J. Eur. Math. Soc. 19, 1159–1187

DOI 10.4171/JEMS/690

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A quasiconformal composition problem for the Q-spaces

Received June 14, 2014

Abstract. Given a quasiconformal mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ with $n \geq 2$, we show that (un-)boundedness of the composition operator $\mathbf{C}_f$ on the spaces $Q_\alpha(\mathbb{R}^n)$ depends on the index α and the degeneracy set of the Jacobian J_f. We establish sharp results in terms of the index α and the local/global self-similar Minkowski dimension of the degeneracy set of J_f. This gives a solution to [3, Problem 8.4] and also reveals a completely new phenomenon, which is totally different from the known results for Sobolev, BMO, Triebel–Lizorkin and Besov spaces. Consequently, Tukia–Väisälä's quasiconformal extension $f : \mathbb{R}^n \to \mathbb{R}^n$ of an arbitrary quasisymmetric mapping $g : \mathbb{R}^{n-p} \to \mathbb{R}^{n-p}$ is shown to preserve $Q_\alpha(\mathbb{R}^n)$ for any $(\alpha, p) \in (0, 1) \times [2, n) \cup (0, 1/2) \times \{1\}$. Moreover, $Q_\alpha(\mathbb{R}^n)$ is shown to be invariant under inversions for all $0 < \alpha < 1$.

Keywords. Quasiconformal mappings, compositions, Q-spaces

Contents

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Mathematics Subject Classification (2010): 42B35, 46E30, 47B38, 30H25

1. Introduction

Quasiconformal mappings can be characterized via invariant function spaces. For example, a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 2$, is quasiconformal if and only if the composition operator $\mathbf{C}_f$ (given by $\mathbf{C}_f(u) = u \circ f$) is bounded on the homogeneous Sobolev space $\dot{W}^{1,n}(\mathbb{R}^n)$ (see for example [5]). The composition property is most easily seen from the usual analytic definition, according to which a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 2$, is quasiconformal if $f \in W^{1,1}_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^n)$ and there is a constant $K \geq 1$ such that

$$|Df(x)|^n \leq K J_f(x) \qquad \text{a.e. } x \in \mathbb{R}^n.$$

Indeed, modulo technicalities, one simply uses the chain rule and a change of variables. It is far less obvious that also the invariance of the Triebel–Lizorkin spaces $\dot{F}^s_{n/s,q}(\mathbb{R}^n)$ with $0 < s < 1$ and $n/(n+s) < q < \infty$ characterizes quasiconformality (see [10, 2, 6, 4]). The difficulty here is that one has to deal with "fractional derivatives" and thus the inequality from the analytic definition is not immediately helpful. For the off-diagonal Besov spaces $\dot{B}^s_{n/s,q}(\mathbb{R}^n)$ with $q \neq n/s$, the situation is different: each homeomorphism f for which $\mathbf{C}_f$ is bounded on $\dot{B}^s_{n/s,q}(\mathbb{R}^n)$ has to be quasiconformal and even bi-Lipschitz; these spaces are clearly bi-Lipschitz invariant (see [4]). Recall here that f is *bi-Lipschitz* if there exists a constant $L \geq 1$ such that

$$\frac{1}{L}|x - y| \leq |f(x) - f(y)| \leq L|x - y|, \qquad \forall x, y \in \mathbb{R}^n.$$

Furthermore, the John–Nirenberg space $\mathrm{BMO}(\mathbb{R}^n)$ is invariant under quasiconformal mappings and each sufficiently regular homeomorphism f for which $\mathbf{C}_f$ is a bounded operator on $\mathrm{BMO}(\mathbb{R}^n)$ is necessarily quasiconformal (see [7, 1]).

In their 2000 paper [3], Essen, Jasson, Peng and Xiao introduced the so-called *Q-spaces* $Q_\alpha(\mathbb{R}^n)$, $0 < \alpha < 1$, that satisfy

$$\dot{W}^{1,n}(\mathbb{R}^n) \subsetneq \dot{F}^{\alpha}_{n/\alpha,n/\alpha}(\mathbb{R}^n) \subsetneq Q_\alpha(\mathbb{R}^n) \subsetneq \mathrm{BMO}(\mathbb{R}^n).$$

Each $Q_\alpha(\mathbb{R}^n)$ consists of all $u \in L^2_{\text{loc}}(\mathbb{R}^n)$ with

$$\|u\|_{Q_\alpha(\mathbb{R}^n)} = \sup_{x_0 \in \mathbb{R}^n,\, r>0} \left(r^{2\alpha-n} \int_{B(x_0,r)} \int_{B(x_0,r)} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\alpha}}\, dx\, dy \right)^{1/2} < \infty.$$

The above definition actually makes perfect sense for all $-\infty < \alpha < \infty$, but the case $\alpha \geq 1$ (when $n \geq 2$) reduces to constant functions and the case $\alpha < 0$ to $\mathrm{BMO}(\mathbb{R}^n)$ (see [3]). These spaces have received considerable interest. In [3], five open problems related to the spaces $Q_\alpha(\mathbb{R}^n)$ were posed. All but the following one of them have by now been solved.

A quasiconformal composition problem for the Q-spaces ([3, Problem 8.4]). *Let f be a quasiconformal mapping. Prove or disprove the boundedness of the composition operator $\mathbf{C}_f$ on $Q_\alpha(\mathbb{R}^n)$ with $\alpha \in (0, 1)$.*

The above string of inclusions of function spaces, all of which except for the Q-spaces are known to be quasiconformally invariant, suggests that the answer should be in the positive.

We show that, surprisingly, the answer to the above question depends on the quasiconformal mapping in question through the shrinking properties of the mapping. For example, the quasiconformal mapping $f(x) = x|x|$ induces a bounded composition operator for all $0 < \alpha < 1$, but if the Jacobian of a quasiconformal mapping decays to zero when we approach a sufficiently large set, then the invariance may fail. Thus, the case of Q-spaces is very different from the other function spaces that we discussed above.

In order to state our results, we need to introduce some terminology whose analogues have appeared in estimating the upper box-counting dimension of the singular set of a suitable weak solution of the Navier–Stokes system [8].

Definition 1.1. For a set $E \subseteq \mathbb{R}^n$ and every $r > 0$, denote by $N_{\text{cov}}(r, E)$ the minimal number of cubes with edge length r required to cover E.

(i) The *local self-similar Minkowski dimension* of E is defined as

$$\overline{\dim}_L E = \liminf_{N\to\infty} \limsup_{r\to 0} \sup_{\substack{B\subset\mathbb{R}^n \\ Nr\le r_B\le 1}} \frac{\log_2 N_{\text{cov}}(r, E\cap B)}{\log_2(r_B/r)}, \tag{1.1}$$

where the supremum is taken over all balls $B = B(x_B, r_B) \subset \mathbb{R}^n$ with $r_B \in [Nr, 1]$.

(ii) The *global self-similar Minkowski dimension* of E is defined as

$$\overline{\dim}_{LG} E = \liminf_{N\to\infty} \sup_{r>0} \sup_{\substack{B\subset\mathbb{R}^n \\ r_B\ge Nr}} \frac{\log_2 N_{\text{cov}}(r, E\cap B)}{\log_2(r_B/r)}, \tag{1.2}$$

where the first supremum is taken over all $r \in (0, \infty)$ and the second is over all balls $B = B(x_B, r_B) \subset \mathbb{R}^n$ with $r_B \in [Nr, \infty)$.

We also need the concept of the local Muckenhoupt class.

Definition 1.2. For a closed set $E \subseteq \mathbb{R}^n$ and a nonnegative function $w : \mathbb{R}^n \to \mathbb{R}$, we say that w belongs to the *local Muckenhoupt class* $A_1(\mathbb{R}^n; E)$ provided there exists a positive constant C such that

$$⨍_B w(z)\,dz \le C \operatorname*{ess\,inf}_{x\in B} w(x) \tag{1.3}$$

for every ball $B = B(x_B, r_B) \subset \mathbb{R}^n$ with $2r_B < d(x_B, E)$. Naturally, $A_1(\mathbb{R}^n; \emptyset)$ stands for the Muckenhoupt class $A_1(\mathbb{R}^n)$. Accordingly, E is called the *degeneracy set* of w when $w \in A_1(\mathbb{R}^n; E)$.

The main result of this paper is the following theorem.

Theorem 1.3. *Given $n \geq 2$, let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a quasiconformal mapping with $J_f \in A_1(\mathbb{R}^n; E)$ for some closed set $E \subseteq \mathbb{R}^n$. If E is a bounded set with $\overline{\dim}_L E \in [0, n)$ or E is an unbounded set with $\overline{\dim}_{LG} E \in [0, n)$, then $\mathbf{C}_f$ is bounded on $Q_\alpha(\mathbb{R}^n)$ for all*

$$0 < \alpha < \begin{cases} \min\{1, (n - \overline{\dim}_L E)/2\} & \text{if } E \text{ is bounded,} \\ \min\{1, (n - \overline{\dim}_{LG} E)/2\} & \text{if } E \text{ is unbounded.} \end{cases} \tag{1.4}$$

In particular, if E is a bounded set with $\overline{\dim}_L E \in [0, n-2]$ or E is an unbounded set with $\overline{\dim}_{LG} E \in [0, n-2]$, then $\mathbf{C}_f$ is bounded on $Q_\alpha(\mathbb{R}^n)$ for all $\alpha \in (0, 1)$.

Theorem 1.3 is essentially sharp—see Theorems 1.6 and 1.7 below.

As a first important consequence of Theorem 1.3, we have the following result.

Corollary 1.4. *Let $\alpha \in (0, 1)$ and $0 \neq \beta \in \mathbb{R}$. If $f(z) = |z|^{\beta-1}z$, then $\mathbf{C}_f$ is bounded on $Q_\alpha(\mathbb{R}^n)$. In particular, $Q_\alpha(\mathbb{R}^n)$ is conformally invariant in the sense that $g \in Q_\alpha(\mathbb{R}^n)$ if and only if $x \mapsto g(x|x|^{-2})$ is in $Q_\alpha(\mathbb{R}^n)$.*

Furthermore, for the Tukia–Väisälä quasiconformal extension $f : \mathbb{R}^n \to \mathbb{R}^n$ of an arbitrary quasiconformal (quasisymmetric) mapping $g : \mathbb{R}^{n-p} \to \mathbb{R}^{n-p}$, we obtain another important consequence of Theorem 1.3.

Corollary 1.5. *Given $1 \leq p < n$, suppose $g : \mathbb{R}^{n-p} \to \mathbb{R}^{n-p}$ is a quasiconformal mapping when $n - p \geq 2$, or a quasisymmetric mapping when $n - p = 1$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the Tukia–Väisälä quasiconformal extension of g as in [9]. Then*

(i) *$J_f, J_{f^{-1}} \in A_1(\mathbb{R}^n; \mathbb{R}^{n-p})$;*
(ii) *$\mathbf{C}_f, \mathbf{C}_{f^{-1}}$ are bounded on $Q_\alpha(\mathbb{R}^n)$ for all*

$$0 < \alpha < \begin{cases} 1/2 & \text{when } p = 1, \\ 1 & \text{when } p \geq 2. \end{cases}$$

Consequently, $u \in Q_\alpha(\mathbb{R}^n)$ if and only if $u \circ f \in Q_\alpha(\mathbb{R}^n)$.

The proof of Theorem 1.3 relies on a new characterization of Q-spaces established in Section 3. This technical result allows us to employ our Muckenhoupt assumption and the control on the number of Whitney-type balls guaranteed by our dimension estimate. We expect that our approach will allow one to handle various other function spaces as well.

Our assumption on the control of the fractal size of the degeneracy set, whether bounded or unbounded, is necessary in the following sense.

Theorem 1.6. *Let $n \geq 2$ and $0 < \alpha_0 < 1$. There is a bounded set E_{α_0} with $\overline{\dim}_L E_{\alpha_0} = n - 2\alpha_0$ and a quasiconformal (Lipschitz) mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ with $J_f \in A_1(\mathbb{R}^n; E_{\alpha_0})$ for which $\mathbf{C}_f$ is not bounded on $Q_\alpha(\mathbb{R}^n)$ for any $\alpha \in (\alpha_0, 1)$.*

The main idea in the constructions for Theorem 1.6 is to patch up suitable pieces of radial stretchings in a family of pairwise disjoint balls. In this manner, we also construct an unbounded set $\widetilde{E}_{\alpha_0} \subset \mathbb{Z}^n$ with $\overline{\dim}_{LG}\, \widetilde{E}_{\alpha_0} = n - 2\alpha$ but $\overline{\dim}_L\, \widetilde{E}_{\alpha_0} = 0$ and an associated quasiconformal mapping as in Theorem 1.6 (see below). This also shows the need for $\overline{\dim}_{LG}$ in Theorem 1.3.

Theorem 1.7. *Let $n \geq 2$ and $0 < \alpha_0 < 1$. There exists an unbounded set $\widetilde{E}_{\alpha_0} \subset \mathbb{Z}^n$ with $\overline{\dim}_{LG}\, \widetilde{E}_{\alpha_0} = n - 2\alpha_0$ but $\overline{\dim}_L\, \widetilde{E}_{\alpha_0} = 0$, and a quasiconformal (Lipschitz) mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ with $J_f \in A_1(\mathbb{R}^n; \widetilde{E}_{\alpha_0})$ for which $\mathbf{C}_f$ is not bounded on $Q_\alpha(\mathbb{R}^n)$ for any $\alpha \in (\alpha_0, 1)$.*

This paper is organized as follows: Section 2 clarifies the relationship between the Minkowski dimension and the local Minkowski dimension $\overline{\dim}_L$ or the global Minkowski dimension $\overline{\dim}_{LG}$, and also computes $\overline{\dim}_L$ and $\overline{\dim}_{LG}$ for the sets in Theorems 1.6 and 1.7; Section 3 explores a new aspect of $Q_\alpha(\mathbb{R}^n)$, which will be used in the proof of Theorem 1.3; in Section 4, we prove Theorem 1.3; Section 5 contains the proofs of Corollaries 1.4 and 1.5; Section 6 is devoted to the proofs of Theorems 1.6 and 1.7.

Finally, as the converse of the above open question, given a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ for which the composition operator $\mathbf{C}_f$ is a bounded on $Q_\alpha(\mathbb{R}^n)$ for some $\alpha \in (0, 1)$, one would like to know if f is necessarily quasiconformal. The answer is actually in the affirmative, at least under suitable regularity assumptions on the homeomorphism in question. Since this requires some work, the details will be given in a forthcoming paper.

Notation. In the following, we denote by C a positive constant which is independent of the main parameters, but may vary from line to line. The symbol $A \lesssim B$ or $B \gtrsim A$ means that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, we write $A \sim B$. For any locally integrable function u and measurable set X, we denote by $⨍_X u$ the average of u on X, that is, $⨍_X u \equiv |X|^{-1} \int_X u\, dx$. For a set Ω and $x \in \mathbb{R}^n$, we use $d(x, \Omega)$ to denote $\inf_{z \in \Omega} |x - z|$, the distance from x to Ω. For two sets $E, F \subset \mathbb{R}^n$, write $\operatorname{dist}(E, F) = \inf_{x \in E,\, y \in F} |x - y|$. By λQ, we mean the cube concentric with Q, with sides parallel to the axes, and with edge length $\ell(\lambda Q) = \lambda \ell(Q)$; similarly, λB denotes the ball concentric with Q with radius λr_B, where r_B is the radius of B.

2. Local and global Minkowski dimensions

In this section, we clarify the relation between the Minkowski dimension and the above dimensions $\overline{\dim}_L$ and $\overline{\dim}_{LG}$. Recall that for a bounded set $E \subset \mathbb{R}^n$, its *Minkowski dimension* $\overline{\dim}_M\, E$ is defined by

$$\overline{\dim}_M\, E = \limsup_{r \to 0} \frac{\log_2 N_{\mathrm{cov}}(r, E)}{\log_2 (1/r)},$$

where $N_{\mathrm{cov}}(r, E)$ is the minimum number of cubes with edge length r required to cover E.

Lemma 2.1. (i) *For every set $E \subset \mathbb{R}^n$ and every $R \geq 1$, we have*

$$\overline{\dim}_L E = \liminf_{N\to\infty} \limsup_{r\to 0} \sup_{\substack{B\subset\mathbb{R}^n \\ Nr\leq r_B\leq R}} \frac{\log_2 N_{\mathrm{cov}}(r, E\cap B)}{\log_2(r_B/r)}.$$

(ii) *For every set $E \subset \mathbb{R}^n$, we always have*

$$0 \leq \sup_B \overline{\dim}_M(E\cap B) \leq \overline{\dim}_L E \leq \overline{\dim}_{LG} E \leq n,$$

where the supremum is taken over all balls in $\mathbb{R}^n$.

(iii) *If $E \subset F$, then $\overline{\dim}_L E \leq \overline{\dim}_L F$ and $\overline{\dim}_{LG} E \leq \overline{\dim}_{LG} F$.*

Proof. (i) From the definition, we always have

$$\overline{\dim}_L E \leq \liminf_{N\to\infty} \limsup_{r\to 0} \sup_{\substack{B\subset\mathbb{R}^n \\ Nr\leq r_B\leq R}} \frac{\log_2 N_{\mathrm{cov}}(r, E\cap B)}{\log_2(r_B/r)}.$$

Towards the reverse inequality, notice that every ball B of radius $1 \leq r_B \leq R$ can be covered by $c_n R^n$ balls B_i of radii 1. So

$$N_{\mathrm{cov}}(r, E\cap B) \leq c_n R^n \sup\{N_{\mathrm{cov}}(r, E\cap \widetilde{B}) : \widetilde{B} \subset \mathbb{R}^n \text{ with } r_{\widetilde{B}} = 1\},$$

and hence for all $r < \min\{r_B/N, 1\}$ we have

$$\frac{\log_2 N_{\mathrm{cov}}(r, E\cap B)}{\log_2(r_B/r)} \leq \frac{\log_2 c_n R^n}{\log_2 N} + \sup_{\substack{\widetilde{B}\subset\mathbb{R}^n \\ r_{\widetilde{B}}=1}} \frac{\log_2 N_{\mathrm{cov}}(r, E\cap \widetilde{B})}{\log_2(1/r)}.$$

Since the first term on the right-hand side tends to 0 as $N \to \infty$, by the definition of $\overline{\dim}_L E$ we obtain the desired inequality.

(ii) Obviously, $\overline{\dim}_{LG} E \leq n$ is obtained from $N_{\mathrm{cov}}(r, E\cap B) \leq (2r_B/r)^n$ for every ball B with radius $r_B \geq Nr$: indeed,

$$\overline{\dim}_{LG} E \leq \liminf_{N\to\infty} \sup_{r>0} \sup_{\substack{B\subset\mathbb{R}^n \\ r_B\geq Nr}} \frac{n\log_2(r_B/r) + n}{n\log_2(r_B/r)} = \liminf_{N\to\infty} \frac{n\log_2 N + n}{n\log_2 N} = n.$$

The other inequalities follow from the definitions and (i) directly.

(iii) These statements are trivial. □

If E is a set of finitely many points, observing that $N_{\mathrm{cov}}(r, E\cap B) \lesssim 1$ for every ball B with radius $r_B \geq Nr$, we obtain

$$\overline{\dim}_{LG} E \leq \liminf_{N\to\infty} \frac{\log_2 C}{\log_2 N} = 0,$$

which implies that

$$\overline{\dim}_M E = \overline{\dim}_L E = \overline{\dim}_{LG} E = 0.$$

However, for a countable set E, $\overline{\dim}_{LG}$, $\overline{\dim}_L$ and $\sup_B \overline{\dim}_M(E \cap B)$ may be very different. Write $\mathbb{N}^n = \mathbb{N} \times \cdots \times \mathbb{N}$ and $(2^{\mathbb{N}})^n = 2^{\mathbb{N}} \times \cdots \times 2^{\mathbb{N}}$ with $2^{\mathbb{N}} = \{2^k : k \in \mathbb{N}\}$. For $\theta \in [0, 1]$, set

$$2^{\mathbb{N}_\theta} := \bigcup_{k\in\mathbb{N}\cup\{0\}} A_{k,\theta} := \bigcup_{k\in\mathbb{N}\cup\{0\}} \{2^k, 2^k + 1, \dots, 2^k + 2^{[\theta k]}\}. \tag{2.1}$$

where $[\theta k]$ is the largest integer less than or equal to θk. Write $(2^{\mathbb{N}_\theta})^n = 2^{\mathbb{N}_\theta} \times \cdots \times 2^{\mathbb{N}_\theta}$. Observe that

$$\begin{cases} 2^{\mathbb{N}_\theta} = \mathbb{N} \cup \{0\} & \text{when } \theta = 1, \\ 2^{\mathbb{N}_\theta} = 2^{\mathbb{N}} \cup \{1\} & \text{when } \theta = 0. \end{cases}$$

We always have

$$\overline{\dim}_M((2^{\mathbb{N}_\theta})^n \cap B) = \overline{\dim}_L((2^{\mathbb{N}_\theta})^n \cap B) = \overline{\dim}_{LG}((2^{\mathbb{N}_\theta})^n \cap B) = 0$$

for all balls B and all $\theta \in [0, 1]$ since $(2^{\mathbb{N}_\theta})^n \cap B$ only contains finitely many points.

Lemma 2.2. *Let $\theta \in [0, 1]$. Then*

$$\overline{\dim}_{LG}\, (2^{\mathbb{N}_\theta})^n = \theta n; \tag{2.2}$$

in particular, $\overline{\dim}_{LG}\, (2^{\mathbb{N}})^n = 0$ *and* $\overline{\dim}_{LG}\, \mathbb{N}^n = n$. *But* $\overline{\dim}_L\, (2^{\mathbb{N}_\theta})^n = 0$.

Proof. We first show that $\overline{\dim}_L\, (2^{\mathbb{N}_\theta})^n = 0$. Observe that each $B \subset \mathbb{R}^n$ with $r_B \le 1$ contains at most a uniform number of points in $\mathbb{Z}^n$. So for each $N \ge 1$ and $r \in (0, r_B/N)$, we can cover $B \cap (2^{\mathbb{N}_\theta})^n$ by a uniform number of balls of radii r, that is, $N_{\mathrm{cov}}(1, (2^{\mathbb{N}_\theta})^n \cap B) \lesssim 1$, which implies that $\dim_L\, (2^{\mathbb{N}_\theta})^n \le 0$ by definition. So by Lemma 2.1, $\overline{\dim}_L\, (2^{\mathbb{N}_\theta})^n = 0$.

To show (2.2), we first consider the easy cases $\overline{\dim}_{LG}\, \mathbb{N}^n = n$ and $\overline{\dim}_{LG}\, (2^{\mathbb{N}})^n = 0$. Indeed, for every ball $B \subset \mathbb{R}^n$ with $r_B = N$, we have

$$N_{\mathrm{cov}}(1, \mathbb{N}^n \cap B) = \sharp(\mathbb{N}^n \cap B) \ge (N/\sqrt{n})^n,$$

which implies that

$$\overline{\dim}_{LG}\, \mathbb{N}^n \ge \liminf_{N\to\infty} \frac{\log_2 (N/\sqrt{n})^n}{\log_2 N} = n,$$

and hence, by Lemma 2.1, $\overline{\dim}_{LG}\, \mathbb{N}^n = n$.

On the other hand, for each N and $r > 0$, if $r \le 1$ and $Nr < r_B$, we have

$$N_{\mathrm{cov}}(r, (2^{\mathbb{N}})^n \cap B) \le (\log_2 r_B)^n;$$

if $2^k < r \le 2^{k+1}$ for some $k \ge 0$, we have

$$N_{\mathrm{cov}}(r, (2^{\mathbb{N}})^n \cap B) \le \sqrt{n}[\log_2(r_B/r + 2)]^n.$$

Hence

$$\overline{\dim}_{LG}\, (2^{\mathbb{N}})^n \le \liminf_{N\to\infty} \sup_{r\ge1} \sup_{B:\, r_B \ge Nr} \frac{n \log_2[\sqrt{n} \log_2(r_B/r + 2)]}{\log_2(r_B/r)} = 0.$$

So by Lemma 2.1, we have $\overline{\dim}_{LG}\, (2^{\mathbb{N}})^n = 0$.

Generally, we let $\theta \in (0, 1)$. For every ball $B = B(0, \sqrt{n}\, 2^{m+1})$ with $m \geq 2/\theta + 1$, we have

$$N_{\mathrm{cov}}(1, (2^{\mathbb{N}_\theta})^n \cap B)) \geq \sharp((2^{\mathbb{N}_\theta})^n \cap B) \geq 2^{n\theta m},$$

and hence

$$\overline{\dim}_{LG}\, (2^{\mathbb{N}_\theta})^n \geq \liminf_{N\to\infty} \sup_{2^{m+1}\geq N} \frac{n\theta m}{m+1} = \liminf_{N\to\infty} \frac{n\theta(\log_2 N) - n\theta}{\log_2 N} = n\theta.$$

The proof of $\overline{\dim}_{LG}\, (2^{\mathbb{N}_\theta})^n \leq \theta n$ is reduced to verifying that for every large N, all $r > 0$ and all balls B with $r_B \geq Nr$, we have

$$N_{\mathrm{cov}}(r, (2^{\mathbb{N}_\theta})^n \cap B) \lesssim (r_B/r)^{\theta n}. \tag{2.3}$$

Indeed, this implies that

$$\begin{aligned}\overline{\dim}_{LG}\, (2^{\mathbb{N}_\theta})^n &\leq \liminf_{N\to\infty} \sup_{r>0} \sup_{B:\, r_B\geq Nr} \frac{\log_2[(r_B/r)^{\theta n}] + \log_2 C}{\log_2(r_B/r)} \\ &= \liminf_{N\to\infty} \frac{\theta n \log_2 N + \theta n \log_2 C}{\log_2 N} = \theta n.\end{aligned}$$

To prove (2.3), we consider two cases under the assumption $N \geq 2^5$.

Case 1: $0 < r \leq 1$. If $r_B < 2$, then $(2^{\mathbb{N}_\theta})^n \cap B$ contains no more than a uniform number of points, and hence

$$\sharp((2^{\mathbb{N}_\theta})^n \cap B) \lesssim 1 \lesssim (r_B/r)^{\theta n}.$$

If $2^m < r_B \leq 2^{m+1}$ for some $m > 1$, then $(2^{\mathbb{N}_\theta})^n \cap B \subset [0, 2^{m+2}]^n$. Notice that the interval $[0, 2^{m+2}]$ contains at most $\sum_{k=1}^{m+1} 2^{\theta k} \sim 2^{\theta m}$ points of $2^{\mathbb{N}_\theta}$, and so we have

$$\sharp((2^{\mathbb{N}_\theta})^n \cap B) \lesssim 2^{\theta mn} \lesssim (r_B/r)^{\theta n},$$

which implies that

$$N_{\mathrm{cov}}(r, (2^{\mathbb{N}_\theta})^n \cap B) \leq \sharp((2^{\mathbb{N}_\theta})^n \cap B) \lesssim (r_B/r)^{\theta n}.$$

Case 2: $r > 1$. Assume that $2^\ell < r < 2^{\ell+1}$. Given a ball B with $r_B \geq Nr$, assume that $2^m < r_B \leq 2^{m+1}$ for some $m \geq 5+\ell$. Then $(2^{\mathbb{N}_\theta})^n \cap B \subset [0, 2^{m+2}]^n$. Observe that $[0, 2^\ell]$ can be covered by an interval of length r. If $\ell \leq k \leq [\ell/\theta]$, then $\{2^k, 2^k+1, \ldots, 2^k+2^{[\theta k]}\}$ can be covered by an interval of length r. If $k > [\ell/\theta]$, then $\{2^k, 2^k+1, \ldots, 2^k+2^{[\theta k]}\}$ can be covered by $2^{[\theta k]-\ell}+1$ intervals of length r. Thus when $m \leq [\ell/\theta]-2$, $2^{\mathbb{N}_\theta} \cap [0, 2^{m+2}]$ can be covered by $m - \ell + 2$ intervals of length r. If $m > [\ell/\theta] - 2$, then $2^{\mathbb{N}_\theta} \cap [0, 2^{m+2}]$ can be covered by $2^{\theta m - \ell}$ intervals of length r. In both cases, $2^{\mathbb{N}_\theta} \cap [0, 2^{m+2}]$ can be covered by $C2^{\theta(m-\ell)} \leq C(r_B/r)^\theta$ intervals of length r. Therefore

$$N_{\mathrm{cov}}(r, (2^{\mathbb{N}_\theta})^n \cap [0, 2^{m+2}]^n) \lesssim (r_B/r)^{\theta n},$$

which gives (2.2) as desired. $\square$

Remark 2.3. Lemma 2.2 indicates that the dimension $\overline{\dim}_{LG}$ not only measures the local self-similarity and local Minkowski size but also the global self-similarity of E.

By a slight modification of the standard Cantor construction, we obtain a set E_a and its self-similar extension $\mathcal{E}_a$ such that $\overline{\dim}_L \mathcal{E}_a$ and $\overline{\dim}_{LG} \mathcal{E}_a$ are the same and coincide with $\overline{\dim}_M E_a$. Precisely, the sets E_a and $\mathcal{E}_a$ are defined as follows. Let $a \in (0,1)$. Let I_i, $i = 1, 2$, be the two closed intervals obtained by removing the middle open interval of length a from $I_0 = [0, 1]$ ordered from left to right; when $m \geq 2$, the subintervals $I_{i_1 \cdots i_m}$, $i_m = 1, 2$, are the two closed intervals obtained by removing the middle open intervals of length $a[(1-a)/2]^{m-1}$ from $I_{i_1 \cdots i_{m-1}}$ ordered from left to right. Notice that $|I_{i_1 \cdots i_m}| = [(1-a)/2]^m$ for $m \geq 1$. For each $m \geq 1$, set

$$I_a^m = \bigcup_{i_1,\ldots,i_m \in \{1,2\}} I_{i_1 \cdots i_m}, \qquad E_a^m = (I_a^m)^n = I_a^m \times \cdots \times I_a^m.$$

Notice that E_a^m consists of 2^{mn} disjoint cubes $\{Q_{m,j}\}_{j=1}^{2^{mn}}$ with edge length $[(1-a)/2]^m$, and $E_a^{m+1} \subset E_a^m$. Denote by $z_{m,j}$ the center of $Q_{m,j}$, and $z_0 = (1/2, \ldots, 1/2)$ the center of $Q_0 = I_0^n$. Denote by E_a the closure of the collection of all these centers, that is,

$$E_a = \overline{\{z_0, z_{m,j} : m \in \mathbb{N},\ j = 1, \ldots, 2^{mn}\}}. \tag{2.4}$$

Set

$$\mathcal{E}_a = \bigcup_{k \geq 0} \left\{ \left(\frac{2}{1-a} \right)^k x : x \in E_a \right\}. \tag{2.5}$$

In this case, we consider the larger family $\{\widetilde{Q}_{m,j}\}_{m \in \mathbb{Z},\, j \in \mathbb{N}}$ consisting of all

$$\left\{ \left(\frac{2}{1-a} \right)^k x : x \in Q_{m+k,i} \right\}$$

for all possible $k \geq -m$ and $i = 1, \ldots, 2^{(m+k)n}$. Let $\widetilde{z}_{m,j}$ be the center of $\widetilde{Q}_{m,j}$. We also have

$$\mathcal{E}_a = \overline{\{\widetilde{z}_{m,j} : m \in \mathbb{Z},\ j \in \mathbb{N}\}}. \tag{2.6}$$

Lemma 2.4. *For every $a \in (0,1)$,*

$$\overline{\dim}_M E_a = \overline{\dim}_L E_a = \overline{\dim}_L \mathcal{E}_a = \overline{\dim}_{LG} E_a = \overline{\dim}_{LG} \mathcal{E}_a = \frac{n}{\log_2[2/(1-a)]}.$$

Proof. By Lemma 2.1, it suffices to show that

$$\overline{\dim}_M E_a \geq \frac{n}{\log_2[2/(1-a)]} \quad \& \quad \overline{\dim}_L \mathcal{E}_a \leq \frac{n}{\log_2[2/(1-a)]}.$$

To this end, notice that for each $k > m$, we have

$$2^{(k-m)n} < N_{\mathrm{cov}}([1-a)/2]^k, E_a \cap \widetilde{Q}_{m,j}) \leq 2^{(k-m)n} + \sum_{\ell=m}^{k-1} 2^{\ell n} < 2^{(k+1-m)n},$$

where recall that $\ell(\widetilde{Q}_{m,j}) = [(1-a)/2]^m$. For each $r < [(1-a)/2]^{m+2}$, picking $k_r > m$ such that

$$[(1-a)/2]^{k_r} < r \le [(1-a)/2]^{k_r-1},$$

we have

$$N_{\rm cov}([1-a)/2]^{k_r+1}, \mathcal{E}_a \cap \widetilde{Q}_{m,j}) \le N_{\rm cov}(r, \mathcal{E}_a \cap \widetilde{Q}_{m,j}) < N_{\rm cov}([1-a)/2]^{k_r}, \mathcal{E}_a \cap \widetilde{Q}_{m,j}),$$

and hence $N_{\rm cov}(r, \mathcal{E}_a) \sim 2^{(k_r-m)n}$. In particular, $N_{\rm cov}(r, E_a) \gtrsim 2^{k_r n}$, which implies that

$$\begin{aligned}\overline{\dim}_M E_a &\ge \limsup_{r\to 0} \frac{k_r n + \log_2 C}{\log_2(1/r)} = \limsup_{k_r\to\infty} \frac{k_r n + \log_2 C}{k_r \log_2[2/(1-a)] + \log_2 C_1}\\ &= \frac{n}{\log_2[2/(1-a)]}.\end{aligned}$$

Moreover, for each ball B with $r_B \ge [(1-a)/2]^3 r$, there exists a $k_B \le k_\epsilon - 2$ such that

$$[(1-a)/2]^{k_B} < r_B \le [(1-a)/2]^{k_B-1}.$$

Hence

$$N_{\rm cov}(r, \mathcal{E}_a \cap B) \le N_{\rm cov}([1-a)/2]^{k_r}, \mathcal{E}_a \cap B) \lesssim 2^{(k_r-k_B)n}.$$

Thus

$$\begin{aligned}\sup_{B:\, r_B \ge [(1-a)/2]^{-N} r} \frac{\log_2 N_{\rm cov}(r, \mathcal{E}_a \cap B)}{\log_2(r_B/r)} &\le \sup_{0\le m\le k_r-N} \frac{\log_2 C_1 2^{(k_r-m)n}}{\log_2[(1-a)/2]^{m-k_r}}\\ &\le \sup_{0\le m\le k_r-N} \frac{n(k_r-m)+\log_2 C_1}{(k_r-m)\log_2[2/(1-a)]} \le \frac{nN+\log_2 C_1}{N\log_2[2/(1-a)]}\\ &\to n/\log_2[2/(1-a)] \quad \text{as } N\to\infty.\end{aligned}$$

Consequently, we get

$$\overline{\dim}_{LG}\, \mathcal{E}_a \le \frac{n}{\log_2[2/(1-a)]},$$

as desired. $\square$

3. A characterization of Q-spaces

In this section, we characterize membership in Q-spaces via oscillations. To do so, let us introduce a couple of concepts. Let u be a measurable function. For $\alpha \in (0,1)$, $q \in (0,\infty)$, and each ball $B = B(x_0, r) \subset \mathbb{R}^n$, set

$$\Psi_{\alpha,q}(u, B) = \sum_{k\ge 0} 2^{2k\alpha} \fint_{B(x_0,r)} \inf_{c\in\mathbb{R}} \left\{ \fint_{B(x,2^{-k}r)} |u(z)-c|^q \, dz \right\}^{2/q} dx.$$

Define $Q_{\alpha,q}(\mathbb{R}^n)$ as the collection of $u \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|u\|_{Q_{\alpha,q}(\mathbb{R}^n)} = \sup_{x_0\in\mathbb{R}^n,\, r>0} [\Psi_{\alpha,q}(u, B(x_0, r))]^{1/2} < \infty.$$

Also, for every ball $B \subset \mathbb{R}^n$ and each function u on B, set

$$\Phi_\alpha(u, B) = |B|^{2\alpha/n-1} \int_B \int_B \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\alpha}}\, dx\, dy.$$

Then $\|u\|_{Q_\alpha(\mathbb{R}^n)} = \sup_B [\Phi_\alpha(u, B)]^{1/2}$, where the supremum is taken over all balls $B \subset \mathbb{R}^n$.

Proposition 3.1. *Let $\alpha \in (0, 1)$ and $q \in (0, 2]$. There exists a constant C such that for all measurable functions u and all balls $B = B(x_0, r)$ one has*

$$C^{-1}\Phi_\alpha(u, B(x_0, r/16)) \le \Psi_{\alpha,q}(u, B(x_0, r)) \le C\Phi_\alpha(u, B(x_0, 16r)).$$

Consequently,

$$Q_\alpha(\mathbb{R}^n) = Q_{\alpha,q}(\mathbb{R}^n) \quad \textit{with} \quad \|\cdot\|_{Q_\alpha(\mathbb{R}^n)} \sim \|\cdot\|_{Q_{\alpha,q}(\mathbb{R}^n)}.$$

To verify Proposition 3.1, we need the following estimate from [6].

Lemma 3.2. *Let $\sigma \in (0, \infty)$ and $u \in L^\sigma_{\mathrm{loc}}(\mathbb{R}^n)$. Then there is a set E with $|E| = 0$ such that for any $x, y \in \mathbb{R}^n \setminus E$ with $|x - y| \in [2^{-k-1}, 2^{-k})$ one has*

$$|u(x) - u(y)| \lesssim \sum_{j\ge k-2} \left\{ \inf_{c\in\mathbb{R}} \left[\fint_{B(x,2^{-j})} |u(w) - c|^\sigma\, dw \right]^{1/\sigma} + \inf_{c\in\mathbb{R}} \left[\fint_{B(y,2^{-j})} |u(w) - c|^\sigma\, dw \right]^{1/\sigma} \right\}. \tag{3.1}$$

Proof of Proposition 3.1. By Lemma 3.2, we obtain

$$\begin{aligned}
&\int_{B(x,2r)} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\alpha}}\, dy \le \sum_{j=-1}^{\infty} (2^{-j}r)^{-(n+2\alpha)} \int_{B(x,2^{-j}r)\setminus B(x,2^{-j-1}r)} |u(x)-u(y)|^2\, dy \\
&\lesssim \sum_{j=-1}^{\infty} (2^{-j}r)^{-2\alpha} \left[\sum_{k\ge j-2} \inf_{c\in\mathbb{R}} \fint_{B(x,2^{-k}r)} |u(w) - c|^q\, dw \right]^{2/q} \\
&\quad + \sum_{j=-1}^{\infty} (2^{-j}r)^{-2\alpha} \fint_{B(x,2^{-j}r)\setminus B(x,2^{-j-1}r)} \left[\sum_{k\ge j-2} \inf_{c\in\mathbb{R}} \fint_{B(y,2^{-k}r)} |u(w) - c|^q\, dw \right]^{2/q} dy \\
&= J_1(x) + J_2(x).
\end{aligned}$$

Applying Hölder's inequality and changing the order of summation, we obtain

$$\begin{aligned} J_1(x) &\lesssim \sum_{j=-1}^{\infty} (2^{-j}r)^{-2\alpha} 2^{-j\alpha} \sum_{k\geq j-2} 2^{k\alpha} \inf_{c\in\mathbb{R}} \left[\fint_{B(x,2^{-k}r)} |u(w)-c|^q \, dw \right]^{2/q} \\ &\lesssim \sum_{k\geq -3} 2^{k\alpha} \sum_{j=-1}^{k} (2^{-j}r)^{-2\alpha} 2^{-j\alpha} \inf_{c\in\mathbb{R}} \left[\fint_{B(x,2^{-k}r)} |u(w)-c|^q \, dw \right]^{2/q} \\ &\lesssim \sum_{k\geq -3} (2^{-k}r)^{-2\alpha} \inf_{c\in\mathbb{R}} \left[\fint_{B(x,2^{-k}r)} |u(w)-c|^q \, dw \right]^{2/q}. \end{aligned}$$

Thus,

$$\begin{aligned} r^{2\alpha-n} \int_{B(x_0,r)} J_1(x)\, dx &\lesssim \sum_{k\geq -3} 2^{2k\alpha} \fint_{B(x_0,8r)} \inf_{c\in\mathbb{R}} \left[\fint_{B(x,2^{-k}r)} |u(w)-c|^q \, dw \right]^{2/q} dx \\ &\lesssim \Psi_{\alpha,q}(u, B(x_0,8r)). \end{aligned}$$

For J_2, notice that

$$\begin{aligned} &\int_{B(x_0,r)} J_2(x)\, dx \\ &\quad \lesssim \int_{B(x_0,4r)} \sum_{j=-1}^{\infty} (2^{-j}r)^{-2\alpha} \left[\sum_{k\geq j-2} \inf_{c\in\mathbb{R}} \fint_{B(y,2^{-k}r)} |u(w)-c|^q \, dw \right]^{2/q} dy. \end{aligned}$$

Then, applying an argument similar to the above estimate for J_1, we have

$$r^{2\alpha-n} \int_{B(x_0,r)} J_2(x)\, dx \lesssim \Psi_{\alpha,q}(u, B(x_0,8r)).$$

Combining the estimates on J_1 and J_2, we obtain

$$\Phi_\alpha(u, B(x_0,r)) \lesssim \Psi_{\alpha,q}(u, B(x_0,8r)).$$

On the other hand, noticing that for all $x \in \mathbb{R}^n$, $r > 0$ and $k \geq 0$ one has

$$2^{-k}r \leq |x-w| - |x-z| \leq |z-w| \leq |x-w| + |x-z| \leq 2^{-k+3}r$$

whenever

$$z \in B(x, 2^{-k}r) \quad \& \quad w \in B(x, 2^{-k+2}r) \setminus B(x, 2^{-k+1}r),$$

we utilize $q \in (0, 2]$ and the Hölder inequality to deduce

$$\begin{aligned}
\inf_{c\in\mathbb{R}} \left[\fint_{B(x,2^{-k}r)} |u(w) - c|^q \, dw \right]^{2/q} &\lesssim \fint_{B(x,2^{-k}r)} |u(z) - u_{B(x,2^{-k}r)}|^2 \, dz \\
&\lesssim \fint_{B(x,2^{-k}r)} |u(z) - u_{B(x,2^{-k+2}r)\setminus B(x,2^{-k+1}r)}|^2 \, dz \\
&\lesssim \fint_{B(x,2^{-k}r)} \fint_{B(x,2^{-k+2}r)\setminus B(x,2^{-k+1}r)} |u(z) - u(w)|^2 \, dw \, dz \\
&\lesssim (2^{-k}r)^{2\alpha} \fint_{B(x,2^{-k}r)} \int_{B(x,2^{-k+2}r)\setminus B(x,2^{-k+1}r)} \frac{|u(z) - u(w)|^2}{|z - w|^{n+2\alpha}} \, dw \, dz \\
&\lesssim (2^{-k}r)^{2\alpha} \fint_{B(x,2^{-k}r)} \int_{B(z,2^{-k+3}r)\setminus B(z,2^{-k}r)} \frac{|u(z) - u(w)|^2}{|z - w|^{n+2\alpha}} \, dw \, dz.
\end{aligned}$$

Thus, by changing the order of the integrals with respect to dz and dx,

$$\begin{aligned}
&\Psi_{\alpha,q}(u, B(x_0, r)) \\
&\quad\lesssim r^{2\alpha} \sum_{k\geq 0} \fint_{B(x_0,r)} \fint_{B(x,2^{-k}r)} \int_{B(z,2^{-k+3}r)\setminus B(z,2^{-k}r)} \frac{|u(z) - u(w)|^2}{|z - w|^{n+2\alpha}} \, dw \, dz \, dx \\
&\quad\lesssim r^{2\alpha} \sum_{k\geq 0} \fint_{B(x_0,2r)} \fint_{B(z,2^{-k}r)} \int_{B(z,2^{-k+3}r)\setminus B(z,2^{-k}r)} \frac{|u(z) - u(w)|^2}{|z - w|^{n+2\alpha}} \, dw \, dx \, dz \\
&\quad\lesssim r^{2\alpha} \sum_{k\geq 0} \fint_{B(x_0,2r)} \int_{B(z,2^{-k+3}r)\setminus B(z,2^{-k}r)} \frac{|u(z) - u(w)|^2}{|z - w|^{n+2\alpha}} \, dw \, dz \\
&\quad\lesssim r^{2\alpha-n} \int_{B(x_0,2r)} \int_{B(z,8r)} \frac{|u(z) - u(w)|^2}{|z - w|^{n+2\alpha}} \, dw \, dz \lesssim \Phi_\alpha(u, B(x_0, 16r)).
\end{aligned}$$

This completes the proof of Proposition 3.1. □

4. Proof of Theorem 1.3

Here we only prove Theorem 1.3 under the assumption $\operatorname{diam} E < \infty$. The case of infinite diameter is similar. Without loss of generality, we may assume that $\operatorname{diam} E = 1$ and $E \subset B(0, 1)$. By Proposition 3.1, it suffices to show that

$$\Psi_{\alpha,2}(u \circ f, B) \lesssim \|u\|_{Q_\alpha(\mathbb{R}^n)} \qquad \text{for each ball } B = B(x_0, r).$$

We divide the argument into two cases.

Case 1: $d(x_0, E) \geq 4r$. Notice that $B(x, 2r) \cap E = \emptyset$ for all $x \in B(x_0, r)$. Since $J_f \in A_1(\mathbb{R}^n; E)$, and $J_f(f^{-1}(z))J_{f^{-1}}(z) = 1$ for almost all $z \in \mathbb{R}^n$, for all $k \geq 0$ and

$x \in B(x_0, r)$ we have

$$\begin{aligned}\operatorname*{ess\,sup}_{z\in f(B(x,2^{-k}r))} J_{f^{-1}}(z) &= \operatorname*{ess\,sup}_{z\in f(B(x,2^{-k}r))} [J_f(f^{-1}(z))]^{-1}\\ &= \Big[\operatorname*{ess\,inf}_{w\in B(x,2^{-k}r)} J_f(w)\Big]^{-1} \lesssim \frac{|B(x,2^{-k}r)|}{|f(B(x,2^{-k}r))|}. \qquad (4.1)\end{aligned}$$

Thus,

$$\begin{aligned}\fint_{B(x,2^{-k}r)} |u\circ f(z)-c|^2\,dz &= \frac{|f(B(x,2^{-k}r))|}{|B(x,2^{-k}r)|}\fint_{f(B(x,2^{-k}r))} |u(z)-c|^2 J_{f^{-1}}(z)\,dz\\ &\lesssim \fint_{f(B(x,2^{-k}r))} |u(z)-c|^2 dz.\end{aligned}$$

Hence we have

$$\begin{aligned}\Psi_{\alpha,2}(u\circ f, B(x_0,r)) &\lesssim \sum_{k\geq 0} 2^{2k\alpha}\fint_{B(x_0,r)} \inf_{c\in\mathbb{R}} \fint_{f(B(x,2^{-k}r))} |u(z)-c|^2\,dz\,dx\\ &\lesssim \sum_{k\geq 0}\int_{B(x_0,r)} \frac{|B(x,r)|^{2\alpha/n-1}}{|B(x,2^{-k}r)|^{2\alpha/n}} \inf_{c\in\mathbb{R}} \fint_{f(B(x,2^{-k}r))} |u(z)-c|^2\,dz\,dx.\end{aligned}$$

Observe that $J_f \in A_1(\mathbb{R}^n; E)$ also implies that

$$\begin{aligned}\frac{|f(B(x,2^{-k}r))|}{|B(x,2^{-k}r)|} &= \fint_{B(x,2^{-k}r)} J_f(z)\,dz\\ &\lesssim \operatorname*{ess\,inf}_{z\in B(x,2^{-k}r)} J_f(z) \lesssim J_f(x) \quad \text{for almost all } x\in B,\end{aligned}$$

that is,

$$|B(x,2^{-k}r)|^{-1} \lesssim J_f(x)|f(B(x,2^{-k}r))|^{-1}.$$

Therefore, by a change of variables again,

$$\begin{aligned}&\Psi_{\alpha,2}(u\circ f, B(x_0,r))\\ &\lesssim \sum_{k\geq 0}\int_{B(x_0,r)} \frac{|B(x,r)|^{2\alpha/n-1}}{|f(B(x,2^{-k}r))|^{2\alpha/n}} \inf_{c\in\mathbb{R}} \fint_{f(B(x,2^{-k}r))} |u(z)-c|^2\,dz\,[J_f(x)]^{2\alpha/n}\,dx\\ &\lesssim \sum_{k\geq 0}\int_{f(B(x_0,r))} \frac{|B(f^{-1}(x),r)|^{2\alpha/n-1}}{|f(B(f^{-1}(x),2^{-k}r))|^{2\alpha/n}}\\ &\quad\times \inf_{c\in\mathbb{R}} \fint_{f(B(f^{-1}(x),2^{-k}r))} |u(z)-c|^2\,dz\,J_{f^{-1}}(x)[J_f(f^{-1}(x))]^{2\alpha/n}\,dx\\ &\lesssim \sum_{k\geq 0}\int_{f(B(x_0,r))} \frac{|B(f^{-1}(x),r)|^{2\alpha/n-1}}{|f(B(f^{-1}(x),2^{-k}r))|^{2\alpha/n}}\\ &\quad\times \inf_{c\in\mathbb{R}} \fint_{f(B(f^{-1}(x),2^{-k}r))} |u(z)-c|^2\,dz\,[J_{f^{-1}}(x)]^{1-2\alpha/n}\,dx.\end{aligned}$$

Now, by (4.1) with $k = 0$ and $x = x_0$, we have

$$\operatorname*{ess\,sup}_{x\in f(B(x_0,r))} J_{f^{-1}}(x) \lesssim \frac{|B(x_0,r)|}{|f(B(x_0,r))|} \sim \frac{|B(w,r)|}{|f(B(w,r))|} \quad \forall w \in B(x_0,r),$$

which further yields

$$\begin{aligned}
&\Psi_{\alpha,2}(u\circ f, B(x_0,r))\\
&\lesssim \sum_{k\geq 0} \fint_{f(B(x_0,r))} \left(\frac{|f(B(f^{-1}(x),r))|}{|f(B(f^{-1}(x),2^{-k}r))|}\right)^{2\alpha/n} \inf_{c\in\mathbb{R}} \fint_{f(B(f^{-1}(x),2^{-k}r))} |u(z)-c|^2\,dz\,dx\\
&\lesssim \fint_{f(B(x_0,r))} \sum_{k\geq 0} \left(\frac{L_f(f^{-1}(x),r)}{L_f(f^{-1}(x),2^{-k}r)}\right)^{2\alpha} \inf_{c\in\mathbb{R}} \fint_{f(B(f^{-1}(x),2^{-k}r))} |u(z)-c|^2\,dz\,dx, \quad (4.2)
\end{aligned}$$

where

$$L_f(z,r) = \sup\{|f(z)-f(w)| : |z-w|\leq r\} \quad \& \quad L_f(z,r)^n \sim |f(B(z,r))|.$$

Moreover, by quasisymmetry of f, for all $j \in \mathbb{Z}$ and $z \in \mathbb{R}^n$ we have

$$\sharp\{k \in \mathbb{Z} : L_f(z,2^{-k}r) \in [2^{-j-1}L_f(z,r), 2^{-j}L_f(z,r))\} \lesssim 1. \quad (4.3)$$

Recalling that

$$f(B(x_0,r)) \subset B(f(x_0), L_f(x_0,r)) \quad \& \quad L_f(f^{-1}(x),r) \leq 2^{N_2} L_f(x_0,r)$$

for some constant $N_2 \geq 1$ (independent of x_0, r; see [5]), we arrive at

$$\begin{aligned}
\Psi_{\alpha,2}(u\circ f, B(x_0,r)) &\lesssim \sum_{j\geq 0} 2^{2j\alpha} \fint_{B(f(x_0),L_f(x_0,r))} \fint_{B(x,2^{-j}2^{N_2}L_f(x_0,r))} |u(z)-c|^2\,dz\,dx\\
&\lesssim \Psi_{\alpha,2}(u, B(f(x_0), 2^{N_2}L_f(x_0,r))), \quad (4.4)
\end{aligned}$$

which together with Proposition 3.1 gives

$$\Psi_{\alpha,2}(u\circ f, B(x_0,r)) \lesssim \|u\|^2_{Q_\alpha(\mathbb{R}^n)}$$

as desired.

Case 2: $d(x_0,E) < 4r \leq 4$. Recall that each domain Ω admits a Whitney decomposition. In particular, for $\Omega = \mathbb{R}^n \setminus E$, there exists a collection $W_\Omega = \{S_j\}_{j\in\mathbb{N}}$ of countably many dyadic (closed) cubes such that

(i) $\Omega = \bigcup_{j\in\mathbb{N}} S_j$ and $(S_k)^\circ \cap (S_j)^\circ = \emptyset$ for all $j,k \in \mathbb{N}$ with $j \neq k$;
(ii) $2^7\sqrt{n}\,\ell(S_j) \leq \operatorname{dist}(S_j,\partial\Omega) \leq 2^9\sqrt{n}\,\ell(S_j)$;
(iii) $\frac{1}{4}\ell(S_k) \leq \ell(S_j) \leq 4\ell(S_k)$ whenever $S_k \cap S_j \neq \emptyset$.

Assume that $2^{-k_0-1} \le 16r < 2^{-k_0}$ for $k \in \mathbb{N}$. For each $k \in \mathbb{Z}$ and ball B, write

$$\mathscr{S}_k(16B) = \{S_j \in W_\Omega : S_j \cap 16B \ne \emptyset, 2^{-k} \le \ell(S_j) < 2^{-k+1}\} \equiv \{S_{k,i}\}_i.$$

Notice that there exists an integer N_0 such that if $k \le k_0 - N_0$, then $\mathscr{S}_k(16B) = \emptyset$. Indeed, by

$$\mathrm{dist}(S_{k,j}, E) \le 16r + d(x_0, E) \le 20r$$

and (ii) above, we have $2^{-k} \lesssim 2^{-k_0}$, which is as desired.

Moreover, letting $\epsilon \in (0, n - \dim_L E - 2\alpha)$, we claim that for all $k \ge k_0 - N$,

$$\sharp\mathscr{S}_k(16B) \lesssim 2^{(k-k_0)(\overline{\dim}_L E+\epsilon)}.$$

To see this, by the definition of $\overline{\dim}_L E$ there exist constants $N_1 \ge 8$ and $k_1 \in \mathbb{N}$ such that for all $k \ge k_1 + k_0 + N_1$, we have

$$\frac{\log_2 N_{\mathrm{cov}}(2^{-k}, E \cap 32B)}{\log_2(32r/2^{-k})} \le \overline{\dim}_L E + \epsilon,$$

which implies that

$$N_{\mathrm{cov}}(2^{-k}, E \cap 32B) \lesssim 2^{(k-k_0)(\overline{\dim}_L E+\epsilon)}. \tag{4.5}$$

For every $\delta > 0$, denote by $\mathscr{N}_{\mathrm{cov}}(\delta, E \cap 32B)$ the collection of cubes of edge length δ required to cover $E \cap 32B$ and have

$$\sharp\mathscr{N}_{\mathrm{cov}}(\delta, E \cap 32B) = N_{\mathrm{cov}}(\delta, E \cap 32B).$$

For $k \ge -N_0$ and $S_{k,i} \in \mathscr{S}_k(16B)$, we have $2^{11}\sqrt{n}\, S_{k,i} \cap E \ne \emptyset$, and hence $S_{k,i}$ intersects some cube $Q \in \mathscr{N}_{\mathrm{cov}}(2^{-k}, E \cap 32B)$, which implies that $S_{k,i} \subset 2^{13} n Q$. Also notice that for each cube $Q \in \mathscr{N}_{\mathrm{cov}}(2^{-k}, 16B \cap E)$, the cube $2^{13} n Q$ can only contain a uniformly bounded number of $S_{k,i} \in \mathscr{S}_k(16B)$. We conclude that for $k \ge -N_0$,

$$\sharp\mathscr{S}_k(16B) \lesssim N_{\mathrm{cov}}(2^{-k}, E \cap 32B).$$

This together with (4.5) implies that for $k \ge k_1 + k_0 + N_1$,

$$\sharp\mathscr{S}_k(16B) \lesssim 2^{(k-k_0)(\overline{\dim}_L E+\epsilon)}.$$

On the other hand, if $k_0 - N_0 \le k \le k_1 + k_0 + N_1$, then from $2^{k-k_0} \le 2^{k_1+N_1+N_0} \lesssim 1$ we always have

$$\sharp\mathscr{S}_k(16B) \lesssim 2^{n(k-k_0)} \lesssim 2^{(k-k_0)(\overline{\dim}_L E+\epsilon)}.$$

This gives the above claim.

By Proposition 3.1, we have

$$\begin{aligned}
\Psi_{\alpha,2}(u\circ f, B(x_0,r)) &\lesssim \Phi_\alpha(u\circ f, B(x_0,16r))\\
&\lesssim r^{2\alpha-n}\sum_{k\ge k_0-N_0}\sum_{i=1}^{\sharp\mathscr{S}_k(16B)}\int_{S_{k,i}}\int_{B(x_0,16r)}\frac{|u\circ f(x)-u\circ f(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy\\
&\lesssim r^{2\alpha-n}\sum_{k\ge k_0-N_0}\sum_{i=1}^{\sharp\mathscr{S}_k(16B)}\int_{S_{k,i}}\int_{2S_{k,i}}\frac{|u\circ f(x)-u\circ f(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy\\
&\quad + r^{2\alpha-n}\sum_{k\ge k_0-N_0}\sum_{i=1}^{\sharp\mathscr{S}_k(16B)}\int_{S_{k,i}}\int_{B(x_0,16r)\setminus 2S_{k,i}}\frac{|u\circ f(x)-u\circ f(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy\\
&= P_1+P_2.
\end{aligned}$$

For each $S_{k,i}$, let $B_{k,i}$ be the ball centered at $x_{k,i}$ (the center of $S_{k,i}$) and of radius $2\sqrt{n}\,\ell(S_{k,i})$. Then

$$2S_{k,i}\subset B_{k,i}\quad \&\quad \operatorname{dist}(x_{k,i},E)\ge 4\cdot 16\cdot 2\sqrt{n}\,\ell(S_{k,i}).$$

So applying the above Case 1 to $16B_{k,i}$, we have

$$\Phi_\alpha(u\circ f, B_{k,i})\lesssim \Psi_{\alpha,2}(u\circ f, 16B_{k,i})\lesssim \|u\|^2_{Q_\alpha(\mathbb{R}^n)}.$$

This, together with $n-2\alpha-\overline{\dim}_L E-\epsilon>0$, gives

$$\begin{aligned}
P_1 &\lesssim r^{2\alpha-n}\sum_{k\ge k_0-N_0}\sum_{i=1}^{\sharp\mathscr{S}_k(16B)}|B_{k,i}|^{1-2\alpha/n}\Phi_\alpha(u\circ f, B_{k,i})\\
&\lesssim r^{2\alpha-n}\sum_{k\ge k_0-N_0}\sum_{i=1}^{\sharp\mathscr{S}_k(16B)}2^{-(n-2\alpha)k}\|u\|^2_{Q_\alpha(\mathbb{R}^n)}\\
&\lesssim \sum_{k\ge k_0-N_0}2^{(k-k_0)(\overline{\dim}_L E+\epsilon)}2^{(n-2\alpha)(k_0-k)}\|u\|^2_{Q_\alpha(\mathbb{R}^n)}\lesssim \|u\|^2_{Q_\alpha(\mathbb{R}^n)}.
\end{aligned}$$

To estimate P_2, write

$$\begin{aligned}
&\int_{S_{k,i}}\int_{B(x_0,16r)\setminus 2S_{k,i}}\frac{|u\circ f(x)-u\circ f(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy\\
&\quad\lesssim \sum_{\ell=1}^{k-k_0+5}2^{(\ell-k)(-n-2\alpha)}\int_{S_{k,i}}\int_{2^{\ell+1}S_{k,i}\setminus 2^\ell S_{k,i}}|u\circ f(x)-u\circ f(y)|^2\,dx\,dy\\
&\quad\lesssim \sum_{\ell=1}^{k-k_0+5}2^{-2\alpha(\ell-k)}2^{-kn}\left\{\fint_{S_{k,i}}|u\circ f(x)-(u\circ f)_{2^{\ell+1}S_{k,i}}|^2\,dx\right.\\
&\qquad\left.+\fint_{2^{\ell+1}S_{k,i}}|u\circ f(y)-(u\circ f)_{2^{\ell+1}S_{k,i}}|^2\,dy\right\}.
\end{aligned}$$

Observing that

$$\left\{ \fint_{S_{k,i}} |u \circ f(x) - (u \circ f)_{2^{\ell+1} S_{k,i}}|^2 \, dx \right\}^{1/2} \lesssim \sum_{j=1}^{\ell+1} \left\{ \fint_{2^j S_{k,i}} |u \circ f(x) - (u \circ f)_{2^j S_{k,i}}|^2 \, dx \right\}^{1/2} \lesssim (\ell+1) \|u \circ f\|_{\mathrm{BMO}(\mathbb{R}^n)},$$

we obtain

$$\int_{S_{k,i}} \int_{B(x_0,16r)\setminus 2S_{k,i}} \frac{|u \circ f(x) - u \circ f(y)|^2}{|x-y|^{n+2\alpha}} \, dx \, dy \lesssim \sum_{\ell=1}^{k-k_0+5} 2^{-2\alpha(\ell-k)} 2^{-kn} (\ell+1)^2 \|u \circ f\|^2_{\mathrm{BMO}(\mathbb{R}^n)} \lesssim 2^{(2\alpha-n)k} \|u \circ f\|^2_{\mathrm{BMO}(\mathbb{R}^n)}.$$

Therefore, since $n - 2\alpha - \overline{\dim}_L E - \epsilon > 0$, one gets

$$P_2 \lesssim r^{2\alpha-n} \sum_{k \geq k_0 - N_0} \sum_{i=1}^{\sharp \mathscr{S}_k(16B)} 2^{(2\alpha-n)k} \|u \circ f\|^2_{\mathrm{BMO}(\mathbb{R}^n)} \lesssim \sum_{k \geq k_0 - N_0} 2^{(k-k_0)(\overline{\dim}_L E + \epsilon)} 2^{(n-2\alpha)(k_0-k)} \|u \circ f\|^2_{\mathrm{BMO}(\mathbb{R}^n)} \lesssim \|u \circ f\|^2_{\mathrm{BMO}(\mathbb{R}^n)}.$$

Recall that it was proved by Reimann [7] that $\|u \circ f\|_{\mathrm{BMO}(\mathbb{R}^n)} \lesssim \|u\|_{\mathrm{BMO}(\mathbb{R}^n)}$, and also in [3] that $\|u\|_{\mathrm{BMO}(\mathbb{R}^n)} \lesssim \|u\|_{Q_\alpha(\mathbb{R}^n)}$. Thus $P_2 \lesssim \|u\|^2_{Q_\alpha(\mathbb{R}^n)}$.

Combining the estimates for P_1 and P_2, we arrive at $\Psi_{\alpha,2}(u \circ f, B(x_0, r)) \lesssim \|u\|^2_{Q_\alpha(\mathbb{R}^n)}$ for all x_0 and r, as desired.

Case 3: $d(x_0, E) \leq 2r$ and $r > 1$. Without loss of generality, we may assume that $x_0 = 0$. Denote by M the minimum number of balls, centered in $B(0,1) \setminus B(0,1/2)$ and of radius 2^{-9}, required to cover $B(0,1) \setminus B(0,1/2)$. Let $\{B_j\}_{j=1}^M$ be a sequence of such balls and write their centers as $\{x_j\}_{j=1}^M$. Write

$$B_{k,j} = B(2^k x_j, 2^{k-9}) \quad \text{for } k \geq 2 \text{ and } j = 1, \ldots, M.$$

Notice that

$$2^{k-9} = 2^{k-2} 2^{-7} \leq 2^{-7} d(2^k x_j, E). \tag{4.6}$$

Assume that $2^{k_0-1} \leq r < 2^{k_0}$. Then $k_0 \geq 1$, and $B(x_0, 16r) \setminus B(x_0, 2)$ can be covered by the family $\{B_{k,j} : 2 \leq k \leq k_0 + 4, 0 \leq j \leq M\}$. Write $B_{1,j} = B(0, 2)$. Then we have

$$
\begin{aligned}
\Psi_{\alpha,2}(u\circ f, B(x_0,r)) &\lesssim \Phi_\alpha(u\circ f, B(x_0,16r))\\
&\lesssim \sum_{k=1}^{k_0+4}\sum_{j=1}^{M} r^{2\alpha-n}\int_{B_{k,j}}\int_{16B}\frac{|u\circ f(x)-u\circ f(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy\\
&\lesssim \sum_{k=1}^{k_0+4} r^{2\alpha-n}2^{k(n-2\alpha)}\Phi_\alpha(u\circ f, 2B_{k,j})\\
&\quad+\sum_{k=1}^{k_0+4} r^{2\alpha-n}\int_{B_{k,j}}\int_{16B\setminus 2B_{k,j}}\frac{|u\circ f(x)-u\circ f(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy\\
&= P_3+P_4.
\end{aligned}
$$

By Proposition 3.1 and the result of Case 1 applied to $32B_{k,j}$, we have

$$\Phi_\alpha(u\circ f, 2B_{k,j}) \lesssim \Psi_{\alpha,2}(u\circ f, 32B_{k,j}) \lesssim \|u\|^2_{Q_\alpha(\mathbb{R}^n)}$$

where

$$32\cdot 2^{k-9} = 32\cdot 2^{-7}d(2^k x_j, E) \le d(2^k x_j, E)/4$$

due to (4.6), and hence

$$P_3 \lesssim \|u\|^2_{Q_\alpha(\mathbb{R}^n)}\sum_{k=1}^{k_0+4} r^{2\alpha-n}2^{k(n-2\alpha)} \lesssim \|u\|^2_{Q_\alpha(\mathbb{R}^n)}.$$

For P_4, an argument similar to P_2 in Case 2 leads to $P_4 \lesssim \|u\|^2_{Q_\alpha(\mathbb{R}^n)}$. This finishes the proof of Theorem 1.3.

5. Proofs of Corollaries 1.4 and 1.5

Proof of Corollary 1.4. Notice that if $\beta > 0$, then f is a quasiconformal mapping from $\mathbb{R}^n \to \mathbb{R}^n$, and that

$$\begin{cases} J_f \in A_1(\mathbb{R}^n;\{0\}) & \text{when } \beta > 1,\\ J_f \in A_1(\mathbb{R}^n) & \text{when } 0<\beta<1.\end{cases}$$

By Theorem 1.3, if $\beta > 0$, then $\mathbf{C}_f$ is bounded on $Q_\alpha(\mathbb{R}^n)$ for all $\alpha \in (0,1)$. If $\beta < 0$, then f is not a quasiconformal mapping from $\mathbb{R}^n \to \mathbb{R}^n$, so we cannot apply Theorem 1.3 directly. However, f is a quasiconformal mapping from $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{R}^n$ with $J_f(x) \sim |x|^{\beta-1}$, yielding $J_f \in A_1(\mathbb{R}^n;\{0\})$. Thus, an argument similar to but easier than that for Theorem 1.3 will lead to the boundedness of $\mathbf{C}_f$ on $Q_\alpha(\mathbb{R}^n)$ for all $\alpha \in (0,1)$.

Indeed, let $u \in Q_\alpha(\mathbb{R}^n)$ and $B = B(x_0,r)$ be an arbitrary ball in $\mathbb{R}^n$. If $r < |x_0|/4$, then

$$J_f(x) \sim |x_0|^{\beta-1} \quad \forall x \in B(x_0,3r).$$

From this and $J_f \in A_1(\mathbb{R}^n; \{0\})$, similarly to Case 1 in the proof of Theorem 1.3, we obtain (4.2)–(4.4). This implies

$$\Psi_{\alpha,2}(u \circ f, B(x_0, r)) \lesssim \|u\|^2_{Q_\alpha(\mathbb{R}^n)}.$$

If $r \geq |x_0|/4$, then from $B(x_0, r) \subset B(0, 2r)$ and Proposition 3.1, we have

$$\Psi_{\alpha,2}(u \circ f, B(x_0, r)) \lesssim \Psi_{\alpha,2}(u \circ f, B(0, 2r)) \lesssim \Phi_\alpha(u \circ f, B(0, 32r)).$$

Similarly to Case 3 in the proof of Theorem 1.3, denote by M the minimum number of balls (centered in $B(0, 1) \setminus B(0, 1/2)$ and of radius 2^{-9}) that are required to cover $B(0, 1) \setminus B(0, 1/2)$. Let $\{B_j\}_{j=1}^M$ be a collection of such balls and write their centers as $\{x_j\}_{j=1}^M$. Write

$$B_{k,j} = B(2^{-k}2^5 r x_j, 2^{-k-9}2^5 r) \quad \text{for } k \geq 0 \text{ and } j = 1, \ldots, M.$$

Then $B(0, 32r) \setminus \{0\}$ is covered by the family of balls $\{B_{k,j} : k \geq 0, 0 \leq j \leq M\}$. Therefore,

$$\begin{aligned}
\Psi_{\alpha,2}(u \circ f, B(x_0, r)) &\lesssim \sum_{k \geq 0} \sum_{j=1}^M r^{2\alpha-n} \int_{B_{k,j}} \int_{B(0,32r)} \frac{|u \circ f(x) - u \circ f(y)|^2}{|x - y|^{n+2\alpha}}\, dx\, dy \\
&\lesssim \sum_{k \geq 0} r^{2\alpha-n} (2^{-k} r)^{n-2\alpha} \Phi_\alpha(u \circ f, 2B_{k,j}) \\
&\quad + \sum_{k \geq 0} r^{2\alpha-n} \int_{B_{k,j}} \int_{B(0,32r)\setminus 2B_{k,j}} \frac{|u \circ f(x) - u \circ f(y)|^2}{|x - y|^{n+2\alpha}}\, dx\, dy \\
&= P_5 + P_6.
\end{aligned}$$

Similarly to the estimate on P_3, we have $P_5 \lesssim \|u\|^2_{Q_\alpha(\mathbb{R}^n)}$; and similarly to but more easily than for P_2, we obtain $P_6 \lesssim \|u\|^2_{Q_\alpha(\mathbb{R}^n)}$. Putting all together gives

$$\Psi_{\alpha,2}(u \circ f, B(x_0, r)) \lesssim \|u\|^2_{Q_\alpha(\mathbb{R}^n)},$$

as desired, finishing the proof of Corollary 1.4. □

Proof of Corollary 1.5. For convenience, let $\mathbb{R}^n_+ = \{z = (x, y) : x \in \mathbb{R}^{n-1}\ \&\ y > 0\}$. We also write $\mathbb{H}^n = \mathbb{R}^n_+ \setminus \mathbb{R}^{n-1}$ and equip it with the hyperbolic distance $d_{\mathbb{H}^n}$, that is,

$$d_{\mathbb{H}^n}(w, w') = \inf_\gamma \int_\gamma \frac{|dz|}{y} \quad \forall w, w' \in \mathbb{H}^n,$$

where the infimum is taken over all rectifiable curves γ in $\mathbb{H}^n$ joining w and w'.

Suppose that $g : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ is a quasiconformal mapping when $n \geq 3$, or a quasisymmetric mapping when $n = 2$. According to Tukia–Väisälä [9, Theorem 3.11], g can be extended to a quasiconformal mapping $f : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ such that

(i) $f|_{\mathbb{R}^{n-1}} = g$;
(ii) $f|_{\mathbb{H}^n}$ is L-bi-Lipschitz with respect to $d_{\mathbb{H}^n}$ for some constant $L \geq 1$, i.e.,

$$\frac{1}{L} d_{\mathbb{H}^n}(z, w) \leq d_{\mathbb{H}^n}(f(z), f(w)) \leq L d_{\mathbb{H}^n}(z, w) \quad \forall w, w' \in \mathbb{H}^n.$$

Obviously, such an f can be further extended to a quasiconformal mapping $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}^n$ by reflection, that is,

$$\widetilde{f}(z) = \begin{cases} f(z_1, \ldots, z_{n-1}, -z_n) & \text{for } z \in \mathbb{R}^n \setminus \mathbb{R}^n_+, \\ f(z) & \text{for } z \in \mathbb{R}^n_+. \end{cases}$$

For simplicity, we write $\widetilde{f}$ as f, and generally set

$$\begin{cases} n \geq 3; \\ 2 \leq p < n; \\ \mathbb{H}^{n,p} = \mathbb{R}^n \setminus \mathbb{R}^p = \{z = (x, y) : x \in \mathbb{R}^{n-p} \ \& \ 0 \neq y \in \mathbb{R}^p\}. \end{cases}$$

We equip $\mathbb{H}^{n,p}$ with the distance $d_{\mathbb{H}^{n,p}}$, an analog of the hyperbolic distance, via

$$d_{\mathbb{H}^{n,p}}(w, w') = \inf_{\gamma} \int_{\gamma} \frac{|dz|}{|(0, y)|} \quad \forall w, w' \in \mathbb{H}^{n,p},$$

where the infimum is taken over all rectifiable curves γ in $\mathbb{H}^{n,p}$ joining w and w'. Suppose that $g : \mathbb{R}^{n-p} \to \mathbb{R}^{n-p}$ is a quasiconformal mapping when $n - p \geq 2$, or a quasisymmetric mapping when $n - p = 1$. In accordance with Tukia–Väisälä's [9, Section 3.13], g can be extended to a quasiconformal mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ such that

(i) $f|_{\mathbb{R}^{n-p}} = g$;
(ii) $f|_{\mathbb{H}^{n,p}}$ is L-bi-Lipschitz with respect to $d_{\mathbb{H}^{n,p}}$ for some constant $L \geq 1$.

Notice that both f and f^{-1} are bi-Lipschitz with respect to $d_{\mathbb{H}^{n,p}}$. We show that $\mathbf{C}_f$ is bounded; the case of $\mathbf{C}_{f^{-1}}$ is analogous. By Theorem 1.3, it suffices to verify that $J_f \in A_1(\mathbb{R}^n; \mathbb{R}^{n-p})$. In what follows, we only consider the case $p = 1$; the argument can easily be modified to handle $p \geq 2$.

First observe that

$$J_f(z) \sim [d(f(z), \mathbb{R}^{n-1})]^n / |y|^n \quad \text{a.e. } z = (x, y) \in \mathbb{R}^n \setminus \mathbb{R}^{n-1},$$

where $d(f(z), \mathbb{R}^{n-1})$ stands for the Euclidean distance from $f(z)$ to $\mathbb{R}^{n-1}$. Indeed, upon taking $r > 0$ small enough such that

$$r < |y|/2 \quad \& \quad L_f(z, r) \leq d(f(z), \mathbb{R}^{n-1})/2,$$

we get

$$d(w, \mathbb{R}^{n-1}) \sim d(z, \mathbb{R}^{n-1}) \sim |y| \quad \& \quad d(f(w), \mathbb{R}^{n-1})/2 \sim d(f(z), \mathbb{R}^{n-1})/2 \qquad \forall w \in B(z, r),$$

which in turn implies

$$d_{\mathbb{H}^n}(z,w) \sim \frac{|z-w|}{|y|} \quad \& \quad d_{\mathbb{H}^n}(f(z), f(w)) \sim \frac{|f(z)-f(w)|}{d(f(z),\mathbb{R}^{n-1})} \quad \forall w \in B(z,r).$$

Therefore

$$J_f(z) \sim |Df(z)|^n \sim [d(f(z),\mathbb{R}^{n-1})]^n/|y|^n \quad \text{a.e. } z \in \mathbb{R}^n,$$

as desired.

Now let $B(x_0, r)$ be an arbitrary ball with radius $r \le |y_0|/2$, and $z_0 = (x_0, y_0)$. Obviously,

$$|y|/2 \le |y_0| \le 2|y| \quad \forall z = (x, y) \in B(z_0, r).$$

Then, it is enough to prove that

$$d(f(z_0),\mathbb{R}^{n-1}) \sim d(f(z),\mathbb{R}^{n-1}) \quad \text{a.e. } z \in B(z_0, r). \tag{5.1}$$

Assuming this holds for the moment, we have

$$J_f(z) \sim [d(f(z_0),\mathbb{R}^{n-1})]^n/|y_0|^n \quad \text{a.e. } z \in B(z_0, r),$$

and further

$$⨍_{B(x_0,r)} J_f(z)\,dz \sim [d(f(z_0),\mathbb{R}^{n-1})]^n/|y_0|^n \sim \operatorname*{ess\,inf}_{z\in B(x_0,r)} J_f(z),$$

that is, $J_f \in A_1(\mathbb{R}^n; \mathbb{R}^{n-1})$, as desired.

Towards (5.1), note that f is a quasisymmetric mapping, so there exists a homeomorphism $\eta : [0,\infty) \to [0,\infty)$ such that

$$\frac{|f(z)-f(w)|}{|f(z_0)-f(w)|} \lesssim \eta\left(\frac{|z-w|}{|z_0-w|}\right) \quad \forall w \in \mathbb{R}^n.$$

Observe that

$$\tfrac{1}{2}|z_0-w| \le |z_0-w|-|z-z_0| \le |z-w| \le |z-z_0|+|z_0-w| \le 2|z_0-w| \quad \forall w \in \mathbb{R}^{n-1}.$$

Thus, by taking a point $w \in \mathbb{R}^{n-1}$ such that $|f(z_0) - f(w)| = d(f(z_0), \mathbb{R}^{n-1})$, we have

$$d(f(z),\mathbb{R}^{n-1}) \le |f(z)-f(w)| \le \eta(2)|f(z_0)-f(w)| \lesssim d(f(z_0),\mathbb{R}^{n-1}).$$

Upon changing the roles of z and z_0, we also have

$$d(f(z_0),\mathbb{R}^{n-1}) \lesssim d(f(z),\mathbb{R}^{n-1}).$$

Hence (5.1) holds. This completes the proof of Corollary 1.5. □

6. Proofs of Theorems 1.6 and 1.7

Proof of Theorem 1.6. Fix $\alpha_0 \in (0,1)$. Let $a = 1 - 2^{-2\alpha_0/(n-2\alpha_0)} \in (0,1)$, and let the sets E_a be as in (2.4). Then $n - 2\alpha_0 = n/\log_2[2/(1-a)]$ and by Lemma 2.4, $\overline{\dim}_L E_a = n - 2\alpha_0$. The set E_a is exactly what we want in the statement of Theorem 1.6.

Now we are going to construct a quasiconformal (Lipschitz) mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $J_f \in A_1(\mathbb{R}^n; E_a)$, and hence $J_f \in A_1(\mathbb{R}^n; \mathcal{E}_a)$, but $\mathbf{C}_f$ is unbounded on $Q_\alpha(\mathbb{R}^n)$ for any $\alpha \in (\alpha_0, 1)$.

Recall that $\{z_{m,j}\}$ are the centers of $\{Q_{m,j}\}$, and $\{Q_{m,j}\}$ are the pre-cubes appearing in the Cantor construction E_a (see Section 2). Let $\beta \in (0,\infty)$ and define the map f by setting

$$f(x) = \left(\tfrac{1}{2}a[(1-a)/2]^m\right)^{-\beta}|x - z_{m,j}|^\beta (x - z_{m,j}) + z_{m,j}$$

if

$$|x - z_{m,j}| < \tfrac{1}{2}a[(1-a)/2]^m \qquad \text{for some } m \in \mathbb{N} \text{ and } j = 1, \dots, 2^{mn},$$

and $f(x) = x$ otherwise. Indeed, we only perturb the identity mapping on all balls

$$B\left(z_{m,j}, \tfrac{1}{2}a[(1-a)/2]^m\right) \subset Q_{m,j}$$

by making "radial" stretchings with respect to their centers, where $|Q_{m,j}| = [(1-a)/2]^{mn}$. Notice that

$$\begin{aligned} J_f(x) \sim |Df(x)|^n &\sim \left(\tfrac{1}{2}a[(1-a)/2]^m\right)^{-n\beta}|x - z_{m,j}|^{n\beta} \\ &\lesssim 1 \quad \text{when } |x - z_{m,j}| < \tfrac{1}{2}a[(1-a)/2]^m, \end{aligned}$$

and $J_f(x) = |Df(x)|^n = 1$ otherwise. Thus f is a quasiconformal mapping. Moreover, it is easy to check that $J_f \in A_1(\mathbb{R}^n; E_a)$ and $J_f \notin A_1(\mathbb{R}^n)$.

Set

$$\beta_0 = 1 + \frac{n - 2\alpha}{n}\log_2\left(\frac{1-a}{2}\right).$$

Then $\beta_0 > 0$ since $n - 2\alpha < n - 2\alpha_0 = n/\log_2[2/(1-a)]$. Set also

$$\ell = \begin{cases} mn\beta/(n-2\alpha) & \text{if } 0 < \beta \le \beta_0, \\ mn\beta_0/(n-2\alpha) & \text{if } \beta > \beta_0. \end{cases}$$

With each $z_{m,j} \in E$, we associate a ball $B_{m,j}$ such that

$$B_{m,j} \subset \tfrac{17}{64}2^{-\ell}aQ_{m,j} \quad \& \quad r_{m,j} = \tfrac{1}{64}2^{-\ell}a[(1-a)/2]^m$$

and so that the center $x_{m,j}$ of $B_{m,j}$ satisfies

$$|x_{m,j} - z_{m,j}| = \tfrac{1}{4}2^{-\ell}a[(1-a)/2]^m.$$

For each m, set

$$u_m = \sum_{j=1}^{2^{mn}} u_{m,j}, \quad \text{where} \quad u_{m,j}(x) = \chi_{B_{m,j}}(x)d(x, \partial B_{m,j}) \quad \text{for all possible } j.$$

Obviously, $u_{m,j}$ is a Lipschitz function.

We make two claims:

$$\|u_m\|^2_{Q_\alpha(\mathbb{R}^n)} \lesssim 2^{mn}2^{-\ell(n+2-2\alpha)}[(1-a)/2]^{m(n+2-2\alpha)}, \tag{6.1}$$

$$\|u_m \circ f\|^2_{Q_\alpha(\mathbb{R}^n)} \gtrsim 2^{mn}2^{-\ell(n-2\alpha)/(\beta+1)}2^{-2\ell}[(1-a)/2]^{m(n-2\alpha+2)}. \tag{6.2}$$

Assuming that both (6.1) and (6.2) hold for the moment, we arrive at

$$\frac{\|u_m \circ f\|^2_{Q_\alpha(\mathbb{R}^n)}}{\|u_m\|^2_{Q_\alpha(\mathbb{R}^n)}} \gtrsim \frac{2^{mn}2^{-\ell(n-2\alpha)/(\beta+1)}2^{-2\ell}[(1-a)/2]^{m(n-2\alpha+2)}}{2^{mn}2^{-\ell(n+2-2\alpha)}[(1-a)/2]^{m(n+2-2\alpha)}} \gtrsim 2^{\ell(n-2\alpha)\beta/(\beta+1)},$$

which tends to ∞ as $m \to \infty$ since $\beta > 0$ and $\ell \sim m$. This gives Theorem 1.6 under (6.1)–(6.2).

Finally, we verify (6.1)–(6.2).

Proof of (6.1). Let $B = B(x_B, r_B)$ be an arbitrary ball. If $r_B \le r_{m,j}$, since

$$|u_m(x) - u_m(y)| \le |x-y| \quad \forall x, y \in \mathbb{R}^n$$

one has

$$\begin{aligned}\Phi_\alpha(u_m, 2B) &\lesssim r_B^{2\alpha-n}\int_{2B}\int_{2B}\frac{1}{|x-y|^{n-2(1-\alpha)}}\,dx\,dy\\ &\lesssim r_B^{2\alpha-n}\int_{2B}\int_{B(y,2r_B)}\frac{1}{|x-y|^{n-2(1-\alpha)}}\,dx\,dy \lesssim r_B^2 \lesssim r_{m,j}^2.\end{aligned} \tag{6.3}$$

In particular, $\Phi_\alpha(u_m, 2B_{m,j}) \lesssim r_{m,j}^2$.

If $r_B > r_{m,j}$, one writes

$$\begin{aligned}\Phi_\alpha(u_m, 2B) &\le 2|B|^{2\alpha/n-1}\sum_{B_{m,j}\cap 2B\neq\emptyset}\int_{B_{m,j}}\int_{2B}\frac{|u_m(x)-u_m(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy\\ &\le |B|^{2\alpha/n-1}\sum_{B_{m,j}\cap 2B\neq\emptyset}|B_{m,j}|^{1-2\alpha/n}\Phi_\alpha(u_m, 2B_{m,j})\\ &\quad + |B|^{2\alpha/n-1}\sum_{B_{m,j}\cap 2B\neq\emptyset}\int_{B_{m,j}}\int_{2B\setminus 2B_{m,j}}\frac{|u_m(x)-u_m(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy.\end{aligned}$$

Notice that

$$\begin{aligned}\int_{B_{m,j}}\int_{2B\setminus 2B_{m,j}}\frac{|u_m(x)-u_m(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy &\lesssim r_{m,j}^2|B_{m,j}|\int_{2B\setminus 2B_{m,j}}\frac{1}{|y-z_{m,j}|^{n+2\alpha}}\,dy\\ &\lesssim r_{m,j}^{2-2\alpha}|B_{m,j}|.\end{aligned}$$

So, by (6.3) one has

$$\Phi_\alpha(u_m, 2B) \lesssim |B|^{2\alpha/n-1}\sum_{B_{m,j}\cap 2B\neq\emptyset} r_{m,j}^{2-2\alpha+n}. \tag{6.4}$$

Below we consider three subcases.

First, if $r_{m,j} < r_B \le \frac{1}{64}a[(1-a)/2]^m$, there are a uniformly bounded number of balls $B_{m,j}$ such that $B_{m,j} \cap 2B \neq \emptyset$, and hence

$$\Phi_\alpha(u_m, 2B) \lesssim |B|^{2\alpha/n-1}\{2^{-\ell}[(1-a)/2]^m\}^{2-2\alpha+n} \lesssim 2^{-2\ell}[(1-a)/2]^{2m}.$$

Second, if $\frac{1}{64}a[(1-a)/2]^{m-k} < r_B \le \frac{1}{64}a[(1-a)/2]^{m-k-1}$ for some $1 \le k \le m$, there are at most 2^{kn}, up to a constant multiplier, balls $B_{m,j}$ such that $B \cap B_{m,j} \neq \emptyset$, and hence

$$\Phi_\alpha(u_m, 2B) \lesssim [(1-a)/2]^{(m-k)(2\alpha-n)}2^{kn}2^{-\ell(n+2-2\alpha)}[(1-a)/2]^{m(n+2-2\alpha)}.$$

Since $2^n[(1-a)/2]^{(n-2\alpha)} > 1$ due to $n-(n-2\alpha)\log_2[2/(1-a)] > 0$, we obtain

$$\Phi_\alpha(u_m, 2B) \lesssim 2^{mn}2^{-\ell(n+2-2\alpha)}[(1-a)/2]^{m(n+2-2\alpha)}.$$

Third, if $r_B > \frac{1}{64}a[(1-a)/2]$, there are at most 2^{mn}, up to a constant multiplier, balls $B_{m,j}$ such that $B \cap B_{m,j} \neq \emptyset$, and hence

$$\Phi_\alpha(u_m, 2B) \lesssim 2^{mn}2^{-\ell(n+2-2\alpha)}[(1-a)/2]^{m(n+2-2\alpha)}.$$

To sum up, one obtains

$$\|u_m\|_{Q_\alpha(\mathbb{R}^n)} \lesssim \max\{2^{-2\ell}[(1-a)/2]^{2m}, 2^{mn}2^{-\ell(n+2-2\alpha)}[(1-a)/2]^{m(n+2-2\alpha)}\}.$$

So (6.1) will follow once we show

$$2^{-2\ell}[(1-a)/2]^{2m} \le 2^{mn}2^{-\ell(n+2-2\alpha)}[(1-a)/2]^{m(n+2-2\alpha)}.$$

Obviously, this is equivalent to $2^{\ell(n-2\alpha)} \le 2^{mn}[(1-a)/2]^{m(n-2\alpha)}$, and hence to

$$\ell(n-2\alpha) \le mn + m(n-2\alpha)\log_2[(1-a)/2].$$

But this last estimate follows from our choice of ℓ, namely,

$$\ell = \frac{mn}{n-2\alpha}\min\{\beta, \beta_0\} \le \frac{mn}{n-2\alpha}\beta_0 = \frac{mn}{n-2\alpha} + m\log_2\left(\frac{1-a}{2}\right).$$

Thus (6.1) holds.

Proof of (6.2). Indeed, we have

$$\begin{aligned}\|u_m \circ f\|^2_{Q_\alpha(\mathbb{R}^n)} &\ge \Phi_\alpha(u_m \circ f, f^{-1}(B(0,2)))\\ &\gtrsim \sum_{j=1}^{2^{mn}} \int_{f^{-1}(B_{m,j})}\int_{f^{-1}(B_{m,j})} \frac{|u_{m,j}\circ f(x) - u_{m,j}\circ f(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy\\ &\gtrsim \sum_{j=1}^{2^{mn}} |f^{-1}(B_{m,j})|^{1-2\alpha/n}\Phi_\alpha(u_{m,j}\circ f, f^{-1}(B_{m,j})).\end{aligned}$$

It suffices to estimate $|f^{-1}(B_{m,j})|$ and $\Phi_\alpha(u_{m,j}\circ f, f^{-1}(B_{m,j}))$ from below. We first notice that if $|x-z_{m,j}|<\frac{1}{2}a[(1-a)/2]^m$, then

$$f^{-1}(x)=\left(\tfrac{1}{2}a[(1-a)/2]^m\right)^{\beta/(\beta+1)}|x-z_{m,j}|^{-\beta/(\beta+1)}(x-z_{m,j})+z_{m,j},$$

and hence

$$J_{f^{-1}}(x)\sim\left(\tfrac{1}{2}a[(1-a)/2]^m\right)^{n\beta/(\beta+1)}|x-z_{m,j}|^{-n\beta/(\beta+1)}.$$

For every $y\in B_{m,j}$, if $|x-z_{m,j}|=2r_{m,j}$ then

$$r_{m,j}\le|z_{m,j}-y_{m,j}|-|y-y_{m,j}|\le|y-z_{m,j}|\le|z_{m,j}-y_{m,j}|+|y-y_{m,j}|\le 3r_{m,j},$$

and hence

$$J_{f^{-1}}(y)\sim\left(\tfrac{1}{2}a[(1-a)/2]^m\right)^{n\beta/(\beta+1)}r_{m,j}^{-n\beta/(\beta+1)}\sim 2^{\ell n\beta/(\beta+1)}.$$

Therefore,

$$|f^{-1}(B_{m,j})|\sim 2^{-\ell n/(\beta+1)}[(1-a)/2]^{mn}$$

and

$$\fint_{B_{m,j}} J_f(y)\,dy\sim 2^{\ell n\beta/(\beta+1)}\lesssim \operatorname*{ess\,inf}_{y\in B_{m,j}} J_f(y).$$

Moreover, since $J_{f^{-1}}\in A_1(\mathbb{R}^n)$, similarly to (4.4) we have

$$\begin{aligned}\Phi_\alpha(u_{m,j}\circ f, f^{-1}(B_{m,j}))&\gtrsim\Psi_{\alpha,2}(u_{m,j}\circ f, f^{-1}(2^{-4}B_{m,j}))\\&\gtrsim\Psi_{\alpha,2}(u_{m,j},2^{-4-N_2}B_{m,j})\gtrsim\Phi_\alpha(u_{m,j},2^{-8-N_2}B_{m,j}).\end{aligned}$$

Notice that for all $x\in 2^{-12-N_2}B_{m,j}$ and $y\in 2^{-8-N_2}B_{m,j}\setminus 2^{-9-N_2}B_{m,j}$, we have

$$|x-y|\sim r_{m,j}\quad\&\quad |u_{m,j}(x)-u_{m,j}(y)|\ge 2^{-9-N_2}r_{m,j}-2^{-12-N_2}r_{m,j}\ge 2^{-10-N_2}r_{m,j}.$$

Hence,

$$\begin{aligned}&\Phi_\alpha(u_{m,j},2^{-8}B_{m,j})\\&\quad\gtrsim r_{m,j}^{2\alpha-n}\fint_{2^{-12-N_2}B_{m,j}}\int_{2^{-8-N_2}B_{m,j}\setminus 2^{-9-N_2}B_{m,j}}\frac{|u_{m,j}(x)-u_{m,j}(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy\\&\quad\gtrsim r_{m,j}^{2\alpha-n}r_{m,j}^{2n}r_{m,j}^{-n-2\alpha+2}\gtrsim r_{m,j}^2.\end{aligned}$$

Therefore

$$\Phi_\alpha(u_{m,j}\circ f, f^{-1}(B_{m,j}))\gtrsim r_{m,j}^2. \tag{6.5}$$

This together with (6.3) implies that

$$\begin{aligned}\|u_m\circ f\|^2_{Q_\alpha(\mathbb{R}^n)}&\gtrsim 2^{mn}2^{-\ell(n-2\alpha)/(\beta+1)}[(1-a)/2]^{m(n-2\alpha)}2^{-2\ell}[(1-a)/2]^{2m}\\&\sim 2^{mn}2^{-\ell(n-2\alpha)/(\beta+1)}2^{-2\ell}[(1-a)/2]^{m(n-2\alpha+2)},\end{aligned}$$

as desired. □

Proof of Theorem 1.7. Fix $\alpha_0 \in (0,1)$. Let $\theta = (n-2\alpha_0)/n \in (0,1)$ and $\widetilde{E}_{\alpha_0} = (2^{\mathbb{N}_\theta})^n$ be as in (2.1). By Lemma 2.2, $\overline{\dim}_{LG}(2^{\mathbb{N}_\theta})^n = n - 2\alpha_0$ but $\dim_L(2^{\mathbb{N}_\theta})^n = 0$.

Now we need to construct a quasiconformal (Lipschitz) mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $J_f \in A_1(\mathbb{R}^n; (2^{\mathbb{N}_\theta})^n)$ but $\mathbf{C}_f$ is unbounded on $Q_\alpha(\mathbb{R}^n)$ for each $\alpha \in (\alpha_0, 1)$. The idea is similar to the construction of Theorem 1.6. We divide the argument into two cases.

Case 1: $\alpha_0 = 1$. Let $\beta > 0$ and define

$$f(x) = \begin{cases} |x-\vec{k}|^\beta (x-\vec{k}) + \vec{k} & \text{if } x \in B(\vec{k},1) \text{ with } \vec{k} \in (3\mathbb{N})^n, \\ x & \text{if } x \notin \bigcup_{\vec{k}\in\mathbb{N}^n} B(\vec{k},1). \end{cases}$$

Then f is a quasiconformal mapping and

$$\begin{cases} J_f(x) \sim |x-\vec{k}|^{n\beta} & \text{if } x \in B(\vec{k},1) \text{ for some } \vec{k} \in (3\mathbb{N})^n, \\ J_f(x) = 1 & \text{otherwise.} \end{cases}$$

Now we show that $\mathbf{C}_f$ is unbounded on $Q_\alpha(\mathbb{R}^n)$ for each $\alpha \in (0,1)$. Indeed, for each $\vec{k} \in (3\mathbb{N})^n$, we take a ball $B_{\vec{k}}$ such that $|x_{B_{\vec{k}}} - \vec{k}| = 2^{-m}$ and $r_{B_{\vec{k}}} = 2^{-m-5}$. Set

$$u_{\vec{k}}(x) = \chi_{B_{\vec{k}}}(x) d(x, \partial B_{\vec{k}}).$$

For each m, set

$$u_m = \sum_{|\vec{k}|\le 2^\ell} u_{\vec{k}} \quad \text{with} \quad \ell = m(n-2\alpha)/2\alpha.$$

Observe that if $x \in B_{\vec{k}}$, then

$$f^{-1}(x) = |x-\vec{k}|^{-\beta/(\beta+1)}(x - x_{\vec{k}}) + x_{\vec{k}},$$

and hence

$$J_{f^{-1}}(x) \sim |x-\vec{k}|^{-n\beta/(\beta+1)} \sim 2^{m\beta/(\beta+1)}.$$

Thus, one gets $|f^{-1}(B_{\vec{k}})| \sim 2^{-[1-\beta/(\beta+1)]mn}$.

By an argument similar to (6.5) for $\Phi_\alpha(u_{m,j} \circ f, f^{-1}(B_{m,j}))$, we have

$$\Phi_\alpha(u_{\vec{k}} \circ f, f^{-1}(B_{\vec{k}})) \gtrsim 2^{-2m}.$$

This leads to

$$\begin{aligned} \|u_m \circ f\|^2_{Q_\alpha(\mathbb{R}^n)} &\ge \Phi_\alpha(u_m \circ f, f^{-1}(B(0, 2^{\ell+1}))) \\ &\gtrsim 2^{\ell(2\alpha-n)} \sum_{|\vec{k}|\le 2^\ell} |f^{-1}(B_{\vec{k}})|^{1-2\alpha/n} \Phi_\alpha(u_{\vec{k}} \circ f, f^{-1}(B_{\vec{k}})) \\ &\gtrsim 2^{\ell(2\alpha-n)} \sum_{|\vec{k}|\le 2^\ell} 2^{-2m} 2^{-[1-\beta/(\beta+1)]m(n-2\alpha)} \\ &\gtrsim 2^{2\alpha\ell} 2^{-2m} 2^{-[1-\beta/(\beta+1)]m(n-2\alpha)} \gtrsim 2^{-2m} 2^{m(n-2\alpha)\beta/(\beta+1)}, \end{aligned}$$

where $\ell = m(n-2\alpha)/2\alpha$.

On the other hand, we claim that $\|u_m\|^2_{Q_\alpha(\mathbb{R}^n)} \lesssim 2^{-2m}$. The proof of this estimate is similar to that of (6.1) . Five situations have to be handled.

If $r_B \le 2^{-m-5}$, by an argument similar to (6.3) we have $\Phi_\alpha(u_m, 2B) \lesssim 2^{-2m}$.

If $r_B > 2^{-m-5}$, similarly to (6.4) we also have

$$\Phi_\alpha(u_m, 2B) \lesssim |B|^{2\alpha/n-1} \sum_{B_{\vec{k}} \cap 2B \neq \emptyset} 2^{-m(2-2\alpha+n)}.$$

If $2^{-m-5} < r_B \le 1$, there is at most one $B_{\vec{k}}$ such that $B \cap B_{\vec{k}} \neq \emptyset$, and hence $\Phi_\alpha(u_m, 2B) \lesssim 2^{-2m}$.

If $1 \le r_B \le 2^\ell$, then there are at most $2^{n+2} r_B^n$ balls $B_{\vec{k}}$ such that $B \cap B_{\vec{k}} \neq \emptyset$, and hence

$$\Phi_\alpha(u_m, 2B) \lesssim r_B^n r_B^{2\alpha-n} 2^{-m(2-2\alpha+n)} \lesssim 2^{2\alpha\ell} 2^{-m(2-2\alpha+n)} \lesssim 2^{-2m},$$

where $\ell = m(n-2\alpha)/2\alpha$.

If $r_B > 2^\ell$, then there are at most $2^{n+2} 2^{\ell n}$ balls $B_{\vec{k}}$ such that $B \cap B_{\vec{k}} \neq \emptyset$, and hence

$$\Phi_\alpha(u_m, 2B) \lesssim r_B^{2\alpha-n} 2^{\ell n} 2^{-m(2-2\alpha+n)} \lesssim 2^{2\alpha\ell} 2^{-m(2-2\alpha+n)} \lesssim 2^{-2m},$$

where $\ell = m(n-2\alpha)/2\alpha$.

Finally, we have

$$\frac{\|u_m \circ f\|_{Q_\alpha(\mathbb{R}^n)}}{\|u_m\|_{Q_\alpha(\mathbb{R}^n)}} \gtrsim 2^{m(n-2\alpha)\beta/(\beta+1)} \to \infty$$

as $m \to \infty$ since $\beta > 0$.

Case 2: $\alpha_0 \in (0, 1)$. Similarly to Case 1, we can first construct quasiconformal mappings $f : \mathbb{R}^n \to \mathbb{R}^n$ with $J_f \in A_1(\mathbb{R}^n; (2^{\mathbb{N}_\theta})^n)$, and then construct the critical function u_m, but the key parameter ℓ there is now taken as $m(n-2\alpha)/(2\alpha - n + \theta n)$ where

$$2\alpha - n + \theta n > 0 \Leftrightarrow 2\alpha > n - \theta n = \alpha_0.$$

Such a $\mathbf{C}_f$ is unbounded on $Q_\alpha(\mathbb{R}^n)$ for all $\alpha \in (\alpha_0, 1)$, and hence satisfies our requirement; we omit the details. □

Acknowledgements. The third and fourth authors would like to thank Professor Jacques Peyrière for kind discussions on the Minkowski type dimension. Part of this research was done during the first and fourth authors' visit at IPAM, UCLA; both authors wish to thank for the support.

P. Koskela was supported by the Academy of Finland grant 120972; J. Xiao was supported by NSERC of Canada (# 202979463102000) and URP of Memorial University (# 208227463102000); Y. Zhou was supported by National Natural Science Foundation of China (#11522102, # 11201015), and New Teachers' Fund for Doctor Stations (# 20121102120031) and Program for New Century Excellent Talents in University (# NCET-11-0782) of Ministry of Education of China.

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