphic character triples

In this section we consider how one can obtain new pairs of N-block isomorphic character triples by using direct products (Theorem 5.1) and wreath products (Theorem 5.2). In addition we prove that the equivalence relation between character triples only depends on the automorphisms induced (see Theorem 5.3).

In the first statement we consider character triples coming from direct products. Recall that the irreducible characters of a group $N_1 \times N_2$ can be written as $\chi_1 \times \chi_2$ with $\chi_1 \in \mathrm{Irr}(N_1)$ and $\chi_2 \in \mathrm{Irr}(N_2)$ (see [Is76, (4.21)]).

Theorem 5.1. *For $j = 1, 2$ let $(H_{j,1}, M_{j,1}, \theta_{j,1}) \sim_{N_j} (H_{j,2}, M_{j,2}, \theta_{j,2})$. Then*

$$(H_1, M_1, \theta_1) \sim_N (H_2, M_2, \theta_2),$$

where $N := N_1 \times N_2$, $H_i := H_{1,i} \times H_{2,i}$, $M_i := M_{1,i} \times M_{2,i}$ and $\theta_i := \theta_{1,i} \times \theta_{2,i} \in \mathrm{Irr}(M_i)$.

Proof. Let $G_j := N_j H_{j,1} = N_j H_{j,2}$ and $D_{j,i}$ be a defect group of $\mathrm{bl}(\theta_{j,i})$. Hence $\mathrm{C}_{G_j}(D_{j,i}) \leq H_{j,i}$. Straightforward calculations show that $D_i := D_{1,i} \times D_{2,i}$ is a defect group of $\mathrm{bl}(\theta_i)$, and hence $\mathrm{C}_G(D_i) \leq H_i$ for $G = G_1 \times G_2$. Hence the group-theoretic assumptions of Theorem 4.3 are satisfied.

Let $\mathcal{P}_{j,1}$ and $\mathcal{P}_{j,2}$ be projective representations associated with $\theta_{j,1}$ and $\theta_{j,2}$ that give an N_j-block isomorphism $(\iota_j, \sigma_j) : (H_{j,1}, M_{j,1}, \theta_{j,1}) \to (H_{j,2}, M_{j,2}, \theta_{j,2})$. Note that then the projective representations $\mathcal{P}_{j,i}$ satisfy the conditions from Theorem 4.3(i) and $\mathcal{P}_i := \mathcal{P}_{1,i} \otimes \mathcal{P}_{2,i}$ is a projective representation of H_i associated with θ_i.

For the proof of the statement using Theorem 4.3 we have to check the conditions from Theorem 4.3(i) for $\mathcal{P}_1$ and $\mathcal{P}_2$: by the definition of $\mathcal{P}_i$ the factor sets α_i of $\mathcal{P}_i$ $(i = 1, 2)$ satisfy

$$\alpha_i((h_1, h_2), (h_1', h_2')) = \alpha_{1,i}(h_1, h_1')\alpha_{2,i}(h_2, h_2')$$

for all $h_j, h_j' \in H_{j,i}$, where $\alpha_{j,i}$ is the factor set of $\mathcal{P}_{j,i}$. Let $\iota : H_1/M_1 \to H_2/M_2$ be the canonical isomorphism. Since then

$$\iota(\overline{h}_1, \overline{h}_2) = (\iota_1(\overline{h}_1), \iota_2(\overline{h}_2))$$

for all $\overline{h}_j \in H_{j,1}/M_{j,1}$, the factor sets α_1 and α_2 coincide via ι. Hence $\mathcal{P}_1$ and $\mathcal{P}_2$ satisfy 4.3(i.a).

Let $x \in \mathrm{C}_G(N)$ and $x_i \in \mathrm{C}_{G_i}(N_i)$ with $x = (x_1, x_2)$. Let $\xi_j \in \mathbb{C}$ be such that $\mathcal{P}_{j,i}(x_j)$ is a scalar matrix associated with ξ_j. (Because of 4.3(i) for $\mathcal{P}_{j,i}$, ξ_j is well-defined.) Then $\mathcal{P}_1(x)$ and $\mathcal{P}_2(x)$ are scalar matrices associated with $\xi_1\xi_2$. This implies that $\mathcal{P}_1$ and $\mathcal{P}_2$ satisfy (i.b) of Theorem 4.3.

It remains to check that for $x \in G$ and $J := \langle N, x\rangle$ the matrices $\mathcal{P}_1((\mathfrak{Cl}_J(x) \cap H_1)^+)^*$ and $\mathcal{P}_2((\mathfrak{Cl}_J(x) \cap H_2)^+)^*$ are associated with the same scalar. Let $x \in G$, $x_1 \in G_1$ and $x_2 \in G_2$ with $x = (x_1, x_2)$. Then straightforward computations show that

$$\mathfrak{Cl}_J(x) = \{(c_1, c_2) \mid c_1 \in \mathfrak{Cl}_{J_1}(x_1) \text{ and } c_2 \in \mathfrak{Cl}_{J_2}(x_2)\},$$

where $J_1 := \langle N_1, x_1\rangle$ and $J_2 := \langle N_2, x_2\rangle$. For $i = 1, 2$ we obtain

$$\mathfrak{Cl}_J(x) \cap H_i = \{(c_1, c_2) \mid c_1 \in \mathfrak{Cl}_{J_1}(x_1) \cap H_{1,i} \quad \text{and} \quad c_2 \in \mathfrak{Cl}_{J_2}(x_2) \cap H_{2,i}\}$$

and

$$\mathcal{P}_i((\mathfrak{Cl}_J(x) \cap H_i)^+) = \mathcal{P}_{1,i}((\mathfrak{Cl}_J(x) \cap H_{1,i})^+) \otimes \mathcal{P}_{2,i}((\mathfrak{Cl}_J(x) \cap H_{2,i})^+).$$

This implies that in all cases condition 4.3(i.c) holds for $\mathcal{P}_1$ and $\mathcal{P}_2$, since the $\mathcal{P}_{i,j}$ have the analogous property. □

We now construct new N^r-block isomorphic character triples using wreath products.

Theorem 5.2. *Let r be a positive integer and $(H_1, M_1, \theta_1) \sim_N (H_2, M_2, \theta_2)$. Then*

$$(H_1 \wr \mathfrak{S}_r, M_1^r, \theta_1^r) \sim_{N^r} (H_2 \wr \mathfrak{S}_r, M_2^r, \theta_2^r).$$

Proof. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be projective representations associated with θ_1 and θ_2 which give an N-block isomorphism of character triples and hence satisfy the conditions from 4.3(i).

Let $G := NH_1 = NH_2$ and D_i a defect group of $\mathrm{bl}(\theta_i)$ for $i \in \{1, 2\}$ with $\mathrm{C}_G(D_i) \leq H_i$. By straightforward computations one sees that $\widetilde{D}_i := D_i^r$ is a defect group of $\mathrm{bl}(\theta_i^r)$.

Assume that a defect group of $\mathrm{bl}(\theta_1)$ or $\mathrm{bl}(\theta_2)$ is contained in $\mathrm{Z}(N)$. Without loss of generality we may assume $D_1 \leq \mathrm{Z}(N)$ and hence $\mathrm{C}_N(D_1) = N \leq H_1$ and $M_1 = N$. Because of $\mathrm{bl}(\theta_2)^N = \mathrm{bl}(\theta_1)^N = \mathrm{bl}(\theta_1)$ this implies $D_2 = D_1$, since according to [NT89, Lemma 5.3.3] some defect group of $\mathrm{bl}(\theta_2)$ is contained in a defect group of $\mathrm{bl}(\theta_2)^N = \mathrm{bl}(\theta_1)$. This implies $M_2 = N$ and $H_2 = G$. Now as $\mathrm{bl}(\theta_1) = \mathrm{bl}(\theta_2)$ is a block of central defect and $\mathrm{Irr}(\theta_{1,\mathrm{Z}(N)}) = \mathrm{Irr}(\theta_{2,\mathrm{Z}(N)})$, the two characters coincide (see [Na98, Theorem (9.12)]).

Accordingly in the following we assume $D_1, D_2 \not\leq \mathrm{Z}(N)$. Again we apply Theorem 4.3 by constructing projective representations with the required properties. For $i \in \{1, 2\}$ let $\widetilde{\mathcal{P}}_i$ be the projective representation of $H_i \wr \mathfrak{S}_r$ defined using $\mathcal{P}_i$ as in [JK81, 4.3].

Let $\iota : H_1/M_1 \to H_2/M_2$ and $\widetilde{\iota} : \widetilde{H}_1/\widetilde{M}_1 \to \widetilde{H}_2/\widetilde{M}_2$ be the canonical isomorphisms for $\widetilde{H}_i := H_i \wr \mathfrak{S}_r \leq G \wr \mathfrak{S}_r$ and $\widetilde{M}_i := M_i^r \leq \widetilde{H}_i$. Note that the factor sets of $\widetilde{\mathcal{P}}_1$ and $\widetilde{\mathcal{P}}_2$ then coincide via $\widetilde{\iota}$. Hence $\widetilde{\mathcal{P}}_1$ and $\widetilde{\mathcal{P}}_2$ satisfy condition (i.a) of Theorem 4.3.

Moreover by definition $(\widetilde{\mathcal{P}}_i)_{H_i^r}$ coincides with the projective representation

$$\mathcal{P}_i \otimes \cdots \otimes \mathcal{P}_i \qquad (r \text{ factors})$$

of H_i^r. By the arguments in the proof of Theorem 5.1 we see that:

- for every $x \in \mathrm{C}_{G \wr \mathfrak{S}_r}(N)$, $\widetilde{\mathcal{P}}_1(x)$ and $\widetilde{\mathcal{P}}_2(x)$ are associated with the same scalar,
- $\widetilde{\mathcal{P}}_1((\mathfrak{Cl}_J(x) \cap \widetilde{H}_1)^+)^*$ and $\widetilde{\mathcal{P}}_2((\mathfrak{Cl}_J(x) \cap \widetilde{H}_2)^+)^*$ are associated with the same scalar whenever $x \in G^r$ and $J := \langle N, x\rangle$.

Since $D_1, D_2 \not\leq \mathrm{Z}(N)$, $\mathrm{C}_{G \wr \mathfrak{S}_r}(\widetilde{D}_i) = \mathrm{C}_G(D_i)^r \leq H_i^r$. Hence condition (i.c) of Theorem 4.3 has to be checked only for elements $x \in N^r \mathrm{C}_{G \wr \mathfrak{S}_r}(D_1) \cap N^r \mathrm{C}_{G \wr \mathfrak{S}_r}(D_2) \leq G^r$. The above arguments prove that $\widetilde{\mathcal{P}}_1$ and $\widetilde{\mathcal{P}}_2$ satisfy (i.c), and hence they give an N^r-block isomorphism of character triples. □

In the following statement one considers character triples where the characters but not the groups coincide. This statement (and its proof) is a generalization of some ideas that led to [Spä13a, Theorem 7.9] and [Spä13b, Proposition 4.6].

Theorem 5.3. *Let* $(H_1, M_1, \theta_1) \sim_N (H_2, M_2, \theta_2)$ *and* $G = NH_1 = NH_2$. *Let* $\widehat{G}$ *be a group with* $N \lhd \widehat{G}$, *and let* $\epsilon : G \to \mathrm{Aut}(N)$ *and* $\widehat{\epsilon} : \widehat{G} \to \mathrm{Aut}(N)$ *be the canonical morphisms. Assume that* $\epsilon(G) = \widehat{\epsilon}(\widehat{G})$. *Let* $\widehat{H}_1 := \widehat{\epsilon}^{-1}(\epsilon(H_1))$ *and* $\widehat{H}_2 := \widehat{\epsilon}^{-1}(\epsilon(H_2))$. *Then*

$$(\widehat{H}_1, M_1, \theta_1) \sim_N (\widehat{H}_2, M_2, \theta_2).$$

Note that $M_i \leq \widehat{H}_i$ since the group of automorphisms of N induced by H_i clearly contains the automorphisms of N induced by M_i. Hence the character triples considered in the above statement are well-defined.

Basically this theorem tells us that the existence of an N-block isomorphism of character triples is a property governed by the group of induced automorphisms, the actual structure of the groups only playing a minor rôle. This statement is used as a key step of the proof of Theorem 1.3.

The proof of this statement is based on Theorem 4.3 and seems quite involved. The rest of this section is devoted to it. We see in Lemma 5.4 that the group-theoretic assumptions of Theorem 4.3 are satisfied. Accordingly it remains to construct projective representations $\widehat{\mathcal{P}}_1$ and $\widehat{\mathcal{P}}_2$ of $\widehat{H}_1$ and $\widehat{H}_2$ associated with θ_1 and θ_2 with the properties from 4.3(i). Their construction uses the notation from 5.5 and is given in Subsection 5.6. Afterwards the properties required in 4.3(i) are successively checked for those projective representations $\widehat{\mathcal{P}}_1$ and $\widehat{\mathcal{P}}_2$, and we conclude the proof of Theorem 5.3 after Proposition 5.11.

Lemma 5.4. *For* $N \lhd G$, *the character triples* $(\widehat{H}_1, M_1, \theta_1)$ *and* $(\widehat{H}_2, M_2, \theta_2)$ *satisfy the group-theoretic assumptions from Theorem* 4.3, *i.e.,*

(a) $\widehat{H}_1 N = \widehat{H}_2 N = \widehat{G}$, $M_1 = N \cap \widehat{H}_1$ *and* $M_2 = N \cap \widehat{H}_2$,
(b) $\mathrm{C}_{\widehat{G}}(D_1) \leq \widehat{H}_1$ *and* $\mathrm{C}_{\widehat{G}}(D_2) \leq \widehat{H}_2$, *where* D_1 *is a defect group of* $\mathrm{bl}(\theta_1)$ *and* D_2 *a defect group of* $\mathrm{bl}(\theta_2)$,
(c) $(\widehat{H}_1, M_1, \theta_1)$ *and* $(\widehat{H}_2, M_2, \theta_2)$ *are character triples.*

Proof. Since $\widehat{\epsilon}(\widehat{G}) = \epsilon(G)$, the definition of $\widehat{H}_1$ and $\widehat{H}_2$ implies the equalities in (a). Part (b) follows analogously from $\mathrm{C}_G(D_1) \leq H_1$ and $\mathrm{C}_G(D_2) \leq H_2$. For $i = 1, 2$ the groups $\widehat{H}_i$ and H_i induce the same automorphisms on M_i, hence θ_i is $\widehat{H}_i$-invariant. This proves (c). □

For the proof of Theorem 5.3, in the following paragraph we explicitly recall the consequences of $(H_1, M_1, \theta_1) \sim_N (H_2, M_2, \theta_2)$.

Notation 5.5. By the assumption of Theorem 5.3 we have $(H_1, M_1, \theta_1) \sim_N (H_2, M_2, \theta_2)$, and hence there exist projective representations $\mathcal{P}_1$ and $\mathcal{P}_2$ associated with θ_1 and θ_2 of H_1 and H_2 respectively that satisfy the statements in Theorem 4.3(i). Since the properties of $\mathcal{P}_1$ and $\mathcal{P}_2$ are crucial for the remaining parts of the proof, we give them here in detail:

(i.a) the factor sets α_1 and α_2 of $\mathcal{P}_1$ and $\mathcal{P}_2$ coincide via the canonical isomorphism $\iota : H_1/M_1 \to H_2/M_2$, i.e.,

$$\overline{\alpha}_1(h, h') = \overline{\alpha}_2(\iota(h), \iota(h')) \quad \text{for all } h, h' \in H_1/M_1,$$

(i.b) there exists a map $\mu : \mathrm{C}_G(N) \to \mathbb{C}$ such that $\mathcal{P}_1(x) = \mu(x)\,\mathrm{Id}_{\theta_1(1)}$ and $\mathcal{P}_2(x) = \mu(x)\,\mathrm{Id}_{\theta_2(1)}$, where $\mathrm{Id}_{\theta_1(1)}$ and $\mathrm{Id}_{\theta_1'(1)}$ denote the appropriate identity matrices,

(i.c) $\mathcal{P}_1((\mathfrak{Cl}_J(x) \cap H_1)^+)^*$ and $\mathcal{P}_2((\mathfrak{Cl}_J(x) \cap H_2)^+)^*$ are scalar matrices associated with the same scalar for every $x \in N\mathrm{C}_G(D_1) \cap N\mathrm{C}_G(D_2)$ and $J := \langle N, x\rangle$.

5.6. Construction of $\widehat{\mathcal{P}}_1$ and $\widehat{\mathcal{P}}_2$

Let $\mathbb{T}$ be a full representative set of $\mathrm{C}_G(N)N$-cosets in G with $1_N \in \mathbb{T}$, i.e., every $x \in G$ can be written as tnc for some $t \in \mathbb{T}$, $n \in N$ and $c \in \mathrm{C}_G(N)$, where t is unique and c and n are unique up to simultaneous $\mathrm{Z}(N)$-multiplication.

Let $1_N \in \widehat{\mathbb{T}} \subset \widehat{G}$ be such that there exists a bijection $\mathbb{T} \to \widehat{\mathbb{T}}$ given by $t \mapsto \widehat{t}$ with $\epsilon(t) = \widehat{\epsilon}(\widehat{t})$. By definition we have

$$n^t = n^{\widehat{t}} \quad \text{for every } n \in N \text{ and } t \in \mathbb{T}.$$

Hence $\widehat{\mathbb{T}}$ is a complete representative set of $\mathrm{C}_{\widehat{G}}(N)N$-cosets in $\widehat{G}$, i.e., every $\widehat{x} \in \widehat{G}$ may be written as $\widehat{t}nc$ for a unique $\widehat{t} \in \widehat{\mathbb{T}}$ and some $n \in N$ and $c \in \mathrm{C}_{\widehat{G}}(N)$. In this context t is unique, and n and c are unique up to simultaneous $\mathrm{Z}(N)$-multiplication.

Notation 5.7 ($\mathbb{T}N$ and $\widehat{\mathbb{T}}N$). Let

$$\mathbb{T}N := \{tn \mid t \in \mathbb{T} \text{ and } n \in N\} \quad \text{and} \quad \widehat{\mathbb{T}}N := \{\widehat{t}n \mid \widehat{t} \in \widehat{\mathbb{T}} \text{ and } n \in N\}.$$

There is a bijection $\widehat{\ }: \mathbb{T}N \to \widehat{\mathbb{T}}N$ with $tn \mapsto \widehat{t}n$ for every $t \in \mathbb{T}$ and $n \in N$. For every $x \in \mathbb{T}N$ the image is denoted by $\widehat{x}$. Then for every $y \in \mathbb{T}N$ and $n \in N$,

$$\widehat{yn} = \widehat{y}\widehat{n} = \widehat{y}n \quad \text{and} \quad \widehat{ny} = \widehat{n}\widehat{y} = n\widehat{y}. \tag{5.1}$$

This bijection maps $H_{i,\mathrm{rep}} := \mathbb{T}N \cap H_i$ to $\widehat{H}_{i,\mathrm{rep}} := \widehat{\mathbb{T}}N \cap \widehat{H}_i$.

Let $\mu_0 \in \mathrm{Irr}((\theta_1)_{\mathrm{Z}(N)})$ and $\widehat{\mu} : \mathrm{C}_{\widehat{G}}(N) \to \mathbb{C}$ be a map such that any $\widehat{\mu}(c)$ ($c \in \mathrm{C}_{\widehat{G}}(N)$) is a root of unity and

$$\widehat{\mu}(cz) = \widehat{\mu}(c)\mu_0(z) \quad \text{for every } c \in \mathrm{C}_{\widehat{G}}(N) \text{ and } z \in \mathrm{Z}(N). \tag{5.2}$$

For $i = 1, 2$ let $\widehat{\mathcal{P}}_i : \widehat{H}_i \to \mathrm{GL}_{\theta_i(1)}(\mathbb{C})$ be given by

$$\widehat{\mathcal{P}}_i(\widehat{h}c) := \mathcal{P}_i(h)\widehat{\mu}(c) \quad \text{for every } \widehat{h} \in \widehat{H}_{i,\mathrm{rep}} \text{ and } c \in \mathrm{C}_{\widehat{G}}(N).$$

Note that by the choice of $\widehat{H}_{i,\mathrm{rep}}$ and the definition of $\widehat{H}_i$ this defines a map on $\widehat{H}_i$ and by (5.2), $\widehat{\mathcal{P}}_i$ is well-defined and does not depend on the actual choice of $\widehat{h}$ and c.

Lemma 5.8. *For $i = 1, 2$ the map $\widehat{\mathcal{P}}_i$ is a projective representation of $\widehat{H}_i$ associated with θ_i, and the factor set $\widehat{\alpha}_i : \widehat{H}_i \times \widehat{H}_i \to \mathbb{C}$ satisfies*

$$\widehat{\alpha}_i(\widehat{h}c, \widehat{h}'c') = \mu(c''')\alpha_i(h'', c''')^{-1}\alpha_i(h, h')\widehat{\mu}(c)\widehat{\mu}(c')\widehat{\mu}(c'')^{-1}$$

for all $\widehat{h}, \widehat{h}' \in \widehat{H}_{i,\mathrm{rep}}$ and $c, c' \in \mathrm{C}_{\widehat{G}}(N)$, where $c'' \in \mathrm{C}_{\widehat{G}}(N)$, $h'' \in H_{i,\mathrm{rep}}$ and $c''' \in \mathrm{C}_G(N)$ satisfy $\widehat{h}c\widehat{h}'c' = \widehat{h}''c''$ and $hh' = h''c'''$.

Note that c'', h'' and c''' are only determined up to $\mathrm{Z}(N)$-multiplication. Nevertheless $\mu(c''')\widehat{\mu}(c'')^{-1}$ and $\alpha_i(h'', c''')$ are constant. Hence the given expression for $\widehat{\alpha}_i(\widehat{h}c, \widehat{h}'c')$ is uniquely determined.

Proof of Lemma 5.8. First we prove that $\widehat{\mathcal{P}}_i$ is a projective representation of $\widehat{H}_i$ associated with θ_i for $i = 1, 2$. By definition it is clear that $(\widehat{\mathcal{P}}_i)_{M_i}$ is a linear representation affording θ_i. Let $x \in \widehat{H}_i$, $m \in M_i$, $\widehat{h} \in \widehat{H}_{i,\mathrm{rep}}$ and $c \in \mathrm{C}_{\widehat{G}}(N)$ with $x = \widehat{h}c$. Then $xm = \widehat{h}cm = \widehat{h}mc$ with $\widehat{h}m \in \widehat{H}_{i,\mathrm{rep}}$ and $mx = m\widehat{h}c$ with $m\widehat{h} \in \widehat{H}_{i,\mathrm{rep}}$. The equalities in (5.1) imply

$$\begin{aligned}\widehat{\mathcal{P}}_i(xm) &= \widehat{\mathcal{P}}_i(\widehat{hm}c) = \mathcal{P}_i(hm)\widehat{\mu}(c) = \mathcal{P}_i(h)\mathcal{P}_i(m)\widehat{\mu}(c)\\ &= \widehat{\mathcal{P}}_i(\widehat{h}c)\widehat{\mathcal{P}}_i(m) = \widehat{\mathcal{P}}_i(x)\widehat{\mathcal{P}}_i(m).\end{aligned}$$

Analogously one obtains $\widehat{\mathcal{P}}_i(mx) = \widehat{\mathcal{P}}_i(m)\widehat{\mathcal{P}}_i(x)$.

In the next step we show that $\widehat{\mathcal{P}}_i$ is a projective representation of $\widehat{H}_i$ and compute its factor set. Let $\widehat{h}, \widehat{h}', \widehat{h}'' \in H_{i,\mathrm{rep}}$ and $c, c', c'' \in \mathrm{C}_{\widehat{G}}(N)$ be such that

$$\widehat{h}c\widehat{h}'c' = \widehat{h}''c''.$$

Note that there is some $c''' \in \mathrm{C}_G(N)$ such that $hh' = h''c'''$. We obtain

$$\begin{aligned}\widehat{\mathcal{P}}_i(\widehat{h}c)\widehat{\mathcal{P}}_i(\widehat{h}'c') &= \mathcal{P}_i(h)\mathcal{P}_i(h')\widehat{\mu}(c)\widehat{\mu}(c'') = \mathcal{P}_i(hh')\alpha_i(h, h')\widehat{\mu}(c)\widehat{\mu}(c')\\ &= \mathcal{P}_i(h''c''')\alpha_i(h, h')\widehat{\mu}(c)\widehat{\mu}(c')\\ &= \mathcal{P}_i(h'')\mathcal{P}_i(c''')\alpha_i(h'', c''')^{-1}\alpha_i(h, h')\widehat{\mu}(c)\widehat{\mu}(c')\\ &= \mathcal{P}_i(h'')\mu(c''')\alpha_i(h'', c''')^{-1}\alpha_i(h, h')\widehat{\mu}(c)\widehat{\mu}(c').\end{aligned}$$

Since $\widehat{\mathcal{P}}_i(\widehat{h}''c'') = \mathcal{P}_i(h'')\widehat{\mu}(c'')$, this implies that the factor set $\widehat{\alpha}_i : \widehat{H}_i \times \widehat{H}_i \to \mathbb{C}$ satisfies

$$\widehat{\alpha}_i(\widehat{h}c, \widehat{h}'c') = \mu(c''')\alpha_i(h'', c''')^{-1}\alpha_i(h, h')\widehat{\mu}(c)\widehat{\mu}(c')\widehat{\mu}(c'')^{-1}.$$

This proves that $\widehat{\mathcal{P}}_i$ is a projective representation of $\widehat{H}_i$ associated with θ_i. □

Lemma 5.9. *The factor sets of $\widehat{\mathcal{P}}_1$ and $\widehat{\mathcal{P}}_2$ coincide via the canonical isomorphism $\widehat{\iota} : \widehat{H}_1/M_1 \to \widehat{H}_2/M_2$.*

Proof. Let $x_1, x_1' \in \widehat{H}_1$ and $x_2, x_2' \in \widehat{H}_2$ with $x_2M_2 = \widehat{\iota}(x_1M_1)$ and $x_2'M_2 = \widehat{\iota}(x_1'M_1)$.

Then there exist $\widehat{h}_1, \widehat{h}_1' \in \widehat{H}_{1,\mathrm{rep}}$ and $c, c' \in \mathrm{C}_{\widehat{G}}(N)$ with $x_1 = \widehat{h}_1c$ and $x_1' = \widehat{h}_1'c'$. According to Lemma 5.8 we have

$$\widehat{\alpha}_1(x_1, x_1') = \mu(c_1''')\alpha_1(h_1'', c_1''')^{-1}\alpha_1(h_1, h_1')\widehat{\mu}(c)\widehat{\mu}(c')\widehat{\mu}(c'')^{-1},$$

where $c'' \in \mathrm{C}_{\widehat{G}}(N)$, $h_1'' \in H_{1,\mathrm{rep}}$ and $c''' \in \mathrm{C}_G(N)$ satisfy $x_1x_1' = \widehat{h}_1''c''$ and $h_1h_1' = h_1''c'''$.

By the choice of $x_2, x_2' \in \widehat{H}_2$ there exist $n, n' \in N$ with $x_2 = x_1n$ and $x_2' = x_1'n'$. Hence $\widehat{h}_2 := \widehat{h}_1n$ and $\widehat{h}_2' := \widehat{h}_1'n'$ are contained in $\widehat{H}_{2,\mathrm{rep}}$. Moreover $x_2 = \widehat{h}_2c$ and $x_2' = \widehat{h}_2'c'$. For $h_2'' := h_1''(n^{\widehat{h}_1'})n' \in H_{2,\mathrm{rep}}$ we get

$$x_2x_2' = x_1nx_1'n' = x_1x_1'(n^{x_1'})n' = (\widehat{h}_1''c'')(n^{x_1'})n' = \widehat{h}_1''(n^{\widehat{h}_1'})n'c'' = \widehat{h}_2''c''.$$

Analogously one sees $h_2h'_2 = h''_2c'''$, where h_i, h'_i and h''_i are the preimages of $\widehat{h}_i$, $\widehat{h}'_i$ and $\widehat{h}''_i$ under the bijection introduced in 5.7.

The arguments from the proof of Lemma 5.8 apply and prove

$$\widehat{\alpha}_2(x_2, x'_2) = \mu(c''')\alpha_2(h''_2, c''')^{-1}\alpha_2(h_2, h'_2)\widehat{\mu}(c)\widehat{\mu}(c')\widehat{\mu}(c'')^{-1}.$$

By 5.5(i.a) we know that the preimages h_1 and h_2 of $\widehat{h}_1$ and $\widehat{h}_2$ satisfy $\alpha_1(h_1, h'_1) = \alpha_2(h_2, h'_2)$. Analogously we see that $\alpha_1(h''_1, c''') = \alpha_2(h''_2, c''')$. Together with the above, this implies the equality of $\widehat{\alpha}_1$ and $\widehat{\alpha}_2$ via $\widehat{\iota}$. □

The above statement proves that condition (i.a) of 4.3 is satisfied by $\widehat{\mathcal{P}}_1$ and $\widehat{\mathcal{P}}_2$. The next lemma ensures that also 4.3(i.b) holds for $\widehat{\mathcal{P}}_1$ and $\widehat{\mathcal{P}}_2$.

Lemma 5.10. *For every $x \in \mathrm{C}_{\widehat{G}}(N)$, $\widehat{\mathcal{P}}_1(x)$ and $\widehat{\mathcal{P}}_2(x)$ are scalar matrices associated with $\widehat{\mu}(x)$.*

Proof. This directly follows from the definitions of $\widehat{\mathcal{P}}_1$ and $\widehat{\mathcal{P}}_2$. □

To verify 4.3(i.c) we compare $\widehat{\mathcal{P}}_1((\mathfrak{Cl}_{\widehat{J}}(x) \cap \widehat{H}_1)^+)^*$ and $\widehat{\mathcal{P}}_2((\mathfrak{Cl}_{\widehat{J}}(x) \cap \widehat{H}_2)^+)^*$ for $x \in N\mathrm{C}_{\widehat{G}}(D_1) \cap N\mathrm{C}_{\widehat{G}}(D_2)$ and $\widehat{J} := \langle N, x\rangle$.

Proposition 5.11. *For $x \in N\mathrm{C}_{\widehat{G}}(D_1) \cap N\mathrm{C}_{\widehat{G}}(D_2)$ and $\widehat{J} := \langle N, x\rangle$, $\widehat{\mathcal{P}}_1((\mathfrak{Cl}_{\widehat{J}}(x) \cap \widehat{H}_1)^+)^*$ and $\widehat{\mathcal{P}}_2((\mathfrak{Cl}_{\widehat{J}}(x) \cap \widehat{H}_2)^+)^*$ are associated with the same scalar.*

Proof. Let $x \in N\mathrm{C}_{\widehat{G}}(D_1) \cap N\mathrm{C}_{\widehat{G}}(D_2)$. We observe that $\mathfrak{Cl}_{\widehat{J}}(x)$ is the N-orbit in $\widehat{J}$ containing x. Accordingly $\mathfrak{Cl}_{\widehat{J}}(x)$ is contained in xN. More concretely, $\mathfrak{Cl}_{\widehat{J}}(x)$ coincides with $\mathcal{L}_x(N)x$, where for $y \in G \cup \widehat{G}$ we define $\mathcal{L}_y : N \to N$ by $n \mapsto n^{-1}n^{y^{-1}}$. Note that $\mathcal{L}_y$ only depends on the automorphism of N induced by y. If $x = \widehat{h}c$ for $h \in \widehat{\mathbb{T}}N$ and $c \in \mathrm{C}_{\widehat{G}}(N)$, we see that $\mathcal{L}_x(N) = \mathcal{L}_{\widehat{h}}(N) = \mathcal{L}_h(N)$ and furthermore $\mathfrak{Cl}_J(h) = \mathcal{L}_h(N)h$ for $J := \langle N, h\rangle$. For $j \in \{1, 2\}$ we obtain

$$\begin{aligned}
\widehat{\mathcal{P}}_j((\mathfrak{Cl}_{\widehat{J}}(x) \cap \widehat{H}_j)^+) &= \sum_{\substack{y\in\mathfrak{Cl}_{\widehat{J}}(x)\\ y\in\widehat{H}_j}} \widehat{\mathcal{P}}_j(y) = \sum_{\substack{l\in\mathcal{L}_x(N)\\ lx\in\widehat{H}_j}} \widehat{\mathcal{P}}_j(lx)\\
&= \sum_{\substack{l\in\mathcal{L}_x(N)\\ lx\in\widehat{H}_j}} \widehat{\mathcal{P}}_j(l\widehat{h}c) = \sum_{\substack{l\in\mathcal{L}_{\widehat{h}}(N)\\ lx\in\widehat{H}_j}} \widehat{\mathcal{P}}_j(l\widehat{h}c)\\
&= \sum_{\substack{l\in\mathcal{L}_h(N)\\ l\widehat{h}c\in\widehat{H}_j}} \mathcal{P}_j(lh)\widehat{\mu}(c) = \widehat{\mu}(c)\Big(\sum_{\substack{l\in\mathcal{L}_h(N)\\ lh\in H_j}} \mathcal{P}_j(lh)\Big)\\
&= \widehat{\mu}(c)\mathcal{P}_j((\mathfrak{Cl}_J(h) \cap H_j)^+).
\end{aligned}$$

Note that $h \in N\mathrm{C}_G(D_1) \cap N\mathrm{C}_G(D_2)$. Hence by 5.5(i.c) we know that $\mathcal{P}_1((\mathfrak{Cl}_J(h) \cap H_1)^+)^*$ and $\mathcal{P}_2((\mathfrak{Cl}_J(h) \cap H_2)^+)^*$ are associated with the same scalar in $\mathbb{F}$. Using the above equation this implies the statement. □

This was the final step necessary for the proof of Theorem 5.3.

Proof of Theorem 5.3. According to Lemma 5.4 we can apply Theorem 4.3. The maps $\widehat{\mathcal{P}}_1$ and $\widehat{\mathcal{P}}_2$ from 5.6 are projective representations of $\widehat{H}_1$ and $\widehat{H}_2$ associated with θ_1 and θ_2 according to Lemma 5.8. Moreover $\widehat{\mathcal{P}}_1$ and $\widehat{\mathcal{P}}_2$ satisfy the conditions from 4.3(i) (see Lemmas 5.9–5.11. This proves

$$(\widehat{H}_1, M_1, \theta_1) \sim_N (\widehat{H}_2, M_2, \theta_2). \qquad \square$$

6. A new version of Dade's conjecture

In this section we propose a new version of Dade's conjecture (see Conjecture 6.3). Central to this conjecture is that there exists a bijection between certain character sets such that associated characters determine block isomorphic character triples in the sense of Definition 3.6. As seen in the previous sections, this allows a precise control of the Clifford theory of those characters and counting the characters "lying above".

We start by introducing some notation related to p-chains and blocks.

Definition 6.1 (p-chains). For any finite group H we denote by $\mathrm{O}_p(H)$ the largest normal p-subgroup of H. Let G be a finite group. We denote by $\mathfrak{P}(G)$ the set of chains of strictly increasing p-subgroups of G starting with $\{1\}$, and by $\mathfrak{P}(G|Z)$ those starting with a given p-subgroup Z. For a chain

$$\mathbb{D} = (P_0 \lneq P_1 \lneq \cdots \lneq P_n)$$

of nontrivial p-subgroups of G we set $|\mathbb{D}| = n$, the *length* of $\mathbb{D}$, and denote by $\mathrm{N}_G(\mathbb{D})$ or $G_{\mathbb{D}}$ the group $\bigcap_{i=0}^{n} \mathrm{N}_G(P_i)$.

Let $B \in \mathrm{Bl}(G)$. Then by $B_{\mathbb{D}}$ one denotes the sum of blocks $b \in \mathrm{Bl}(G_{\mathbb{D}})$ with $b^G = B$. For an integer d and $\epsilon \in \{+, -\}$ let

$$\mathcal{C}^d(B)_\epsilon := \{(\mathbb{D}, \theta) \mid \mathbb{D} \in \mathfrak{P}(G|\mathrm{O}_p(G))_\epsilon \text{ and } \theta \in \mathrm{Irr}^d(B_{\mathbb{D}})\}$$

and $\mathcal{C}^d(B) := \mathcal{C}^d(B)_+ \cup \mathcal{C}^d(B)_-$.

For $\sigma \in \mathrm{Aut}(G)$, $\mathbb{D} \in \mathfrak{P}(G)$ and $\theta \in \mathrm{Irr}(\mathrm{N}_G(\mathbb{D}))$ we define $(\mathbb{D}, \theta)^\sigma$ to be $(\mathbb{D}^\sigma, \theta^\sigma)$. This determines an action of $\mathrm{Aut}(G)$ on $\mathcal{C}^d(B)$ preserving the length of the p-chains involved. Hence G acts on $\mathcal{C}^d(B)_+$ and $\mathcal{C}^d(B)_-$ by conjugation. For $(\mathbb{D}, \theta) \in \mathcal{C}(B)$ we denote by $\overline{(\mathbb{D}, \theta)}$ its G-orbit and by $\overline{\mathcal{C}^d(B)_+}$ and $\overline{\mathcal{C}^d(B)_-}$ the associated sets of G-orbits.

As mentioned in the introduction, we consider Dade's conjectures as the existence of a certain bijection. Recall that $\mathrm{Bl}_{\mathrm{nc}}(G)$ is the set of blocks of G with noncentral defect groups.

Proposition 6.2. *Let G be a finite group, p a prime, d a nonnegative integer and $B \in \mathrm{Bl}_{\mathrm{nc}}(G)$. Assume that $\mathrm{O}_p(G) \leq \mathrm{Z}(G)$. Then the following are equivalent:*

(i) *Dade's Projective Conjecture holds for B and d.*

(ii) *There exists a bijection*

$$\Omega : \overline{\mathcal{C}^d(B)_+} \to \overline{\mathcal{C}^d(B)_-}$$

such that $\theta_{\mathrm{O}_p(G)}$ and $\theta'_{\mathrm{O}_p(G)}$ are multiples of the same irreducible character whenever $(\mathbb{D}, \theta) \in \mathcal{C}^d(B)_+$ and $(\mathbb{D}', \theta') \in \Omega(\overline{(\mathbb{D}, \theta)})$.

Recall that Dade's Projective Conjecture for B and d—in the version we are considering here—claims

$$\sum_{\mathbb{D}\in\mathfrak{P}(G|\mathrm{O}_p(G))/\sim_G} (-1)^{|\mathbb{D}|}|\mathrm{Irr}^d(B_{\mathbb{D}} \mid \nu)| = 0 \quad \text{for every } \nu \in \mathrm{Irr}(\mathrm{O}_p(G)),$$

where $\mathbb{D}$ runs over a full set of representatives of the G-orbits in $\mathfrak{P}(G|\mathrm{O}_p(G))$.

Proof of Proposition 6.2. Let $\mathbb{D}^{(1)}, \dots, \mathbb{D}^{(r)}$ be a full set of representatives of the G-orbits in $\mathfrak{P}(G|\mathrm{O}_p(G))_+$, and $\mathbb{D}^{(-1)}, \dots, \mathbb{D}^{(-r')}$ one in $\mathfrak{P}(G|\mathrm{O}_p(G))_-$. Then part (i) is equivalent to

$$\sum_{i=1}^{r} |\mathrm{Irr}^d(B_{\mathbb{D}^{(i)}} \mid \nu)| = \sum_{i=1}^{r'} |\mathrm{Irr}^d(B_{\mathbb{D}^{(-i)}} \mid \nu)| \quad \text{for every } \nu \in \mathrm{Irr}(\mathrm{O}_p(G)).$$

We analogously rephrase part (ii). For $\nu \in \mathrm{Irr}(\mathrm{O}_p(G))$ let $\overline{\mathcal{C}^d(B \mid \nu)_+}$ be the set of G-orbits in $\{(\mathbb{D}, \theta) \in \mathcal{C}^d(B)_+ \mid \theta \in \mathrm{Irr}^d(B_{\mathbb{D}} \mid \nu)\}$, and define $\overline{\mathcal{C}^d(B \mid \nu)_-}$ analogously. The existence of the bijection Ω in part (ii) is equivalent to

$$|\overline{\mathcal{C}^d(B \mid \nu)_+}| = |\overline{\mathcal{C}^d(B \mid \nu)_-}| \quad \text{for every } \nu \in \mathrm{Irr}(\mathrm{O}_p(G)).$$

Since G acts on the pairs, there is a one-to-one correspondence between

$$\{(\mathbb{D}^{(i)}, \theta) \mid 1 \le i \le r \text{ and } \theta \in \mathrm{Irr}^d(B_{\mathbb{D}^{(i)}} \mid \nu)\}$$

and $\overline{\mathcal{C}^d(B \mid \nu)_+}$. Analogously the elements of $\{(\mathbb{D}^{(-i)}, \theta) \mid 1 \le i \le r' \text{ and } \theta \in \mathrm{Irr}^d(B_{\mathbb{D}^{(-i)}} \mid \nu)\}$ correspond to those of $\overline{\mathcal{C}^d(B \mid \nu)_-}$. This proves that (i) and (ii) are equivalent. □

Analogously other forms of Dade's conjecture can then be seen as different requirements on this bijection. We strengthen this conjecture by requiring that characters associated with each other by the bijection determine character triples satisfying the equivalence relation from Definition 3.6.

Conjecture 6.3 (Character Triple Conjecture). *Let G be a finite group, p a prime, d a nonnegative integer and $B \in \mathrm{Bl}_{\mathrm{nc}}(G)$. Assume that $\mathrm{O}_p(G) \le \mathrm{Z}(G)$. Suppose that $G \lhd A$. Then there exists an A_B-equivariant bijection*

$$\Omega : \overline{\mathcal{C}^d(B)_+} \to \overline{\mathcal{C}^d(B)_-}$$

such that for every $(\mathbb{D}, \theta) \in \mathcal{C}^d(B)_+$, some $(\mathbb{D}', \theta') \in \Omega(\overline{(\mathbb{D}, \theta)})$ defines G-block isomorphic character triples in the sense of Definition 3.6, i.e.,

$$(A_{\mathbb{D},\theta}, G_{\mathbb{D}}, \theta) \sim_G (A_{\mathbb{D}',\theta'}, G_{\mathbb{D}'}, \theta').$$

(*The last statement is independent of the choice of $(\mathbb{D}', \theta')$ because of Lemma 3.8(c).*)

From Definition 3.6 we see that this is already a stronger form of [Da97, Conjecture 4.10].

Proposition 6.4. *Conjecture 6.3 implies Dade's Extended Projective Conjecture from* [Da97, 4.10].

Thus Conjecture 6.3 is formally stronger than [Da92, Conjecture 6.3], [Da94, Conjecture 15.5] and Dade's Projective Conjecture that are implied by Dade's Extended Projective Conjecture.

Proof of Proposition 6.4. For the proof we make free use of the notation introduced in [Da94] for twisted group algebras.

Let $G \lhd E$ be finite groups with $\mathrm{O}_p(G) = 1$ and $\widehat{\epsilon} : \widehat{E} \to E$ a central extension of E by a cyclic group Z, and let $\lambda \in \mathrm{Irr}(Z)$ be a faithful character. Then $\mathrm{O}_p(\widehat{G})$ is central in $\widehat{E}$. Note that $\widehat{E}$ together with λ determines a totally split twisted group algebra $\mathfrak{A}$ of G over $\mathbb{C}$ and vice versa (see [Da94, Theorem 6.20]). Analogously any block B of $\widehat{G}$ with noncentral defect group that contains a character of $\mathrm{Irr}(\widehat{G} \mid \lambda)$ corresponds to a block $\overline{B}$ of $\mathfrak{A}$ with nontrivial defect (see [Da94, Theorems 8.6 and 9.6]). There is a bijection $\mathrm{Irr}(B \mid \lambda) \to \mathrm{Irr}(\overline{B})$ with $\chi \mapsto \chi^*$ such that χ and χ^* have the same height and the defect of χ is the sum of the defect of χ^* and z_0, where $p^{z_0} = |Z|_p$ (see [Da94, Propositions 9.2 and 9.10]). For any subgroup $U \leq E$ the group $\widehat{U} := \epsilon^{-1}(U)$ is associated with the twisted group algebra $\mathfrak{A}[U]$ as in [Da94, (5.4)]. Furthermore block induction for blocks and subgroups of $\widehat{G}$ corresponds naturally via the correspondence of blocks from [Da94, Theorem 8.6] to the block induction in the totally split group algebra context introduced in [Da94, Definition 10.5] (see [Da94, Theorem 10.10]).

Dade's Projective Conjecture [Da94, 15.5] for the p-block $\overline{B}$ of $\mathfrak{A}[G]$ and any positive integers d is equivalent to the statement that for the corresponding p-block B of $\widehat{G}$ there exists a bijection

$$\Omega_0 : \overline{\mathcal{C}^{d+z_0}(B \mid \lambda)_+} \to \overline{\mathcal{C}^{d+z_0}(B \mid \lambda)_-},$$

where $\overline{\mathcal{C}^{d+z_0}(B \mid \lambda)_\epsilon}$ is defined as the set of $\overline{(\mathbb{D}, \theta)} \in \overline{\mathcal{C}^{d+z_0}(B)_\epsilon}$ with $\theta \in \mathrm{Irr}(\widehat{G}_{\mathbb{D}} \mid \lambda)$ for $\epsilon \in \{+, -\}$. This is a consequence of Conjecture 6.3 since two characters can only be in $\widehat{G}$-block isomorphic character triples if they cover the same character of Z (see Lemma 3.4).

If Ω_0 is given by Conjecture 6.3, the bijection is $\widehat{E}_{B,\lambda}$-equivariant and for every $(\mathbb{D}, \theta) \in \mathcal{C}^{d+z_0}(B \mid \lambda)_+$ every $(\mathbb{D}', \theta') \in \Omega_0(\overline{(\mathbb{D}, \theta)})$ satisfies

$$(\widehat{E}_{\mathbb{D},\theta}, \widehat{G}_{\mathbb{D}}, \theta) \sim_{\widehat{G}} (\widehat{E}_{\mathbb{D}',\theta'}, \widehat{G}_{\mathbb{D}'}, \theta').$$

Since Ω_0 is $\widehat{E}_B$-equivariant, Dade's Invariant Projective Conjecture [Da97, 4.7] holds for $\overline{B}$. Furthermore in the above situation there exist projective representations of $\widehat{E}_{\mathbb{D},\theta}$ and $\widehat{E}_{\mathbb{D}',\theta'}$ associated with θ and θ' such that the factor sets coincide via the canonical isomorphism of $\widehat{E}_{\mathbb{D},\theta}/\widehat{G}_{\mathbb{D},\theta}$ and $\widehat{E}_{\mathbb{D}',\theta'}/\widehat{G}_{\mathbb{D}',\theta'}$. Accordingly also the Extended Projective Conjecture [Da97, 4.10] holds. □

The sets $\mathcal{C}^d(B)$ are also defined for blocks with central defect. In that case they can be described explicitly.

Lemma 6.5 (Blocks with central defect). *Let G be a finite group and $B \in \mathrm{Bl}(G)$ with central defect. Then $\overline{\mathcal{C}^d(B)_-} = \emptyset$ and*

$$\overline{\mathcal{C}^d(B)_+} = \begin{cases} \{(\mathbb{D}^{(0)}, \chi) \mid \chi \in \mathrm{Irr}(B)\}, & d = z_0, \\ \emptyset, & d \neq z_0, \end{cases}$$

where $\mathbb{D}^{(0)} = (\mathrm{O}_p(G))$ is the chain of length 0 and $|\mathrm{O}_p(G)| = p^{z_0}$.

Accordingly Conjecture 6.3 cannot be generalized to those blocks. In the case of cyclic A/G Conjecture 6.3 can be reformulated.

Proposition 6.6. *Let G be a finite group, p a prime and $B \in \mathrm{Bl}_{\mathrm{nc}}(G)$. Assume that $\mathrm{O}_p(G) \leq \mathrm{Z}(G)$. Suppose that $G \lhd A$ is such that A_B/G is cyclic. Assume there exists an A_B-equivariant bijection*

$$\Omega : \overline{\mathcal{C}^d(B)_+} \to \overline{\mathcal{C}^d(B)_-}$$

such that for every $(\mathbb{D}, \theta) \in \mathcal{C}^d(B)_+$, some $(\mathbb{D}', \theta') \in \Omega(\overline{(\mathbb{D}, \theta)})$ satisfies $\mathrm{Irr}(\theta_{\mathrm{Z}(G)}) = \mathrm{Irr}(\theta'_{\mathrm{Z}(G)})$. Then Conjecture 6.3 holds for B, $G \lhd A$ and d.

If A_B/G is cyclic the assumptions are equivalent to Dade's Invariant Projective Conjecture for B in the reformulation of [Ro02]. This can be seen using the same arguments as in the proof of Proposition 6.2.

Proof of Proposition 6.6. It only remains to check that the associated character triples satisfy the given equivalence relation, but this follows from Proposition 3.11. □

The above statement will be essential in checking Conjecture 6.3 for groups related to sporadic groups (see Theorem 9.2).

Definition 6.7 (Inductive Condition for Dade's Conjecture). Let S be a nonabelian simple group, $\widehat{S}$ its universal covering group, $B \in \mathrm{Bl}_{\mathrm{nc}}(\widehat{S})$ and d a positive integer. We say that *the Inductive Condition for Dade's Conjecture holds for B and d* if Conjecture 6.3 holds for B' with respect to $X \lhd A$ and integers $d' \leq d + z_0$ whenever X is a quotient of $\widehat{S}$ by a central subgroup, $B' \in \mathrm{Bl}(X)$ is contained in B, $A := X \rtimes \mathrm{Aut}(X)$ and $p^{z_0} = |\mathrm{Z}(X)|_p$. If this holds for all $B \in \mathrm{Bl}_{\mathrm{nc}}(\widehat{S})$ and $d' \leq d$, then we say that *the Inductive Condition for Dade's Conjecture holds for S and d*.

In the verification of this condition we use the following equivalent (more technical) reformulation.

Proposition 6.8. *Let S be a nonabelian simple group, $\widehat{S}$ its universal covering group, $B \in \mathrm{Bl}_{\mathrm{nc}}(\widehat{S})$ and d a positive integer. Then the Inductive Condition for Dade's Conjecture holds for B and d if and only if for $A_0 := \widehat{S} \rtimes \mathrm{Aut}(\widehat{S})$ and the integer z_0 with $p^{z_0} = |\mathrm{Z}(\widehat{S})|_p$ there exists a defect preserving $A_{0,B}$-equivariant bijection*

$$\Omega : \overline{\mathcal{C}^{\leq d+z_0}(B)_+} \to \overline{\mathcal{C}^{\leq d+z_0}(B)_-},$$

where for every $(\mathbb{D}, \theta) \in \mathcal{C}^{\leq d+z_0}(B)_+$, $(\mathbb{D}', \theta') \in \Omega(\overline{(\mathbb{D}, \theta)})$ and $Z := \ker(\theta_{\mathrm{Z}(\widehat{S})})$ we have $Z = \ker(\theta'_{\mathrm{Z}(\widehat{S})})$ and

$$(A_{\mathbb{D},\theta}/Z, \widehat{S}_{\mathbb{D}}/Z, \overline{\theta}) \sim_{\widehat{S}/Z} (A_{\mathbb{D}',\theta'}/Z, \widehat{S}_{\mathbb{D}'}/Z, \overline{\theta}').$$

(Here $\overline{\theta}$ and $\overline{\theta}'$ denote the characters of $\widehat{S}_{\mathbb{D}}/Z$ and $\widehat{S}_{\mathbb{D}'}/Z$ that are associated with θ and θ'.)

Note that according to Theorem 5.3 it is clear that in the above statement A_0 can be replaced by any group L containing $\widehat{S}$ as a normal subgroup with $L/\mathrm{C}_L(\widehat{S}) = \mathrm{Aut}(\widehat{S})$.

Proof of Proposition 6.8. Here we concentrate on one direction and construct Ω assuming that the Inductive Condition for Dade's Conjecture holds. The other direction follows from similar arguments. By definition, for every covering group X of S and every block $\overline{B} \in \mathrm{Bl}(X)$ contained in B there exists a bijection $\Omega_X : \overline{\mathcal{C}_{\mathrm{nc}}^{\leq d+z_0}(\overline{B})_+} \to \overline{\mathcal{C}^{\leq d+z_0}(\overline{B})_-}$ such that corresponding pairs give X-block isomorphic character triples. Using those maps one can construct a map Ω with the required properties. □

In Section 9 we give some examples of blocks and simple groups for which the Inductive Condition for Dade's Conjecture holds.

An important property of Dade's conjectures is that the type of chains can be varied and thereby adapted to the groups considered. Also the refinement introduced here allows such flexibility.

We recall the terminology introduced in [Da92, Section 3] and adapt it to our situation.

Notation 6.9. A chain $\mathbb{D} = (P_0 \lneq P_1 \lneq \cdots \lneq P_n) \in \mathfrak{P}(G)$ is called *radical* if

(a) $P_0 = \mathrm{O}_p(G)$,
(b) $P_k = \mathrm{O}_p(\bigcap_{i=0}^k \mathrm{N}_G(P_i))$ for every $1 \leq k \leq n$.

We denote by $\mathfrak{R}(G)$ the set of radical p-chains of G. In addition let $\mathfrak{E}(G|\mathrm{O}_p(G))$ be the set of elementary abelian chains of G starting with $\mathrm{O}_p(G)$ (see also [Da94, Definition 1.5]).

Let $\kappa \in \{p, \mathrm{rad}, \mathrm{elem}\}$, G a finite group, $B \in \mathrm{Bl}(G)$ and d a nonnegative integer. We define $\mathcal{C}_\kappa^d(B)_+$ and $\mathcal{C}_\kappa^d(B)_-$ to be the sets of pairs $(\mathbb{D}, \theta)$ where $\mathbb{D} \in \mathfrak{P}(G|\mathrm{O}_p(G))$, $\mathbb{D} \in \mathfrak{R}(G)$ or $\mathbb{D} \in \mathfrak{E}(G|\mathrm{O}_p(G))$ is of even length or odd length respectively, and $\theta \in \mathrm{Irr}^d(B_\mathbb{D})$. The action of G on those sets allows us to define $\overline{\mathcal{C}_\kappa^d(B)_+}$ and $\overline{\mathcal{C}_\kappa^d(B)_-}$ as the sets of G-orbits in $\mathcal{C}_\kappa^d(B)_+$ and $\mathcal{C}_\kappa^d(B)_-$.

Proposition 6.10. *Let $G \lhd A$ with $\mathrm{O}_p(G) \leq \mathrm{Z}(G)$, let $B \in \mathrm{Bl}(G)$ be an A-invariant block with a noncentral defect group and let d be a nonnegative integer. If for some $\kappa_0 \in \{p, \mathrm{rad}, \mathrm{elem}\}$ there exists an A-equivariant bijection*

$$\Omega_{\kappa_0} : \overline{\mathcal{C}_{\kappa_0}^d(B)_+} \to \overline{\mathcal{C}_{\kappa_0}^d(B)_-}$$

such that for each $(\mathbb{D}, \theta) \in \mathcal{C}_\kappa^d(B)_+$, every $(\mathbb{D}', \theta') \in \Omega_\kappa(\overline{(\mathbb{D}, \theta)})$ satisfies

$$(A_{\mathbb{D},\theta}, G_\mathbb{D}, \theta) \sim_G (A_{\mathbb{D}',\theta'}, G_{\mathbb{D}'}, \theta'),$$

then there exists for every $\kappa \in \{p, \mathrm{rad}, \mathrm{elem}\}$ an A-equivariant bijection

$$\Omega_\kappa : \overline{\mathcal{C}_\kappa^d(B)_+} \to \overline{\mathcal{C}_\kappa^d(B)_-}$$

such that for each $(\mathbb{D}, \theta) \in \mathcal{C}_\kappa^d(B)_+$, every $(\mathbb{D}', \theta') \in \Omega(\overline{(\mathbb{D}, \theta)})$ satisfies

$$(A_{\mathbb{D},\theta}, G_\mathbb{D}, \theta) \sim_G (A_{\mathbb{D}',\theta'}, G_{\mathbb{D}'}, \theta').$$

In other words, the statement of Conjecture 1.2 is independent of the type of the underlying p-chains considered there.

Proof of Proposition 6.10. Let $(\mathbb{D}_0,\theta_0)\in\mathcal{C}^d_p(B)_+\cup\mathcal{C}^d_p(B)_-$. For $\kappa\in\{p,\mathrm{rad},\mathrm{elem}\}$ let

$$M_{+,(\mathbb{D}_0,\theta_0),\kappa}:=\{\overline{(\mathbb{D}',\theta')}\in\overline{\mathcal{C}^d_\kappa(B)_+}\mid (A_{\mathbb{D}',\theta'},G_{\mathbb{D}'},\theta')\sim_G(A_{\mathbb{D}_0,\theta_0},G_{\mathbb{D}_0},\theta_0)\},$$
$$M_{-,(\mathbb{D}_0,\theta_0),\kappa}:=\{\overline{(\mathbb{D}',\theta')}\in\overline{\mathcal{C}^d_\kappa(B)_-}\mid (A_{\mathbb{D}',\theta'},G_{\mathbb{D}'},\theta')\sim_G(A_{\mathbb{D}_0,\theta_0},G_{\mathbb{D}_0},\theta_0)\}.$$

Let $\mathrm{Stab}_{A,\sim_G}(\mathbb{D}_0,\theta_0)$ be the stabilizer in A of the $\sim_G$-equivalence class in $\mathcal{C}^d_\kappa(B)$ containing $(\mathbb{D}_0,\theta_0)$, i.e., the group of elements $y\in A$ with

$$(A_{\mathbb{D}_0^y,\theta_0^y},G_{\mathbb{D}_0^y},\theta^y)\sim_G(A_{\mathbb{D}_0,\theta_0},G_{\mathbb{D}_0},\theta_0).$$

Note that according to Lemma 3.8, $\mathrm{Stab}_{A,\sim_G}(\mathbb{D}_0,\theta_0)$ is an actual group. It further acts on $M_{+,(\mathbb{D}_0,\theta_0),\kappa}$ and $M_{-,(\mathbb{D}_0,\theta_0),\kappa}$.

The required bijection exists for $\kappa\in\{p,\mathrm{rad},\mathrm{elem}\}$ if for every $(\mathbb{D}_0,\theta_0)\in\mathcal{C}^d_p(B)_+\cup\mathcal{C}^d_p(B)_-$ the sets $M_{+,(\mathbb{D}_0,\theta_0),\kappa}$ and $M_{-,(\mathbb{D}_0,\theta_0),\kappa}$ are equivalent $\mathrm{Stab}_{A,\sim_G}(\mathbb{D}_0,\theta_0)$-sets. According to [Is08, Lemma 3.33] it is sufficient to check that for every subgroup $H\leq\mathrm{Stab}_{A,\sim_G}(\mathbb{D}_0,\theta_0)$ with $G\leq H$ the set of H-fixed points in $M_{+,(\mathbb{D}_0,\theta_0),\kappa}$, denoted by $(M_{+,(\mathbb{D}_0,\theta_0),\kappa})^H$, has the same cardinality as the analogously defined set $(M_{-,(\mathbb{D}_0,\theta_0),\kappa})^H$.

Let $f_{(\mathbb{D}_0,\theta_0)}:\mathfrak{P}(G)\to\mathbb{Z}$ be the map defined by

$$f_{(\mathbb{D}_0,\theta_0)}(\mathbb{D}):=|\{\theta\in\mathrm{Irr}^d(B_{\mathbb{D}})\mid\overline{(\mathbb{D},\theta)}\in(M_{+,(\mathbb{D}_0,\theta_0),\kappa})^H\cup(M_{-,(\mathbb{D}_0,\theta_0),\kappa})^H\}|$$

for every $\mathbb{D}\in\mathfrak{P}(G)$. This map is constant on H-orbits, and $f_{(\mathbb{D}_0,\theta_0)}(\mathbb{D})=f_{(\mathbb{D}_0,\theta_0)}(\mathbb{D}')$ whenever $\mathrm{N}_A(\mathbb{D})=\mathrm{N}_A(\mathbb{D}')$. Then, according to [Da94, Proposition 2.10],

$$\begin{aligned}\sum_{\mathbb{D}\in\mathfrak{P}(G|\mathrm{O}_p(G))/\sim_H}(-1)^{|\mathbb{D}|}f_{(\mathbb{D}_0,\theta_0)}(\mathbb{D})&=\sum_{\mathbb{D}\in\mathfrak{E}(G|\mathrm{O}_p(G))/\sim_H}(-1)^{|\mathbb{D}|}f_{(\mathbb{D}_0,\theta_0)}(\mathbb{D})\\&=\sum_{\mathbb{D}\in\mathfrak{R}(G)/\sim_H}(-1)^{|\mathbb{D}|}f_{(\mathbb{D}_0,\theta_0)}(\mathbb{D}).\end{aligned}$$

Note that whenever $\overline{(\mathbb{D},\theta)}\in(M_{+,(\mathbb{D}_0,\theta_0),\kappa})^H\cup(M_{-,(\mathbb{D}_0,\theta_0),\kappa})^H$, the p-chain satisfies $GH_{\mathbb{D}}=H$, accordingly

$$\begin{aligned}\sum_{\mathbb{D}\in\mathfrak{P}(G|\mathrm{O}_p(G))/\sim_G}(-1)^{|\mathbb{D}|}f_{(\mathbb{D}_0,\theta_0)}(\mathbb{D})&=\sum_{\mathbb{D}\in\mathfrak{E}(G|\mathrm{O}_p(G))/\sim_G}(-1)^{|\mathbb{D}|}f_{(\mathbb{D}_0,\theta_0)}(\mathbb{D})\\&=\sum_{\mathbb{D}\in\mathfrak{R}(G)/\sim_G}(-1)^{|\mathbb{D}|}f_{(\mathbb{D}_0,\theta_0)}(\mathbb{D}).\end{aligned}$$

Since $|(M_{+,(\mathbb{D}_0,\theta_0),\kappa_0})^H|=|(M_{-,(\mathbb{D}_0,\theta_0),\kappa_0})^H|$, this implies that $|(M_{+,(\mathbb{D}_0,\theta_0),\kappa})^H|=|(M_{-,(\mathbb{D}_0,\theta_0),\kappa})^H|$ for every $\kappa\in\{p,\mathrm{rad},\mathrm{elem}\}$. Accordingly $M_{+,(\mathbb{D}_0,\theta_0),\kappa}$ and $M_{-,(\mathbb{D}_0,\theta_0),\kappa}$ are equivalent as $\mathrm{Stab}_{A,\sim_G}(\mathbb{D}_0,\theta_0)$-sets. As this applies to all pairs $(\mathbb{D},\theta)\in\mathcal{C}^d(B)_+\cup\mathcal{C}^d(B)_-$, this proves the statement. □

A comparison with other inductive conditions in this area seems helpful to understand the origin of Conjecture 6.3 and the Inductive Condition for Dade's Conjecture.

6.11. Comparison with inductive conditions for other global/local conjectures

We compare here mainly with the results and statements around the McKay and the Alperin–McKay conjecture. A similar link to results around Alperin's weight conjecture seems possible but requires more adaptations since an equivalence relation on *modular* character triples from [Na98, Definition (8.25)] has to be introduced first.

In order to see the parallel between the above and the results in [Spä13a] and [NS14] we translate the inductive conditions into the language used here.

Recall that for a finite group G and a p-subgroup D of G we denote by $\mathrm{Irr}_0(G \mid D)$ the height zero characters of G that belong to a block with defect group D.

Proposition 6.12. *Let S be a nonabelian simple group, $\widehat{S}$ its universal covering group and $A := \widehat{S} \rtimes \mathrm{Aut}(\widehat{S})$. Then the following statements are equivalent:*

- (i) *The Inductive AM Condition from* [Spä13a, *Definition* 7.2] *holds for S and a prime p with respect to a noncentral defect group D.*
- (ii) *There exists an A_D-stable subgroup M with $\mathrm{N}_{\widehat{S}}(D) \leq M \lneq \widehat{S}$ and an A_D-equivariant bijection*

$$\Omega : \mathrm{Irr}_0(\widehat{S} \mid D) \to \mathrm{Irr}_0(M \mid D)$$

such that for every $\chi \in \mathrm{Irr}_0(\widehat{S} \mid D)$ and $\chi' := \Omega(\chi)$ the groups $Z := \ker(\chi_{\mathrm{Z}(\widehat{S})})$ and $\ker(\chi'_{\mathrm{Z}(\widehat{S})})$ coincide and

$$(A_\chi/Z, \widehat{S}/Z, \overline{\chi}) \sim_{\widehat{S}/Z} (MA_{D,\chi'}/Z, M/Z, \overline{\chi}'),$$

where $\overline{\chi}$ and $\overline{\chi}'$ are the characters corresponding to χ and χ'.

Proof. We use the reformulation of the Inductive AM Condition given in [KS16a, Definition 6.2], which is equivalent to the one in [Spä13a, Definition 7.2].

To prove that (i) implies (ii), we assume that the group M and the bijection Ω are given by the Inductive AM Condition. Furthermore for every $\chi \in \mathrm{Irr}_0(M \mid D)$ and $\chi' := \Omega(\chi)$ the groups $Z := \ker(\chi_{\mathrm{Z}(\widehat{S})})$ and $\ker(\chi'_{\mathrm{Z}(\widehat{S})})$ coincide. Moreover, for every $\chi \in \mathrm{Irr}_0(\widehat{S} \mid D)$ there exists a group $L := L(\chi)$ and characters $\widetilde{\theta}$ and $\widetilde{\theta}'$ such that:

- (a) For $Z := \ker(\chi_{\mathrm{Z}(X)})$ and $\overline{G} := \widehat{S}/Z$ the group L satisfies $\overline{G} \lhd L$, $A/\mathrm{C}_A(\overline{G}) = \mathrm{Aut}(\widehat{S})_\chi$ and $\mathrm{C}_L(\overline{X}) = \mathrm{Z}(L)$.
- (b) $\widetilde{\theta} \in \mathrm{Irr}(L)$ is an extension of the character $\theta \in \mathrm{Irr}(\overline{G})$ determined by χ.
- (c) For $\overline{D} := DZ/Z$ and $\overline{M} := MZ/Z$ let $\theta' \in \mathrm{Irr}(\overline{M})$ be the character defined by $\Omega(\chi) \in \mathrm{Irr}_0(M \mid D)$. Then $\widetilde{\theta}' \in \mathrm{Irr}(\overline{M}\mathrm{N}_L(\overline{D}))$ is an extension of θ'.
- (d) The characters satisfy

$$\mathrm{Irr}(\widetilde{\theta}_{\mathrm{C}_L(\overline{G})}) = \mathrm{Irr}(\widetilde{\theta}'_{\mathrm{C}_L(\overline{G})}),$$
$$\mathrm{bl}(\widetilde{\theta}_J) = \mathrm{bl}(\widetilde{\theta}'_{M\mathrm{N}_J(D)})^J \quad \text{for every } J \text{ with } \overline{G} \leq J \leq L.$$

According to Lemma 3.10 we see that then

$$(L, \overline{G}, \theta) \sim_{\overline{G}} (M\mathrm{N}_L(\overline{D}), \overline{M}, \theta').$$

In view of the description of L in (a) this implies

$$(A_\chi/Z, \widehat{S}/Z, \overline{\chi}) \sim_{\widehat{S}/Z} (MA_{D,\chi'}/Z, M/Z, \overline{\chi}')$$

according to Theorem 5.3. This proves that (i) implies (ii).

Assume now that (ii) holds for $\widehat{S}$, a group M and a bijection Ω. Using Theorem 4.1 together with Proposition 3.13(b) it is easy to construct, for every character $\chi \in \mathrm{Irr}_0(M \mid D)$, a group $L(\chi)$ with the above properties (a)–(d). $\square$

Moreover we like to mention that in [NS14, Theorem 7.1] a strengthening of the Alperin–McKay conjecture is shown to be a consequence of the Inductive AM Conditions. The above Character Triple Conjecture strengthens Dade's Conjecture in the same fashion.

7. p-chains in wreath products

The Inductive Condition for Dade's Conjecture for a simple nonabelian group S from Definition 6.7 is a statement on the representation theory of normalizers of p-chains in quasisimple groups associated with S. The aim of this section is to prove Theorem 7.2 and thereby show that the Inductive Condition for Dade's Conjecture for S implies the Character Triple Conjecture for covering groups of S^r (for some positive integer r) (see Theorem 7.1).

Theorem 7.1 is rephrased in terms of the universal covering group $\widehat{S}^r$ that is embedded as a normal subgroup in $\widehat{S}^r \rtimes (\mathrm{Aut}(\widehat{S}) \wr \mathfrak{S}_r)$ (see Theorem 7.2). The Character Triple Conjecture, as well as its reformulation in Theorem 7.2, can be divided into two parts: by the first part there exists an equivariant bijection, and according to the second part, associated character triples satisfy the equivalence relation (see (7.1)).

For the first part, the construction of the bijection, one determines all radical p-chains in $\widehat{S}^r$ (see Lemma 7.5). Note that radical p-chains of direct products are more practical than other kinds of p-chains and we can concentrate on them according to Proposition 6.10. The radical p-chains of $\widehat{S}^r$ are deduced from $\mathfrak{R}(\widehat{S})$ using a combinatorial description of those chains (see Proposition 7.8). This combinatorial tool is applied to prove (in Corollary 7.11) the existence of an equivariant bijection. In its construction, particular care is necessary to determine and compare normalizers of the chains involved.

At the end of this section one uses Theorems 5.1 and 5.2 to prove the required equality of the associated character triples.

Theorem 7.1. *Let S be a nonabelian simple group, r a positive integer, K a covering group of S^r, $\widetilde{z}$ the integer with $|\mathrm{Z}(K)|_p = p^{\widetilde{z}}$, $\overline{B} \in \mathrm{Bl}_{\mathrm{nc}}(K)$ and d an integer. Assume that the Inductive Condition for Dade's Conjecture from Definition 6.7 holds for S and d. Suppose that $K \lhd L$ for some finite group L. Let d' be an integer with $d' \leq d + \widetilde{z}$. Then the Character Triple Conjecture from 1.2 holds for $\overline{B}$ and d' with respect to $K \lhd L$.*

Note that K is a quotient of $\widehat{S}^r$, where $\widehat{S}$ is the universal covering group of S. The automorphisms of L induced on K correspond to ones in $\mathrm{Aut}(S^r) = \mathrm{Aut}(S) \wr \mathfrak{S}_r$. Hence the Character Triple Conjecture for $\overline{B}$ with respect to $K \lhd L$ is proven by considering the block B

of $\widehat{S}^r$ containing $\overline{B}$ and considering $\widehat{S}^r$ as a normal subgroup of $\widetilde{A} := \widehat{S}^r \rtimes \mathrm{Aut}(\widehat{S}^r)$. Then the above Theorem 7.1 is a direct consequence of the following more technical statement.

Since we concentrate on characters arising from radical p-chains we use the set

$$\mathcal{C}^d_{\mathrm{rad}}(B)_\epsilon := \{(\mathbb{D}, \theta) \mid \mathbb{D} \in \mathfrak{R}(G)_\epsilon \text{ and } \theta \in \mathrm{Irr}^d(B_{\mathbb{D}})\}$$

for any integer d, $\epsilon \in \{+, -\}$ and $B \in \mathrm{Bl}(G)$. In order to consider characters with various defects simultaneously we study the sets

$$\mathcal{C}^{\leq d}_{\mathrm{rad}}(B)_\epsilon := \bigcup_{d' \leq d} \mathcal{C}^{d'}_{\mathrm{rad}}(B)_\epsilon \quad \text{and} \quad \overline{\mathcal{C}^{\leq d}_{\mathrm{rad}}(B)_\epsilon} := \bigcup_{d' \leq d} \overline{\mathcal{C}^{d'}_{\mathrm{rad}}(B)_\epsilon}$$

for any integer d, $\epsilon \in \{+, -\}$ and $B \in \mathrm{Bl}(G)$.

Theorem 7.2. *Let S be a nonabelian simple group, r a positive integer, $\widehat{S}$ its universal covering group, $B \in \mathrm{Bl}_{\mathrm{nc}}(\widehat{S}^r)$, d an integer, z_0 the integer with $p^{z_0} = |\mathrm{Z}(\widehat{S})|_p$ and $\widetilde{A} := \widehat{S}^r \rtimes \mathrm{Aut}(\widehat{S}^r)$. Assume that the Inductive Condition for Dade's Conjecture from Definition 6.7 holds for S and d. Then there exists an $\widetilde{A}_B$-equivariant defect preserving bijection*

$$\Omega : \overline{\mathcal{C}^{\leq d + r z_0}(B)_+} \to \overline{\mathcal{C}^{\leq d + r z_0}(B)_-}$$

such that for every $(\mathbb{D}, \theta) \in \mathcal{C}^{\leq d + r z_0}(B)_+$, the pair $(\mathbb{D}', \theta') \in \Omega(\overline{(\mathbb{D}, \theta)})$ and the group $Z := \ker(\theta_{\mathrm{Z}(\widehat{S}^r)})$ satisfy $Z = \ker(\theta'_{\mathrm{Z}(\widehat{S}^r)})$ and

$$(\widetilde{A}_{\mathbb{D},\theta}/Z, (\widehat{S}^r)_{\mathbb{D}}/Z, \overline{\theta}) \sim_{\widehat{S}^r/Z} (\widetilde{A}_{\mathbb{D}',\theta'}/Z, \widehat{S}^r_{\mathbb{D}'}/Z, \overline{\theta}'), \tag{7.1}$$

where $\overline{\theta}$ and $\overline{\theta}'$ are the characters of $\widehat{S}^r_{\mathbb{D}}/Z$ and $\widehat{S}^r_{\mathbb{D}'}/Z$ lifting to θ and θ'.

While for $r = 1$ such a bijection is given by assumption, for higher r the construction of Ω is more involved and requires additional arguments. The proof will be given after Corollary 7.11. As mentioned above, there are two main difficulties to address. On the one hand, one has to understand the radical p-chains of $\widehat{S}^r$. This is done by using a combinatorial description of those chains in terms of paths in lattices. On the other hand, some effort is needed to see the equivalence of the character triples given in (7.1).

The arguments on paths in $\widehat{S}^r$ is based on the following observation on radical p-chains of direct products made by Eaton and Höfling [EH02].

Lemma 7.3. *Let G_1, G_2 be finite groups, $\mathbb{D} = (D_1 \lneq \cdots \lneq D_{|\mathbb{D}|}) \in \mathfrak{R}(G_1 \times G_2)$, and $\mathrm{Pr}_i : G_1 \times G_2 \to G_i$ be the canonical projection for $i = 1, 2$. Set $\mathbb{D}^{(i)} := \mathrm{Pr}_i(\mathbb{D})$ to be the chain $\mathrm{Pr}_i(D_1) \leq \ldots \leq \mathrm{Pr}_i(D_{|\mathbb{D}|})$, and define $\mathrm{Pr}^\circ_i(\mathbb{D})$ as the associated strictly increasing chain of p-groups obtained by deleting groups that occur twice. Then*

(a) $D_i = \mathrm{Pr}_1(D_i) \times \mathrm{Pr}_2(D_i)$ *for* $i = 1, \ldots, |\mathbb{D}|$,
(b) $\mathrm{Pr}^\circ_1(\mathbb{D}) \in \mathfrak{R}(G_1)$, $\mathrm{Pr}^\circ_2(\mathbb{D}) \in \mathfrak{R}(G_2)$,
(c) $\mathrm{N}_{G_1 \times G_2}(\mathbb{D}) = \mathrm{N}_{G_1}(\mathrm{Pr}^\circ_1(\mathbb{D})) \times \mathrm{N}_{G_2}(\mathrm{Pr}^\circ_2(\mathbb{D}))$.

Proof. This follows directly from [EH02, Lemma 3.1(a)]. □

Notation 7.4. Let $\widehat{S}$ be the universal covering group of S and $A := \widehat{S} \rtimes \mathrm{Aut}(\widehat{S})$. Moreover, for $\epsilon \in \{+,-\}$ let $\mathcal{C}^{\leq d}_{\mathrm{nc,rad}}(\widehat{S})_\epsilon := \bigcup_{B\in \mathrm{Bl}_{\mathrm{nc}}(\widehat{S})} \mathcal{C}^{\leq d}_{\mathrm{rad}}(B)_\epsilon$ and $\overline{\mathcal{C}^{\leq d}_{\mathrm{nc,rad}}(\widehat{S})_\epsilon} := \bigcup_{B\in \mathrm{Bl}_{\mathrm{nc}}(\widehat{S})} \overline{\mathcal{C}^{\leq d}_{\mathrm{rad}}(B)_\epsilon}$. Since S satisfies the Inductive Condition for Dade's Conjecture, using the arguments of Proposition 6.10 we see that there exists a defect preserving A-equivariant bijection

$$\Omega_0 : \overline{\mathcal{C}^{\leq d}_{\mathrm{nc,rad}}(\widehat{S})_+} \to \overline{\mathcal{C}^{\leq d}_{\mathrm{nc,rad}}(\widehat{S})_-}$$

such that for every $(\mathbb{D},\theta) \in \mathcal{C}^{\leq d}_{\mathrm{nc,rad}}(\widehat{S})_+$ and for $Z := \ker(\theta_{Z(\widehat{S})})$ every $(\mathbb{D}',\theta') \in \Omega_0(\overline{(\mathbb{D},\theta)})$ satisfies $Z = \ker(\theta'_{Z(\widehat{S})})$ and

$$(\widetilde{A}_{\mathbb{D},\theta}/Z, \widehat{S}_{\mathbb{D}}/Z, \overline{\theta}) \sim_{\widehat{S}/Z} (\widetilde{A}_{\mathbb{D}',\theta'}/Z, \widehat{S}_{\mathbb{D}'}/Z, \overline{\theta}'), \tag{7.2}$$

where $\overline{\theta}$ and $\overline{\theta}'$ are the characters of $\widehat{S}_{\mathbb{D}}/Z$ and $\widehat{S}_{\mathbb{D}'}/Z$ corresponding to θ and θ'.

Let $\mathbb{T}_\epsilon \subseteq \mathcal{C}^{\leq d}_{\mathrm{nc,rad}}(\widehat{S})_\epsilon$ be a complete set of $\mathrm{Aut}(\widehat{S})$-representatives in $\mathcal{C}^{\leq d}_{\mathrm{nc,rad}}(\widehat{S})_\epsilon$ such that for every $x \in \mathbb{T}_+$ there exists a unique $x' \in \mathbb{T}_-$ with $\Omega_0(\overline{x}) = \overline{x}'$. We set $|x|$ to be $|\mathbb{D}|$ whenever $x = (\mathbb{D},\theta) \in \mathbb{T}_+ \cup \mathbb{T}_-$.

Using $\mathbb{T}_+$ and $\mathbb{T}_-$ one can now describe some complete set of $\mathrm{Aut}(\widehat{S})^r$-orbit representatives in a certain subset of $\mathcal{C}^{\leq d}_{\mathrm{nc,rad}}(\widehat{S}^r)$.

Lemma 7.5. *For every $c = (c_1,\ldots,c_r) \in (\mathbb{T}_+)^r$ let $c' = (c'_1,\ldots,c'_r) \in (\mathbb{T}_-)^r$ with $c'_i \in \Omega_0(\overline{c}_i)$. Let $(\mathbb{D}^{(i)},\theta_i) = c_i$ and $(\mathbb{D}^{(-i)},\theta_{-i}) = c'_i$ for $1 \leq i \leq r$. In addition let $\mathcal{C}_{c,c'}$ be the set of chains $(\widetilde{\mathbb{D}},\widetilde{\theta}) \in \mathcal{C}^{\leq d}_{\mathrm{nc,rad}}(\widehat{S}^r)$ such that*

- $\mathrm{Pr}^\circ_i(\widetilde{\mathbb{D}}) \in \{\mathbb{D}^{(i)},\mathbb{D}^{(-i)}\}$ *for every* $1 \leq i \leq r$,
- $\psi_i \in \{\theta_i,\theta_{-i}\}$, *where* $\psi_i \in \mathrm{Irr}(\mathrm{N}_{\widehat{S}}(\mathrm{Pr}^\circ_i(\widetilde{\mathbb{D}})))$ *is defined by* $\widetilde{\theta} = \psi_1 \times \cdots \times \psi_r$,
- $(\mathrm{Pr}^\circ_i(\widetilde{\mathbb{D}}),\psi_i) \in \{c_i,c'_i\}$.

Then $\dot{\bigcup}_{c\in(\mathbb{T}_+)^r} \mathcal{C}_{c,c'}$ is a complete set of $\mathrm{Aut}(\widehat{S})^r$-representatives in $\mathcal{C}^{\leq d}_{\mathrm{nc,rad}}(\widehat{S}^r)'$, where

$$\mathcal{C}^{\leq d}_{\mathrm{nc,rad}}(\widehat{S}^r)' = \bigcup_{(b_1,\ldots,b_r)\in(\mathrm{Bl}_{\mathrm{nc}}(\widehat{S}))^r} \mathcal{C}^{\leq d}_{\mathrm{nc,rad}}(b_1\times\cdots\times b_r).$$

Proof. This follows from Lemma 7.3. □

In the next step we introduce certain combinatorial objects, which we later show to be in bijection with the elements of $\mathcal{C}_{c,c'}$.

Notation 7.6. Let $\overline{k} \in (\mathbb{Z}_{\geq 0})^r$. We call

$$\mathbb{P} = \{\overline{x} \in \mathbb{Z}^r \mid 0 \leq x_i \leq k_i\}$$

the *standard polyhedron associated with* $\overline{k}$. Let $\overline{a},\overline{a}' \in (\mathbb{Z}_{\geq 0})^r$ with $2 \mid a_i$ and $2 \nmid a'_i$ for all $1 \leq i \leq r$. A *standard* $\overline{a},\overline{a}'$*-polyhedron* is a standard polyhedron associated with $\overline{k} \in \mathbb{Z}^r$ where $k_i \in \{a_i,a'_i\}$ for every $1 \leq i \leq r$.

A *lattice path* $\overline{p}$ *in the standard polyhedron* $\mathbb{P}$ *associated with* $\overline{k}$ is a sequence of pairwise distinct points $\overline{x}^{(0)} = (0, \dots, 0), \dots, \overline{x}^{(l)} = \overline{k} \in \mathbb{P}$ such that $0 \le x_i^{(j+1)} - x_i^{(j)} \le 1$ for all $1 \le i \le r$ and $1 \le j \le l-1$. The integer l is called the *length* of $\overline{p}$, abbreviated $|\overline{p}|$. (Note that this might differ from other definitions of such paths.) We call such a path $\overline{p}$ *odd* if $2 \nmid |\overline{p}|$, and *even* if $2 \mid |\overline{p}|$. For a given standard polyhedron $\mathbb{P}$ we define $\mathcal{L}_{\mathbb{P}}$, $\mathcal{L}_{\mathbb{P},+}$, and $\mathcal{L}_{\mathbb{P},-}$ as the sets of all, all even and all odd lattice paths in $\mathbb{P}$ respectively. Furthermore the lattice path in $\mathbb{P}$ that contains the points $(0, 0, \dots, 0)$, $(k_1, 0, \dots, 0)$, $(k_1, k_2, 0, \dots, 0), \dots, (k_1, k_2, \dots, k_{r-1}, 0)$ and $\overline{k}$ is called the *representative lattice path* of $\mathbb{P}$, denoted by $\overline{p}_{\text{rep}}$. (This name will be justified in Lemma 7.9.) Note that the length of this path is congruent to $\sum_i k_i$ in $\mathbb{Z}/2\mathbb{Z}$.

Analogously let $\mathcal{L}_{\overline{a},\overline{a}',+}$ be the set of all even lattice paths of $\overline{a}, \overline{a}'$-polyhedra, and $\mathcal{L}_{\overline{a},\overline{a}',-}$ be the set of all odd lattice paths of $\overline{a}, \overline{a}'$-polyhedra. Let $\mathcal{L}_{\overline{a},\overline{a}'} := \mathcal{L}_{\overline{a},\overline{a}',+} \cup \mathcal{L}_{\overline{a},\overline{a}',-}$.

Example 7.7. Let $r = 2$ and $\overline{k} = (1, 1) \in \mathbb{Z}^2$. Then the standard polyhedron associated with $\overline{k}$ has only the following paths:

- $\overline{p}_1$: $(0, 0), (0, 1), (1, 1)$,
- $\overline{p}_2$: $(0, 0), (1, 0), (1, 1)$,
- $\overline{p}_3$: $(0, 0), (1, 1)$.

In general one can picture paths as in Figure 1 where the path has length 6.

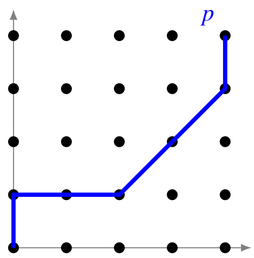

Fig. 1. A lattice path in the standard polyhedron associated with (4, 4) of length 6

Proposition 7.8. *Let* $c \in (\mathbb{T}_+)^r$, *and let* $c' \in (\mathbb{T}_-)^r$ *be associated with* c *as in Lemma 7.5. Let* $\overline{a} = (a_1, \dots, a_r) \in \mathbb{Z}^r$ *with* $a_i = |c_i|$ *and* $\overline{a}' = (a'_1, \dots, a'_r) \in \mathbb{Z}^r$ *with* $a'_i = |c'_i|$. *Then there is a bijection*

$$\Upsilon : \mathcal{C}_{c,c'} \to \mathcal{L}_{\overline{a},\overline{a}'}$$

such that for $(\widetilde{\mathbb{D}}, \widetilde{\theta})$ *the associated path* $\overline{p} := \Upsilon((\widetilde{\mathbb{D}}, \widetilde{\theta}))$ *passes through the points* $\overline{x}^{(j)} \in \mathbb{Z}^r$ $(0 \le j \le |\widetilde{\mathbb{D}}|)$ *with*

$$\mathrm{Pr}_i(\widetilde{D}_j) = D^{(i)}_{x_i^{(j)}} \quad \text{for every } 1 \le i \le r,$$

where $\widetilde{\mathbb{D}} = (\widetilde{D}_0 \lneq \widetilde{D}_1 \lneq \cdots \lneq \widetilde{D}_{|\widetilde{\mathbb{D}}|})$ *and* $\widetilde{\mathbb{D}}^{(i)} := \mathrm{Pr}_i^\circ(\widetilde{\mathbb{D}}) = (D_0^{(i)} \lneq D_1^{(i)} \lneq \cdots \lneq D^{(i)}_{|\widetilde{\mathbb{D}}^{(i)}|})$. *The length of* $\widetilde{\mathbb{D}}$ *coincides with the length of* $\Upsilon(\overline{(\widetilde{\mathbb{D}}, \widetilde{\theta})})$.

Proof. This follows from the definitions. □

The number of lattice paths in a fixed polyhedron $\mathbb{P}$ is given by Delannoy numbers for $r = 2$ and $\overline{k} = (k_1, k_1)$ (see for example [KR91, Section 2]). But we are more interested here in the comparison of the numbers of odd and even lattice paths in a standard polyhedron.

Lemma 7.9. *Let $\mathbb{P}$ be the standard polyhedron associated with $\overline{k} \in (\mathbb{Z}_{\geq 0})^r$, and $\overline{p}_{\mathrm{rep}}$ its representative lattice path. Then*

$$|\mathcal{L}_{\mathbb{P},+}| - |\mathcal{L}_{\mathbb{P},-}| = (-1)^{\sum_i k_i} = (-1)^{|\overline{p}_{\mathrm{rep}}|}.$$

Proof. This follows by induction on $\sum_i k_i$. By induction the number of even lattice paths minus the number of odd lattice paths whose penultimate point differs from the last in exactly t fixed entries is $(-1)^{(\sum k_i)-t-1}$. Thus the difference we are looking for is

$$\sum_{t=1}^{r} \binom{r}{t} (-1)^{(\sum k_i)-t-1} = (-1)^{\sum k_i} \left(\sum_{t=1}^{r} \binom{r}{t} (-1)^{t-1} \right) = (-1)^{\sum k_i},$$

since $\sum_{t=0}^{r} \binom{r}{t} (-1)^{t-1} = 0$. □

There is a natural action of $\mathfrak{S}_r$ on $\mathbb{Z}^r$. For given $\overline{a}, \overline{a}' \in (\mathbb{Z}_{\geq 0})^r$, any group $\mathfrak{Y} \leq (\mathfrak{S}_r)_{\overline{a},\overline{a}'}$ acts on $\mathcal{L}_{\overline{a},\overline{a}',+}$ and $\mathcal{L}_{\overline{a},\overline{a}',-}$. For $\overline{p} \in \mathcal{L}$ we denote by $(\mathfrak{Y})_{\overline{p}}$ its stabilizer in $\mathfrak{Y}$, which is a Young subgroup in $\mathfrak{Y}$ if $\mathfrak{Y}$ is a Young subgroup of $\mathfrak{S}_r$.

Proposition 7.10. *Let $\overline{a}, \overline{a}' \in (\mathbb{Z}_{\geq 0})^r$ with $2 \mid a_i$ and $2 \nmid a_i'$ for every $1 \leq i \leq r$. For every Young subgroup $\mathfrak{Y}$ of $(\mathfrak{S}_r)_{\overline{a},\overline{a}'}$ there exists a $\mathfrak{Y}$-equivariant bijection*

$$\Pi_{\overline{a},\overline{a}'} : \mathcal{L}_{\overline{a},\overline{a}',+} \to \mathcal{L}_{\overline{a},\overline{a}',-}.$$

In particular $(\mathfrak{Y})_{\overline{p}} = (\mathfrak{Y})_{\Pi_{\overline{a},\overline{a}'}(\overline{p})}$ for every $\overline{p} \in \mathcal{L}_{\overline{a},\overline{a}',+}$.

Proof. Let $\mathcal{L} := \mathcal{L}_{\overline{a},\overline{a}'}$, $\mathcal{L}_+ := \mathcal{L}_{\overline{a},\overline{a}',+}$ and $\mathcal{L}_- := \mathcal{L}_{\overline{a},\overline{a}',-}$. According to [Is08, Lemma 3.33] it is sufficient to prove that every subgroup $Y \leq \mathfrak{Y}$ has the same number of fixed points on $\mathcal{L}_+$ and on $\mathcal{L}_-$. Since the stabilizer of a lattice path in $\mathfrak{Y}$ is always a Young subgroup, the number of fixed points of a group $Y \leq \mathfrak{Y}$ is exactly the one for the minimal Young subgroup that contains Y. Accordingly it is sufficient to compute the fixed points of Y on $\mathcal{L}_+$ and on $\mathcal{L}_-$ for Young subgroups $Y \leq \mathfrak{Y}$.

In a first step we prove $|\mathcal{L}_+| = |\mathcal{L}_-|$. According to Lemma 7.9 one has only to show that the number of $\overline{a}, \overline{a}'$-polyhedra with an even representative path coincides with the number of $\overline{a}, \overline{a}'$-polyhedra having an odd representative path. The $\overline{a}, \overline{a}'$-polyhedra with an even representative path correspond to vectors $\overline{k} \in \mathbb{Z}^r$ with $k_i \in \{a_i, a_i'\}$ and $(-1)^{\sum_i k_i} = 1$, i.e., the set $\{i \mid k_i = a_i'\}$ has even cardinality. There are $\sum_{0 \leq 2j \leq r} \binom{r}{2j}$ such polyhedra. On the other hand, the number of $\overline{a}, \overline{a}'$-polyhedra with an odd representative path is $\sum_{0 \leq 2j-1 \leq r} \binom{r}{2j-1} = -(1-1)^r + \sum_{0 \leq 2j \leq r} \binom{r}{2j}$. This implies $|\mathcal{L}_+| = |\mathcal{L}_-|$.

We prove the proposition by induction on r. For $r = 1$ the statement is obviously true since $\mathcal{L}$ is formed by one path of length a_1 and one of length a_1' respectively.

For $r > 1$ we consider the paths that are invariant under a given Young subgroup $Y \leq \mathfrak{Y}$. Let $j_1, \ldots, j_{r'}$ be a full set of representatives of Y-orbits on $\{1, \ldots, r\}$ and let

$\mathcal{L}^Y$ be the set of Y-invariant paths in $\mathcal{L}$. For $\overline{b} \in \mathbb{Z}^{r'}$ with $b_i = a_{j_i}$ and $\overline{b}' \in \mathbb{Z}^{r'}$ with $b'_i = a'_{j_i}$, let $f : \mathcal{L}^Y \to \mathcal{L}_{\overline{b},\overline{b}'}$ be the map naturally induced by

$$(x_1, \dots, x_r) \mapsto (x_{j_1}, \dots, x_{j_{r'}}).$$

One sees that f is bijective. In addition it preserves parity.

As shown above, the number of even paths in $\mathcal{L}_{\overline{b},\overline{b}'}$ coincides with the number of odd paths in this set. This proves that $|\mathcal{L}_+ \cap \mathcal{L}^Y| = |\mathcal{L}_- \cap \mathcal{L}^Y|$. □

Proposition 7.10 translates into the following statement on $\mathcal{C}_{c,c'}$.

Corollary 7.11. *Let $c \in (\mathbb{T}_+)^r$, and let $c' \in (\mathbb{T}_-)^r$ be associated with c as in Lemma 7.5. Let $\mathfrak{Y} := (\mathfrak{S}_r)_{c,c'}$ be the stabilizer of c and c'.*

(a) *There is a $\mathfrak{Y}$-equivariant defect preserving bijection*

$$\Omega_{c,c'} : \mathcal{C}_{c,c',+} \to \mathcal{C}_{c,c',-}.$$

(b) *Every $(\widetilde{\mathbb{D}}, \widetilde{\theta}) \in \mathcal{C}_{c,c',+}$ and $(\widetilde{\mathbb{D}}', \widetilde{\theta}') = \Omega_{c,c'}((\widetilde{\mathbb{D}}, \widetilde{\theta}))$ satisfy $\ker(\widetilde{\theta}_{Z(\widehat{S})}) = \ker(\widetilde{\theta}'_{Z(\widehat{S})})$ and*

$$((A \wr \mathfrak{S}_r)_{\widetilde{\mathbb{D}},\widetilde{\theta}}/Z_0, \widehat{S}^r_{\mathbb{D}}/Z_0, \overline{\widetilde{\theta}}) \sim_{\widehat{S}^r/Z_0} ((A \wr \mathfrak{S}_r)_{\widetilde{\mathbb{D}}',\widetilde{\theta}'}/Z_0, \widehat{S}^r_{\mathbb{D}'}/Z_0, \overline{\widetilde{\theta}'}),$$

where $Z_0 = \langle \ker(\widetilde{\theta}_{Z(\widehat{S}^{(i)})}) \mid 1 \le i \le r \rangle$ and $\widehat{S}^{(i)}$ denotes the subgroup

$$\langle 1_{\widehat{S}} \rangle \times \cdots \times \langle 1_{\widehat{S}} \rangle \times \widehat{S} \times \langle 1_{\widehat{S}} \rangle \times \cdots \times \langle 1_{\widehat{S}} \rangle$$

of $\widehat{S}^r$, where the ith factor is isomorphic to $\widehat{S}$.

Proof. Let $\overline{a} = (a_1, \dots, a_r) \in \mathbb{Z}^r$ with $a_i = |c_i|$, and $\overline{a}' = (a'_1, \dots, a'_r) \in \mathbb{Z}^r$ with $a'_i = |c'_i|$.

By definition, the bijection Υ from Proposition 7.8 is $\mathfrak{Y}$-equivariant and preserves parity. Hence the bijection $\Pi_{\overline{a},\overline{a}'}$ from Proposition 7.10 gives the bijection $\Omega_{c,c'}$ required in (a).

In part (b) the character $\widetilde{\theta}$ is of the form $\psi_1 \times \cdots \times \psi_r \in \mathrm{Irr}(\mathrm{N}_{\widehat{S}}(\mathbb{D}^{(1)}) \times \cdots \times \mathrm{N}_{\widehat{S}}(\mathbb{D}^{(r)}))$ with $(\mathbb{D}^{(i)}, \psi_i) \in \{c_i, c'_i\}$ for every $1 \le i \le r$ and $\mathbb{D}^{(i)} := \mathrm{Pr}^\circ_i(\widetilde{\mathbb{D}})$ (see Lemma 7.5). Furthermore according to Lemma 7.3 we have

$$\mathrm{N}_{\widehat{S}^r}(\widetilde{\mathbb{D}}) = \mathrm{N}_{\widehat{S}}(\mathbb{D}^{(1)}) \times \cdots \times \mathrm{N}_{\widehat{S}}(\mathbb{D}^{(r)}).$$

Analogously $\widetilde{\theta}'$ is of the form $\psi'_1 \times \cdots \times \psi'_r \in \mathrm{Irr}(\mathrm{N}_{\widehat{S}}(\mathbb{D}'^{(1)}) \times \cdots \times \mathrm{N}_{\widehat{S}}(\mathbb{D}'^{(r)}))$ with $(\mathbb{D}'^{(i)}, \psi'_i) \in \{c_i, c'_i\}$ for every $1 \le i \le r$, where $\mathbb{D}'^{(i)} := \mathrm{Pr}^\circ_i(\widetilde{\mathbb{D}}')$. Lemma 7.3 implies

$$\mathrm{N}_{\widehat{S}^r}(\widetilde{\mathbb{D}}) = \mathrm{N}_{\widehat{S}}(\mathbb{D}'^{(1)}) \times \cdots \times \mathrm{N}_{\widehat{S}}(\mathbb{D}'^{(r)}).$$

By the choice of c', for every $1 \le i \le r$ and $Z_i := \ker(\psi_{i,Z(\widehat{S})})$ we have the equality $Z_i = \ker(\psi'_{i,Z(\widehat{S})})$ and the equivalence

$$(A_{(\mathbb{D}^{(i)},\psi_i)}/Z_i, \widehat{S}_{\mathbb{D}^{(i)}}/Z_i, \overline{\psi}_i) \sim_{\widehat{S}/Z_i} (A_{(\mathbb{D}'^{(i)},\psi'_i)}/Z_i, \widehat{S}_{\mathbb{D}'^{(i)}}/Z_i, \overline{\psi}'_i),$$

where $\overline{\psi}_i$ and $\overline{\psi}'_i$ are the characters of $\widehat{S}_{\mathbb{D}(i)}/Z_i$ and $\widehat{S}_{\mathbb{D}'(i)}/Z_i$ corresponding to ψ_i and ψ'_i. By the choice of $\mathbb{T}_+$ and $\mathbb{T}_-$ we see that

$$(A \wr \mathfrak{S}_r)_{(\widetilde{\mathbb{D}},\widetilde{\theta})} = A^r_{(\widetilde{\mathbb{D}},\widetilde{\theta})} \rtimes (\mathfrak{S}_r)_{(\widetilde{\mathbb{D}},\widetilde{\theta})}.$$

Note that for $Z_0 = \langle Z_i \mid 1 \leq i \leq r\rangle$, $Z_0 \lhd (A \wr \mathfrak{S}_r)_{(\widetilde{\mathbb{D}},\widetilde{\theta})}$ and $Z_0 \lhd (A \wr \mathfrak{S}_r)_{(\widetilde{\mathbb{D}}',\widetilde{\theta}')}$. The combination of Theorems 5.1 and 5.2 implies that then

$$((A \wr \mathfrak{S}_r)_{\widetilde{\mathbb{D}},\widetilde{\theta}}/Z_0, \widehat{S}^r_{\widetilde{\mathbb{D}}}/Z_0, \overline{\widetilde{\theta}}) \sim_{\widehat{S}^r/Z_0} ((A \wr \mathfrak{S}_r)_{\widetilde{\mathbb{D}}',\widetilde{\theta}'}/Z_0, \widehat{S}^r_{\widetilde{\mathbb{D}}'}/Z_0, \overline{\widetilde{\theta}'}),$$

where $\overline{\widetilde{\theta}}$ and $\overline{\widetilde{\theta}'}$ are the characters of $\widehat{S}^r_{\widetilde{\mathbb{D}}}/Z_0$ and $\widehat{S}^r_{\widetilde{\mathbb{D}}'}/Z_0$ corresponding to $\widetilde{\theta}$ and $\widetilde{\theta}'$. □

This finally enables us to construct the bijection from Theorem 7.2. We first prove it in a simplified situation.

Theorem 7.12. *Let S be a nonabelian simple group, r a positive integer, $\widehat{S}$ its universal covering group, $B \in \mathrm{Bl}_{\mathrm{nc}}(\widehat{S}^r)$, d an integer, and $\widetilde{A} := \widehat{S}^r \rtimes \mathrm{Aut}(\widehat{S}^r)$. Assume that the Inductive Condition for Dade's Conjecture from Definition 6.7 holds for S and d. If $p \nmid |\mathrm{Z}(\widehat{S})|$, then there exists an $\widetilde{A}_B$-equivariant defect preserving bijection*

$$\Omega : \overline{\mathcal{C}^{\leq d+rz_0}(B)_+} \to \overline{\mathcal{C}^{\leq d+rz_0}(B)_-}$$

such that for every $(\mathbb{D},\theta) \in \mathcal{C}^{\leq d+rz_0}_{\mathrm{rad}}(B)_+$, the pair $(\mathbb{D}',\theta') \in \Omega(\overline{(\mathbb{D},\theta)})$ and the group $Z := \ker(\theta_{\mathrm{Z}(\widehat{S}^r)})$ satisfy $Z = \ker(\theta'_{\mathrm{Z}(\widehat{S}^r)})$ and

$$(\widetilde{A}_{\mathbb{D},\theta}/Z, (\widehat{S}^r)_{\mathbb{D}}/Z, \overline{\theta}) \sim_{\widehat{S}^r/Z} (\widetilde{A}_{\mathbb{D}',\theta'}/Z, \widehat{S}^r_{\mathbb{D}'}/Z, \overline{\theta}'), \tag{7.3}$$

where $\overline{\theta}$ and $\overline{\theta}'$ are the characters of $\widehat{S}^r_{\mathbb{D}}/Z$ and $\widehat{S}^r_{\mathbb{D}'}/Z$ lifting to θ and θ'.

Proof. Note that the arguments of the proof of Proposition 6.10 imply that it is sufficient to construct an $\widetilde{A}_B$-equivariant defect preserving bijection

$$\Omega : \overline{\mathcal{C}^{\leq d+rz_0}_{\mathrm{rad}}(B)_+} \to \overline{\mathcal{C}^{\leq d+rz_0}_{\mathrm{rad}}(B)_-}$$

with the requirements as above.

Let $b_i \in \mathrm{Bl}(\widehat{S})$ be such that $B = b_1 \times \cdots \times b_r$. Without loss of generality assume $b_i \in \mathrm{Bl}_{\mathrm{nc}}(\widehat{S})$ for $1 \leq i \leq r'$ and $b_i \notin \mathrm{Bl}_{\mathrm{nc}}(\widehat{S})$ for $i > r'$ for some r'. Let $A := \widehat{S} \rtimes \mathrm{Aut}(\widehat{S})$.

First we assume that $r' = r$. Then using the bijections $\Omega_{c,c'}$ for $c \in (\mathbb{T}_-)^r$ and $c' \in (\mathbb{T}_+)^r$ associated with c as in Lemma 7.5 we obtain an $A \wr \mathfrak{S}_r$-equivariant defect preserving bijection

$$\Omega' : \overline{\mathcal{C}^{\leq d}_{\mathrm{nc,rad}}(\widehat{S}^r)_+} \to \overline{\mathcal{C}^{\leq d}_{\mathrm{nc,rad}}(\widehat{S}^r)_-}$$

such that every $(\widetilde{\mathbb{D}}_1,\widetilde{\theta}_1) \in \mathcal{C}^{\leq d}_{\mathrm{nc,rad}}(\widehat{S}^r)_+$ and every $(\widetilde{\mathbb{D}}_2,\widetilde{\theta}_2) \in \Omega'(\overline{(\widetilde{\mathbb{D}}_1,\widetilde{\theta}_1)})$ satisfy

$$((A \wr \mathfrak{S}_r)_{\widetilde{\mathbb{D}}_1,\widetilde{\theta}_1}/Z', \widehat{S}^r_{\widetilde{\mathbb{D}}_1}/Z', \overline{\widetilde{\theta}}_1) \sim_{\widehat{S}^r/Z'} ((A \wr \mathfrak{S}_r)_{\widetilde{\mathbb{D}}_2,\widetilde{\theta}_2}/Z', \widehat{S}^r_{\widetilde{\mathbb{D}}_2}/Z', \overline{\widetilde{\theta}}_2), \tag{7.4}$$

where Z', $\widetilde{\widetilde{\theta}}_1$ and $\widetilde{\widetilde{\theta}}_2$ are defined as in 7.11. Note that for any block $B \in \mathrm{Bl}(\widehat{S}^r)$ that has a defect group D with $D \cap \widehat{S}^{(i)} \not\leq \mathrm{Z}(\widehat{S})$ for every $1 \leq i \leq r$, the bijections Ω' can be restricted to $\overline{\mathcal{C}^{\leq d}_{\mathrm{rad}}(B)_+}$ and $\Omega'(\overline{\mathcal{C}^{\leq d}_{\mathrm{rad}}(B)_+}) = \overline{\mathcal{C}^{\leq d}_{\mathrm{rad}}(B)_-}$. This proves the existence of the bijection in the case where $r' = r$, but it remains to show that (7.1) holds with this choice of Ω'.

In the next step we assume $r' \neq r$. Let a_0 be the integer with $p^{a_0} = |\mathrm{Z}(\widehat{S})|_p$. Recall that according to Lemma 6.5 the set $\mathcal{C}^{\leq d}(b_i)_-$ is empty and $\mathcal{C}^{\leq d}(b_i)_+ = \mathcal{C}^{a_0}(b_i)_+$ contains just the characters of b_i whenever $r' < i \leq r$. Let $b'' := b_{r'+1} \times \cdots \times b_r$ and let

$$\Omega'' : \overline{\mathcal{C}^{\leq d}(b'')_+} \to \overline{\mathcal{C}^{\leq d}(b'')_+}$$

be the identity map. For $r'' := r - r'$, $(\widetilde{\mathbb{D}}_1, \widetilde{\theta}_1) \in \mathcal{C}^{\leq d}(b'')_+$ and $(\widetilde{\mathbb{D}}_2, \widetilde{\theta}_2) := \Omega'(\overline{(\widetilde{\mathbb{D}}_1, \widetilde{\theta}_1)})$ we have

$$((A \wr \mathfrak{S}_{r''})_{\widetilde{\mathbb{D}}_1, \widetilde{\theta}_1}/Z'', (\widehat{S}^{r''})_{\widetilde{\mathbb{D}}_1/Z''}, \widetilde{\widetilde{\theta}}_1) \sim_{\widehat{S}^{r''}/Z''} ((A \wr \mathfrak{S}_{r''})_{\widetilde{\mathbb{D}}_2, \widetilde{\theta}_2}/Z'', (\widehat{S}^{r''})_{\widetilde{\mathbb{D}}_2}/Z'', \widetilde{\widetilde{\theta}}_2), \quad (7.5)$$

where $Z'' := \ker(\widetilde{\theta}_{\mathrm{Z}(\widehat{S}^{r''})})$ and the characters $\widetilde{\widetilde{\theta}}_1$ and $\widetilde{\widetilde{\theta}}_2$ of $(\widehat{S}^{r''})_{\mathbb{D}}/Z''$ and $(\widehat{S}^{r''})_{\mathbb{D}'}/Z''$ correspond to $\widetilde{\theta}_1$ and $\widetilde{\theta}_2$. (The equivalence of the character triples follows from Lemma 3.8.)

Let $b' := b_1 \times \cdots \times b_{r''}$ and $d' := d - r''a_0$. Now the elements of $\mathcal{C}^{\leq d}_{\mathrm{rad}}(B)$ can be built from $\mathcal{C}^{\leq d'}_{\mathrm{rad}}(b')$ and $\mathcal{C}^{r''a_0}(b'')_+$: Let $(\mathbb{D}', \theta') \in \mathcal{C}^{\leq d'}_{\mathrm{rad}}(b')$ and $(\mathbb{D}'', \theta'') \in \mathcal{C}^{r''a_0}_{\mathrm{rad}}(b'')_+$. Then

$$\mathbb{D}' = (D'_0 \lneq D'_1 \lneq \cdots \lneq D'_{|\mathbb{D}|})$$

and $\mathbb{D}'' = (\mathrm{O}_p(\widehat{S}^{r''}))$ for suitable p-groups D'_i $(1 \leq i \leq |\mathbb{D}'|)$.

Let $\widetilde{D}_i := D'_i \times \mathrm{O}_p(\widehat{S}^{r''})$ for $0 \leq i \leq |\mathbb{D}'|$. Then

$$\widetilde{\mathbb{D}} := (\widetilde{D}_0 \lneq \widetilde{D}_1 \lneq \cdots \lneq \widetilde{D}_{|\mathbb{D}'|}) \in \mathfrak{R}(\widehat{S}^r)$$

with $|\widetilde{\mathbb{D}}| = |\mathbb{D}'|$. This chain satisfies

$$\mathrm{N}_{\widehat{S}^r}(\widetilde{\mathbb{D}}) = \mathrm{N}_{\widehat{S}^{r'}}(\mathbb{D}') \times \widehat{S}^{r''}.$$

The character $\widetilde{\theta} := \theta' \times \theta''$ is well-defined and $\widetilde{\theta} \in \mathrm{Irr}(\mathrm{N}_{\widehat{S}^r}(\widetilde{\mathbb{D}}))$. Straightforward calculations show $(\widetilde{\mathbb{D}}, \widetilde{\theta}) \in \mathcal{C}_{\mathrm{rad}}(B)$. Moreover $d(\widetilde{\theta}) = d(\theta') + d(\theta'')$. In the following we denote $(\widetilde{\mathbb{D}}, \widetilde{\theta})$ by $(\mathbb{D}', \theta') \times (\mathbb{D}'', \theta'')$. This gives a bijection

$$\mathcal{C}^{\leq d'}_{\mathrm{rad}}(b') \times \mathcal{C}^{r''a_0}(b'')_+ \to \mathcal{C}^{\leq d}_{\mathrm{rad}}(B) \quad \text{with} \quad ((\mathbb{D}', \theta'), (\mathbb{D}'', \theta'')) \mapsto (\mathbb{D}', \theta') \times (\mathbb{D}'', \theta''),$$

that preserves the length of chains and adds the defect of the characters. It also induces a bijection

$$\overline{\mathcal{C}^{\leq d'}_{\mathrm{rad}}(b')} \times \overline{\mathcal{C}^{r''a_0}(b'')_+} \to \overline{\mathcal{C}^{\leq d}_{\mathrm{rad}}(B)} \quad \text{with} \quad (\overline{(\mathbb{D}', \theta')}, \overline{(\mathbb{D}'', \theta'')}) \mapsto \overline{(\mathbb{D}', \theta') \times (\mathbb{D}'', \theta'')}.$$

In such a situation we denote $\overline{(\mathbb{D}', \theta') \times (\mathbb{D}'', \theta'')}$ also by $\overline{(\mathbb{D}', \theta')} \times \overline{(\mathbb{D}'', \theta'')}$.

Let $\Omega : \overline{\mathcal{C}^{\leq d}_{\mathrm{rad}}(B)_+} \to \overline{\mathcal{C}^{\leq d}_{\mathrm{rad}}(B)_-}$ with

$$\overline{(\mathbb{D}', \theta')} \times \overline{(\mathbb{D}'', \theta'')} \mapsto \Omega'(\overline{(\mathbb{D}', \theta')}) \times \Omega''(\overline{(\mathbb{D}'', \theta'')}).$$

Then by definition we see that Ω is $(A \wr \mathfrak{S}_r)_B$-equivariant and defect preserving.

Let $(\widetilde{\mathbb{D}}_1, \widetilde{\theta}_1) \in \mathcal{C}^{\leq d}_{\mathrm{rad}}(B)$ and $(\widetilde{\mathbb{D}}_2, \widetilde{\theta}_2) \in \Omega(\overline{(\widetilde{\mathbb{D}}_1, \widetilde{\theta}_1)})$. Let $(\mathbb{D}'_1, \theta'_1) \in \mathcal{C}^{\leq d'}_{\mathrm{rad}}(b')$ and $(\mathbb{D}''_1, \theta''_1) \in \mathcal{C}^{r'' a_0}(b'')$ be such that $(\mathbb{D}'_1, \theta'_1) \times (\mathbb{D}''_1, \theta''_1) = (\widetilde{\mathbb{D}}_1, \widetilde{\theta}_1)$. Let $Z'_0 \leq \ker(\theta'_1)$ be associated with θ'_1 as in Lemma 7.11 and $Z'' := \ker(\theta''_1)$. Let $Z_0 := Z' \times Z''$. According to Theorem 5.1, (7.4) and (7.5),

$$(\widetilde{A}_{\widetilde{\mathbb{D}}_1, \widetilde{\theta}_1}/Z_0, (\widehat{S}^r)_{\widetilde{\mathbb{D}}_1}/Z_0, \overline{\widetilde{\theta}}_1) \sim_{\widehat{S}^r/Z_0} (\widetilde{A}_{\widetilde{\mathbb{D}}_2, \widetilde{\theta}_2}/Z_0, (\widehat{S}^r)_{\widetilde{\mathbb{D}}_2}/Z_0, \overline{\widetilde{\theta}}_2), \tag{7.6}$$

where $\widetilde{A} := (\widehat{S} \rtimes \mathrm{Aut}(\widehat{S}))^r \rtimes \mathfrak{S}_r$, and $\overline{\widetilde{\theta}}_1$ and $\overline{\widetilde{\theta}}_2$ are the characters of $(\widehat{S}^r)_{\widetilde{\mathbb{D}}_1}/Z_0$ and $(\widehat{S}^r)_{\widetilde{\mathbb{D}}_2}/Z_0$ corresponding to $\widetilde{\theta}_1$ and $\widetilde{\theta}_2$.

It remains to prove that with the given bijection, equality (7.3) holds. Let $Z := \ker(\widetilde{\theta}_{\mathrm{Z}(\widehat{S}^r)})$, and let $\phi_1 \in \mathrm{Irr}(\widehat{S}^r_{\widetilde{\mathbb{D}}_1}/Z)$ and $\phi_2 \in \mathrm{Irr}(\widehat{S}^r_{\widetilde{\mathbb{D}}_2}/Z)$ be the characters lifting to $\widetilde{\theta}_1$ and $\widetilde{\theta}_2$ respectively.

Since $p \nmid |\mathrm{Z}(\widehat{S})|$ we have $p \nmid |Z/Z_0|$. In order to apply Corollary 4.5 we have to check that $\mathrm{C}_{(\widehat{S}^r(\widetilde{A})_{(\widetilde{\mathbb{D}}_1, \widetilde{\theta}_1)})/Z_0}(\widehat{S}^r/Z_0)/(Z/Z_0)$ coincides with $\mathrm{C}_{(\widehat{S}^r(\widetilde{A})_{(\widetilde{\mathbb{D}}_1, \widetilde{\theta}_1)})/Z}(\widehat{S}^r/Z)$. Let $\overline{x} \in \mathrm{C}_{(\widehat{S}^r(\widetilde{A})_{(\widetilde{\mathbb{D}}_1, \widetilde{\theta}_1)})/Z}(\widehat{S}^r/Z)$ and $x \in (A \wr \mathfrak{S}_r)_{\widetilde{\mathbb{D}}_1, \widetilde{\theta}_1}/Z_0$ with $x(Z/Z_0) = \overline{x}$. Straightforward calculations show that x defines a morphism $\nu : \widehat{S}^r/Z_0 \to Z/Z_0$ via $s \mapsto [s, x]$. Since $\widehat{S}^r$ and hence $\widehat{S}^r/Z_0$ are perfect and Z/Z_0 is abelian, the map ν is trivial. Accordingly x centralizes $\widehat{S}^r/Z_0$. Now according to Corollary 4.5, equality (7.6) implies

$$((\widetilde{A})_{(\widetilde{\mathbb{D}}_1, \widetilde{\theta}_1)}/Z, (\widehat{S}^r)_{\widetilde{\mathbb{D}}_1}/Z, \phi_1) \sim_{\widehat{S}^r/Z} (\widetilde{A}_{\widetilde{\mathbb{D}}_2, \widetilde{\theta}_2}/Z, (\widehat{S}^r)_{\widetilde{\mathbb{D}}_2}/Z, \phi_2).$$

This concludes the proof of Theorem 7.2. □

We now consider the general case and verify that the above constructed bijection has all of the properties required for 7.2.

Proof of Theorem 7.2. Let Ω be the bijection constructed in the proof of Theorem 7.12. Let $(\widetilde{\mathbb{D}}_1, \widetilde{\theta}_1) \in \mathcal{C}^{\leq d'}(B)$ and $(\widetilde{\mathbb{D}}_2, \widetilde{\theta}_2) \in \Omega(\overline{(\widetilde{\mathbb{D}}_1, \widetilde{\theta}_1)})$. It remains to verify that (7.1) holds, i.e., the groups $Z := \ker(\widetilde{\theta}_{1,\mathrm{Z}(\widehat{S}^r)})$ and $\ker(\widetilde{\theta}_{2,\mathrm{Z}(\widehat{S}^r)})$ coincide and

$$(\widetilde{A}_{\widetilde{\mathbb{D}}_1, \widetilde{\theta}_1}/Z, (\widehat{S}^r)_{\widetilde{\mathbb{D}}_1}/Z, \phi_1) \sim_{\widehat{S}^r/Z} (\widetilde{A}_{\widetilde{\mathbb{D}}_2, \widetilde{\theta}_2}/Z, (\widehat{S}^r)_{\widetilde{\mathbb{D}}_2}/Z, \phi_2),$$

where $\widetilde{A} := \widehat{S}^r \rtimes \mathrm{Aut}(\widehat{S}^r) = (\widehat{S} \rtimes \mathrm{Aut}(\widehat{S})) \wr \mathfrak{S}_r = (\widehat{S} \rtimes \mathrm{Aut}(\widehat{S}))^r \rtimes \mathfrak{S}_r$ and the characters ϕ_1 and ϕ_2 are the characters of $(\widehat{S}^r)_{\widetilde{\mathbb{D}}_1}/Z$ and $(\widehat{S}^r)_{\widetilde{\mathbb{D}}_2}/Z$ associated with $\widetilde{\theta}_1$ and $\widetilde{\theta}_2$.

By the proof of Theorem 7.12 we already have

$$(\widetilde{A}_{\widetilde{\mathbb{D}}_1, \widetilde{\theta}_1}/Z_0, (\widehat{S}^r)_{\widetilde{\mathbb{D}}_1}/Z_0, \overline{\widetilde{\theta}}_1) \sim_{\widehat{S}^r/Z_0} (\widetilde{A}_{\widetilde{\mathbb{D}}_2, \widetilde{\theta}_2}/Z_0, (\widehat{S}^r)_{\widetilde{\mathbb{D}}_2}/Z_0, \overline{\widetilde{\theta}}_2), \tag{7.7}$$

where Z_0 is determined as before (7.6) and $\overline{\widetilde{\theta}}_1$ and $\overline{\widetilde{\theta}}_2$ are the characters of $(\widehat{S}^r)_{\widetilde{\mathbb{D}}}/Z_0$ and $(\widehat{S}^r)_{\widetilde{\mathbb{D}}'}/Z_0$ corresponding to $\widetilde{\theta}_1$ and $\widetilde{\theta}_2$.

It remains to prove that (7.1) holds with the given bijection. Let $Z := \ker(\widetilde{\theta}_{1,Z(\widehat{S}^r)})$, $\mathcal{P}_1$ a projective representation of $\widetilde{A}_{\widetilde{\mathbb{D}}_1,\widetilde{\theta}_1}/Z_0$ associated with $\widetilde{\widetilde{\theta}}_1$, and $\mathcal{P}_2$ a projective representation of $\widetilde{A}_{\widetilde{\mathbb{D}}_2,\widetilde{\theta}_2}/Z_0$ associated with $\widetilde{\widetilde{\theta}}_2$, such that they give an $\widehat{S}^r/Z_0$-block isomorphism of character triples. Then $\phi_1 \in \mathrm{Irr}(\widehat{S}^r_{\widetilde{\mathbb{D}}_1}/Z)$ and $\phi_2 \in \mathrm{Irr}(\widehat{S}^r_{\widetilde{\mathbb{D}}_2}/Z)$ lift to $\widetilde{\widetilde{\theta}}_1$ and $\widetilde{\widetilde{\theta}}_2$.

By definition $\mathcal{P}_1$ and $\mathcal{P}_2$ define projective representations $\overline{\mathcal{P}}_1$ and $\overline{\mathcal{P}}_2$ of $\widetilde{A}_{\widetilde{\mathbb{D}}_1,\widetilde{\theta}_1}/Z$ and $\widetilde{A}_{\widetilde{\mathbb{D}}_2,\widetilde{\theta}_2}/Z$ associated with ϕ_1 and ϕ_2. We use the following notation: $\overline{H}_i := \widetilde{A}_{\widetilde{\mathbb{D}}_i,\widetilde{\theta}_i}/Z$, $\overline{M}_i := (\widehat{S}^r)_{\widetilde{\mathbb{D}}_i}/Z$, $H_i := \widetilde{A}_{\widetilde{\mathbb{D}}_i,\widetilde{\theta}_i}/Z_0$, $M_i := (\widehat{S}^r)_{\widetilde{\mathbb{D}}_i}/Z_0$, $N := \widehat{S}^r/Z_0$, $\overline{N} := \widehat{S}^r/Z$, $G := (\widetilde{A}_{\widetilde{\mathbb{D}}_i,\widetilde{\theta}_i}/Z_0)N$ and $\overline{G} := ((\widetilde{A})_{(\widetilde{\mathbb{D}}_i,\widetilde{\theta}_i)}/Z)\overline{N}$. With this we check that $\overline{\mathcal{P}}_1$ and $\overline{\mathcal{P}}_2$ satisfy the assumptions in Theorem 4.3(i). As in the proof of Corollary 4.5 we see that the required group-theoretic assumptions are satisfied. For any defect group $\overline{D}_1$ of $\mathrm{bl}(\phi_1)$ we have

$$\mathrm{C}_G(D_1)N \leq (A^{r'} \times A \wr \mathfrak{S}_{r''})/Z. \tag{7.8}$$

Moreover the assumption from 4.3(i.a) on the factor set is satisfied by the definition of $\overline{\mathcal{P}}_1$ and $\overline{\mathcal{P}}_2$.

Now we check $\overline{\mathcal{P}}_1(\overline{x})$ and $\overline{\mathcal{P}}_2(\overline{x})$ for $\overline{x} \in \mathrm{C}_{\overline{G}}(\overline{N})$. Let $x \in H_1 = (A \wr \mathfrak{S}_r)_{\widetilde{\mathbb{D}}_1,\widetilde{\theta}_1}/Z_0$ with $x(Z/Z_0) = \overline{x}$. By the arguments in the proof of Theorem 7.12 we see that $x \in \mathrm{C}_G(N)$. Accordingly $\overline{\mathcal{P}}_1(\overline{x}) = \mathcal{P}_1(x)$ and $\overline{\mathcal{P}}_2(\overline{x}) = \mathcal{P}_2(x)$ are scalar matrices associated with the same scalar.

Next we verify that $\overline{\mathcal{P}}_1$ and $\overline{\mathcal{P}}_2$ satisfy assumption 4.3(i.c): for every $\overline{x} \in \overline{N}\mathrm{C}_{\overline{G}}(\overline{D}_1)$ the scalar matrices $\overline{\mathcal{P}}_1((\mathfrak{Cl}_{\overline{J}}(\overline{x}) \cap \overline{H}_1)^+)^*$ and $(\overline{\mathcal{P}}_2((\mathfrak{Cl}_{\overline{J}}(\overline{x}) \cap \overline{H}_2)^+))^*$ are associated with the same scalar, where $\overline{D}_1$ is a defect group of $\mathrm{bl}(\phi_1)$ and $\overline{J} := \langle \overline{N}, \overline{x} \rangle$. The equality obviously holds if $\mathfrak{Cl}_{\overline{J}}(\overline{x}) \cap \overline{H}_1 = \emptyset$ and $\mathfrak{Cl}_{\overline{J}}(\overline{x}) \cap \overline{H}_2 = \emptyset$. Without loss of generality we assume in the following that $\overline{x} \in \overline{H}_1$.

Let $x \in N\mathrm{C}_G(\overline{D}_1)$ and $J := \langle N, x \rangle$. Straightforward calculations prove that

$$k^*\overline{\mathcal{P}}_i((\mathfrak{Cl}_{\overline{J}}(\overline{x}) \cap \overline{H}_i)^+)^* = \mathcal{P}_i((\mathfrak{Cl}_J(x) \cap H_i)^+)^*,$$

where $k = |\mathfrak{Cl}_J(x) \cap x(Z/Z_0)|$.

If k^* is invertible or equivalently $p \nmid k$, Theorem 4.3(i.c) for $\mathcal{P}_1$ and $\mathcal{P}_2$ implies $\mathcal{P}_1((\mathfrak{Cl}_J(x) \cap H_1)^+)^* = \mathcal{P}_2((\mathfrak{Cl}_J(x) \cap H_2)^+)^*$, and hence

$$\overline{\mathcal{P}}_1((\mathfrak{Cl}_{\overline{J}}(\overline{x}) \cap \overline{H}_1)^+)^* = \overline{\mathcal{P}}_2((\mathfrak{Cl}_{\overline{J}}(\overline{x}) \cap \overline{H}_2)^+)^*.$$

Otherwise there exists some nontrivial element $z \in Z/Z_0$ such that xz and x are J-conjugate, hence there exists some $s \in \widehat{S}^r/Z_0$ with $x^s = xz$. According to (7.8) we can write x as (x', x'') where $x' \in A^{r'}$ and $x'' \in A \wr \mathfrak{S}_{r''}$. Moreover z may be written as $(z_1, \ldots, z_{r'}, z'')$ with $z_i \in \mathrm{Z}(\widehat{S})/Z_i$ and $z'' \in \mathrm{Z}(\widehat{S}^{r''})$. Let $s_i \in \widehat{S}$ with $s = (s_1, \ldots, s_r)$ and $s_i' = (1, \ldots, 1, s_i, 1, \ldots, 1) \in \widehat{S}^{(i)}$. Then $x^{s_i} = (x'^{s_i})x'' = x'x''z_i$ for every $1 \leq i \leq r'$ and

$$x^{s_{r'+1}\cdots s_r} = xz''.$$

This proves that xz_i and xz'' are J-conjugate to x for every $1 \leq i \leq r'$. Since z is a nontrivial element of Z/Z_0, either z_i is nontrivial for some $1 \leq i \leq r''$ or z'' is nontrivial. By the definition of Z_0 this implies that

$$\overline{\widetilde{\theta}}_1(z_i) = \zeta\overline{\widetilde{\theta}}_1(1) \quad \text{or} \quad \overline{\widetilde{\theta}}_1(z'') = \zeta\overline{\widetilde{\theta}}_1(1)$$

for some root of unity $\zeta \neq 1$. Let $\underline{\widetilde{\eta}_1}$ be an extension of ϕ_1 to $\overline{J} \cap \overline{H}_1$ and $\overline{\mathcal{Q}}$ a projective representation of $\overline{J}$ with $\overline{N} \leq \ker(\overline{\mathcal{Q}})$ such that $\overline{\mathcal{Q}}_{\overline{J}\cap\overline{H}_1} \otimes \overline{\mathcal{P}}_{1,J\cap H_1}$ affords $\widetilde{\eta}_1$.

The block $\mathrm{bl}(\widetilde{\eta}_1)^{\overline{J}}$ covers $\mathrm{bl}(\phi_1)^{\overline{N}}$. Let $\tau \in \mathrm{Irr}(\mathrm{bl}(\widetilde{\eta}_1)^{\overline{J}})$ with $\mathrm{Irr}(\tau_{\mathrm{Z}(N)}) = \mathrm{Irr}(\phi_{1,\mathrm{Z}(N)})$. (Such a character exists by [KS15, Theorem B] and [Na98, Theorem (9.2)].) Since the elements $\overline{x}$, $\overline{x}z_i Z/Z_0$ and $\overline{x}z''Z/Z_0$ are $\overline{J}$-conjugate, the above arguments imply $\tau(\overline{x}) = 0$ and $\lambda_\tau(\mathfrak{Cl}_{\overline{J}}(\overline{x})^+) = 0$, and therefore

$$\overline{\mathcal{Q}}_{\overline{J}\cap\overline{H}_1} \otimes \overline{\mathcal{P}}_1((\mathfrak{Cl}_{\overline{J}}(\overline{x}) \cap \overline{H}_1)^+)^* = 0.$$

Since $\overline{\mathcal{Q}}$ is a one-dimensional projective representation, this proves

$$\overline{\mathcal{P}}_1((\mathfrak{Cl}_{\overline{J}}(\overline{x}) \cap \overline{H}_1)^+)^* = 0.$$

Analogously one sees $\overline{\mathcal{P}}_2((\mathfrak{Cl}_{\overline{J}}(\overline{x}) \cap \overline{H}_2)^+) = 0$. We see that $\overline{\mathcal{P}}_1$ and $\overline{\mathcal{P}}_2$ also satisfy 4.3(i.c), and hence

$$(\widetilde{A}_{\widetilde{\mathbb{D}}_1,\widetilde{\theta}_1}/Z, (\widehat{S}^r)_{\widetilde{\mathbb{D}}_1}/Z, \phi_1) \sim_{\widehat{S}^r/Z} (\widetilde{A}_{\widetilde{\mathbb{D}}_2,\widetilde{\theta}_2}/Z, (\widehat{S}^r)_{\widetilde{\mathbb{D}}_2}/Z, \phi_2).$$

Hence (7.1) holds with the bijection Ω constructed above. □

8. About a minimal counterexample to Dade's Projective Conjecture

In this section we show how one can apply Theorem 7.2 to prove our main theorem. The first two statements are almost immediate consequences of Theorem 7.2. Then we connect those statements with earlier results of [Ro02] and [ER02].

Theorem 8.1. *Let d be a nonnegative integer and S a nonabelian simple group satisfying the Inductive Condition for Dade's Conjecture from Definition 6.7 with respect to d. Let L be a finite group and suppose there is some $K \lhd L$ with $K = [K, K]$ such that $K/(\mathrm{Z}(L) \cap K)$ is isomorphic to S^r for some $r \geq 1$. Let $B \in \mathrm{Bl}_{\mathrm{nc}}(K)$. Then there exists a defect preserving L_B-equivariant bijection $\Omega : \overline{\mathcal{C}^{\leq d}(B)_+} \to \overline{\mathcal{C}^{\leq d}(B)_-}$ with*

$$(L_{\mathbb{D},\theta}, K_{\mathbb{D}}, \theta) \sim_K (L_{\mathbb{D}',\theta'}, K_{\mathbb{D}'}, \theta')$$

for every $(\mathbb{D}, \theta) \in \mathcal{C}^d(B)_+$ and $(\mathbb{D}', \theta') \in \Omega(\overline{(\mathbb{D}, \theta)})$.

Proof. Let $\widehat{S}$ be the universal covering group of S. By the assumptions on K it is clear that there exists an epimorphism $\epsilon : \widehat{S}^r \to K$, since $\widehat{S}^r$ is the universal covering of S^r by [As86, Chapter 11, Exercise 2]. Let $\widetilde{A} := \widehat{S}^r \rtimes \mathrm{Aut}(\widehat{S}^r)$. Via ϵ the block B is contained in some $\widehat{B} \in \mathrm{Bl}(\widehat{S}^r)$ and $\widehat{B}$ contains just this block (see [NT89, Theorems 5.8.8 and 5.8.11]).

Let z_0 be the integer with $p^{z_0} = |\mathrm{Z}(\widehat{S})|_p$. According to Theorem 7.2 there exists for $\widehat{S}^r$ an $\widetilde{A}_{\widehat{B}}$-equivariant bijection

$$\widehat{\Omega} : \overline{\mathcal{C}^{\leq d+rz_0}(\widehat{B})_+} \to \overline{\mathcal{C}^{\leq d+rz_0}(\widehat{B})_-}$$

such that for every $(\mathbb{D}_1, \theta_1) \in \mathcal{C}^{\leq d+rz_0}(\widehat{B})_+$ and $(\mathbb{D}_2, \theta_2) \in \widehat{\Omega}(\overline{(\mathbb{D}_1, \theta_1)})$ the groups $Z := \ker(\widetilde{\theta}_{1,\mathrm{Z}(\widehat{S}^r)})$ and $\ker(\widetilde{\theta}_{2,\mathrm{Z}(\widehat{S}^r)})$ coincide and the pairs satisfy

$$(\widetilde{A}_{\mathbb{D}_1,\theta_1}/Z, \widehat{S}^r_{\mathbb{D}_1}/Z, \phi_1) \sim_{\widehat{S}^r/Z} (\widetilde{A}_{\mathbb{D}_2,\theta_2}/Z, \widehat{S}^r_{\mathbb{D}_2}/Z, \phi_2), \tag{8.1}$$

where ϕ_1 and ϕ_2 are the characters of $\widehat{S}^r_{\mathbb{D}_1}$ and $\widehat{S}^r_{\mathbb{D}_2}$ corresponding to θ_1 and θ_2. Hence θ_1 is the lift of a character of a subgroup of K if and only if θ_2 is.

Let $(\mathbb{D}_1, \theta_1) \in \mathcal{C}^{\leq d+rz_0}(\widehat{B})_+$ with $\ker(\theta_1) \geq \ker(\epsilon)$ and $(\mathbb{D}_2, \theta_2) \in \widehat{\Omega}(\overline{(\mathbb{D}_1, \theta_1)})$. Since $Z_0 := \ker(\epsilon) \leq \mathrm{Z}(\widehat{S}^r)$, we have $\epsilon(\mathrm{N}_{\widehat{S}^r}(\mathbb{D}_i)) = \mathrm{N}_K(\epsilon(\mathbb{D}_i))$ and $\epsilon(\mathbb{D}_i) \in \mathfrak{P}(K|\mathrm{O}_p(K))$. Analogously, in $\widetilde{A}' := \mathrm{N}_{\widetilde{A}}(Z_0)$ we have $\epsilon(\mathrm{N}_{\widetilde{A}'}(\mathbb{D}_i)) = \mathrm{N}_{\widetilde{A}'/Z_0}(\overline{\mathbb{D}}_i)$, where $\overline{\mathbb{D}}_i := \epsilon(\mathbb{D}_i)$.

For $i = 1, 2$ let $\overline{\theta}_i \in \mathrm{Irr}(\mathrm{N}_{\widehat{S}^r/Z_0}(\overline{\mathbb{D}}_i))$ be the lift of ϕ_i, hence a character that lifts to θ_i. According to Corollary 4.4 in combination with Lemma 3.8(b), (8.1) implies

$$(\widetilde{A}'_{\overline{\mathbb{D}}_1,\overline{\theta}_1}/Z_0, \mathrm{N}_{\widehat{S}^r/Z_0}(\overline{\mathbb{D}}_1), \overline{\theta}_1) \sim_{\widehat{S}^r/Z_0} (\widetilde{A}'_{\overline{\mathbb{D}}_2,\overline{\theta}_2}, \mathrm{N}_{\widehat{S}^r/Z_0}(\overline{\mathbb{D}}_2), \overline{\theta}_2).$$

By Theorem 5.3 in combination with Lemma 3.8(b) this implies that

$$(L_{\overline{\mathbb{D}}_1,\overline{\theta}_1}, \mathrm{N}_K(\overline{\mathbb{D}}_1), \overline{\theta}_1) \sim_K (L_{\overline{\mathbb{D}}_2,\overline{\theta}_2}, \mathrm{N}_K(\overline{\mathbb{D}}_2), \overline{\theta}_2).$$

Let z_1 be the integer with $p^{z_1} = |Z_0|_p$. Since $\widehat{\Omega}$ is an $\mathrm{Aut}(\widehat{S}^r)_{\widehat{B}}$-equivariant defect preserving bijection, it induces an L_B-equivariant defect preserving bijection

$$\Omega : \overline{\mathcal{C}^{\leq d+rz_0-z_1}(B)_+} \to \overline{\mathcal{C}^{\leq d+rz_0-z_1}(B)_-}.$$

Moreover $(\mathbb{D}_1, \theta_1) \in \mathcal{C}^{\leq d+rz_0-z_1}(B)_+$ and $(\mathbb{D}_2, \theta_2) \in \Omega(\overline{(\mathbb{D}_1, \theta_1)})$ satisfy

$$(L_{\mathbb{D}_1,\theta_1}, \mathrm{N}_K(\mathbb{D}_1), \theta_1) \sim_K (L_{\mathbb{D}_2,\theta_2}, \mathrm{N}_K(\mathbb{D}_2), \theta_2).$$

Since $rz_0 - z_1 \geq 0$, this proves the statement. □

Corollary 8.2. *Let d be a nonnegative integer and S a nonabelian simple group such that S satisfies the Inductive Condition for S and d (see Definition 6.7). Let L be a finite group and suppose there is some $K \lhd L$ with $K = [K, K]$ such that $K/(\mathrm{Z}(L) \cap K)$ is isomorphic to S^r for some $r \geq 1$. Let $C \in \mathrm{Bl}(L)$ be a block with a defect group D such that $D \cap K \not\leq \mathrm{Z}(K)$. Then there exists a bijection*

$$\Pi : \dot{\bigcup_{\mathbb{D}\in\mathfrak{P}(K|\mathrm{O}_p(K))_+/\sim_L}} \mathrm{Irr}^d(C_{\mathbb{D}}) \to \dot{\bigcup_{\mathbb{D}'\in\mathfrak{P}(K|\mathrm{O}_p(K))_-/\sim_L}} \mathrm{Irr}^d(C_{\mathbb{D}'})$$

such that $\mathrm{Irr}(\chi_{\mathrm{Z}(L)}) = \mathrm{Irr}(\Pi(\chi)_{\mathrm{Z}(L)})$ *for every* $\chi \in \dot\bigcup_{\mathbb{D}\in\mathfrak{P}(K|\mathrm{O}_p(K))_+/\sim_L} \mathrm{Irr}^d(C_{\mathbb{D}})$.

Proof. Let B be a p-block of K covered by C. Any defect group D_0 of B is L-conjugate to $D \cap K$ (see [Na98, Theorem (9.26)]). By assumption $D_0 \not\leq \mathrm{Z}(K)$.

According to Theorem 8.1 there exists a defect preserving L_B-equivariant bijection

$$\Omega : \overline{\mathcal{C}^{\leq d}(B)_+} \to \overline{\mathcal{C}^{\leq d}(B)_-}$$

with

$$(L_{\mathbb{D}_1,\theta_1}, K_{\mathbb{D}_1}, \theta_1) \sim_K (L_{\mathbb{D}_2,\theta_2}, K_{\mathbb{D}_2}, \theta_2)$$

for every $(\mathbb{D}_1, \theta_1) \in \mathcal{C}^{\leq d}(B)_+$ where $(\mathbb{D}_2, \theta_2) \in \Omega(\overline{(\mathbb{D}_1, \theta_1)})$. Let $\mathbb{D}_1, \ldots, \mathbb{D}_s$ be a full set of L-orbit representatives in $\mathfrak{P}_+(L)$. Then one can choose in $\mathcal{C}^{\leq d}(B)_+$ a full set $\mathbb{T}_+$ of L-orbit representatives such that for every $(\mathbb{D}, \theta) \in \mathbb{T}_+$ the chain $\mathbb{D}$ coincides with $\mathbb{D}_i$ for some $1 \leq i \leq s$. For fixed $1 \leq i \leq s$ the characters $\psi_1^{(i)}, \ldots, \psi_{t_i}^{(i)}$ with $(\mathbb{D}_i, \psi_j^{(i)}) \in \mathbb{T}_+$ form a full set of $\mathrm{N}_{L_B}(\mathbb{D}_i)$-orbit representatives in $\mathrm{Irr}(B_{\mathbb{D}_i})$. By standard Clifford theory we know that $\mathrm{Irr}^d(C_{\mathbb{D}_i})$ can be seen as the disjoint union

$$\mathrm{Irr}^d(C_{\mathbb{D}_i}) = \dot{\bigcup_{k=1}^{t_i}} \mathrm{Irr}^d(C_{\mathbb{D}_i} \mid \psi_k^{(i)})$$

where $\mathrm{Irr}^d(C_{\mathbb{D}_i} \mid \psi_k^{(i)}) := \mathrm{Irr}^d(C_{\mathbb{D}_i}) \cap \mathrm{Irr}(L_{\mathbb{D}_i} \mid \psi_k^{(i)})$ for every $1 \leq i \leq s$.

Let $\mathbb{D}_{s+1}, \ldots, \mathbb{D}_{s'}$ be a full set of L-orbit representatives in $\mathfrak{P}(L)_-$. For every $(\mathbb{D}, \theta)$ in $\mathbb{T}_+$ we choose an element $(\mathbb{D}', \theta') \in \Omega(\overline{(\mathbb{D}, \theta)})$ such that $\mathbb{D}' \in \{\mathbb{D}_{s+1}, \ldots, \mathbb{D}_{s'}\}$. Let $\mathbb{T}_-$ be the set of all such pairs. Since $\mathbb{T}_+$ is a full set of L_B-orbit representatives, $\mathbb{T}_-$ is a full set of L_B-orbit representatives as well. As above, for fixed $s + 1 \leq i \leq s'$ the characters $\psi_1^{(i)}, \ldots, \psi_{t_i}^{(i)}$ with $(\mathbb{D}_i, \psi_j^{(i)}) \in \mathbb{T}_-$ form a full set of $\mathrm{N}_{L_B}(\mathbb{D}_i)$-orbit representatives in $\mathrm{Irr}(B_{\mathbb{D}_i})$. Again we obtain

$$\mathrm{Irr}^d(C_{\mathbb{D}_i}) = \dot{\bigcup_{k=1}^{t_i}} \mathrm{Irr}^d(C_{\mathbb{D}_i} \mid \psi_k^{(i)}) \quad \text{for every } s+1 \leq i \leq s'.$$

This implies

$$\dot{\bigcup_{\mathbb{D} \in \mathfrak{P}(K)_+/\sim_L}} \mathrm{Irr}^d(C_{\mathbb{D}}) = \dot{\bigcup_{(\mathbb{D},\theta) \in \mathbb{T}_+}} \mathrm{Irr}^d(C_{\mathbb{D}} \mid \theta),$$

$$\dot{\bigcup_{\mathbb{D} \in \mathfrak{P}(K)_-/\sim_L}} \mathrm{Irr}^d(C_{\mathbb{D}}) = \dot{\bigcup_{(\mathbb{D},\theta) \in \mathbb{T}_-}} \mathrm{Irr}^d(C_{\mathbb{D}} \mid \theta).$$

By the construction we can associate to every $(\mathbb{D}, \theta) \in \mathbb{T}_+$ a pair $(\mathbb{D}', \theta') \in \mathbb{T}_-$ such that $(\mathbb{D}', \theta') \in \Omega(\overline{(\mathbb{D}, \theta)})$. Hence for the associated character triples we know

$$(L_{\mathbb{D},\theta}, K_{\mathbb{D}}, \theta) \sim_K (L_{\mathbb{D}',\theta'}, K_{\mathbb{D}'}, \theta')$$

from the construction using Theorem 8.1. According to Proposition 3.9(c) we also have $|\mathrm{Irr}^d(C_{\mathbb{D},\theta} \mid \theta)| = |\mathrm{Irr}^d(C_{\mathbb{D}',\theta'} \mid \theta')|$. Now Clifford correspondence gives a defect preserving bijection between $\mathrm{Irr}^d(C_{\mathbb{D},\theta} \mid \theta)$ and $\mathrm{Irr}^d(C_{\mathbb{D}} \mid \theta)$, and between $\mathrm{Irr}^d(C_{\mathbb{D}',\theta'} \mid \theta')$ and $\mathrm{Irr}^d(C_{\mathbb{D}'} \mid \theta')$. This implies $|\mathrm{Irr}^d(C_{\mathbb{D}} \mid \theta)| = |\mathrm{Irr}^d(C_{\mathbb{D}'} \mid \theta')|$. Together with the above equalities this implies the existence of a bijection

$$\Pi : \dot{\bigcup_{\mathbb{D} \in \mathfrak{P}(K)_+/\sim_L}} \mathrm{Irr}^d(C_{\mathbb{D}}) \to \dot{\bigcup_{\mathbb{D}' \in \mathfrak{P}(K)_-/\sim_L}} \mathrm{Irr}^d(C_{\mathbb{D}'}).$$

Since the bijection is given by the associated character triple isomorphisms in combination with Clifford correspondence, it satisfies in addition

$$\operatorname{Irr}(\chi_{\mathrm{Z}(L)}) = \operatorname{Irr}(\Pi(\chi)_{\mathrm{Z}(L)}) \quad \text{for every } \chi \in \dot{\bigcup_{\mathbb{D}\in\mathfrak{P}(K)_+/\sim_L}} \operatorname{Irr}^d(C_{\mathbb{D}}). \qquad \square$$

Now we turn to the proof of Theorem 1.3. We use here the results from [Ro02] and [ER02], and consider a given finite group G with $\mathrm{O}_p(G) \leq \mathrm{Z}(G)$ and a p-block C of G with noncentral defect.

Assumption 8.3. *Let G be a finite group with $\mathrm{O}_p(G) \leq \mathrm{Z}(G)$ and $C \in \mathrm{Bl}_{\mathrm{nc}}(G)$. Assume that Dade's Projective Conjecture holds for all groups H with $\mathrm{O}_p(H) \leq \mathrm{Z}(H)$ whenever*

- *all the nonabelian simple groups involved in H are involved in G,*
- *either*
  - $|H : \mathrm{Z}(H)| < |G : \mathrm{Z}(G)|$, *or*
  - $|H : \mathrm{Z}(H)| = |G : \mathrm{Z}(G)|$ *and* $|H| < |G|$.

We apply results from [Ro02] and [ER02]. Note that in both cases the results stated there hold for G under the above assumption. As explained in the remark before [Ro02, Theorem 2] the actual assumption used in [ER02] is the one given in 8.3, namely that one assumes that Dade's Projective Conjecture holds in every central extension H of a section of G, where $|H| < |G|$ or $|H : \mathrm{Z}(H)| < |G : \mathrm{Z}(G)|$. An analogous statement can be made about the proof of [ER02], which describes properties of a finite group G satisfying Assumption 8.3 in the case where G has a block not satisfying Dade's Projective Conjecture.

Theorem 8.4. *Let $\mathcal{S}$ be a set of nonabelian simple groups S satisfying the Inductive Condition for Dade's Conjecture from Definition 6.7. Assume that every nonabelian simple component of G is contained in $\mathcal{S}$, and suppose that Assumption 8.3 holds. Then Dade's Projective Conjecture holds for the block C and d.*

Proof. According to [ER02, Theorem 1] and the succeeding remarks we may assume that G has only one component up to isomorphism, i.e., for the generalized Fitting subgroup $\mathrm{F}^*(G)$ of G, $\mathrm{F}^*(G)/\mathrm{Z}(G)$ is the direct product of groups isomorphic to a nonabelian simple group S. Let $K := [\mathrm{F}^*(G), \mathrm{F}^*(G)]$. According to [Ro02, Theorem 1] the block C covers a block B of $\mathrm{F}^*(G)$ such that B has a defect group that is not central in $\mathrm{F}^*(G)$. Let $\lambda \in \operatorname{Irr}(\mathrm{Z}(G))$. From Corollary 8.2 we know

$$\sum_{\mathbb{D}\in\mathfrak{P}(K\,\mathrm{Z}(G)|\mathrm{O}_p(K))/\sim_G} (-1)^{|\mathbb{D}|} |\operatorname{Irr}^d(B_{\mathbb{D}} \mid \lambda)| = 0,$$

where $\operatorname{Irr}^d(B_{\mathbb{D}} \mid \lambda) = \operatorname{Irr}^d(B_{\mathbb{D}}) \cap \operatorname{Irr}(G_{\mathbb{D}} \mid \lambda)$. By [Ro02, Theorem 1] this implies

$$\sum_{\mathbb{D}\in\mathfrak{P}(G|\mathrm{O}_p(K))/\sim_G} (-1)^{|\mathbb{D}|} |\operatorname{Irr}^d(B_{\mathbb{D}} \mid \lambda)| = \sum_{\mathbb{D}\in\mathfrak{P}(K\,\mathrm{Z}(G)|\mathrm{O}_p(K))/\sim_G} (-1)^{|\mathbb{D}|} |\operatorname{Irr}^d(B_{\mathbb{D}} \mid \lambda)|$$
$$= 0.$$

(Note that by an application of [Da92, Proposition 3.7], Theorem 1 of [Ro02] is also valid when the underlying p-chains are not assumed to be normal.) Hence Dade's Projective Conjecture holds for C and d. □

The above implies Theorem 1.3.

9. The Character Triple Conjecture for quasisimple groups

In this section we verify the Character Triple Conjecture for some quasisimple groups. Recall that Proposition 6.10 simplifies the checking of Dade's Conjecture since the type of the p-chains under consideration can be varied. For example it seems that for certain blocks of groups of Lie type and in the nondefining characteristic, elementary abelian p-chains have good properties (see [Br06]). Proposition 6.10 is applied later in the verification of the Character Triple Conjecture for blocks with cyclic defect and for p-blocks of $\mathrm{SL}_2(q)$.

In [Da96] and [Ma05] it is proven that various strong versions of Dade's conjectures hold for blocks with cyclic defect groups. Since the relation of our Character Triple Conjecture to Dade's Final Conjecture is not fully established, we give here an independent proof and use the above proposition in order to deduce from [KS16a, KS16b] that our Character Triple Conjecture holds for all blocks with cyclic defect groups.

Proposition 9.1. *Let G be a finite group and $B \in \mathrm{Bl}(G)$ a block with a nonnormal, cyclic defect group D. Then the Character Triple Conjecture 6.3 holds for B.*

Proof. According to Proposition 6.10 it is sufficient to prove that the Character Triple Conjecture holds with respect to elementary abelian p-chains. Let D be a defect group of B and $(\mathbb{D}, \theta) \in \overline{\mathcal{C}^d(B)}$ with $\mathbb{D} \in \mathfrak{C}(G)$. Then $\mathbb{D}_{|\mathbb{D}|}$ is contained in a defect group of $\mathrm{bl}(\theta)$, and after suitable G-conjugation, $\mathbb{D}_{|\mathbb{D}|} \leq D$. Since D is cyclic, there exists a unique subgroup E of D with $\mathrm{O}_p(G) \lneq E$ such that $E/\mathrm{O}_p(G)$ is elementary abelian. So the chains $\mathbb{D}_0 := (\mathrm{O}_p(G))$ and $\mathbb{D}_1 := (\mathrm{O}_p(G) \lneq E)$ are the only chains that occur in $\mathcal{C}^d_{\mathrm{elem}}(B)$ up to G-conjugacy. Accordingly it is sufficient to find a defect preserving $\mathrm{Aut}(G)_B$-equivariant bijection $\Lambda : \mathrm{Irr}(B) \to \mathrm{Irr}(b)$ with

$$((G \rtimes \mathrm{Aut}(G))_\theta, G, \theta) \sim_G (\mathrm{N}_{G \rtimes \mathrm{Aut}(G)}(E)_{\Lambda(\theta)}, \mathrm{N}_G(E), \Lambda(\theta)) \quad \text{for every } \theta \in \mathrm{Irr}(B),$$

where $b \in \mathrm{Bl}(\mathrm{N}_G(E))$ is the block with $b^G = B$.

Since the defect group is cyclic, all ordinary irreducible characters of B and b have height zero. By [KS16a, Theorem 1.1] and [KS16b, Theorem 1.1] the Inductive AM Condition holds for b via a bijection $\Lambda : \mathrm{Irr}(B) \to \mathrm{Irr}(b)$. According to Proposition 6.12 the bijection satisfies

$$((G \rtimes \mathrm{Aut}(G))_\theta, G, \theta) \sim_G (\mathrm{N}_{G \rtimes \mathrm{Aut}(G)}(M)_{\Lambda(\theta)}, M, \Lambda(\theta)) \quad \text{for every } \theta \in \mathrm{Irr}_0(B).$$

This proves the statement. □

Theorem 9.2. *Let p be a prime. Let S be either a simple group $\mathrm{SL}_2(2^n)$, ${}^2\mathrm{B}_2(2^{2n+1})$ or a simple sporadic group such that p is odd or S is not the Baby Monster or the Monster group. Then the Inductive Condition for Dade's Conjecture from Definition 6.7 holds for S.*

Proof. All listed groups have cyclic outer automorphism group. Hence Proposition 6.6 can be applied here. For the listed groups, respectively their universal covering groups, Dade's Invariant Projective Conjecture from [Da97, 4.9] holds: see [Da99] for the groups $\mathrm{SL}_2(2^n)$ and ${}^2\mathrm{B}_2(2^{2n+1})$, [AC95, Hu97] for the Matthieu groups, [Da92, Theorem 10.1] and [Ko97, AOW03] for the Janko groups, [AO04, AO98, An99] for the three simple Conway groups, [AO99, AO05, AOU08] for the Fischer sporadic groups, [HH99] for the Higman–Sims group, [EP99, Mu98] for the McLaughlin group, [An97] for the Held group, [Hi99] for the Suzuki group, [AO02] for the O'Nan and Rudvalis sporadic simple groups, [AO03] for the Harada–Norton group, [SU03] for the Lyons group, [Uno04] for the Thompson group, [AW04] for the Baby Monster, and [AW10] for the Fischer–Griess Monster (the latter two with $p \neq 2$). □

The above results allow us to verify that the blocks of $\mathrm{SL}_2(q)$ with nonnormal defect groups satisfy the Character Triple Conjecture.

Proposition 9.3. *Let p a prime, q a prime power, $G := \mathrm{SL}_2(q)$ and $B \in \mathrm{Bl}_{\mathrm{nc}}(G)$ be a p-block. Then the Character Triple Conjecture holds for B and for the p-block of $G/\mathrm{Z}(G)$ contained in B. Moreover the Inductive Condition for Dade's Conjecture holds for $G/\mathrm{Z}(G)$.*

Proof. Note that if G is the simple group $\mathrm{SL}_2(4)$ as mentioned above the statement follows from [Da99, Theorem 6.13] since the group of outer automorphisms of G is then cyclic. For G a covering group of $\mathrm{PSL}_2(9)$ computer calculations prove the statement.

Otherwise G is the universal covering group of $G/\mathrm{Z}(G)$ (see [GLS98, Table 6.1.3]). Hence the first part of the statement implies the second.

We now distinguish the various possible values of p with respect to the characteristic of the underlying field $\mathbb{F}_q$.

If $p \mid q$ we verify the Character Triple Conjecture using radical chains. Let P be a Sylow p-subgroup of G and $A := G \rtimes \mathrm{Aut}(G)$. Then the Inductive AM Condition holds for B according to [Spä13a, Theorem 8.4] whenever G is the universal covering group of $\mathrm{PSL}_2(q)$. According to Proposition 6.12 this gives an A_P-equivariant bijection

$$\Pi : \mathrm{Irr}_0(G \mid P) \to \mathrm{Irr}_0(\mathrm{N}_G(P) \mid P)$$

such that for every $\theta \in \mathrm{Irr}_0(G \mid P)$ the character $\theta' := \Omega(\theta)$ satisfies $Z := \ker(\theta_Z) = \ker(\theta'_Z)$ and

$$(A_\theta/Z, G/Z, \overline{\theta}) \sim_{G/Z} (A_{P,\theta}/Z, \mathrm{N}_G(P)/Z, \overline{\theta}).$$

The only radical p-chains of G are $\mathbb{D}_0 := (\{1_G\})$ and $\mathbb{D}_1 := (\{1_G\} \lneq P)$. Accordingly Π induces a height preserving A_B-equivariant bijection

$$\Omega : \overline{\mathcal{C}^{\leq d}_{\mathrm{rad}}(B)_+} \to \overline{\mathcal{C}^{\leq d}_{\mathrm{rad}}(B)_-}$$

for all blocks $B \in \mathrm{Bl}(G)$. Hence it is also defect preserving. Moreover the bijection has the properties described in Proposition 6.12.

Let $\mathbf{G} := \mathrm{SL}_2(\overline{\mathbb{F}_q})$ and $F : \mathbf{G} \to \mathbf{G}$ be a Frobenius endomorphism defining an $\mathbb{F}_q$-structure of $\mathbf{G}$. If $p \nmid q$, then either B is the principal 2-block of G, or B has cyclic defect (see [Bo11, Theorem 7.1.1]). Assume that $p = 2$ and q is odd. Let B the principal 2-block of G, $Z = \mathrm{Z}(G)$ and P be a Sylow 2-subgroup of G. Then $\overline{P} := P/Z$ is isomorphic to $\mathrm{C}_{(q+\epsilon)_2/2} \rtimes \mathrm{C}_2$, where $\epsilon \in \{\pm 1\}$ is such that $4 \mid q+\epsilon$ and the group C_2 acts on $\mathrm{C}_{(q+\epsilon)_2/2}$ by inverting. Let q_0 be the prime with $q_0 \mid q$ and F_0 be a field automorphism of $\mathrm{GL}_2(q)$ that corresponds to a field automorphism of $\mathbb{F}_q$ that generates the Galois group of $\mathbb{F}_q$ over $\mathbb{F}_{q_0}$; let $\mathbf{T}$ be an F_0-stable maximal torus of $\mathbf{G} := \mathrm{SL}_2(\overline{\mathbb{F}_q})$ with $|\mathbf{T}^F| = q+\epsilon$ and $\mathbf{T}'$ be an F_0-stable maximal torus of $\mathbf{G}$ with $|(\mathbf{T}')^F| = q-\epsilon$.

Assume $8 \nmid q+\epsilon$ and hence $\overline{P}$ is a Klein four-group. Then the only elementary abelian noncyclic subgroup of $\overline{P}$ is $\overline{P}$. In G/Z all involutions are G-conjugate, and hence all cyclic subgroups P of order 4 are conjugate in G. This leads to the following elementary abelian chains of G starting with Z up to G-conjugation:

$$\mathbb{D}_0 := (Z), \quad \mathbb{D}_1 := (Z \lneq \mathrm{C}_4), \quad \mathbb{D}_2 := (Z \lneq \mathrm{C}_4 \lneq P), \quad \mathbb{D}_3 := (Z \lneq P).$$

Straightforward calculations in $\mathrm{SL}_2(q)$ lead to $G_{\mathbb{D}_0} = G$, $G_{\mathbb{D}_1} = M$, $G_{\mathbb{D}_2} = P$ and $G_{\mathbb{D}_3} = \mathrm{N}_G(P)$, where $M = \mathrm{N}_G(\mathbf{T})$ or $M = \mathrm{N}_G(\mathbf{T}')$. Furthermore the group $\mathrm{N}_G(P)$ is isomorphic to the semidirect product of P with an automorphism of order 3, permuting the cyclic subgroups of order 4.

Let $\widetilde{G}$ be the subgroup of $\mathrm{GL}_2(q)$ with $\widetilde{G} \geq G$ and $\widetilde{G}/G \in \mathrm{Syl}_2(\mathrm{GL}_2(q)/\mathrm{SL}_2(q))$. Let F_0 be a field automorphism of $\mathrm{GL}_2(q)$ that corresponds to a field automorphism of $\mathbb{F}_q$ that generates the Galois group of $\mathbb{F}_q$. Note that $8 \nmid q+\epsilon$ implies by straightforward calculations that F_0 has odd order. Hence for $A := \widetilde{G} \rtimes \langle F_0 \rangle$ the quotient A/G is cyclic and induces on G the group $\mathrm{Aut}(G)$. The aim here is to verify the Character Triple Conjecture for the principal block b_0 of G for $G \lhd A$, since this implies the statement for $G \lhd (G \rtimes \mathrm{Aut}(G))$ according to Theorem 5.3. By Proposition 6.6 it is therefore sufficient to construct a defect preserving A-equivariant bijection

$$\Omega : \overline{\mathcal{C}^{\leq d}_{\mathrm{elem}}(b_0)_+} \to \overline{\mathcal{C}^{\leq d}_{\mathrm{elem}}(b_0)_-}$$

such that pairs $(\mathbb{D}, \theta)$ with faithful $\theta_{\mathrm{Z}(G)}$ are sent to a G-orbit of some $(\mathbb{D}', \theta')$ with faithful $\theta'_{\mathrm{Z}(G)}$, where d is the integer with $2^d = |G|_2$.

In the following we describe successively the groups $G_{\mathbb{D}_i}$ for $0 \leq i \leq 4$, the characters of $b_{0,\mathbb{D}_i}$ and their stabilizers in $A_{\mathbb{D}_i}$.

The characters of $G_{\mathbb{D}_0}$ belonging to b_0 are described in [Bo11, Theorem 7.1.1]. We use the notation for those characters as given there: Let $\mathbf{T}$ be an F_0-stable maximal torus of $\mathbf{G} := \mathrm{SL}_2(\overline{\mathbb{F}_q})$ with $q+\epsilon = |\mathbf{T}^F|$, and $\mathbf{T}'$ be an F_0-stable maximal torus of $\mathbf{G}$ with $q-\epsilon = |(\mathbf{T}')^F|$.

Characters of b_0	$\chi(1)$	$d(\chi)$	A_χ	# char.
1_G	1	3	A	1
St_G	q	3	A	1
$\mathrm{R}^{\mathbf{G}}_{\mathbf{T}}(\xi)_\pm$ ($\xi \in \mathrm{Irr}(\mathbf{T}^F)$, $o(\xi) = 2$)	$(q-\epsilon)/2$	3	$G \rtimes \langle F_0 \rangle$	2
$\mathrm{R}^{\mathbf{G}}_{\mathbf{T}}(\xi)$ ($\xi \in \mathrm{Irr}(\mathbf{T}^F)$, $o(\xi) = 4$)	$q-\epsilon$	2	A	1
$\mathrm{R}^{\mathbf{G}}_{\mathbf{T}'}(\xi)_\pm$ ($\xi \in \mathrm{Irr}((\mathbf{T}')^F)$, $o(\xi) = 2$)	$(q+\epsilon)/2$	2	$G \rtimes \langle F_0 \rangle$	2

For $\mathbb{D}_1 = (Z \lneq C_4)$ the group $G_{\mathbb{D}_1} = \mathrm{N}_{\mathbf{G}}(\mathbf{T})^F$ has $\mathbf{T}^F$ as an abelian normal subgroup of index 2. Since $G_{\mathbb{D}_1} \geq \mathrm{N}_G(P)$, the block $b_{0,\mathbb{D}_1}$ is the principal one. Accordingly $\mathrm{Irr}(b_{0,\mathbb{D}_1})$ is the union of characters $\mathrm{Irr}(G_{\mathbb{D}_1} \mid \xi)$, where $\xi \in \mathrm{Irr}(\mathbf{T}^F)$ has 2-power order. Moreover $GA_{\mathbb{D}_1} = A$.

Characters of $b_{0,\mathbb{D}_1}$	$\chi(1)$	$d(\chi)$	$A_{\mathbb{D}_1,\chi}$	# char.
$\chi \in \mathrm{Irr}(G_{\mathbb{D}_1} \mid 1_{\mathbf{T}^F})$	1	3	$A_{\mathbb{D}_1}$	2
$\chi \in \mathrm{Irr}(G_{\mathbb{D}_1} \mid \xi)$ $(\xi \in \mathrm{Irr}(\mathbf{T}^F), o(\xi) = 2)$	1	3	$(G \rtimes \langle F_0 \rangle)_{\mathbb{D}_1}$	2
$\chi \in \mathrm{Irr}(G_{\mathbb{D}_1} \mid \xi)$ $(\xi \in \mathrm{Irr}(\mathbf{T}^F), o(\xi) = 4)$	2	2	$A_{\mathbb{D}_1}$	1

The group $G_{\mathbb{D}_2}$ coincides with P, and $\mathrm{Irr}(P) = \mathrm{Irr}(b_{0,\mathbb{D}_2})$. The group $\mathrm{Z}(\mathrm{SL}_2(q))$ is a normal subgroup of P and $P/\mathrm{Z}(\mathrm{SL}_2(q))$ is a Klein four-group.

Characters of $b_{0,\mathbb{D}_2}$	$\chi(1)$	$d(\chi)$	$A_{\mathbb{D}_2,\chi}$	# char.
$\chi_1, \chi_2 \in \mathrm{Irr}(G_{\mathbb{D}_2} \mid 1_{\mathrm{Z}(\mathrm{SL}_2(q))})$	1	3	$A_{\mathbb{D}_2}$	2
$\chi_3, \chi_4 \in \mathrm{Irr}(G_{\mathbb{D}_2} \mid 1_{\mathrm{Z}(\mathrm{SL}_2(q))})$	1	3	$(G \rtimes \langle F_0 \rangle)_{\mathbb{D}_2}$	2
$\chi \in \mathrm{Irr}(G_{\mathbb{D}_2}) \setminus \mathrm{Irr}(G_{\mathbb{D}_2} \mid 1_{\mathrm{Z}(\mathrm{SL}_2(q))})$	2	2	$A_{\mathbb{D}_2}$	1

The group $\mathrm{N}_G(P)$ has P as a normal subgroup of index 3. All characters of $b_{0,\mathbb{D}_3}$ belong to the principal block of $\mathrm{N}_G(P)$. Straightforward computations determine the characters, their defects and their stabilizers.

Characters of $b_{0,\mathbb{D}_3}$	$\chi(1)$	$d(\chi)$	$A_{\mathbb{D}_3,\chi}$	# char.
χ_1	1	3	$A_{\mathbb{D}_3}$	1
χ_2	3	3	$A_{\mathbb{D}_3}$	1
χ_3, χ_4	1	3	$(G \rtimes \langle F_0 \rangle)_{\mathbb{D}_3}$	2
χ_5	2	2	$A_{\mathbb{D}_2}$	1
χ_6, χ_7	2	2	$(G \rtimes \langle F_0 \rangle)_{\mathbb{D}_3}$	2

A thorough inspection of the above tables shows that a bijection Ω with the necessary properties exists.

Assume $8 \mid q + \epsilon$. We show that there are six G-orbits of elementary abelian 2-chains: Let $o = \log_2(q + \epsilon)$, let $\mathbf{T}$ be a maximal F-stable torus of $\mathbf{G}$ such that $q + \epsilon = |\mathbf{T}^F|$ and let T_2 be the Sylow 2-subgroup of $\mathbf{T}^F$. Recall that Z denotes the centre of $\mathrm{SL}_2(q)$.

Then the Sylow 2-subgroup P of G coincides with $\langle T_2, x \rangle$ for some element $x \in G$ of order 4 acting on T_2 by inverting. Let $a' \in T_2$ be an element of order 8. From the description of the fusion system in [Li07, Example 8.8] straightforward computations show that the 2-chains in $\mathfrak{P}_{\mathrm{elem}}(G)$ up to G-conjugation are exactly the following:

$$\mathbb{D}_0 := (Z), \quad \mathbb{D}_1 := (Z \lneq \langle a'^2 \rangle), \quad \mathbb{D}_2 := (Z \lneq \langle a'^2, x \rangle),$$
$$\mathbb{D}_3 := (Z \lneq \langle a'^2, a'x \rangle), \quad \mathbb{D}_4 := (Z \lneq \langle a'^2 \rangle \lneq \langle a'^2, x \rangle),$$
$$\mathbb{D}_5 := (Z \lneq \langle a'^2 \rangle \lneq \langle a'^2, a'x \rangle).$$

The chains $\mathbb{D}_4$ and $\mathbb{D}_5$ are $\widetilde{G}$-conjugate, and $\mathbb{D}_2$ and $\mathbb{D}_3$ are as well. The other chains satisfy $GA_{\mathbb{D}_i} = A$. Accordingly the characters from $b_{0,\mathbb{D}_3}$ and $b_{0,\mathbb{D}_5}$ can be studied using the results on the other blocks.

As above, we use the description of $\mathrm{Irr}(b_0)$ from [Bo11, Theorem 7.1.1] with the same notation.

Characters of b_0	$\chi(1)$	$d(\chi)$	A_χ	# char.
1_G	1	$o+1$	A	1
St_G	q	$o+1$	A	1
$\mathrm{R}_{\mathbf{T}}^{\mathbf{G}}(\xi)_\pm$ ($\xi \in \mathrm{Irr}(\mathbf{T}^F)$, $o(\xi) = 2$)	$(q-\epsilon)/2$	$o+1$	$G\,\mathrm{Z}(\widetilde{G}) \rtimes \langle F_0\rangle$	2
$\mathrm{R}_{\mathbf{T}}^{\mathbf{G}}(\xi)$ ($\xi \in \mathrm{Irr}(\mathbf{T}^F)$, $2 < o(\xi)$ 2-power)	$q-\epsilon$	o	$GA_{\mathbf{T}^F,\{\xi,\xi^{-1}\}}$	$2^{o-1}-1$
$\mathrm{R}_{\mathbf{T}'}^{\mathbf{G}}(\xi)_\pm$ ($\xi \in \mathrm{Irr}((\mathbf{T}')^F)$, $o(\xi) = 2$)	$(q+\epsilon)/2$	2	$G\,\mathrm{Z}(\widetilde{G}) \rtimes \langle F_0\rangle$	2

For $\mathbb{D}_1 = (Z \lneq \langle a'^2\rangle)$ the group $G_{\mathbb{D}_1} = \mathrm{N}_{\mathbf{G}}(\mathbf{T})^F$ has $\mathbf{T}^F$ as an abelian normal subgroup of index 2. As above, $\mathrm{Irr}(b_{0,\mathbb{D}_1})$ is the union of characters $\mathrm{Irr}(G_{\mathbb{D}_1} \mid \xi)$, where $\xi \in \mathrm{Irr}(\mathbf{T}^F)$ has 2-power order.

Characters of $b_{0,\mathbb{D}_1}$	$\chi(1)$	$d(\chi)$	$A_{\mathbb{D}_1,\chi}$	# char.
$\chi \in \mathrm{Irr}(G_{\mathbb{D}_1} \mid 1_{\mathbf{T}^F})$	1	$o+1$	$A_{\mathbb{D}_1}$	2
$\chi \in \mathrm{Irr}(G_{\mathbb{D}_1} \mid \xi)$ ($\xi \in \mathrm{Irr}(\mathbf{T}^F)$, $o(\xi) = 2$)	1	$o+1$	$(G\,\mathrm{Z}(\widetilde{G}) \rtimes \langle F_0\rangle)_{\mathbb{D}_1}$	2
$\chi \in \mathrm{Irr}(G_{\mathbb{D}_1} \mid \xi)$ ($\xi \in \mathrm{Irr}(\mathbf{T}^F)$, $4 \le o(\xi)$ 2-power)	2	o	$A_{\mathbb{D}_1,\mathbf{T}^F,\{\xi,\xi^{-1}\}}$	$2^{o-1}-1$

The group $G_{\mathbb{D}_2}$ has $P := \langle a'^2, x\rangle$ as a normal subgroup of index 6, and all characters of $\mathrm{Irr}(G_{\mathbb{D}_2})$ belong to the principal block of $G_{\mathbb{D}_2}$, hence $\mathrm{Irr}(G_{\mathbb{D}_2}) = \mathrm{Irr}(b_{0,\mathbb{D}_2})$. Let ν be a linear nontrivial character of P. Let $\psi \in \mathrm{Irr}(P)$ be the character of degree 2. The characters of $\mathrm{Irr}(G_{\mathbb{D}_2} \mid \nu)$ of degree 3 can be obtained by induction from two linear characters of $\langle a', x\rangle$.

Characters of $b_{0,\mathbb{D}_2}$	$\chi(1)$	$d(\chi)$	$A_{\mathbb{D}_2,\chi}$	# char.
$\chi \in \mathrm{Irr}(G_{\mathbb{D}_2} \mid 1_P)$ (of degree 1)	1	4	$(G\,\mathrm{Z}(\widetilde{G}) \rtimes \langle F_0\rangle)_{\mathbb{D}_2}$	2
$\chi \in \mathrm{Irr}(G_{\mathbb{D}_2} \mid 1_P)$ (of degree 2)	2	3	$(G\,\mathrm{Z}(\widetilde{G}) \rtimes \langle F_0\rangle)_{\mathbb{D}_2}$	1
$\chi \in \mathrm{Irr}(G_{\mathbb{D}_2} \mid \nu)$	3	4	$(G\,\mathrm{Z}(\widetilde{G}) \rtimes \langle F_0\rangle)_{\mathbb{D}_2}$	2
$\chi \in \mathrm{Irr}(G_{\mathbb{D}_2} \mid \psi_2)$ (of degree 2)	2	3	$G_{\mathbb{D}_2}\,\mathrm{Z}(\widetilde{G})(G \rtimes \langle F_0\rangle)_{\mathbb{D}_2,\mathbf{T}^F,\{a',a'^{-1}\}}$	2
$\chi \in \mathrm{Irr}(G_{\mathbb{D}_2} \mid \psi_2)$ (of degree 4)	4	2	$(G \rtimes \langle F_0\rangle)_{\mathbb{D}_2}$	1

The group $G_{\mathbb{D}_4} = \langle a', x\rangle$ has the cyclic subgroup $\langle a'\rangle$ of order 8 as normal subgroup of index 2, hence $\mathrm{Irr}(G_{\mathbb{D}_4})$ has four linear characters and three characters of degree 2.

Characters of $b_{0,\mathbb{D}_4}$	$\chi(1)$	$d(\chi)$	$A_{\mathbb{D}_4,\chi}$	# char.
$\chi \in \mathrm{Irr}(G_{\mathbb{D}_2} \mid \xi)$ ($\xi \in \mathrm{Irr}(\langle a'\rangle)$, $o(\xi) \mid 2$)	1	4	$(G\,\mathrm{Z}(\widetilde{G}) \rtimes \langle F_0\rangle)_{\mathbb{D}_2}$	4
$\chi \in \mathrm{Irr}(G_{\mathbb{D}_2} \mid \xi)$ ($\xi \in \mathrm{Irr}(\langle a'\rangle)$, $o(\xi) = 4$)	2	3	$(G\,\mathrm{Z}(\widetilde{G}) \rtimes \langle F_0\rangle)_{\mathbb{D}_2}$	1
$\chi \in \mathrm{Irr}(G_{\mathbb{D}_2} \mid \xi)$ ($\xi \in \mathrm{Irr}(\langle a'\rangle)$, $o(\xi) = 4$)	2	3	$G_{\mathbb{D}_2}\,\mathrm{Z}(\widetilde{G})(G \rtimes \langle F_0\rangle)_{\mathbf{T}^F,\{a',a'^{-1}\}}$	2

Reviewing the given lists one can see that there exists an A-equivariant defect preserving bijection

$$\Omega : \overline{\mathcal{C}^{\le d_0}(b_0)_+} \to \overline{\mathcal{C}^{\le d_0}(b_0)_-}.$$

This map can be chosen such that for any $(\mathbb{D}, \theta) \in \overline{\mathcal{C}^{\le d_0}(b_0)_+}$ and $(\mathbb{D}', \theta') \in \Omega(\overline{(\mathbb{D}, \theta)})$ the groups $\ker(\theta_Z)$ and $\ker(\theta'_Z)$ coincide. Let $(\mathbb{D}, \theta) \in \overline{\mathcal{C}^{\le d_0}(b_0)_+}$ and $Z_0 := \ker(\theta_Z)$. If $A_{\mathbb{D},\theta} \le G\,\mathrm{Z}(\widetilde{G}) \rtimes \langle F_0\rangle$ then Proposition 6.6 applies, and hence for $(\mathbb{D}', \theta') \in \Omega(\overline{(\mathbb{D}, \theta)})$ we get the equivalence

$$((G \rtimes \langle F_0\rangle)_{(\mathbb{D},\theta)}/Z_0, G_{\mathbb{D}}/Z_0, \overline{\theta}) \sim_{G/Z_0} ((G \rtimes \langle F_0\rangle)_{(\mathbb{D}',\theta')}/Z_0, G_{\mathbb{D}}/Z_0, \overline{\theta}'),$$

where $\overline{\theta}$ and $\overline{\theta}'$ are the characters corresponding to θ and θ'.

It remains to prove the analogous property in the case of $A_{\mathbb{D},\theta} \not\le G\,\mathrm{Z}(\widetilde{G}) \rtimes \langle F_0\rangle$. The character $\theta \in \{1_G, \mathrm{St}_G, \mathrm{R}_{\mathbf{T}}^{\mathbf{G}}(\xi) \mid \xi \in \mathrm{Irr}(\mathbf{T}^F)$ with $2 < o(\theta) < 2^o\}$ can be considered as a character of G/Z and has an extension $\widetilde{\theta} \in A_\theta/\mathrm{Z}(\widetilde{G})$ such that for every $G\,\mathrm{Z}(\widetilde{G})/\mathrm{Z}(\widetilde{G}) \le J \le A_\theta/\mathrm{Z}(\widetilde{G})$ the character $\widetilde{\theta}_J$ belongs to the principal block of J.

Every $\theta' \in \mathrm{Irr}(G_{\mathbb{D}_1} \mid 1_{\mathbf{T}^F}) \cup \mathrm{Irr}(G_{\mathbb{D}_1} \mid \xi)$ ($\xi \in \mathrm{Irr}(\mathbf{T}^F)$ with $2 < o(\xi) < 2^o$) seen as character of $G_{\mathbb{D}_1}\,\mathrm{Z}(\widetilde{G})/\mathrm{Z}(\widetilde{G})$ has an extension $\widetilde{\theta}' \in \mathrm{Irr}(A_{\mathbb{D}_1,\theta}/\mathrm{Z}(\widetilde{G}))$ such that $\widetilde{\theta}'_J$ belongs to the principal block for every J with $G_{\mathbb{D}_1}\,\mathrm{Z}(\widetilde{G})/\mathrm{Z}(\widetilde{G}) \le J \le A_{\mathbb{D}_1,\theta'}/\mathrm{Z}(\widetilde{G})$.

If $\theta \in \mathrm{Irr}(G_{\mathbb{D}_0})$ is a character of the form $\mathrm{R}_{\mathbf{T}}^{\mathbf{G}}(\xi)$ ($\xi \in \mathrm{Irr}(\mathbf{T}^F)$ with $o(\xi) = 2^o$) or $\theta \in \mathrm{Irr}(G_{\mathbb{D}_1} \mid \xi)$ ($\xi \in \mathrm{Irr}(\mathbf{T}^F)$ with $o(\xi) = 2^o$), then θ extends to some $\widetilde{\theta} \in \mathrm{Irr}(A_{\mathbb{D}_0,\theta})$ and $\widetilde{\theta} \in \mathrm{Irr}(A_{\mathbb{D}_1,\theta})$ respectively, by an adaptation of the proof given in [IMN07, Theorem (14.1)]. For $\mathrm{R}_{\mathbf{T}}^{\mathbf{G}}(\xi)$ and the character $\theta \in \mathrm{Irr}(G_{\mathbb{D}_1} \mid \xi)$ the extension can be chosen to lie over the same faithful character of $\mathrm{Z}(\widetilde{G})$.

Let $i \in \{1, 2\}$ be such that $\theta \in G_{\mathbb{D}_i}$. There exists a linear character $\mu \in \mathrm{Irr}(A_{\mathbb{D}_i,\theta})$ with $\ker(\mu) \ge \widetilde{G}_{\mathbb{D}_i}$ such that $(\mu\widetilde{\theta})_J$ belongs to the principal block for every J with $\widetilde{G}_{\mathbb{D}_i} \le J \le A_{\mathbb{D}_i,\theta}$ (see [KS15, Theorem C (a)(2)]). Straightforward calculations show that $(\mu\widetilde{\theta})_J$ belongs to the principal block for every J with $G_{\mathbb{D}_i} \le J \le A_{\mathbb{D}_i,\theta}$.

This implies that for those characters $\theta = \mathrm{R}_{\mathbf{T}}^{\mathbf{G}}(\xi)$ and $\theta' \in \mathrm{Irr}(G_{\mathbb{D}_1} \mid \xi)$ we have the equivalence

$$(A_{(\theta)}, G_{\mathbb{D}}, \theta) \sim_G (A_{(\mathbb{D}_1,\theta')}, G_{\mathbb{D}_1}, \theta'). \qquad \square$$

In the above proof we have simultaneously proven the following statement.

Corollary 9.4. *Let q be an odd prime power such that $G := \mathrm{SL}_2(q)$ is the universal covering group of the simple group $G/\mathrm{Z}(G)$. Then the Inductive AM Condition from* [Spä13a, *Definition* 7.2] *is satisfied for the principal block of $G/\mathrm{Z}(G)$.*

Proof. Let p be a prime and P a Sylow p-subgroup of G. Let d be the integer such that $p^d = |G|_p$. Then the bijection

$$\Omega : \overline{\mathcal{C}^{\le d}(B)_+} \to \overline{\mathcal{C}^{\le d}(B)_-}$$

constructed above maps every G-orbit $\overline{(\mathbb{D}, \theta)}$ with $\theta \in \mathrm{Irr}_0(G)$ to some $\overline{(\mathbb{D}', \theta')}$ with $\theta' \in \mathrm{Irr}_0(M)$, where M is a fixed $\mathrm{Aut}(G)_P$-stable group M with $\mathrm{N}_G(P) \le M \lneq G$.

According to Proposition 6.12 this implies that the Inductive AM Condition holds for $G/\mathrm{Z}(G)$. $\square$

Acknowledgments. The author thanks Gunter Malle for a careful reading and the referee for his numerous suggestions that improved the readability of the paper.

This research has been supported by the European Research Council Advanced Grant 291512.

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