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Mirror symmetry for exceptional unimodular singularities

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Abstract. In this paper, we prove the mirror symmetry conjecture between the Saito–Givental theory of exceptional unimodular singularities on the Landau–Ginzburg B-side and the Fan–Jarvis–Ruan–Witten theory of their mirror partners on the Landau–Ginzburg A-side. On the B-side, we develop a perturbative method to compute the genus-0 correlation functions associated to the primitive forms. This is applied to the exceptional unimodular singularities, and we show that the numerical invariants match the orbifold-Grothendieck–Riemann–Roch and WDVV calculations in FJRW theory on the A-side. The coincidence of the full data at all genera is established by reconstruction techniques. Our result establishes the first examples of LG-LG mirror symmetry for weighted homogeneous polynomials of central charge greater than one (i.e. which contain negative degree deformation parameters).

Keywords. Landau–Ginzburg model, mirror symmetry, singularity, FJRW theory, primitive form

1. Introduction

Mirror symmetry is a fascinating geometric phenomenon discovered in string theory. The rise of mathematical interest in it dates back to the early 1990s, when Candelas, de la Ossa, Green and Parkes [6] successfully predicted the number of rational curves on the quintic 3-fold in terms of period integrals on the mirror quintics. Since then, one popular mathematical formulation of mirror symmetry has been the equivalence on the mirror pairs between the Gromov–Witten theory of counting curves and the theory of variation of Hodge structures. This was proved in [20, 33] for a large class of mirror examples via toric geometry. Mirror symmetry has also deep extensions to open strings incorpo-

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rating D-brane constructions [27, 48]. In our paper, we will focus on closed string mirror symmetry.

Gromov–Witten theory presents the mathematical counterpart of A-twisted supersymmetric nonlinear σ-models, borrowing the name of A-model from physics terminology. Its mirror theory is called the B-model. On either side, there is a closely related linearized model, called the $N = 2$ Landau–Ginzburg model (or LG model), describing the quantum geometry of singularities. There exist deep connections in physics between nonlinear sigma models on Calabi–Yau manifolds and Landau–Ginzburg models (see [25] for related literature).

In this paper, we will study the LG-LG mirror symmetry conjecture, which asserts the equivalence of two nontrivial theories of singularities for mirror pairs (W, G), (W^T, G^T). Here W is an *invertible weighted homogeneous polynomial* on $\mathbb{C}^n$ with an isolated critical point at the origin, and G is a finite abelian symmetry group of W. The mirror weighted homogeneous polynomial W^T was introduced by Berglund and Hübsch [5] in the early 1990s. For an invertible polynomial $W = \sum_{i=1}^n \prod_{j=1}^n x_j^{a_{ij}}$, the mirror polynomial is $W^T = \sum_{i=1}^n \prod_{j=1}^n x_j^{a_{ji}}$. The mirror group G^T was introduced by Berglund and Henningson [4] and Krawitz [28] independently. Krawitz also constructed a ring isomorphism between the two models. Now the mirror symmetry between these LG pairs is also called *Berglund–Hübsch–Krawitz mirror* [11]. When $G = G_W$ is the group of diagonal symmetries of W, the dual group $G_W^T = \{1\}$ is trivial. In order to formulate the conjecture, let us introduce the theories on both sides first. We remark that one of the most general mirror constructions of LG models was proposed by Hori and Vafa [26].

A geometric candidate for LG A-model is Fan–Jarvis–Ruan–Witten theory (or FJRW theory) constructed by Fan, Jarvis and Ruan [14, 15], based on a proposal of Witten [50]. Several purely algebraic versions of LG A-model have been worked out [7, 36]. FJRW theory is closely related to Gromov–Witten theory, in terms of the so-called Landau–Ginzburg/Calabi–Yau correspondence [10, 37]. The purpose of FJRW theory is to solve the moduli problem for the Witten equations of a LG model (W, G) (G is an appropriate subgroup of G_W). The outputs are the *FJRW invariants*. Analogous to the Gromov–Witten invariants, the FJRW invariants are defined via the intersection theory of appropriate *virtual fundamental cycles* with tautological classes on the moduli space of stable curves. These invariants virtually count the solutions of the Witten equations on orbifold curves. For our purpose later, we consider $G = G_W$, and summarize the main ingredients of FJRW theory as follows (see Section 2 for more details):

- An *FJRW ring* $(H_W, \bullet)$. Here H_W is the *FJRW state space* given by the G_W-invariant relative cohomology of W, and the multiplication $\bullet$ is defined by an intersection pairing together with the *genus*-0 *primary FJRW invariants* with three marked points.
- A *prepotential* $\mathcal{F}_{0,W}^{\mathrm{FJRW}}$ of a formal Frobenius manifold structure on H_W, whose coefficients are all the *genus*-0 *primary FJRW invariants* $\langle \cdots \rangle_0^W$.
- A *total ancestor potential* $\mathscr{A}_W^{\mathrm{FJRW}}$ that collects the FJRW invariants at all genera.

A geometric candidate of the LG B-model of (W^T, G^T) for general G^T is still missing. When $G = G_W$, then $G^T = \{1\}$ and a candidate comes from the third author's theory of primitive forms [41]. The starting point here is a germ of holomorphic function

($f = W^T$ here)

$$f(\mathbf{x}) : (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}, 0), \qquad \mathbf{x} = \{x_i\}_{i=1,\dots,n},$$

with an isolated singularity at the origin $\mathbf{0}$. We consider its universal unfolding

$$(\mathbb{C}^n \times \mathbb{C}^\mu, \mathbf{0} \times \mathbf{0}) \to (\mathbb{C} \times \mathbb{C}^\mu, 0 \times \mathbf{0}), \qquad (\mathbf{x}, \mathbf{s}) \mapsto (F(\mathbf{x}, \mathbf{s}), \mathbf{s}),$$

where $\mu = \dim_{\mathbb{C}} \mathrm{Jac}(f)_{\mathbf{0}}$ is the Milnor number, and $\mathbf{s} = \{s_\alpha\}_{\alpha=1,\dots,\mu}$ parametrize the deformation. Roughly speaking, a primitive form is a relative holomorphic volume form

$$\zeta = P(\mathbf{x}, \mathbf{s}) d^n\mathbf{x}, \qquad d^n\mathbf{x} = dx_1 \cdots dx_n,$$

at the germ $(\mathbb{C}^n \times \mathbb{C}^\mu, \mathbf{0} \times \mathbf{0})$, which induces a Frobenius manifold structure (called the flat structure in [41]) at the germ $(\mathbb{C}^\mu, \mathbf{0})$. This gives the genus-0 invariants in the LG B-model. At higher genus, Givental [19] proposed a remarkable formula (its uniqueness was established by Teleman [49]) for the total ancestor potential for semisimple Frobenius manifold structures, which can be extended to some nonsemisimple boundary points [12, 34] including $\mathbf{s} = \mathbf{0}$ of our interest. The whole package is now referred to as Saito–Givental theory. We will call the extended total ancestor potential at $\mathbf{s} = \mathbf{0}$ the *Saito–Givental potential* and denote it by $\mathscr{A}_f^{\mathrm{SG}}$.

For $G = G_W$, the LG-LG mirror conjecture (for all genera) is well-formulated [11]:

Conjecture 1.1. *For a mirror pair (W, G_W) and $(W^T, \{1\})$, there exists a ring isomorphism $(H_W, \bullet) \cong \mathrm{Jac}(W^T)$ together with a choice of primitive forms ζ such that the FJRW potential $\mathscr{A}_W^{\mathrm{FJRW}}$ is identified with the Saito–Givental potential $\mathscr{A}_{W^T}^{\mathrm{SG}}$.*

For the weighted homogeneous polynomial $W = W(x_1, \dots, x_n)$, we have

$$W(\lambda^{q_1} x_1, \dots, \lambda^{q_n} x_n) = \lambda W(x_1, \dots, x_n), \qquad \forall \lambda \in \mathbb{C}^*,$$

with each weight q_i being a unique rational number satisfying $0 < q_i \le \frac{1}{2}$ [38]. There is a partial classification of W using the *central charge* [43]

$$\hat{c}_W := \sum_i (1 - 2q_i).$$

So far, Conjecture 1.1 has only been proved for $\hat{c}_W < 1$ (i.e., ADE singularities) by Fan, Jarvis and Ruan [14] and for $\hat{c}_W = 1$ (i.e., simple elliptic singularities) by Krawitz, Milanov and Shen [30, 35]. However, it has been open for $\hat{c}_W > 1$, including exceptional unimodular modular singularities and a wide class of those related to K3 surfaces and CY 3-folds. One of the major obstacles is that computations in the LG B-model require concrete information about the primitive forms. The existence of the primitive forms for a general isolated singularity has been proved by M. Saito [46]. However, explicit formulas were only known for weighted homogeneous polynomials with $\hat{c}_W \le 1$ [41]. This is due to the difficulty of mixing between positive and negative degree deformations when $\hat{c}_W > 1$.

The main objective of the present paper is to prove that Conjecture 1.1 is true when W^T is one of the exceptional unimodular singularities as in Table 1. Here we use variables

Table 1. Exceptional unimodular singularities.

	Polynomial		Polynomial		Polynomial		Polynomial
E_{12}	x^3+y^7	W_{12}	x^4+y^5	U_{12}	$x^3+y^3+z^4$		
Q_{12}	$x^2y+xy^3+z^3$	Z_{12}	x^3y+y^4x	S_{12}	$x^2y+y^2z+z^3x$		
E_{14}	$x^2+xy^4+z^3$	E_{13}	x^3+xy^5	Z_{13}	$x^2+xy^3+yz^3$	W_{13}	$x^2+xy^2+yz^4$
Q_{10}	$x^2y+y^4+z^3$	Z_{11}	x^3y+y^5	Q_{11}	$x^2y+y^3z+z^3$	S_{11}	$x^2y+y^2z+z^4$

x, y, z instead of the conventional $x_1, \ldots, x_n$. These polynomials are all of central charge larger than 1, providing the first nontrivial examples with the existence of negative degree deformation (i.e., irrelevant deformation) parameters.

Originally, the 14 exceptional unimodular singularities given by Arnold [3] are one-parameter families of singularities with three variables. Each family contains a weighted homogeneous singularity characterized by the existence of only one negative degree but no zero-degree deformation parameter [43]. In this paper, we consider the stable equivalence class of a singularity, and always choose polynomial representatives of the class with no square terms for additional variables. The FJRW theory with the group of diagonal symmetries is invariant when adding square terms for additional variables.

LG-LG mirror symmetry for exceptional unimodular singularities

Let us explain how we achieve the goal in more detail. Following [28], we can specify a ring isomorphism $\mathrm{Jac}(W^T) \cong (H_W, \bullet)$. Then we calculate certain FJRW invariants, by an orbifold-Grothendieck–Riemann–Roch formula and WDVV equations. More precisely, we have

Proposition 1.2. *Let W^T be one of the* 14 *singularities. Then*

$$\Psi : \mathrm{Jac}(W^T) \to (H_W, \bullet),$$

defined in (2.20) *and* (2.23)*, generates a ring isomorphism. Let M_i^T be the i-th monomial of W^T, and ϕ_μ be of the highest degree among the specified basis of* $\mathrm{Jac}(W^T)$ *in Table* 2*. Let q_i be the weight of x_i with respect to W. For each i, we have genus-*0 *FJRW invariants*

$$\langle \Psi(x_i), \Psi(x_i), \Psi(M_i^T/x_i^2), \Psi(\phi_\mu)\rangle_0^W = q_i \quad \textit{whenever } M_i^T \neq x_i^2. \tag{1.1}$$

Surprisingly, if W^T belongs to Q_{11} or S_{11}, then the ring isomorphism $\mathrm{Jac}(W^T) \cong (H_W, \bullet)$ was not known in the literature. The difficulty comes from the fact that if some q_j is $1/2$, then one of the ring generators is a so-called *broad element* in FJRW theory, and invariants with broad generators are hard to compute. We overcome this difficulty for the two cases, using Getzler's relation on $\overline{\mathcal{M}}_{1,4}$. It is quite interesting that the higher genus structure detects the ring structure. We expect that our method works for general unknown cases of $(H_W, \bullet)$ as well.

On the B-side, recently there has appeared a perturbative way to compute the primitive forms for arbitrary weighted homogeneous singularities [32]. In this paper, we develop a perturbative method for the whole package of the associated Frobenius manifolds, and describe a recursive algorithm to compute the associated flat coordinates and the potential function $\mathcal{F}^{\mathrm{SG}}_{0,W^T}$ (see Section 3.2). We apply this perturbative method to compute genus-0 invariants of LG B-model associated to the unique primitive forms [23, 32] of the exceptional unimodular singularities, and show that it coincides with the A-side FJRW invariants for W in Proposition 1.2 (up to sign).

In the next step, we establish a reconstruction theorem in such cases (Lemma 4.2), showing that the WDVV equations are powerful enough to determine the full prepotentials for both sides from those invariants in (1.1). This gives the main result of our paper:

Theorem 1.3. *Let W^T be one of the* 14 *exceptional unimodular singularities in Table 1. Then the specified ring isomorphism Ψ induces an isomorphism of Frobenius manifolds between* $\mathrm{Jac}(W^T)$ (*which comes from the primitive form of W^T*) *and H_W* (*which comes from the FJRW theory of (W, G_W)*)*. That is, the prepotentials are equal to each other:*

$$\mathcal{F}^{\mathrm{SG}}_{0,W^T} = \mathcal{F}^{\mathrm{FJRW}}_{0,W}. \tag{1.2}$$

In general, the computations of FJRW invariants are challenging due to our very little understanding of virtual fundamental cycles, especially at higher genus. However, according to Teleman [49] and Milanov [34], the nonsemisimple limit $\mathscr{A}^{\mathrm{SG}}_{W^T}$ is fully determined by the genus-0 data on the semisimple points nearby. As a consequence, we upgrade our mirror symmetry statement to higher genus and prove Conjecture 1.1 for the exceptional unimodular singularities.

Corollary 1.4. *Conjecture 1.1 is true for W^T being one of the* 14 *exceptional unimodular singularities in Table 1. The specified ring isomorphism Ψ induces the following coincidence of total ancestor potentials:*

$$\mathscr{A}^{\mathrm{SG}}_{W^T} = \mathscr{A}^{\mathrm{FJRW}}_{W}. \tag{1.3}$$

The choice in Table 1 has the property that the mirror weighted homogeneous polynomials are again representatives of the exceptional unimodular singularities. Arnold discovered a strange duality among the 14 exceptional unimodular singularities, which says that the Gabrielov numbers of each coincide with the Dolgachev numbers of its strange dual [2]. The strange duality is also reproved algebraically in [44]. The choices in Table 1 also represent Arnold's strange duality: the first two rows are strange dual to themselves, and the last two rows are dual to each other. For example, E_{14} is strange dual to Q_{10}. Beyond the choices in Table 1, we also discuss the LG-LG mirror symmetry for other invertible polynomial representatives (some of whose mirrors may no longer be exceptional singularities) where equality (1.3) still holds. The results are summarized in Theorem 4.3 and Remark 4.5. Our method has the advantage of being applicable to general invertible polynomials with more involved WDVV techniques developed.

The present paper is organized as follows. In Section 2, we give a brief review of FJRW theory and compute the initial FJRW invariants as in Proposition 1.2. In Section 3, we develop the perturbative method for computing the Frobenius manifolds in the LG B-model following [32]. In Section 4, we prove Conjecture 1.1 when the B-side is given by one of the exceptional unimodular singularities. We also discuss the more general case when either side is given by an arbitrary weighted homogeneous polynomial representative of the exceptional unimodular singularities. Finally, in the appendix, we provide detailed descriptions of the specified isomorphism Ψ as well as a complete list of the genus-0 4-point functions on the B-side for all the exceptional unimodular singularities. We would like to point out that Sections 2 and 3 are completely independent of each other. The reader can choose either section to start from.

2. A-model: FJRW theory

2.1. FJRW theory

In this section, we give a brief review of FJRW theory. For more details, we refer the readers to [14, 15]. We start with a *nondegenerate weighted homogeneous polynomial* $W = W(x_1, \dots, x_n)$, where the nondegeneracy means that W has an isolated critical point at the origin $\mathbf{0} \in \mathbb{C}^n$ and contains no monomial of the form $x_i x_j$ for $i \neq j$. This implies that each x_i has a unique weight $q_i \in \mathbb{Q} \cap (0, 1/2]$ [38]. Let G_W be the *group of diagonal symmetries*,

$$G_W := \{(\lambda_1, \dots, \lambda_n) \in (\mathbb{C}^*)^n \mid W(\lambda_1 x_1, \dots, \lambda_n x_n) = W(x_1, \dots, x_n)\}. \tag{2.1}$$

In this paper, we will only consider FJRW theory for the pair (W, G_W). In general, FJRW theory also works for any subgroup $G \subset G_W$ where G contains the *exponential grading element*

$$J = \big(\exp(2\pi\sqrt{-1}\, q_1), \dots, \exp(2\pi\sqrt{-1}\, q_n)\big) \in G_W. \tag{2.2}$$

Definition 2.1. The *FJRW state space* H_W for (W, G_W) is defined to be the direct sum of all G_W-invariant relative cohomology:

$$H_W := \bigoplus_{\gamma \in G_W} H_\gamma, \qquad H_\gamma := H^{N_\gamma}(\mathrm{Fix}(\gamma), W_\gamma^\infty, \mathbb{C})^{G_W}. \tag{2.3}$$

Here $\mathrm{Fix}(\gamma)$ is the fixed point set of γ, and $\mathbb{C}^{N_\gamma} \cong \mathrm{Fix}(\gamma) \subset \mathbb{C}^n$; W_γ is the restriction of W to $\mathrm{Fix}(\gamma)$; and $W_\gamma^\infty := (\mathrm{Re}\, W_\gamma)^{-1}(M, \infty)$ with $M \gg 0$, where $\mathrm{Re} W_\gamma$ is the real part of W_γ.

Each $\gamma \in G_W$ has a unique form

$$\gamma = \big(\exp(2\pi\sqrt{-1}\, \Theta_1^\gamma), \dots, \exp(2\pi\sqrt{-1}\, \Theta_n^\gamma)\big) \in (\mathbb{C}^*)^n, \qquad \Theta_i^\gamma \in [0, 1) \cap \mathbb{Q}. \tag{2.4}$$

Thus H_W is a graded vector space, where to each nonzero $\alpha \in H_\gamma$, we assign the degree

$$\deg \alpha = N_\gamma/2 + \sum_{i=1}^{n} (\Theta_i^\gamma - q_i).$$

We call H_γ a *narrow sector* if $\mathrm{Fix}(\gamma)$ consists of $\mathbf{0} \in \mathbb{C}^n$ only, and a *broad sector* otherwise.

The FJRW vector space H_W is equipped with a nondegenerate symmetric pairing

$$\langle\ ,\ \rangle := \sum_{\gamma\in G_W} \langle\ ,\ \rangle_\gamma,$$

where each $\langle\ ,\ \rangle_\gamma : H_\gamma \times H_{\gamma^{-1}} \to \mathbb{C}$ is induced from the intersection pairing of Lefschetz thimbles. The pairing between H_{γ_1} and H_{γ_2} is nonzero only if $\gamma_1\gamma_2 = 1$. Moreover, there is a canonical isomorphism (see [14, Section 5.1], [9, Appendix A], and references therein)

$$(H_W, \langle\ ,\ \rangle) \cong \Big(\bigoplus_{\gamma\in G_W} (\mathrm{Jac}(W_\gamma)\omega_\gamma)^{G_W}, \sum_{\gamma\in G_W} \langle\ ,\ \rangle_{\mathrm{res},\gamma} \Big). \tag{2.5}$$

Here ω_γ is a volume form on $\mathrm{Fix}(\gamma)$ of the type $dx_{j_1}\wedge\cdots\wedge dx_{j_{N_\gamma}}$, where we mean $\omega_\gamma = 1$ if $N_\gamma = 0$. G_W acts on both x_i and dx_i. Let $(\mathrm{Jac}(W_\gamma)\omega_\gamma)^{G_W}$ be the G_W-invariant part of the action. We choose a generator

$$\mathbf{1}_\gamma := \omega_\gamma \in H^{N_\gamma}(\mathrm{Fix}(\gamma), W_\gamma^\infty, \mathbb{C}). \tag{2.6}$$

If H_γ is narrow, then $H_\gamma \cong (\mathrm{Jac}(W_\gamma)\omega_\gamma)^{G_W} \cong \mathbb{C}$ is generated by $\mathbf{1}_\gamma$. If H_γ is broad, we denote generators of H_γ by $\phi\,\mathbf{1}_\gamma$ via $\phi\,\omega_\gamma \in (\mathrm{Jac}(W_\gamma)\omega_\gamma)^{G_W}$. Finally, the residue pairing $\langle\ ,\ \rangle_{\mathrm{res},\gamma}$ is defined from the standard residue Res_{W_γ} of W_γ,

$$\langle f\omega_\gamma, g\omega_\gamma\rangle_{\mathrm{res},\gamma} := \mathrm{Res}_{W_\gamma}(fg) := \mathrm{Residue}_{x=0} \frac{fg\omega_\gamma}{\frac{\partial W_\gamma}{\partial x_{j_1}}\cdots\frac{\partial W_\gamma}{\partial x_{j_{N_\gamma}}}}.$$

It is highly nontrivial to construct a virtual cycle for the moduli of solutions of Witten equations. For the details of the construction, we refer to the original paper of Fan, Jarvis and Ruan [15]. Let $\mathscr{C} := \mathscr{C}_{g,k}$ be a stable genus-g orbifold curve with marked points $p_1, \ldots, p_k$ (where $2g-2+k>0$). We only allow orbifold points at marked points and nodals. Near each orbifold point p, a local chart is given by $\mathbb{C}/G_p$ with $G_p \cong \mathbb{Z}/m\mathbb{Z}$ for some positive integer m. Let $\mathscr{L}_1, \ldots, \mathscr{L}_n$ be orbifold line bundles over $\mathscr{C}$. Let σ_i be a C^∞-section of $\mathscr{L}_i$. We consider the W-structures, which can be thought of as the background data to be used to set up the Witten equations

$$\bar\partial\sigma_i + \overline{\frac{\partial W}{\partial\sigma_i}} = 0.$$

For simplicity, we only discuss cases where $W = M_1 + \cdots + M_n$ with $M_i = \prod_{j=1}^n x_j^{a_{ij}}$. Let K_C be the canonical bundle for the underlying curve C and $\rho : \mathscr{C} \to C$ be the forgetful morphism. A *W-structure* $\mathfrak{L}$ consists of $(\mathscr{C}, \mathscr{L}_1, \ldots, \mathscr{L}_n, \varphi_1, \ldots, \varphi_n)$ where φ_i is an isomorphism of orbifold line bundles

$$\varphi_i : \bigotimes_{j=1}^n \mathscr{L}_j^{\otimes a_{i,j}} \to \rho^*(K_{C,\log}), \qquad K_{C,\log} := K_C \otimes \bigotimes_{j=1}^k \mathscr{O}(p_j).$$

A W-structure induces a representation $r_p : G_p \to G_W$ at each point $p \in \mathscr{C}$. We require it to be faithful. The moduli space of pairs $\mathfrak{C} = (\mathscr{C}, \mathfrak{L})$ is called the *moduli of stable*

W-orbicurves and denoted by $\overline{\mathscr{W}}_{g,k}$. According to [14], $\overline{\mathscr{W}}_{g,k}$ is a Deligne–Mumford stack, and the forgetful morphism $\mathrm{st} : \overline{\mathscr{W}}_{g,k} \to \overline{\mathcal{M}}_{g,k}$ to the moduli space of stable curves is flat, proper and quasi-finite. $\overline{\mathscr{W}}_{g,k}$ can be decomposed into open and closed stacks by decorations on each marked point,

$$\overline{\mathscr{W}}_{g,k} = \sum_{(\gamma_1,\ldots,\gamma_k)\in(G_W)^k} \overline{\mathscr{W}}_{g,k}(\gamma_1,\ldots,\gamma_k), \qquad \gamma_j := r_{p_j}(1).$$

Furthermore, let Γ be the dual graph of the underlying curve C. Each vertex of Γ represents an irreducible component of C, each edge represents a node, and each half-edge represents a marked point. Let $\sharp E(\Gamma)$ be the number of edges in Γ. We decorate the half-edge representing the point p_j by an element $\gamma_j \in G_W$. We denote the decoration by $\Gamma_{\gamma_1,\ldots,\gamma_k}$ and call it a *G_W-decorated dual graph*. We further call it *fully G_W-decorated* if we also assign some $\gamma_+ \in G_W$ and $\gamma_- = (\gamma_+)^{-1}$ on the two sides of each edge. The stack $\overline{\mathscr{W}}_{g,k}(\gamma_1,\ldots,\gamma_k)$ is stratified, where each closure in $\overline{\mathscr{W}}_{g,k}(\gamma_1,\ldots,\gamma_k)$ of the stack of stable W-orbicurves with fixed decorations $(\gamma_1,\ldots,\gamma_k)$ on Γ is denoted by $\overline{\mathscr{W}}_{g,k}(\Gamma_{\gamma_1,\ldots,\gamma_k})$.

If $\overline{\mathscr{W}}_{g,k}(\gamma_1,\ldots,\gamma_k)$ is nonempty, then the *line bundle criterion* follows [14, Proposition 2.2.8]:

$$\deg(\rho_*\mathscr{L}_i) = (2g-2+k)q_i - \sum_{j=1}^{k}\Theta_i^{\gamma_j} \in \mathbb{Z}, \qquad i = 1,\ldots,n. \tag{2.7}$$

In [15], Fan, Jarvis and Ruan perturb the polynomial W to polynomials of Morse type and construct virtual cycles from the solutions of the perturbed Witten equations. Those virtual cycles transform in the same way as the Lefschetz thimbles attached to the critical points of the perturbed polynomials. As a consequence, they construct a virtual cycle

$$[\overline{\mathscr{W}}_{g,k}(\Gamma_{\gamma_1,\ldots,\gamma_k})]^{\mathrm{vir}} \in H_*(\overline{\mathscr{W}}_{g,k}(\Gamma_{\gamma_1,\ldots,\gamma_k}),\mathbb{C}) \otimes \prod_{j=1}^{k} H_{N_{\gamma_j}}(\mathrm{Fix}(\gamma_j), W_{\gamma_j}^{\infty},\mathbb{C})^{G_W},$$

which has total degree

$$2\Big((\hat{c}_W-3)(1-g)+k-\sharp E(\Gamma)-\sum_{j=1}^{k}\sum_{i=1}^{n}(\Theta_i^{\gamma_j}-q_i)\Big). \tag{2.8}$$

Based on this, they obtain a cohomological field theory $\{\Lambda_{g,k}^W : (H_W)^{\otimes k} \to H^*(\overline{\mathcal{M}}_{g,k},\mathbb{C})\}$ with a flat identity. Each $\Lambda_{g,k}^W$ is defined by extending the following map linearly to H_W:

$$\Lambda_{g,k}^W(\alpha_1,\ldots,\alpha_k) := \frac{|G_W|^g}{\deg(\mathrm{st})}\mathrm{PD}\ \mathrm{st}_*\Big([\overline{\mathscr{W}}_{g,k}(\gamma_1,\ldots,\gamma_k)]^{\mathrm{vir}} \cap \prod_{j=1}^{k}\alpha_j\Big), \qquad \alpha_j \in H_{\gamma_j}.$$

Definition 2.2. Let ψ_j be the j-th psi class in $H^*(\overline{\mathcal{M}}_{g,k})$. Define *FJRW invariants* (or *correlators*)

$$\langle\tau_{\ell_1}(\alpha_1),\ldots,\tau_{\ell_k}(\alpha_k)\rangle_{g,k}^W = \int_{\overline{\mathcal{M}}_{g,k}} \Lambda_{g,k}^W(\alpha_1,\ldots,\alpha_k)\prod_{j=1}^{k}\psi_j^{\ell_j}, \qquad \alpha_j \in H_{\gamma_j}. \tag{2.9}$$

The FJRW invariants in (2.9) are called *primary* if all ℓ_j are zero. We then simply denote them by $\langle\alpha_1,\dots,\alpha_k\rangle_g^W$. We call $(H_W,\bullet)$ an *FJRW ring* where the multiplication $\bullet$ on H_W is defined by

$$\langle\alpha\bullet\beta,\gamma\rangle=\langle\alpha,\beta,\gamma\rangle_0^W. \tag{2.10}$$

If the invariant in (2.9) is nonzero, the integrand should be a top degree element in $H^*(\overline{\mathcal{M}}_{g,k})$. Then using the total degree formula (2.8) and the definition of the cohomological field theory, it is not hard to see that

$$\sum_{j=1}^k\deg\alpha_j+\sum_{j=1}^k\ell_j=(\hat{c}_W-3)(1-g)+k. \tag{2.11}$$

Let us fix a basis $\{\alpha_j\}_{j=1}^{\mu}$ of H_W, with α_1 being the identity. Let $\mathbf{t}(z)=\sum_{m\geq 0}\sum_{j=1}^{\mu}t_{m,\alpha_j}\,\alpha_j\,z^m$. The *FJRW total ancestor potential* is defined to be

$$\mathscr{A}_W^{\mathrm{FJRW}}=\exp\Big(\sum_{g\geq 0}\hbar^{g-1}\sum_{k\geq 0}\frac{1}{k!}\langle\mathbf{t}(\psi_1)+\psi_1,\dots,\mathbf{t}(\psi_k)+\psi_k\rangle_{g,k}^W\Big). \tag{2.12}$$

There is a formal Frobenius manifold structure on H_W, in the sense of Dubrovin [13]. Its *prepotential* is given by

$$\mathcal{F}_{0,W}^{\mathrm{FJRW}}=\sum_{k\geq 3}\frac{1}{k!}\langle\mathbf{t}_0,\dots,\mathbf{t}_0\rangle_{0,k}^W,\qquad \mathbf{t}_0=\sum_{j=1}^{\mu}t_{0,\alpha_j}\,\alpha_j.$$

The prepotential satisfies the WDVV (Witten–Dijkgraaf–Verlinde–Verlinde) equations

$$\sum_{i,j}\frac{\partial^3\mathcal{F}_{0,W}^{\mathrm{FJRW}}}{\partial t_{\alpha_a}\partial t_{\alpha_d}\partial t_{\alpha_i}}\eta^{ij}\frac{\partial^3\mathcal{F}_{0,W}^{\mathrm{FJRW}}}{\partial t_{\alpha_j}\partial t_{\alpha_b}\partial t_{\alpha_c}}=\sum_{i,j}\frac{\partial^3\mathcal{F}_{0,W}^{\mathrm{FJRW}}}{\partial t_{\alpha_a}\partial t_{\alpha_b}\partial t_{\alpha_i}}\eta^{ij}\frac{\partial^3\mathcal{F}_{0,W}^{\mathrm{FJRW}}}{\partial t_{\alpha_j}\partial t_{\alpha_c}\partial t_{\alpha_d}},\qquad t_\alpha:=t_{0,\alpha}, \tag{2.13}$$

where (η^{ij}) is the inverse of the matrix $(\langle\alpha_i,\alpha_j\rangle)$. It implies [14, Lemma 6.2.6]

$$\begin{aligned}\langle\dots,\alpha_a,\alpha_b\bullet\alpha_c,\alpha_d\rangle_{0,k}=S_k&+\langle\dots,\alpha_a\bullet\alpha_b,\alpha_c,\alpha_d\rangle_{0,k}+\langle\dots,\alpha_a,\alpha_b,\alpha_c\bullet\alpha_d\rangle_{0,k}\\&-\langle\dots,\alpha_a\bullet\alpha_d,\alpha_b,\alpha_c\rangle_{0,k}.\end{aligned} \tag{2.14}$$

where $k\geq 3$, S_k is a linear combination of products of correlators with the number of marked points no greater than $k-1$. Moreover, $S_3=S_4=0$.

Another important tool is the Concavity Axiom [14, Theorem 4.1.8]. Consider the universal W-structure $(\mathscr{L}_1,\dots,\mathscr{L}_n)$ on the universal curve $\pi:\mathscr{C}\to\overline{\mathscr{W}}_{g,k}(\Gamma_{\gamma_1,\dots,\gamma_k})$. If

$$\text{all } H_{\gamma_i} \text{ are narrow and } \pi_*\Big(\bigoplus_{i=1}^n\mathscr{L}_i\Big)=0, \tag{2.15}$$

then $R^1\pi_*(\bigoplus_{i=1}^n\mathscr{L}_i)$ is a vector bundle of constant rank, denoted by D, and

$$[\overline{\mathscr{W}}_{g,k}(\Gamma_{\gamma_1,\dots,\gamma_k})]^{\mathrm{vir}}\cap\prod_{i=1}^k\mathbf{1}_{\gamma_i}=(-1)^D c_D\Big(R^1\pi_*\Big(\bigoplus_{i=1}^n\mathscr{L}_i\Big)\Big)\cap[\overline{\mathscr{W}}_{g,k}(\Gamma_{\gamma_1,\dots,\gamma_k})]. \tag{2.16}$$

This can be calculated by the orbifold Grothendieck–Riemann–Roch formula [8, Theorem 1.1.1]. As a consequence, if the codimension D is 1, we have

$$\Lambda^W_{0,4}(\mathbf{1}_{\gamma_1},\ldots,\mathbf{1}_{\gamma_4}) = \sum_{i=1}^{n}\left(\frac{B_2(q_i)}{2}\kappa_1 - \sum_{j=1}^{4}\frac{B_2(\Theta_i^{\gamma_j})}{2}\psi_j + \sum_{\Gamma_{\mathrm{cut}}}\frac{B_2(\Theta_i^{\gamma_{\Gamma_{\mathrm{cut}}}})}{2}[\Gamma_{\mathrm{cut}}]\right). \quad (2.17)$$

Here $B_2(x) := x^2 - x + 1/6$ is the second Bernoulli polynomial, and κ_1 is the first kappa class on $\overline{\mathcal{M}}_{0,4}$. Here the graphs Γ_{cut} are fully G_W-decorated on the boundary of $\overline{\mathscr{W}}_{0,4}(\gamma_1,\ldots,\gamma_4)$. Each Γ_{cut} has exactly one edge which seperates the graph into two components. Two sides of the edge are decorated by some $\gamma_+ \in G_W$ and $\gamma_- := (\gamma_+)^{-1}$ so that each component of Γ_{cut} satisfies the line bundle criterion (2.7). Finally, $[\Gamma_{\mathrm{cut}}]$ denotes the boundary class in $H^*(\overline{\mathcal{M}}_{0,4},\mathbb{C})$ that corresponds to the underlying undecorated graph of Γ_{cut}.

We call a correlator *concave* if it satisfies (2.15). Otherwise we call it *nonconcave*. A nonconcave correlator may contain broad sectors. In this paper, we will use WDVV to compute the nonconcave correlators. Some other methods are described in [7, 22].

2.2. *FJRW invariants*

In this subsection, we will prove Proposition 1.2. Let us first describe the construction of the mirror polynomial W^T. Let $W = M_1 + \cdots + M_n$ with $M_i = \prod_{j=1}^n x_j^{a_{ij}}$. We call such a polynomial W *invertible* because its *exponent matrix* $E_W := (a_{ij})$ is invertible. Berglund and Hübsch [5] introduced a mirror polynomial W^T,

$$W^T := \sum_{i=1}^{n}\prod_{j=1}^{n} x_j^{a_{ji}}. \quad (2.18)$$

Its exponent matrix E_{W^T} is just the transpose matrix of E_W, i.e. $E_{W^T} = (E_W)^T$. In [31], Kreuzer and Skarke proved that every invertible W is a direct sum of three *atomic types* of singularities: Fermat, chain and loop. If W is of atomic type, then W^T belongs to the same atomic type. We list the three atomic types (with $q_i \le 1/2$) and a $\mathbb{C}$-basis of their Jacobi algebra as follows. The table also contains an element ϕ_μ of highest degree.

Table 2. Invertible singularities.

	Polynomial f	$\mathbb{C}$-basis of Jac(f)	ϕ_μ
m-Fermat	$x_1^{a_1} + \cdots + x_m^{a_m}$	$\prod_{i=1}^m x_i^{k_i}$, $k_i < a_i - 1$	$\prod_{i=1}^m x_i^{a_i-2}$
m-Chain:	$x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_m^{a_m}$	$\{\prod_{i=1}^m x_i^{k_i}\}_{\mathbf{k}}$	$x_1^{a_1-2}\prod_{i=2}^m x_i^{a_i-1}$
m-Loop:	$x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_m^{a_m}x_1$	$\prod_{i=1}^m x_i^{k_i}$, $k_i < a_i$	$\prod_{i=1}^m x_i^{a_i-1}$

Here in the case of m-Chain, $\mathbf{k} = (k_1,\ldots,k_m)$ satisfies (1) $k_j \le a_j - 1$ for all j, and (2) $\mathbf{k}$ is not of the form $(a_1-1, 0, a_3-1, 0, \ldots, a_{2l-1}-1, i, *, \ldots, *)$ with $i \ge 1$.

A first step towards the LG-LG mirror symetry Conjecture 1.1 is a ring isomorphism between $(H_W, \bullet)$ and $\mathrm{Jac}(W^T)$. For computation convenience later, we use the following normalized residue defined by the normalized residue pairing η_{W^T} (to be explained in (3.1)):

$$\widetilde{\mathrm{Res}}_{W^T}(\phi_\mu) := \eta_{W^T}(dx_1 \cdots dx_n, \phi_\mu dx_1 \cdots dx_n) = 1. \tag{2.19}$$

The ring isomorphism has been studied in [1, 14, 16, 28, 29] for various examples. According to the Axiom of Sums of singularities [14, Theorem 4.1.8(8)] in FJRW theory, the FJRW ring $(H_W, \bullet)$ is a tensor product of the FJRW ring of each direct summand. Krawitz constructed a ring isomorphism for each atomic type if $q_i < 1/2$ for all i [28]. For our purpose, if W is a polynomial in Table 1, then it is already known that $(H_W, \bullet)$ is isomorphic to $\mathrm{Jac}(W^T)$ except for $W = x^2 + xy^q + yz^r$, $(q, r) = (3, 3), (2, 4)$. We will give new constructions for the two exceptional cases, and will also briefly introduce the earlier constructions for the other 12 cases.

Since E_W is invertible, we can write E_W^{-1} using column vectors ρ_k,

$$E_W^{-1} = (\rho_1 | \cdots | \rho_n), \qquad \rho_k := (\varphi_1^{(k)}, \ldots, \varphi_n^{(k)})^T, \qquad \varphi_i^{(k)} \in \mathbb{Q}.$$

We can view ρ_k as an element in G_W by defining the action

$$\rho_k = \left(\exp(2\pi\sqrt{-1}\,\varphi_1^{(k)}), \ldots, \exp(2\pi\sqrt{-1}\,\varphi_n^{(k)})\right) \in G_W.$$

Thus $\rho_i J \in G_W$, with J the exponential grading element in (2.2).

Proposition 2.3 ([28]). *For any n-variable invertible polynomial W with each degree $q_i < 1/2$, there is a degree-preserving ring isomorphism $\Psi : \mathrm{Jac}(W^T) \to (H_W, \bullet)$. In particular, if $\rho_i J$ is narrow for $i = 1, \ldots, n$, then Ψ is generated by*

$$\Psi(x_i) = \mathbf{1}_{\rho_i J}, \qquad i = 1, \ldots, n, \tag{2.20}$$

Example 2.4. Let $W = x^p + y^q$, $p, q > 2$. Denote

$$\gamma_{i,j} = \left(\exp(2\pi\sqrt{-1}\, i/p), \exp(2\pi\sqrt{-1}\, j/q)\right).$$

The FJRW ring $(H_W, \bullet)$ is generated by $\{\mathbf{1}_{\gamma_{2,1}}, \mathbf{1}_{\gamma_{1,2}}\}$. Then $W^T = W$ and the ring isomorphism $\Psi : \mathrm{Jac}(W^T) \xrightarrow{\cong} (H_W, \bullet)$ generated by (2.20) extends as

$$\Psi(x^{i-1}y^{j-1}) = \mathbf{1}_{\gamma_{i,j}}, \qquad 1 \le i < p, 1 \le j < q. \tag{2.21}$$

For 2-Loop singularities, $\rho_i J$ may not be narrow for some $i \in \{1, 2\}$. However, ring isomorphisms still exist. According to [1, 28], we have

Example 2.5. For $W = x^2y + xy^3 + z^3 \in Q_{12}$, we have $G_W \cong \mu_{15}$. A ring isomorphism $\Psi : \mathrm{Jac}(W^T) \xrightarrow{\cong} (H_W, \bullet)$ is obtained by extending (2.20) from

$$\Psi(x) = x\mathbf{1}_{J^{10}}. \tag{2.22}$$

The corresponding vector space isomorphism $\Psi : \mathrm{Jac}(W^T) \to H_W$ is as follows:

H_W	$\mathbf{1}_J$	$\mathbf{1}_{J^{13}}$	$\mathbf{1}_{J^{11}}$	$x\mathbf{1}_{J^{10}}$	$y^2\mathbf{1}_{J^{10}}$	$\mathbf{1}_{J^8}$	$\mathbf{1}_{J^7}$	$x\mathbf{1}_{J^5}$	$y^2\mathbf{1}_{J^5}$	$\mathbf{1}_{J^4}$	$\mathbf{1}_{J^2}$	$\mathbf{1}_{J^{14}}$
$\mathrm{Jac}(W^T)$	1	y	z	x	y^2	yz	xy	xz	y^2z	xy^2	xyz	xy^2z

Now we discuss if there exists $q_i = 1/2$ for W. Without loss of generality, we assume W is of the atomic type: $W = x_1^2 + x_1 x_2^{a_2} + \cdots + x_{m-1} x_m^{a_m}$. Then $\mathrm{Fix}(\rho_1 J) = \{(x_1, \ldots, x_m) \in \mathbb{C}^k \mid x_i = 0, i > 2\}$. Thus $H_{\rho_1 J}$ is generated by a broad element $x_2^{a_2-1}\mathbf{1}_{\rho_1 J}$, which is a ring generator of H_W. If $m = 2$, it is known [16] that Ψ : $\mathrm{Jac}(W^T) \to (H_W, \bullet)$ generates a ring isomorphism by $\Psi(x_1) = a_2 x_2^{a_2-1}\mathbf{1}_{\rho_1 J}$ and $\Psi(x_2) = \mathbf{1}_{\rho_2 J}$. The key point is that the residue formula in (2.5) implies

$$\langle x_2^{a_2-1}\mathbf{1}_{\rho_1 J}, x_2^{a_2-1}\mathbf{1}_{\rho_1 J}, \mathbf{1}_{\rho_2^{1-a_2} J^{-1}}\rangle_0^W = \langle x_2^{a_2-1}\mathbf{1}_{\rho_1 J}, x_2^{a_2-1}\mathbf{1}_{\rho_1 J}, \mathbf{1}_J\rangle_0^W = -1/a_2.$$

Inspired by this, for $m \geq 3$, we consider

$$K := \langle x_2^{a_2-1}\mathbf{1}_{\rho_1 J}, x_2^{a_2-1}\mathbf{1}_{\rho_1 J}, \mathbf{1}_{\rho_2^{1-a_2}\rho_3^{-1} J^{-1}}\rangle_0^W.$$

If $K \neq 0$, then it is possible to define

$$\Psi(x_1) = \sqrt{-a_2/K}\, x_2^{a_2-1}\mathbf{1}_{\rho_1 J}. \tag{2.23}$$

In Section 2.3, using Getzler's relation, we will prove the following nonvanishing lemma:

Lemma 2.6. *Let* $W = x^2 + xy^q + yz^r$ *with* $(q, r) = (3, 3), (2, 4)$. *Then*

$$K_{q,r} := \langle y^{q-1}\mathbf{1}_{\rho_1 J}, y^{q-1}\mathbf{1}_{\rho_1 J}, \mathbf{1}_{\rho_2^{1-q}\rho_3^{-1} J^{-1}}\rangle_0^W \neq 0. \tag{2.24}$$

As a direct consequence of Lemma 2.6, it is not hard to check the following.

Proposition 2.7. *Let* W^T *be one of the exceptional unimodular singularities in Table 1. Then the map* Ψ *in* (2.20) *and* (2.23) *generates a degree-preserving ring isomorphism*

$$\Psi : \mathrm{Jac}(W^T) \cong (H_W, \bullet).$$

Proof. We only need to consider $W = x^2 + xy^q + yz^r$ with $(q, r) = (3, 3), (2, 4)$. We will check that Ψ gives a vector space isomorphism which preserves the degree and the pairing on both sides. We will also check that the generators in H_W satisfy exactly the algebra relations in $\mathrm{Jac}(W^T)$, by computing all the genus-0 3-point correlators. We remark that we use the normalized residue in $\mathrm{Jac}(W^T)$, i.e.,

$$\widetilde{\mathrm{Res}}_{W^T}(y^{q-1}z^{r-1}) = 1.$$

Lemma 2.6 allows us to extend Ψ by defining $\Psi(x)$ as in (2.23). Then we can check directly that

$$\Psi(x) \bullet \Psi(x) = \left(-\frac{q}{K_{q,r}}\langle y^{q-1}\mathbf{1}_{\rho_1 J}, y^{q-1}\mathbf{1}_{\rho_1 J}, \mathbf{1}_{\rho_2^{1-q}\rho_3^{-1} J^{-1}}\rangle_0^W\right)\mathbf{1}_{\rho_2^{q-1}\rho_3 J} = -q\Psi(y^{q-1}z).$$

This coincides with $x^2 + qy^{q-1}z = 0$ in $\mathrm{Jac}(W^T)$. We notice that the product $\Psi(x) \bullet \Psi(z)$ can be computed via

$$\langle \Psi(x), \Psi(x), \Psi(z^{r-2})\rangle_{0,3}^W = \langle \Psi(x) \bullet \Psi(x), \Psi(z^{r-2})\rangle = -q.$$

For $r = 4$, we use the WDVV equation once to get $\Psi(x) \bullet \Psi(z)$. The preimages of the broad sectors are of the form cxz^j, $j = 1, \ldots, r-2$, where the constant c is fixed by the constant in (2.24) and the normalized residue pairing.

We have $\Psi(x) \bullet \Psi(y) = 0$ by simply checking the formula (2.11). This coincides with $xy = 0$ in $\mathrm{Jac}(W^T)$.

The rest of the proof is the same as in [28, Lemmas 4.5–4.7]. For the reader's convenience, we sketch a proof for $W = x^2 + xy^3 + yz^3$. The other case can be treated similarly. By (2.20), we get

$$\Psi(y) = \mathbf{1}_{J^{15}}, \qquad \Psi(z) = \mathbf{1}_{J^{13}}.$$

According to (2.11), the nonzero $\langle \cdots \rangle^W_{0,3}$ with narrow insertions only is one of the following:

$$\langle \mathbf{1}_J, \mathbf{1}_{J^j}, \mathbf{1}_{J^{18-j}} \rangle^W_{0,3}, \qquad j \text{ odd}, \tag{2.25}$$

or

$$\langle \mathbf{1}_{J^{15}}, \mathbf{1}_{J^{15}}, \mathbf{1}_{J^7} \rangle^W_{0,3}, \quad \langle \mathbf{1}_{J^{15}}, \mathbf{1}_{J^{13}}, \mathbf{1}_{J^9} \rangle^W_{0,3}, \quad \langle \mathbf{1}_{J^{13}}, \mathbf{1}_{J^{13}}, \mathbf{1}_{J^{11}} \rangle^W_{0,3}, \quad \langle \mathbf{1}_{J^{15}}, \mathbf{1}_{J^{11}}, \mathbf{1}_{J^{11}} \rangle^W_{0,3}. \tag{2.26}$$

All the correlators listed above are concave. Furthermore, we apply (2.7) to get the line bundle degrees. Except for the last correlator in (2.26), we have

$$\deg(\rho_* \mathscr{L}_i) = -1, \qquad i = 1, 2, 3.$$

This implies all the bundles $R^1\pi_*(\bigoplus_{i=1}^n \mathscr{L}_i)$ have rank zero. By (2.16) for $D = 0$, the values of those correlators all equal 1. We use those correlators to get, for example,

$$\Psi(y) \bullet \Psi(y) = \langle \mathbf{1}_{J^{15}}, \mathbf{1}_{J^{15}}, \mathbf{1}_{J^7} \rangle^W_{0,3}\, \eta^{\mathbf{1}_{J^7}, \mathbf{1}_{J^{11}}}\, \mathbf{1}_{J^{11}} = \mathbf{1}_{J^{11}}.$$

Here $\eta^{i,j}$ is defined in (2.13). Similarly, we obtain

$$\Psi(yz) = \mathbf{1}_{J^9}, \quad \Psi(z^2) = \mathbf{1}_{J^7}, \quad \Psi(y^2 z) = \mathbf{1}_{J^5}, \quad \Psi(yz^2) = \mathbf{1}_{J^3}, \quad \Psi(y^2 z^2) = \mathbf{1}_{J^{17}}.$$

The correlators in (2.25) match the normalized residue pairing. For the last correlator $\langle \mathbf{1}_{J^{15}}, \mathbf{1}_{J^{15}}, \mathbf{1}_{J^{11}} \rangle^W_{0,3}$, we have

$$\deg(\rho_* \mathscr{L}_1) = -1, \qquad \deg(\rho_* \mathscr{L}_2) = -2, \qquad \deg(\rho_* \mathscr{L}_3) = 0.$$

Thus for each fiber (isomorphic to $\mathbb{CP}^1$) of the universal curve $\mathscr{C}$ over $\overline{\mathscr{W}}_{0,3}(J^{15}, J^{15}, J^{11})$, we have

$$H^0\Big(\mathbb{CP}^1, \bigoplus \mathscr{L}_i\Big) = 0 \oplus 0 \oplus \mathbb{C}, \qquad H^1\Big(\mathbb{CP}^1, \bigoplus \mathscr{L}_i\Big) = 0 \oplus \mathbb{C} \oplus 0.$$

According to the Index Zero axiom in [14, Theorem 4.1.8], this corrlelator equals the degree of the so-called Witten map from H^0 to H^1, which sends (x, y, z) to $\big(\overline{\frac{\partial W}{\partial x}}, \overline{\frac{\partial W}{\partial y}}, \overline{\frac{\partial W}{\partial z}}\big)$. In this case,

$$\langle \mathbf{1}_{J^{15}}, \mathbf{1}_{J^{11}}, \mathbf{1}_{J^{11}} \rangle^W_{0,3} = -3.$$

From this, we check that

$$\Psi(y) \bullet \Psi(y^2) = -3\Psi(z^2).$$

This coincides with the last relation in $\mathrm{Jac}(W^T)$, i.e., $y^3 + 3z^2 = 0$.

Finally, we list the table for each vector space isomorphism.

If $W = x^2 + xy^3 + yz^3$, the vector space isomorphism $\Psi : \mathrm{Jac}(W^T) \xrightarrow{\cong} H_W$ is

H_W	$\mathbf{1}_J$	$\mathbf{1}_{J^{15}}$	$\mathbf{1}_{J^{13}}$	$\sqrt{-3/K_{3,3}}\, y^2 \mathbf{1}_{J^{12}}$	$\mathbf{1}_{J^{11}}$	$\mathbf{1}_{J^9}$	$\mathbf{1}_{J^7}$	$\sqrt{-3^3 K_{3,3}}\, y^2 \mathbf{1}_{J^6}$	$\mathbf{1}_{J^5}$	$\mathbf{1}_{J^3}$	$\mathbf{1}_{J^{17}}$
$\mathrm{Jac}(W^T)$	1	y	z	x	y^2	yz	z^2	xz	y^2z	yz^2	y^2z^2

If $W = x^2 + xy^2 + yz^4$, then the vector space isomorphism is given by

H_W	$\mathbf{1}_J$	$\mathbf{1}_{J^{13}}$	$\sqrt{-2/K_{2,4}}\, y \mathbf{1}_{J^{12}}$	$\mathbf{1}_{J^{11}}$	$\mathbf{1}_{J^9}$	$y\mathbf{1}_{J^8}$	$\mathbf{1}_{J^7}$	$\mathbf{1}_{J^5}$	$\sqrt{-2^3 K_{2,4}}\, y \mathbf{1}_{J^4}$	$\mathbf{1}_{J^3}$	$\mathbf{1}_{J^{15}}$
$\mathrm{Jac}(W^T)$	1	z	x	y	z^2	xz	yz	z^3	xz^2	yz^2	yz^3

□

We will give explicit formulas for the isomorphism Ψ in all other cases in the appendix. Those isomorphisms Ψ turn out to identify the ancestor total potential of the FJRW theory of (W, G_W) with that of the Saito–Givental theory of W^T up to rescaling.

Next we compute the FJRW invariants in Proposition 1.2. We introduce a new notation

$$\mathbf{1}_\phi := \Psi(\phi), \qquad \phi \in \mathrm{Jac}(W^T). \tag{2.27}$$

Due to the above conventions, the second part of Proposition 1.2 is simplified as follows.

Proposition 2.8. *Let M_i^T be the i-th monomial of W^T with the ordering in Table 1. Then*

$$\langle \mathbf{1}_{x_i}, \mathbf{1}_{x_i}, \mathbf{1}_{M_i^T/x_i^2}, \mathbf{1}_{\phi_\mu} \rangle_0^W = q_i, \qquad \forall i = 1, \ldots, n. \tag{2.28}$$

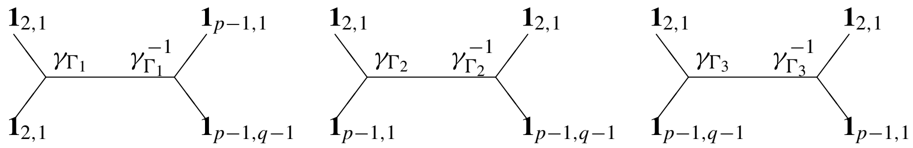
Proof. We classify the correlators in (2.28) into concave and nonconcave ones. For the concave correlators, we use (2.17). For the nonconcave correlators, we use WDVV to reconstruct them from concave correlators and again use (2.17). We will freely interchange the notation

$$(x_1, x_2, x_3) = (x, y, z). \tag{2.29}$$

Let us start with concave correlators. As an example, we compute $\langle \mathbf{1}_x, \mathbf{1}_x, \mathbf{1}_{x^{p-2}}, \mathbf{1}_{\phi_\mu} \rangle_0^W$ for $W = x^p + y^q$. The computation of all other concave corrlators in (2.28) is similar. For $W = x^p + y^q$, we recall that for $\gamma_{i,j} \in G_W \cong \mu_p \times \mu_q$, we have $\Theta_1^{\gamma_{i,j}} = i/p$ and $\Theta_2^{\gamma_{i,j}} = j/q$. All the sectors are narrow and $\mathbf{1}_{\gamma_{i,j}} = \mathbf{1}_{x^{i-1}y^{j-1}}$ with our notation conventions. According to the line bundle criterion (2.7), we know that for $\langle \mathbf{1}_x, \mathbf{1}_x, \mathbf{1}_{x^{p-2}}, \mathbf{1}_{\phi_\mu} \rangle_0^W$,

$$\deg(\rho_* \mathscr{L}_1) = -2, \qquad \deg(\rho_* \mathscr{L}_2) = -1.$$

Thus $\pi_* \mathscr{L}_1 = \pi_* \mathscr{L}_2 = 0$ and the correlator is concave. Moreover, $R^1\pi_*\mathscr{L}_2 = 0$ and the nonzero contribution of the virtual cycle only comes from $R^1\pi_*\mathscr{L}_1$. Now we can apply (2.17). There are three decorated dual graphs in Γ_{cut}, where we simply denote $\mathbf{1}_{i,j} := \mathbf{1}_{\gamma_{i,j}}$:



The decorations of the boundary classes are $\Theta_1^{\gamma_{\Gamma_i}} = (p-3)/p, 0, 0$ for $i = 1, 2, 3$. We obtain

$$\begin{aligned}\langle \mathbf{1}_{2,1}, \mathbf{1}_{2,1}, \mathbf{1}_{p-1,1}, \mathbf{1}_{p-1,q-1}\rangle_0^W &= \int_{\overline{\mathcal{M}}_{0,4}} \Lambda_{0,4}^W(\mathbf{1}_{2,1}, \mathbf{1}_{2,1}, \mathbf{1}_{p-1,1}, \mathbf{1}_{p-1,q-1}) \\ &= \frac{1}{2}\left(B_2\left(\frac{1}{p}\right) - 2B_2\left(\frac{2}{p}\right) - 2B_2\left(\frac{p-1}{p}\right) + 2B_2(0) + B_2\left(\frac{p-3}{p}\right)\right) = \frac{1}{p}.\end{aligned}$$

All the nonconcave correlators in (2.28) are listed as follows:

- $\langle \mathbf{1}_y, \mathbf{1}_y, \mathbf{1}_z, \mathbf{1}_{\phi_\mu}\rangle_0^W$ for the 3-Chain $W = x^2 + xy^2 + yz^4$.
- $\langle \mathbf{1}_x, \mathbf{1}_x, \mathbf{1}_y, \mathbf{1}_{\phi_\mu}\rangle_0^W$ for the 3-Chain $W = x^2 + xy^q + yz^r$, $(q, r) = (3, 3)$ or $(2, 4)$.
- $\langle \mathbf{1}_x, \mathbf{1}_x, \mathbf{1}_y, \mathbf{1}_{\phi_\mu}\rangle_0^W$ and $\langle \mathbf{1}_y, \mathbf{1}_y, \mathbf{1}_z, \mathbf{1}_{\phi_\mu}\rangle_0^W$ for the 3-Loop $W = x^2z + xy^2 + yz^3$.
- $\langle \mathbf{1}_x, \mathbf{1}_x, \mathbf{1}_y, \mathbf{1}_{\phi_\mu}\rangle_0^W$ for $W = x^2 + xy^4 + z^3$.
- $\langle \mathbf{1}_x, \mathbf{1}_x, \mathbf{1}_y, \mathbf{1}_{\phi_\mu}\rangle_0^W$ for $W = x^2y + xy^3 + z^3$.
- $\langle \mathbf{1}_x, \mathbf{1}_y, \mathbf{1}_y, \mathbf{1}_{\phi_\mu}\rangle_0^W$ for $W = x^2y + y^2 + z^4$.

For the nonconcave correlators, we will use the WDVV equations and the ring relations to reconstruct them from the concave correlators. Let us start with the value of $\langle \mathbf{1}_y, \mathbf{1}_y, \mathbf{1}_{y^{q-2}z}, \mathbf{1}_{\phi_\mu}\rangle_0^W$ in the 3-Chain $W = x^2 + xy^2 + yz^4$. Since $\phi_\mu = yz^3 \in \mathrm{Jac}(W^T)$ and $\mathbf{1}_y \bullet \mathbf{1}_{yz} = 0$, we get

$$\begin{aligned}\langle \mathbf{1}_z, \mathbf{1}_y, \mathbf{1}_{yz} \bullet \mathbf{1}_{z^2}, \mathbf{1}_y\rangle_0^W &= \langle \mathbf{1}_z, \mathbf{1}_y, \mathbf{1}_{yz}, \mathbf{1}_{z^2} \bullet \mathbf{1}_y\rangle_0^W - \langle \mathbf{1}_z, \mathbf{1}_y \bullet \mathbf{1}_y, \mathbf{1}_{yz}, \mathbf{1}_{z^2}\rangle_0^W \\ &= 0 - (-4)\tfrac{1}{16} = \tfrac{1}{4}.\end{aligned}$$

The first equality follows from the WDVV equation (2.14). We also use $\mathbf{1}_y \bullet \mathbf{1}_{yz} = 0$. Both $\langle \mathbf{1}_z, \mathbf{1}_y, \mathbf{1}_{yz}, \mathbf{1}_{z^2} \bullet \mathbf{1}_y\rangle_0^W$ and $\langle \mathbf{1}_z, \mathbf{1}_y \bullet \mathbf{1}_y, \mathbf{1}_{yz}, \mathbf{1}_{z^2}\rangle_0^W$ are concave correlators and can be computed from (2.17). For other nonconcave correlators, we will list the WDVV equations. The concavity computation is checked easily. For the 3-Chain $W = x^2 + xy^q + yz^r$, $(q, r) = (3, 3)$ or $(2, 4)$, $\mathbf{1}_{\phi_\mu} = \mathbf{1}_{y^{q-1}z^{r-1}}$, and we have

$$\begin{aligned}\langle \mathbf{1}_y, \mathbf{1}_x, \mathbf{1}_{\phi_\mu}, \mathbf{1}_x\rangle_0^W &= -\langle \mathbf{1}_y, \mathbf{1}_x \bullet \mathbf{1}_x, \mathbf{1}_y, \mathbf{1}_{y^{q-2}z^{r-1}}\rangle_0^W \\ &= q\langle \mathbf{1}_y, \mathbf{1}_{y^{q-1}z}, \mathbf{1}_y, \mathbf{1}_{y^{q-2}z^{r-1}}\rangle_0^W = \tfrac{1}{2}.\end{aligned}$$

For the 3-Loop $W = x^2z + xy^2 + yz^3$, $\mathbf{1}_{\phi_\mu} = \mathbf{1}_{xyz^2}$, and we get

$$\begin{aligned}\langle \mathbf{1}_y, \mathbf{1}_x, \mathbf{1}_{xy} \bullet \mathbf{1}_{z^2}, \mathbf{1}_x\rangle_0^W &= \langle \mathbf{1}_y, \mathbf{1}_x, \mathbf{1}_{xy}, \mathbf{1}_{z^2} \bullet \mathbf{1}_x\rangle_0^W - \langle \mathbf{1}_y, \mathbf{1}_x \bullet \mathbf{1}_x, \mathbf{1}_{xy}, \mathbf{1}_{z^2}\rangle_0^W \\ &= \tfrac{1}{13} - (-2)\tfrac{2}{13} = \tfrac{5}{13}. \\ \langle \mathbf{1}_z, \mathbf{1}_y, \mathbf{1}_z \bullet \mathbf{1}_{xyz}, \mathbf{1}_y\rangle_0^W &= \langle \mathbf{1}_z, \mathbf{1}_y \bullet \mathbf{1}_z, \mathbf{1}_{xyz}, \mathbf{1}_y\rangle_0^W - \langle \mathbf{1}_z, \mathbf{1}_y \bullet \mathbf{1}_y, \mathbf{1}_z, \mathbf{1}_{xyz}\rangle_0^W \\ &= \tfrac{1}{13} - (-3)\tfrac{1}{13} = \tfrac{4}{13}.\end{aligned}$$

For $W = x^2 + xy^4 + z^3$, $\mathbf{1}_x$ is broad. However,

$$\langle \mathbf{1}_y, \mathbf{1}_x, \mathbf{1}_{\phi_\mu}, \mathbf{1}_x\rangle_0^W = -\langle \mathbf{1}_y, \mathbf{1}_x \bullet \mathbf{1}_x, \mathbf{1}_y, \mathbf{1}_{y^2z}\rangle_0^W = 4\langle \mathbf{1}_y, \mathbf{1}_{y^3}, \mathbf{1}_y, \mathbf{1}_{y^2z}\rangle_0^W = \tfrac{1}{2}.$$

For $W = x^2y + xy^3 + z^3$, we get

$$\begin{aligned}
\langle \mathbf{1}_y, \mathbf{1}_x, \mathbf{1}_{xy} \bullet \mathbf{1}_{yz}, \mathbf{1}_x\rangle_0^W + \langle \mathbf{1}_y, \mathbf{1}_x \bullet \mathbf{1}_x, \mathbf{1}_{xy}, \mathbf{1}_{yz}\rangle_0^W &= \langle \mathbf{1}_y, \mathbf{1}_x, \mathbf{1}_{xy}, \mathbf{1}_{yz} \bullet \mathbf{1}_x\rangle_0^W \\
&= -\tfrac{1}{2}\langle \mathbf{1}_y, \mathbf{1}_x, \mathbf{1}_y \bullet \mathbf{1}_{y^2z}, \mathbf{1}_{xy}\rangle_0^W = -\tfrac{1}{2}\langle \mathbf{1}_y, \mathbf{1}_{xy}, \mathbf{1}_{xy}, \mathbf{1}_{y^2z}\rangle_0^W \\
&= -\tfrac{1}{2}\langle \mathbf{1}_y, \mathbf{1}_{xy}, \mathbf{1}_y \bullet \mathbf{1}_{yz}, \mathbf{1}_{xy}\rangle_0^W = -\langle \mathbf{1}_y, \mathbf{1}_{xy}, \mathbf{1}_{yz}, \mathbf{1}_{xy^2}\rangle_0^W.
\end{aligned}$$

The first, third and last equalities are WDVV equations. Finally, we get

$$\begin{aligned}
\langle \mathbf{1}_x, \mathbf{1}_x, \mathbf{1}_y, \mathbf{1}_{xy^2z}\rangle_0^W &= -\langle \mathbf{1}_y, \mathbf{1}_x \bullet \mathbf{1}_x, \mathbf{1}_{xy}, \mathbf{1}_{yz}\rangle_0^W - \langle \mathbf{1}_y, \mathbf{1}_{xy}, \mathbf{1}_{yz}, \mathbf{1}_{xy^2}\rangle_0^W \\
&= -\left(-\tfrac{1}{5}\right) - \left(-\tfrac{1}{5}\right) = \tfrac{2}{5}.
\end{aligned}$$

For $W = x^2y + y^2 + z^4$, we get

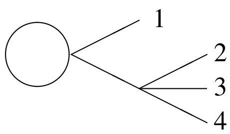
$$\begin{aligned}
&\langle \mathbf{1}_x, \mathbf{1}_y, \mathbf{1}_{xy} \bullet \mathbf{1}_{z^2}, \mathbf{1}_y\rangle_0^W = \langle \mathbf{1}_x, \mathbf{1}_y, \mathbf{1}_{xy}, \mathbf{1}_{z^2} \bullet \mathbf{1}_y\rangle_0^W - \langle \mathbf{1}_x, \mathbf{1}_y \bullet \mathbf{1}_y, \mathbf{1}_{xy}, \mathbf{1}_{z^2}\rangle_0^W \\
&\quad = \left(\langle \mathbf{1}_y, \mathbf{1}_x, \mathbf{1}_x, \mathbf{1}_y \bullet \mathbf{1}_{yz^2}\rangle_0^W - \langle \mathbf{1}_y, \mathbf{1}_x \bullet \mathbf{1}_{yz^2}, \mathbf{1}_x, \mathbf{1}_y\rangle_0^W\right) - \langle \mathbf{1}_x, \mathbf{1}_y \bullet \mathbf{1}_y, \mathbf{1}_{xy}, \mathbf{1}_{z^2}\rangle_0^W.
\end{aligned}$$

Combining this equation and $y^2 = -2x$, we get

$$\begin{aligned}
\langle \mathbf{1}_x, \mathbf{1}_y, \mathbf{1}_{xyz^2}, \mathbf{1}_y\rangle_0^W &= -\langle \mathbf{1}_y, \mathbf{1}_x, \mathbf{1}_x, \mathbf{1}_{xz^2}\rangle_0^W + \langle \mathbf{1}_x, \mathbf{1}_x, \mathbf{1}_{xy}, \mathbf{1}_{z^2}\rangle_0^W \\
&= -\left(-\tfrac{1}{8}\right) + \tfrac{1}{4} = \tfrac{3}{8}.
\end{aligned}$$ □

2.3. Nonvanishing invariants

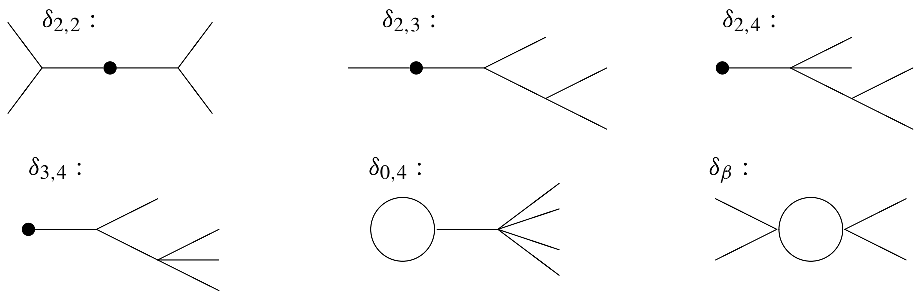
In this subsection, we will prove Lemma 2.6. Our tool is the Getzler relation [17], which is a linear relation between codimension two cycles in $H_*(\overline{\mathcal{M}}_{1,4}, \mathbb{Q})$. Let us briefly introduce this relation here. Consider the dual graph,

$\Delta_0 \cdot \Delta_{\{234\}} :=$ 

This graph represents a codimension-2 stratum in $\overline{\mathcal{M}}_{1,4}$: A vertex represents a genus-0 component. An edge connecting two vertices (including a circle connecting the same vertex) represents a node, and a tail (or half-edge) represents a marked point on the component of the corresponding vertex. Let $\Delta_{0,3}$ be the S_4-invariant of the codimension-2 stratum in $\overline{\mathcal{M}}_{1,4}$,

$$\Delta_{0,3} = \Delta_0 \cdot \Delta_{\{123\}} + \Delta_0 \cdot \Delta_{\{124\}} + \Delta_0 \cdot \Delta_{\{134\}} + \Delta_0 \cdot \Delta_{\{234\}}.$$

We denote by $\delta_{0,3} = [\Delta_{0,3}]$ the corresponding cycle in $H_4(\overline{\mathcal{M}}_{1,4}, \mathbb{Q})$. We list the corresponding unordered dual graphs for other strata below. A filled circle (as a vertex) represents a genus-1 component. See [17] for more details.



In [17], Getzler found the following identity:

$$12\delta_{2,2}+4\delta_{2,3}-2\delta_{2,4}+6\delta_{3,4}+\delta_{0,3}+\delta_{0,4}-2\delta_{\beta}=0\in H_4(\overline{\mathcal{M}}_{1,4},\mathbb{Q}). \tag{2.30}$$

Proof of Lemma 2.6. We start with $W=x^2+xy^2+yz^4$. We normalize

$$u=y\mathbf{1}_{J^{12}},\qquad v=\sqrt{-2}y\mathbf{1}_{J^8},\qquad w=-2y\mathbf{1}_{J^4}.$$

The nonvanishing pairings between these broad elements are $\langle u,w\rangle=1$ and $\langle v,v\rangle=1$.

We integrate $\Lambda^W_{1,4}(\mathbf{1}_{J^9},\mathbf{1}_{J^9},\mathbf{1}_{J^9},\mathbf{1}_{J^9})$ over the Getzler relation (2.30). The Composition law [14, Theorem 4.1.8(6)] in FJRW theory implies

$$\begin{aligned}\int_{\delta_{0,3}}\Lambda^W_{1,4}(\mathbf{1}_{J^9},\mathbf{1}_{J^9},\mathbf{1}_{J^9},\mathbf{1}_{J^9})&=4\langle\mathbf{1}_{J^9},\mathbf{1}_{J^9},\mathbf{1}_{J^9},\mathbf{1}_{J^7}\rangle^W_0\Big(\sum_{\alpha,\beta}\eta^{\alpha,\beta}\langle\mathbf{1}_{J^9},\mathbf{1}_{J^9},\alpha,\beta\rangle^W_0\Big)\\&=4\langle\mathbf{1}_{J^9},\mathbf{1}_{J^9},\mathbf{1}_{J^9},\mathbf{1}_{J^7}\rangle^W_0\big(2\langle\mathbf{1}_{J^9},\mathbf{1}_{J^9},\mathbf{1}_{J^{13}},\mathbf{1}_{J^3}\rangle^W_0+2\langle\mathbf{1}_{J^9},\mathbf{1}_{J^9},\mathbf{1}_{J^{11}},\mathbf{1}_{J^5}\rangle^W_0\\&\qquad+2\langle\mathbf{1}_{J^9},\mathbf{1}_{J^9},\mathbf{1}_{J^9},\mathbf{1}_{J^7}\rangle^W_0+2\langle\mathbf{1}_{J^9},\mathbf{1}_{J^9},u,w\rangle^W_0+\langle\mathbf{1}_{J^9},\mathbf{1}_{J^9},v,v\rangle^W_0\big).\end{aligned}$$

The factor 4 comes from the fact that there are four strata in $\Delta_{0,3}$ which contribute. We have the factor 2 for $\langle\mathbf{1}_{J^9},\mathbf{1}_{J^9},\mathbf{1}_{J^{13}},\mathbf{1}_{J^3}\rangle^W_0$ since both $\alpha=\mathbf{1}_{J^{13}}$ and $\alpha=\mathbf{1}_{J^3}$ give the same correlator. Finally, $\mathbf{1}_J$ is the identity, and the string equation implies $\langle\mathbf{1}_{J^9},\mathbf{1}_{J^9},\mathbf{1}_{J^{15}},\mathbf{1}_J\rangle^W_0=0$. There are two correlators containing broad sectors; we simply denote

$$C_1:=\langle\mathbf{1}_{J^9},\mathbf{1}_{J^9},v,v\rangle^W_0,\qquad C_2:=\langle\mathbf{1}_{J^9},\mathbf{1}_{J^9},u,w\rangle^W_0.$$

We can calculate the concave correlators using the orbifold-GRR formula (2.17) to get

$$\langle\mathbf{1}_{J^9},\mathbf{1}_{J^9},\mathbf{1}_{J^{13}},\mathbf{1}_{J^3}\rangle^W_0=\tfrac{1}{4},\quad\langle\mathbf{1}_{J^9},\mathbf{1}_{J^9},\mathbf{1}_{J^{11}},\mathbf{1}_{J^5}\rangle^W_0=-\tfrac{1}{8},\quad\langle\mathbf{1}_{J^9},\mathbf{1}_{J^9},\mathbf{1}_{J^9},\mathbf{1}_{J^7}\rangle^W_0=\tfrac{1}{8}.$$

This implies

$$\int_{\delta_{0,3}}\Lambda^W_{1,4}(\mathbf{1}_{J^9},\mathbf{1}_{J^9},\mathbf{1}_{J^9},\mathbf{1}_{J^9})=C_2+\tfrac{1}{2}C_1+\tfrac{1}{4}.$$

Similarly, we get

$$\int_{\delta_\beta}\Lambda^W_{1,4}(\mathbf{1}_{J^9},\mathbf{1}_{J^9},\mathbf{1}_{J^9},\mathbf{1}_{J^9})=6C_2^2+3C_1^2+\tfrac{9}{16},\qquad\int_{\delta_{0,4}}\Lambda^W_{1,4}(\mathbf{1}_{J^9},\mathbf{1}_{J^9},\mathbf{1}_{J^9},\mathbf{1}_{J^9})=\tfrac{165}{128}.$$

The last equality requires the computation for a genus-0 correlator with five marked points. It is reconstructed from some known 4-point correlators by the WDVV equations. On the other hand, using the homological degree (2.8), we deduce the vanishing of the integral of $\Lambda^W_{1,4}(\mathbf{1}_{J^9}, \mathbf{1}_{J^9}, \mathbf{1}_{J^9}, \mathbf{1}_{J^9})$ over those strata which contain the genus-1 component. Thus

$$\int_{12\delta_{2,2}+4\delta_{2,3}-2\delta_{2,4}+6\delta_{3,4}} \Lambda^W_{1,4}(\mathbf{1}_{J^9}, \mathbf{1}_{J^9}, \mathbf{1}_{J^9}, \mathbf{1}_{J^9}) = 0.$$

Now apply the Getzler relation (2.30) to get

$$-12C_2^2 + C_2 - 6C_1^2 + \tfrac{1}{2}C_1 + \tfrac{53}{128} = 0. \tag{2.31}$$

On the other hand, since $\mathbf{1}_{J^9} = \mathbf{1}_{J^{13}} \bullet \mathbf{1}_{J^{13}}$, we apply WDVV equations to get

$$\begin{cases} \langle u, u, \mathbf{1}_{J^9}\rangle^W_0 = (\langle \mathbf{1}_{J^{13}}, u, v\rangle^W_0)^2, \\ \langle \mathbf{1}_{J^9}, \mathbf{1}_{J^9}, v, v\rangle^W_0 + \langle \mathbf{1}_{J^{13}}, \mathbf{1}_{J^{13}}, \mathbf{1}_{J^9}, \mathbf{1}_{J^{15}}\rangle^W_0 = 2\langle \mathbf{1}_{J^9}, \mathbf{1}_{J^{13}}, v, w\rangle^W_0 \langle \mathbf{1}_{J^{13}}, u, v\rangle^W_0, \\ \langle \mathbf{1}_{J^9}, \mathbf{1}_{J^9}, u, w\rangle^W_0 + \langle \mathbf{1}_{J^{13}}, \mathbf{1}_{J^{13}}, \mathbf{1}_{J^9}, \mathbf{1}_{J^{15}}\rangle^W_0 = \langle \mathbf{1}_{J^9}, \mathbf{1}_{J^{13}}, v, w\rangle^W_0 \langle \mathbf{1}_{J^{13}}, u, v\rangle^W_0. \end{cases}$$

If $\langle u, u, \mathbf{1}_{J^9}\rangle^W_0 = 0$, then $\langle \mathbf{1}_{J^{13}}, u, v\rangle^W_0 = 0$, and the other two equations above imply

$$C_1 = C_2 = -\langle \mathbf{1}_{J^{13}}, \mathbf{1}_{J^{13}}, \mathbf{1}_{J^9}, \mathbf{1}_{J^{15}}\rangle^W_0 = -\tfrac{3}{16},$$

where the last equality follows from (2.17). However, this contradicts (2.31).

Next we consider $W = x^2 + xy^3 + yz^3$. We denote

$$\begin{cases} u := y^2\mathbf{1}_{J^{12}}, \quad w := -3y^2\mathbf{1}_{J^6}, \\ C_1 := \langle \mathbf{1}_{J^{13}}, \mathbf{1}_{J^{13}}, w, w\rangle^W_0, \quad C_2 := \langle \mathbf{1}_{J^7}, \mathbf{1}_{J^{13}}, u, w\rangle^W_0, \quad C_3 := \langle \mathbf{1}_{J^7}, \mathbf{1}_{J^7}, u, u\rangle^W_0. \end{cases}$$

We integrate $\Lambda^W_{1,4}(\mathbf{1}_{J^{13}}, \mathbf{1}_{J^{13}}, \mathbf{1}_{J^7}, \mathbf{1}_{J^7})$ over the Getzler relation (2.30) to get

$$-8C_2^2 - \tfrac{2}{3}C_2 - 2C_1C_3 + \tfrac{8}{81} = 0. \tag{2.32}$$

On the other hand, since $\mathbf{1}_{J^7} = \mathbf{1}_{J^{13}} \bullet \mathbf{1}_{J^{13}}$, the WDVV equations imply

$$\begin{cases} \langle \mathbf{1}_{J^7}, \mathbf{1}_{J^{13}}, u, w\rangle^W_0 + \langle \mathbf{1}_{J^{13}}, \mathbf{1}_{J^{13}}, \mathbf{1}_{J^{13}}, \mathbf{1}_{J^{17}}\rangle^W_0 = \langle \mathbf{1}_{J^{13}}, \mathbf{1}_{J^{13}}, w, w\rangle^W_0 \langle \mathbf{1}_{J^{13}}, u, u\rangle^W_0, \\ \langle \mathbf{1}_{J^7}, \mathbf{1}_{J^7}, u, u\rangle^W_0 = \langle \mathbf{1}_{J^7}, \mathbf{1}_{J^{13}}, u, w\rangle^W_0 \langle \mathbf{1}_{J^{13}}, u, u\rangle^W_0. \end{cases}$$

Now $\langle \mathbf{1}_{J^{13}}, u, u\rangle^W_0 = 0$ implies $C_2 = -\frac{5}{18}$ and $C_3 = 0$. This contradicts (2.32). □

3. B-model: Saito's theory of primitive form

Throughout this section, we consider the Landau–Ginzburg B-model defined by

$$f : X = \mathbb{C}^n \to \mathbb{C},$$

where f is a weighted homogeneous polynomial with isolated singularity at the origin:

$$f(\lambda^{q_1}x_1, \ldots, \lambda^{q_n}x_n) = \lambda f(x_1, \ldots, x_n).$$

Recall that q_i are called the *weights* of x_i, and the *central charge* of f is defined by

$$\hat{c}_f = \sum_i (1 - 2q_i).$$

Associated to f, the third author [41] has introduced the concept of a primitive form, which, in particular, induces a Frobenius manifold structure (sometimes called a flat structure) on the local universal deformation space of f. This gives rise to the genus zero correlation functions in the Landau–Ginzburg B-model, which are conjectured to be equivalent to the FJRW invariants on the mirror singularities.

The general existence of primitive forms for local isolated singularities is proved by M. Saito [46] via Deligne's mixed Hodge theory. For f being a weighted homogeneous polynomial, the existence problem is greatly simplified due to the semisimplicity of monodromy [41, 46]. However, explicit formulas for primitive forms have only been known for ADE and simple elliptic singularities [41] (i.e., for $\hat{c}_f \leq 1$). This led to the difficulty of computing correlation functions in the Landau–Ginzburg B-model, and has become one of the main obstacles toward proving mirror symmetry between Landau–Ginzburg models.

Based on the recent idea of perturbative approach to primitive forms [32], in this section we will develop a general perturbative method of computing the Frobenius manifolds in the Landau–Ginzburg B-model. This is applied to the 14 exceptional unimodular singularities. With the help of a certain reconstruction type theorem for the WDVV equations (see e.g. Lemma 4.2), it completely solves the computation problem in the Landau–Ginzburg B-model at genus zero.

3.1. Higher residue and good basis

Let $\mathbf{0} \in X = \mathbb{C}^n$ be the origin. Let $\Omega^k_{X,\mathbf{0}}$ be the germs of holomorphic k-forms at $\mathbf{0}$. In this paper we will work with the following space [42]:

$$\mathcal{H}^{(0)}_f := \Omega^n_{X,\mathbf{0}}[[z]]/(df + zd)\Omega^{n-1}_{X,\mathbf{0}},$$

which is a formally completed version of the *Brieskorn lattice* associated to f. Given a differential form $\varphi \in \Omega^n_{X,\mathbf{0}}$, we will use $[\varphi]$ to represent its class in $\mathcal{H}^{(0)}_f$.

There is a natural semi-infinite Hodge filtration on $\mathcal{H}^{(0)}_f$ given by $\mathcal{H}^{(-k)}_f := z^k \mathcal{H}^{(0)}_f$, with graded pieces

$$\mathcal{H}^{(-k)}_f / \mathcal{H}^{(-k-1)}_f \cong \Omega_f, \quad \text{where} \quad \Omega_f := \Omega^n_{X,\mathbf{0}}/df \wedge \Omega^{n-1}_{X,\mathbf{0}}.$$

In particular, $\mathcal{H}^{(0)}_f$ is a free $\mathbb{C}[[z]]$-module of rank $\mu = \dim_{\mathbb{C}} \operatorname{Jac}(f)_0$, the Milnor number of f. We will also denote the extension to Laurent series by

$$\mathcal{H}_f := \mathcal{H}^{(0)}_f \otimes_{\mathbb{C}[[z]]} \mathbb{C}((z)).$$

There is a natural $\mathbb{Q}$-grading on $\mathcal{H}^{(0)}_f$ defined by assigning the degrees

$$\deg x_i = q_i, \quad \deg(dx_i) = q_i, \quad \deg z = 1.$$

Then for a homogeneous element of the form $\varphi = z^k g(x_i) dx_1 \wedge \cdots \wedge dx_n$, we have

$$\deg \varphi = \deg g + k + \sum_i q_i.$$

In [42], the third author constructed a *higher residue pairing*

$$K_f : \mathcal{H}_f^{(0)} \otimes \mathcal{H}_f^{(0)} \to z^n \mathbb{C}[[z]]$$

which satisfies the following properties:

1. K_f is equivariant with respect to the $\mathbb{Q}$-grading, i.e.,

$$\deg(K_f(\alpha, \beta)) = \deg \alpha + \deg \beta$$

for homogeneous elements $\alpha, \beta \in \mathcal{H}_f^{(0)}$.
2. $K_f(\alpha, \beta) = (-1)^n \overline{K_f(\beta, \alpha)}$, where the $^-$ operator takes z to $-z$.
3. $K_f(v(z)\alpha, \beta) = K_f(\alpha, v(-z)\beta) = v(z) K_f(\alpha, \beta)$ for $v(z) \in \mathbb{C}[[z]]$.
4. The leading z-order of K_f defines a pairing

$$\mathcal{H}_f^{(0)} / z\mathcal{H}_f^{(0)} \otimes \mathcal{H}_f^{(0)} / z\mathcal{H}_f^{(0)} \to \mathbb{C}, \qquad \alpha \otimes \beta \mapsto \lim_{z \to 0} z^{-n} K_f(\alpha, \beta),$$

which coincides with the usual residue pairing $\eta_f : \Omega_f \otimes \Omega_f \to \mathbb{C}$.

We remark that the classical residue pairing η_f is intrinsically defined up to a nonzero constant. In the case of weighted homogeneous singularities (for instance for the exceptional unimodular singularities), we will always specify a top degree element ϕ_μ in a weighted homogeneous basis of $\mathrm{Jac}(f)$, and will fix the constant such that

$$\eta_f(dx_1 \cdots dx_n, \phi_\mu dx_1 \cdots dx_n) = 1. \tag{3.1}$$

We will call it the *normalized* residue pairing.

The last property implies that K_f defines a semi-infinite extension of the residue pairing, which explains the name “higher residue”. It is naturally extended to

$$K_f : \mathcal{H}_f \otimes \mathcal{H}_f \to \mathbb{C}((z)),$$

which we denote by the same symbol. This defines a symplectic pairing ω_f on $\mathcal{H}_f$ by

$$\omega_f(\alpha, \beta) := \mathrm{Res}_{z=0}\, z^{-n} K_f(\alpha, \beta) dz,$$

with $\mathcal{H}_f^{(0)}$ being a maximal isotropic subspace. Following [41], we introduce

Definition 3.1. A *good section* σ is defined by a splitting of the quotient $\mathcal{H}_f^{(0)} \to \Omega_f$,

$$\sigma : \Omega_f \to \mathcal{H}_f^{(0)},$$

such that (1) σ preserves the $\mathbb{Q}$-grading; (2) $K_f(\mathrm{Im}(\sigma), \mathrm{Im}(\sigma)) \subset z^n \mathbb{C}$. A basis of $\mathrm{Im}(\sigma)$ will be referred to as a *good basis* of $\mathcal{H}_f^{(0)}$.

Definition 3.2. A *good opposite filtration* $\mathcal{L}$ is defined by a splitting

$$\mathcal{H}_f = \mathcal{H}_f^{(0)} \oplus \mathcal{L}$$

such that (1) $\mathcal{L}$ preserves the $\mathbb{Q}$-grading; (2) $\mathcal{L}$ is an isotropic subspace; (3) $z^{-1} : \mathcal{L} \to \mathcal{L}$.

Remark 3.3. Here for f being weighted homogeneous, (1) is a convenient statement equivalent to the conventional condition that $\nabla^{GM}_{z\partial_z}$ preserves $\mathcal{L}$ (see e.g. [32]).

The above two definitions are equivalent. In fact, a good opposite filtration $\mathcal{L}$ defines the splitting $\sigma : \Omega_f \stackrel{\cong}{\to} \mathcal{H}_f^{(0)} \cap z\mathcal{L}$. Conversely, a good section σ gives rise to the good opposite filtration $\mathcal{L} = z^{-1}\operatorname{Im}(\sigma)[z^{-1}]$. As shown in [41,46], the primitive forms associated to the weighted homogeneous singularities are in one-to-one correspondence with good sections (up to a nonzero scalar). Therefore, we only introduce the notion of good sections, and refer our readers to loc. cit. for the precise notion of primitive forms. We remark that for general isolated singularities, we need the notion of *very good sections* [46,47] in order to incorporate monodromy.

3.2. *The perturbative equation*

We start with a good basis $\{[\phi_\alpha d^n\mathbf{x}]\}_{\alpha=1}^{\mu}$ of $\mathcal{H}_f^{(0)}$, where $d^n\mathbf{x} := dx_1\cdots dx_n$. In this subsection, we will formulate the perturbative method of [32] for computing its associated primitive form, flat coordinates and the potential function. The construction works for general f after the replacement of a good basis by a very good one (see also [47]). We will focus on f being weighted homogeneous since in that case it leads to a very effective computation algorithm. In the following discussion we will then assume $\{\phi_\alpha\}_{\alpha=1}^{\mu}$ to be weighted homogeneous polynomials in $\mathbb{C}[\mathbf{x}]$ that represent a basis of the Jacobi algebra $\operatorname{Jac}(f)$ and $\phi_1 = 1$.

3.2.1. The exponential map. Let F be a local universal unfolding of $f(\mathbf{x})$ around $\mathbf{0} \in \mathbb{C}^\mu$:

$$F : \mathbb{C}^n \times \mathbb{C}^\mu \to \mathbb{C}, \qquad F(\mathbf{x}, \mathbf{s}) := f(\mathbf{x}) + \sum_{\alpha=1}^{\mu} s_\alpha \phi_\alpha(\mathbf{x}), \qquad \mathbf{s} = (s_1, \ldots, s_\mu).$$

The polynomial F becomes weighted homogeneous of total degree 1 after the assignment

$$\deg s_\alpha := 1 - \deg \phi_\alpha.$$

The higher residue pairing is also defined for F as the family version, but we will not use it explicitly in our discussion (although implicitly it is used in an essential way).

Let $B := \operatorname{span}_{\mathbb{C}}\{[\phi_\alpha d^n\mathbf{x}]\} \subset \mathcal{H}_f^{(0)}$ be spanned by the chosen good basis. Then

$$\mathcal{H}_f^{(0)} = B[[z]], \qquad \mathcal{H}_f = B((z)).$$

Let $B_F := \operatorname{span}_{\mathbb{C}}\{\phi_\alpha d^n\mathbf{x}\}$ be another copy of the vector space spanned by the forms $\phi_\alpha d^n\mathbf{x}$. We use a different notation to distinguish it from B, since B_F should be viewed as a subspace of the Brieskorn lattice for the unfolding F. See [32] for more details.

Consider the exponential operator [32]

$$e^{(F-f)/z} : B_F \to B((z))[[\mathbf{s}]]$$

defined as a $\mathbb{C}$-linear map on the basis of B_F as follows. Let $\mathbb{C}[\mathbf{s}]_k := \mathrm{Sym}^k(\mathrm{span}_{\mathbb{C}}\{s_1, \ldots, s_\mu\})$ denote the space of k-homogeneous polynomial in $\mathbf{s}$ (not to be confused with the weighted homogeneous polynomials). As elements in $\mathcal{H}_f \otimes \mathbb{C}[\mathbf{s}]_k$, we can decompose

$$[z^{-k}(F-f)^k \phi_\alpha d^n\mathbf{x}] = \sum_{m\geq -k} \sum_\beta h^{(k)}_{\alpha\beta,m} z^m [\phi_\beta d^n \mathbf{x}],$$

where $h^{(k)}_{\alpha\beta,m} \in \mathbb{C}[\mathbf{s}]_k$. Then we define

$$e^{(F-f)/z}(\phi_\alpha d^n\mathbf{x}) := \sum_{k=0}^{\infty} \sum_\beta \sum_{m\geq -k} h^{(k)}_{\alpha\beta,m} \frac{z^m}{k!}[\phi_\beta d^n\mathbf{x}] \in B((z))[[\mathbf{s}]].$$

Proposition 3.4. *The exponential map extends to a* $\mathbb{C}((z))[[\mathbf{s}]]$*-linear isomorphism*

$$e^{(F-f)/z} : B_F((z))[[\mathbf{s}]] \to B((z))[[\mathbf{s}]].$$

Proof. Clearly, $e^{(F-f)/z}$ extends to a $\mathbb{C}((z))[[\mathbf{s}]]$-linear map on $B_F((z))[[\mathbf{s}]]$. The statement follows by noticing $e^{(F-f)/z} \equiv 1 \bmod (\mathbf{s})$ under the manifest identification between B and B_F. □

We will use the same symbol

$$K_f : B((z))[[\mathbf{s}]] \times B((z))[[\mathbf{s}]] \to \mathbb{C}((z))[[\mathbf{s}]]$$

to denote the $\mathbb{C}[[\mathbf{s}]]$-linear extension of the higher residue pairing to $\mathcal{H}_f[[\mathbf{s}]] = B((z))[[\mathbf{s}]]$.

Lemma 3.5. *For any* $\varphi_1, \varphi_2 \in B_F$*, we have*

$$K_f(e^{(F-f)/z}\varphi_1, e^{(F-f)/z}\varphi_2) \in z^n\mathbb{C}[[z, \mathbf{s}]].$$

In particular, $e^{(F-f)/z}$ *maps* $B_F[[z]]$ *to an isotropic subspace of* $\mathcal{H}_f[[\mathbf{s}]]$.

Proof. Let K_F denote the higher residue pairing for the unfolding F [42]. The exponential operator $e^{(F-f)/z}$ gives an isometry (with respect to the higher residue pairing) between the Brieskorn lattice for the unfolding F and the trivial unfolding f [32,47]. That is, $K_f(e^{(F-f)/z}\varphi_1, e^{(F-f)/z}\varphi_2) = K_F(\varphi_1, \varphi_2) \in z^n\mathbb{C}[[z, \mathbf{s}]]$, where φ_1, φ_2 are treated as elements of the Brieskorn lattice for the unfolding F. □

Remark 3.6. The above lemma can also be proved directly via an explicit formula for K_f described in [32]. By that formula, there exists a compactly supported differential operator $P\left(\frac{\partial}{\partial \bar{x}_i}, z\frac{\partial}{\partial x_i}, \lrcorner\partial_{x_i}, \wedge d\bar{x}_i\right)$ on smooth differential forms composed of $\frac{\partial}{\partial \bar{x}_i}, z\frac{\partial}{\partial x_i}, \lrcorner\partial_{x_i}, \wedge d\bar{x}_i$ and some cut-off function such that

$$K_f(e^{(F-f)/z}\varphi_1, e^{(F-f)/z}\varphi_2) = z^n \int_X e^{(F-f)/z}\varphi_1 \wedge P\left(\frac{\partial}{\partial \bar{x}_i}, z\frac{\partial}{\partial x_i}, \lrcorner\partial_{x_i}, \wedge d\bar{x}_i\right)(e^{-(F-f)/z}\varphi_2).$$

Since P will not introduce negative powers of z when passing through $e^{(f-F)/z}$, the lemma follows.

Theorem 3.7. *Given a good basis* $\{[\phi_\alpha d^n\mathbf{x}]\}_{\alpha=1}^{\mu} \subset \mathcal{H}_f^{(0)}$, *there exists a unique pair* $(\zeta, \mathcal{J})$ *satisfying* (1) $\zeta \in B_F[[z]][[\mathbf{s}]]$, (2) $\mathcal{J} \in [d^n\mathbf{x}] + z^{-1}B[z^{-1}][[\mathbf{s}]] \subset \mathcal{H}_f[[s]]$, *and*

$$e^{(F-f)/z}\zeta = \mathcal{J}. \tag{$\star$}$$

Moreover, both ζ *and* $\mathcal{J}$ *are weighted homogeneous.*

Proof. We will find $\zeta(\mathbf{s})$ recursively with respect to the order in $\mathbf{s}$. Let

$$\zeta = \sum_{k=0}^{\infty} \zeta_{(k)} = \sum_{k=0}^{\infty}\sum_{\alpha} \zeta_{(k)}^{\alpha}\phi_\alpha d^n\mathbf{x}, \qquad \zeta_{(k)}^{\alpha} \in \mathbb{C}[[z]] \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{s}]_k.$$

Since $e^{(F-f)/z} \equiv 1 \bmod (\mathbf{s})$, the leading order of ($\star$) is

$$\zeta_{(0)} \in [d^n\mathbf{x}] + z^{-1}B[z^{-1}],$$

which is uniquely solved by $\zeta_{(0)} = \phi_1 d^n\mathbf{x}$. Suppose we have solved ($\star$) up to order N, i.e. $\zeta_{(\le N)} := \sum_{k=0}^{N} \zeta_{(k)}$ such that

$$e^{(F-f)/z}\zeta_{(\le N)} \in [d^n\mathbf{x}] + z^{-1}B[z^{-1}][[\mathbf{s}]] \bmod (\mathbf{s}^{N+1}).$$

Let $R_{N+1} \in B((z)) \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{s}]_{(N+1)}$ be the $(N+1)$-th order component of $e^{(F-f)/z}\zeta_{(\le N)}$. Let

$$R_{N+1} = R_{N+1}^{+} + R_{N+1}^{-}$$

where $R_{N+1}^{+} \in B[[z]] \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{s}]_{(N+1)}$ and $R_{N+1}^{-} \in z^{-1}B[z^{-1}] \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{s}]_{(N+1)}$. Let $\tilde{R}_{N+1}^{+} \in B_F[[z]] \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{s}]_{(N+1)}$ correspond to R_{N+1}^{+} under the manifest identification between B and B_F. Then

$$\zeta_{(\le N+1)} := \zeta_{(\le N)} - \tilde{R}_{N+1}^{+}$$

gives the unique solution of ($\star$) up to order $N+1$. This algorithm allows us to solve $\zeta, \mathcal{J}$ perturbatively to arbitrary order. The weighted homogeneity follows from the fact that ($\star$) respects the weighted degree. □

Remark 3.8. In [32], it is shown that the volume form

$$\sum_{k=0}^{\infty}\sum_{\alpha} \zeta_{(k)}^{\alpha}\phi_\alpha d^n\mathbf{x}$$

gives the power series expansion of a representative of the primitive form associated to the good basis $\{[\phi_\alpha d^n\mathbf{x}]\}_{\alpha=1}^{\mu}$. In particular, this is a perturbative way to compute the primitive form via a formal solution of the Riemann–Hilbert–Birkhoff problem.

3.2.2. Flat coordinates and potential function. Let $(\zeta, \mathcal{J})$ be the unique solution of $(\star)$. As shown in [32], ζ represents the power series expansion of a primitive form. However, for the purpose of mirror symmetry, it is more convenient to work with $\mathcal{J}$, which plays the role of Givental's J-function (see [21] for an introduction). This allows us to read off the flat coordinates and the potential function of the associated Frobenius manifold structure.

With the natural embedding $z^{-1}\mathbb{C}[z^{-1}][[\mathbf{s}]] \hookrightarrow z^{-1}\mathbb{C}[[z^{-1}]][[\mathbf{s}]]$, we decompose

$$\mathcal{J} = [d^n x] + \sum_{m=-1}^{-\infty} z^m \mathcal{J}_m, \quad \text{where} \quad \mathcal{J}_m = \sum_\alpha \mathcal{J}_m^\alpha [\phi_\alpha d^n \mathbf{x}],\ \mathcal{J}_m^\alpha \in \mathbb{C}[[\mathbf{s}]].$$

We denote the z^{-1}-term by

$$t_\alpha(\mathbf{s}) := \mathcal{J}_{-1}^\alpha(\mathbf{s}).$$

It is easy to see that t_α is weighted homogeneous of the same degree as s_α such that $t_\alpha = s_\alpha + O(\mathbf{s}^2)$. Therefore t_α defines a set of new homogeneous local coordinates on the (formal) deformation space of f.

Proposition 3.9. *The function $\mathcal{J} = \mathcal{J}(\mathbf{s}(\mathbf{t}))$ in coordinates t_α satisfies*

$$\partial_{t_\alpha}\partial_{t_\beta}\mathcal{J} = z^{-1}\sum_\gamma A_{\alpha\beta}^\gamma(\mathbf{t})\partial_{t_\gamma}\mathcal{J}$$

for some homogeneous $A_{\alpha\beta}^\gamma(\mathbf{t}) \in \mathbb{C}[[\mathbf{t}]]$ of weighted degree $\deg\phi_\alpha + \deg\phi_\beta - \deg\phi_\gamma$. Moreover, for any $\alpha, \beta, \gamma, \delta$,

$$\partial_{t_\alpha} A_{\beta\gamma}^\delta = \partial_{t_\beta} A_{\alpha\gamma}^\delta, \qquad \sum_\sigma A_{\alpha\sigma}^\delta A_{\beta\gamma}^\sigma = \sum_\sigma A_{\beta\sigma}^\delta A_{\alpha\gamma}^\sigma.$$

Proof. Consider the splitting

$$\mathcal{H}_f[[\mathbf{s}]] = B((z))[[\mathbf{s}]] = \mathcal{H}_+ \oplus \mathcal{H}_-,$$

where

$$\mathcal{H}_+ := e^{(F-f)/z}(B_F[[z]][[\mathbf{s}]]) \subset B((z))[[\mathbf{s}]], \qquad \mathcal{H}_- := z^{-1}B[z^{-1}][[\mathbf{s}]].$$

Let $\mathfrak{B}_F := \mathcal{H}_+ \cap z\mathcal{H}_-$. Equation $(\star)$ implies that $z\partial_{t_\alpha}\mathcal{J} \in \mathfrak{B}_F$, with z-leading term of constant coefficient

$$z\partial_{t_\alpha}\mathcal{J} \in [\phi_\alpha d^n \mathbf{x}] + \mathcal{H}_-.$$

In particular, $\{z\partial_{t_\alpha}\mathcal{J}\}$ form a $\mathbb{C}[[\mathbf{s}]]$-basis of $\mathfrak{B}_F$.

Similarly, $z^2\partial_{t_\alpha}\partial_{t_\beta}\mathcal{J} = z^2\partial_{t_\alpha}\partial_{t_\beta}(e^{(F-f)/z}\zeta) \in \mathcal{H}_+$, and $z^2\partial_{t_\alpha}\partial_{t_\beta}\mathcal{J} \in z\mathcal{H}_-$ by the above property of the leading constant coefficient. Therefore $z^2\partial_{t_\alpha}\partial_{t_\beta}\mathcal{J} \in \mathfrak{B}_F$. This implies the existence of functions $A_{\alpha\beta}^\gamma = A_{\alpha\beta}^\gamma(\mathbf{s}(\mathbf{t}))$ such that

$$z^2\partial_{t_\alpha}\partial_{t_\beta}\mathcal{J} = \sum_\gamma zA_{\alpha\beta}^\gamma(\mathbf{s}(\mathbf{t}))\partial_{t_\gamma}\mathcal{J}$$

The homogeneous degree follows from the fact that $\mathcal{J}$ is weighted homogeneous.

Let $\mathcal{A}_\alpha$ denote the linear transformation on $\mathfrak{B}_F$ given by

$$\mathcal{A}_\alpha : z\partial_\beta \mathcal{J} \mapsto \sum_\gamma A^\gamma_{\alpha\beta} z\partial_{t_\gamma}\mathcal{J}.$$

We can rewrite the above equation as $(\partial_{t_\alpha} - z^{-1}\mathcal{A}_\alpha)\partial_{t_\beta}\mathcal{J} = 0$. We notice that

$$[\partial_{t_\alpha} - z^{-1}\mathcal{A}_\alpha, \partial_{t_\beta} - z^{-1}\mathcal{A}_\beta] = 0 \quad \text{on } \mathfrak{B}_F \text{ for all } \alpha, \beta.$$

Therefore the last equations in the proposition hold. □

Lemma 3.10. *In terms of the coordinates t_α, we have*

$$K_f(z\partial_{t_\alpha}\mathcal{J}, z\partial_{t_\beta}\mathcal{J}) = z^n g_{\alpha\beta}.$$

Here $g_{\alpha\beta}$ is the constant equal to the residue pairing $\eta_f(\phi_\alpha d^n\mathbf{x}, \phi_\beta d^n\mathbf{x})$.

Proof. We adopt the same notation as in the above proof. Since $z\partial_{t_\alpha}\mathcal{J} \in \mathcal{H}_+$,

$$K_f(z\partial_{t_\alpha}\mathcal{J}, z\partial_{t_\beta}\mathcal{J}) \in z^n\mathbb{C}[[z]][[\mathbf{s}]]$$

by Lemma 3.5. Since also $z\partial_{t_\alpha}\mathcal{J} = [\phi_\alpha d^n\mathbf{x}] + \mathcal{H}_- \in z\mathcal{H}_-$, we have

$$K_f(z\partial_{t_\alpha}\mathcal{J}, z\partial_{t_\beta}\mathcal{J}) \in z^n g_{\alpha\beta} + z^{n-1}\mathbb{C}[z^{-1}][[\mathbf{s}]].$$

The lemma follows from the above two properties. □

Corollary 3.11. *Let $A_{\alpha\beta\gamma}(\mathbf{t}) := \sum_\delta A^\delta_{\alpha\beta} g_{\delta\gamma}$. Then $A_{\alpha\beta\gamma}$ is symmetric in α, β, γ.*

Proof. By Lemma 3.10, $\partial_{t_\gamma} K_f(z\partial_{t_\alpha}\mathcal{J}, z\partial_{t_\beta}\mathcal{J}) = 0$. Now apply Proposition 3.9. □

The above propositions can be summarized as follows. The triple $(\partial_{t_\alpha}, A^\gamma_{\alpha\beta}, g_{\alpha\beta})$ defines a (formal) Frobenius manifold structure on a neighborhood S of the origin with $\{t_\alpha\}$ being the flat coordinates, together with the potential function $\mathcal{F}_0(\mathbf{t})$ satisfying

$$A_{\alpha\beta\gamma}(\mathbf{t}) = \partial_{t_\alpha}\partial_{t_\beta}\partial_{t_\gamma}\mathcal{F}_0(\mathbf{t}).$$

It is not hard to see that $\mathcal{F}_0(\mathbf{t})$ is homogeneous of degree $3-\hat{c}_f$. As in the next proposition, the potential function $\mathcal{F}_0(\mathbf{t})$ can also be computed perturbatively. Let

$$\mathcal{F}_0(\mathbf{t}) = \mathcal{F}_{0,(\le N)}(\mathbf{t}) + O(\mathbf{t}^{N+1}).$$

Proposition 3.12. *The potential function $\mathcal{F}_0$ associated to the unique pair $(\zeta, \mathcal{J})$ satisfies*

$$\partial_{t_\alpha}\mathcal{F}_0(\mathbf{t}) = \sum_\beta g_{\alpha\beta}\mathcal{J}^\beta_{-2}(\mathbf{s}(\mathbf{t})).$$

Moreover, $\mathcal{F}_0^{(\le N)}(\mathbf{t})$ is determined by $\zeta_{(\le N-3)}(\mathbf{s})$.

Proof. The first statement follows directly from Proposition 3.9.

Recall $\zeta(\mathbf{s}) = \zeta_{(\le N)}(\mathbf{s}) + O(\mathbf{s}^{N+1})$. Let $\mathcal{J}^\alpha_m(\mathbf{s}) = \mathcal{J}^\alpha_{m,(\le N)}(\mathbf{s}) + O(\mathbf{s}^{N+1})$. It is easy to see that $\mathcal{F}_0^{(\le N)}(\mathbf{t})$ only depends on $\mathcal{J}^\alpha_{-1,(\le N-2)}(\mathbf{s})$, $\mathcal{J}^\alpha_{-2,(\le N-1)}(\mathbf{s})$, and $\mathcal{J}^\alpha_{m,(\le N)}(\mathbf{s})$ only depends on $\zeta_{(\le N+m)}(\mathbf{s})$. Hence, the second statement follows. □

Remark 3.13. By Remark 3.8, ζ is in fact an analytic primitive form. Therefore, both t_α and $\mathcal{F}_0(\mathbf{t})$ are in fact analytic functions of $\mathbf{s}$ at the germ $\mathbf{s} = 0$.

3.3. Computation for exceptional unimodular singularities

We start with the next proposition, which follows from a related statement for Brieskorn lattices [23]. An explicit calculation of the moduli space of good sections for general weighted homogenous polynomials is also given in [32,47]. For exposition, we include a proof here.

Proposition 3.14. *If f is one of the* 14 *exceptional unimodular singularities, then there exists a unique good section $\{[\phi_\alpha d^n\mathbf{x}]\}_{\alpha=1}^{\mu}$, where $\{\phi_\alpha\} \subset \mathbb{C}[\mathbf{x}]$ are (arbitrary) weighted homogeneous representatives of a basis of the Jacobi algebra* $\mathrm{Jac}(f)$.

Proof. We give the details for the E_{12}-singularity. The other 13 types are established similarly.

The E_{12}-singularity is given by $f = x^3 + y^7$ with $\deg x = \frac{1}{3}$, $\deg y = \frac{1}{7}$, and central charge $\hat{c}_f = \frac{22}{21}$. We consider the weighted homogeneous monomials

$$\{\phi_1, \ldots, \phi_{12}\} = \{1, y, y^2, x, y^3, xy, y^4, xy^2, y^5, xy^3, xy^4, xy^5\} \subset \mathbb{C}[x, y]$$

which represent a basis of $\mathrm{Jac}(f)$. The normalized residue pairing $g_{\alpha\beta}$ between ϕ_α, ϕ_β is equal to 1 if $\alpha + \beta = 13$, and 0 otherwise. Since K_f preserves the $\mathbb{Q}$-grading,

$$\deg K_f([\phi_\alpha dxdy], [\phi_\beta dxdy]) = \deg\phi_\alpha + \deg\phi_\beta + 2 - \hat{c}_f,$$

which has to be an integer for a nonzero pairing. A simple degree count implies that

$$K_f([\phi_\alpha dxdy], [\phi_\beta dxdy]) = z^2 g_{\alpha\beta},$$

and therefore $\{[\phi_\alpha dxdy]\}$ constitutes a good basis.

Let $\{\phi'_\alpha\}$ be another set of weighted homogeneous polynomials such that $\{[\phi'_\alpha dxdy]\}$ gives a good basis. We can assume $\phi'_\alpha \equiv \phi_\alpha$ as elements in $\mathrm{Jac}(f)$ and $\deg\phi'_\alpha = \deg\phi_\alpha$. Since $[\phi_\alpha dxdy]$ forms a $\mathbb{C}[[z]]$-basis of $\mathcal{H}_f^{(0)}$, we can decompose

$$[\phi'_\alpha dxdy] = \sum_\beta R_\alpha^\beta[\phi_\beta dxdy], \qquad R_\alpha^\beta \in \mathbb{C}[[z]].$$

By weighted homogeneity, R_α^β is homogeneous of degree $\deg\phi_\alpha - \deg\phi_\beta$, which is not an integer unless $\alpha = \beta$. Thus $[\phi'_\alpha dxdy] = [\phi_\alpha dxdy]$, which proves uniqueness. □

Let $\mathcal{F}_0$ be the potential function of the associated Frobenius manifold structure. Then $\mathcal{F}_0$ is an analytic function, as an immediate consequence of the above uniqueness together with the existence of the (analytic) primitive form. As will be shown in Lemma 4.2, we only need to compute $\mathcal{F}_{0,(\le 4)}$ to prove mirror symmetry.

We illustrate the perturbative calculation for the E_{12}-singularity $f = x^3 + y^7$. The full result is summarized in the appendix by similar calculations. We adopt the same notation as in the proof of Proposition 3.14. By Proposition 3.12, we only need $\zeta_{(\le 1)}$ to compute $\mathcal{F}_{0,(\le 4)}$, which is

$$\zeta_{(\le 1)} = dxdy.$$

Using the equivalence relation in $\mathcal{H}_f$, we can expand

$$e^{(F-f)/z}(\zeta_{(\leq 1)}) = \sum_{k=0}^{3} \frac{(F-f)^k}{k!} z^{-k} \zeta_{(\leq 1)} + O(\mathbf{s}^4)$$

in terms of the good basis $\{\phi_\alpha\}$. We find the flat coordinates up to order 2:

$$\begin{aligned}
&t_1 \doteq s_1 - \tfrac{1}{7}s_5s_7 - \tfrac{1}{7}s_3s_9, \quad t_2 \doteq s_2 - \tfrac{1}{7}s_7^2 - \tfrac{2}{7}s_5s_9, \quad t_3 \doteq s_3 - \tfrac{3}{7}s_7s_9,\\
&t_4 \doteq s_4 - \tfrac{1}{7}s_8s_9 - \tfrac{1}{7}s_7s_{10} - \tfrac{1}{7}s_5s_{11} - \tfrac{1}{7}s_3s_{12}, \quad t_5 \doteq s_5 - \tfrac{2}{7}s_9^2,\\
&t_6 \doteq s_6 - \tfrac{2}{7}s_9s_{10} - \tfrac{2}{7}s_7s_{11} - \tfrac{2}{7}s_5s_{12}, \quad t_7 \doteq s_7, \quad t_8 \doteq s_8 - \tfrac{3}{7}s_9s_{11} - \tfrac{3}{7}s_7s_{12},\\
&t_9 \doteq s_9, \quad t_{10} \doteq s_{10} - \tfrac{4}{7}s_9s_{12}, \quad t_{11} \doteq s_{11}, \quad t_{12} \doteq s_{12}.
\end{aligned}$$

This allows us to solve the inverse function $s_\alpha = s_\alpha(\mathbf{t})$ up to order 2. A straightforward but tedious computation of the z^{-2}-term shows that in terms of flat coordinates

$$\mathcal{F}_{0,(\leq 4)} = \mathcal{F}_0^{(3)} + \mathcal{F}_0^{(4)},$$

where $\mathcal{F}_0^{(3)}$ is the third order term representing the algebraic structure of $\mathrm{Jac}(f)$,

$$\partial_{t_\alpha}\partial_{t_\beta}\partial_{t_\gamma}\mathcal{F}_0^{(3)} = \eta_f([\phi_\alpha\phi_\beta\phi_\gamma dxdy], [dxdy]).$$

The fourth order term $\mathcal{F}_0^{(4)}$, which we call the 4-*point function*, is computed by

$$\begin{aligned}
-\mathcal{F}_0^{(4)} = {}& \tfrac{1}{14}t_5t_6t_7^2 + \tfrac{1}{18}t_6^3t_8 + \tfrac{1}{7}t_5^2t_7t_8 + \tfrac{1}{7}t_3t_7^2t_8 + \tfrac{1}{6}t_4t_6t_8^2 + \tfrac{1}{14}t_5^2t_6t_9 + \tfrac{1}{7}t_3t_6t_7t_9\\
&+ \tfrac{1}{7}t_3t_5t_8t_9 + \tfrac{1}{7}t_2t_7t_8t_9 + \tfrac{1}{14}t_2t_6t_9^2 + \tfrac{1}{14}t_5^3t_{10} + \tfrac{1}{6}t_4t_6^2t_{10} + \tfrac{2}{7}t_3t_5t_7t_{10} + \tfrac{1}{14}t_2t_7^2t_{10}\\
&+ \tfrac{1}{6}t_4^2t_8t_{10} + \tfrac{1}{14}t_3^2t_9t_{10} + \tfrac{1}{7}t_2t_5t_9t_{10} + \tfrac{1}{7}t_3t_5^2t_{11} + \tfrac{1}{6}t_4^2t_6t_{11} + \tfrac{1}{7}t_3^2t_7t_{11} + \tfrac{1}{7}t_2t_5t_7t_{11}\\
&+ \tfrac{1}{7}t_2t_3t_9t_{11} + \tfrac{1}{18}t_4^3t_{12} + \tfrac{1}{14}t_3^2t_5t_{12} + \tfrac{1}{14}t_2t_5^2t_{12} + \tfrac{1}{7}t_2t_3t_7t_{12} + \tfrac{1}{14}t_2^2t_9t_{12}.
\end{aligned}$$

In particular, for our later use, we can read off

$$\partial_{t_4}\partial_{t_4}\partial_{t_4}\partial_{t_{12}}\mathcal{F}_0|_{\mathbf{t}=\mathbf{0}} = -\tfrac{1}{3}, \qquad \partial_{t_2}\partial_{t_2}\partial_{t_9}\partial_{t_{12}}\mathcal{F}_0|_{\mathbf{t}=\mathbf{0}} = -\tfrac{1}{7}.$$

4. Mirror symmetry for exceptional unimodular singularities

In this section, we use two reconstruction results to prove the mirror symmetry conjecture between the 14 exceptional unimodular singularities and their FJRW mirrors both at genus zero and at higher genera.

4.1. Mirror symmetry at genus zero

Throughout this subsection, we assume W^T to be one of the 14 exceptional unimodular singularities in Table 1. We will consider the ring isomorphism $\Psi : \mathrm{Jac}(W^T) \to (H_W, \bullet)$ defined in Proposition 2.7. We will also denote the basis of $\mathrm{Jac}(W^T)$ specified therein by $\{\phi_1, \ldots, \phi_\mu\}$ with $\deg\phi_1 \leq \cdots \leq \deg\phi_\mu$. As mentioned, there is a formal Frobenius manifold structure on the FJRW ring $(H_W, \bullet)$ with a prepotential $\mathcal{F}_{0,W}^{\mathrm{FJRW}}$. We have also shown in the previous section that there is a Frobenius manifold structure with flat coordinates $(t_1, \ldots, t_\mu)$ associated to (the primitive form) ζ therein, whose prepotential will be denoted by $\mathcal{F}^{\mathrm{SG}}_{0,W^T}$ from now on. We introduce the primary correlators $\langle\cdots\rangle^{W^T,\mathrm{SG}}_{0,k}$ associated to the Frobenius manifold structure on B-side. The primary correlators, up to linear combinations, are given by

$$\langle\phi_{i_1},\ldots,\phi_{i_k}\rangle^{W^T,\mathrm{SG}}_{0,k}=\frac{\partial^k\mathcal{F}^{\mathrm{SG}}_{0,W^T}}{\partial t^{i_1}\cdots\partial t^{i_k}}(0). \tag{4.1}$$

From the specified ring isomorphism Ψ and (2.10), we have

$$\langle\mathbf{1}_{\phi_i},\mathbf{1}_{\phi_j},\mathbf{1}_{\phi_k}\rangle^{W}_{0,3}=\langle\phi_i,\phi_j,\phi_k\rangle^{W^T,\mathrm{SG}}_{0,3}.$$

From Proposition 2.8 and the computation in Section 3.3 and in the appendix, we have

$$\langle\mathbf{1}_{x_i},\mathbf{1}_{x_i},\mathbf{1}_{M_i^T/x_i^2},\mathbf{1}_{\phi_\mu}\rangle^{W}_{0,4}=-\langle x_i,x_i,M_i^T/x_i^2,\phi_\mu\rangle^{W^T,\mathrm{SG}}_{0,4}.$$

To deal with the sign, we will make the following modifications, as in [14, Section 6.5]. We simply denote $(-1)^r:=e^{\pi\sqrt{-1}\,r}$. Let $\tilde{\mathcal{F}}^{\mathrm{SG}}_0$ denote the potential function of the Frobenius manifold structure $\tilde{\zeta}:=(-1)^{-\hat{c}_{W^T}}\zeta$. Set $\tilde{\phi}_j:=(-1)^{-\deg\phi_j}\phi_j$ and define a map $\tilde{\Psi}:\mathrm{Jac}(W^T)\to H_W$ by $\tilde{\Psi}(\tilde{\phi}_j):=\Psi(\phi_j)$. Let $\tilde{\mathbf{t}}$ denote the flat coordinate of $\tilde{\mathcal{F}}^{\mathrm{SG}}_0$, namely

$$\tilde{t}_j=(-1)^{1-\deg t_j}t_j. \tag{4.2}$$

As a consequence, we have $\tilde{\mathcal{F}}^{(3),\mathrm{SG}}_{0,W^T}=\mathcal{F}^{(3),\mathrm{SG}}_{0,W^T}$ and $\tilde{\mathcal{F}}^{(4),\mathrm{SG}}_{0,W^T}=-\mathcal{F}^{(4),\mathrm{SG}}_{0,W^T}$. Denote $\tilde{\mathbf{1}}_{\tilde{\phi}_j}:=\tilde{\Psi}(\tilde{\phi}_j)$. Then $\tilde{\Psi}$ defines a pairing-preserving ring isomorphism, which is read off from the identities $\langle\tilde{\mathbf{1}}_{\tilde{\phi}_i},\tilde{\mathbf{1}}_{\tilde{\phi}_j},\tilde{\mathbf{1}}_{\tilde{\phi}_k}\rangle^{W}_{0,3}=\langle\tilde{\phi}_i,\tilde{\phi}_j,\tilde{\phi}_k\rangle^{W^T,\tilde{\zeta},\mathrm{SG}}_{0,3}$, Moreover,

$$\langle\tilde{\mathbf{1}}_{\tilde{x}_i},\tilde{\mathbf{1}}_{\tilde{x}_i},\tilde{\mathbf{1}}_{\widetilde{M_i^T/x_i^2}},\tilde{\mathbf{1}}_{\tilde{\phi}_\mu}\rangle^{W}_{0,4}=\langle\tilde{x}_i,\tilde{x}_i,\widetilde{M_i^T/x_i^2},\tilde{\phi}_\mu\rangle^{W^T,\tilde{\zeta},\mathrm{SG}}_{0,4}. \tag{4.3}$$

From now on, we simplify the notation by dropping the symbol $\tilde{}$ and the superscript $\tilde{\zeta}$. In addition, we simply denote both H_W and $\mathrm{Jac}(W^T)$ as H, and we denote the correlators on both sides as $\langle\phi_{i_1},\ldots,\phi_{i_k}\rangle_{0,k}$ (or $\langle\phi_{i_1},\ldots,\phi_{i_k}\rangle$) whenever there is no risk of confusion. We have the following "selection rule" for primary correlators.

Lemma 4.1. *A primary correlator $\langle\phi_{i_1},\ldots,\phi_{i_k}\rangle_{0,k}$ on either A-side or B-side is nonzero only if*

$$\sum_{j=1}^{k}\deg\phi_{i_j}=\hat{c}_{W^T}-3+k. \tag{4.4}$$

Proof. The A-side case follows from (2.11) and $\hat{c}_W=\hat{c}_{W^T}$. The primary correlator on B-side is given by $\partial_{t_{i_1}}\cdots\partial_{t_{i_k}}\mathcal{F}^{\mathrm{SG}}_{0,W^T}(0)$, where $\deg\phi_{i_j}=1-\deg t_{i_j}$. Then the statement follows, by noticing that $\mathcal{F}^{\mathrm{SG}}_{0,W^T}(0)$ is weighted homogenous of degree $3-\hat{c}_{W^T}$. $\square$

A homogeneous $\alpha\in H$ is called a *primitive class* with respect to the specified basis $\{\phi_j\}$ if it cannot be written as $\alpha=\alpha_1\bullet\alpha_2$ for $0<\deg\alpha_i<\deg\alpha$. A primary correlator $\langle\phi_{i_1},\ldots,\phi_{i_k}\rangle_{0,k}$ is called *basic* if at least $k-2$ insertions ϕ_{i_j} are primitive classes. Now Theorem 1.3 is a direct consequence of the equalities (4.3) and the following statement:

Lemma 4.2 (Reconstruction Lemma). *If W^T is one of the* 14 *exceptional singularities, then all the following hold.*

(1) *The prepotential $\mathcal{F}_0$ is uniquely determined from the basic correlators $\langle \ldots \rangle_{0,k}$ with $k \leq 5$.*
(2) *All basic correlators $\langle \phi_{i_1}, \ldots, \phi_{i_5} \rangle_{0,5}$ vanish.*
(3) *All the* 4*-point basic correlators are uniquely determined from* (1.1).

Proof of (1). The potential function $\mathcal{F}_0$ satisfies the WDVV equation (2.13) (hence (2.14)). We can assume that $\langle \cdots \rangle_{0,k}$ is not of type $\langle 1, \ldots \rangle_{0,k}$, $k \geq 4$ (otherwise it vanishes according to the string equation, or the invariance of the primitive form along the ϕ_1-direction where we notice $\phi_1 = 1$). Consider a correlator $\langle \ldots, \alpha_a, \alpha_b \bullet \alpha_c, \alpha_d \rangle_{0,k}$, with the last three insertions nonprimitive. By (2.14), such a correlator is the sum of S_k together with three terms whose insertion replaces $\alpha_b \bullet \alpha_c$ with lower degree ones α_b or α_c at the same position. Repeating this will turn $\alpha_b \bullet \alpha_c$ into a primitive class, up to a product of correlators with fewer insertions. By induction both on the degree of nonprimitive classes and on k, we can reduce any correlator to a linear combination of basic correlators.

Now we assume that $\langle \phi_{i_1}, \ldots, \phi_{i_k} \rangle_{0,k}$ is a nonzero basic correlator. Then we can write $\phi_{i_1} \bullet \cdots \bullet \phi_{i_k} = x^a y^b z^c$. It follows from the degree constraint (4.4) that

$$\hat{c}_{W^T} - 3 + k = \sum_{j=1}^{k} \deg \phi_{i_j} = a q_x + b q_y + c q_z. \tag{4.5}$$

Let P be the maximal degree of a generator x, y and z (or x and y if $W^T = W^T(x, y)$ is in two variables x, y only). By direct calculations, we conclude

$$k \leq \frac{\hat{c}_{W^T} + 1}{1 - P} + 2 < 6.$$

Proof of (2). For $W^T = x^p + y^q$, x, y are generators for the ring structure H. The multiplications for all the insertions will be in the form of $x^a \bullet y^b$. By the degree constraint, a nonzero basic correlator $\langle \phi_{i_1}, \ldots, \phi_{i_k} \rangle_{0,k}$ satisfies

$$aq + bp = (k - 1)pq - 2p - 2q. \tag{4.6}$$

On the other hand, we assume the first $k - 2$ insertions are primitive classes, so that they are either x or y. The top degree class $\phi_\mu = x^{p-2} \bullet y^{q-2}$ is of degree $2 - 2/p - 2/q$. Therefore we have the following inequalities required for a nonvanishing correlator:

$$a \leq k - 2 + 2(p - 2), \quad b \leq k - 2 + 2(q - 2), \quad a + b \leq k - 2 + 2(p - 2 + q - 2). \tag{4.7}$$

It is easy to see that there is no (a, b) satisfying both (4.6) and (4.7) if $k = 5$. Hence $\langle \phi_{i_1}, \ldots, \phi_{i_5} \rangle_{0,5} = 0$. The arguments for the remaining W^T on B-side and all the W on A-side are all similar and elementary, details of which are left to the readers.

Proof of (3). Let us start with $W^T = x^p + y^q$, where we notice that p, q are coprime. The degree constraint (4.6) with $k = 4$ implies that $(a, b) = (2p - 2, q - 2)$ or $(p - 2, 2q - 2)$. Thus the possibly nonzero basic correlators are

$\langle x, x, x^{p-2}y^i, x^{p-2}y^{q-2-i}\rangle_{0,4}$, $i = 0, \ldots, q-2$. On the other hand, if formula (1.1) holds, then by the WDVV equation (2.14), we have

$$\begin{aligned}\langle x, x, x^{p-2} \bullet y^i, x^{p-2}y^{q-2-i}\rangle &= -\langle x, x^{p-2}, y^i, x \bullet x^{p-2}y^{q-2-i}\rangle \\ &\quad + \langle x, x \bullet x^{p-2}, y^i, x^{p-2}y^{q-2-i}\rangle + \langle x, x, x^{p-2}, y^i \bullet x^{p-2}y^{q-2-i}\rangle \\ &= \langle x, x, x^{p-2}, y^i \bullet x^{p-2}y^{q-2-i}\rangle = 1/p.\end{aligned}$$

For the 2-Chain $W^T = x^p y + y^q$, the degree constraint (4.5) tells us

$$a\frac{q-1}{pq} + b\frac{1}{q} = \hat{c}_W - 3 + k.$$

For $k = 4$, this implies that $(a, b) = (2p-2, q)$ or $(p-2, 2q-1)$. The basic correlators are $\langle x, x, x^{p-2}y^{1+i}, x^{p-2}y^{q-1-i}\rangle$ with $0 \le i \le q-1$, $\langle y, y, x^i y^{q-2}, x^{p-2-i}y^{q-1}\rangle$ with $0 \le i \le p-2$ and $\langle x, y, x^i y^{q-1}, x^{p-3-i}y^{q-1}\rangle$ with $0 \le i \le p-3$. The first two types are uniquely determined from the correlators which are listed in Proposition 1.2. For example, if $0 < i < q-1$, since $p x^{p-1}y = \partial_x W^T = 0$ in $\mathrm{Jac}(W^T)$, we have

$$\langle x, x, x^{p-2}y \bullet y^i, x^{p-2}y^{q-1-i}\rangle = \langle x, x, x^{p-2}y, x^{p-2}y^{q-1-i} \bullet y^i\rangle.$$

The last type is determined by

$$\begin{aligned}\langle x, y, x^i y^{q-1}, x^{p-3-i}y^{q-1}\rangle &= -\frac{1}{q}\langle x, y, x^i y^{q-1}, x^{p-2-i} \bullet x^{p-1}\rangle \\ &= -\frac{1}{q}(\langle x, y, x^{p-1}, x^{p-2}y^{q-1}\rangle + \langle x, y \bullet x^{p-1}, x^{p-2-i}, x^i y^{q-1}\rangle) \\ &= -\frac{1}{q}\langle x, y, x \bullet x^{p-2}, x^{p-2}y^{q-1}\rangle = -\frac{1}{q}\langle x, x, x^{p-2} \bullet y, x^{p-2}y^{q-1}\rangle = -\frac{1}{pq}.\end{aligned}$$

Here we use the relation $x^p + qy^{q-1} = \partial_y W^T = 0$ in $\mathrm{Jac}(W^T)$ in the first equality.

For the 2-Loop $W^T = x^3 y + xy^4$, the degree constraint (4.6) with $k = 4$ implies that $(a, b) = (5, 4)$ or $(3, 7)$. If (1.1) holds, namely if

$$\langle x, x, xy, x^2y^3\rangle = \tfrac{3}{11}, \qquad \langle y, y, xy^2, x^2y^3\rangle = \tfrac{2}{11},$$

then we conclude $\langle x, y, x^2, x^2y^3\rangle = \frac{3}{11}$, $\langle x, y, y^2, x^2y^3\rangle = \frac{2}{11}$ and $\langle x, x, x^2y^2, xy^2\rangle = \frac{2}{11}$ from a single WDVV equation for each correlator. For the rest, we conclude $\langle x, x, xy^3, x^2y\rangle = \frac{1}{11}$ and $\langle x, y, xy^3, xy^3\rangle = -\frac{1}{11}$ by solving the following linear equations which come from the WDVV equation:

$$\begin{cases}-3\langle x, x, x^2y, xy^3\rangle + \langle x, x \bullet xy^3, y^3, y\rangle = \langle x, x \bullet y^3, y, xy^3\rangle, \\ -4\langle x, y, xy^3, xy^3\rangle = \langle x, y, x^2, x \bullet xy^3\rangle + \langle x, y \bullet x^2, x, xy^3\rangle.\end{cases}$$

Here the coefficient -3 (resp. -4) comes from $3x^2y + y^4 = 0$ (resp. $x^3 + 4xy^3 = 0$) in $\mathrm{Jac}(W^T)$. Similarly, we conclude $\langle x, y, x^2y^2, x^2y\rangle = -\frac{1}{11}$ and $\langle y, y, x^2y^2, xy^3\rangle = \frac{1}{11}$.

For $W^T = x^2y + y^4 + z^3 \in Q_{10}$, the number of 4-point basic correlators is 10. Three of them are the initial correlators in (1.1), $\langle x, x, y, y^3z\rangle$, $\langle y, y, y^2, y^3z\rangle$, $\langle z, z, z, y^3z\rangle$, the rest are $\langle y, y, y^2z, y^3\rangle$, $\langle y, z, y^3, y^3\rangle$, $\langle z, z, yz, y^2z\rangle$, $\langle x, x, yz, y^3\rangle$, $\langle x, x, y^2, y^2z\rangle$, $\langle x, y, xz, y^3\rangle$, and $\langle z, z, xz, xz\rangle$. We have seven WDVV equations to reconstruct them from the initial correlators:

$$\begin{cases} 4\langle y, y, y^2z, y^3\rangle = \langle x, x, y, y^3z\rangle, \quad \langle y, z, y^3, y^3\rangle = \langle y, y, y^2z, y^3\rangle - \langle y, y, y^2, y^3z\rangle, \\ \langle z, z, yz, y^2z\rangle = \langle z, z, z, y^3z\rangle, \quad \langle x, x, yz, y^3\rangle = \langle x, x, y, y^3z\rangle, \\ \langle x, x, y^2, y^2z\rangle = \langle x, x, y, y^3z\rangle, \quad \langle x, y, xz, y^3\rangle = \langle x, x, y, y^3z\rangle, \\ \langle z, z, xz, xz\rangle = -4\langle z, z, z, y^3z\rangle. \end{cases}$$

For other singularities of three variables, all the basic 4-point correlators are uniquely determined from the initial correlators in (1.1), by the same technique. However, the discussion is more tedius. For example, there are 21 4-point basic correlators for the type S_{12} singularity $W^T = x^2y+y^2z+z^3x$. We can write down 18 WDVV equations carefully to determine all the 21 basic correlators from the three correlators in (1.1). The details are skipped here. □

4.2. *Mirror symmetry at higher genus*

In Section 2, we already constructed the total ancestor FJRW potential $\mathscr{A}_W^{\mathrm{FJRW}}$ for a pair (W, G_W). Now we give the B-model total ancestor Saito–Givental potential $\mathscr{A}_{W^T}^{\mathrm{SG}}$. Let S be the universal unfolding of the isolated singularity W^T. For a semisimple point $\mathbf{s} \in S$, Givental [19] constructed the following formula containing higher genus information on the Landau–Ginzburg B-model of f (see [12, 18, 19] for more details):

$$\mathscr{A}_f^{\mathrm{SG}}(\mathbf{s}) := \exp\left(-\frac{1}{48}\sum_{i=1}^{\mu} \log \Delta^i(\mathbf{s})\right)\widehat{\Psi_{\mathbf{s}}}\widehat{\mathbf{R}_{\mathbf{s}}}(\mathcal{T}).$$

Here $\mathcal{T}$ is the product of μ copies of the Witten–Kontsevich τ-function; $\Delta^i(\mathbf{s})$, $\Psi_{\mathbf{s}}$ and $R_{\mathbf{s}}$ are data coming from the Frobenius manifold; and the operators $\widehat{\cdot}$ are the so-called quantization operators. We call $\mathscr{A}_f^{\mathrm{SG}}(\mathbf{s})$ the Saito–Givental potential for f at $\mathbf{s}$. Teleman [49] proved that $\mathscr{A}_f^{\mathrm{SG}}(\mathbf{s})$ is uniquely determined by the genus zero data on the Frobenius manifold. By definition, the coefficients in each genus-g generating function of $\mathscr{A}_f^{\mathrm{SG}}(\mathbf{s})$ are just meromorphic near the nonsemisimple point $\mathbf{s} = \mathbf{0}$. Recently, using Eynard–Orantin recursion, Milanov [34] proved $\mathscr{A}_{W^T}^{\mathrm{SG}}(\mathbf{t})$ extends holomorphically at $\mathbf{t} = \mathbf{0}$. We denote such an extension by $\mathscr{A}_{W^T}^{\mathrm{SG}}$, and Corollary 1.4 follows from Theorem 1.3 and Teleman's theorem.

4.3. *Alternative representatives and the other direction*

Although the theory of primitive forms depends only on the stable equivalence class of the singularity, FJRW theory definitely depends on the choice of the polynomial together with the group. For the exceptional unimodular singularities, in the following we list all the additional invertible weighted homogeneous polynomial representatives without quadratic terms x_k^2 in additional variables x_k (up to permutation symmetry among variables):

$$\begin{cases} E_{14}: x^3+y^8, & W_{12}: x^2y+y^2+z^5, \quad W_{13}: x^4y+y^4; \\ Q_{12}: x^2y+y^5+z^3, & Z_{13}: x^3y+y^6, \quad U_{12}: x^2y+y^3+z^4; \\ U_{12}: x^2y+xy^2+z^4. \end{cases} \tag{4.8}$$

It is quite natural to investigate Conjecture 1.1 for all the weighted homogeneous polynomial representatives on the B-side.

Theorem 4.3. *Conjecture 1.1 is true if W^T is given by any weighted homogenous polynomial representative of the exceptional unimodular singularities such that W^T is not $x^2y+xy^2+z^4$. That is, there exists a mirror map such that*

$$\mathscr{A}_W^{\mathrm{FJRW}} = \mathscr{A}_{W^T}^{\mathrm{SG}}.$$

Sketch of the proof. Thanks to Corollary 1.4, it remains to consider the case when W^T is given by (4.8). By Proposition 3.14, there is a unique good section. Let us specify a weighted homogeneous basis $\{\phi_1,\dots,\phi_\mu\}$ of $\mathrm{Jac}(W^T)$ as in Table 2 for each atomic type and take the product of such bases for mixed types. Then we could obtain the 4-point function by direct calculations (see the link in the appendix for precise output). An isomorphism $\Psi: \mathrm{Jac}(W^T) \to H_W$ is chosen much as in Section 2. We compute the corresponding 4-point FJRW correlators as in Proposition 2.8. If W^T is not $x^2y+xy^2+z^4$, then the 4-point FJRW correlators turn out to be the same as the B-side 4-point correlator up to sign. These invariants completely determine the full data of the generating function at all genera on both sides, by exactly the same reconstruction technique as in the previous two subsections. Therefore, the statement follows. □

Remark 4.4. If $W^T = x^2y+xy^2+z^4$, then H_W has broad ring generators $x\mathbf{1}_{J^8}$ and $y\mathbf{1}_{J^8}$. Our method does not apply to compute

$$\langle x\mathbf{1}_{J^8}, x\mathbf{1}_{J^8}, y\mathbf{1}_{J^8}, \mathbf{1}_{J^{15}}\rangle_0^W, \quad \langle y\mathbf{1}_{J^8}, y\mathbf{1}_{J^8}, x\mathbf{1}_{J^8}, \mathbf{1}_{J^{15}}\rangle_0^W, \quad \mathbf{1}_{\phi_\mu} = \mathbf{1}_{J^{15}}.$$

If $W^T = x^2y+y^2+z^5$, we may need a further rescaling on $\Psi(x)$ since we only know

$$(\langle \Psi(x), \Psi(x), \Psi(y), \Psi(yz^3)\rangle_0^W)^2 = 2\langle \Psi(y), \Psi(y), \Psi(y), \Psi(y), \Psi(yz^3)\rangle_0^W = \tfrac{1}{4}.$$

The first equality is a consequence of the WDVV equation, and the second is a consequence of the orbifold GRR calculation with codimension $D=2$ (i.e., (2.16)).

The other direction. Among all the representatives W on the A-side, there are in total three cases for which W^T is no longer exceptional unimodular. The corresponding W^T is given by x^3+xy^6, $x^2+xy^5+z^3$, or $x^2+xy^3+z^4$. Let us end this section by the following remark, which gives a positive answer to Conjecture 1.1 for those representatives.

Remark 4.5. 1. For the remaining three cases, W^T is no longer given by any one of the exceptional unimodular singularities.

2. A similar calculation to the one in Proposition 3.14 shows that there exists a unique primitive form (up to constant) for $x^2+xy^5+z^3$. However, for the other two cases x^3+xy^6 and $x^2+xy^3+z^4$, there is a whole one-dimensional family of choices of primitive forms.

3. Let us specify a basis $\{\phi_1,\dots,\phi_\mu\}$ of $\mathrm{Jac}(W^T)$ following Table 2. It is easy to check that $\{[\phi_1 d^n\mathbf{x}],\dots,[\phi_\mu d^n\mathbf{x}]\}$ form a good basis and specifies a choice of primitive form. A similar calculation shows that the B-side 4-point function coincides with the A-side one (up to sign as before), and they completely determine the full data of the generating functions at all genera by the same reconstruction technique again.

5. Appendix

5.1. The vector space isomorphisms

Here we list the vector space isomorphisms $\Psi:\mathrm{Jac}(W^T)\to(H_W)$ for the remaining cases of W in Table 1.

(1) 3-**Fermat type.** Let $W=W^T=x^3+y^3+z^4\in U_{12}$. We denote $\mathbf{1}_{i,j,k}:=1\in H_\gamma$ for $\gamma=(\exp(2\pi\sqrt{-1}\,i/3),\exp(2\pi\sqrt{-1}\,j/3,\exp(2\pi\sqrt{-1}\,k/4))\in G_W$. The isomorphism Ψ is given by

$$\Psi(x^{i-1}y^{j-1}z^{k-1})=\mathbf{1}_{i,j,k},\qquad 1\le i,j\le 3,\ 1\le k\le 4.$$

(2) **Chain type.** Let $W=x^3+xy^5$. The mirror W^T is of type Z_{11}. Note $G_W\cong\mu_{15}$.

H_W	$\mathbf{1}_J$	$\mathbf{1}_{J^{13}}$	$\mathbf{1}_{J^{11}}$	$\mathbf{1}_{J^{10}}$	$\mathbf{1}_{J^8}$	$\mp5y^4\mathbf{1}_{J^0}$	$\mathbf{1}_{J^7}$	$\mathbf{1}_{J^5}$	$\mathbf{1}_{J^4}$	$\mathbf{1}_{J^2}$	$\mathbf{1}_{J^{14}}$
$\mathrm{Jac}(W^T)$	1	y	x	y^2	xy	x^2	y^3	xy^2	y^4	xy^3	xy^4

Let $W=x^3y+y^5$. The mirror W^T is of type E_{13}. Note $G_W\cong\mu_{15}$.

H_W	$\mathbf{1}_J$	$\mathbf{1}_{J^{13}}$	$\mathbf{1}_{J^{12}}$	$\mathbf{1}_{J^{11}}$	$\mathbf{1}_{J^9}$	$\mathbf{1}_{J^8}$	$\mp3y^2\mathbf{1}_{J^0}$	$\mathbf{1}_{J^7}$	$\mathbf{1}_{J^6}$	$\mathbf{1}_{J^5}$	$\mathbf{1}_{J^4}$	$\mathbf{1}_{J^2}$	$\mathbf{1}_{J^{14}}$
$\mathrm{Jac}(W^T)$	1	y	y^2	x	y^3	xy	y^4	xy^2	x^2	xy^3	x^2y	x^2y^2	x^2y^3

Let $W=x^2y+y^3z+z^3$. The mirror W^T is of type Z_{13}. Note $G_W\cong\mu_{18}$.

H_W	$\mathbf{1}_J$	$\mathbf{1}_{J^{16}}$	$\mathbf{1}_{J^{14}}$	$\mathbf{1}_{J^{13}}$	$\mathbf{1}_{J^{11}}$	$\mathbf{1}_{J^{10}}$	$\mp3y^2\mathbf{1}_{J^9}$	$\mathbf{1}_{J^8}$	$\mathbf{1}_{J^7}$	$\mathbf{1}_{J^5}$	$\mathbf{1}_{J^4}$	$\mathbf{1}_{J^2}$	$\mathbf{1}_{J^{17}}$
$\mathrm{Jac}(W^T)$	1	y	z	y^2	yz	x	z^2	y^2z	xy	xz	xy^2	xyz	xy^2z

Let $W=x^2y+y^2z+z^4$. The mirror W^T is of type W_{13}. Note $G_W\cong\mu_{16}$.

H_W	$\mathbf{1}_J$	$\mathbf{1}_{J^{14}}$	$\mathbf{1}_{J^{13}}$	$\mathbf{1}_{J^{11}}$	$\mathbf{1}_{J^{10}}$	$\mathbf{1}_{J^9}$	$\mp2y\mathbf{1}_{J^8}$	$\mathbf{1}_{J^7}$	$\mathbf{1}_{J^6}$	$\mathbf{1}_{J^5}$	$\mathbf{1}_{J^3}$	$\mathbf{1}_{J^2}$	$\mathbf{1}_{J^{15}}$
$\mathrm{Jac}(W^T)$	1	z	y	z^2	yz	x	z^3	yz^2	xz	xy	xz^2	xyz	xyz^2

(3) **Loop type.** There is one 2-Loop of type Z_{12}: $W = W^T = x^3y + xy^4$ with $G_W \cong \mu_{11}$.

H_W	$\mathbf{1}_J$	$\mathbf{1}_{J^8}$	$\mathbf{1}_{J^6}$	$\mathbf{1}_{J^4}$	$\mathbf{1}_{J^2}$	$x^2\mathbf{1}_{J^0}$	$y^3\mathbf{1}_{J^0}$	$\mathbf{1}_{J^9}$	$\mathbf{1}_{J^7}$	$\mathbf{1}_{J^5}$	$\mathbf{1}_{J^3}$	$\mathbf{1}_{J^{10}}$
$\mathrm{Jac}(W^T)$	1	y	x	y^2	xy	x^2	y^3	xy^2	x^2y	xy^3	x^2y^2	x^2y^3

There is one 3-Loop with W^T of type S_{12}: $W = x^2z + xy^2 + yz^3$ with $G_W \cong \mu_{13}$.

H_W	$\mathbf{1}_J$	$\mathbf{1}_{J^{11}}$	$\mathbf{1}_{J^{10}}$	$\mathbf{1}_{J^9}$	$\mathbf{1}_{J^8}$	$\mathbf{1}_{J^7}$	$\mathbf{1}_{J^6}$	$\mathbf{1}_{J^5}$	$\mathbf{1}_{J^4}$	$\mathbf{1}_{J^3}$	$\mathbf{1}_{J^2}$	$\mathbf{1}_{J^{12}}$
$\mathrm{Jac}(W^T)$	1	z	x	y	z^2	xz	yz	xy	xz^2	yz^2	xyz	xyz^2

(4) **Mixed type.** Let $W = x^2 + xy^4 + z^3$. The mirror W^T is of type Q_{10}. Note $G_W \cong \mu_{24}$.

H_W	$\mathbf{1}_J$	$\mathbf{1}_{J^{19}}$	$\mathbf{1}_{J^{17}}$	$\mp 4y^3\mathbf{1}_{J^{16}}$	$\mathbf{1}_{J^{13}}$	$\mathbf{1}_{J^{11}}$	$\mp 4y^3\mathbf{1}_{J^8}$	$\mathbf{1}_{J^7}$	$\mathbf{1}_{J^5}$	$\mathbf{1}_{J^{23}}$
$\mathrm{Jac}(W^T)$	1	y	z	x	y^2	yz	xz	y^3	y^2z	y^3z

Let $W = x^2y + y^4 + z^3$. The mirror W^T is of type E_{14}. Note $G_W \cong \mu_{24}$.

H_W	$\mathbf{1}_J$	$\mathbf{1}_{J^{22}}$	$\mathbf{1}_{J^{19}}$	$\mathbf{1}_{J^{17}}$	$\mp 2y\mathbf{1}_{J^{16}}$	$\mathbf{1}_{J^{14}}$	$\mathbf{1}_{J^{13}}$	$\mathbf{1}_{J^{11}}$	$\mathbf{1}_{J^{10}}$	$\mp 2y\mathbf{1}_{J^8}$	$\mathbf{1}_{J^7}$	$\mathbf{1}_{J^5}$	$\mathbf{1}_{J^2}$	$\mathbf{1}_{J^{23}}$
$\mathrm{Jac}(W^T)$	1	z	z^2	x	y^2	xz	x	yz^2	yz	y^2z	yz^2	xy	xyz	xyz^2

5.2. *Four-point functions for exceptional unimodular singularities*

In the following, we provide the 4-point functions $\mathcal{F}_0^{(4)}(\mathbf{t})$ of the Frobenius manifold structure associated to the primitive form ζ for all the remaining 13 cases in Table 1. We mark the terms that give the B-side 4-point invariants corresponding to (1.1) by using boxes. We also provide the expression of ζ up to order 3. We remind the reader of $\tilde{\mathcal{F}}_0^{(4)}(\tilde{\mathbf{t}}) = -\mathcal{F}_0^{(4)}(\mathbf{t})$ as discussed in Section 4.1. We obtain the list with the help of a computer. The codes are written in Mathematica 8, and are available at http://member.ipmu.jp/changzheng.li/index.htm.

- Type E_{13}: $f = x^3 + xy^5$.

$$\{\phi_i\}_i = \{1, y, y^2, x, y^3, xy, y^4, y^2x, x^2, y^3x, yx^2, x^2y^2, y^3x^2\},$$

$$\zeta = 1 - \tfrac{4}{75}s_{12}s_{13} - \tfrac{1}{25}xs_{13}^2 + O(\mathbf{s}^4),$$

$$\begin{aligned}
-\mathcal{F}_0^{(4)} = & -\tfrac{3}{10}t_6t_7^3 - \tfrac{3}{5}t_5t_7^2t_8 + \tfrac{1}{10}t_5t_6t_8^2 + \tfrac{1}{15}t_3t_8^3 + \tfrac{1}{90}t_6^3t_9 + \tfrac{3}{5}t_5t_6t_7t_9 + \tfrac{2}{5}t_4t_7^2t_9 + \tfrac{1}{5}t_5^2t_8t_9 \\
& + \tfrac{1}{15}t_4t_6t_8t_9 + \tfrac{2}{5}t_3t_7t_8t_9 - \tfrac{1}{10}t_4t_5t_9^2 - \tfrac{1}{15}t_3t_6t_9^2 - \tfrac{1}{30}t_2t_8t_9^2 + \tfrac{1}{10}t_5t_6^2t_{10} \\
& - \tfrac{3}{10}t_5^2t_7t_{10} - \tfrac{3}{10}t_3t_7^2t_{10} + \tfrac{1}{5}t_3t_6t_8t_{10} + \tfrac{1}{10}t_2t_8^2t_{10} + \tfrac{1}{30}t_4^2t_9t_{10} + \tfrac{1}{5}t_3t_5t_9t_{10} \\
& + \tfrac{1}{5}t_2t_7t_9t_{10} + \tfrac{1}{10}t_2t_6t_{10}^2 + \tfrac{3}{10}t_5^2t_6t_{11} + \tfrac{1}{15}t_4t_6^2t_{11} + \tfrac{3}{5}t_4t_5t_7t_{11} + \tfrac{2}{5}t_3t_6t_7t_{11} \\
& + \tfrac{1}{15}t_4^2t_8t_{11} + \tfrac{2}{5}t_3t_5t_8t_{11} + \tfrac{1}{5}t_2t_7t_8t_{11} - \tfrac{2}{15}t_3t_4t_9t_{11} - \tfrac{1}{15}t_2t_6t_9t_{11} + \tfrac{1}{10}t_3^2t_{10}t_{11} \\
& + \tfrac{1}{5}t_2t_5t_{10}t_{11} - \tfrac{1}{30}t_2t_4t_{11}^2 + \tfrac{1}{5}t_4t_5^2t_{12} + \tfrac{1}{10}t_4^2t_6t_{12} + \tfrac{2}{5}t_3t_5t_6t_{12} + \tfrac{2}{5}t_3t_4t_7t_{12} \\
& + \tfrac{1}{5}t_2t_6t_7t_{12} + \tfrac{1}{5}t_3^2t_8t_{12} + \tfrac{1}{5}t_2t_5t_8t_{12} - \tfrac{1}{15}t_2t_4t_9t_{12} + \tfrac{1}{5}t_2t_3t_{10}t_{12} \boxed{+\tfrac{2}{45}t_4^3t_{13}} \\
& + \tfrac{1}{5}t_3t_4t_5t_{13} + \tfrac{1}{10}t_3^2t_6t_{13} + \tfrac{1}{5}t_2t_5t_6t_{13} + \tfrac{1}{5}t_2t_4t_7t_{13} + \tfrac{1}{5}t_2t_3t_8t_{13} \boxed{+\tfrac{1}{10}t_2^2t_{10}t_{13}}.
\end{aligned}$$

• Type E_{14}: $f = x^2 + xy^4 + z^3$.

$$
\begin{aligned}
\{\phi_i\}_i &= \{1, y, x, y^2, xy, y^3, xy^2, z, yz, xz, y^2z, xyz, y^3z, xy^2z\},\\
\zeta &= 1 + \tfrac{1}{64}s_{12}^2 s_{14} + \tfrac{1}{64}s_{10}s_{14}^2 + \tfrac{1}{48}ys_{12}s_{14}^2 + \tfrac{1}{192}y^2 s_{14}^3 + O(\mathbf{s}^4),\\
-\mathcal{F}_0^{(4)} &= -\tfrac{1}{16}t_3^2t_5t_9 + \tfrac{1}{8}t_5^2t_6t_9 + \tfrac{1}{4}t_4t_5t_7t_9 + \tfrac{1}{4}t_3t_6t_7t_9 + \tfrac{1}{8}t_2t_7^2t_9 - \tfrac{1}{8}t_3^2t_4t_{10} - \tfrac{1}{8}t_2t_3t_5t_{10}\\
&\quad + \tfrac{1}{2}t_4t_5t_6t_{10} + \tfrac{3}{8}t_3t_6^2t_{10} + \tfrac{1}{8}t_4^2t_7t_{10} + \tfrac{1}{4}t_2t_6t_7t_{10} + \tfrac{1}{6}t_8t_9^2t_{10} - \tfrac{1}{24}t_3^3t_{11} + \tfrac{1}{4}t_4t_5^2t_{11}\\
&\quad + \tfrac{1}{2}t_3t_5t_6t_{11} + \tfrac{1}{4}t_3t_4t_7t_{11} + \tfrac{1}{4}t_2t_5t_7t_{11} - \tfrac{1}{4}t_6^2t_7t_{11} + \tfrac{1}{6}t_8^2t_{10}t_{11} - \tfrac{1}{6}t_9^2t_{11}^2 - 1/9t_8t_{11}^3\\
&\quad - \tfrac{1}{16}t_2t_3^2t_{12} + \tfrac{1}{4}t_4^2t_5t_{12} + \tfrac{1}{2}t_3t_4t_6t_{12} + \tfrac{1}{4}t_2t_5t_6t_{12} - \tfrac{1}{6}t_6^3t_{12} + \tfrac{1}{4}t_2t_4t_7t_{12} + \tfrac{1}{6}t_8^2t_9t_{12}\\
&\quad + \tfrac{1}{2}t_3t_4t_5t_{13} + \tfrac{1}{8}t_2t_5^2t_{13} + \tfrac{3}{8}t_3^2t_6t_{13} - \tfrac{1}{2}t_5t_6^2t_{13} + \tfrac{1}{4}t_2t_3t_7t_{13} - \tfrac{1}{9}t_9^3t_{13} - \tfrac{1}{6}t_8^2t_{13}^2\\
&\quad \boxed{+\tfrac{1}{18}t_8^3t_{14}} - \tfrac{2}{3}t_8t_9t_{11}t_{13} + \tfrac{1}{8}t_3t_4^2t_{14} + \tfrac{1}{4}t_2t_4t_5t_{14} + \tfrac{1}{4}t_2t_3t_6t_{14} - \tfrac{1}{4}t_4t_6^2t_{14}\\
&\quad \boxed{+\tfrac{1}{8}t_2^2t_7t_{14}} - \tfrac{1}{2}t_4t_6t_7t_{13}.
\end{aligned}
$$

• Type Z_{11}: $f = x^3y + y^5$.

$$
\begin{aligned}
\{\phi_i\}_i &= \{1, y, x, y^2, xy, x^2, y^3, xy^2, y^4, xy^3, xy^4\},\\
\zeta &= 1 + \tfrac{17}{675}s_{10}s_{11}^2 + \tfrac{2}{81}ys_{11}^3 + O(\mathbf{s}^4),\\
-\mathcal{F}_0^{(4)} &= -\tfrac{5}{18}t_5t_6^3 + \tfrac{1}{3}t_4t_6^2t_7 + \tfrac{1}{15}t_4t_5t_7^2 - \tfrac{1}{90}t_3t_7^3 + \tfrac{1}{18}t_5^3t_8 + \tfrac{2}{15}t_4^2t_7t_8 + \tfrac{1}{3}t_3t_6t_7t_8\\
&\quad + \tfrac{1}{10}t_2t_7^2t_8 + \tfrac{1}{6}t_3t_5t_8^2 + \tfrac{1}{30}t_4^2t_5t_9 + \tfrac{1}{3}t_3t_5t_6t_9 + \tfrac{1}{3}t_2t_6^2t_9 - \tfrac{1}{15}t_3t_4t_7t_9 + \tfrac{1}{15}t_2t_5t_7t_9\\
&\quad + \tfrac{1}{15}t_2t_4t_8t_9 - \tfrac{1}{30}t_2t_3t_9^2 + \tfrac{1}{18}t_4^3t_{10} + \tfrac{1}{6}t_3t_5^2t_{10} + \tfrac{1}{3}t_3t_4t_6t_{10} + \tfrac{1}{5}t_2t_4t_7t_{10}\\
&\quad + \tfrac{1}{6}t_3^2t_8t_{10} + \tfrac{1}{30}t_2^2t_9t_{10} + \tfrac{1}{15}t_2t_4^2t_{11} \boxed{+\tfrac{1}{6}t_3^2t_5t_{11}} + \tfrac{1}{3}t_2t_3t_6t_{11} \boxed{+\tfrac{1}{15}t_2^2t_7t_{11}}.
\end{aligned}
$$

• Type Z_{12}: $f = x^3y + xy^4$.

$$
\begin{aligned}
\{\phi_i\}_i &= \{1, y, x, y^2, xy, x^2, y^3, xy^2, x^2y, xy^3, x^2y^2, x^2y^3\},\\
\zeta &= 1 - \tfrac{6}{121}s_{11}s_{12} - \tfrac{5}{121}ys_{12}^2 + \tfrac{29}{1331}s_{10}s_{12}^2 + \tfrac{9}{1331}xs_{12}^3 + O(\mathbf{s}^4),\\
-\mathcal{F}_0^{(4)} &= -\tfrac{10}{33}t_5t_6^3 + \tfrac{5}{22}t_5t_6^2t_7 + \tfrac{7}{22}t_5t_6t_7^2 - \tfrac{7}{22}t_5t_7^3 + \tfrac{3}{11}t_4t_6^2t_8 + \tfrac{4}{11}t_4t_6t_7t_8\\
&\quad - \tfrac{6}{11}t_4t_7^2t_8 + \tfrac{1}{11}t_4t_5t_8^2 + \tfrac{2}{11}t_3t_6t_8^2 - \tfrac{1}{22}t_3t_7t_8^2 + \tfrac{1}{22}t_2t_8^3 + \tfrac{1}{66}t_5^3t_9\\
&\quad - \tfrac{2}{11}t_4t_5t_6t_9 - \tfrac{2}{11}t_3t_6^2t_9 + \tfrac{6}{11}t_4t_5t_7t_9 + \tfrac{1}{11}t_3t_6t_7t_9 + \tfrac{4}{11}t_3t_7^2t_9 + \tfrac{2}{11}t_4^2t_8t_9\\
&\quad + \tfrac{1}{11}t_3t_5t_8t_9 - \tfrac{1}{11}t_2t_6t_8t_9 + \tfrac{3}{11}t_2t_7t_8t_9 - \tfrac{1}{11}t_3t_4t_9^2 - \tfrac{1}{22}t_2t_5t_9^2 + \tfrac{1}{22}t_4t_5^2t_{10}\\
&\quad + \tfrac{1}{22}t_4^2t_6t_{10} + \tfrac{4}{11}t_3t_5t_6t_{10} + \tfrac{7}{22}t_2t_6^2t_{10} - \tfrac{3}{22}t_4^2t_7t_{10} - \tfrac{1}{11}t_3t_5t_7t_{10} + \tfrac{1}{11}t_2t_6t_7t_{10}\\
&\quad - \tfrac{3}{22}t_2t_7^2t_{10} - \tfrac{1}{11}t_3t_4t_8t_{10} + \tfrac{1}{11}t_2t_5t_8t_{10} + \tfrac{1}{22}t_3^2t_9t_{10} + \tfrac{1}{11}t_2t_4t_9t_{10} - \tfrac{1}{22}t_2t_3t_{10}^2\\
&\quad + \tfrac{5}{22}t_4^2t_5t_{11} + \tfrac{1}{11}t_3t_5^2t_{11} + \tfrac{2}{11}t_3t_4t_6t_{11} - \tfrac{1}{11}t_2t_5t_6t_{11} + \tfrac{5}{11}t_3t_4t_7t_{11} + \tfrac{3}{11}t_2t_5t_7t_{11}\\
&\quad + \tfrac{1}{11}t_3^2t_8t_{11} + \tfrac{3}{11}t_2t_4t_8t_{11} - \tfrac{1}{11}t_2t_3t_9t_{11} + \tfrac{1}{22}t_2^2t_{10}t_{11} + \tfrac{1}{11}t_3t_4^2t_{12}\\
&\quad \boxed{+\tfrac{3}{22}t_3^2t_5t_{12}} + \tfrac{2}{11}t_2t_4t_5t_{12} + \tfrac{3}{11}t_2t_3t_6t_{12} + \tfrac{2}{11}t_2t_3t_7t_{12} \boxed{+\tfrac{1}{11}t_2^2t_8t_{12}}.
\end{aligned}
$$

• Type Z_{13}: $f = x^2 + xy^3 + yz^3$.

$$\begin{aligned}
\{\phi_i\}_i &= \{1, y, z, y^2, yz, x, z^2, y^2z, xy, xz, xy^2, xyz, xy^2z\},\\
\zeta &= 1 + \tfrac{7}{486}s_{12}^2 s_{13} + \tfrac{7}{486}s_{10}s_{13}^2 + \tfrac{5}{243}ys_{12}s_{13}^2 + \tfrac{2}{729}y^2 s_{13}^3 + O(\mathbf{s}^4),\\
-\mathcal{F}_0^{(4)} &= -\tfrac{1}{12}t_6^2t_7^2 - \tfrac{1}{6}t_5t_7^3 - \tfrac{5}{108}t_6^3t_8 - \tfrac{1}{6}t_5^2t_8^2 - \tfrac{1}{9}t_3t_8^3 - \tfrac{1}{18}t_5t_6^2t_9 + \tfrac{1}{3}t_4t_7^2t_9 + \tfrac{4}{9}t_4t_6t_8t_9\\
&\quad + \tfrac{1}{3}t_3t_7t_8t_9 + \tfrac{1}{9}t_4t_5t_9^2 + \tfrac{1}{36}t_3t_6t_9^2 + \tfrac{1}{6}t_2t_8t_9^2 + \tfrac{1}{18}t_5^3t_{10} - \tfrac{1}{6}t_4t_6^2t_{10} - \tfrac{1}{6}t_3t_6t_7t_{10}\\
&\quad + \tfrac{1}{3}t_3t_5t_8t_{10} - \tfrac{1}{6}t_2t_6t_9t_{10} + \tfrac{1}{9}t_4t_5t_6t_{11} + \tfrac{1}{36}t_3t_6^2t_{11} + \tfrac{1}{3}t_3t_5t_7t_{11} + \tfrac{2}{9}t_2t_4t_6t_{13}\\
&\quad + \tfrac{1}{3}t_2t_4t_9t_{12} - \tfrac{1}{9}t_4^2t_8t_{11} + \tfrac{1}{9}t_2t_6t_8t_{11} - \tfrac{1}{9}t_3t_4t_9t_{11} + \tfrac{1}{9}t_2t_5t_9t_{11} + \tfrac{1}{9}t_2t_4t_{10}t_{11}\\
&\quad + \tfrac{1}{3}t_2t_7^2t_{11} - \tfrac{1}{24}t_3^2t_{10}^2 + \tfrac{1}{6}t_3t_5^2t_{12} + \tfrac{5}{18}t_4^2t_6t_{12} - \tfrac{1}{12}t_2t_6^2t_{12} + \tfrac{1}{3}t_3t_4t_7t_{12} + \tfrac{1}{6}t_3^2t_8t_{12}\\
&\quad + \tfrac{1}{18}t_2^2t_{11}t_{12} - \tfrac{2}{27}t_4^3t_{13} \boxed{+\tfrac{1}{6}t_3^2t_5t_{13}} + \tfrac{1}{3}t_2t_3t_7t_{13} \boxed{+\tfrac{1}{9}t_2^2t_9t_{13}} - \tfrac{1}{18}t_2t_3t_{11}^2\\
&\quad + \frac{2}{9}t_4^2t_9t_{10}.
\end{aligned}$$

• Type W_{12}: $f = x^4 + y^5$.

$$\begin{aligned}
\{\phi_i\}_i &= \{1, y, x, y^2, xy, x^2, y^3, xy^2, x^2y, xy^3, x^2y^2, x^2y^3\},\\
\zeta &= 1 - \tfrac{1}{20}s_{11}s_{12} - \tfrac{1}{20}ys_{12}^2 + O(\mathbf{s}^4),\\
-\mathcal{F}_0^{(4)} &= \tfrac{1}{20}t_5^2t_7^2 + \tfrac{1}{8}t_5t_6^2t_8 + \tfrac{1}{5}t_4t_5t_7t_8 + \tfrac{1}{10}t_4^2t_8^2 + \tfrac{1}{10}t_2t_7t_8^2 + \tfrac{1}{8}t_5^2t_6t_9\\
&\quad + \tfrac{1}{10}t_4^2t_7t_9 + \tfrac{1}{10}t_2t_7^2t_9 + \tfrac{1}{4}t_3t_6t_8t_9 + \tfrac{1}{8}t_3t_5t_9^2 + \tfrac{1}{10}t_4^2t_5t_{10} + \tfrac{1}{8}t_3t_6^2t_{10}\\
&\quad + \tfrac{1}{5}t_2t_5t_7t_{10} + \tfrac{1}{5}t_2t_4t_8t_{10} + \tfrac{1}{20}t_2^2t_{10}^2 + \tfrac{1}{15}t_4^3t_{11} + \tfrac{1}{4}t_3t_5t_6t_{11}\\
&\quad + \tfrac{1}{5}t_2t_4t_7t_{11} + \tfrac{1}{8}t_3^2t_9t_{11} + \tfrac{1}{10}t_2t_4^2t_{12} \boxed{+\tfrac{1}{8}t_3^2t_6t_{12}} \boxed{+\tfrac{1}{10}t_2^2t_7t_{12}}.
\end{aligned}$$

• Type W_{13}: $f = x^2 + xy^2 + yz^4$.

$$\begin{aligned}
\{\phi_i\}_i &= \{1, z, y, z^2, yz, x, z^3, yz^2, xz, xy, xz^2, xyz, xyz^2\},\\
\zeta &= 1 - \tfrac{5}{64}s_{11}s_{13} + \tfrac{15}{1024}s_{12}^2s_{13} - \tfrac{1}{1}28ys_{13}^2 + \tfrac{3}{256}s_{10}s_{13}^2 + \tfrac{11}{512}zs_{12}s_{13}^2 + \tfrac{3}{512}z^2s_{13}^3\\
&\quad + O(\mathbf{s}^4),\\
-\mathcal{F}_0^{(4)} &= -\tfrac{3}{32}t_6^2t_7^2 - \tfrac{1}{6}t_5t_7^3 - \tfrac{1}{48}t_6^3t_8 - \tfrac{1}{4}t_4t_7^2t_8 - \tfrac{1}{8}t_5^2t_8^2 - \tfrac{3}{32}t_5t_6^2t_9 - \tfrac{1}{4}t_4t_6t_7t_9 + \tfrac{1}{4}t_4t_5t_8t_9\\
&\quad + \tfrac{1}{8}t_2t_8^2t_9 - \tfrac{1}{16}t_4^2t_9^2 - \tfrac{1}{8}t_3t_6t_9^2 - \tfrac{1}{16}t_2t_7t_9^2 + \tfrac{1}{16}t_5^2t_6t_{10} + \tfrac{1}{16}t_4t_6^2t_{10} + \tfrac{1}{2}t_4t_5t_7t_{10}\\
&\quad + \tfrac{3}{8}t_3t_7^2t_{10} + \tfrac{1}{8}t_4^2t_8t_{10} + \tfrac{1}{8}t_3t_6t_8t_{10} + \tfrac{1}{4}t_2t_7t_8t_{10} + \tfrac{1}{8}t_3t_5t_9t_{10} + \tfrac{1}{16}t_2t_6t_9t_{10}\\
&\quad - \tfrac{1}{8}t_3t_4t_{10}^2 - \tfrac{1}{16}t_2t_5t_{10}^2 + \tfrac{1}{8}t_4t_5^2t_{11} - \tfrac{1}{16}t_4^2t_6t_{11} - \tfrac{1}{8}t_3t_6^2t_{11} - \tfrac{1}{8}t_2t_6t_7t_{11}\\
&\quad + \tfrac{1}{4}t_2t_5t_8t_{11} - \tfrac{1}{8}t_2t_4t_9t_{11} + \tfrac{1}{16}t_3^2t_{10}t_{11} - \tfrac{1}{32}t_2^2t_{11}^2 + \tfrac{1}{4}t_4^2t_5t_{12} + \tfrac{1}{4}t_3t_5t_6t_{12}\\
&\quad + \tfrac{1}{32}t_2t_6^2t_{12} + \tfrac{1}{2}t_3t_4t_7t_{12} + \tfrac{1}{4}t_2t_5t_7t_{12} + \tfrac{1}{4}t_2t_4t_8t_{12} + \tfrac{1}{8}t_3^2t_9t_{12} - \tfrac{1}{8}t_2t_3t_{10}t_{12}\\
&\quad + \tfrac{1}{8}t_3t_4^2t_{13} + \tfrac{1}{4}t_2t_4t_5t_{13} \boxed{+\tfrac{3}{16}t_3^2t_6t_{13}} + \tfrac{1}{4}t_2t_3t_7t_{13} \boxed{+\tfrac{1}{8}t_2^2t_8t_{13}}.
\end{aligned}$$

- Type Q_{10}: $f = x^2y + y^4 + z^3$.

$$\begin{aligned}
\{\phi_i\}_i &= \{1, y, z, x, y^2, yz, xz, y^3, y^2z, y^3z\},\\
\zeta &= 1 + \tfrac{3}{128}s_9s_{10}^2 + \tfrac{11}{384}ys_{10}^3 + O(\mathbf{s}^4),\\
-\mathcal{F}_0^{(4)} &= \tfrac{1}{24}t_5^3t_6 + \tfrac{1}{18}t_3t_6^3 + \tfrac{1}{4}t_4t_5^2t_7 - \tfrac{1}{3}t_3^2t_7^2 + \tfrac{1}{4}t_4^2t_6t_8 + \tfrac{1}{8}t_2t_5t_6t_8 + \tfrac{1}{2}t_2t_4t_7t_8\\
&\quad + \tfrac{1}{4}t_4^2t_5t_9 + \tfrac{1}{8}t_2t_5^2t_9 + \tfrac{1}{6}t_3^2t_6t_9 + \tfrac{1}{16}t_2^2t_8t_9 \boxed{+\tfrac{1}{18}t_3^3t_{10}} \boxed{+\tfrac{1}{4}t_2t_4^2t_{10}} \boxed{+\tfrac{1}{16}t_2^2t_5t_{10}}.
\end{aligned}$$

- Type Q_{11}: $f = x^2y + y^3z + z^3$.

$$\begin{aligned}
\{\phi_i\}_i &= \{1, y, z, x, y^2, yz, z^2, xz, y^2z, yz^2, y^2z^2\},\\
\zeta &= 1 - \tfrac{5}{108}s_{10}s_{11} - \tfrac{1}{24}ys_{11}^2 + \tfrac{13}{648}s_9s_{11}^2 + \tfrac{25}{1944}zs_{11}^3 + O(\mathbf{s}^4),\\
-\mathcal{F}_0^{(4)} &= \tfrac{1}{36}t_5t_6^3 + \tfrac{1}{4}t_5^2t_6t_7 + \tfrac{1}{36}t_3t_6^2t_7 - \tfrac{1}{24}t_4^2t_7^2 - \tfrac{1}{9}t_3t_5t_7^2 - \tfrac{1}{18}t_2t_6t_7^2 + \tfrac{1}{2}t_4t_5t_6t_8\\
&\quad - \tfrac{1}{6}t_3t_4t_7t_8 - \tfrac{1}{4}t_3^2t_8^2 - \tfrac{1}{12}t_5^3t_9 + \tfrac{1}{4}t_4^2t_6t_9 + \tfrac{1}{12}t_2t_6^2t_9 + \tfrac{1}{36}t_3^2t_7t_9\\
&\quad + \tfrac{1}{6}t_2t_5t_7t_9 + \tfrac{1}{2}t_2t_4t_8t_9 + \tfrac{1}{4}t_4^2t_5t_{10} + \tfrac{1}{4}t_3t_5^2t_{10} + \tfrac{1}{12}t_3^2t_6t_{10} + \tfrac{1}{3}t_2t_5t_6t_{10}\\
&\quad - \tfrac{1}{9}t_2t_3t_7t_{10} + \tfrac{1}{12}t_2^2t_9t_{10} \boxed{+\tfrac{5}{108}t_3^3t_{11}} \boxed{+\tfrac{1}{4}t_2t_4^2t_{11}} + \tfrac{1}{6}t_2t_3t_5t_{11} \boxed{+\tfrac{1}{12}t_2^2t_6t_{11}}.
\end{aligned}$$

- Type Q_{12}: $f = x^2y + xy^3 + z^3$.

$$\begin{aligned}
\{\phi_i\}_i &= \{1, x, y, xy, y^2, xy^2, z, xz, yz, xyz, y^2z, xy^2z\},\\
\zeta &= 1 + \tfrac{1}{75}s_{10}^2s_{12} + \tfrac{1}{75}s_8s_{12}^2 + \tfrac{1}{50}ys_{10}s_{12}^2 + \tfrac{1}{25}s_{11}s_{12}^2 + O(\mathbf{s}^4),\\
-\mathcal{F}_0^{(4)} &= -\tfrac{3}{10}t_2^2t_4t_8 - \tfrac{1}{10}t_3t_4^2t_8 + \tfrac{1}{5}t_2t_4t_5t_8 + \tfrac{3}{10}t_4t_5^2t_8 + \tfrac{2}{5}t_2t_3t_6t_8 + \tfrac{1}{5}t_3t_5t_6t_8 \boxed{+\tfrac{1}{10}t_3^2t_4t_{12}}\\
&\quad - \tfrac{1}{10}t_2t_4^2t_9 + \tfrac{1}{5}t_4^2t_5t_9 + \tfrac{1}{5}t_2^2t_6t_9 + \tfrac{1}{5}t_3t_4t_6t_9 + \tfrac{1}{5}t_2t_5t_6t_9 - \tfrac{1}{5}t_5^2t_6t_9 + \tfrac{1}{6}t_7t_8t_9^2\\
&\quad - \tfrac{1}{36}t_9^4 + \tfrac{1}{5}t_2t_3t_5t_{12} - \tfrac{1}{5}t_2t_3t_4t_{10} + \tfrac{1}{10}t_2^2t_5t_{10} + \tfrac{2}{5}t_3t_4t_5t_{10} + \tfrac{3}{10}t_2t_5^2t_{10} - \tfrac{1}{5}t_5^3t_{10}\\
&\quad + \tfrac{1}{10}t_3^2t_6t_{10} - \tfrac{1}{10}t_2^3t_{10} + \tfrac{1}{10}t_2^2t_4t_{11} + \tfrac{1}{5}t_3t_4^2t_{11} + \tfrac{3}{5}t_2t_4t_5t_{11} - \tfrac{3}{5}t_4t_5^2t_{11}\\
&\quad + \tfrac{1}{5}t_2t_3t_6t_{11} - \tfrac{2}{5}t_3t_5t_6t_{11} + \tfrac{1}{6}t_7^2t_8t_{11} - \tfrac{1}{3}t_7t_9^2t_{11} - \tfrac{1}{6}t_7^2t_{11}^2 \boxed{+\tfrac{1}{5}t_2^2t_3t_{12}}\\
&\quad - \tfrac{1}{5}t_3t_5^2t_{12} \boxed{+\tfrac{1}{18}t_7^3t_{12}} + \tfrac{1}{6}t_7^2t_9t_{10} - \tfrac{1}{4}t_7^2t_8^2
\end{aligned}$$

- Type S_{11}: $f = x^2y + y^2z + z^4$.

$$\begin{aligned}
\{\phi_i\}_i &= \{1, z, x, y, z^2, xz, yz, z^3, xz^2, yz^2, yz^3\},\\
\zeta &= 1 - \tfrac{3}{64}s_{10}s_{11} - \tfrac{7}{128}zs_{11}^2 + \tfrac{9}{512}s_8s_{11}^2 + \tfrac{5}{1024}ys_{11}^3 + O(\mathbf{s}^4),\\
-\mathcal{F}_0^{(4)} &= -\tfrac{5}{32}t_5^2t_6^2 + \tfrac{1}{48}t_5^3t_7 + \tfrac{1}{4}t_4t_6^2t_7 + \tfrac{1}{4}t_3t_6t_7^2 - \tfrac{1}{16}t_4t_5^2t_8 - \tfrac{1}{8}t_3t_5t_6t_8\\
&\quad - \tfrac{1}{16}t_2t_6^2t_8 + \tfrac{1}{8}t_4^2t_7t_8 + \tfrac{1}{16}t_2t_5t_7t_8 - \tfrac{1}{32}t_3^2t_8^2 - \tfrac{1}{16}t_2t_4t_8^2 - \tfrac{1}{16}t_3t_5^2t_9\\
&\quad - \tfrac{1}{2}t_2t_5t_6t_9 + \tfrac{1}{2}t_3t_4t_7t_9 - \tfrac{1}{8}t_2t_3t_8t_9 - \tfrac{1}{8}t_2^2t_9^2 + \tfrac{1}{8}t_4^2t_5t_{10} + \tfrac{1}{8}t_2t_5^2t_{10}\\
&\quad + \tfrac{1}{2}t_3t_4t_6t_{10} + \tfrac{1}{4}t_3^2t_7t_{10} + \tfrac{1}{32}t_2^2t_8t_{10} \boxed{+\tfrac{1}{4}t_3^2t_4t_{11}} \boxed{+\tfrac{1}{8}t_2t_4^2t_{11}} \boxed{+\tfrac{3}{32}t_2^2t_5t_{11}}.
\end{aligned}$$

• Type S_{12}: $f = x^2y + y^2z + xz^3$.

$$\{\phi_i\}_i = \{1, z, x, y, z^2, xz, yz, xy, xz^2, yz^2, xyz, xyz^2\},$$

$$\zeta = 1 - \tfrac{12}{169}s_{10}s_{12} + \tfrac{30}{2197}s_{11}^2s_{12} - \tfrac{2}{169}xs_{12}^2 + \tfrac{20}{2197}s_8s_{12}^2 + \tfrac{93}{4394}zs_{11}s_{12}^2 + \tfrac{9}{2197}z^2s_{12}^3 + O(\mathbf{s}^4),$$

$$\begin{aligned}-\mathcal{F}_0^{(4)} = {} & -\tfrac{5}{156}t_6^4 + \tfrac{1}{13}t_5t_6^2t_7 - \tfrac{1}{13}t_5^2t_7^2 - \tfrac{1}{13}t_4t_6t_7^2 - \tfrac{1}{26}t_3t_7^3 + \tfrac{5}{26}t_5^2t_6t_8 \\ & + \tfrac{1}{26}t_4t_6^2t_8 + \tfrac{2}{13}t_4t_5t_7t_8 + \tfrac{1}{13}t_3t_6t_7t_8 + \tfrac{1}{26}t_2t_7^2t_8 - \tfrac{3}{52}t_4^2t_8^2 - \tfrac{2}{13}t_3t_5t_8^2 \\ & - \tfrac{1}{13}t_2t_6t_8^2 - \tfrac{1}{39}t_5^3t_9 - \tfrac{2}{13}t_4t_5t_6t_9 - \tfrac{1}{13}t_3t_6^2t_9 + \tfrac{1}{13}t_4^2t_7t_9 + \tfrac{1}{13}t_3t_5t_7t_9 \\ & + \tfrac{1}{13}t_2t_6t_7t_9 + \tfrac{1}{13}t_3t_4t_8t_9 + \tfrac{1}{13}t_2t_5t_8t_9 - \tfrac{1}{26}t_3^2t_9^2 - \tfrac{1}{13}t_2t_4t_9^2 - \tfrac{1}{13}t_4t_5^2t_{10} \\ & + \tfrac{1}{26}t_4^2t_6t_{10} + \tfrac{1}{13}t_3t_5t_6t_{10} + \tfrac{2}{13}t_2t_6^2t_{10} - \tfrac{3}{13}t_3t_4t_7t_{10} - \tfrac{2}{13}t_2t_5t_7t_{10} + \tfrac{1}{26}t_3^2t_8t_{10} \\ & + \tfrac{1}{13}t_2t_4t_8t_{10} + \tfrac{1}{13}t_2t_3t_9t_{10} - \tfrac{1}{26}t_2^2t_{10}^2 + \tfrac{5}{26}t_4^2t_5t_{11} + \tfrac{7}{26}t_3t_5^2t_{11} + \tfrac{3}{13}t_3t_4t_6t_{11} \\ & + \tfrac{4}{13}t_2t_5t_6t_{11} + \tfrac{3}{26}t_3^2t_7t_{11} + \tfrac{1}{13}t_2t_4t_7t_{11} - \tfrac{2}{13}t_2t_3t_8t_{11} + \tfrac{1}{26}t_2^2t_9t_{11} \boxed{+\tfrac{5}{26}t_3^2t_4t_{12}} \\ & \boxed{+\tfrac{2}{13}t_2t_4^2t_{12}} + \tfrac{3}{13}t_2t_3t_5t_{12} \boxed{+\tfrac{3}{26}t_2^2t_6t_{12}}.\end{aligned}$$

• Type U_{12}: $f = x^3 + y^3 + z^4$.

$$\{\phi_i\}_i = \{1, z, x, y, z^2, xz, yz, xy, xz^2, yz^2, xyz, xyz^2\},$$

$$\zeta = 1 + \tfrac{1}{72}s_{11}^2s_{12} + \tfrac{1}{72}s_8s_{12}^2 + \tfrac{1}{36}zs_{11}s_{12}^2 + \tfrac{1}{72}z^2s_{12}^3 + O(\mathbf{s}^4),$$

$$\begin{aligned}-\mathcal{F}_0^{(4)} = {} & \tfrac{1}{8}t_5^2t_6t_7 + \tfrac{1}{6}t_3t_6^2t_8 + \tfrac{1}{6}t_4t_7^2t_8 + \tfrac{1}{4}t_2t_5t_7t_9 + \tfrac{1}{6}t_3^2t_8t_9 + \tfrac{1}{4}t_2t_5t_6t_{10} + \tfrac{1}{6}t_4^2t_8t_{10} \\ & + \tfrac{1}{8}t_2^2t_9t_{10} + \tfrac{1}{8}t_2t_5^2t_{11} + \tfrac{1}{6}t_3^2t_6t_{11} + \tfrac{1}{6}t_4^2t_7t_{11} \boxed{+\tfrac{1}{18}t_3^3t_{12}} \boxed{+\tfrac{1}{18}t_4^3t_{12}} \boxed{+\tfrac{1}{8}t_2^2t_5t_{12}}.\end{aligned}$$

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