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Linear groups with Borel's property

Abstract. When does Borel's theorem on free subgroups of semisimple groups generalize to other groups? We initiate a systematic study of this question and find positive and negative answers for it. In particular, we fully classify fundamental groups of surfaces and von Dyck groups that satisfy Borel's theorem. Further, as a byproduct of this theory, we make headway on a question of Breuillard, Green, Guralnick, and Tao concerning double word maps.

Keywords. Free groups, linear groups, Borel's property

Contents

1. Introduction

Let Γ be a group. *What group-theoretic properties of Γ can we infer from the flexibility of its representation theory?* To systematically approach this basic question, we focus on the following property: Γ has *Borel's property* if for every connected semisimple group G,

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every proper subvariety V of G, and every nontorsion $\gamma \neq 1$ in Γ, there exists a homomorphism $\phi \colon \Gamma \to G(\mathbb{C})$ such that $\phi(\gamma) \notin V(\mathbb{C})$. Loosely speaking, groups with Borel's property have so many representations that not only can every element be detected in any semisimple group, but any element can be made to miss every proper subvariety.

Question 1. *Which classes of groups have Borel's property?*

In 1983, Armand Borel demonstrated the remarkable fact that free groups have Borel's property. Question 1 also sheds light on double word maps (see §4) and, in fact, gives a partial answer to [BGGT12, Problem 2]. That is, we obtain

Theorem 1.1. *Let w_1, w_2 be two words in a free group, F_k, of rank k with w_2 not in the normal closure of w_1, and $\Gamma_{w_1} := F_k/\langle\langle w_1 \rangle\rangle$ a torsion-free group with Borel's property. If $\mathrm{Hom}(\Gamma_{w_1}, G)$ is integral for every semisimple group G, then the double word map, $G^k \to G \times G$, defined by w_1, w_2 is dominant. In particular, if k is even, one can choose w_1 to be a word of the form*

$$[x_1, x_2] \cdots [x_{k-1}, x_k],$$

where $k \geq 2$.

The remainder of our main results revolve around applications of tools we have developed for determining whether a group has Borel's property. Let $\mathscr{L}$ be the class of groups that satisfy Borel's property. Let $\mathscr{B}$ be the class of torsion-free groups in $\mathscr{L}$. We start with a complete classification of fundamental groups of surfaces that are in $\mathscr{L}$, which indicates that our new line of study is not an empty theory.

Theorem 1.2. *Let S be a compact surface without boundary. Then $\pi_1(S, \cdot)$ is in $\mathscr{L}$ if and only if S is not the Klein bottle. In particular, $\pi_1(S, \cdot)$ is in $\mathscr{B}$ if and only if S is neither the Klein bottle nor the real projective plane.*

The examples in Theorem 1.2 are handled in different parts of the paper. The Klein bottle group is handled in Corollary 3.18. In studying this case, we discovered a Tits alternative for $\mathscr{B}$:

Theorem 1.3. *Let Γ be a finitely generated group that is in $\mathscr{B}$. Then Γ contains a nonabelian free group or is a free abelian group.*

See §2.3 for a proof. Since the fundamental group of an oriented surface injects into a direct product of free groups [Bau62], all such groups are in $\mathscr{B}$ (see Lemma 2.4), and the same can be said for connected sums of four or more real projective planes. We are left with the fundamental group of the connected sum of three projective planes. This group, which has presentation

$$\pi_1 := \langle a, b, c : a^2 b^2 c^2 = 1 \rangle,$$

does not inject into a direct product of free groups [LS62] (for another example of a group in $\mathscr{B}$ that is not residually free, see Theorem 3.12). Handling π_1 requires new machinery that we develop in §3.1 (see also Proposition 3.10).

In §2, we show that in the context of finitely generated groups

$$\text{residually free} \subseteq \mathscr{B} \subseteq \mathscr{L} \cap \text{linear} \subseteq \mathscr{L}. \tag{1}$$

The group π_1, discussed above, shows that the first inclusion is strict. The last two inclusions are shown to be strict in §2. Significant examples of linear groups in $\mathscr{L}$ which are not in $\mathscr{B}$ are supplied by the next two theorems.

Theorem 1.4. *For $\ell \equiv 1 \pmod 3$ prime, the group $\mathbb{Z}/\ell * \mathbb{Z}/\ell$ is in $\mathscr{L}$.*

Theorem 1.5. *Let $\ell \geq 19$ be a prime that is $\equiv 1 \pmod 3$. The group*

$$\langle x, y, z, t : x^\ell = y^\ell = z^\ell = t^\ell = xyzt = 1 \rangle$$

is in $\mathscr{L}$.

The proofs of Theorems 1.4 and 1.5 appear in §3.2. The proof of Theorem 1.4 relies on a delicate strengthening of Borel's original proof. The proof of the latter theorem follows a similar track while relying, in addition, on a method developed by Avraham Aizenbud and Nir Avni [AA16] and a new character theory estimate established in Appendix B. The three previous theorems might lead to some hope that all Fuchsian groups are in $\mathscr{L}$. However, this is certainly not the case:

Theorem 1.6. *No group containing infinitely many elements of order 2 is in $\mathscr{L}$, and therefore no group containing the infinite dihedral group is in $\mathscr{L}$. Moreover, groups such as a nonoriented Fuchsian group or an oriented Fuchsian group with an elliptic point of order 2, cannot be in $\mathscr{L}$.*

It turns out that even subgroups consisting of orientation-preserving isometries in triangle groups are not in $\mathscr{L}$. Recall that a hyperbolic von Dyck group has presentation

$$\Gamma = \langle x, y, z : x^a = y^b = z^c = xyz = 1 \rangle,$$

where a, b, c are positive integers with $1/a + 1/b + 1/c < 1$.

Theorem 1.7. *No hyperbolic von Dyck group is in $\mathscr{L}$.*

The proofs of Theorems 1.6 and 1.7 appear in §3.3. It would be interesting to understand, in general, which Fuchsian groups are in $\mathscr{L}$.

This article is organized as follows. In §2 basic notions are defined, groups in $\mathscr{B}$ are shown to be linear, and a Tits alternative for $\mathscr{B}$ is established. Sections 3.1 and 3.2 give general methods for proving a group is in $\mathscr{B}$ or $\mathscr{L}$, respectively. Section 3.3 discusses obstructions to membership in $\mathscr{B}$ or $\mathscr{L}$. In §4 we give a partial answer to a question of Emmanuel Breuillard, Ben Green, Robert Guralnick, and Terence Tao concerning double word maps. For the convenience of the reader, Appendix A collects some basic definitions and facts from algebraic geometry which are used in the paper. Appendix B gives some bounds on irreducible characters of certain finite groups that are used in §3.2.

2. General theory

In this section, we present some basic results and examples that we hope will cast light on Question 1. Before we begin, we need some notation that allows us to succinctly describe groups that satisfy Borel's property to varying degrees for a given linear algebraic group G.

2.1. Notation and terminology

For notation and terminology regarding algebraic geometry, see Appendix A. Here we note only that we do not assume that our varieties are either irreducible or reduced, but for our purposes infinitesimal structure will never matter. Algebraic groups will always be linear and will be assumed to be defined over $\mathbb{C}$ unless some other field is specified explicitly.

Let Γ be a finitely generated group. Denote by $\Gamma^\bullet$ the set $\Gamma \setminus \{1\}$. Let $\mathcal{P}$ be a class of groups. We say that $S \subseteq \Gamma^\bullet$ is *detected by* $\mathcal{P}$ if there exists a homomorphism $\phi : \Gamma \to P$ with $P \in \mathcal{P}$ such that $\phi(S) \cap \{1\} = \emptyset$. We say that $S \subseteq \Gamma$ is *almost detected by* $\mathcal{P}$ if there exists a homomorphism $\phi : \Gamma \to P$ with $P \in \mathcal{P}$ such that $\phi(s)$ is nontorsion for every nontorsion $s \in S$. When $S = \{\gamma\}$ with γ nontrivial, we sometimes say that γ *is [almost] detected by* $\mathcal{P}$ instead of $\{\gamma\}$ is [almost] detected by $\mathcal{P}$. If every element in $\Gamma^\bullet$ is [almost] detected by $\mathcal{P}$ we say that Γ is [*almost*] *detectable by* $\mathcal{P}$ or is [*almost*] *residually* $\mathcal{P}$.

Let G be a linear algebraic group defined over $\mathbb{C}$. Let V be a subvariety of G. We say $S \subseteq \Gamma$ is *detected by G rel V* if there exists a homomorphism $\phi : \Gamma \to G(\mathbb{C})$ such that $\phi(S) \cap V = \emptyset$. If $S = \{\gamma\}$ we sometimes say γ *is detected by G rel V* instead of S is detected by G rel V. If every element in $\Gamma^\bullet$ is detected by G rel V for every proper closed subvariety V of G, then we say that Γ is G-*free*. If every nontorsion element in Γ is detected by G rel V for every proper closed subvariety V of G, then we say that Γ is *almost G-free*. Equivalently, Γ is G-free (resp. almost G-free) if and only if the *evaluation map*

$$e_{G,\gamma} : \operatorname{Hom}(\Gamma, G(\mathbb{C})) \to G(\mathbb{C})$$

has dense image for all $\gamma \neq 1$ (resp. all nontorsion γ). In the definition of [almost] G-free, if the representations can be taken to be faithful, we say that the group is [*almost*] G-*faithful*.

2.2. Some basic results

We start by showing that groups in $\mathscr{B}$ are precisely those in $\mathscr{L}$ that are G-free for some connected semisimple G.

Lemma 2.1. *Let G be a connected semisimple algebraic group. Then any G-free group must be torsion-free.*

Proof. Let Γ be a group with an element γ of order k, where $k \in \mathbb{Z}^{>0}$. For any map $\phi : \Gamma \to G(\mathbb{C})$, we have $\phi(\gamma) \in \{A \in G : A^k = 1\}$. If e_{G,x^k} denotes the kth power map on G, then $e_{G,x^k}^{-1}(G)$ is a proper closed subvariety of G, and γ cannot be detected by G rel $e_{G,x^k}^{-1}(G)$. $\qquad\square$

The following lemma is a slight strengthening of the fact that free groups are in $\mathscr{B}$. While the result is known, we include a proof for completeness.

Lemma 2.2. *Finitely generated free groups are G-faithful for all connected semisimple linear algebraic G.*

Proof. Let F_d be a finitely generated free group of rank d. Let V be a subvariety of G, where G is a connected semisimple linear algebraic G. By Borel's Theorem [Bor83], we have that

$$L_\gamma := \left\{ \phi \in \mathrm{Hom}(F_d, G(\mathbb{C})) : \phi(\gamma) \notin V \cup \{1\} \right\}$$

is nonempty for every $\gamma \in F_d^\bullet$. Each complement of L_γ consists of the set of closed points of a proper closed subvariety of $\mathrm{Hom}(F_d, G) \cong G^d$ and is therefore a closed subset of $G(\mathbb{C})^d$ without interior points. As F_d is countable,

$$\bigcap_{\gamma \in F_d^\bullet} L_\gamma$$

is nonempty by the Baire Category Theorem (Theorem A.5). Any representation lying in this intersection is faithful and satisfies $\phi(F_d^\bullet) \cap V = \emptyset$, so F is G-faithful for any semisimple G. $\square$

The following lemmas are useful tools for constructing elements in $\mathscr{L}$ or $\mathscr{B}$. In particular, by the next lemma, it follows that any residually free group is in $\mathscr{B}$.

Lemma 2.3. *Let $\mathcal{P}$ be a class of [almost] G-free groups. If Γ is [almost] detectable by $\mathcal{P}$, then Γ is [almost] G-free.*

Proof. Let $\gamma \in \Gamma^\bullet$ be a given [torsion-free] element. Let V be an arbitrary subvariety of G. Since Γ is [almost] detectable by $\mathcal{P}$, there exists a homomorphism $\phi : \Gamma \to P \in \mathcal{P}$ with $\phi(\gamma) \neq 1$ [$\phi(\gamma)$ torsion-free]. Since P is [almost] G-free, there exists a homomorphism $\psi : P \to G$ with $\psi(\phi(\gamma)) \notin V$. Thus, the map $\psi \circ \phi : \Gamma \to G$ has $\psi \circ \phi(\gamma) \notin V$, as desired. $\square$

The next lemmas demonstrate that $\mathscr{L}$, like the class of residually free groups, is closed under direct products and passage to subgroups.

Lemma 2.4. *Finite direct products of [almost] G-free groups are [almost] G-free.*

Proof. Let G be a linear algebraic group and V an arbitrary subvariety of G. Let $\Gamma = \prod_{i=1}^N \Gamma_i$ be a direct product of [almost] G-free groups. Set $\mathcal{P} = \{\Gamma_i\}_{i=1}^N$. Using the natural projections onto Γ_i, we see that Γ is [almost] detectable by $\mathcal{P}$. Lemma 2.3 then implies that Γ is [almost] G-free. $\square$

Lemma 2.5. *Subgroups of [almost] G-free groups are [almost] G-free.*

Proof. Let Γ be an [almost] G-free group and $\Delta \leq \Gamma$. By using the injection map $\phi : \Delta \to \Gamma$ induced by $\Delta \leq \Gamma$, we see that Δ is [almost] residually $\{\Gamma\}$. As Γ is [almost] G-free, we conclude that Δ is [almost] G-free by Lemma 2.3. $\square$

The next lemma will be used in §3.

Lemma 2.6. *Let Γ be a G_i-free group for each $i = 1, \ldots, k$. Then Γ is $\prod_{i=1}^{k} G_i$-free.*

Proof. Let $G = \prod_i G_i$. Then

$$\mathrm{Hom}(\Gamma, G) = \prod_i \mathrm{Hom}(\Gamma, G_i), \quad e_{G,\gamma}(\mathrm{Hom}(\Gamma, G)) = \prod_i e_{G_i,\gamma}(\mathrm{Hom}(\Gamma, G_i)).$$

Given a finite collection of topological spaces X_i and dense subsets $D_i \subset X_i$, by definition of product topology, $\prod_i D_i$ is dense in $\prod_i X_i$. Applying this to $D_i := e_{G_i,\gamma}$ and $X_i := G_i$, we obtain the lemma. □

2.3. Connections with linearity and a Tits alternative

The next proposition, coupled with the fact that finite groups are never G-free (Lemma 2.1), shows that for finitely generated groups, being G-free is a stronger condition than being linear.

Remark 1. It is not true that an almost G-free group is necessarily linear. The first Grigorchuk group is a finitely generated [dlH00, Corollary VIII.15], residually finite [dlH00, Proposition VIII.6], nonlinear [dlH00, Corollary VIII.19] group consisting only of torsion elements [dlH00, Theorem VIII.17]. Hence, it is almost G-free for any semisimple G, but is not linear. It now follows that the rightmost inclusion in (1) cannot be reversed.

Proposition 2.7. *If Γ is a finitely generated group that is G-free, then Γ is linear. If Γ is a finitely generated group that is almost G-free, then Γ is an extension of a linear group by a torsion group.*

Proof. Since Γ is finitely generated, its representation variety is a subvariety of $G^{\mathrm{rank}(\Gamma)}$, and thus has finitely many irreducible components (see Appendix A). Let Φ be the finite collection of irreducible components of $\mathrm{Hom}(\Gamma, G)$. For each nontorsion $\gamma \in \Gamma$, there exists some $\Omega \in \Phi$ such that $e_{G,\gamma}(\Omega)$ is dense in G. For any $\Omega \in \Phi$, let S_Ω denote the collection of γ for which this density condition holds. Since Γ is almost G-free, $\bigcup_{\Omega \in \Phi} S_\Omega$ consists of all nontorsion elements of Γ.

By the Baire Category Theorem (Theorem A.5), the intersection

$$\bigcap_{\gamma \in S_\Omega} \{\phi \in \Omega : \phi(\gamma) \neq 1\}$$

is always a nonempty subset (note that by convention, $\bigcap_{\gamma \in \emptyset} \{\phi \in \Omega : \phi(\gamma) \neq 1\} = \Omega$). For each $\Omega \in \Phi$, we select a single ϕ_Ω from this nonempty set. The kernel of the natural homomorphism from Γ to

$$\prod_{\Omega \in \Phi} \Gamma/\mathrm{ker}(\phi_\Omega) \tag{2}$$

contains only torsion elements since no $\gamma \in S_\Omega$ lies in $\mathrm{ker}(\phi_\Omega)$. On the other hand, each $\Gamma/\mathrm{ker}(\phi_\Omega)$ can be realized as a subgroup of G via ϕ_Ω. Thus, Γ is an extension of a linear group by a torsion group, and if Γ is torsion-free, it is linear. □

The next lemma demonstrates that virtually solvable groups that are in $\mathscr{B}$ are actually virtually free abelian. It is possible, however, for a solvable group that is not virtually abelian to be in $\mathscr{L}$:

Remark 2. Let Γ be the lamplighter group $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$. Set $\Delta = \bigoplus_{i \in \mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ to be the base group of Γ so $\Gamma/\Delta \cong \mathbb{Z}$. Every element in Δ is of order 2. It follows from Lemma 2.3 that the lamplighter group is almost G-free for any semisimple group G.

Lemma 2.8. *Let Γ be a finitely generated group. If Γ is virtually solvable and G-free for some semisimple group G then it is virtually free abelian.*

Proof. The unipotent elements in G form a proper closed subvariety as G is semisimple. Following the proof of Proposition 2.7, there exist homomorphisms $\phi_\Omega : \Gamma \to G(\mathbb{C})$, one for each component of $\mathrm{Hom}(\Gamma, G)$, such that $\phi_\Omega(\gamma)$ is not unipotent for all $\gamma \in S_\Omega$. If Q_Ω denotes the Zariski closure of $\phi_\Omega(\Gamma)$, then each Q_Ω is virtually solvable, so each identity component Q_Ω° is connected solvable and therefore contained in a Borel subgroup $B_\Omega \subset G$. If

$$\Gamma^\circ := \Gamma \cap \bigcap_\Omega \phi_\Omega^{-1}(Q_\Omega^\circ(\mathbb{C})),$$

then Γ° is of finite index in Γ. If $\gamma \in [\Gamma^\circ, \Gamma^\circ]$, then

$$\phi_\Omega(\gamma) \in [B_\Omega(\mathbb{C}), B_\Omega(\mathbb{C})]$$

is unipotent, so $\gamma \notin S_\Omega$ for all Ω. It follows that $\gamma = 1$, which means that Γ is virtually abelian. $\square$

We now present a version of Tits alternative for groups in $\mathscr{B}$. Note that the condition that Γ be finitely generated is needed. For instance, $\mathbb{Q}$ is in $\mathscr{B}$ (the image of any element is dense in a maximal torus) but is not virtually free abelian.

Proof of Theorem 1.3. By Tits' alternative and Proposition 2.7, Γ is virtually solvable or contains a nonabelian free group. We can therefore assume that Γ is virtually solvable. By Lemma 2.8, the group Γ is virtually free abelian. Since Γ is torsion-free by Lemma 2.1, we are done by Theorem 3.17 below. $\square$

3. Determining when a group satisfies Borel's Theorem

3.1. Conditions for torsion-free groups

In this section, we present several variants of Borel's original proof that free groups are G-free for all semisimple groups G.

Lemma 3.1. *Let Γ be a finitely generated group. If Γ is SL_n-free for all $n \geq 2$, then Γ is G-free for all semisimple G.*

Proof. Let $e_{G,\gamma} : \mathrm{Hom}(\Gamma, G) \to G$ denote the evaluation map at γ. For any homomorphism $\phi : H \to G$, $e_{G,\gamma}$ contains $\phi(e_{H,\gamma}(\mathrm{Hom}(\Gamma, H)))$ and is closed under conjugation by G. Therefore, if $e_{H,\gamma}$ has Zariski-dense image in H, and the union of G-conjugates of $\phi(H)$ is Zariski-dense in G, then $e_{G,\gamma}(\mathrm{Hom}(\Gamma, G))$ is Zariski-dense in G. Also, if $e_{H_i,\gamma}$ has Zariski-dense image for $i = 1, \ldots, n$, then $e_{\prod_i H_i, \gamma}$ has Zariski-dense image (see Lemma 2.6). Every semisimple group admits a surjective homomorphism from

a product of simply connected groups which are simple modulo center, and every semi-simple group G which is simple modulo center admits a homomorphism from a group of the form $\prod_i \mathrm{SL}_{n_i}$ whose image contains a maximal torus of G, and therefore has the property that the union of G-conjugates is Zariski-dense. (See, e.g., [Lar01, Lemma 4] for the fact that every simple algebraic group admits a subgroup of type A of maximal rank.)   □

Theorem 3.2. *Let Γ be a finitely generated group such that*

(1) $\mathrm{Hom}(\Gamma, \mathrm{SL}_n)$ *is irreducible for all $n \geq 4$.*
(2) Γ *is SL_2-free.*
(3) Γ *is SL_3-free.*

Then Γ is G-free for all semisimple groups G.

Proof. Let $e_{n,\gamma}\colon \mathrm{Hom}(\Gamma, \mathrm{SL}_n) \to \mathrm{SL}_n$ denote the evaluation map at γ. Let $X_{n,\gamma}$ denote the closure of the image of $e_{n,\gamma}$. For $\gamma \neq 1$, by (2) and (3), $X_{2,\gamma} = \mathrm{SL}_2$ and $X_{3,\gamma} = \mathrm{SL}_3$. We use induction on n to prove $X_{n,\gamma} = \mathrm{SL}_n$ for all $n \geq 2$.

If $n \geq 4$, the obvious embedding $\mathrm{SL}_{n-1} \subset \mathrm{SL}_n$ and the induction hypothesis imply $X_{n,\gamma}$ contains SL_{n-1}, and of course it is invariant under conjugation in SL_n. The Zariski closure of the set of all SL_n-conjugates of SL_{n-1} is the codimension 1 subvariety of SL_n consisting of elements for which 1 is an eigenvalue. As $X_{n,\gamma}$ is irreducible, it consists either of this subvariety or of all SL_n. Applying the induction hypothesis to the embedding $\mathrm{SL}_{n-2} \times \mathrm{SL}_2 \subset \mathrm{SL}_n$, we obtain $\mathrm{SL}_{n-2} \times \mathrm{SL}_2 \subset X_{\gamma,n}$, which proves $X_{\gamma,n} = \mathrm{SL}_n$. The theorem now follows from Lemma 3.1.   □

The conditions of Theorem 3.2 are in general not easy to check. For condition (1), we have the following proposition:

Proposition 3.3. *Suppose Γ is a group with d generators and r relations and for all $n \geq 2$ and $m \geq 1$, and for each prime p sufficiently large,*

$$|\mathrm{Hom}(\Gamma, \mathrm{SL}_n(\mathbb{F}_{p^m}))| = (1 + o(1)) p^{m(n^2-1)(d-r)}.$$

Then condition (1) *of Theorem 3.2 holds for Γ.*

Proof. By Theorem A.7, it suffices to prove that the characteristic p representation variety $\mathrm{Hom}(\Gamma, \mathrm{SL}_{n,\mathbb{F}_p})$ is geometrically irreducible for all p sufficiently large. By Theorem A.8 and the estimate for $|\mathrm{Hom}(\Gamma, \mathrm{SL}_{n,\mathbb{F}_p})(\mathbb{F}_{p^m})|$, it follows that there is a unique geometric component of $\mathrm{Hom}(\Gamma, \mathrm{SL}_{n,\mathbb{F}_p})$ of dimension $(d-r)\dim \mathrm{SL}_n$ and that all other geometric components are of lower dimension.

Let

$$\Gamma = \langle x_1, \ldots, x_d : R_1 = \cdots = R_r = 1 \rangle$$

be a presentation of Γ with d generators and r relations. Applying Theorem A.4 to the multiword map

$$(R_1, \ldots, R_r)\colon \mathrm{SL}_{n,\mathbb{F}_p}^d \to \mathrm{SL}_{n,\mathbb{F}_p}^r,$$

we see that the minimum dimension of a geometric component of $\mathrm{Hom}(\Gamma, \mathrm{SL}_{n,\mathbb{F}_p})$ is at least $(d-r)\dim \mathrm{SL}_{n,\mathbb{F}_p}$, and we are done.   □

The point-counting hypothesis of Proposition 3.3 can be verified for some interesting 1-relator groups.

Proposition 3.4. *Let $\Gamma = S_g$, the fundamental group of an oriented surface of genus $g \geq 2$. Then for each integer $m \geq 1$ and prime p sufficiently large,*

$$|\mathrm{Hom}(\Gamma, \mathrm{SL}_n(\mathbb{F}_{p^m}))| = (1 + o(1))\, p^{m(n^2-1)(2g-1)}.$$

Proof. By a theorem of Frobenius [Sha07, Proposition 4.1], the number of ways of representing an element h of a finite group H as $xyx^{-1}y^{-1}$ for $x, y \in H$ is

$$|H| \sum_\chi \frac{\chi(h)}{\chi(1)},$$

where the sum is taken over all irreducible characters χ of H. By the generalized orthogonality relation [Isa76, Th. 2.13] and induction on g, we obtain

$$|\mathrm{Hom}(S_g, H)| = |H|^{2g-1} \sum_\chi \frac{1}{\chi(1)^{2g-2}}.$$

Following Liebeck and Shalev [LS05a], for each finite group H we define

$$\zeta^H(s) := \sum_\chi \chi(1)^{-s},$$

where the sum is taken over the irreducible characters of H. By a result of the same authors [LS05b, Th. 1.1], we have

$$\lim \zeta^{G(\mathbb{F}_q)}(s) - 1 = 0 \tag{3}$$

for $s > 2/h$, where the limit is taken over any sequence of groups of the form $G(\mathbb{F}_q)$, where each G is a simply connected simple group over $\mathbb{F}_q$ of Coxeter number h, and q tends to ∞. In particular,

$$\lim \sum_{\chi \neq 1} \frac{1}{\chi(1)^{2g-2}} = 0.$$

This implies the result of Li [Li93] that $\mathrm{Hom}(S_g, G)$ is irreducible not only for G of the form $\mathrm{SL}_{n,\mathbb{C}}$ but for all simply connected semisimple groups G. $\qquad\square$

Likewise, we obtain

Proposition 3.5. *If $\Gamma = \langle x_1, \ldots, x_k : x_1^2 \cdots x_k^2 = 1 \rangle$, $k \geq 3$, $n \geq 2$, $(k, n) \neq (3, 2)$, and $m \geq 1$, then for each prime p sufficiently large,*

$$|\mathrm{Hom}(\Gamma, \mathrm{SL}_n(\mathbb{F}_{p^m}))| = (1 + o(1))\, p^{m(n^2-1)(k-1)}.$$

Proof. The formula counting homomorphisms from Γ to a finite group H is

$$|\mathrm{Hom}(\Gamma, H)| = |H|^{k-1} \sum_\chi \frac{\iota(\chi)^k}{\chi(1)^{k-2}},$$

where ι is the Frobenius–Schur indicator. The proof is essentially the same as before, the starting point being the classical theorem of Frobenius and Schur [Isa76, Th. 4.5] that for any finite group H, the number of solutions in H of $x^2 = h$ is $\sum_\chi \iota(\chi)\chi(h)$. For $k \geq 4$ or $k = 3$ and $n \geq 3$, the proposition follows from equation (3) and the fact that the Coxeter number of SL_n is n. □

Proposition 3.6. *In Theorem* 3.2, *assuming* $\mathrm{Hom}(\Gamma, \mathrm{SL}_2)$ *is irreducible, we can deduce hypothesis* (2) *from the assertion that there exists a homomorphism* $i_2 \colon \Gamma \to \mathrm{SL}_2(\mathbb{C})$ *such that* $i_2(\gamma)$ *is not unipotent for* $\gamma \neq 1$.

Proof. The condition on i_2 is equivalent to $\mathrm{trace}(i_2(\gamma)) \neq 2$, so that $\mathrm{trace} \circ e_{2,\gamma}$ is a nonconstant function in $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}_2)$. As $\mathrm{Hom}(\Gamma, \mathrm{SL}_2)$ is irreducible, it follows that the image of $\mathrm{trace} \circ e_{2,\gamma}$ is Zariski-dense in the affine line. Any closed subvariety of SL_2 which is invariant under conjugation and has a Zariski-dense set of traces is all of SL_2. Note that if i_2 maps Γ to $\mathrm{SU}(2)$, it suffices to assume that it is injective. □

If D is a central division algebra of degree n over a field K, following standard notation [Tit66, Table II], we denote by $\mathrm{SL}_1(D)$ the algebraic group over K whose K-points give the elements of $D^\times$ of reduced norm 1, while its $\bar{K}$-points give $\mathrm{SL}_n(\bar{K})$. We recall that reduced trace gives a map $D \to K$, and applying reduced trace to all integer powers of an element of D, we see that the power sums of the eigenvalues of any element of $\mathrm{SL}_1(D) \subset \mathrm{SL}_n(\bar{K})$ lie in K. If K is of characteristic zero, this implies that the characteristic polynomial of every element of $\mathrm{SL}_1(D)$ has coefficients in K. If any element of $\lambda \in K$ is an eigenvalue of $\alpha \in \mathrm{SL}_1(D)$, then $\alpha - \lambda$ is not invertible, so it is zero, and $\alpha = \lambda$ lies in the center of $\mathrm{SL}_1(D)$. Moreover, if D is of degree 3 and the characteristic polynomial of $\alpha \in \mathrm{SL}_1(D)$ has a multiple root r, then $r \in K$. Thus every element of $\mathrm{SL}_1(D)$ is central or regular semisimple.

In particular, $1 \in K$, so 1 cannot be an eigenvalue of any $\alpha \in \mathrm{SL}_1(D)^\bullet$. Using this observation, we can replace (3) in Theorem 3.2 as follows:

Proposition 3.7. *If* $\mathrm{Hom}(\Gamma, \mathrm{SL}_n)$ *is irreducible for all* $n \geq 3$, Γ *is* SL_2-*free, and there exists a degree 3 division algebra* $D \subset M_3(\mathbb{C})$ *such that* $\mathrm{Hom}(\Gamma, \mathrm{SL}_1(D)) \subset \mathrm{Hom}(\Gamma, \mathrm{SL}_3(\mathbb{C}))$ *is Zariski-dense in* $\mathrm{Hom}(\Gamma, \mathrm{SL}_3)$, *then hypothesis* (3) *follows and therefore* Γ *is* G-*free for all semisimple groups* G.

Proof. Let $\gamma \in \Gamma$ be a nontrivial element. As $X_{2,\gamma} = \mathrm{SL}_2$, we see that $X_{3,\gamma}$ contains the codimension 1 subvariety of SL_3 consisting of matrices for which 1 is an eigenvalue. It suffices to prove that there is at least one point of $X_{3,\gamma}$ for which 1 is not an eigenvalue. However, for $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}_1(D)) \subset \mathrm{Hom}(\Gamma, \mathrm{SL}_3(\mathbb{C}))$, 1 can be an eigenvalue if and only if $\rho(\gamma) = 1$. If $\rho(\gamma) = 1$ for all $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}_1(D))$, Zariski density implies the same for all $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}_3(\mathbb{C}))$, contrary to the nontriviality of $X_{3,\gamma}$. □

A related criterion for (3) is the following:

Proposition 3.8. *If* $\mathrm{Hom}(\Gamma, \mathrm{SL}_n)$ *is irreducible for all* $n \geq 3$, Γ *is* SL_2-*free, and there exists a degree* 3 *division algebra* D *over a characteristic zero local field* K *such that* $\mathrm{Hom}(\Gamma, \mathrm{SL}_1(D))$ *contains a regular point of* $\mathrm{Hom}(\Gamma, \mathrm{SL}_3(\bar{K}))$, *then hypothesis* (3) *follows, and therefore* Γ *is* G-*free for all semisimple groups* G.

Proof. Applying Theorem A.10 to $X = X_{3,\gamma}$ and identifying $\bar{K}$ and $\mathbb{C}$ by the axiom of choice, the proposition now follows from Proposition 3.7. □

Proposition 3.9. *Let* π_1 *be the fundamental group of the connected sum of three projective planes. If* $\mathrm{Hom}(\pi_1, \mathrm{SL}_n)$ *is irreducible for all* $n \geq 3$, *and* π_1 *is* SL_2-*free, then* π_1 *is* SL_3-*free.*

Proof. The Brauer group of $\mathbb{Q}_p$ is canonically isomorphic to $\mathbb{Z}/\mathbb{Q}$ [Ser79, XIII Prop. 6], so we can define D to be the (degree 3) division algebra over $\mathbb{Q}_p$ with invariant $1/3$. To apply Proposition 3.8, we observe that the trivial representation $\pi_1 \to \mathrm{SL}_3(\mathbb{C})$ is a nonsingular point of $\mathrm{Hom}(\pi_1, \mathrm{SL}_3)$.

Indeed, identifying $\mathrm{Hom}(\Gamma, \mathrm{SL}_3)$ with

$$e^{-1}_{\mathrm{SL}_3, x_1^2 x_2^2 x_3^2}(1),$$

it suffices by Theorem A.11 to note that the morphism $\mathrm{SL}_3^3 \to \mathrm{SL}_3$ given by the word $x_1^2 x_2^2 x_3^2$ induces a surjective map on tangent spaces at $(1, 1, 1)$. The induced map on tangent spaces $\mathfrak{sl}_3^3 \to \mathfrak{sl}_3$ sends (X_1, X_2, X_3) to $2X_1 + 2X_2 + 2X_3$ and is therefore surjective. □

We can now apply the previous results to obtain a new class of groups in $\mathscr{B}$.

Proposition 3.10. *Let* S *be the connected sum of three or more projective planes. Then* $\pi_1(S, \cdot)$ *is in* $\mathscr{B}$.

Proof. Let S be the connected sum of k projective planes, where $k \geq 3$. Let $\pi_1 = \pi_1(S, \cdot)$. We break the proof into two cases, depending on k.

(1) If $k = 3$, then $\pi_1 = \langle x, y : x^2 y^2 z^2 = 1 \rangle$. This group is known not to be residually free [LS62]. By Theorem 3.2 and Propositions 3.3, 3.5, 3.8, and 3.9 it suffices to show that there exists some map $\phi : \pi_1 \to \mathrm{SL}_2(\mathbb{C})$ such that $\phi(\pi_1)$ does not contain any unipotent elements. To do this, identify π_1 with a compact subgroup P of the isometries of the hyperbolic plane which is the universal covering space of the connected sum of three real projective planes, regarded as a hyperbolic surface. Let π denote the homomorphism from

$$\mathrm{SL}_2(\mathbb{C}) \cap \langle i \rangle \mathrm{GL}_2(\mathbb{R})$$

to the group of all Möbius transformations given by Seppälä and Sorvali [SS93, §6]. Then there exists a lift $\tilde{P}$ of P so that π gives an isomorphism $\tilde{P} \to P$ [SS93, Theorem 6]. We claim that $\tilde{P}$ has no nontrivial unipotent elements. Indeed, P contains, as a subgroup of index 2, a discrete and cocompact subgroup of $\mathrm{PSL}_2(\mathbb{R})$ (the orientation-preserving isometries of the hyperbolic plane). Let X be a nontrivial unipotent element in $\tilde{P}$. Then $\pi(X^2) = \pi(X)^2$ is unipotent and lies inside $\mathrm{PSL}_2(\mathbb{R})$ and so $\pi(X^2)$ is parabolic. This is

impossible, as no discrete and cocompact subgroup of $\mathrm{PSL}_2(\mathbb{R})$ contains a parabolic element [Kat92, Theorem 4.2.1]. It follows that $\tilde{P}$ does not contain any nontrivial unipotent elements, as desired.

(2) If $k > 3$, then by [Bau67], π_1 is residually free. Thus, by Lemma 2.3, $\pi_1 \in \mathscr{B}$. $\square$

3.2. Conditions for general groups

In this section, we present some conditions for a finitely generated group Γ to be almost G-free for all semisimple groups G. Our main theorem is a variant of the results in the previous section. This variation is forced on us. We cannot expect that $\mathrm{Hom}(\Gamma, \mathrm{SL}_n)$ will be connected if Γ has nontrivial torsion elements. Indeed, there are typically several different conjugacy classes of elements of $\mathrm{SL}_n(\mathbb{C})$ of given order $m > 1$. We therefore try to pin down the class of the image of each torsion conjugacy class. We assume that Γ has finitely many classes of nontrivial elements of finite order, and we denote by $x_1, \ldots, x_k$ representatives of each class.

If G is a semisimple group defined over $\mathbb{C}$ and $y \in G(\mathbb{C})$ is of finite order, it is semisimple, and its conjugacy class is therefore closed. Since G is irreducible, its conjugacy classes are likewise irreducible. If $\mathbf{y} = (y_1, \ldots, y_m)$ is an m-tuple of semisimple elements of $G(\mathbb{C})$, we denote by $V(G, \mathbf{y})$ the closed subvariety

$$V(G, \mathbf{y}) := e_{x_1,\ldots,x_k}^{-1}(C_1 \times \cdots \times C_k) \subset \mathrm{Hom}(\Gamma, G),$$

where C_i is the conjugacy class of y_i, and

$$e_{x_1,\ldots,x_k} \colon \mathrm{Hom}(\Gamma, G) \to G^k$$

is the multiword evaluation map. If $G = \mathrm{SL}_n$, we denote $V(\mathrm{SL}_n, \mathbf{y})$ by $V_n(\mathbf{y})$ for brevity.

Theorem 3.11. *Let Γ be a finitely generated group with finitely many conjugacy classes of nontrivial elements of finite order, represented by elements $x_1, \ldots, x_k$. For each $n \geq 2$, let $Y_n \subset \mathrm{SL}_n(\mathbb{C})^k$ be a nonempty set of k-tuples of semisimple elements. Suppose:*

(1) *For each $n \geq 2$ and $\mathbf{y} \in Y_n$, the variety $V_n(\mathbf{y})$ is irreducible.*
(2) *For each $n \geq 3$ and each $\mathbf{y} \in Y_n$, there exists $\mathbf{y}' \in Y_{n-1}$ and a 1-dimensional character χ of Γ such that $\chi(x_i)y_i' \oplus \chi(x_i)^{1-n}$ is conjugate in $\mathrm{SL}_n(\mathbb{C})$ to y_i for $i = 1, \ldots, k$.*
(3) *For each $n \geq 4$ and each $\mathbf{y} \in Y_n$, there exist $\mathbf{y}^1 \in Y_{n-2}$, $\mathbf{y}^2 \in Y_2$ and 1-dimensional characters χ_1, χ_2 of Γ such that $\chi_1(x_i)y_i^1 \oplus \chi_2(x_i)y_i^2$ is conjugate in $\mathrm{SL}_n(\mathbb{C})$ to y_i for $i = 1, \ldots, k$.*
(4) *For each $\mathbf{y} \in Y_2$, there exists an injective homomorphism $\Gamma \to \mathrm{SL}_2(\mathbb{C})$ in $V_2(\mathbf{y})$ such that $\rho(\Gamma)$ contains no nontrivial unipotent element.*
(5) *For each $\mathbf{y} \in Y_3$, there exists a regular point in $V_3(\mathbf{y})$ corresponding to a homomorphism $\Gamma \to \mathrm{SL}_3(\bar{K})$ whose image lies in $\mathrm{SL}_1(D)$ for some degree 3 division algebra D over an ℓ-adic field K.*

Then Γ is almost G-free for all semisimple G.

Proof. For any $\gamma \in \Gamma$, the evaluation map $e_{G,\gamma}$ restricts to a map $V(G, \mathbf{y}) \to G$ which we denote $e_{G,\mathbf{y},\gamma}$. For any homomorphism $\phi \colon G \to H$ we have a commutative diagram

$$
\begin{array}{ccc}
V(G, \mathbf{y}) & \xrightarrow{\ \phi\ } & V(H, \phi(\mathbf{y})) \\
{\scriptstyle e_{G,\mathbf{y},\gamma}} \big\downarrow & & \big\downarrow {\scriptstyle e_{H,\phi(\mathbf{y}),\gamma}} \\
G & \xrightarrow{\ \phi\ } & H
\end{array}
$$

so the closure $X_{H,\phi(\mathbf{y}),\gamma}$ of the image of $e_{H,\phi(\mathbf{y}),\gamma}$ contains $\phi(X_{G,\mathbf{y},\gamma})$. Also, $V(G, \mathbf{y})$ depends only on the conjugacy classes of the y_i, and therefore admits a conjugacy action by G which the evaluation maps respect. Thus $X_{G,\mathbf{y},\gamma}$ is a closed, conjugation-invariant subvariety of G. It follows that if for all $n \geq 2$ exists $\mathbf{y} \in Y_n$ such that $X_{\mathrm{SL}_n,\mathbf{y},\gamma} = \mathrm{SL}_n$, then $e_{G,\gamma}(\mathrm{Hom}(\Gamma, G))$ is dense in G for all semisimple G.

For $n \geq 2$ and $\mathbf{y} \in Y_{n+1}$, there exist $\mathbf{y}'$ and χ such that the homomorphism

$$\mathrm{Hom}(\Gamma, \mathrm{SL}_n) \to \mathrm{Hom}(\Gamma, \mathrm{SL}_{n+1})$$

defined by

$$\rho_n \mapsto \rho_n \otimes \chi \oplus \chi^{\otimes -n}$$

and condition (2) guarantees that $V_n(\mathbf{y}')$ maps to $V_{n+1}(\mathbf{y})$. Thus, we have commutative diagrams

$$
\begin{array}{ccc}
V_n(\mathbf{y}') & \longrightarrow & V_{n+1}(\mathbf{y}) \\
{\scriptstyle e_{n,\gamma}} \big\downarrow & & \big\downarrow {\scriptstyle e_{n+1,\gamma}} \\
\mathrm{SL}_n & \longrightarrow & \mathrm{SL}_{n+1}
\end{array}
$$

where the bottom row sends

$$M \mapsto \chi(\gamma)M \oplus \chi(\gamma)^{-n}.$$

Let S be the union of all conjugacy classes of the image of this function. Note that any element in S has at least one eigenvalue which has order n.

For each $\mathbf{y} \in Y_{n+2}$ we have homomorphisms

$$\mathrm{Hom}(\Gamma, \mathrm{SL}_n) \times \mathrm{Hom}(\Gamma, \mathrm{SL}_2) \to \mathrm{Hom}(\Gamma, \mathrm{SL}_{n+2})$$

defined by

$$(\rho_n, \rho_2) \mapsto \rho_n \otimes \chi_1 \oplus \rho_2 \otimes \chi_2$$

which map $V_n(\mathbf{y}^1) \times V_2(\mathbf{y}^2) \to V_{n+2}(\mathbf{y})$, and there is a commutative diagram

$$
\begin{array}{ccc}
V_n(\mathbf{y}^1) \times V_2(\mathbf{y}^2) & \longrightarrow & V_{n+2}(\mathbf{y}) \\
{\scriptstyle e_{n,\gamma} \times e_{2,\gamma}} \big\downarrow & & \big\downarrow {\scriptstyle e_{n+2,\gamma}} \\
\mathrm{SL}_n \times \mathrm{SL}_2 & \longrightarrow & \mathrm{SL}_{n+2}
\end{array}
$$

where the bottom row sends

$$M_1 \times M_2 \mapsto \chi_1(\gamma)M_1 \oplus \chi_2(\gamma)M_2.$$

By conditions (4) and (5), the image of this map contains an element with no eigenvalues which are roots of unity. It follows that this image contains an element which is not in S. Thus, we can use induction on n to prove that the $e_{n,\gamma}$ all have Zariski-dense image, provided we can treat the base cases $n = 2$ and $n = 3$.

For $n = 2$, we use (4) together with the fact that an irreducible closed subvariety of SL_2 which is a union of conjugacy classes and contains both 1 and a nonunipotent element is all of SL_2. For $n = 3$, we use the fact that a conjugation-invariant closed irreducible subvariety of SL_3 which contains SL_2 and some element without eigenvalue 1 is all of SL_3. Although the particular homomorphism ρ_3 whose existence is guaranteed by (5) might have a nontorsion element γ in its kernel, the homomorphisms in an ℓ-adic neighborhood of ρ_3 cannot be identitically trivial on γ. Indeed, they are Zariski-dense in $V_3(\mathbf{y})$, and $V_3(\mathbf{y})$ contains at least one injective representation, namely the representation coming via condition (1) from the injective SL_2-representations of Γ guaranteed by condition (4).   □

We can now prove Theorem 1.4: if ℓ is a prime which is 1 (mod 3), then $\Gamma := \mathbb{Z}/\ell\mathbb{Z} * \mathbb{Z}/\ell\mathbb{Z}$ is almost G-free for all semisimple G.

Proof of Theorem 1.4. Let γ_1 and γ_2 denote generators of the two free factors $\mathbb{Z}/\ell\mathbb{Z}$. By [MKS04, Cor. 4.1.4 and Cor. 4.1.5], there are $2(\ell - 1)$ different conjugacy classes of nontrivial elements of Γ of finite order, and they are represented by

$$x_1 = \gamma_1, \ x_2 = \gamma_1^2, \ \ldots, \ x_{\ell-1} = \gamma_1^{\ell-1}, \ x_\ell = \gamma_2, \ \ldots, \ x_{2\ell-2} = \gamma_2^{\ell-1}.$$

All of our $\mathbf{y}$ will be of the form

$$(y_1, y_1^2, \ldots, y_1^{\ell-1}, y_2, y_2^2, \ldots, y_2^{\ell-1}),$$

where $y_1, y_2 \in SL_n(\mathbb{C})$ are of order ℓ, so $V_n(\mathbf{y}) = C_1 \times C_2$, where C_1 and C_2 denote the conjugacy classes of y_1 and y_2 respectively. This implies condition (1) of Theorem 3.11.

In order to define Y_n precisely, we first define for each integer $k \in [0, \ell - 1]$ a set B_k of subsets $S \subset \mathbb{F}_\ell$. Since $\ell \equiv 1 \pmod 3$, there exists a unique 3-element subgroup $\mu_3 \subset \mathbb{F}_\ell^\times \subset \mathbb{F}_\ell$. We let B_k consist of all $S \subset \mathbb{F}_\ell$ of cardinality k which sum to 0 and satisfy the additional condition for $k \geq 3$ that S contains the image of μ_3 under some affine transformation. For any integer $n \geq 2$, we define A_n to be the set of all functions

$$f : \mathbb{F}_\ell \to \{\lfloor n/\ell \rfloor, \lceil n/\ell \rceil\}$$

such that

$$\{x \in \mathbb{F}_\ell : f(x) > n/\ell\} \in B_k,$$

where $k \in [0, \ell - 1]$ is the mod ℓ reduction of n.

We fix an injective homomorphism ψ from $(\mathbb{F}_\ell, +)$ to $\mathbb{C}^\times$, and to any $f \in A_n$ we associate the conjugacy class $C_f \subset SL_n(\mathbb{C})$ consisting of those ℓth roots of the identity for

which the eigenvalue $\psi(x)$ occurs with multiplicity $f(x)$. For each conjugacy class C_f, we select any element $\mathbf{y}^f$ for which $y_1 = y_2$ belongs to C_f. We let

$$Y_n = \{\mathbf{y}^f : f \in A_n\}.$$

Since there exists a character which takes the value $\psi(1)$ on both γ_1 and γ_2, to prove (2) it suffices to show that every element of A_n is the sum of a translate of an element of A_{n-1} and a translate of an element of A_1. When $\ell \nmid n$, it suffices to prove that for $1 \leq k \leq \ell - 1$, every element of B_k is the union of a single element of $\mathbb{F}_\ell$ and an additive translate of an element of B_{k-1}. Clearly, every k-element subset of $\mathbb{F}_\ell$ has a translate which sums to zero, which proves the claim. To finish (2), we note that when $\ell \mid n$, A_n consists of the single element $(n/\ell, \ldots, n/\ell)$ which decomposes as the sum of an element of A_{n-1} and an element of A_1:

$$(n/\ell, n/\ell, \ldots, n/\ell) = (n/\ell - 1, n/\ell, \ldots, n/\ell) + (1, 0, \ldots, 0).$$

Likewise, we can prove (3) when n reduces to $k \geq 2 \pmod{\ell}$ by showing that every element of B_k is the union of a translate of an element of B_{k-2} and a translate of an element of B_2. As every 2-element set is a translate of an element of B_2, this is clear. So we must deal with two cases: $k = 0$ and $k = 1$. In these two cases, A_n has only one element, and we use the decompositions

$$(n/\ell, n/\ell, \ldots, n/\ell) = (n/\ell - 1, n/\ell, \ldots, n/\ell, n/\ell - 1) + (1, 0, \ldots, 0, 1),$$
$$(n/\ell + 1, n/\ell, \ldots, n/\ell) = (n/\ell, n/\ell, \ldots, n/\ell, n/\ell - 1) + (1, 0, \ldots, 0, 1).$$

For condition (4), it suffices to prove that for any primitive ℓth root of unity ζ_ℓ, there exists an injective homomorphism from Γ to $\mathrm{SL}_2(\mathbb{C})$ sending γ_1 and γ_2 to matrices with eigenvalues $\zeta_\ell^{\pm 1}$ and with no nontrivial unipotents in the image. If we realize Γ as a Fuchsian group of the second kind with signature $(1; p, p)$, we achieve such an embedding in $\mathrm{SL}_2(\mathbb{R})$ for $\zeta_p = e^{2\pi i/p}$, and all other cases can be achieved by composing the resulting homomorphism $\Gamma \hookrightarrow \mathrm{SL}_2(\mathbb{C})$ with a suitable automorphism of $\mathbb{C}$.

For condition (5), $V_3(\mathbf{y})$ is nonsingular, so it is just a matter of showing that some homomorphism $\Gamma \to \mathrm{SL}_3(\mathbb{C})$ in $V_3(\mathbf{y})$ has image contained in a suitable $\mathrm{SL}_1(D) \leq \mathrm{SL}_3(\mathbb{C})$. As $\mathrm{Gal}(\mathbb{Q}(\zeta_\ell)/\mathbb{Q}) \cong \mathbb{F}_\ell^\times$, there exists an intermediate field $E := \mathbb{Q}(\zeta_\ell)^{\mu_3}$ such that $[\mathbb{Q}(\zeta_\ell) : E] = 3$. A rational prime p splits completely in E if and only if p reduces mod ℓ to an element of μ_3; it splits completely in $\mathbb{Q}(\zeta_\ell)$ if and only if it reduces mod ℓ to 1. By Dirichlet's theorem, there exists a prime p which splits in E but not in $\mathbb{Q}(\zeta_\ell)$. It follows that $E \subset \mathbb{Q}_p$ but ζ_ℓ is algebraic of degree 3 over $\mathbb{Q}_p$. The Brauer group of $\mathbb{Q}_p$ is canonically isomorphic to $\mathbb{Z}/\mathbb{Q}$ [Ser79, XIII Prop. 6], and we define D to be the (degree 3) division algebra over $\mathbb{Q}_p$ with invariant $1/3$. Every degree 3 extension of $\mathbb{Q}_p$ can be embedded in D [Ser79, XIII Prop. 7]. In particular, there exists an injective $\mathbb{Q}_p$-homomorphism $i : \mathbb{Q}_p(\zeta_\ell) \to D$, and it follows that

$$D \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\zeta_\ell) \cong M_3(\mathbb{Q}_p(\zeta_\ell)).$$

If $\alpha \in \mathbb{Q}_p(\zeta_\ell)$ has minimal polynomial $P(x)$ over $\mathbb{Q}_p$, then $P(i(\alpha)) = 0$, but viewed as an element of $M_3(\mathbb{Q}_p(\zeta_\ell))$, $i(\alpha)$ has a characteristic polynomial with coefficients in $\mathbb{Q}_p$,

which must then be $P(x)$ as well. It follows that the eigenvalues of $i(\alpha) \in M_3(\bar{\mathbb{Q}}_p)$ are α and its conjugates over $\mathbb{Q}_p$. In particular, if ζ_ℓ is a primitive ℓth root of unity, its conjugates over E (and therefore over $\mathbb{Q}_p$) are ζ_ℓ^a and $\zeta_\ell^{a^2}$, where the image of a in $\mathbb{F}_\ell$ generates μ_3.

Any element $S \in B_3$ is a coset of the order 3 subgroup $\mu_3 \subset \mathbb{F}_\ell^\times$. On identifying $\bar{\mathbb{Q}}_p$ and $\mathbb{C}$, $\psi(S)$ is therefore the Galois orbit of an element of $\mathbb{Q}_p(\zeta_l)$, and it follows that there exists an element $e_S \in D \subset M_3(\mathbb{C})$ with eigenvalues $\psi(S)$. As $\sum_{s \in S} s = 0$, we have $e_S \in \mathrm{SL}_1(D)$. The homomorphism sending γ_1 and γ_2 to e_S is therefore of the desired kind. $\qquad\square$

The same strategy can be used to prove Theorem 1.5: if $\ell \geq 19$ is a prime that is $\equiv 1$ (mod 3), then

$$\Gamma := \langle x, y, z, t : x^\ell = y^\ell = z^\ell = t^\ell = xyzt = 1 \rangle$$

is in $\mathscr{L}$.

Proof of Theorem 1.5. Let x, y, z, and t be as in the presentation of Γ. By [MKS04, Cor. 4.4.5 and Th. 4.5], there are $4(\ell - 1)$ different conjugacy classes of nontrivial elements of Γ of finite order, and they are represented by

$$x, x^2, \ldots, x^{\ell-1}, y, y^2, \ldots, y^{\ell-1}, \ldots, z, z^2, \ldots, z^{\ell-1}, t, t^2, \ldots, t^{\ell-1}.$$

All of our **y** will be of the form

$$(y_1, y_1^2, \ldots, y_1^{\ell-1}, y_1^{\ell-1}, y_1^{\ell-2}, \ldots, y_1, y_1, y_1^2, \ldots, y_1^{\ell-1}, y_1^{\ell-1}, y_1^{\ell-2}, \ldots, y_1),$$

where $y_1, y_2 \in \mathrm{SL}_n(\mathbb{C})$ are of order ℓ. Thus, $V_n(\mathbf{y})$ is a subvariety of $\mathrm{SL}_n(\mathbb{C})^4$ of the form

$$W := \{(X, Y, Z, T) \in C_1 \times C_2 \times C_3 \times C_4 : XYZT = 1\}, \qquad (4)$$

where C_i are specified conjugacy classes of semisimple elements in $\mathrm{SL}_n(\mathbb{C})$ which satisfy the condition $C_1 = C_2^{-1} = C_3 = C_4^{-1}$. Before showing condition (1) of Theorem 3.11, we need to define Y_n precisely.

We define B_k, A_n, ψ, and C_f exactly as in the proof of Theorem 1.4. For each C_f, we select any element $\mathbf{y}^f$ for which $y_1 = y_2^{-1}$ belongs to C_f. We let

$$Y_n = \{\mathbf{y}^f : f \in A_n\}.$$

Thus, the conjugacy classes appearing in W corresponding to $V_n(\mathbf{y})$ have semisimple elements with multiplicity at most $\lceil n/\ell \rceil$.

We now show condition (1) of Theorem 3.11 by showing that varieties of the form (4) are geometrically irreducible. By [Ser92, Th. 7.2.1] and the fact that two semisimple elements in $\mathrm{SL}_n(\mathbb{F}_p)$ are conjugate if and only if they are conjugate in $\mathrm{GL}_n(\mathbb{F}_p)$, the number of elements in $W(\mathbb{F}_p)$ is

$$\frac{1}{|\mathrm{GL}_n(\mathbb{F}_p)|}|C_1| \cdots |C_4| \sum_\chi \frac{\chi(x_1) \cdots \chi(x_4)}{\chi(1)^2} = \frac{1}{|\mathrm{GL}_n(\mathbb{F}_p)|}|C_1|^4 \sum_\chi \frac{|\chi(x_1)|^4}{\chi(1)^2}, \qquad (5)$$

where x_i is a representative of the conjugacy class C_i in $\mathrm{GL}_n(\mathbb{F}_p)$ and χ runs through all irreducible characters of $\mathrm{GL}_n(\mathbb{F}_p)$. As there are exactly $q - 1$ characters of $\mathrm{GL}_n(\mathbb{F}_q)$ of degree 1, namely those characters which factor through the determinant map $\mathrm{GL}_n(\mathbb{F}_q) \to \mathbb{F}_q^{\times}$, it follows that the quantity (5) is given by

$$\frac{|C_1|^4}{|\mathrm{GL}_n(\mathbb{F}_p)|}\left(q - 1 + \sum_{\chi(1)>1} \frac{|\chi(x_1)|^4}{\chi(1)^2}\right).$$

Note that C_1 is the conjugacy class of a semisimple element in a simply connected semi-simple group, so by Steinberg's theorem [Hum95, Th. 2.11], it is the quotient of $\mathrm{SL}_n(\mathbb{F}_q)$ by the group of $\mathbb{F}_q$-points of a geometrically connected group over $\mathbb{F}_q$, so

$$|C_1| = q^{\dim \mathrm{SL}_n - \dim C_{\mathrm{SL}_n}(x_1)}(1 + o_q(1)).$$

In the special case where x_1 is regular, this is $q^{n^2-n}(1 + o_q(1))$.

Let P_q denote the set of ordered pairs (χ, χ') consisting of an irreducible character χ of $\mathrm{SL}_n(\mathbb{F}_q)$ and an irreducible character χ' of $\mathrm{GL}_n(\mathbb{F}_q)$ such that χ is an irreducible constituent of the restriction of χ', or (equivalently, by Frobenius reciprocity) χ' is a constituent of the induced character of χ. Thus, P_q projects onto the set of irreducible representations of $\mathrm{SL}_n(\mathbb{F}_q)$ and likewise onto the set of irreducible representations of $\mathrm{GL}_n(\mathbb{F}_q)$. For $(\chi, \chi') \in P_q$, we have

$$\chi(1) \leq \chi'(1) \leq [\mathrm{GL}_n(\mathbb{F}_q) : \mathrm{SL}_n(\mathbb{F}_q)]\chi(1) = (q - 1)\chi(1).$$

In particular, the number of characters χ' associated to a single χ is at most $q - 1$. The characters of $\mathrm{GL}_n(\mathbb{F}_q)$ associated to the trivial character of $\mathrm{SL}_n(\mathbb{F}_q)$ are precisely the $q-1$ characters of degree 1. All other characters of $\mathrm{GL}_n(\mathbb{F}_q)$ have degree at least $(q^{n-1} - 1)/2$ by the bound of Landazuri and Seitz for degrees of nontrivial projective characters of $\mathrm{PSL}_n(\mathbb{F}_q)$ [LS74]. The total number of characters of $\mathrm{GL}_n(\mathbb{F}_q)$ is $O(q^n)$ by a result of Liebeck and Pyber [LP97]. If x_1 is regular semisimple, then $|\chi(x_1)|$ is bounded above by a constant depending only on n [GLL12]. Thus, (5) is given by

$$q^{3n^2-4n+1}(1 + o_q(1)).$$

We would like to achieve a similar upper bound when x_1 has an eigenvalue with multiplicity greater than one. Let $\alpha = 1/10$. If n is divisible by $\ell \geq 19$, each eigenvalue has multiplicity $n/\ell < \alpha n$. Otherwise, writing $n = a\ell + k$, $1 \leq k < \ell$, we have $a \geq 1$, so each eigenvalue has multiplicity at most

$$a + 1 \leq \frac{(a + 1)n}{a\ell + 1} \leq \frac{2n}{\ell + 1} \leq \alpha n.$$

Let $\beta = 4/9$. By Theorem B.1, we have $|\chi(x_1)| \leq \chi(1)^{\beta}$ for all p sufficiently large and any irreducible character χ of $\mathrm{SL}_n(\mathbb{F}_p)$.

By (3),

$$\zeta^{\mathrm{SL}_n(\mathbb{F}_p)}(s) = 1 + o_p(1)$$

if $s > 2/n$. It follows that

$$\frac{\zeta^{\mathrm{GL}_n(\mathbb{F}_p)}(s)}{q-1} - 1 \le \sum_{\chi(1)\neq 1} \frac{\sum_{\{\chi' : (\chi,\chi')\in P_p\}} \chi'(1)^{-s}}{p-1} \le \zeta_{\mathrm{SL}_n(\mathbb{F}_p)}(s) - 1 = o_p(1).$$

Thus,

$$|W(\mathbb{F}_p)| = \frac{1}{|\mathrm{SL}_n(\mathbb{F}_p)|} |C_1|^4 (1 + o_p(1)) = p^{3\dim\mathrm{SL}_n - 4\dim C_{\mathrm{SL}_n}(x_1)}(1 + o_p(1)). \quad (6)$$

By Theorem A.4, every geometric component of W has dimension at least

$$3\dim\mathrm{SL}_n - 4\dim C_{\mathrm{SL}_n}(x_1).$$

Coupling this with (6) and with Theorem A.8, we deduce that $W/\mathbb{F}_p$ is geometrically irreducible, as desired.

The proofs that conditions (2), (3), and (4) of Theorem 3.11 are satisfied exactly parallel those in the proof of Theorem 1.4. For condition (5), we need an additional argument to verify that $V_3(\mathbf{y})$ has a regular point. The last result in [Wei64] gives a sufficient condition for a homomorphism from an oriented Fuchsian group Γ to an algebraic group G in characteristic zero to be a regular point of $\mathrm{Hom}(\Gamma, G)$; it suffices that the space of coinvariants of the adjoint action of Γ on the Lie algebra $\mathfrak{g}$ of G is zero. Equivalently, it suffices that the space of invariants of the coadjoint representation is zero, and if G is semisimple, the adjoint and coadjoint representations are isomorphic, so it suffices that the centralizer of the image of Γ in G is 0-dimensional.

We identify $\overline{\mathbb{Q}}_p$ with $\mathbb{C}$ as in the proof of Theorem 1.4 and conclude by showing that we may find a (noninjective) homomorphism $\Gamma \to \mathrm{SL}_3(\mathbb{C})$ such that the centralizer of the image is 0-dimensional. Fixing $\mathbf{y} \in Y_3$ fixes a regular semisimple conjugacy class C_1 in $\mathrm{SL}_1(D)$. We will choose a homomorphism $\gamma_{s,t} : \Gamma \to \mathrm{SL}_1(D)$ in $V(\mathbf{y})$ defined by

$$(x, y, z, t) \mapsto (s, s^{-1}, t, t^{-1}),$$

for $s, t \in C_1$: Set s to be any element in C_1 and let S denote the unique maximal torus in SL_3 containing s. By Zariski density of $\mathrm{SL}_1(D)$ in SL_3, there exists $g \in \mathrm{SL}_1(D)$ that does not lie in the normalizer of S (a proper subvariety of SL_3). Set $t = gsg^{-1}$. We claim that the centralizer of $\gamma_{s,t}(\Gamma)$ in $\mathrm{SL}_1(D)$ is contained in the center of $\mathrm{SL}_3(\mathbb{C})$. Let $z \in \mathrm{SL}_1(D)$ be an element in this centralizer. Then because z commutes with s, we have $z \in S$. Further, z commutes with t, so $g^{-1}zg$ commutes with s and therefore lies in S. Suppose that z is noncentral and in $\mathrm{SL}_1(D)$. Then z is regular (see the observation before Proposition 3.7), so it belongs to a unique maximal torus, which must be S. Thus, g normalizes S, which is impossible by our choice of g. Hence, the only elements in $\mathrm{SL}_1(D)$ that commute with every element in $\gamma_{s,t}(\Gamma)$ are in the center of $\mathrm{SL}_3(\mathbb{C})$. Since the centralizer of $\gamma_{s,t}(\Gamma)$ in $\mathrm{SL}_1(D)$ is finite, it follows that the centralizer of $\gamma_{s,t}(\Gamma)$ in $\mathrm{SL}_3(\mathbb{C})$ is 0-dimensional, as desired. □

We finish the section by showing there exist G-free groups that are not residually free.

Theorem 3.12. *The group*

$$\langle a_1, \ldots, a_7, b : ba_1b^{-1} = a_2, \ ba_2b^{-1} = a_3, \ldots, \ ba_6b^{-1} = a_7, \ ba_7b^{-1} = a_1 \rangle$$

is in $\mathscr{B}$ but is not residually free.

Proof. By construction, Γ is a semidirect product of F_7 with $\mathbb{Z}$. It maps onto the group

$$\Delta = \langle a, b_1, b_2, \ldots, b_7 : ab_1a^{-1} = b_2, \ldots, ab_6a^{-1} = b_7, \ ab_7a^{-1} = b_1, \ a^7 = 1 \rangle,$$

which is contained inside $\mathbb{Z}/7 * \mathbb{Z}/7$ and is almost G-free by Theorem 1.4. The entire set $F_7^\bullet$ is almost detected by Δ. Any element of $\Gamma \setminus F_7$ is detected by $\mathbb{Z}$. Since Δ and $\mathbb{Z}$ are both G-free, it follows that Γ is so by Lemma 2.3.

On the other hand, Γ is not residually free: Since free groups are residually 2-finite, any residually free group must be residually 2-finite. However, the element $b_1 b_2^{-1}$ must vanish in any 2-group quotient of Γ. Indeed, the action of a on F_7 has order 7, and so if $a^{2^k} = 1$ for any k, the action of a on F_7 must be trivial. $\qquad\square$

3.3. Groups that fail to satisfy Borel's Theorem

Here, we give examples of groups which fail to be G-free for some semisimple group G. We start by showing that it is not always true that free products of G-free groups are G-free.

Proposition 3.13. *The group $\Gamma := (F_2 \times \mathbb{Z}) * \mathbb{Z}$ is not residually $\{\mathrm{SL}_2(\mathbb{C})\}$ but is the free product of two groups that are in $\mathscr{B}$.*

Proof. The groups $\mathbb{Z}$ and $F_2 \times \mathbb{Z}$ are G-free for any semisimple G by Borel's Theorem [Bor83] and Lemma 2.4.

We now show that Γ, which has presentation

$$\Gamma = \langle a_1, a_2, b, c : [a_1, b] = 1, \ [a_2, b] = 1 \rangle,$$

is not residually $\{\mathrm{SL}_2(\mathbb{C})\}$. Supposing, for the sake of contradiction, that the element

$$[[a_1, a_2], [b, c]]$$

does not vanish in the image of some homomorphism $\phi : \Gamma \to \mathrm{SL}_2(\mathbb{C})$, the elements $\phi(a_1)$ and $\phi(a_2)$ do not commute. Since $[\phi(b), \phi(c)] \neq 1$ and $\mathrm{SL}_2(\mathbb{C})$ is commutative transitive away from its center, it follows that $\phi(a_1)$ and $\phi(a_2)$ must commute. Thus, $\phi([[a_1, a_2], [b, c]]) = 1$. $\qquad\square$

For our next result, note that the lamplighter group $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$, mentioned in Remark 2, below is not linear but has an element of order two with infinite conjugacy class. Thus, dropping the linearity hypothesis from Theorem 3.14 is not possible. Further, we note that as a consequence of the following theorem, neither $\mathrm{SL}_n(\mathbb{Z})$, nor the infinite dihedral group, nor any triangle group is almost $\mathrm{SL}_2(\mathbb{C})$-free.

Theorem 3.14. *Let Γ be a finitely generated linear group which has infinitely many elements of order two. Then Γ is not almost $\mathrm{SL}_2(\mathbb{C})$-free.*

Proof. Let X be the set of all order two elements in Γ. Any image of Γ in $\mathrm{SL}_2(\mathbb{C})$ must take $\langle X \rangle$ into $Z(\mathrm{SL}_2(\mathbb{C}))$, and so if $\langle X \rangle$ contains a nontorsion element, the group Γ cannot be almost $\mathrm{SL}_2(\mathbb{C})$-free. Thus, for the remainder of the proof, we assume that $\langle X \rangle$ does not contain any nontorsion element.

Since Γ is a finitely generated linear group, by Selberg's lemma, Γ contains a finite-index normal subgroup $\Delta \leq \Gamma$ that is torsion-free. Since $\langle X \rangle$ does not contain any non-torsion elements, we have $\langle X \rangle \cap \Delta = \{1\}$. Thus $\langle X \rangle$ embeds into the finite group Γ/Δ. This is impossible as X contains infinitely many distinct elements. $\qquad\square$

Before proving our next result concerning G-freeness of torsion-free virtually abelian groups, we need two technical lemmas.

Lemma 3.15. *Let Γ be a torsion-free and virtually abelian group that is not abelian. Then Γ contains a solvable subgroup that is not abelian.*

Proof. If Γ is solvable, then there is nothing to prove. Hence, we assume that Γ is not solvable. In any group, the intersection of all possible conjugates of any finite-index subgroup is finite-index. Thus, since Γ is virtually abelian, it follows that Γ contains a normal subgroup, N, of finite-index that is abelian. Let M be a maximal normal solvable subgroup of Γ that contains N. Thus, Γ/M is not solvable. By Feit–Thompson, it must have an element of order two. If all order two elements commute in Γ/M, then the group they generate is a nontrivial, proper (because Γ/M is not abelian), abelian, and normal subgroup of Γ/M, which is impossible by maximality of M. Thus, there exist two order two elements in Γ/M that do not commute. We lift these elements to Γ and let D be the group they generate. There is a short exact sequence,

$$1 \to D \cap M \to D \to S \to 1,$$

where S is solvable, as it is generated by two order two elements, and hence is the image of an infinite dihedral group, which is solvable. It follows that D is solvable and nonabelian, so we are done. $\qquad\square$

Lemma 3.16. *Let Γ be a group with normal abelian torsion-free subgroup $\Delta \leq \Gamma$. Let $t \in \Gamma$ and $a \in \Delta$ be such that $S = \{t^k a t^{-k} : k \in \mathbb{Z}\}$ is of cardinality r with $1 < r < \infty$. Then Γ is not $\mathrm{SL}_n(\mathbb{C})$-free for any $n > r$.*

Proof. We use additive notation for Δ and write a^t for tat^{-1} when $a \in \Delta$ and $t \in \Gamma$. Set $S' = \{a^{t^k} - a^{t^{k+1}} : k \in \mathbb{Z}\}$. Thus S' forms a single orbit under $\langle t \rangle$, and

$$m := a - a^t \neq 0,$$

so all elements of the orbit are of infinite order (see Lemma 2.1). We have

$$\sum_{b \in S'} b = \sum_{c \in S}(c - c^t) = \sum_{c \in S} c - \sum_{c \in S} c = 0. \tag{7}$$

With this in hand, suppose, for the sake of contradiction, that Γ is $\mathrm{SL}_n(\mathbb{C})$-free for some $n > p$. Let V be the union of the subvariety of $\mathrm{SL}_n(\mathbb{C})$ consisting of elements that are not regular semisimple and the variety of elements $A \in \mathrm{SL}_n(\mathbb{C})$ with the following property: there exists a collection of complex numbers, $x_1, \ldots, x_r$, each of which is an eigenvalue of A, with $x_1 x_2 \cdots x_r = 1$. Since V is a proper subvariety (guaranteed as $r < n$), there exists some $\phi : \Gamma \to \mathrm{SL}_n(\mathbb{C})$ such that $\phi(m) \notin V$. However, $\prod_{A \in \phi(S')} A = 1$ by (7). Since all the elements in $\phi(S')$ commute with one another, they are simultaneously diagonalizable, so $\phi(m) \in V$, which is impossible. $\qquad\square$

Theorem 3.17. *Let Γ be a torsion-free and virtually abelian group that is not abelian. Then Γ is not $\mathrm{SL}_n(\mathbb{C})$-free for some n.*

Proof. By Lemmas 3.15 and 2.5 we reduce to the case of Γ a solvable group. Thus, we have a short exact sequence

$$1 \to A \to \Gamma \to H \to 1$$

where A is abelian and H is a finite nontrivial solvable group. Let A' be a maximal normal abelian subgroup containing A, and let H' be the quotient Γ/A'. Since Γ is not abelian, H' is nontrivial. If A' is non-central, we are in the situation of Lemma 3.16, and we are done. We therefore assume it is central. Let D be a nontrivial abelian normal subgroup of H' (which exists because H' is solvable). By maximality of A', there exist $t \in D$ and $a \in A'$ such that $tat^{-1} \neq a$. Since A' is of finite index in H, we are in the situation of Lemma 3.16. In fact, if A' is noncentral, then we are in the same situation. We assume, then, that A' is in the center of Γ. Then D acts trivially on A', so by maximality of A', there exist $d_1, d_2 \in D$ such that $[d_1, d_2] \in A' \setminus \{1\}$. Suppose, for the sake of a contradiction, that Γ is $\mathrm{SL}_2(\mathbb{C})$-free. Then we may find some image of Γ that detects $[d_1, d_2]$ in $\mathrm{SL}_2(\mathbb{C})$ rel $Z(\mathrm{SL}_2(\mathbb{C}))$. Since $\mathrm{SL}_2(\mathbb{C})$ is commutative transitive away from its center, this is impossible. Thus, Γ is not $\mathrm{SL}_2(\mathbb{C})$-free. $\qquad\square$

Corollary 3.18. *Let S be the Klein bottle. Then $\pi_1(S, \cdot)$ is not in $\mathscr{B}$. In fact, $\pi_1(S, \cdot)$ is not $\mathrm{SL}_2(\mathbb{C})$-free.*

Proof. We have $\pi_1 = \langle x, y : x^2 y^2 = 1 \rangle$. This group is torsion-free and virtually abelian but nonabelian, so it cannot be in $\mathscr{B}$ by Theorem 3.17.

In fact, we can prove directly that π_1 is not $\mathrm{SL}_2(\mathbb{C})$-free. Note first that the element x^2 is central and nontrivial in π_1. Suppose that some image of π_1 detects $[x, y]$. Then x and y do not commute, but both commute with x^2. By commutative transitivity of $\mathrm{SL}_2(\mathbb{C})$, it follows that the image of x^2 is in $Z(\mathrm{SL}_2(\mathbb{C}))$. But every noncentral square root of an element in $Z(\mathrm{SL}_2(\mathbb{C}))$ has trace 0, so x must map to the proper closed subvariety of trace 0 elements in SL_2. $\qquad\square$

Theorem 3.19. *If $\Gamma \in \mathscr{B}$ is finitely generated, nontrivial, and not isomorphic to $\mathbb{Z}$, then*

$$\dim \mathrm{Hom}(\Gamma, G) \geq \dim G + \mathrm{rk}\, G \qquad (8)$$

for all simply connected semisimple G.

Proof. If Γ is abelian, it must be free abelian of rank $r \geq 2$. If γ_1, γ_2 are two generators, then e_{G,γ_1} is surjective, and the fiber over $g \in G(\mathbb{C})$ maps onto $C_G(g)$ under e_{G,γ_2}. Since the dimension of every centralizer is at least rk G [Hum95, §1.6], this implies (8). We can therefore, by Lemma 2.8 and Theorem 3.17, assume that Γ is not virtually solvable, and every finitely generated subgroup of Γ which is virtually solvable is abelian.

We say $g \in G(\mathbb{C})$ is *torus-generic* if it is regular semisimple and generates a Zariski-dense subgroup of $T := C_G(g)$ (which is a maximal torus by Steinberg's theorem [Hum95, Th. 2.11]). We claim that the complement of the subset of torus-generic elements lies in the union of countably many proper closed subvarieties of G. Indeed, there are countably many closed subgroups S of a maximal torus T [Bor91, Cor. 8.3], and for each such S, the set of conjugates of elements of S is contained in the proper closed subvariety of G which is the Zariski closure of the conjugation map $\xi \colon G \times S \to G$. All fibers of this morphism have dimension $\geq \dim T$ because if $(h, s) \in \xi^{-1}(g)$, then $(hT, s) \subset \xi^{-1}(g)$. By Theorem A.4,

$$\overline{\xi(G \times S)} \leq \dim G + \dim S - \dim T < \dim G.$$

Suppose that there exists a homomorphism $\phi \colon \Gamma \to G(\mathbb{C})$ and $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$ such that:

(i) ϕ belongs to a unique irreducible component Ω of $\mathrm{Hom}(\Gamma, G)$;

(ii) the restriction of e_{G,γ_i} to Ω is dominant for $i \in \{1, 2, 3\}$;

(iii) $\phi(\gamma_i)$ is torus-generic for $i \in \{1, 2, 3\}$;

(iv) γ_3 lies in the derived group of $\langle \gamma_1, \gamma_2 \rangle$.

We claim that these conditions imply (8). Any element of $G(\mathbb{C})$ which commutes with $\phi(\gamma_1)$ and $\phi(\gamma_2)$ is semisimple by (iii). Thus the identity component S of $\mathrm{Stab}_G(\phi(\gamma_1), \phi(\gamma_2))$ is a torus. If S is nontrivial, let s be an element which generates a Zariski-dense subgroup of S. Then the centralizer $H = C_G(S)$ equals $C_G(s)$, and by Steinberg's theorem, it is connected and reductive, and S belongs to the center of H. Thus the derived group D of H has rank strictly smaller than rk H, and therefore strictly smaller than rk G. However, $\gamma_3 \in D(\mathbb{C})$ is torus-generic, so this is impossible.

We conclude that the orbit of $\phi(\gamma_2)$ under conjugation by the maximal torus $T_1 := C_G(\phi(\gamma_1))$ has dimension $\dim T_1 = $ rk G. Since conjugation by T_1 fixes $\phi(\gamma_1)$, the fiber of the restriction of e_{G,γ_1} to Ω has dimension at least rk G. By Theorem A.4, $\dim \Omega \geq \dim G + $ rk G.

To construct γ_i as above, we use induction on n to prove the following claim: if Δ denotes the free group on two generators, x and y, and $\Delta_1, \ldots, \Delta_n \subset \Delta$ have union $\Delta^\bullet$, then for some i there exist $\gamma_1, \gamma_2 \in \Delta_i$ and some element γ_3 in the intersection of Δ_i and the derived group of $\langle \gamma_1, \gamma_2 \rangle$. The case $n = 1$ is trivial. If the statement is true for n and $\Delta = \Delta_1 \cup \cdots \cup \Delta_{n+1}$, then we observe that as no two of the elements $x, xy, xy^2, \ldots$ commute with one another, at least one of the sets Δ_i, without loss of generality Δ_{n+1}, contains elements α, β which fail to commute. If Δ_{n+1} contains some element of the derived group of $\langle \alpha, \beta \rangle$, then we are done. If not, we replace Δ by any subgroup Δ' of $[\langle \alpha, \beta \rangle, \langle \alpha, \beta \rangle]$ generated by two noncommuting elements and replace $\Delta_1, \ldots, \Delta_n$ by $\Delta_i' := \Delta_i \cap \Delta'$. The claim now follows by induction.

By Theorem 1.3, Γ contains a subgroup Δ isomorphic to the free group on two generators. We define the Δ_i to be the intersections of Δ with S_Ω, as defined in Proposition 2.7, as Ω ranges over the irreducible components of $\mathrm{Hom}(\Gamma, G)$. We fix Ω such that there exist $\gamma_1, \gamma_2, \gamma_3 \in S_\Omega \cap \Delta$, with γ_3 in the derived group of $\langle \gamma_1, \gamma_2 \rangle$. Conditions (i) and (ii) on ϕ are nonempty and open, and condition (iii) is satisfied on the complement of a countable union of proper closed subvarieties of Ω. By Theorem A.5, all three can be satisfied simultaneously, and the theorem follows. $\qquad\square$

Corollary 3.20. *If $w \in F_2$ is a word such that for some simply connected semisimple group G, the word map $G^2 \to G$ is flat over some neighborhood of the identity in G, then the one-relator group Γ determined by w is not in $\mathscr{B}$.*

Proof. Applying Theorem A.4 to $e_{G,w} \colon G^2 \to G$, we deduce that the dimension of $w^{-1}(1) = \mathrm{Hom}(\Gamma, G)$ is $\dim G < \dim G + \mathrm{rk}\, G$. $\qquad\square$

We also have a version of Theorem 3.19 for $\mathscr{L}$:

Theorem 3.21. *Suppose $\Gamma \in \mathscr{L}$ is finitely generated and linear over $\mathbb{C}$. If every virtually solvable subgroup of Γ is cyclic, but Γ itself is not, then*

$$\dim \mathrm{Hom}(\Gamma, G) \geq \dim G + \mathrm{rk}\, G$$

for all simply connected semisimple G.

Proof. By the Tits alternative, every noncyclic subgroup of Γ contains a free subgroup on two generators x and y. For any semisimple G, the union of the sets S_Ω associated with the irreducible components Ω of $\mathrm{Hom}(\Gamma, G)$ consists of all nontorsion elements of Γ. Therefore, we can find two elements of the form xy^i, xy^j which belong to the same S_Ω. The proof now finishes in exactly the same way as the proof of Theorem 3.19. $\qquad\square$

With a little work, we can now deduce Theorem 1.7.

Proof of Theorem 1.7. Every hyperbolic von Dyck group is of the form

$$\Gamma = \langle x_1, x_2, x_3 : x_1^a = x_2^b = x_3^c = x_1 x_2 x_3 = 1 \rangle,$$

where a, b, c are positive integers with $1/a + 1/b + 1/c < 1$. As Γ embeds in $\mathrm{PSL}_2(\mathbb{R})$, it is linear, so it suffices to show that all of its virtually solvable subgroups are cyclic and $\dim \mathrm{Hom}(\Gamma, \mathrm{SL}_2) \leq 3$.

Let $\Delta \subset \Gamma$ be a virtually solvable subgroup, and let $H \subset \mathrm{PGL}_2$ denote its Zariski closure. As H is virtually solvable, it is a proper closed subgroup of PGL_2, and therefore either 0-dimensional, contained in a Borel subgroup, or contained in the normalizer of a maximal torus. Since every finite subgroup of a von Dyck group is cyclic, we need only consider the remaining two cases. Since Γ is a discrete and cocompact subgroup of $\mathrm{PSL}_2(\mathbb{R})$, it has no nontrivial unipotent elements [Kat92, Theorem 4.2.1], so if H is contained in a Borel, it is contained in a maximal torus. Thus, we can assume a subgroup Δ° of Δ of index ≤ 2 is contained in a maximal torus.

The fact that Γ contains a hyperbolic surface group as a subgroup of finite index implies that Δ° contains a finite-index subgroup which is abelian and embeds in a hyperbolic surface group. This subgroup must be isomorphic to $\mathbb{Z}$. Thus Δ° is an abelian group containing $\mathbb{Z}$ as a subgroup of finite index. Since the elements of finite order in Γ all have finite centralizers, Γ° is torsion-free and therefore isomorphic to $\mathbb{Z}$. The only groups which contain $\mathbb{Z}$ as a subgroup of index ≤ 2 are $\mathbb{Z}$ itself, $\mathbb{Z} \times \mathbb{Z}/2$, and the infinite dihedral group. Theorem 3.14 rules out the second and third cases. Thus, Δ is, indeed, cyclic.

For the dimension computation, we note that for all homomorphisms ϕ with $\phi(x_1) = I$, ϕ is determined by $\phi(x_2)$, so the subvariety of the representation variety satisfying this condition has dimension ≤ 3. Likewise for x_2 and x_3, and likewise if some $\phi(x_i)$ is $-I$. Therefore, we can assume that $\phi(x_i)$ has order ≥ 3 for all i, and therefore the eigenvalues $\lambda_i^{\pm 1}$ are pairwise distinct. There are finitely many possibilities for $\lambda_1, \lambda_2, \lambda_3$, given the values a, b, c. We fix these values and show that the resulting subvariety X has dimension ≤ 3. Indeed, e_{x_1, SL_2} maps X to the (2-dimensional) conjugacy class Y of the diagonal matrix

$$D_{\lambda_1} := \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}.$$

As this morphism respects the conjugation action of SL_2, and Y consists of a single SL_2-orbit, the dimension of the fibers X_y does not depend on y. We therefore consider the fiber $X_{D_{\lambda_1}}$. By Theorem A.4, it suffices to prove that this dimension is 1. To do this, we note that if

$$\phi(x_2) = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$$

then z_{11} and z_{22} are uniquely determined by the conditions $\mathrm{trace}(\phi(x_2)) = \lambda_2 + \lambda_2^{-1}$ and $\mathrm{trace}(\phi(x_3)) = \lambda_3 + \lambda_3^{-1}$. The product $z_{12}z_{21}$ is then uniquely determined by the determinant 1 condition on $\phi(x_2)$. As ϕ is determined by $\phi(x_1)$ and $\phi(x_2)$, this implies that $\dim X_{D_{\lambda_1}} = 1$. $\qquad\square$

We remark that one could bypass Theorem 3.21 here by proving directly that there are finitely many $\mathrm{SL}_2(\mathbb{C})$ orbits of homomorphisms $\Gamma \to \mathrm{SL}_2(\mathbb{C})$. However, the argument above gives a nontrivial example in which the hypotheses of Theorem 3.21 can be checked.

4. An application to double word maps

In [BGGT12], Breuillard, Green, Guralnick, and Tao prove the following:

Theorem 4.1. *Let w_1, w_2 be elements in a free group of rank 2 that do not commute. Let a, b be generic elements of a semisimple Lie group G over an algebraically closed field. Then $w_1(a, b)$ and $w_2(a, b)$ generate a Zariski-dense subgroup of G.*

In light of this result, they ask the following:

Question 2 ([BGGT12, Problem 2]). *Can one characterize the set of pairs of words (w_1, w_2) in the free group F_2 such that the double word map $G \times G \to G \times G$ given by*

$$e_{w_1, w_2}(a, b) = (w_1(a, b), w_2(a, b))$$

is dominant?

Theorem 1.1 provides an interesting class of examples for which the double word map is, indeed, dominant.

Proof of Theorem 1.1. It suffices to prove the theorem for G simply connected and simple. Let w_1 and w_2 be fixed, and let $\phi := e_{w_1, w_2}$ denote the double word map associated to (w_1, w_2). It suffices to find a point $\mathbf{a} := (a_1, \ldots, a_k) \in G^k(\mathbb{C})$ at which the map ϕ_* of tangent spaces is surjective. By Theorem A.11, this implies that ϕ is dominant.

Let Γ denote the quotient of the free group on k generators x_i by the normal subgroup generated by the conjugacy class of w_1, and let $\bar{w}_2 \neq 1$ denote the image of w_2 in this group. We can identify the homomorphism variety $\mathrm{Hom}(\Gamma, G)$ with the variety $e_{w_1}^{-1}(1)$. By assumption, this is an integral variety, and at every regular point $\mathbf{a} \in w_1^{-1}(1) \subset G^k(\mathbb{C})$, by Theorem A.12, $e_{w_1 *} \colon T_{\mathbf{a}} G^k \to T_1 G$ is surjective. As Γ is assumed to be G-free, $e_{\bar{w}_2}$ is dominant. By Theorem A.11, there is an nonempty open set on which $e_{\bar{w}_2}$ induces a surjection of tangent spaces, so we may choose $\mathbf{a}$ to belong to this set as well.

We consider the map

$$\phi_* \colon T_{\mathbf{a}}(G^k) \to T_{(1, w_2(\mathbf{a}))}(G \times G) = T_1(G) \oplus T_{w_2(\mathbf{a})}(G).$$

As composition with projection onto the first summand gives the map $e_{w_1 *}$, to prove the surjectivity of ϕ_*, it suffices to prove that at $\mathbf{a}$, $\phi_* \ker e_{w_1 *}$ projects onto $T_{w_2(\mathbf{a})}(G)$. It is clear that $T_{\mathbf{a}} e_{w_1}^{-1}(1)$ lies in the kernel of $e_{w_1 *}$, and we have chosen $\mathbf{a}$ such that $e_{\bar{w}_2 *}$ is surjective. The theorem follows.

To finish, note that oriented surface groups are residually free, and therefore G-free. Further, for an oriented surface group Γ, $\mathrm{Hom}(\Gamma, G)$ is integral by [Wei64, Li93]. Thus, when k is even, we can pick $w_1 = [x_1, x_2] \cdots [x_{k-1}, x_k]$.   □

It would be interesting to find other significant examples of words w_1 satisfying the hypotheses of the theorem. For instance, do they hold in general for words of the form

$$w_1 := h(x_1, \ldots, x_k) h(x_{k+1}, \ldots, x_{2k})?$$

Appendix A. Algebraic geometry background

We collect here some basic terminology and known facts regarding algebraic varieties, with special reference to representation varieties.

By a *variety* X over a field K, we mean in this paper an affine scheme of finite type over K, i.e. $X = \mathrm{Spec}\, A$, where A is a finitely generated K-algebra, the *coordinate ring*

of X. Recall that as a set, X consists of the prime ideals of A. It is endowed with the Zariski topology, where the closed sets

$$V(I) = \{P \in \text{Spec } A : I \subset P\}$$

are in one-to-one correspondence with radical ideals $I = \text{rad}(I)$ of A. Maximal ideals $\mathfrak{m}$ of A correspond to points of X which are closed in the Zariski topology. The set of closed points of X is Zariski-dense in X [Eis95, Th. 4.19]; note that this is a property of spectra of finitely generated algebras over a field and is not true for general affine schemes. The *Zariski tangent space* $T_x(X)$ at a closed point x in X corresponding to a maximal ideal $\mathfrak{m}$ of A is the linear dual of $\mathfrak{m}/\mathfrak{m}^2$ regarded as a vector space over $A/\mathfrak{m}$.

A topological space is *irreducible* if it cannot be realized as a finite union of proper closed subsets. The closed set $V(I)$ (where I is assumed to be radical) is irreducible if and only I is a prime ideal. By the Hilbert Basis Theorem, A is a Noetherian ring; the ascending chain condition on ideals implies the descending chain condition on closed subsets, so the process of decomposing closed subsets of Spec A into finite unions of proper closed subsets must terminate, and Spec A can be written as a finite union of irreducible components. Each component can be regarded as a variety in its own right, namely Spec A/P, where P is the prime ideal associated to the component. Every prime ideal of A contains the nilradical rad A, and X is irreducible if and only if rad A is prime, in which case it is the unique minimal prime ideal, which will be denoted $\eta \in X$. If $I \subset \text{rad } A$, every prime ideal of A contains I, so that from a topological point of view there is no difference between Spec A and Spec A/I.

If L is an extension field of K, we denote by $X(L)$ the set of L-points of X, i.e., the set of K-homomorphisms $A \to L$. If K is algebraically closed, then by Hilbert's Nullstellensatz, $X(K)$ is identified with the closed points of X, so $X(K)$ is Zariski-dense in X. We will usually consider the case $K = \mathbb{C}$, and we sometimes fail to distinguish between X and $X(\mathbb{C})$ when this is unlikely to cause confusion.

We say Spec A is *integral* if A is an integral domain. This is the case if and only if Spec A is irreducible and rad $A = (0)$.

A *morphism of varieties* $f : X = \text{Spec } A \to \text{Spec } B = Y$ is determined by a homomorphism $\phi : B \to A$ of K-algebras. If $y \in Y$ corresponds to the prime ideal P of B, then the *fiber* of f at y, denoted by X_y, is the variety over $\text{Frac}(B/P)$ associated to $A \otimes_K \text{Frac}(B/P)$, where Frac denotes the field of fractions of an integral domain.

Theorem A.1. *If* $f : X \to Y$ *is a morphism of varieties and* X *is irreducible, then the Zariski closure of* $f(X)$ *is irreducible.*

Proof. As X is irreducible, the point η, corresponding to the nilradical of the coordinate ring of A, is Zariski-dense in X. The closure of any point of a topological space is irreducible, so $\overline{f(\eta)}$ is an irreducible subset of Y. As

$$\{f(\eta)\} \subset f(X) \subset \overline{f(\eta)},$$

taking closures, we obtain $\overline{f(X)} = \overline{f(\eta)}$. □

Let Γ be a group, K a field, and G an algebraic group over K, i.e., a group object in the category of varieties over K. If X is a K-variety with coordinate ring A, $\Phi\colon \Gamma \to G(A)$ is a homomorphism, and B is a K-algebra, then every element of $X(B)$ determines a homomorphism $A \to B$, therefore a homomorphism $G(A) \to G(B)$, and by composition with Φ, a homomorphism $\Gamma \to G(B)$. If Γ is finitely generated, then there exists a finitely generated K-algebra A and a homomorphism $\Phi\colon \Gamma \to G(A)$ which is universal in the sense that for all K-algebras B, there is a one-to-one correspondence between homomorphisms $\Gamma \to G(B)$ and elements of $X(B)$. (For a proof, see [LM85, Prop. 1.2], where a valid general argument is given under the unnecessary hypotheses that K is algebraically closed and of characteristic zero and $G = \mathrm{GL}_n$.) When Γ is a free group on d generators, $X = G^d$. More generally, if Γ is a quotient of F_d, we can regard X as the closed subvariety of G^d given by the conditions that each relation word $\gamma \in \ker F_d \to \Gamma$ maps to the identity in G. In particular, for a group Γ given by a single relation $\gamma \in F_d$, we can identify $\mathrm{Hom}(\Gamma, G)$ with the fiber of the evaluation morphism $e_{F_d,\gamma}\colon G^d \to G$ at the identity $1 \in G(K)$.

We observe (see [LM85]) that this construction works over a general commutative ground ring R, so if G is an affine group scheme over R, then $\mathrm{Hom}(\Gamma, G)$ is represented by $\mathrm{Spec}\, A$ for some finitely generated R-algebra A. This construction respects change of base ring so for example if $R = \mathbb{Z}$, $G = \mathrm{SL}_{n,\mathbb{Z}}$, and $\mathrm{Hom}(\Gamma, G)$ is represented by a $\mathbb{Z}$-algebra A, then for all primes p, $\mathrm{Hom}(\Gamma, \mathbb{F}_p)$ is represented by $A \otimes \mathbb{F}_p$.

If Y is an irreducible variety, we say a morphism $f\colon X \to Y$ is *dominant* if $f(X)$ is Zariski-dense in Y. This is equivalent to the statement that $f(\Omega)$ is dense in Y for some irreducible component Ω of X. The following theorem is an immediate consequence of Chevalley's Theorem [Eis95, Cor. 14.7]:

Theorem A.2. *The image of a dominant morphism $X \to Y$ contains a nonempty open subset of Y.*

A morphism $X = \mathrm{Spec}\, A \to Y = \mathrm{Spec}\, B$ corresponding to a K-algebra homomorphism $\phi\colon B \to A$ is *flat* if A is flat when regarded as a B-module via ϕ. We say $X \to Y$ is *flat in a neighborhood of a point* $x \in X$ corresponding to a prime ideal P if there exists $b \in B$ such that $\phi(b) \notin P$ and $A[1/\phi(b)]$ is flat as a $B[1/b]$-module. In particular, if X is irreducible, we say $X \to Y$ is *generically flat* if it is flat in a neighborhood of the generic point η of X. Note that *generic* here means generically in X.

Theorem A.3. *If A and B are integral domains, then* $\mathrm{Spec}\, A \to \mathrm{Spec}\, B$ *is dominant if and only if it is generically flat.*

Proof. As B is an integral domain, if $\mathrm{Spec}\, A \to \mathrm{Spec}\, B$ is dominant then $\phi\colon B \to A$ is injective. Grothendieck's generic freeness lemma [Eis95, Th. 14.4] says that there exists a nonzero b such that $A[1/\phi(b)]$ is free, and therefore flat, as a $B[1/b]$-module. Conversely, if there exists $b \in B$ such that $A[1/\phi(b)]$ is flat as $B[1/b]$-module, then every nonzero element of $B[1/b]$ maps to a nonzero element of $A[1/\phi(b)]$. Since B is an integral domain, $B \to A[1/\phi(b)]$ is injective [Eis95, Cor. 6.3]. As this homomorphism factors through $B \to A$, the latter morphism is injective, and the generic point of $\mathrm{Spec}\, A$ maps to the generic point of $\mathrm{Spec}\, B$. $\square$

By the *dimension* of $X = \operatorname{Spec} A$, we mean the Krull dimension of A, which is the maximum possible length n of a chain of prime ideals $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ of A. Note that the dimension of a variety is the maximum of the dimensions of its components. For an irreducible variety $\operatorname{Spec} A$, the dimension of X equals the transcendence degree of $\operatorname{Frac}(A/\operatorname{rad} A)$ [Eis95, §8.2.1, Th. A]. Given a morphism $X \to Y$, the function $y \mapsto \dim X_y$ is not in general constant on irreducible components of Y. However, we have the following facts:

Theorem A.4. *Let $f \colon X \to Y$ be a morphism of irreducible varieties. Then:*

(1) *The function which assigns to each $x \in X$ the dimension of the irreducible component of $X_{f(x)}$ to which x belongs is upper semicontinuous.*

(2) *If f is dominant, there exists a nonempty open set of Y on which every irreducible component of X_y has dimension $\dim X - \dim Y$.*

(3) *If f is dominant and flat in a neighborhood of y, then $\dim X_y = \dim X - \dim Y$.*

(4) *Every component of X_y has dimension at least $\dim X - \dim Y$.*

Proof. The first part is a special case of [Eis95, Th. 14.8a]. For the remaining statements, let us first assume that f is dominant, i.e., we have a homomorphism $\phi \colon B \to A$ for which $f^{-1}(\operatorname{rad} A) = \operatorname{rad} B$. If we replace A and B by $A/\operatorname{rad} A$ and $B/\operatorname{rad} B$ respectively and ϕ by the induced homomorphism $A/\operatorname{rad} A \to B/\operatorname{rad} B$, then A and B become integral domains and ϕ becomes an injection, but at the level of points nothing changes. So, we can apply [Eis95, Cor. 13.5] and deduce that the dimension of the generic fiber (which is irreducible since X is) satisfies $\dim X_\eta = \dim X - \dim Y$. Item (1) now implies (2) and (4) in the dominant case. Item (3) follows from (2) and [Eis95, Th. 10.10]. For (4), if f is not dominant, we can replace B by $B/\ker \phi$, which means, topologically, that Y is replaced by $\overline{f(X)}$. The general statement follows.   □

The following result is an algebraic analogue of the Baire Category Theorem.

Theorem A.5. *Let K be an algebraically closed field and J a set whose cardinality is strictly less than that of K. If X is an irreducible variety over K and $\{X_j : j \in J\}$ a collection of proper closed subvarieties, then*

$$X(K) \setminus \bigcup_{j \in J} X_j(K) \neq \emptyset.$$

Proof. Without loss of generality we can assume that X is irreducible over K. Replacing A by $A/\operatorname{rad} A$ does not change the underlying set, so without loss of generality we can assume A is an integral domain. We use induction on $\dim X$. If $\dim X = 1$, every proper subvariety has a finite number of 0-dimensional components, so it suffices to prove that $|X(K)| \geq |K|$. Let a denote any element of its fraction field which is not in K. Then $t \mapsto a$ defines a morphism $\operatorname{Spec} A \to \operatorname{Spec} K[t]$. The Zariski closure of the image is irreducible, hence either a point or all of $\operatorname{Spec} K[t]$. As a is not constant, Theorem A.2 implies that the image contains all but finitely many elements of K. Since K is infinite, the cardinality of the image equals that of K, which finishes the base case.

For the induction step, we again use a morphism $X \to \operatorname{Spec} K[t]$ given by a nonconstant element a. Again the image contains all but finitely many points of $\operatorname{Spec} K[t]$, and

by Theorem A.4 all components of all nonempty fibers of this morphism have dimension $\dim X - 1$. Therefore, no irreducible proper subvariety of X can contain a component of more than one fiber, and no proper subvariety of X can contain components of more than a finite number of fibers. It follows that some nonempty fiber has an irreducible component X' not contained in any of the X_j. By replacing X by X' and each X_j by $X' \cap X_j$, the theorem follows by induction. □

In particular, we apply this theorem in the case where $K = \mathbb{C}$ and J is countable.

Theorem A.6. *The following three conditions on a finitely generated K-algebra A are equivalent:*

(1) $\operatorname{Spec} A \otimes_K L$ *is irreducible for all finite K-extensions L.*
(2) $\operatorname{Spec} A \otimes_K \bar{K}$ *is irreducible for some algebraic closure $\bar{K}$ of K.*
(3) $\operatorname{Spec} A \otimes_K L$ *is irreducible for all K-extensions L.*

Proof. By [Gro65, Prop. 4.5.9], (1) implies (3). It is trivial that (3) implies (2). To see that (2) implies (1), we note that $A_L := A \otimes_K L$ fails to be irreducible for some finite extension L if and only if there exist $f, g \in A_L$ which are not nilpotent and such that $fg = 0$. Realizing L as a K-subextension of $\bar{K}$, we conclude that $A_{\bar{K}}$ also contains nonnilpotent elements which multiply to zero. □

We say a variety $X = \operatorname{Spec} A$ over K is *geometrically irreducible* if A satisfies any of these equivalent conditions. From Theorem A.6 and the finiteness of the set of irreducible components $\operatorname{Spec} A_{\bar{K}}$, which is a variety over $\bar{K}$, it follows that K has some finite extension L such that A_L is a finite union of geometrically irreducible components. By the proof of [Gro65, Cor. 4.5.11], the field of fractions of $A_{\bar{K}}/P$, where P is any minimal prime ideal, contains a finite extension of $\operatorname{Frac}(A/\operatorname{rad} A)$, so the transcendence degrees over K are equal. It follows that the dimension of every irreducible component of $\operatorname{Spec} A_L$ equals $\dim A$.

Theorem A.7. *Let A be a finitely generated $\mathbb{Z}$-algebra. If there exist infinitely many primes p such that $A \otimes \mathbb{F}_p$ is a geometrically irreducible $\mathbb{F}_p$-algebra of dimension n, then $A \otimes \mathbb{C}$ is irreducible of dimension n.*

Proof. This is a special case of [Gro66, Th. 9.7.9]. □

Replacing X by $X^{\mathrm{red}} := \operatorname{Spec} A/\operatorname{rad} A$ does not change $X(L)$ for any field extension L of K, so if K is a finite field $\mathbb{F}_q$, we have $|X(\mathbb{F}_{q^n}) = X^{\mathrm{red}}(\mathbb{F}_{q^n})$ for all $n \in \mathbb{N}$. Now, X^{red} is a variety in the sense of Lang–Weil [LW54], so by the result of that paper,

$$|X(\mathbb{F}_{q^n})| = q^{n \dim X}(1 + o(1)).$$

If X is irreducible, then for some $m \in \mathbb{N}$,

$$X_m := \operatorname{Spec} A \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m}$$

decomposes as a union of c irreducible varieties of dimension $\dim X$, and

$$|X(\mathbb{F}_{q^{mn}})| = |X_m(\mathbb{F}_{q^{mn}})| = cq^{mn \dim X}(1 + o(1)).$$

For any variety, we can take m sufficiently divisible that X_m is a union of geometrically irreducible components, and

$$|X(\mathbb{F}_{q^{mn}})| = c_{\dim X} q^{mn \dim X}(1 + o(1)),$$

where c_k is the number of irreducible components of dimension k. From this we deduce:

Theorem A.8. *If $|X(\mathbb{F}_{q^n})| = (1 + o(1))q^{kn}$, then X has a single component of dimension k, it is geometrically irreducible, and all other components have lower dimension.*

A local ring B with maximal ideal M is *regular* if $\dim_{B/M} M/M^2 = \dim B$. In the special case of $B = A_{\mathfrak{m}}$ where $\mathfrak{m}$ is the maximal ideal corresponding to a closed point x, this is equivalent to the condition $\dim T_x X = \dim X$, and we say that X is *regular at x*. If all local rings of A are regular, we say that X is *nonsingular*.

Theorem A.9. *Suppose K is a perfect field and a K-variety X is regular at a closed point $x \in X(K)$. Then there exists an element $f \in A$ which does not lie in the maximal ideal associated to x, a morphism*

$$Y = \mathrm{Spec}[y_1, \ldots, y_m] \to Z = \mathrm{Spec}[z_1, \ldots, z_{m-n}],$$

and an isomorphism $i \colon \mathrm{Spec}\, A[1/f] \to Y_z$, where z corresponds to the ideal

$$(z_1, \ldots, z_{m-n}),$$

such that the induced map of Zariski tangent spaces $T_{i(x)}Y \to T_z Z$ is surjective. Moreover, there exists a morphism from X to $\mathrm{Spec}\, K[t_1, \ldots, t_n]$ which induces an isomorphism of tangent spaces at x.

Proof. Conversely, since K is perfect, if X is regular at x, then $Y_z \to \mathrm{Spec}\, K$ is smooth at $i(x)$ in the sense of Grothendieck [Gro67, Cor. 17.15.3]. By [Mil80, I Prop. 3.24], for some $m \geq n$ and some $P_1, \ldots, P_{m-n} \in K[y_1, \ldots, y_m]$, there exists $f \notin \mathfrak{m}$ such that

$$A[1/f] \cong K[y_1, \ldots, y_m]/(P_1, \ldots, P_{m-n})$$

and

$$\det\left(\frac{\partial P_i}{\partial y_j}\right)_{1 \leq i, j \leq m-n}$$

is a unit in $A[1/f]$. The K-homomorphism $K[z_1, \ldots, z_{m-n}] \to K[y_1, \ldots y_m]$ sending $z_i \mapsto P_i(y_1, \ldots, y_m)$ determines a morphism $Y \to Z$ whose fiber over $(z_1, \ldots, z_{m-n})$ is isomorphic to $\mathrm{Spec}\, A[1/f]$ and which induces a surjection of tangent spaces as claimed. By [Gro67, Cor. 17.15.9], there is a morphism $g \colon X \to \mathrm{Spec}\, K[t_1, \ldots, t_n]$ which is étale at x and therefore smooth. Since the dimensions are equal, g induces an isomorphism of tangent spaces at x [Gro67, Th. 17.11.1]. $\square$

Theorem A.10. *If K is a local field, X/K a variety, and x a regular point of X corresponding to a maximal ideal with $A/\mathfrak{m} \cong K$, then $X(K)$ is Zariski-dense in X.*

Proof. Let g be the morphism $g: X \to \operatorname{Spec} K[t_1, \ldots, x_n]$ of Theorem A.9. By the implicit function theorem [Bou67, Th. 5.7.1], there exists a neighborhood of x in $X(K)$ which is a K-analytic manifold, and by a second application of the same result, g induces an isomorphism from a neighborhood of x in $X(K)$ to a neighborhood of $g(i(x)) = (0, \ldots, 0)$ in K^n. If the Zariski closure of $X(K)$ in X were of dimension $< n$, the Zariski closure Z of its image in $\operatorname{Spec} K[t_1, \ldots, t_n]$ would be a proper closed subvariety of $\operatorname{Spec} K[x_1, \ldots, x_n]$. By induction on n, $Z(K) \subset K^n$ cannot contain a subset of the form $Z_1 \times \cdots \times Z_n$ where the Z_i are all infinite. In particular, it cannot contain a nonempty open subset of K^n. $\qquad\square$

Theorem A.11. *Let $f: X \to Y$ be a morphism of integral varieties over $\mathbb{C}$ and x be a regular point of $X(\mathbb{C})$ such that $f(x)$ is a regular point of $Y(\mathbb{C})$. If $T_x X \to T_{f(x)} Y$ is surjective, then $X \to Y$ is dominant. Conversely, if $X \to Y$ is dominant, there exists a nonempty open subset $U \subset X(\mathbb{C})$ such that for all $x \in U$, $T_x X \to T_{f(x)} Y$ is surjective.*

Proof. As x and $y := f(x)$ are regular points, $X \to \operatorname{Spec} \mathbb{C}$ and $Y \to \operatorname{Spec} \mathbb{C}$ are smooth at x and y respectively. By [Gro67, Th. 17.11.1], surjectivity on the level of tangent spaces now implies that $X \to Y$ is smooth (and therefore flat) at x. By [Gro66, Th. 11.1.1], $X \to Y$ is flat in a neighborhood of x, and hence is generically flat. By Theorem A.3, the morphism is dominant.

For the converse, we use the Jacobian criterion for smoothness in the form [Eis95, Cor. 16.23] to prove that there exists $b \in B$ such that $A[1/b]$ is smooth over $B[1/b]$, where A and B are the coordinate rings of X and Y respectively. $\qquad\square$

Theorem A.12. *If $f: X \to Y$ is a dominant morphism of nonsingular irreducible varieties over $\mathbb{C}$, $y \in Y(\mathbb{C})$, every component of X_y has dimension $\dim X - \dim Y$, and $x \in X_y(\mathbb{C})$ is a nonsingular point, then $T_x X \to T_y Y$ is surjective.*

Proof. By [Gro64, $\mathbf{0}_{\text{IV}}$ Cor. 17.1.3], X and Y are Cohen–Macaulay, and so by [Gro66, Prop. 15.4.2], f is flat at every point of the fiber X_y. By [Gro67, Th. 17.5.1], f is smooth at x, and by [Gro67, Th. 17.11.1], we conclude that $T_x X \to T_y Y$ is surjective. $\qquad\square$

Appendix B. Character bounds

Here, we present a character bound that is used in the proof of Theorem 1.5.

Theorem B.1. *Given an integer $n > 0$ and positive real numbers α and β such that*

$$\alpha < \frac{\beta^2}{1 + 2\beta},$$

for all sufficiently large finite fields $\mathbb{F}_q$, all irreducible characters χ of $G_n := \operatorname{GL}_n(\mathbb{F}_q)$, and all semisimple elements $x \in G_n$ whose maximal eigenvalue multiplicity is $\le \alpha n$, we have

$$|\chi(x)| \le \chi(1)^\beta.$$

We understand that Bezrukavnikov, Liebeck, Shalev, and Tiep have as yet unpublished estimates of character values of groups of Lie types which are both stronger and more general than this result. Our proof, however, is elementary, using only classical results on the character theory of $GL_n(\mathbb{F}_q)$ due to J. A. Green and Robert Steinberg.

Lemma B.2. *Let n be an arbitrary natural number. There exists a real number C_n such that if s is a natural number less than n and $T : \mathbb{F}_q^n \to \mathbb{F}_q^n$ is a semisimple linear transformation whose eigenvalues all have algebraic multiplicity at most m, then the number of s-dimensional $\mathbb{F}_q$-subspaces of $\mathbb{F}_q^n$ which are fixed by T is at most $C_n q^{ms}$.*

Proof. Let $V = \mathbb{F}_q^n$ decompose as a direct sum $V_1^{a_1} \oplus \cdots \oplus V_r^{a_r}$, where the V_i are pairwise nonisomorphic irreducible $\mathbb{F}_q[T]$-modules of $\mathbb{F}_q$-dimension $b_1, \ldots, b_r$ respectively, and $a_1, \ldots, a_r \le m$. We can identify V_i with $\mathbb{F}_{q^{b_i}}$ in such a way that $\mathbb{F}_q[T]$-submodules of $V_i^{a_i}$ correspond to $\mathbb{F}_{q^{b_i}}$-subspaces of $\mathbb{F}_{q^{b_i}}^{a_i}$. Every subspace $W \subset V$ fixed by T is a direct sum $W_1 \oplus \cdots \oplus W_r$, where each W_i is an $\mathbb{F}_q[T]$-submodule of $V_i^{a_i}$. For each r-tuple of nonnegative integers w_i such that $\sum_i b_i w_i = s$, we can classify the subspaces W such that $\dim_{\mathbb{F}_{q^{b_i}}} W_i = w_i$ by a product of r Grassmannians $G(a_i, w_i)(\mathbb{F}_{q^{b_i}})$. As

$$|G(a_i, w_i)(\mathbb{F}_{q^{b_i}})| = \frac{\prod_{j=1}^{a_i}(q^{b_i j} - 1)}{\prod_{k=1}^{w_i}(q^{b_i k} - 1) \prod_{l=1}^{a_i - w_i}(q^{b_i l} - 1)},$$

we have

$$\log_q \prod_{i=1}^r |G(a_i, w_i)(\mathbb{F}_{q^{b_i}})| = \sum_{i=1}^r b_i w_i (a_i - w_i) + o(1)$$

$$\le \sum_{i=1}^r b_i w_i m + o(1) = ms + o(1). \qquad \square$$

Proof of Theorem B.1. Throughout the proof, we can and do assume without loss of generality that q is sufficiently large in terms of n; $o(1)$ means "smaller than any given positive ϵ when q is large enough in terms of n and ϵ." We also assume $\beta < 1$, since the theorem is trivial otherwise.

We follow the notation and terminology of Green [Gre55]. For s a positive integer, an *s-simplex* g is a q-Frobenius-orbit $\{\phi^{q^s} = \phi, \phi^q, \phi^{q^2}, \ldots, \phi^{q^{s-1}}\}$ of length s of complex characters of the multiplicative group $\mathbb{F}_{q^s}^\times$. We write $s = d(g)$ and call it the *degree* of s. By [Gre55, Th. 13], the irreducible characters of G_n are indexed by partition-valued functions ν on the set S of simplices such that

$$\sum_{g \in S} |\nu(g)| d(g) = n,$$

where $|p|$ denotes the sum of the parts of the partition p. Moreover, if $g_1, \ldots, g_k$ are the simplices on which ν is supported, the character associated to ν is obtained by parabolic induction from the characters of

$$G_{|\nu(g_1)|d(g_1)}, \ldots, G_{|\nu(g_k)|d(g_1 k)}$$

associated with the partition-valued functions ν_i supported at g_i and such that $\nu_i(g_i) = \nu(g_i)$. In particular, the degree of χ is at least $|G_n/P|$, where P is the parabolic subgroup associated to the sequence

$$|\nu(g_1)|d(g_1), \ldots, |\nu(g_k)|d(g_k).$$

We can assume the g_i to be chosen in such an order that this sequence is nonincreasing. If ν is supported on a single simplex, we say that the character is *primary*.

Let $\lambda_1 + \cdots + \lambda_r = n$ express n as a sum of positive integers. If P_λ denotes the stabilizer in G_n of a flag of $\mathbb{F}_q$-spaces

$$(0) = V_0 \subset V_1 \subset \cdots \subset V_r = \mathbb{F}_q^n, \tag{9}$$

where $\dim_{\mathbb{F}_q} V_i / V_{i-1} = \lambda_i$, then

$$|G_n/P_\lambda| = \frac{\prod_{i=1}^n (q^i - 1)}{\prod_{i=1}^r \prod_{j=1}^{\lambda_i} (q^j - 1)},$$

so

$$\log_q |G_n/P_\lambda| = \sum_{1 \le i < j \le r} \lambda_i \lambda_j + o(1) = n^2/2 - \sum_{i=1}^r \lambda_i^2/2 + o(1).$$

In particular, if χ is a character of G_n induced from some character of P_λ, then

$$\log_q \chi(1) \ge n^2/2 - \sum_{i=1}^r \lambda_i^2/2 + o(1). \tag{10}$$

If $\lambda_1 \ge \cdots \ge \lambda_r$, then

$$n^2/2 - \sum_{i=1}^r \lambda_i^2/2 \ge \begin{cases} \lambda_1(n - \lambda_1) & \text{if } \lambda_1 \ge n/2, \\ n^2/4 & \text{if } \lambda_1 \le n/2. \end{cases} \tag{11}$$

Let $a_1, \ldots, a_k$ denote the eigenvalue multiplicities of x. As $\max_i a_i \le \alpha n$ and $\sum_i a_i = n$, we have $\sum_i a_i^2 \le \alpha n^2$. Since the centralizer $C(x)$ of x in G_n is the group of $\mathbb{F}_q$-points of a connected reductive group of dimension $\sum_i a_i^2$,

$$\log_q |C(x)| = \sum_i a_i^2 + o(1) \le \alpha n^2 + o(1). \tag{12}$$

By Schur's lemma, $|\chi(x)| \le |C(x)|^{1/2}$, so if $\lambda_1 \le n/2$, then by (10)–(12), $|\chi(x)| > \chi(1)^\beta$ implies

$$\beta n^2/4 \le \beta \log_q \chi(1) + o(1) < \log_q |\chi(x)| + o(1) \le \frac{\log_q |C(x)|}{2} + o(1)$$

$$\le (\alpha/2)n^2 + o(1) < \frac{\beta^2}{2 + 4\beta} n^2 + o(1),$$

impossible since $\beta > 0$. Thus, we assume $\lambda_1 > n/2$.

Let $\gamma := \alpha/\beta$, so that $\gamma < 1/2$. We claim that if q is sufficiently large and $|\chi(x)| > \chi(1)^\beta$, then $\lambda_1 > (1 - \gamma)n$. Indeed, if $n/2 \leq \lambda_1 \leq (1 - \gamma)n$, then

$$\frac{\alpha n^2}{2} \geq \frac{\log_q |C(x)|}{2} + o(1) \geq \log_q |\chi(x)| + o(1) > \beta \log_q \chi(1) + o(1)$$
$$\geq \beta \lambda_1 (n - \lambda_1) + o(1) \geq \beta n^2 \gamma (1 - \gamma) + o(1) > \alpha n^2/2 + o(1),$$

which is impossible for large q. This justifies our claim. We therefore have

$$n - \lambda_1 < \gamma n < 2\gamma \lambda_1.$$

Now χ arises by parabolic induction from an irreducible representation χ' of $G_{n-\lambda_1}$ and a primary irreducible representation χ'' of G_{λ_1}, where $n - \lambda_1 < \gamma n$. Thus,

$$\log_q \chi(1) = \log_q \chi'(1) + \log_q \chi''(1) + \lambda_1(n - \lambda_1) + o(1)$$
$$\geq \log_q \chi''(1) + \lambda_1(n - \lambda_1) + o(1).$$

On the other hand, $\chi(x)$ can be written as a sum of terms of the form $\chi'(x')\chi''(x'')$, where $x' \in G_{n-\lambda_1}$, $x'' \in G_{\lambda_1}$, and $x' \oplus x''$ is conjugate to x. The terms in the sum are indexed by elements of the Grassmannian of $(n - \lambda_1)$-planes $W \subset \mathbb{F}_q^n$ such that x preserves W, the action of x on W is conjugate to x', and the action of x on $\mathbb{F}_q^n/W$ is conjugate to x''.

The dimension estimate

$$\log_q \chi'(1) \leq \binom{n - \lambda_1}{2} + o(1)$$

can be deduced from [Gre55], but can also be found in various forms in the literature. (See, e.g., [Sei90, Th. 2.1], [LMT13, Th. 5.1].) By Lemma B.2, we have

$$\log_q |\chi(x)| \leq \log_q \chi'(1) + \max_{x''} \log_q |\chi''(x'')| + \alpha(n - \lambda_1)n + o(1)$$
$$< (n - \lambda_1)^2/2 + \max_{x''} \log_q |\chi''(x'')| + \alpha(n - \lambda_1)n + o(1)$$
$$< \gamma \lambda_1 (n - \lambda_1) + \max_{x''} \log_q |\chi''(x'')| + \alpha(n - \lambda_1)n + o(1)$$
$$< \gamma(n - \lambda_1)n + \max_{x''} \log_q |\chi''(x'')| + \alpha(n - \lambda_1)n + o(1)$$
$$< \beta(n - \lambda_1)n + \max_{x''} \log_q |\chi''(x'')| + o(1) < \beta \log_q |\chi(1)| + o(1)$$

provided that

$$\log_q |\chi''(x'')| \leq \beta \log_q \chi''(1) + o(1)$$

for all semisimple $x'' \in G_{\lambda_1}$ with eigenvalue multiplicity less than or equal to

$$\alpha n \leq \frac{\alpha \lambda_1}{1 - \gamma} \leq \frac{\beta^2 \lambda_1}{1 + \beta}.$$

Replacing n by λ_1 and χ'' by χ, we have a statement very similar to what we originally set out to prove. The advantage over the original statement is that we can now assume that

χ is primary; the disadvantage is that the upper bound on maximal eigenvalue multiplicity as a fraction of matrix size is $\beta^2/(1 + \beta)$ instead of $\alpha = \beta^2/(1 + 2\beta)$.

From now on we assume that χ is associated to a partition-valued function ν of simplices supported on a single g. Denoting by s the degree of g and setting $v = |\nu(g)|$, we have $n = sv$. By [Gre55, Lemma 7.4], for each partition λ of v, there exists a rational function $\{\lambda : t\}$ such that

$$\chi(1) = \Big(\prod_{i=1}^{n}(q^i - 1)\Big)\{\nu(g) : q^s\}^{-1}.$$

As this takes integer values for all prime powers q, it follows that the pole at $t = \infty$ of $\{\lambda : t\}^{-1}$ has order at most $\binom{v+1}{2}$. Thus,

$$\log_q \chi(1) \geq \binom{n+1}{2} - s\binom{v+1}{2} + o(1) \geq \frac{n^2}{4} + o(1)$$

if $s \geq 2$. If $|\chi(x)| > \chi(1)^{\beta}$, then

$$\frac{\beta^2}{1 + \beta}n^2 \geq \log_q |C(x)| + o(1) > 2\log_q |\chi(x)| + o(1)$$

$$\geq 2\beta \log_q \chi(1) + o(1) > \frac{\beta n^2}{2} + o(1),$$

which is impossible when q is sufficiently large since $\beta < 1$. Thus, we can assume that $s = 1$, which means that after tensoring with a degree 1 character of G_n (which does not affect $|\chi(x)|$, of course), we can assume that ν is supported on the trivial simplex, i.e., χ is a *unipotent* character.

Let $\chi = \chi_\lambda$ denote the unipotent character associated with any partition $\lambda_1 + \cdots + \lambda_r = n$. Let ϕ_λ be the permutation character associated with the same partition, i.e., the character associated to the action of G_n on the set of $\mathbb{F}_q$-flags (9). It is a classical theorem of Steinberg [Ste51, §2, Cor. 1] that

$$\phi_\lambda = \sum_{\mu} K_{\lambda,\mu}\chi_\mu,$$

where $K_{\lambda,\mu}$ is the Kostka number associated to λ and μ, and the sum is taken over all partitions μ of n. In particular, $K_{\lambda,\mu} = 0$ unless $\mu \preceq \lambda$ in the partial order of *majorization*. (This means that $\mu_1 + \cdots + \mu_s \leq \lambda_1 + \cdots + \lambda_s$ for all $s \leq r$, with equality when $s = r$.) It is also known that $K_{\lambda,\lambda} = 1$, and from [Gre55, Lemma 7.4] it follows that $\dim \chi_\mu \leq \dim \chi_\lambda$ if q is sufficiently large and $\mu \preceq \lambda$. We can therefore proceed by induction with respect to the partial order $\preceq$. The base case is trivial, and it suffices to prove

$$\log_q |\phi_\lambda(x)| \leq \beta \log_q |\phi_\lambda(1)| + o(1).$$

As the inner product of ϕ_λ with itself is bounded above independent of q, we have

$$\log_q |\phi_\lambda(x)| \leq \frac{\log_q |C(x)|}{2} + o(1) \leq \frac{\beta^2 n^2}{2(1 + \beta)} + o(1).$$

Thus, by (10) and (11), $\lambda_1 > n/2$. Now $\phi_\lambda(x)$ counts the number of x-stable flags (9) with dim V_i/V_{i-1} giving the parts of the partition λ in some given order. We choose an order such that dim $\mathbb{F}_q^n/V_{r-1} = \lambda_1$. Each such flag determines the combinatorial data of the multiplicities of the various eigenvalues of x on each V_i. This combinatorial data fixes an irreducible component of the variety of x-stable flags. The number of possibilities for the data is bounded independent of q, and each component is a product of flag varieties on spaces of dimensions $\leq \beta^2 n/(1+\beta)$ whose dimensions add up to $n - \lambda_1$. The dimension of a flag variety on a space of dimension a is less than $a^2/2$, and since the eigenspaces of x all have dimension $\leq \beta^2 n/(1+\beta)$, we deduce that

$$\log_q |\phi_\lambda(x)| \leq \frac{\beta^2(n-\lambda_1)n}{2(1+\beta)} + o(1).$$

On the other hand,

$$\log_q \phi_\lambda(1) = \frac{n^2 - \sum_{i=1}^r \lambda_i^2}{2} + o(1) \geq \lambda_1(n-\lambda_1) + o(1)$$
$$\geq (n-\lambda_1)n/2 + o(1),$$

so this implies $\log_q |\phi_\lambda(x)| < \beta \log_q \phi_\lambda(1)$ for all q sufficiently large. □

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