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On concavity of solutions of the Dirichlet problem for the equation $(-\Delta)^{1/2}\varphi = 1$ in convex planar regions

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Abstract. For a sufficiently regular open bounded set $D \subset \mathbb{R}^2$ let us consider the equation $(-\Delta)^{1/2}\varphi(x) = 1$ for $x \in D$ with the Dirichlet exterior condition $\varphi(x) = 0$ for $x \in D^c$. Its solution $\varphi(x)$ is the expected value of the first exit time from D of the Cauchy process in $\mathbb{R}^2$ starting from x. We prove that if $D \subset \mathbb{R}^2$ is a convex bounded domain then φ is concave on D. To do so we study the Hessian matrix of the harmonic extension of φ. The key idea of the proof is based on a deep result of Hans Lewy concerning the determinants of Hessian matrices of harmonic functions.

Keywords. Fractional Laplacian, concavity, Hessian matrix, harmonic function, Cauchy process, first exit time

1. Introduction

Let $D \subset \mathbb{R}^2$ be an open bounded set which satisfies a uniform exterior cone condition on ∂D and consider the following Dirichlet problem for the square root of the Laplacian:

$$(-\Delta)^{1/2}\varphi(x) = 1, \qquad x \in D, \tag{1}$$

$$\varphi(x) = 0, \qquad x \in D^c, \tag{2}$$

where we understand that φ is a continuous function on $\mathbb{R}^2$. The operator $(-\Delta)^{1/2}$ in $\mathbb{R}^2$ is given by

$$(-\Delta)^{1/2} f(x) = \frac{1}{2\pi} \lim_{\varepsilon\to 0^+} \int_{|y-x|>\varepsilon} \frac{f(x) - f(y)}{|y-x|^3}\, dy,$$

whenever the limit exists.

It is well known that (1)–(2) has a unique solution, which has a natural probabilistic interpretation. Let X_t be the Cauchy process in $\mathbb{R}^2$ (that is, a symmetric α-stable process in $\mathbb{R}^2$ with $\alpha = 1$) with transition density $p_t(x) = \frac{1}{2\pi} t(t^2 + |x|^2)^{-3/2}$ and let $\tau_D = \inf\{t \geq 0 : X_t \notin D\}$ be the first exit time of X_t from D. Then $\varphi(x) = E^x(\tau_D)$, $x \in \mathbb{R}^2$, where E^x is the expected value of the process X_t starting from x [18]. The function $E^x(\tau_D)$ plays an important role in the potential theory of symmetric stable processes (see e.g. [5], [4], [11]).

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About 10 years ago R. Bañuelos asked about p-concavity of $E^x(\tau_D)$ for symmetric α-stable processes. The problem was inspired by a beautiful result of Ch. Borell about $1/2$-concavity of $E^x(\tau_D)$ for the Brownian motion.

The main result of this paper is the following theorem. It solves the problem posed by R. Bañuelos for the Cauchy process in $\mathbb{R}^2$.

Theorem 1.1. *If $D \subset \mathbb{R}^2$ is a bounded convex domain then the solution of (1)–(2) is concave on D.*

To the best of the author's knowledge this is the first result concerning concavity of solutions of equations for fractional Laplacians on general convex domains. There is a recent interesting paper of R. Bañuelos and R. D. DeBlassie [1] in which the first eigenfunction of the Dirichlet eigenvalue problem for fractional Laplacians on Lipschitz domains is studied, but in that paper superharmonicity and not concavity of the first eigenfunction is proved (similar results were also obtained by M. Kaßmann and L. Silvestre [22]). In [3] concavity of the first eigenfunction for fractional Laplacians was studied, but only for boxes and not for general convex domains.

Now let $D \subset \mathbb{R}^d$, $d \geq 1$, be an open bounded set which satisfies a uniform exterior cone condition on ∂D, let $\alpha \in (0, 2]$ and consider a more general Dirichlet problem for the fractional Laplacian

$$(-\Delta)^{\alpha/2}\varphi(x) = 1, \qquad x \in D, \tag{3}$$

$$\varphi(x) = 0, \qquad x \in D^c, \tag{4}$$

where we understand that φ is a continuous function on $\mathbb{R}^d$. The operator $(-\Delta)^{\alpha/2}$ in $\mathbb{R}^d$ for $\alpha \in (0, 2)$ is given by

$$(-\Delta)^{\alpha/2} f(x) = \mathcal{A}_{d,-\alpha} \lim_{\varepsilon \to 0^+} \int_{|y-x|>\varepsilon} \frac{f(x) - f(y)}{|y - x|^{d+\alpha}} \, dy,$$

whenever the limit exists, with $\mathcal{A}_{d,-\alpha} = 2^\alpha \Gamma((d+\alpha)/2)/(\pi^{d/2}|\Gamma(-\alpha/2)|)$. For $\alpha = 2$ the operator $(-\Delta)^{\alpha/2}$ is simply $-\Delta$.

It is well known that (3)–(4) has a unique solution. It is the expected value of the first exit time from D of the symmetric α-stable process in $\mathbb{R}^d$.

Remark 1.2. For $\alpha = 2$, i.e. for the Laplacian, it is well known that if $D \subset \mathbb{R}^d$ is a bounded convex domain then the solution of (3)–(4) is $1/2$-concave, that is, $\sqrt{\varphi}$ is concave. This was proved for $d = 2$ in 1969 by L. Makar-Limanov [32], and for $d \geq 3$ in 1983 by Ch. Borell [8] and independently by A. Kennington [23], [24] using ideas of N. Korevaar [25].

Remark 1.3. Let $\alpha \in (0, 2]$ and φ be a solution of (3)–(4) for $D = B(0, r) \subset \mathbb{R}^d$, $d \geq 1$, the open ball with centre 0 and radius $r > 0$. Then φ is given by the explicit formula [18] (see also [21], [17]) $\varphi(x) = C_B(r^2 - |x|^2)^{\alpha/2}$ for $x \in B(0, r)$, where $C_B = \Gamma(d/2)(2^\alpha \Gamma(1 + \alpha/2)\Gamma(d/2 + \alpha/2))^{-1}$. In particular φ is concave on $B(0, r)$.

Remark 1.4. For any $\alpha \in (1, 2)$ and $d \geq 2$ there exists a bounded convex domain $D \subset \mathbb{R}^d$ (a sufficiently narrow bounded cone) such that φ is not concave on D. This is justified in Section 7. In particular, this implies that the assertion of Theorem 1.1 is not true for problem (3)–(4) for $\alpha \in (1, 2)$.

For general $\alpha \in (0, 2)$ and $d \geq 2$ we have the following regularity result.

Theorem 1.5. *Let $\alpha \in (0, 2)$, $d \geq 2$ and let φ be a solution of (3)–(4). If $D \subset \mathbb{R}^d$ is a bounded convex domain then*

(a) *for any $x_0 \in \partial D$, $x \in D$ and $\lambda \in (0, 1)$,*

$$\varphi(\lambda x + (1 - \lambda)x_0) \geq \lambda^\alpha \varphi(x),$$

(b) *for any $x, y \in D$ and $\lambda \in (0, 1)$,*

$$\varphi(\lambda x + (1 - \lambda)y) \geq \max(\lambda^\alpha \varphi(x), (1 - \lambda)^\alpha \varphi(y)).$$

The proof of this theorem is in Section 7. It is based on a tricky observation and is much easier than the proof of Theorem 1.1. Clearly, Theorem 1.5 does not imply p-concavity of φ for any $p \in [-\infty, 1]$. Some conjectures concerning p-concavity of solutions of (3)–(4) are presented in Section 7.

Below we present the idea of the proof of Theorem 1.1. The proof is in the spirit of papers by L. Caffarelli and A. Friedman [9] and N. Korevaar and J. Lewis [26], in which they study the geometric properties of solutions of some PDEs using the constant rank theorem and the method of continuity. In the proof of Theorem 1.1 the role of the constant rank theorem is played by the following result of Hans Lewy from 1968.

Theorem 1.6 (Hans Lewy, [31]). *Let $u(x_1, x_2, x_3)$ be real and harmonic in a domain Ω of $\mathbb{R}^3$ and let $H(u)$ denote the determinant of the Hessian matrix of u. Suppose $H(u)$ vanishes at a point $x_0 \in \Omega$ without vanishing identically in Ω. Then $H(u)$ assumes both positive and negative values near x_0.*

This result is key to the proof of Theorem 1.1. S. Gleason and T. Wolff [20] generalized Theorem 1.6 to higher dimensions. Their result gives some hope that it is also possible to extend Theorem 1.1 to higher dimensions (see Conjecture 7.1).

Let us now present the idea of the proof of Theorem 1.1. We prove the theorem for a sufficiently smooth bounded convex domain $D \subset B(0, 1) \subset \mathbb{R}^2$, whose boundary has a strictly positive curvature (the result for an arbitrary bounded convex domain then follows by approximation and scaling). Let us consider the harmonic extension u of φ. Namely, let

$$K(x) = C_K \frac{x_3}{(x_1^2 + x_2^2 + x_3^2)^{3/2}}, \qquad x \in \mathbb{R}^3_+, \tag{5}$$

where $C_K = 1/(2\pi)$ and $\mathbb{R}^3_+ = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$. Set $u(x_1, x_2, 0) = \varphi(x_1, x_2)$ for $(x_1, x_2) \in \mathbb{R}^2$ and

$$u(x_1, x_2, x_3) = \int_D K(x_1 - y_1, x_2 - y_2, x_3)\varphi(y_1, y_2)\, dy_1\, dy_2, \qquad (x_1, x_2, x_3) \in \mathbb{R}^3_+. \tag{6}$$

Note that $K(x_1 - y_1, x_2 - y_2, x_3)$ is the Poisson kernel of $\mathbb{R}^3_+$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3_+$ and $(y_1, y_2, 0) \in \partial\mathbb{R}^3_+$. We denote $\frac{\partial f}{\partial x_i}$ by f_i and $\frac{\partial^2 f}{\partial x_i \partial x_j}$ by f_{ij}. It is well known that $u_3(x_1, x_2, 0) = -(-\Delta)^{1/2}\varphi(x_1, x_2)$ for $(x_1, x_2) \in D$, so u satisfies

$$\Delta u(x) = 0, \qquad x \in \mathbb{R}^3_+, \tag{7}$$

$$u_3(x) = -1, \qquad x \in D \times \{0\}, \tag{8}$$

$$u(x) = 0, \qquad x \in D^c \times \{0\}, \tag{9}$$

where $\Delta u = u_{11} + u_{22} + u_{33}$.

The idea of studying equations for fractional Laplacians via harmonic extensions is well known. It was used for the first time by F. Spitzer [35] and then by many other authors, e.g. by S. A. Molchanov and E. Ostrovskiĭ [34], R. D. DeBlassie [14], P. Méndez-Hernández [33], R. Bañuelos and T. Kulczycki [2], A. El Hajj, H. Ibrahim and R. Monneau [16] and L. Caffarelli and L. Silvestre [10].

In the next step of the proof we extend u to $\mathbb{R}^3_- = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 < 0\}$ by setting

$$u(x_1, x_2, x_3) = u(x_1, x_2, -x_3) - 2x_3, \qquad (x_1, x_2, x_3) \in \mathbb{R}^3_-. \tag{10}$$

Note that u is continuous on $\mathbb{R}^3$ and for $(x_1, x_2) \in D$ it satisfies

$$\begin{aligned} u_{3^-}(x_1, x_2, 0) &= \lim_{h \to 0^-} \frac{u(x_1, x_2, h) - u(x_1, x_2, 0)}{h} \\ &= \lim_{h \to 0^-} \frac{u(x_1, x_2, -h) - 2h - u(x_1, x_2, 0)}{h} = -1. \end{aligned}$$

By standard arguments, u is harmonic in $\mathbb{R}^3_+ \cup \mathbb{R}^3_- \cup (D \times \{0\}) = \mathbb{R}^3 \setminus (D^c \times \{0\})$.

Let $\mathrm{Hess}(u)$ be the Hessian matrix of u, and $H(u) = \det(\mathrm{Hess}(u))$. The general strategy of the proof is as follows:

1. We show that $H(u)(x) > 0$ for every $x \in \mathbb{R}^3 \setminus (D^c \times \{0\})$.
2. We show that for $x = (x_1, x_2, 0) \in D \times \{0\}$ the Hessian matrix has the form

$$\mathrm{Hess}(u)(x) = \begin{pmatrix} u_{11}(x) & u_{12}(x) & 0 \\ u_{12}(x) & u_{22}(x) & 0 \\ 0 & 0 & u_{33}(x) \end{pmatrix} = \begin{pmatrix} \varphi_{11}(x_1, x_2) & \varphi_{12}(x_1, x_2) & 0 \\ \varphi_{12}(x_1, x_2) & \varphi_{22}(x_1, x_2) & 0 \\ 0 & 0 & u_{33}(x) \end{pmatrix}$$

   and $u_{33}(x) > 0$.

Since $\Delta u(x) = 0$, the two assertions above immediately imply that $\varphi_{11}(x_1, x_2) < 0$ and $\varphi_{22}(x_1, x_2) < 0$ for $(x_1, x_2) \in D$, so φ is strictly concave on D.

The proof is almost entirely the justification of the first assertion. This is done by the continuity method, i.e. by deforming the domain D to the unit ball $B(0, 1)$. The continuity method requires the maximum principle for $H(u)$ (Lewy's theorem), estimates of u_{ij} near $\partial D \times \{0\}$ (see Sections 3 and 4) and the result for the unit ball (Section 5). Roughly speaking, estimates of u_{ij} justify that zeroes of $H(u)$ do not "emerge" from $\partial D \times \{0\}$ along the deformation. Lewy's theorem implies that zeroes of $H(u)$ cannot appear in compact subdomains of $\mathbb{R}^3 \setminus (D^c \times \{0\})$ along the deformation.

Below, we briefly present the main steps in the continuity method. It can be easily shown that $H(u)(x) \to 0$ as $x \to x_0 \in \mathrm{int}(D^c) \times \{0\}$. This causes some technical difficulties in the proof. To deal with this problem we add an auxiliary harmonic function to u. Namely, for any $\varepsilon \geq 0$ we consider $v^{(\varepsilon,D)}(x) = u^{(D)}(x) + \varepsilon(-x_1^2/2 - x_2^2/2 + x_3^2)$ (where $u^{(D)}$ denotes the u corresponding to D). We consider the family $\{D(t)\}_{t\in[0,1]}$ of domains such that $D(0) = D$, $D(1) = B(0, 1)$, all $D(t)$ are smooth bounded convex domains whose boundaries have strictly positive curvature and $\partial D(t) \to \partial D(s)$ as $t \to s$ in an appropriate sense. For large M we set (see Figure 8)

$$\Omega(M, D(t)) = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 < M^2,\ x_3 \in (-M, M)\} \setminus (D(t)^c \times \{0\}).$$

We fix a large M and a sufficiently small $\varepsilon > 0$ ($\varepsilon \in (0, C(M)]$), and define

$$T = \{t \in [0, 1] : H(v^{(\varepsilon,D(t))})(x) > 0 \text{ for all } x \in \Omega(M, D(t))\}.$$

Next, one can show that $1 \in T$ (the result for the unit ball). Then we prove that T is closed, which follows from Lewy's theorem applied to $v^{(\varepsilon,D(t))}$. Next, we show that T is open (relatively in $[0, 1]$), which follows from the fact that for any fixed large M and any fixed $\varepsilon \in (0, C(M)]$ and all $t \in [0, 1]$ we have $H(v^{(\varepsilon,D(t))})(x) > c > 0$ near $\partial\Omega(M, D(t))$, where c does not depend on t (in the proof of this estimate the results from Section 4 are used). This implies that $T = [0, 1]$. By taking $\varepsilon \to 0$ (and again using Lewy's theorem) we deduce that $H(u^{(D)})(x) > 0$ for $x \in \Omega(M, D)$. Letting $M \to \infty$ we conclude that $H(u^{(D)})(x) > 0$ for all $\mathbb{R}^3 \setminus (D^c \times \{0\})$.

The paper is organized as follows. In Section 2 we present notation and collect some known facts needed in the rest of the paper. Sections 3 and 4 are the most technical parts. In Section 3 we estimate $\varphi_{ij}^{(D)}$ near ∂D. This is done by using an explicit formula for the Poisson kernel $P_B(x, y)$ for a ball B corresponding to $(-\Delta)^{1/2}$. Note that due to the nonlocality of $(-\Delta)^{1/2}$ the corresponding harmonic measure $P_B(x, y)\,dy$ is concentrated not on ∂B but on B^c. The results for $\varphi_{ij}^{(D)}$ are obtained by estimating integrals involving the Poisson kernel and its derivatives over different subdomains of D. This method is very technical. Nevertheless, this is a standard method for boundary value problems for fractional Laplacians used by many authors, e.g. K. Bogdan, Z.-Q. Chen, R. Song. It seems that the reason the estimates of $\varphi_i^{(D)}$, $\varphi_{ij}^{(D)}$ are quite long and technical is just the nonlocality of the equation $(-\Delta)^{1/2}\varphi = 1$. The results of Section 3 are used only in Section 4, where estimates of $u_{ij}^{(D)}$ near $\partial D \times \{0\}$ are obtained. These estimates are also quite technical. The reason is that $u_{ij}^{(D)}$ is singular near $\partial D \times \{0\}$ and its behaviour is quite complicated. For example, in an appropriate coordinate system (see Figure 4) in a neighborhood of $0 \in \partial D \times \{0\}$ we have $u_{11}^{(D)}(x) \approx (\mathrm{dist}(x, \partial D \times \{0\}))^{-3/2}$ at some points, $u_{11}^{(D)}(x)$ vanishes at some other points, and $u_{11}^{(D)}(x) \approx -(\mathrm{dist}(x, \partial D \times \{0\}))^{-3/2}$ at some other points. In order to control all six different $u_{ij}^{(D)}$ and ultimately control $H(v^{(\varepsilon,D)})$, we have to consider many cases. The results of Section 4 are used only in the proofs of Proposition 6.2 and Lemma 5.2. Let us point out that the only aim of Sections 3 and 4 is to get control on $H(v^{(\varepsilon,D)})$ and $H(u^{(D)})$ near $\partial D \times \{0\}$.

In Section 5 we prove that $H(u^{(B(0,1))})(x) > 0$ for $x \in \mathbb{R}^3 \setminus (B(0,1)^c \times \{0\})$. The function $u^{(B(0,1))}$ is given by an explicit formula but it seems hard to show $H(u^{(B(0,1))})(x) > 0$ using this formula directly. Instead, the proof is based on an auxiliary function and Lewy's theorem.

The most important part of the paper is Section 6, which contains the proof of the main theorem. In particular, it contains the proof of positivity of $H(u^{(D)})$ via the continuity method, which was briefly described above. It is worth emphasizing that all the derivative estimates obtained in Sections 3 and 4 are used in Section 6 only in the proof of Proposition 6.2. The results of Section 5 are used only in the proof of Proposition 6.5. Corollary 6.6, in which estimates of $H(v^{(\varepsilon,D)})$ near $\partial\Omega(M,D)$ (see Figure 8) and $H(v^{(\varepsilon,B(0,1))})$ in $\Omega(M,B(0,1))$ are formulated, is a direct consequence of Propositions 6.2 and 6.5. Let us point out that the results of Sections 3–5 are invoked in the proof of the main theorem only through Corollary 6.6.

In Section 7 some extensions and conjectures are presented.

2. Preliminaries

For $x \in \mathbb{R}^d$ and $r > 0$ we let $B(x,r) = \{y \in \mathbb{R}^d : |y - x| < r\}$. For $a, b \in \mathbb{R}$ we write $a \wedge b$ for $\min(a,b)$ and $a \vee b$ for $\max(a,b)$. For $x \in \mathbb{R}^d$ and $D \subset \mathbb{R}^d$ we set $\delta_D(x) = \operatorname{dist}(x, \partial D)$. For $\psi : \mathbb{R}^d \to \mathbb{R}$ we denote $\psi_i(x) = \frac{\partial \psi}{\partial x_i}(x)$ and $\psi_{ij}(x) = \frac{\partial^2 \psi}{\partial x_i \partial x_j}(x)$ for $i, j \in \{1, \dots, d\}$. We write $\mathbb{R}^3_+ = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_3 > 0\}$ and $\mathbb{R}^3_- = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_3 < 0\}$. The uniform exterior cone condition is defined e.g. in [19, p. 195].

Let us define a subclass of bounded, convex $C^{2,1}$ domains in $\mathbb{R}^2$ with strictly positive curvature, which will be suitable for our purposes.

Definition 2.1. Let $C_1, R_1 > 0$ and $\kappa_2 \geq \kappa_1 > 0$, and fix a Cartesian coordinate system CS in $\mathbb{R}^2$. We say that a domain $D \subset \mathbb{R}^2$ belongs to the class $F(C_1, R_1, \kappa_1, \kappa_2)$ when:

1. D is convex and in CS coordinates we have
$$\{(y_1,y_2) : y_1^2 + y_2^2 < R_1^2\} \subset D \subset \{(y_1,y_2) : y_1^2 + y_2^2 < 1\}.$$
2. For any $x \in \partial D$ there exists a Cartesian coordinate system CS_x with origin at x obtained by translation and rotation of CS, and there exist $R > 0$ and $f : [-R, R] \to [0,\infty)$ (R, f depend on x) such that $f \in C^{2,1}[-R,R]$, $f(0) = 0$, $f'(0) = 0$ and in CS_x coordinates
$$\{(y_1,y_2) : y_2 \in [-R,R],\ y_1 \in (f(y_2), R]\} = D \cap \{(y_1,y_2) : y_1, y_2 \in [-R,R]\}.$$
3. For any $y \in \partial D$ we have
$$\kappa_1 \leq \kappa(y) \leq \kappa_2,$$
where $\kappa(y)$ denotes the curvature of ∂D at y.
4. For any $y, z \in \partial D$ we have
$$|\kappa(y) - \kappa(z)| \leq C_1 |y - z|.$$

For brevity, we will often use the notation $\Lambda = \{C_1, R_1, \kappa_1, \kappa_2\}$ and write $D \in F(\Lambda)$.

Let $C_1, R_1 > 0, \kappa_2 \geq \kappa_1 > 0$, and $\Lambda = \{C_1, R_1, \kappa_1, \kappa_2\}$. Let $D \in F(\Lambda)$. For any $y \in \partial D$ we denote by $\vec{n}(y)$ the unit inner normal vector at y, and by $\vec{T}(y)$ the unit tangent vector at y which agrees with the negative (clockwise) orientation of ∂D. We set $e_1 = (1, 0)$, $e_2 = (0, 1)$.

It may be easily shown that there exists $\tilde{R} = \tilde{R}(\Lambda)$ such that for any $y \in D$ with $\delta_D(y) \leq \tilde{R}$ there exists a unique $y^* \in \partial D$ such that $|y - y^*| = \delta_D(y)$. For any $y \in D$ such that $\delta_D(y) \leq \tilde{R}$ we define $\vec{n}(y) = \vec{n}(y^*)$ and $\vec{T}(y) = \vec{T}(y^*)$. For any $\psi \in C^2(D)$, $y \in D$, $v_1(y), v_2(y) \in \mathbb{R}$ and $\vec{v}(y) = v_1(y)e_1 + v_2(y)e_2$ we set

$$\frac{\partial \psi}{\partial \vec{v}}(y) = v_1(y)\psi_1(y) + v_2(y)\psi_2(y)$$

(recall that $\psi_i(y) = \frac{\partial \psi}{\partial x_i}(y)$). Similarly, for any $w_1(y), w_2(y) \in \mathbb{R}$ and $\vec{w}(y) = w_1(y)e_1 + w_2(y)e_2$ we write

$$\frac{\partial^2 \psi}{\partial \vec{v} \partial \vec{w}}(y) = v_1(y)w_1(y)\psi_{11}(y) + v_2(y)w_2(y)\psi_{22}(y) + (v_1(y)w_2(y) + v_2(y)w_1(y))\psi_{12}(y).$$

Lemma 2.2. *Let $C_1, R_1 > 0$, $\kappa_2 \geq \kappa_1 > 0$, $\Lambda = \{C_1, R_1, \kappa_1, \kappa_2\}$ and fix a Cartesian coordinate system CS in $\mathbb{R}^2$. Fix $D \in F(\Lambda)$ and $x_0 \in \partial D$. Choose a new Cartesian coordinate system CS_{x_0} with origin at x_0 obtained by translation and rotation of CS such that the positive coordinate halflines y_1, y_2 are in the directions $\vec{n}(x_0)$, $\vec{T}(x_0)$ respectively.*

From now on all points and vectors are in this new coordinate system CS_{x_0}, in particular $\vec{n}(0, 0) = (1, 0) = e_1$, $\vec{T}(0, 0) = (0, 1) = e_2$. For any $y \in \partial D$ define $\alpha(y) \in (-\pi, \pi]$ such that $\vec{T}(y) = \sin\alpha(y)\, e_1 + \cos\alpha(y)\, e_2$ (the angle between e_2 and $\vec{T}(y)$). For any $y \in D$ with $\delta_D(y) < \tilde{R}$ define $\alpha(y) = \alpha(y^*)$, where $y^* \in \partial D$ is the unique point such that $|y - y^*| = \delta_D(y)$.

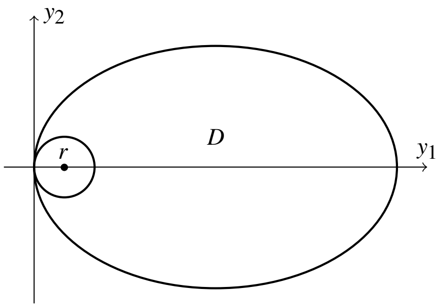


Fig. 1

There exist $r_0 = r_0(\Lambda) \leq \tilde{R} \wedge (1/2)$, $c_1 = c_1(\Lambda)$, $c_2 = c_2(\Lambda)$, $c_3 = c_3(\Lambda)$, $c_4 = c_4(\Lambda)$, $c_5 = c_5(\Lambda)$, $c_6 = c_6(\Lambda)$ and $f : [-r_0, r_0] \to [0, \infty)$ such that $f \in C^{2,1}[-r_0, r_0]$, $f(0) = 0$, $f'(0) = 0$, $c_4 r_0 \leq 1/4$ and for any fixed $r \in (0, r_0]$ we have (see Figure 1):

1. $\{(y_1, y_2) : (y_1 - r)^2 + y_2^2 < r^2\} \subset D$,

$$W := \{(y_1, y_2) : y_2 \in [-r, r], y_1 \in (f(y_2), r]\} = D \cap \{(y_1, y_2) : y_1, y_2 \in [-r, r]\}.$$

2. *For any* $y \in W$ *we have* $\alpha(y) \in [-\pi/4, \pi/4]$ *and*

$$c_1|y_2| \le |\sin\alpha(y)| \le c_2|y_2|,$$
$$\vec{T}(y) = \sin\alpha(y)\, e_1 + \cos\alpha(y)\, e_2, \tag{11}$$
$$\vec{n}(y) = \cos\alpha(y)\, e_1 - \sin\alpha(y)\, e_2. \tag{12}$$

3. *For any* $y_2 \in [-r, r]$ *we have*

$$c_3 y_2^2 \le f(y_2) \le c_4 y_2^2.$$

4. *For any* $y \in W$ *we have* $e_1 = \cos\alpha(y)\,\vec{n}(y) + \sin\alpha(y)\,\vec{T}(y)$, $e_2 = -\sin\alpha(y)\,\vec{n}(y) + \cos\alpha(y)\,\vec{T}(y)$. *For any* $\psi \in C^2(D)$ *and* $y \in W$ *we have*

$$\frac{\partial\psi}{\partial\vec{T}}(y) = \sin\alpha(y)\,\psi_1(y) + \cos\alpha(y)\,\psi_2(y), \tag{13}$$
$$\frac{\partial\psi}{\partial\vec{n}}(y) = \cos\alpha(y)\,\psi_1(y) - \sin\alpha(y)\,\psi_2(y), \tag{14}$$
$$\psi_1(y) = \cos\alpha(y)\,\frac{\partial\psi}{\partial\vec{n}}(y) + \sin\alpha(y)\,\frac{\partial\psi}{\partial\vec{T}}(y),$$
$$\psi_2(y) = -\sin\alpha(y)\,\frac{\partial\psi}{\partial\vec{n}}(y) + \cos\alpha(y)\,\frac{\partial\psi}{\partial\vec{T}}(y),$$
$$\psi_{11}(y) = \cos^2\alpha(y)\,\frac{\partial^2\psi}{\partial\vec{n}^2}(y) + \sin^2\alpha(y)\,\frac{\partial^2\psi}{\partial\vec{T}^2}(y) + 2\sin\alpha(y)\cos\alpha(y)\,\frac{\partial^2\psi}{\partial\vec{n}\partial\vec{T}}(y),$$
$$\psi_{22}(y) = \cos^2\alpha(y)\,\frac{\partial^2\psi}{\partial\vec{T}^2}(y) + \sin^2\alpha(y)\,\frac{\partial^2\psi}{\partial\vec{n}^2}(y) - 2\sin\alpha(y)\cos\alpha(y)\,\frac{\partial^2\psi}{\partial\vec{n}\partial\vec{T}}(y),$$
$$\psi_{12}(y) = (\cos^2\alpha(y) - \sin^2\alpha(y))\frac{\partial^2\psi}{\partial\vec{n}\partial\vec{T}}(y) - \sin\alpha(y)\cos\alpha(y)\left(\frac{\partial^2\psi}{\partial\vec{n}^2}(y) - \frac{\partial^2\psi}{\partial\vec{T}^2}(y)\right).$$

5. *For any* $y \in \{(y_1, y_2) \in W : y_2 > 0\}$ *we have*

$$c_5(f^{-1}(y_1) - y_2)f^{-1}(y_1) \le \delta_D(y) \le c_6(f^{-1}(y_1) - y_2)f^{-1}(y_1),$$

where $f^{-1} : [0, f(r)] \to [0, r]$.

This lemma follows by elementary geometry and its proof is omitted.

Lemma 2.3. *Let* $C_1, R_1 > 0$, $\kappa_2 \ge \kappa_1 > 0$ *and* $\Lambda = \{C_1, R_1, \kappa_1, \kappa_2\}$. *There exists a constant* $c = c(\Lambda)$ *such that for any* $D \in F(\Lambda)$ *we have*

$$\int_D \delta_D^{-1/2}(x)\,dx \le c. \tag{15}$$

Proof. By Definition 2.1 we have $B(0, R_1) \subset D \subset B(0, 1)$. Let $x_0 \in \partial D$. By convexity of D the convex hull of $B(0, R_1) \cup \{x_0\}$ is a subset of $\overline{D}$. Using this fact and $D \subset B(0, 1)$ one can easily show that for every x in the line segment between 0 and x_0 we have $|x - x_0| \le c\delta_D(x)$, where c depends only on R_1. Hence $\delta_D^{-1/2}(x) \le c^{1/2}|x - x_0|^{-1/2}$. Now (15) easily follows by using polar coordinates with centre at 0. □

In what follows we will use the method of continuity (cf. [26, p. 20], [9]). Roughly speaking, we will deform a convex bounded domain D to the ball $B(0, 1)$. To do this we will consider the following construction. Let $C_1, R_1 > 0$ and $\kappa_2 \geq \kappa_1 > 0$. For any $D \in F(C_1, R_1, \kappa_1, \kappa_2)$ and $t \in [0, 1]$ we define

$$D(t) = \{x : \exists y \in D, z \in B(0, 1) \text{ such that } x = (1 - t)y + tz\}. \tag{16}$$

Lemma 2.4. *For any $C_1, R_1 > 0$ and $\kappa_2 \geq \kappa_1 > 0$ there exist $C_1', R_1' > 0$ and $\kappa_2' \geq \kappa_1' > 0$ such that for any $D \in F(C_1, R_1, \kappa_1, \kappa_2)$ and any $t \in [0, 1]$ we have $D(t) \in F(C_1', R_1', \kappa_1', \kappa_2')$.*

This lemma seems to be standard, similar results are well known (cf. [19, Appendix, pp. 381–384] or [9, proof of Theorem 3.1]). Therefore we omit its proof.

Now we state some properties of the solution of (1)–(2) and its harmonic extension which will be needed in the rest of the paper.

Let $D \subset \mathbb{R}^2$ be an open bounded set and $\varphi^{(D)}$ be the solution of (1)–(2) for D. Then the following scaling property is well known [4, (1.61)]:

$$\varphi^{(aD)}(ax) = a\varphi^{(D)}(x), \quad x \in D, \ a > 0. \tag{17}$$

For any open bounded sets $D_1, D_2 \subset \mathbb{R}^2$ set $d(D_1, D_2) = [\sup\{\text{dist}(x, \partial D_2) : x \in \partial D_1\}] \vee [\sup\{\text{dist}(x, \partial D_1) : x \in \partial D_2\}]$.

Lemma 2.5. *Let $\{D_n\}_{n=0}^{\infty}$ be a sequence of bounded convex domains in $\mathbb{R}^2$ and $\varphi^{(D_n)}$ be the solution of (1)–(2) for D_n. If $d(D_n, D_0) \to 0$ as $n \to \infty$ then for any $x \in D_0$ we have $\varphi^{(D_n)}(x) \to \varphi^{(D_0)}(x)$ as $n \to \infty$.*

This lemma seems to be well known and follows easily from (17), so we omit its proof (in fact, it holds not only for convex domains, but we need it only in this case).

Lemma 2.6. *Let $C_1, R_1 > 0$, $\kappa_2 \geq \kappa_1 > 0$ and $\Lambda = \{C_1, R_1, \kappa_1, \kappa_2\}$. There exist a constant $c_1 = c_1(\Lambda)$ and an absolute constant c_2 such that for any $D \in F(\Lambda)$ we have*

$$\varphi(x) \leq 2/\pi, \qquad x \in D,$$
$$c_1\delta_D^{1/2}(x) \leq \varphi(x) \leq c_2\delta_D^{1/2}(x), \qquad x \in D,$$

where φ is the solution of (1)–(2) for D.

Proof. We have $D \subset B(0, 1)$, so for any $x \in D$ we get

$$\varphi(x) = E^x(\tau_D) \leq E^x(\tau_{B(0,1)}) = \frac{2}{\pi}(1 - |x|^2)^{1/2}.$$

Let $x \in D$ and let $x^* \in \partial D$ be such that $|x - x^*| = \delta_D(x)$. Define $z = x^* - \vec{n}(x^*)$, where $\vec{n}(x^*)$ is the unit inner normal vector at x^* (clearly $|z - x^*| = 1$). By convexity of D we get $B(z, 1) \subset D^c$. Set

$$U = \{y \in \mathbb{R}^2 : 1 < |y - z| < 3\}.$$

Since $D \subset B(0,1)$, we get $\mathrm{diam}(D) \le 2$. Clearly, $x^* \in \partial D \cap \partial U$, which implies that $D \subset U$ and $\delta_D(x) = \delta_U(x)$. By [13] there exists an absolute constant c_2 such that

$$\varphi(x) = E^x(\tau_D) \le E^x(\tau_U) \le c_2 \delta_U^{1/2}(x) = c_2 \delta_D^{1/2}(x).$$

Now we will prove the lower bound of φ. Since $D \subset B(0,1)$, we have $\delta_D(x) \le 1$. Let $x \in D$. If $\delta_D(x) \ge r_0$, where $r_0 = r_0(\Lambda)$ is the constant from Lemma 2.2, then

$$\varphi(x) = E^x(\tau_D) \ge E^x(\tau_{B(x,r_0)}) = \frac{2}{\pi} r_0 \ge \frac{2}{\pi} r_0 \delta_D^{1/2}(x).$$

If $\delta_D(x) < r_0$ then we may choose a coordinate system as in Lemma 2.2 (see Figure 1) and assume that $x = (x_1, 0)$ and $\delta_D(x) = x_1$. Set $B = B((r_0,0), r_0)$. By Lemma 2.2 we have $B \subset D$. Clearly $x \in B$ and $\delta_D(x) = \delta_B(x) = x_1$. It follows that

$$\varphi(x) = E^x(\tau_D) \ge E^x(\tau_B) = \frac{2}{\pi}\left(r_0^2 - |(r_0,0) - (x_1,0)|^2\right)^{1/2} \ge \frac{2}{\pi} r_0^{1/2} \delta_D^{1/2}(x). \quad \square$$

Lemma 2.7. *Let $C_1, R_1 > 0$, $\kappa_2 \ge \kappa_1 > 0$, $D \in F(C_1, R_1, \kappa_1, \kappa_2)$, φ be the solution of (1)–(2) for D, and u the harmonic extension of φ given by (6)–(10). For any $(x_1, x_2, x_3) \in \mathbb{R}^3_+$ we have $H(u)(x_1, x_2, -x_3) = H(u)(x_1, x_2, x_3)$.*

Proof. For $x = (x_1, x_2, x_3)$ set $\hat{x} = (x_1, x_2, -x_3)$. For $x \in \mathbb{R}^3_+$ we have $u_{ii}(\hat{x}) = u_{ii}(x)$ for $i = 1,2,3$, $u_{12}(\hat{x}) = u_{12}(x)$, $u_{13}(\hat{x}) = -u_{13}(x)$ and $u_{23}(\hat{x}) = -u_{23}(x)$. Hence $H(u)(\hat{x}) = H(u)(x)$. $\square$

We recall the definition of an α-harmonic function, $\alpha \in (0,2)$. A Borel function h on $\mathbb{R}^d$ is said to be *α-harmonic* on an open set $D \subset \mathbb{R}^d$ if for any $x_0 \in \mathbb{R}^d$ and $r > 0$ such that $\overline{B(x_0,r)} \subset D$ we have

$$h(x) = \int_{B(x_0,r)^c} P_r(x - x_0, y - x_0) h(y)\, dy,$$

where the integral is absolutely convergent and $P_r(x,y)$ is the Poisson kernel for the ball $B(0,r)$ corresponding to $(-\Delta)^{\alpha/2}$. The explicit formula for the Poisson kernel is well known (see e.g. [4, (1.57)]. For $\alpha = 1$ and $d = 2$ the Poisson kernel for $B(z,s)$ is given by (19). It is well known that h is α-harmonic on an open set $D \subset \mathbb{R}^d$ if and only if h is C^2 on D and $(-\Delta)^{\alpha/2} h(x) = 0$ for any $x \in D$. A Borel function h on $\mathbb{R}^d$ is said to be *singular α-harmonic* on an open set $D \subset \mathbb{R}^d$ if it is α-harmonic on D and $h \equiv 0$ on D^c.

We will need the following formulas for derivatives of $K(x) = C_K x_3 (x_1^2 + x_2^2 + x_3^2)^{-3/2}$:

$$\begin{aligned}
K_1(x) &= -3C_K x_3 x_1 (x_1^2 + x_2^2 + x_3^2)^{-5/2},\\
K_2(x) &= -3C_K x_3 x_2 (x_1^2 + x_2^2 + x_3^2)^{-5/2},\\
K_3(x) &= C_K (x_1^2 + x_2^2 - 2x_3^2)(x_1^2 + x_2^2 + x_3^2)^{-5/2};
\end{aligned}$$

$$K_{11}(x) = C_K x_3 (12x_1^2 - 3x_2^2 - 3x_3^2)(x_1^2 + x_2^2 + x_3^2)^{-7/2},$$
$$K_{22}(x) = C_K x_3 (12x_2^2 - 3x_1^2 - 3x_3^2)(x_1^2 + x_2^2 + x_3^2)^{-7/2},$$
$$K_{33}(x) = C_K x_3 (6x_3^2 - 9x_1^2 - 9x_2^2)(x_1^2 + x_2^2 + x_3^2)^{-7/2},$$
$$K_{12}(x) = 15 C_K x_3 x_1 x_2 (x_1^2 + x_2^2 + x_3^2)^{-7/2},$$
$$K_{13}(x) = C_K x_1 (12x_3^2 - 3x_1^2 - 3x_2^2)(x_1^2 + x_2^2 + x_3^2)^{-7/2},$$
$$K_{23}(x) = C_K x_2 (12x_3^2 - 3x_1^2 - 3x_2^2)(x_1^2 + x_2^2 + x_3^2)^{-7/2}.$$

Remark 2.8. All constants appearing in this paper are positive and finite. We write $C = C(a, \dots, z)$ to emphasize that C depends only on $a, \dots, z$. We adopt the convention that constants denoted by c (or c_1, c_2, etc.) may change their value from one use to the next.

Remark 2.9. In Sections 3, 4 and in the proof of Proposition 6.2 we use the following convention. Constants denoted by c (or c_1, c_2, etc.) depend on $\Lambda = \{C_1, R_1, \kappa_1, \kappa_2\}$, which appears in Definition 2.1. We write $f(x) \approx g(x)$ for $x \in A \subset \mathbb{R}^2$ to indicate that there exist constants $c_1 = c_1(\Lambda)$ and $c_2 = c_2(\Lambda)$ such that for any $x \in A$ we have $c_1 g(x) \le f(x) \le c_2 g(x)$ (in particular, it may happen that both f, g are positive on A or both f, g are negative on A).

3. Estimates of derivatives of φ near ∂D

In this section we obtain estimates of φ_i, φ_{ij} near ∂D. These results are used in this paper only in Section 4, where the behaviour of u_{ij} near $\partial D \times \{0\}$ is studied. To obtain the estimates of φ_i, φ_{ij} we use the well known representation (18) below. This formula involves the Poisson kernel $P(x, y)$ for a ball corresponding to $(-\Delta)^{1/2}$. Recall that due to nonlocality of this operator the support of the corresponding harmonic measure $P(x, y)\,dy$ for a ball B is equal to B^c. This makes proofs in this section quite long and complicated because we have to obtain estimates of integrals involving the Poisson kernel and its derivatives over different subdomains of D. Most of the techniques used in this section are similar to the standard methods used by Z.-Q. Chen and R. Song [12], T. Kulczycki [28], and K. Bogdan, T. Kulczycki and A. Nowak [6]. These methods were used in estimates of the Green function corresponding to $(-\Delta)^{\alpha/2}$, $\alpha \in (0, 2)$, on smooth domains [12], [28] and in estimates of gradients of α-harmonic functions [6].

It should be mentioned that similar estimates for derivatives of α-harmonic functions were simultaneously obtained by the author's student G. Żurek in his Master Thesis [36].

The most difficult part of this section is the proof of Lemma 3.7. In this lemma estimates of $\varphi_{22}(x_1, 0)$ are obtained (the y_2 axis is tangent to the boundary of D at $(0, 0) \in \partial D$, see Figure 3). To the best of the author's knowledge the idea of that proof is new. Roughly speaking, the proof is based on the representation

$$\varphi_{22}(x_1, 0) = \int_{D \setminus B} P_2((x_1, 0), y)\varphi_2(y)\,dy$$

and the precise control of the derivatives of φ in normal and tangent directions in a small neighbourhood of $(0, 0)$.

In the whole section we fix $C_1, R_1 > 0$, $\kappa_2 \geq \kappa_1 > 0$, $D \in F(C_1, R_1, \kappa_1, \kappa_2)$ and $x_0 \in \partial D$. We write $\Lambda = \{C_1, R_1, \kappa_1, \kappa_1\}$, and φ is the solution of (1)–(2) for D. Unless otherwise stated we fix the coordinate system CS_{x_0} and notation as in Lemma 2.2 (see Figure 1). In particular, x_0 is $(0, 0)$ in CS_{x_0} coordinates.

Let $r \in (0, r_0]$, $z = (r, 0)$, $s \in (0, r]$ and $B = B(z, s)$ (where r_0 is the constant from Lemma 2.2). It is well known (see e.g. [4, (1.50), (1.56), (1.57)]) that

$$\varphi(x) = h(x) + \int_{B^c} P(x, y)\varphi(y)\, dy, \qquad x \in B, \tag{18}$$

where $h(x) = C_B(s^2 - |x - z|^2)^{1/2}$ for $x \in B$ and

$$P(x, y) = C_P \frac{(s^2 - |x - z|^2)^{1/2}}{(|y - z|^2 - s^2)^{1/2}|x - y|^2}, \qquad x \in B,\ y \in (\overline{B})^c, \tag{19}$$

with $C_B = 2/\pi$ and $C_P = \pi^{-2}$.

We have $h_1(x) = C_B(r - x_1)(s^2 - |x - z|^2)^{-1/2}$ for $x \in B$. Write $P_i(x, y) = \frac{\partial}{\partial x_i} P(x, y)$, $i = 1, 2$. For any $x \in B$ and $y \in (\overline{B})^c$ we have $P_1(x, y) = A(x, y) + E(x, y)$ where

$$A(x, y) = -C_P \frac{(s^2 - |x - z|^2)^{-1/2}(x_1 - r)}{(|y - z|^2 - s^2)^{1/2}|x - y|^2}, \tag{20}$$

$$E(x, y) = -2C_P \frac{(s^2 - |x - z|^2)^{1/2}(x_1 - y_1)}{(|y - z|^2 - s^2)^{1/2}|x - y|^4}. \tag{21}$$

In this section we use only those geometric properties of the domain D which are stated in Lemmas 2.2 and 2.3, and additionally the facts that $D \subset B(0, 1)$ and D is convex. Recall that all constants in the assertions of Lemmas 2.2 and 2.3 depend only on Λ. Hence all constants in the estimates of this section also depend only on Λ. In the whole section we use the convention stated in Remark 2.9.

Lemma 3.1. *There exists $r_1 = r_1(\Lambda) \in (0, r_0/4]$ such that $\varphi_1(x_1, 0) \approx x_1^{-1/2}$ for any $x_1 \in (0, r_1]$.*

Proof. Set $r = r_0$. We will use (18) for $s = r$, in particular $B = B(z, r)$. Note that for $x = (x_1, 0)$ we have $r^2 - |x - z|^2 = x_1(r + |x_1 - r|) \leq 2rx_1$. Define

$$k(x) = 1_B(x) \int_{B^c} P(x, y)\varphi(y)\, dy + 1_{B^c}(x)\varphi(x), \quad x \in \mathbb{R}^2.$$

We have $k(x) \geq 0$ on $\mathbb{R}^2$, by (18) $k(x) \leq \varphi(x)$ on B, and k is 1-harmonic on B. For the definition and basic properties of α-harmonic functions see Section 2 and [4, pp. 20–21, 61]. The fact that k is 1-harmonic follows from [4, p. 61]. By [6, Lemma 3.2] (cf. also [30]) and Lemma 2.6,

$$k_1(x_1, 0) \leq 2\frac{k(x_1, 0)}{x_1} \leq 2\frac{k\varphi(x_1, 0)}{x_1} \leq cx_1^{-1/2} \quad \text{for } x_1 \in (0, r].$$

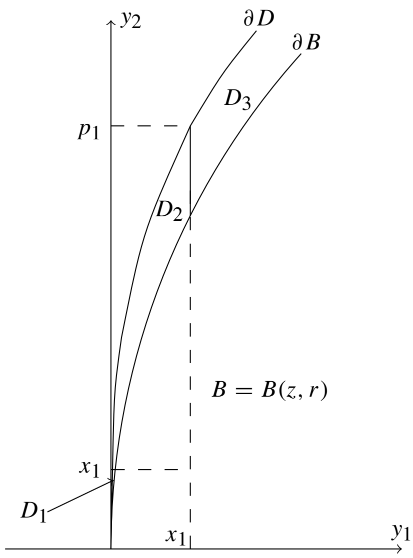


Fig. 2

By the formula for h_1 and the formula for $r^2 - |x - z|^2$ we get $h_1(x_1, 0) = C_B(r - x_1) \times (2r - x_1)^{-1/2} x_1^{-1/2} \le C_B r^{1/2} x_1^{-1/2}$. Hence $\varphi_1(x_1, 0) = h_1(x_1, 0) + k_1(x_1, 0) \le c x_1^{-1/2}$ for $x_1 \in (0, r/4]$.

What remains is to show that $\varphi_1(x_1, 0) \ge c x_1^{-1/2}$. For $x_1 \in (0, r]$ we have $\varphi_1(x_1, 0) = \int_{B^c} P_1((x_1, 0), y)\varphi(y)\, dy + h_1(x_1, 0)$. We will estimate $\int_{B^c} P_1\varphi$.

Let $x_1 \in (0, f(r/2) \wedge f(-r/2)]$. By Lemma 2.2 we have $f(r/2) \le c_4(r/2)^2 \le r/16$ (because $c_4 r \le 1/4$), so $x_1 \in (0, r/16]$. Note that $f(r/2) \wedge f(-r/2) \ge c_3 r^2/4$, where c_3 and $r = r_0$ are the constants from Lemma 2.2, and $c_3 r^2/4$ depends only on Λ. Let $p_1 \in (0, r/2]$ be such that $f(p_1) = x_1$, and $p_2 \in [-r/2, 0)$ be such that $f(p_2) = x_1$ (recall that f is defined in Lemma 2.2). By Lemma 2.2, $f(x_1) < c_4 x_1^2 \le (1/2)x_1$ and $f(-x_1) \le (1/2)x_1$, so $p_1 > x_1$ and $|p_2| > x_1$. Let $f_1 : [-r, r] \to \mathbb{R}$ be defined by $f_1(y_2) = r - (r^2 - y_2^2)^{1/2}$. Denote (see Figure 2)

$$
\begin{aligned}
D_1 &= \{(y_1, y_2) : y_2 \in [-x_1, x_1],\ y_1 \in (f(y_2), f_1(y_2))\},\\
D_2 &= \{(y_1, y_2) : y_2 \in (x_1, p_1] \cup [p_2, -x_1),\ y_1 \in (f(y_2), f_1(y_2) \wedge x_1)\},\\
D_3 &= D \setminus (D_1 \cup D_2 \cup B).
\end{aligned}
$$

Note that $\int_{D\setminus B} A((x_1, 0), y)\varphi(y)\, dy > 0$ and $\int_{D_3} E((x_1, 0), y)\varphi(y)\, dy > 0$, because we have $A((x_1, 0), y) > 0$ for $y \in D \setminus B$ and $E((x_1, 0), y) > 0$ for $y \in D_3$.

Recall that we use (18) for $s = r$. We have $f_1(y_2) \le y_2^2/r = c y_2^2$. By Lemma 2.6, $\varphi(y) \le c\delta_D^{1/2}(y)$. For $y \in D_1 \cup D_2$ we also have $\delta_D(y) \le y_1 \le f_1(y_2) \le c y_2^2$. It follows that $\varphi(y) \le c|y_2|$ for $y \in D_1 \cup D_2$. Note that for $y \in D_1$ we have $|y_2| \le x_1$, so $\varphi(y) \le c x_1$. Note also that $|y - z|^2 - r^2 = (|y - z| + r)(|y - z| - r)$. This is bounded from above by $3r(f_1(y_2)-y_1)$ and from below by $r(f_1(y_2)-y_1)/2$. Hence for $y \in D_1 \cup D_2$ we have $|y-z|^2-r^2 \approx f_1(y_2)-y_1$. For $y \in D_1$ we obtain

$$0 < y_1 \le f_1(x_1) = \frac{x_1^2}{r+(r^2-x_1^2)^{1/2}} \le \frac{x_1^2}{r} \le \frac{x_1}{16},$$

because $x_1 \in (0, r/16]$. Hence for $y \in D_1$ we have $|x-y| \ge |x_1-y_1| \ge 15x_1/16$ and $|x_1-y_1| \le x_1$. It follows that

$$\begin{aligned} \left|\int_{D_1} E((x_1,0),y)\varphi(y)\,dy\right| &\le c x_1^{-3/2}\int_{D_1}\frac{dy}{(|y-z|^2-r^2)^{1/2}} \\ &\approx x_1^{-3/2}\int_{-x_1}^{x_1} dy_2 \int_{f(y_2)}^{f_1(y_2)} (f_1(y_2)-y_1)^{-1/2}\,dy_1 \\ &= 2x_1^{-3/2}\int_{-x_1}^{x_1}(f_1(y_2)-f(y_2))^{1/2}\,dy_2 \le c x_1^{1/2}. \end{aligned}$$

For $y \in D_2$ we have $|x-y| = ((x_1-y_1)^2+y_2^2)^{1/2} \ge |y_2|$ and $|x_1-y_1| \le |x_1|+|y_1| \le 2x_1$. Note also that by Lemma 2.2 we have $p_1 \le c\sqrt{x_1} \wedge (r/2)$ and $|p_2| \le c\sqrt{x_1} \wedge (r/2)$, so

$$\begin{aligned} &\left|\int_{D_2} E((x_1,0),y)\varphi(y)\,dy\right| \\ &\quad \le c x_1^{3/2}\int_{x_1}^{c\sqrt{x_1}\wedge(r/2)} dy_2\, y_2^{-3}\int_{f(y_2)}^{f_1(y_2)\wedge x_1}(f_1(y_2)-y_1)^{-1/2}\,dy_1 \\ &\quad \le c x_1^{3/2}\int_{x_1}^{c\sqrt{x_1}\wedge(r/2)} y_2^{-3}(f_1(y_2)-f(y_2))^{1/2}\,dy_2 \le c x_1^{1/2} \end{aligned}$$

(here we omit $\int_{p_2}^{-x_1}\dots$ because it can be estimated in the same way).

We have

$$\varphi_1(x_1,0) = h_1(x_1,0) + \int_{D\setminus B} A\varphi + \int_{D_1} E\varphi + \int_{D_2} E\varphi + \int_{D_3} E\varphi.$$

By the formula for h_1 we easily get $h_1(x_1,0) \ge (2\sqrt{2})^{-1} C_B r^{1/2} x_1^{-1/2}$. It follows that

$$\varphi_1(x_1,0) \ge (2\sqrt{2})^{-1} C_B r^{1/2} x_1^{-1/2} - c x_1^{1/2} = x_1^{-1/2}\big((2\sqrt{2})^{-1} C_B r^{1/2} - c x_1\big).$$

Set $c_1 = (2\sqrt{2})^{-1} C_B r^{1/2}$. For sufficiently small x_1 we have $c_1 - c x_1 \ge c_1/2$ and $\varphi_1(x_1,0) \ge (c_1/2) x_1^{-1/2}$ (one can take $x_1 \le r_1 := (c_1/(2c)) \wedge (r/4)$). □

Lemma 3.2. *Set $r_1 = r_0/4$. For any $x_1 \in (0, r_1]$ we have $|\varphi_2(x_1,0)| \le c x_1^{1/2} |\log x_1|$.*

Proof. Set $r = r_0$. We will use (18) for $s = r$, in particular $B = B(z,r)$. Let $x_1 \in (0, r/4]$. We have $\varphi_2(x_1,0) = \int_{B^c} P_2((x_1,0),y)\varphi(y)\,dy + h_2(x_1,0)$, $h_2(x_1,0) = 0$ and

$$P_2((x_1,0),y) = 2C_P \frac{(r^2-|x-z|^2)^{1/2} y_2}{(|y-z|^2-r^2)^{1/2}|x-y|^4}, \qquad y \in (\overline{B})^c.$$

Let f_1 be as in the proof of Lemma 3.1. Define

$$D_1 = \{(y_1, y_2) : y_2 \in [-x_1, x_1],\ y_1 \in (f(y_2), f_1(y_2))\},$$
$$D_2 = \{(y_1, y_2) : y_2 \in (x_1, r/2] \cup [-r/2, -x_1),\ y_1 \in (f(y_2), f_1(y_2))\},$$
$$D_3 = D \setminus (D_1 \cup D_2 \cup B).$$

By the same arguments as in the proof of Lemma 3.1, for $x = (x_1, 0)$ we have $r^2 - |x-z|^2 \le 2rx_1$ and for $y \in D_1 \cup D_2$ we have $|y-z|^2 - r^2 \approx f_1(y_2) - y_1$. Note also that for $y \in D_1 \cup D_2$ we have $\delta_D(y) \le y_1 \le f_1(y_2) \le cy_2^2$, so $\varphi(y) \le c|y_2|$ (by Lemma 2.6). For $y \in D_1$ we have $|y_2| \le x_1$, so $\varphi(y) \le cx_1$ and $|x - y| \ge 3x_1/4$. Hence

$$\left|\int_{D_1} P_2((x_1, 0), y)\varphi(y)\,dy\right| \le cx_1^{-3/2} \int_{D_1} \frac{dy}{(|y-z|^2 - r^2)^{1/2}}.$$

By the same estimates as in the proof of Lemma 3.1 this is bounded by $cx_1^{1/2}$.

For $x = (x_1, 0)$ and $y \in D_2$ we have $|x - y| \ge y_2$ and $f_1(y_2) \le cy_2^2$. It follows that

$$\left|\int_{D_2} P_2((x_1, 0), y)\varphi(y)\,dy\right| \le cx_1^{1/2} \int_{x_1}^{r/2} dy_2 y_2^{-2} \int_{f(y_2)}^{f_1(y_2)} (f_1(y_2) - y_1)^{-1/2}\,dy_1$$
$$\le cx_1^{1/2} \int_{x_1}^{r/2} y_2^{-2}(f_1(y_2) - f(y_2))^{1/2}\,dy_2 \le cx_1^{1/2}|\log x_1|.$$

For $x = (x_1, 0)$ and $y \in D_3$ we have $|y-z|^2 - r^2 = (|y-z| + r)\delta_B(y) \ge r\delta_B(y)$ and $y_2/|x-y|^4 \le |x-y|^{-3} \le (r/2)^{-3}$. Set $B_1 = \{w \notin B : \delta_B(w) \le 2\}$. Since $D \subset B(0, 1)$, we have $D \setminus B \subset B_1$. Hence

$$\left|\int_{D_3} P_2((x_1, 0), y)\varphi(y)\,dy\right| \le cx_1^{1/2} \int_{B_1} \delta_B^{-1/2}(y)\,dy$$
$$= cx_1^{1/2} \int_r^2 \frac{\rho}{(\rho - r)^{1/2}}\,d\rho = cx_1^{1/2}.$$

It follows that $|\varphi_2(x_1, 0)| \le cx_1^{1/2}|\log x_1|$. □

In the following corollary we simply restate Lemmas 3.1 and 3.2 for an arbitrary point $y \in D$ (with $\delta_D(y) \le r_1$). Recall that $\vec{T}(y)$, $\vec{n}(y)$ are given by (11), (12), and $\frac{\partial \psi}{\partial \vec{T}}(y)$, $\frac{\partial \psi}{\partial \vec{n}}(y)$ are given by (13), (14).

By Lemmas 3.1, 3.2 and 2.2 we obtain

Corollary 3.3. *There exists $r_1 = r_1(\Lambda) \in (0, r_0/4]$ such that for any $y \in D$ with $\delta_D(y) \le r_1$ we have*

$$\frac{\partial \varphi}{\partial \vec{n}}(y) \approx \delta_D^{-1/2}(y), \tag{22}$$
$$\left|\frac{\partial \varphi}{\partial \vec{T}}(y)\right| \le c\delta_D^{1/2}(y)|\log \delta_D(y)|, \tag{23}$$
$$|\nabla \varphi(y)| \le c\delta_D^{-1/2}(y). \tag{24}$$

Lemma 3.4. *For any $y \in D$ we have $|\nabla \varphi(y)| \le c\delta_D^{-1/2}(y)$.*

Proof. Let $r_1 = r_1(\Lambda)$ be the constant from Corollary 3.3. If $y \in D$ satisfies $\delta_D(y) \le r_1$ then the assertion follows from Corollary 3.3. Fix $y_0 \in D$ such that $\delta_D(y_0) > r_1$ and write $B = B(y_0, r_1)$. We are going to estimate $|\nabla \varphi(y_0)|$. For $y \in B$ we have $\varphi(y) = h(y) + k(y)$, where $h(y) = C_B(r_1^2 - |y - y_0|^2)^{1/2}$ and $k(y) = 1_B(y) \int_{D \setminus B} P(y - y_0, z - y_0)\varphi(z)\, dz + 1_{B^c}(y)\varphi(y)$, where P is given by (19) with $s = r_1$. Clearly $\nabla h(y_0) = 0$. Now, k is a 1-harmonic function on B and $k(y) \le \varphi(y) \le 2/\pi$ (the last inequality follows from Lemma 2.6). By [6, Lemma 3.2], $|\nabla k(y_0)| \le 2k(y_0)/r_1 \le 4/(\pi r_1) \le 4\delta_D^{-1/2}(y)/(\pi r_1)$. □

The definition of α-harmonic functions (see Section 2) on an open set $U \subset \mathbb{R}^d$ demands that the function be defined on the whole $\mathbb{R}^d$. The functions φ_1, φ_2 are well defined on D and also on $D^c \setminus \partial D$. They are not well defined on ∂D but ∂D has Lebesgue measure zero. One may formally define $\varphi_1 = \varphi_2 = 0$ on ∂D. For the definition of singular α-harmonic functions, see Section 2.

Lemma 3.5. *φ_1, φ_2 are singular 1-harmonic on D.*

The proof of this lemma is omitted. By standard arguments (translation invariance and regularity of φ) it can be easily shown that $(-\Delta)^{1/2}(\frac{\partial \varphi}{\partial x_i})(x) = \frac{\partial}{\partial x_i}((-\Delta)^{1/2}\varphi)(x) = 0$ for $x \in D$.

Remark 3.6. φ_{11}, φ_{22} are not 1-harmonic on D because they are not locally integrable on $\mathbb{R}^2$ (see Corollary 3.10).

Lemma 3.7. *There exists $r_2 = r_2(\Lambda) \in (0, r_0/4]$ such that $\varphi_{22}(x_1, 0) \approx -x_1^{-1/2}$ for any $x_1 \in (0, r_2]$.*

Proof. Set $r = r_0$. Let r_1 be the constant from Corollary 3.3. In this proof we take $s \in (r - (r_1/2)^2, r)$, i.e. $0 < r - s < (r_1/2)^2$. Recall that $z = (r, 0)$, $B = B(z, s)$ and P is given by (19). For any $x_1 \in (r - s, r]$ by Lemma 3.5 we have $\varphi_2(x_1, 0) = \int_{D \setminus B} P((x_1, 0), y)\varphi_2(y)\, dy$. It follows that $\varphi_{22}(x_1, 0) = \int_{D \setminus B} P_2((x_1, 0), y)\varphi_2(y)\, dy$. We have $P_2((x_1, 0), y) = 2C_P \frac{(s^2 - |x - z|^2)^{1/2} y_2}{(|y - z|^2 - s^2)^{1/2}|x - y|^4}$. Take $x_1 = \sqrt{r - s}$ (we have $\sqrt{r - s} < r_1/2$). Let $f_1 : [-s, s] \to \mathbb{R}$ be defined by $f_1(y_2) = r - \sqrt{s^2 - y_2^2}$. Set (see Figure 3)

$$D_1 = \{(y_1, y_2) : y_2 \in [-x_1, x_1],\ y_1 \in (f(y_2), f_1(y_2))\},$$
$$D_2 = \{(y_1, y_2) : y_2 \in (x_1, r_1/2] \cup [-r_1/2, -x_1),\ y_1 \in (f(y_2), f_1(y_2))\},$$
$$D_3 = D \setminus (D_1 \cup D_2 \cup B).$$

By Lemma 2.2, for $y \in D_1 \cup D_2$ we have

$$\varphi_2(y) = \cos \alpha(y)\, \frac{\partial \varphi}{\partial \vec{T}}(y) - \sin \alpha(y)\, \frac{\partial \varphi}{\partial \vec{n}}(y).$$

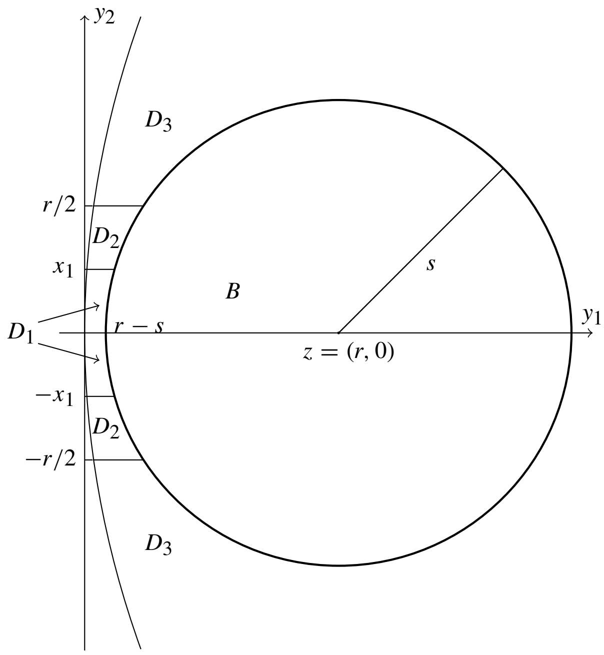


Fig. 3

Note that by definition of s we have $\delta_D(y) < r_1$ for $y \in D_1 \cup D_2$. For such y, by Corollary 3.3,

$$\left|\frac{\partial\varphi}{\partial\vec{T}}(y)\right| \le c(y_1 - f(y_2))^{1/2}|\log(y_1 - f(y_2))|,$$
$$\frac{\partial\varphi}{\partial\vec{n}}(y) \approx (y_1 - f(y_2))^{-1/2}.$$

Hence

$$\left|\cos\alpha(y)\,\frac{\partial\varphi}{\partial\vec{T}}(y)\right| \le c(y_1 - f(y_2))^{1/2}|\log(y_1 - f(y_2))|,$$
$$-\sin\alpha(y)\,\frac{\partial\varphi}{\partial\vec{n}}(y) \approx -y_2(y_1 - f(y_2))^{-1/2}.$$

Note also that for $y \in D_1 \cup D_2$ we have $(|y-z|^2 - s^2)^{1/2} \approx (-y_1 + f_1(y_2))^{1/2}$. Recall that we have chosen $x_1 = \sqrt{r-s}$. It follows that

$$-\int_{D_1} P_2((x_1,0),y)\sin\alpha(y)\,\frac{\partial\varphi}{\partial\vec{n}}(y)\,dy$$
$$\approx -x_1^{-7/2}\int_{-x_1}^{x_1} dy_2\, y_2^2 \int_{f(y_2)}^{f_1(y_2)} dy_1\,(-y_1 + f_1(y_2))^{-1/2}(y_1 - f(y_2))^{-1/2} \approx -x_1^{1/2},$$

because $\int_a^b (x-a)^{-1/2}(b-x)^{-1/2}\,dx = \text{const}$. Similarly,

$$-\int_{D_2} P_2((x_1,0),y)\sin\alpha(y)\,\frac{\partial\varphi}{\partial\vec{n}}(y)\,dy$$
$$\approx -x_1^{1/2}\int_{x_1}^{r_1/2} dy_2\, y_2^{-2}\int_{f(y_2)}^{f_1(y_2)} dy_1\,(-y_1+f_1(y_2))^{-1/2}(y_1-f(y_2))^{-1/2} \approx -x_1^{1/2}.$$

On the other hand,

$$\left|\int_{D_1} P_2((x_1,0),y)\cos\alpha(y)\,\frac{\partial\varphi}{\partial\vec{T}}(y)\,dy\right|$$
$$\le cx_1^{-7/2}\int_{-x_1}^{x_1} dy_2\, y_2\int_{f(y_2)}^{f_1(y_2)} dy_1\,(-y_1+f_1(y_2))^{-1/2}(y_1-f(y_2))^{1/2}|\log(y_1-f(y_2))|$$
$$\le cx_1^{1/2}|\log x_1|,$$
$$\left|\int_{D_2} P_2((x_1,0),y)\cos\alpha(y)\,\frac{\partial\varphi}{\partial\vec{T}}(y)\,dy\right|$$
$$\le cx_1^{1/2}\int_{x_1}^{r_1/2} dy_2\, y_2^{-3}\int_{f(y_2)}^{f_1(y_2)} dy_1\,(-y_1+f_1(y_2))^{-1/2}(y_1-f(y_2))^{1/2}|\log(y_1-f(y_2))|$$
$$\le cx_1^{1/2}|\log x_1|^2.$$

By Lemmas 2.3 and 3.4 we obtain

$$\left|\int_{D_3} P_2((x_1,0),y)\varphi_2(y)\,dy\right| \le cx_1^{1/2}\int_{D_3}\delta_B^{-1/2}(y)\delta_D^{-1/2}(y)\,dy \le cx_1^{1/2}.$$

It follows that

$$-c_1x_1^{-1/2}-c_2x_1^{1/2}|\log x_1|^2 \le \varphi_{22}(x_1,0) \le -c_3x_1^{-1/2}+c_4x_1^{1/2}|\log x_1|^2,$$

where $x_1=\sqrt{r-s}$. It is important that c_1, c_2, c_3, c_4 do not depend on s. Hence there exists $r_2=r_2(\Lambda)\in(0,r/4]$ such that $\varphi_{22}(x_1,0)\approx -x_1^{-1/2}$ for any $x_1\in(0,r_2]$. □

Lemma 3.8. *There exists $r_2=r_2(\Lambda)\in(0,r_0/4]$ such that $\varphi_{11}(x_1,0)\approx -x_1^{-3/2}$ for any $x_1\in(0,r_2]$.*

Proof. First we show that $|\varphi_{11}(x_1,0)|\le cx_1^{-3/2}$ for $x_1\in(0,r_2]$. We will use similar notation to that in Lemma 3.7. Set $r=r_0$. Let r_1 be the constant from Corollary 3.3. We take $s\in(r-(r_1/2)^2,r)$, $z=(r,0)$, $B=B(z,s)$, and P is given by (19). For any $x_1\in(r-s,r]$ by Lemma 3.5 we have $\varphi_1(x_1,0)=\int_{D\setminus B}P((x_1,0),y)\varphi_1(y)\,dy$. It follows that

$$\varphi_{11}(x_1,0)=\int_{D\setminus B}P_1((x_1,0),y)\varphi_1(y)\,dy$$
$$=\int_{D\setminus B}A((x_1,0),y)\varphi_1(y)\,dy+\int_{D\setminus B}E((x_1,0),y)\varphi_1(y)\,dy,$$

where A, E are given by (20), (21).

Take $x_1 = \sqrt{r-s}$ (we have $\sqrt{r-s} < r_1/2 \le r/8$). By (24), $|\varphi_1(y)| \le c\delta_D^{-1/2}(y)$ for $y \in D$. We have

$$\int_{D\setminus B} A((x_1,0),y)\varphi_1(y)\,dy = \frac{r-x_1}{s^2-(x_1-r)^2}\int_{D\setminus B} P((x_1,0),y)\varphi_1(y)\,dy,$$

$$\left|\int_{D\setminus B} P((x_1,0),y)\varphi_1(y)\,dy\right| = |\varphi_1(x_1,0)| \le cx_1^{-1/2}$$

and $\frac{r-x_1}{s^2-(x_1-r)^2} \approx x_1^{-1}$, so

$$\left|\int_{D\setminus B} A((x_1,0),y)\varphi_1(y)\,dy\right| \le cx_1^{-3/2}$$

for $x_1 = \sqrt{r-s}$.

Let f_1, D_1, D_2, D_3 be as in the proof of Lemma 3.7. Using $|\varphi_1(y)| \le c\delta_D^{-1/2}(y)$ and similar arguments to the proof of Lemma 3.7 we get the estimates

$$\begin{aligned}&\left|\int_{D_1} E((x_1,0),y)\varphi_1(y)\,dy\right| \\ &\le cx_1^{-5/2}\int_{-x_1}^{x_1} dy_2 \int_{f(y_2)}^{f_1(y_2)} dy_1\,(-y_1+f_1(y_2))^{-1/2}(y_1-f(y_2))^{-1/2} \le cx_1^{-3/2},\end{aligned} \tag{25}$$

$$\begin{aligned}&\left|\int_{D_2} E((x_1,0),y)\varphi_1(y)\,dy\right| \\ &\le cx_1^{1/2}\int_{x_1}^{r_1/2} dy_2\,y_2^{-4}\int_{f(y_2)}^{f_1(y_2)} dy_1\,(-y_1+f_1(y_2))^{-1/2}(y_1-f(y_2))^{-1/2}(x_1+y_1) \\ &\le cx_1^{-3/2}\end{aligned} \tag{26}$$

(here we have used the estimate $y_1 \le cy_2^2$). By Lemmas 2.3 and 3.4 we obtain

$$\left|\int_{D_3} E((x_1,0),y)\varphi_1(y)\,dy\right| \le cx_1^{1/2}\int_{D_3}\delta_B^{-1/2}(y)\delta_D^{-1/2}(y)\,dy \le cx_1^{1/2}.$$

It follows that $|\varphi_{11}(x_1,0)| \le cx_1^{-3/2}$, where c does not depend on s and $x_1 = \sqrt{r-s}$. Since $s \in (r-(r_1/2)^2, r)$ we get $|\varphi_{11}(x_1,0)| \le cx_1^{-3/2}$ for $x_1 \in (0, r_1/2]$.

Now we will show that $\varphi_{11}(x_1,0) \le -cx_1^{-3/2}$ for $x_1 \in (0, r_2]$. Here we will use notation similar to the notation used in the proof of Lemma 3.1. We will use (18) for $s = r$, in particular $B = B(z,r)$. By (18), for $x_1 \in (0,r]$ we get

$$\begin{aligned}\varphi_{11}(x_1,0) &= h_{11}(x_1,0) + \int_{D\setminus B} P_{11}((x_1,0),y)\varphi(y)\,dy \\ &= h_{11}(x_1,0) + \int_{D\setminus B}\frac{\partial A}{\partial x_1}((x_1,0),y)\varphi(y)\,dy + \int_{D\setminus B}\frac{\partial E}{\partial x_1}((x_1,0),y)\varphi(y)\,dy.\end{aligned}$$

One easily gets $h_{11}(x_1, 0) \approx -x_1^{-3/2}$ for $x_1 \in (0, r/4]$. For $x \in B$ and $y \in (\overline{B})^c$ we have

$$\begin{aligned}\frac{\partial A}{\partial x_1}(x, y) &= \frac{-C_P(r^2 - |x-z|^2)^{-3/2}(x_1 - r)^2}{(|y-z|^2 - r^2)^{1/2}|x-y|^2} + \frac{-C_P(r^2 - |x-z|^2)^{-1/2}}{(|y-z|^2 - r^2)^{1/2}|x-y|^2}\\ &\quad + \frac{-2C_P(r^2 - |x-z|^2)^{-1/2}(r - x_1)(x_1 - y_1)}{(|y-z|^2 - r^2)^{1/2}|x-y|^4}\\ &= A^{(1)}(x, y) + A^{(2)}(x, y) + A^{(3)}(x, y),\end{aligned}$$

$$\begin{aligned}\frac{\partial E}{\partial x_1}(x, y) &= \frac{-2C_P(r^2 - |x-z|^2)^{-1/2}(r - x_1)(x_1 - y_1)}{(|y-z|^2 - r^2)^{1/2}|x-y|^4} + \frac{-2C_P(r^2 - |x-z|^2)^{1/2}}{(|y-z|^2 - r^2)^{1/2}|x-y|^4}\\ &\quad + \frac{8C_P(r^2 - |x-z|^2)^{1/2}(x_1 - y_1)^2}{(|y-z|^2 - r^2)^{1/2}|x-y|^6}\\ &= E^{(1)}(x, y) + E^{(2)}(x, y) + E^{(3)}(x, y).\end{aligned}$$

Let $x_1 \in (0, r/8]$ and $y \in (\overline{B})^c$. We have $A^{(1)}(x, y), A^{(2)}(x, y) \le 0$. Moreover $A^{(3)}(x, y) \ge 0$ iff $y_1 \ge x_1$. Let f_1 be as in the proof of Lemma 3.1. Let $p_1' > 0$ be such that $f_1(p_1') = x_1$, and $p_2' < 0$ be such that $f_1(p_2') = x_1$ (we have $p_2' = -p_1'$). Note that $p_1' \approx \sqrt{x_1}$ and $|p_2'| \approx \sqrt{x_1}$. Furthermore $f_1(r/2) = r(1 - \sqrt{3}/2) > r/8$ and $f_1(p_1') = x_1 \le r/8$, so $p_1' < r/2$. Define

$$\begin{aligned}D_1' &= \{(y_1, y_2) : y_2 \in [p_2', p_1'],\ y_1 \in (f(y_2), f_1(y_2))\},\\ D_2' &= \{(y_1, y_2) : y_2 \in (p_1', r/2] \cup [-r/2, p_2'),\ y_1 \in (f(y_2), f_1(y_2))\},\\ D_3' &= D \setminus (D_1' \cup D_2' \cup B).\end{aligned}$$

We have $\int_{D_1'} A^{(3)}((x_1, 0), y)\varphi(y)\, dy \le 0$. Note that for $y \in D_2'$ we have $y_1 \le f_1(y_2) \le c y_2^2$, which gives $\varphi(y) \le c\delta_D^{1/2}(y) \le c(y_2^2)^{1/2} = c y_2$ by Lemma 2.6. Hence

$$\begin{aligned}\int_{D_2'} &A^{(3)}((x_1, 0), y)\varphi(y)\, dy\\ &\le c x_1^{-1/2} \int_{c\sqrt{x_1}}^{r/2} dy_2\, y_2^{-4} \int_{f(y_2)}^{f_1(y_2)} dy_1\, (y_1 - f_1(y_2))^{-1/2} y_1 \varphi(y)\\ &\le c x_1^{-1/2} \int_{c\sqrt{x_1}}^{r/2} dy_2 \le c x_1^{-1/2},\\ \left|\int_{D_3'} \right.&\left. A^{(3)}((x_1, 0), y)\varphi(y)\, dy\right| \le c x_1^{-1/2} \int_{D_3'} \delta_B^{-1/2}(y)\, dy \le c x_1^{-1/2}.\end{aligned}$$

Note that $E^{(1)}(x, y) = A^{(3)}(x, y)$ and $E^{(2)}(x, y) \le 0$. To estimate $\int_{D \setminus B} E^{(3)}\varphi$ we set

$$\begin{aligned}D_1'' &= \{(y_1, y_2) : y_2 \in [-x_1, x_1],\ y_1 \in (f(y_2), f_1(y_2))\},\\ D_2'' &= \{(y_1, y_2) : y_2 \in (x_1, r/2] \cup [-r/2, -x_1),\ y_1 \in (f(y_2), f_1(y_2))\},\\ D_3'' &= D \setminus (D_1'' \cup D_2'' \cup B).\end{aligned}$$

Note that for $y \in D_1''$ we have $(x_1 - y_1)^2 \le x_1^2$, which gives $\varphi(y) \le c\delta_D^{1/2}(y) \le cx_1$ by Lemma 2.6. Hence

$$\int_{D_1''} E^{(3)}((x_1,0),y)\varphi(y)\,dy \le cx_1^{-7/2}\int_{-x_1}^{x_1} dy_2 \int_{f(y_2)}^{f_1(y_2)} dy_1\,(y_1 - f_1(y_2))^{-1/2}\varphi(y)$$
$$\le cx_1^{-1/2}.$$

Moreover, for $y \in D_2''$ we have $(x_1 - y_1)^2 \le x_1^2 + y_1^2 \le x_1^2 + cy_2^4$ and $\varphi(y) \le c\delta_D^{1/2}(y) \le cy_2$, so

$$\int_{D_2''} E^{(3)}((x_1,0),y)\varphi(y)\,dy$$
$$\le cx_1^{1/2}\int_{x_1}^{r/2} dy_2\, y_2^{-6}(x_1^2 + y_2^4)\int_{f(y_2)}^{f_1(y_2)} dy_1\,(y_1 - f_1(y_2))^{-1/2}\varphi(y)$$
$$\le cx_1^{5/2}\int_{x_1}^{r/2} y_2^{-4}\,dy_2 + cx_1^{1/2}\int_{x_1}^{r/2} dy_2 \le cx_1^{-1/2}.$$

We also have $\int_{D_3''} E^{(3)}((x_1,0),y)\varphi(y)\,dy \le cx_1^{1/2}$.

It follows that for sufficiently small x_1 we have $\varphi_{11}(x_1,0) \le -cx_1^{-3/2}$. □

Lemma 3.9. *There exists* $r_2 = r_2(\Lambda) \in (0, r_0/4]$ *such that* $|\varphi_{12}(x_1,0)| \le cx_1^{-1/2}|\log x_1|$ *for any* $x_1 \in (0, r_2]$.

Proof. We will use similar notation to that in Lemma 3.7. Set $r = r_0$. Let r_1 be the constant from Corollary 3.3. We take $s \in (r - (r_1/2)^2, r)$. Recall that $z = (r,0)$, $B = B(z,s)$, and P is given by (19). For any $x_1 \in (r-s, r]$ by Lemma 3.5 we have $\varphi_2(x_1,0) = \int_{D\setminus B} P((x_1,0),y)\varphi_2(y)\,dy$. It follows that

$$\varphi_{12}(x_1,0) = \int_{D\setminus B} P_1((x_1,0),y)\varphi_2(y)\,dy$$
$$= \int_{D\setminus B} A((x_1,0),y)\varphi_2(y)\,dy + \int_{D\setminus B} E((x_1,0),y)\varphi_2(y)\,dy.$$

Take $x_1 = \sqrt{r-s}$ (we have $\sqrt{r-s} < r_1/2 \le r/8$). We obtain

$$\int_{D\setminus B} A((x_1,0),y)\varphi_2(y)\,dy = \frac{r - x_1}{(s^2 - (x_1 - r)^2)}\int_{D\setminus B} P((x_1,0),y)\varphi_2(y)\,dy.$$

By Lemma 3.2,

$$\left|\int_{D\setminus B} P((x_1,0),y)\varphi_2(y)\,dy\right| = |\varphi_2(x_1,0)| \le cx_1^{1/2}|\log x_1|.$$

Since $(r - x_1)(s^2 - (x_1 - r)^2)^{-1} \approx x_1^{-1}$, we obtain

$$\left| \int_{D \setminus B} A((x_1, 0), y)\varphi_2(y)\, dy \right| \le c x_1^{-1/2} |\log x_1|,$$

for $x_1 = \sqrt{r - s}$.

Let f_1, D_1, D_2, D_3 be as in the proof of Lemma 3.7. By Lemma 2.2, for $y \in D_1 \cup D_2$ we have

$$\varphi_2(y) = \cos\alpha(y) \frac{\partial \varphi}{\partial \vec{T}}(y) - \sin\alpha(y) \frac{\partial \varphi}{\partial \vec{n}}(y).$$

By the arguments in the proof of Lemma 3.7, for such y,

$$\left| \cos\alpha(y) \frac{\partial \varphi}{\partial \vec{T}}(y) \right| \le c(y_1 - f(y_2))^{1/2} |\log(y_1 - f(y_2))| \le c y_1^{1/2} |\log y_1|,$$

$$\left| \sin\alpha(y) \frac{\partial \varphi}{\partial \vec{n}}(y) \right| \le c y_2 (y_1 - f(y_2))^{-1/2}.$$

Much as in the proofs of Lemmas 3.7 and 3.8, we obtain

$$\left| \int_{D_1} E((x_1, 0), y) \cos\alpha(y) \frac{\partial \varphi}{\partial \vec{T}}(y)\, dy \right|$$
$$\le c x_1^{-5/2} \int_{-x_1}^{x_1} dy_2 \int_{f(y_2)}^{f_1(y_2)} dy_1\, (-y_1 + f_1(y_2))^{-1/2} y_1^{1/2} |\log y_1| \le c x_1^{1/2} |\log x_1|.$$

Here we have used the inequalities $y_1^{1/2}|\log y_1| \le c y_2 |\log y_2| \le c x_1 |\log x_1|$ and $\int_{f(y_2)}^{f_1(y_2)} (-y_1 + f_1(y_2))^{-1/2}\, dy_1 \le c f_1^{1/2}(y_2) \le c y_2 \le c x_1$.

Using similar arguments we get

$$\left| \int_{D_2} E((x_1, 0), y) \cos\alpha(y) \frac{\partial \varphi}{\partial \vec{T}}(y)\, dy \right|$$
$$\le c x_1^{1/2} \int_{x_1}^{r/2} dy_2\, y_2^{-4} \int_{f(y_2)}^{f_1(y_2)} dy_1\, (-y_1 + f_1(y_2))^{-1/2} y_1^{1/2} |\log y_1| (x_1 + y_1)$$
$$\le c x_1^{1/2} |\log x_1|.$$

By the same arguments as in (25), (26) one can easily obtain

$$\left| \int_{D_1} E((x_1, 0), y) y_2 (y_1 - f(y_2))^{-1/2}\, dy \right| \le c x_1^{-1/2},$$

$$\left| \int_{D_2} E((x_1, 0), y) y_2 (y_1 - f(y_2))^{-1/2}\, dy \right| \le c x_1^{-1/2} + c x_1^{1/2} |\log x_1|,$$

By Lemmas 2.3 and 3.4,

$$\left| \int_{D_3} E((x_1, 0), y)\varphi_2(y)\, dy \right| \le c x_1^{1/2} \int_{D_3} \delta_B^{-1/2}(y) \delta_D^{-1/2}(y)\, dy \le c x_1^{1/2}.$$

It follows that $|\varphi_{12}(x_1, 0)| \le c x_1^{-1/2} |\log x_1|$, where c does not depend on s, and $x_1 = \sqrt{r-s}$. Since $s \in (r - (r_1/2)^2, r)$ we get $|\varphi_{12}(x_1, 0)| \le c x_1^{-1/2} |\log x_1|$ for all $x_1 \in (0, r_1/2]$. □

By Lemmas 2.2, 3.7, 3.8, 3.9 and Corollary 3.3 we obtain

Corollary 3.10. *There exists $r_2 = r_2(\Lambda) \in (0, r_0/4]$ such that for any $y \in D$ with $\delta_D(y) \le r_2$ we have (22)–(24) and*

$$\frac{\partial^2 \varphi}{\partial \vec{n}^2}(y) \approx -\delta_D^{-3/2}(y),$$
$$\frac{\partial^2 \varphi}{\partial \vec{T}^2}(y) \approx -\delta_D^{-1/2}(y),$$
$$\left| \frac{\partial^2 \varphi}{\partial \vec{n} \partial \vec{T}}(y) \right| \le c \delta_D^{-1/2}(y) |\log(\delta_D(y))|.$$

Lemma 3.11. *There exists $r_3 = r_3(\Lambda) \in (0, r_0/4]$ such that for any $y = (y_1, y_2) \in B((r_3, 0), r_3)$ we have*

$$|\varphi_2(y)| \le c(y_1^{1/2} |\log y_1| + |y_2| y_1^{-1/2}), \tag{27}$$
$$|\varphi_{12}(y)| \le c(y_1^{-1/2} |\log y_1| + |y_2| y_1^{-3/2}), \tag{28}$$
$$|\varphi_{22}(y)| \approx -y_1^{-1/2}, \tag{29}$$

and for any $y = (y_1, y_2) \in W_{r_3} = \{(y_1, y_2) : y_2 \in [-r_3, r_3],\ y_1 \in (f(y_2), r_3]\}$ we have

$$\varphi_1(y) \approx \delta_D^{-1/2}(y). \tag{30}$$

Proof. We may assume that $y_2 > 0$. Let $r \in (0, r_2]$ where r_2 is the constant from Corollary 3.10 (recall that $r_2 \le r_0/4$). Let $y = (y_1, y_2) \in B((r, 0), r)$ with $y_2 > 0$. By Lemma 2.2 we have $\sin \alpha(y) \approx y_2$, $\cos \alpha(y) \approx c$. Moreover, $\delta_D(y) \approx y_1$ and $y_2^2 \le c y_1$.

By Corollary 3.10 we get

$$\frac{\partial \varphi}{\partial \vec{n}}(y) \approx -\delta_D^{-1/2}(y) \approx -y_1^{-1/2}, \qquad \left| \frac{\partial \varphi}{\partial \vec{T}}(y) \right| \le c \delta_D^{1/2}(y) |\log(\delta_D(y))| \le c y_1^{1/2} |\log y_1|.$$

Using this and the formula for φ_2 from Lemma 2.2 we get (27).

By Corollary 3.10 we have

$$\left| \frac{\partial^2 \varphi}{\partial \vec{n} \partial \vec{T}}(y) \right| \le c \delta_D^{-1/2}(y) |\log(\delta_D(y))| \le c y_1^{-1/2} |\log y_1|,$$
$$\left| \frac{\partial^2 \varphi}{\partial \vec{n}^2}(y) - \frac{\partial^2 \varphi}{\partial \vec{T}^2}(y) \right| \le c \delta_D^{-3/2}(y) \le c y_1^{-3/2}.$$

Using this and the formula for φ_{12} from Lemma 2.2 we get (28).

By Corollary 3.10 we have $\frac{\partial^2\varphi}{\partial\vec{T}^2}(y) \approx -\delta_D^{-1/2}(y) \approx -y_1^{-1/2}$, $\frac{\partial^2\varphi}{\partial\vec{n}^2}(y) \approx -\delta_D^{-3/2}(y) \approx -y_1^{-3/2}$, $\sin^2\alpha(y) \approx y_2^2 \le cy_1$ and

$$\left|\sin\alpha(y)\cos\alpha(y)\frac{\partial^2\varphi}{\partial\vec{n}\partial\vec{T}}(y)\right| \le cy_2 y_1^{-1/2}|\log y_1| \le c|\log y_1|.$$

Using this and the formula for φ_{22} from Lemma 2.2 we get (29) for sufficiently small r.

By (22), (23) and the formula for φ_1 from Lemma 2.2 we deduce (30) for sufficiently small r. □

We have $(-\Delta)^{1/2}\varphi(x) = 1$ for $x \in D$. We need to estimate $(-\Delta)^{1/2}\varphi(x)$ for $x \in (\overline{D})^c$. For such x we have $(-\Delta)^{1/2}\varphi(x) = -(2\pi)^{-1}\int_D \frac{\varphi(y)}{|y-x|^3}\,dy$.

Lemma 3.12. *Let $x = (-x_1, 0)$ with $x_1 > 0$. We have*

$$|(-\Delta)^{1/2}\varphi(x)| \approx \delta_D^{-1/2}(x)(1+|x|)^{-5/2}.$$

Proof. Set $r = r_0$. When $x_1 \in (-\infty, -r/2)$ we have

$$\int_D \frac{\varphi(y)}{|y-x|^3}\,dy \approx |x|^{-3} \approx \delta_D^{-1/2}(x)(1+|x|)^{-5/2}.$$

When $x_1 \in [-r/2, 0)$, using Lemma 2.6 we obtain

$$\int_D \frac{\varphi(y)}{|y-x|^3}\,dy \approx \int_{D\cap B(0,\delta_D(x))} \delta_D^{-5/2}(x)\,dy + \int_{D\cap(B(0,r/2)\setminus B(0,\delta_D(x)))} |y|^{-5/2}\,dy + \int_{D\cap B^c(0,r/2)} |y|^{-5/2}\,dy \approx \delta_D^{-1/2}(x).$$ □

Lemma 3.12 immediately yields

Corollary 3.13. *For any $x \in (\overline{D})^c$ we have*

$$|(-\Delta)^{1/2}\varphi(x)| \approx \delta_D^{-1/2}(x)(1+|x|)^{-5/2}.$$

4. Estimates of derivatives of u near $\partial D \times \{0\}$

In this section we study the behaviour of u_{ij} near $\partial D \times \{0\}$. The ultimate aim of these estimates is to control the determinants of the Hessian matrices of u and $v^{(\varepsilon,D)}$ (which is equal to u plus a small auxiliary harmonic function; for a precise definition see Section 6) near $\partial D \times \{0\}$. The estimates are quite long and technical because the u_{ij} are singular near $\partial D \times \{0\}$ and their behaviour is quite complicated.

In the whole section we fix $C_1, R_1 > 0$, $\kappa_2 \ge \kappa_1 > 0$, $D \in F(C_1, R_1, \kappa_1, \kappa_2)$ and $x_0 \in \partial D$. We write $\Lambda = \{C_1, R_1, \kappa_1, \kappa_1\}$; φ is the solution of (1)–(2) for D and u is the harmonic extension of φ given by (6)–(10). Unless otherwise stated, we fix a 2-dimensional coordinate system CS_{x_0} and notation as in Lemma 2.2 (see Figure 1). In

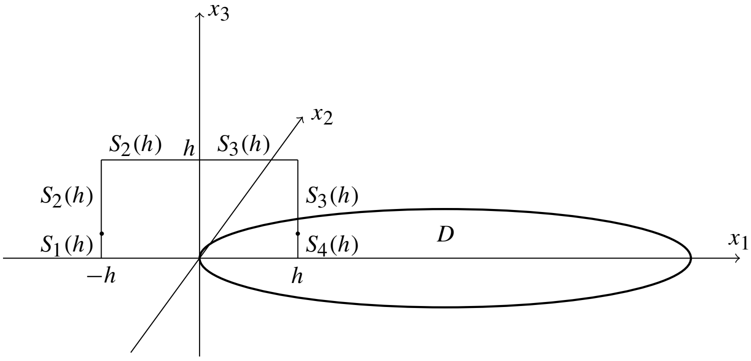


Fig. 4

particular x_0 is (0, 0) in CS_{x_0} coordinates. To study u we also use a 3-dimensional Cartesian coordinate system $0x_1x_2x_3$ (see Figure 4), which is formed (roughly speaking) by adding the $0x_3$ axis to the above 2-dimensional coordinate system. Recall that in the whole section we use the convention stated in Remark 2.9.

Set $r = r_1 \wedge r_2 \wedge r_3 \wedge f(r_0/4) \wedge f(-r_0/4)$, where r_0, r_1, r_2, r_3 are the constants from Lemma 2.2, Corollary 3.3, Corollary 3.10 and Lemma 3.11. Note that $f(r_0/4) \wedge f(-r_0/4) \geq c_3 r_0^2/16$, where c_3 is the constant from Lemma 2.2; here $c_3 r_0^2/16$ depends only on Λ. Define $f_1 : [-r, r] \to \mathbb{R}$ by $f_1(y_2) = r - \sqrt{r^2 - y_2^2}$ and $g_1 : [0, r] \to \mathbb{R}$ by $g_1(y_1) = \sqrt{r^2 - (y_1 - r)^2}$ (the graphs of f_1, g_1 are parts of the circle $\{(y_1, y_2) : (y_1 - r)^2 + y_2^2 = r^2\}$). For any $h \in (0, r]$ we denote (see Figure 4)

$$\begin{aligned}
S_1(h) &= \{(x_1, x_2, x_3) : x_1 = -h,\ x_2 = 0,\ x_3 \in (0, h/4]\},\\
S_2(h) &= \{(x_1, x_2, x_3) : x_1 = -h,\ x_2 = 0,\ x_3 \in (h/4, h]\}\\
&\quad \cup \{(x_1, x_2, x_3) : x_1 \in (-h, 0],\ x_2 = 0,\ x_3 = h\},\\
S_3(h) &= \{(x_1, x_2, x_3) : x_1 \in (0, h],\ x_2 = 0,\ x_3 = h\}\\
&\quad \cup \{(x_1, x_2, x_3) : x_1 = h,\ x_2 = 0,\ x_3 \in (h/4, h]\},\\
S_4(h) &= \{(x_1, x_2, x_3) : x_1 = h,\ x_2 = 0,\ x_3 \in (0, h/4]\}.
\end{aligned}$$

The main tool which we use in this section is the formula

$$u(x) = \int_D K(x_1 - y_1, x_2 - y_2, x_3)\varphi(y_1, y_2)\, dy_1\, dy_2.$$

To obtain estimates of u_{ij} we differentiate under the integral sign in the above formula. The results concerning estimates of u_{ij} are divided into six propositions. In the proof of Proposition 4.1 we use the formula

$$u_{22}(x) = \int_D K_2(x_1 - y_1, x_2 - y_2, x_3)\varphi_2(y_1, y_2)\, dy_1\, dy_2$$

(for brevity we simply write $u_{22} = \int_D K_2\varphi_2$), the estimates of $\partial\varphi/\partial\vec{n}$, $\partial\varphi/\partial\vec{T}$ from Corollary 3.3 and the estimate of $|\nabla\varphi|$ from Lemma 3.4. In this proof we also use the formula $\varphi_2(y_1, y_2) - \varphi_2(y_1, -y_2) = 2y_2\varphi_{22}(y_1, \xi)$ and the estimate of φ_{22} from Lemma 3.11. In the proof of Proposition 4.2 (which is the easiest result of this section) we use the formulas $u_{11} = \int_D K_{11}\varphi$, $u_{13} = \int_D K_{13}\varphi$ and the estimate $\varphi(x) \le c\delta_D^{1/2}(x)$. In the proof of Proposition 4.3 we use the formulas $u_{11} = \int_D K_1\varphi_1$, $u_{13} = \int_D K_3\varphi_1$, the estimate of φ_1 from Lemma 3.11 and the estimate of $|\nabla\varphi|$ from Lemma 3.4. The proof of Proposition 4.4 is based on a different idea than the proofs of the previous propositions. Namely, we use the fact that $u_3(y_1, y_2, 0) = -(-\Delta)^{1/2}\varphi(y_1, y_2)$ for $(y_1, y_2) \notin \partial D$. We also use the formulas $u_{13} = \int_{\mathbb{R}^2} K_1 u_3$, $u_{33} = \int_{\mathbb{R}^2} K_3 u_3$ and the estimate of $|(-\Delta)^{1/2}\varphi|$ from Corollary 3.13. In the proof of Proposition 4.5 we use the formulas $u_{12} = \int_D K_{12}\varphi$, $u_{23} = \int_D K_{23}\varphi$, $\varphi(y_1, y_2) - \varphi(y_1, -y_2) = 2y_2\varphi_2(y_1, \xi)$, the estimate of $\varphi(x)$ from Lemma 2.6 and the estimate of φ_2 from Lemma 3.11. Moreover, we apply the formula $\varphi(z_1+h, z_2) - \varphi(-z_1+h, z_2) - \varphi(z_1+h, -z_2) + \varphi(-z_1+h, -z_2) = 4z_1z_2\varphi_{12}(\xi_1+h, \xi_2)$ and the estimate of φ_{12} from Lemma 3.11. The most difficult result of this section is Proposition 4.6. In this proposition we study u_{23} on $S_4(h)$ using two different formulas: $u_{23} = \int_{\mathbb{R}^2} K_2 u_3$ and $u_{23} = \int_D K_{23}\varphi$. We use the estimate of $|(-\Delta)^{1/2}\varphi|$ from Corollary 3.13, the estimates of $\varphi_2, \varphi_{12}, \varphi_{22}$ from Lemma 3.11 and the estimate of $\varphi(x)$ from Lemma 2.6. In Lemma 4.7 we obtain results concerning $u_{i3}(x_1, x_2, 0)$ for $i = 1, 2, 3$ and $(x_1, x_2) \in D$.

In this section we only use those geometric properties of the domain D which are stated in Lemmas 2.2 and 2.3 (and additionally the fact that D is convex and $D \subset B(0, 1)$). Let us recall that all constants in Lemmas 2.2 and 2.3 depend only on Λ. We only use those inequalities for $\varphi, \varphi_i, \varphi_{ij}$ which are stated in Section 3 and in Lemma 2.6. The constants in those inequalities depend only on Λ. Therefore all constants in the estimates of u_{ij} obtained in Section 4 depend only on Λ.

Proposition 4.1. *There exists $h_0 = h_0(\Lambda) \in (0, r/8]$ such that for any $h \in (0, h_0]$ we have $u_{22}(x) \approx -x_3h^{-3/2}$ for $x \in S_1(h) \cup S_2(h) \cup S_3(h)$, $u_{22}(x) \approx -h^{-1/2}$ for $x \in S_4(h)$.*

Proof. Let $h \in (0, r/8]$. We have

$$u_{22}(x) = \int_D K_2(x_1 - y_1, -y_2, x_3)\varphi_2(y_1, y_2)\, dy_1\, dy_2. \tag{31}$$

Denote (see Figure 5)

$$\begin{aligned}
D_1 &= \{(y_1, y_2) : y_1 \in [f_1(h), h],\ y_2 \in [-g_1(y_1), g_1(y_1)]\},\\
D_2 &= \{(y_1, y_2) : y_1 \in (h, r],\ y_2 \in [-g_1(y_1), g_1(y_1)]\},\\
D_3 &= \{(y_1, y_2) : y_2 \in [-h, h],\ y_1 \in (f(y_2), f_1(h))\},\\
D_4 &= \{(y_1, y_2) : y_2 \in [-r/2, -h] \cup [h, r/2],\ y_1 \in (f(y_2), f_1(y_2))\},\\
D_5 &= D \setminus (D_1 \cup D_2 \cup D_3 \cup D_4).
\end{aligned}$$

For $i = 1, 2, 3, 4$ we also set $D_{i+} = \{(y_1, y_2) \in D_i : y_2 > 0\}$, $D_{i-} = \{(y_1, y_2) \in D_i : y_2 < 0\}$.

Note that $f_1(h) \le h^2/r \le h/4$.

We will estimate (31). The most important part is $\int_{D_1 \cup D_2} K_2\varphi_2$. By Lemma 3.11 for $y \in D_{1+} \cup D_{2+}$ we have $\varphi_2(y_1, y_2) - \varphi_2(y_1, -y_2) = 2y_2\varphi_{22}(y_1, \xi) \approx -y_2y_1^{-1/2}$, where

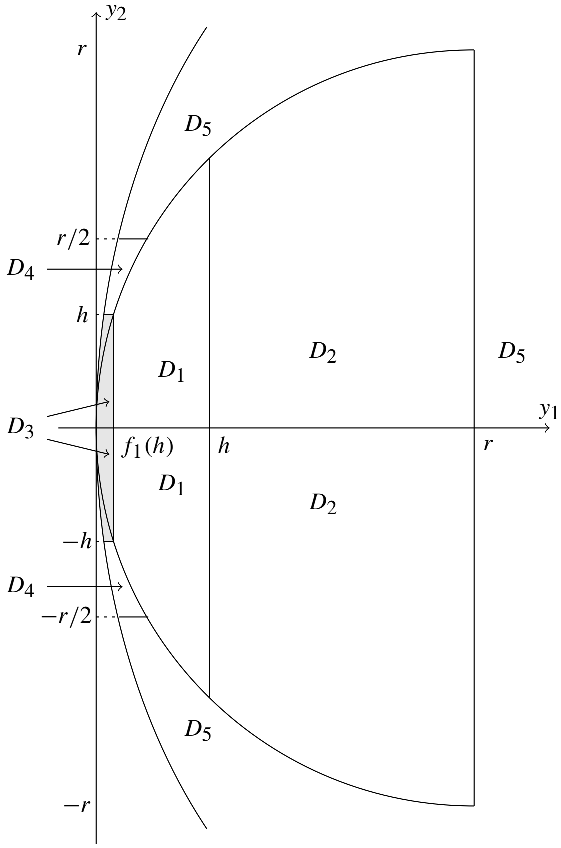


Fig. 5

$\xi \in (-y_2, y_2)$. It follows that

$$\int_{D_1\cup D_2} K_2(x_1 - y_1, -y_2, x_3)\varphi_2(y_1, y_2)\, dy_1\, dy_2$$
$$= cx_3 \int_{D_{1+}\cup D_{2+}} \frac{y_2}{((x_1 - y_1)^2 + y_2^2 + x_3^2)^{5/2}} \big(\varphi_2(y_1, y_2) - \varphi_2(y_1, -y_2)\big)\, dy_1\, dy_2$$
$$\approx cx_3 \int_{D_{1+}\cup D_{2+}} \frac{-y_2^2 y_1^{-1/2}}{((x_1 - y_1)^2 + y_2^2 + x_3^2)^{5/2}}\, dy_1\, dy_2.$$

We have

$$\int_{D_{1+}} \frac{-y_2^2 y_1^{-1/2}}{((x_1 - y_1)^2 + y_2^2 + x_3^2)^{5/2}}\, dy_1\, dy_2$$
$$\approx \frac{1}{h^5} \int_{f_1(h)}^{h} dy_1\, y_1^{-1/2} \int_0^h dy_2\, (-y_2^2) + \int_{f_1(h)}^{h} dy_1\, y_1^{-1/2} \int_h^{g_1(y_1)} dy_2\, \frac{-y_2^2}{y_2^5}.$$

Since $f_1(y_2) = y_2^2(r + (r^2 - y_2^2)^{1/2})^{-1}$ and $g_1(y_1) = y_1^{1/2}(2r - y_1)^{1/2}$, we obtain $c_1 y_2^2 \le f_1(y_2) \le c_2 y_2^2$ and $c_3 y_1^{1/2} \le g_1(y_1) \le c_4 y_1^{1/2}$ and the constants c_1, c_2, c_3, c_4 depend only on Λ. Hence the last expression is comparable to $-h^{-3/2}$ (with constants depending only on Λ).

By similar arguments we have

$$\int_{D_{2+}} \frac{-y_2^2 y_1^{-1/2}}{((x_1 - y_1)^2 + y_2^2 + x_3^2)^{5/2}} \, dy_1 \, dy_2$$
$$\approx \int_h^r dy_1 \int_0^{y_1} dy_2 \frac{-y_2^2 y_1^{-1/2}}{y_1^5} + \int_h^r dy_1 \int_{y_1}^{g_1(y_1)} dy_2 \frac{-y_2^2 y_1^{-1/2}}{y_2^5} \approx -h^{-3/2}.$$

It follows that $\int_{D_1 \cup D_2} K_2 \varphi_2 \approx -x_3 h^{-3/2}$.

Now we will estimate $\int_{D_3 \cup D_4} K_2 \varphi_2$. It is sufficient to estimate $\int_{D_{3+} \cup D_{4+}} K_2 \varphi_2$. The estimate of $\int_{D_{3-} \cup D_{4-}} K_2 \varphi_2$ is the same. By Lemma 2.2 and Corollary 3.3, for $y \in D_{3+} \cup D_{4+}$ we get

$$\begin{aligned}
|\varphi_2(y)| &= \left| \cos\alpha(y) \frac{\partial \varphi}{\partial \vec{T}}(y) - \sin\alpha(y) \frac{\partial \varphi}{\partial \vec{n}}(y) \right| \\
&\le c\delta_D^{1/2}(y) |\log(\delta_D(y))| + c y_2 \delta_D^{-1/2}(y) \\
&\le c(f^{-1}(y_1) - y_2)^{1/2} (f^{-1}(y_1))^{1/2} |\log((f^{-1}(y_1) - y_2) f^{-1}(y_1))| \\
&\quad + c y_2 (f^{-1}(y_1) - y_2)^{-1/2} (f^{-1}(y_1))^{-1/2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left| \int_{D_{3+}} K_2(x_1 - y_1, -y_2, x_3) \varphi_2(y_1, y_2) \, dy_1 \, dy_2 \right| \\
&\quad \le \frac{c x_3}{h^5} \int_0^{f_1(h)} dy_1 \int_0^{f^{-1}(y_1)} dy_2 \, y_2 |\varphi_2(y_1, y_2)| \\
&\quad \le \frac{c x_3}{h^5} \int_0^{f_1(h)} dy_1 \int_0^{f^{-1}(y_1)} dy_2 \, (f^{-1}(y_1) - y_2)^{1/2} (f^{-1}(y_1))^{1/2} \\
&\qquad\qquad \times |\log((f^{-1}(y_1) - y_2) f^{-1}(y_1))| y_2 \\
&\qquad + \frac{c x_3}{h^5} \int_0^{f_1(h)} dy_1 \int_0^{f^{-1}(y_1)} dy_2 \, (f^{-1}(y_1) - y_2)^{-1/2} (f^{-1}(y_1))^{-1/2} y_2^2.
\end{aligned}$$

By substituting $w = f^{-1}(y_1) - y_2$ and using $y_2 = f^{-1}(y_1) - w \le f^{-1}(y_1)$, $f^{-1}(y_1) \approx y_1^{1/2}$ and $f_1(h) \le ch^2$ this is bounded from above by

$$\begin{aligned}
&\frac{c x_3}{h^5} \int_0^{f_1(h)} dy_1 \int_0^{f^{-1}(y_1)} dw \, w^{1/2} (f^{-1}(y_1))^{3/2} |\log(w f^{-1}(y_1))| \\
&\qquad + \frac{c x_3}{h^5} \int_0^{f_1(h)} dy_1 \int_0^{f^{-1}(y_1)} dw \, w^{-1/2} (f^{-1}(y_1))^{3/2} \le c x_3 |\log h| + c x_3 h^{-1}.
\end{aligned}$$

In the above estimate we have used the inequality $f^{-1}(y_1) \le c y_1^{1/2}$, which follows from Lemma 2.2 (property 3), so the constant c depends only on Λ.

In the same way we get

$$\left|\int_{D_{4+}} K_2(x_1-y_1,-y_2,x_3)\varphi_2(y_1,y_2)\,dy_1\,dy_2\right|$$
$$\le cx_3\int_h^{r/2} dy_2 \int_{f(y_2)}^{f_1(y_2)} dy_1\, \frac{y_2}{y_2^5}|\varphi_2(y_1,y_2)|$$
$$\le cx_3\int_{f(h)}^{f_1(r/2)} dy_1 \int_{g_1(y_1)}^{f^{-1}(y_1)} dy_2\, y_2^{-4}(f^{-1}(y_1)-y_2)^{1/2}(f^{-1}(y_1))^{1/2} \times |\log((f^{-1}(y_1)-y_2)f^{-1}(y_1))|$$
$$+ cx_3\int_{f(h)}^{f_1(r/2)} dy_1 \int_{g_1(y_1)}^{f^{-1}(y_1)} dy_2\, y_2^{-3}(f^{-1}(y_1)-y_2)^{-1/2}(f^{-1}(y_1))^{-1/2}.$$

Similarly to the estimate of $\int_{D_{3+}} K_2\varphi_2$, using the substitution $w = f^{-1}(y_1) - y_2$ we find that the above is bounded from above by $cx_3|\log h|^2 + cx_3 h^{-1}$. By Lemma 3.4 we get

$$\left|\int_{D_5} K_2(x_1-y_1,-y_2,x_3)\varphi_2(y_1,y_2)\,dy_1\,dy_2\right| \le cx_3\int_{D_5}\delta_D^{-1/2}(y)\,dy.$$

By Lemma 2.3 this is bounded from above by cx_3. We finally obtain $\int_{D_1\cup D_2} K_2\varphi_2 \approx -x_3h^{-3/2}$ and $|\int_{D_3\cup D_4\cup d_5} K_2\varphi_2| \le cx_3h^{-1}$, where all constants depend only on Λ. It is clear that one can choose $h_0 = h_0(\Lambda)$ such that for any $h \in (0, h_0]$ we have $u_{22}(x) = \int_{D_1\cup\cdots\cup D_5} K_2\varphi_2 \approx -x_3h^{-3/2}$ for $x \in S_1(h)\cup S_2(h)\cup S_3(h)$.

Now we estimate $u_{22}(x)$ for $x \in S_4(h)$. Set $A = B((h,0),h/2)$, $A_+ = \{y \in A : y_2 > 0\}$ and $A_{1+} = \{y \in B((h,0),x_3) : y_2 > 0\}$, $A_{2+} = A_+ \setminus A_{1+}$. By similar arguments to those above we obtain $\int_{D\setminus A} K_2\varphi_2 \approx -x_3h^{-3/2}$ and for $y \in A$ we get $\varphi_2(y_1,y_2) - \varphi_2(y_1,-y_2) \approx -y_2y_1^{-1/2} \approx -y_2h^{-1/2}$. Note that for $x \in S_4(h)$ we have $x = (h,0,x_3)$, where $x_3 \in (0,h/4]$. It follows that

$$\int_A K_2(x_1-y_1,-y_2,x_3)\varphi_2(y_1,y_2)\,dy_1\,dy_2$$
$$= \int_{A_+} K_2(x_1-y_1,-y_2,x_3)(\varphi_2(y_1,y_2)-\varphi_2(y_1,-y_2))\,dy_1\,dy_2$$
$$\approx -x_3h^{-1/2}\int_{A_{1+}\cup A_{2+}} \frac{y_2^2}{((h-y_1)^2+y_2^2+x_3^2)^{5/2}}\,dy_1\,dy_2$$
$$\approx \frac{-h^{-1/2}}{x_3^4}\int_0^{x_3}\rho^3\,d\rho - x_3h^{-1/2}\int_{x_3}^{h/2}\rho^{-2}\,d\rho \approx -h^{-1/2}. \qquad \square$$

Proposition 4.2. *There exists $h_0 = h_0(\Lambda) \in (0, r/8]$ such that $|u_{11}(x)| \le cx_3h^{-5/2}$, $|u_{33}(x)| \le cx_3h^{-5/2}$ and $|u_{13}(x)| \le ch^{-3/2}$ for any $h \in (0,h_0]$ and any $x \in S_1(h)\cup S_2(h)\cup S_3(h)$.*

Proof. Let $h \in (0, r/8]$. We have

$$u_{11}(x) = \int_D K_{11}(x_1 - y_1, -y_2, x_3)\varphi(y_1, y_2)\, dy_1\, dy_2,$$

Set $D_1 = D \cap B(0, h)$. By Lemma 2.6, for $y \in D_1$ we have $\varphi(y) \le ch^{1/2}$, while for $y \in D \setminus D_1$ we have $\varphi(y) \le c(\text{dist}(0, y))^{1/2}$. It follows that

$$\left|\int_{D_1} K_{11}\varphi\right| \le cx_3 \frac{h^2}{h^7} h^{1/2} \int_{D_1} dy \approx cx_3 h^{-5/2},$$

$$\left|\int_{D\setminus D_1} K_{11}\varphi\right| \le cx_3 \int_h^\infty \frac{\rho^2}{\rho^7} \rho^{1/2} \rho\, d\rho \approx cx_3 h^{-5/2}.$$

Since $u_{11}(x) + u_{22}(x) + u_{33}(x) = 0$ and, by Lemma 4.1, $u_{22}(x) \approx -x_3 h^{-3/2}$ for $x \in S_1(h) \cup S_2(h) \cup S_3(h)$, we get $|u_{33}(x)| \le cx_3 h^{-5/2}$.

Similarly we have

$$u_{13}(x) = \int_D K_{13}(x_1 - y_1, -y_2, x_3)\varphi(y_1, y_2)\, dy_1\, dy_2,$$

$$\left|\int_{D_1} K_{13}\varphi\right| \le ch \frac{h^2}{h^7} h^{1/2} \int_{D_1} dy \approx ch^{-3/2},$$

$$\left|\int_{D\setminus D_1} K_{13}\varphi\right| \le c \int_h^\infty \frac{\rho^3}{\rho^7} \rho^{1/2} \rho\, d\rho \approx ch^{-3/2}. \qquad \square$$

Proposition 4.3. *There exists $h_0 = h_0(\Lambda) \in (0, r/8]$ such that for any $h \in (0, h_0]$ we have $u_{13}(x) \approx h^{-3/2}$ for $x \in S_1(h)$, and $u_{11}(x) \approx h^{-3/2}$, $u_{33}(x) \approx -h^{-3/2}$ for $x \in S_2(h)$.*

Proof. Let $h \in (0, r/8]$. We have

$$u_{13}(x) = \int_D K_3(x_1 - y_1, -y_2, x_3)\varphi_1(y_1, y_2)\, dy_1\, dy_2,$$

$$K_3(x_1 - y_1, -y_2, x_3) = C_K \frac{(x_1 - y_1)^2 + y_2^2 - 2x_3^2}{((x_1 - y_1)^2 + y_2^2 + x_3^2)^{5/2}}.$$

Set $D_1 = \{(y_1, y_2) : y_2 \in (-r, r),\ y_1 \in (f(y_2), r)\}$. By Lemma 3.11 we get $\varphi_1(y) \approx \delta_D^{-1/2}(y)$ for $y \in D_1$. We also have $K_3(x_1 - y_1, -y_2, x_3) \ge 0$ for $y \in D_1$ and $x \in S_1(h)$. Let $\beta(y)$ be the acute angle between $0y$ and the y_1 axis. Define $D_2 = \{(y_1, y_2) : |y| \in (h, r), \beta(y) \in [0, \pi/6)\}$. Clearly, $D_2 \subset D_1$. For $y \in D_2$ we have $\varphi_1(y) \approx \delta_D^{-1/2}(y) \approx |y|^{-1/2}$ and $K_3(x_1 - y_1, -y_2, x_3) \ge c|y|^{-3}$. It follows that

$$\int_{D_1} K_3\varphi_1 \ge \int_{D_2} |y|^{-7/2}\, dy \approx h^{-3/2}.$$

By Lemmas 3.4 and 2.3 we get

$$\left|\int_{D\setminus D_1} K_3\varphi_1\right| \le c \int_{D\setminus D_1} \delta_D^{-1/2}(y)\, dy \le c.$$

Hence $u_{13}(x) \geq ch^{-3/2}$ for $x \in S_1(h)$ and sufficiently small h. By Proposition 4.2, $|u_{13}(x)| \leq ch^{-3/2}$, so $u_{13}(x) \approx h^{-3/2}$.

We have

$$u_{11}(x) = \int_D K_1(x_1 - y_1, -y_2, x_3)\varphi_1(y_1, y_2)\, dy_1\, dy_2,$$

$$K_1(x_1 - y_1, -y_2, x_3) = 3C_K \frac{x_3(y_1 - x_1)}{((x_1 - y_1)^2 + y_2^2 + x_3^2)^{5/2}}.$$

Here $K_1(x_1 - y_1, -y_2, x_3) \geq 0$ for $y \in D_1$ and $x \in S_2(h)$. For $y \in D_2$ and $x \in S_2(h)$ we have $K_1(x_1 - y_1, -y_2, x_3) \geq ch|y|^{-4}$. It follows that

$$\int_{D_1} K_1\varphi_1 \geq ch \int_{D_2} |y|^{-9/2}\, dy \approx h^{-3/2}.$$

By Lemmas 3.4 and 2.3 we get $|\int_{D\setminus D_1} K_1\varphi_1| \leq c$. Hence $u_{11}(x) \geq ch^{-3/2}$ for $x \in S_2(h)$ and sufficiently small h. By Proposition 4.2, $|u_{11}(x)| \leq ch^{-3/2}$, so $u_{11}(x) \approx h^{-3/2}$. Since $u_{11}(x) + u_{22}(x) + u_{33}(x) = 0$ and, by Proposition 4.1, $u_{22}(x) \approx -h^{-1/2}$ for $x \in S_2(h)$, we get $u_{33}(x) \approx -h^{-3/2}$. □

Proposition 4.4. *There exists $h_0 = h_0(\Lambda) \in (0, r/8]$ such that for any $h \in (0, h_0]$ we have $|u_{13}(x)| \leq ch^{-3/2}$ for $x \in S_4(h)$, $u_{13}(x) \approx -h^{-3/2}$ for $x \in S_3(h)$, $u_{13}(x) \leq -cx_3h^{-5/2}$ for $x \in S_4(h)$, and $u_{33}(x) \approx h^{-3/2}$, $u_{11}(x) \approx -h^{-3/2}$ for $x \in S_4(h)$.*

Proof. Let $h \in (0, r/8]$. We have

$$u_{13}(x) = \int_{\mathbb{R}^2} K_1(x_1 - y_1, -y_2, x_3)u_3(y_1, y_2, 0)\, dy_1\, dy_2,$$

$$K_1(x_1 - y_1, -y_2, x_3) = 3C_K \frac{x_3(y_1 - x_1)}{((x_1 - y_1)^2 + y_2^2 + x_3^2)^{5/2}}.$$

For $y \in D$ we have $u_3(y_1, y_2, 0) = -1$ and for $y \in (\overline{D})^c$, by Corollary 3.13,

$$u_3(y_1, y_2, 0) = -(-\Delta)^{1/2}\varphi(y) \approx (1 + |y|^{-5/2})\delta_D^{-1/2}(y).$$

Denote (see Figure 6)

$$\begin{aligned}
A_1 &= \{y \in B(0, h) : y_1 \leq 0\},\\
A_2 &= \{y \in B(0, r) \setminus B(0, h) : y_1 < 0,\ |y_2| \leq |y_1|\},\\
A_3 &= \{y \in B(0, r) \setminus B(0, h) : y_1 \leq 0,\ |y_2| \geq |y_1|\},\\
A_4 &= \{y : y_2 \in [-h, h],\ y_1 \in (0, f(y_2))\},\\
A_5 &= \{y : y_2 \in (h, r] \cup [-r, -h),\ y_1 \in (0, f(y_2))\},\\
A_6 &= D^c \setminus (A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5).
\end{aligned}$$

Clearly $A_1, A_2, A_3, A_4, A_5, A_6 \subset D^c$. We also set $D_1 = B((h, 0), h/2)$.

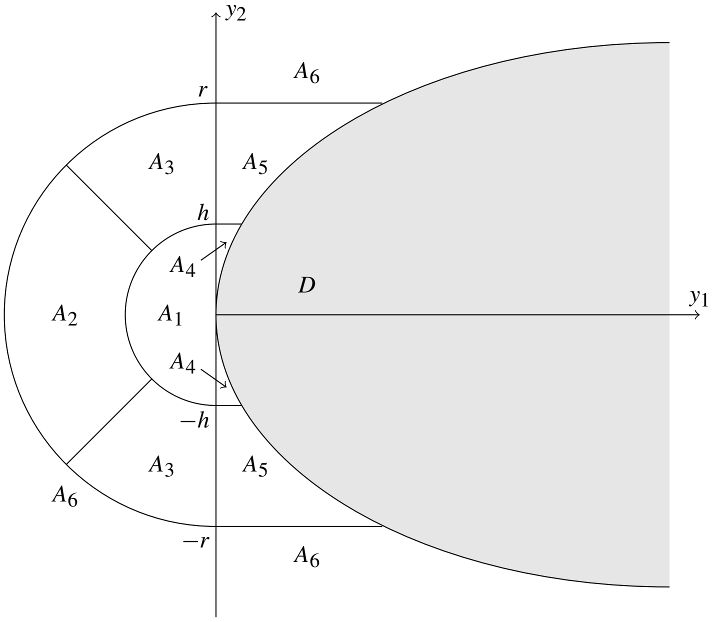


Fig. 6

Let $x \in S_3(h) \cup S_4(h)$. We have

$$\left|\int_{A_1} K_1 u_3\right| \le ch^{-3} \int_{A_1} \delta_D^{-1/2}(y)\,dy \le ch^{-3/2},$$

$$\int_{A_2} K_1 u_3 \approx -x_3 \int_{A_2} |y|^{-9/2}\,dy \approx -x_3 h^{-5/2},$$

$$\left|\int_{A_3} K_1 u_3\right| \le ch \int_{h/\sqrt{2}}^{r} dy_2 \int_{-y_2}^{0} dy_1\, |y_1|^{-1/2} y_2^{-4} \le ch^{-3/2}.$$

For $x \in S_3(h) \cup S_4(h)$ and $y \in A_4$ we estimate $|y_1 - x_1| \le y_1 + h \le ch$, $f(y_2) \le cy_2^2$. Hence

$$\left|\int_{A_4} K_1 u_3\right| \le cx_3 h^{-4} \int_{-h}^{h} dy_2 \int_{0}^{f(y_2)} dy_1\, (-y_1 + f(y_2))^{-1/2} \le cx_3 h^{-2}.$$

For $x \in S_3(h) \cup S_4(h)$ and $y \in A_5$ we estimate $|y_1 - x_1| \le y_1 + h \le c|y_2|$ and $f(y_2) \le cy_2^2$. Hence

$$\left|\int_{A_5} K_1 u_3\right| \le cx_3 \int_{h}^{r} dy_2 \int_{0}^{f(y_2)} dy_1\, (-y_1 + f(y_2))^{-1/2} y_2^{-4} \le cx_3 h^{-2}.$$

Moreover,

$$\left|\int_{A_6} K_1 u_3\right| \le cx_3 \int_{A_6} |y|^{-13/2} \delta_D^{-1/2}(y)\,dy \le cx_3.$$

For $x \in S_3(h)$ we have

$$\left|\int_{D_1} K_1 u_3\right| = \left|\int_{D_1} K_1\right| \le c x_3 h^{-4} \int_{D_1} dy \approx x_3 h^{-2}.$$

For $x \in S_4(h)$ we have

$$\left|\int_{D_1} K_1 u_3\right| = c x_3 \int_{D_1} \frac{y_1 - h}{((y_1 - h)^2 + y_2^2 + x_3^2)^{5/2}}\, dy_1\, dy_2 = 0.$$

For $x \in S_3(h) \cup S_4(h)$ we also have

$$\left|\int_{D\setminus D_1} K_1 u_3\right| \le c x_3 \int_{D\setminus D_1} ((y_1 - h)^2 + y_2^2)^{-2}\, dy \le c x_3 h^{-2}.$$

It follows that for $x \in S_3(h) \cup S_4(h)$,

$$|u_{13}(x)| = \left|\int_{\mathbb{R}^2} K_1 u_3\right| \le c h^{-3/2} \tag{32}$$

(for $x \in S_3(h)$ such an estimate also follows from Proposition 4.2).

Now note that $K_1(x_1 - y_1, -y_2, x_3) \le 0$ and $u_3(y_1, y_2, 0) \ge 0$ for $x \in S_3(h) \cup S_4(h)$ and $y \in A_1 \cup A_3$. So $\int_{A_1 \cup A_3} K_1 u_3 \le 0$. It follows that for $x \in S_3(h) \cup S_4(h)$ we have

$$u_{13}(x) = \int_{\mathbb{R}^2} K_1 u_3 \le \int_{A_2 \cup A_4 \cup A_5 \cup A_6 \cup D} K_1 u_3 \le -c x_3 h^{-5/2} + c_1 x_3 h^{-2}.$$

It is clear that one can choose sufficiently small $h_0 = h_0(\Lambda)$ such that for any $h \in (0, h_0]$ and $x \in S_3(h) \cup S_4(h)$ we have $u_{13}(x) \le -c_2 x_3 h^{-5/2}$. Using this and (32) we also obtain $u_{13}(x) \approx -h^{-3/2}$ for any $h \in (0, h_0]$ and $x \in S_3(h)$.

Now we will estimate $u_{33}(x)$ for $x \in S_4(h)$. We have

$$u_{33}(x) = \int_{\mathbb{R}^2} K_3(x_1 - y_1, -y_2, x_3) u_3(y_1, y_2, 0)\, dy_1\, dy_2,$$
$$K_3(x_1 - y_1, -y_2, x_3) = C_K \frac{(x_1 - y_1)^2 + y_2^2 - 2x_3^2}{((x_1 - y_1)^2 + y_2^2 + x_3^2)^{5/2}}.$$

For $x \in S_4(h)$ and $y \in D^c$ we have $K_3(x_1 - y_1, -y_2, x_3) > 0$ and $u_3(y_1, y_2, 0) \approx (1 + |y|^{-5/2})\delta_D^{-1/2}(y)$. For $y \in D$ we have $u_3(y_1, y_2, 0) = -1$. We obtain

$$\begin{aligned}\left|\int_{A_1 \cup A_4} K_3 u_3\right| &\le \frac{c}{h^3} \int_{A_1 \cup A_4} \delta_D^{-1/2}(y)\, dy \\ &\le \frac{c}{h^3} \int_0^h dy_2 \int_{-h}^{f(y_2)} dy_1\, (-y_1 + f(y_2))^{-1/2} \approx h^{-3/2},\end{aligned}$$
$$\int_{A_2} K_3 u_3 \approx \int_{A_2} |y|^{-7/2}\, dy \approx h^{-3/2},$$

$$\left|\int_{A_3\cup A_5} K_3 u_3\right| \le c\int_{h/\sqrt{2}}^{r} dy_2 \int_{-y_2}^{f(y_2)} dy_1\, \frac{(-y_1+f(y_2))^{-1/2}}{y_2^3} \approx h^{-3/2},$$
$$\left|\int_{A_6} K_3 u_3\right| \le c\int_{A_6} |y|^{-11/2}\delta_D^{-1/2}(y)\,dy \le c,$$
$$\left|\int_{D\setminus D_1} K_3 u_3\right| \le c\int_{D\setminus D_1} ((y_1-h)^2+y_2^2)^{-3/2}\,dy \le ch^{-1}.$$

The integral over D_1 is computed directly. Recall that $D_1 = B((h,0),h/2)$ and $x = (x_1,x_2,x_3) \in S_4(h)$, so $x_1 = h$, $x_2 = 0$ and $x_3 \in (0,h/4]$. We have

$$\int_{D_1} K_3(x_1-y_1,-y_2,x_3)u_3(y_1,y_2,0)\,dy_1\,dy_2$$
$$= C_K\int_{D_1} \frac{(h-y_1)^2+y_2^2-2x_3^2}{((h-y_1)^2+y_2^2+x_3^2)^{5/2}}\,dy_1\,dy_2. \tag{33}$$

Let us introduce polar coordinates $h-y_1 = \rho\cos\theta$, $y_2 = \rho\sin\theta$. Then (33) equals $2\pi C_K\int_0^{h/2}\frac{\rho^2-2x_3^2}{(\rho^2+x_3^2)^{5/2}}\rho\,d\rho$. The substitution $t=\rho^2$ shows that this is equal to $\pi C_K\int_0^{h^2/4}\frac{t-2x_3^2}{(t+x_3^2)^{5/2}}\,dt$. By elementary calculations this in turn equals $\frac{-\pi C_K h^2}{2(h^2/4+x_3^2)^{3/2}}$. Hence $|\int_{D_1} K_3u_3| \le c/h$.

It follows that $|u_{33}(x)| \le ch^{-3/2}$. Since for $x \in S_4(h)$ and $y \in (\overline{D})^c$ we have $K_3(x_1-y_1,-y_2,x_3) > 0$ and $u_3(y_1,y_2,0) > 0$, we get

$$u_{33}(x) = \int_{\mathbb{R}^2} K_3u_3 \ge \int_{A_2\cup D} K_3u_3 \ge \int_{A_2} K_3u_3 - \left|\int_D K_3u_3\right| \ge ch^{-3/2} - c_1h^{-1}.$$

It follows that $u_{33}(x) \approx h^{-3/2}$ for $x \in S_4(h)$ and sufficiently small h. Since $u_{11}(x) + u_{22}(x) + u_{33}(x) = 0$ and, by Proposition 4.1, $u_{22}(x) \approx -h^{-1/2}$ for $x \in S_4(h)$, we get $u_{11}(x) \approx -h^{-3/2}$. □

Proposition 4.5. *There exists $h_0 = h_0(\Lambda) \in (0,r/8]$ such that for any $h \in (0,h_0]$ we have $|u_{12}(x)| \le cx_3h^{-3/2}|\log h|$ for $x \in S_1(h)\cup S_2(h)\cup S_3(h)$, $|u_{12}(x)| \le ch^{-1/2}|\log h|$ for $x \in S_4(h)$, and $|u_{23}(x)| \le ch^{-1/2}|\log h|$ for $x \in S_1(h)\cup S_2(h)\cup S_3(h)$.*

Proof. Let $h \in (0,r/8]$. We have

$$u_{12}(x) = \int_D K_{12}(x_1-y_1,-y_2,x_3)\varphi(y_1,y_2)\,dy_1\,dy_2, \tag{34}$$
$$K_{12}(x_1-y_1,-y_2,x_3) = -15C_K\frac{x_3(x_1-y_1)y_2}{((x_1-y_1)^2+y_2^2+x_3^2)^{7/2}}.$$

Let D_1, D_2, D_3, D_4, D_5 and D_{i+}, D_{i-} for $i = 1,2,3,4$ be as in the proof of Proposition 4.2. We have

$$\int_{D_1\cup D_2} K_{12}\varphi$$
$$= -cx_3 \int_{D_{1+}\cup D_{2+}} \frac{(x_1-y_1)y_2}{((x_1-y_1)^2+y_2^2+x_3^2)^{7/2}} (\varphi(y_1,y_2)-\varphi(y_1,-y_2))\,dy_1\,dy_2.$$

For $y \in D_{1+}\cup D_{2+}$ by Lemma 3.11 we get $|\varphi(y_1,y_2)-\varphi(y_1,-y_2)| = |2y_2\varphi_2(y_1,\xi)| \le cy_2(y_2y_1^{-1/2}+y_1^{1/2}|\log y_1|)$ for some $\xi \in (-y_2,y_2)$. Hence

$$\left|\int_{D_1} K_{12}\varphi\right| \le cx_3 \int_{D_{1+}} \frac{|x_1-y_1|}{((x_1-y_1)^2+y_2^2+x_3^2)^{7/2}} (y_2^3y_1^{-1/2}+y_2^2y_1^{1/2}|\log y_1|)\,dy_1\,dy_2$$
$$\le cx_3h^{-6}\int_0^h dy_1 \int_0^h dy_2\,(y_2^3y_1^{-1/2}+y_2^2y_1^{1/2}|\log y_1|)$$
$$+ cx_3h\int_0^h dy_1 \int_h^{c_1y_1^{1/2}} dy_2\,(y_2^{-4}y_1^{-1/2}+y_2^{-5}y_1^{1/2}|\log y_1|)$$
$$\le cx_3h^{-3/2}|\log h|.$$

Note that for $y \in D_2$ we have $|x_1-y_1| \le cy_1$. We obtain

$$\left|\int_{D_2} K_{12}\varphi\right| \le cx_3 \int_{D_{2+}} \frac{|x_1-y_1|}{((x_1-y_1)^2+y_2^2+x_3^2)^{7/2}} (y_2^3y_1^{-1/2}+y_2^2y_1^{1/2}|\log y_1|)\,dy_1\,dy_2$$
$$\le cx_3\int_h^r dy_1 \int_0^{y_1} dy_2\,(y_2^3y_1^{-13/2}+y_2^2y_1^{-11/2}|\log y_1|)$$
$$+ cx_3\int_h^r dy_1 \int_{y_1}^{r} dy_2\,(y_2^{-4}y_1^{1/2}+y_2^{-5}y_1^{3/2}|\log y_1|)$$
$$\le cx_3h^{-3/2}|\log h|.$$

By Lemma 2.6 for $y \in D_3\cup D_4$ we have $\varphi(y) \le c\delta_D^{1/2}(y) \le cy_2$. Note also that $|x_1-y_1| \le 2h$ for $y \in D_3$ and $|x_1-y_1| \le h+y_1$ for $y \in D_4$. We get

$$\left|\int_{D_3} K_{12}\varphi\right| \le cx_3h^{-5}\int_0^h dy_2 \int_0^{f_1(h)} dy_1 y_2 \le cx_3h^{-1},$$
$$\left|\int_{D_{4+}} K_{12}\varphi\right| \le cx_3\int_h^r dy_2 \int_0^{c_1y_2^2} dy_1 (h+y_1)y_2^{-5} \le cx_3h^{-1}.$$

The estimate of $|\int_{D_{4-}} K_{12}\varphi|$ is the same, so $|\int_{D_4} K_{12}\varphi| \le cx_3h^{-1}$. Note that for $y \in D_5$ we have $|x_1-y_1| \le cy_1$ and $\varphi(y) \le c$. Hence

$$\left|\int_{D_5} K_{12}\varphi\right| \le cx_3\int_{B^c(0,c_1r^2)} \frac{y_1|y_2|}{(y_1^2+y_2^2)^{7/2}}\,dy_1\,dy_2 \le cx_3.$$

For $x \in S_1(h)\cup S_2(h)\cup S_3(h)$ we obtain

$$u_{23}(x) = \int_D K_{23}(x_1-y_1,-y_2,x_3)\varphi(y_1,y_2)\,dy_1\,dy_2.$$

The proof of $|\int_D K_{23}\varphi| \le ch^{-1/2}|\log h|$ is very similar to that of the estimate $|\int_D K_{12}\varphi| \le cx_3 h^{-3/2}|\log h|$ and is omitted.

We have

$$u_{12}(x) = \int_D K_{12}(x_1 - y_1, -y_2, x_3)\varphi(y_1, y_2)\, dy_1\, dy_2.$$

Set $A = B((h, 0), h/2)$. By the same argument as above we obtain $|\int_{D\setminus A} K_{12}\varphi| \le cx_3 h^{-3/2}|\log h|$. We have

$$\left|\int_A K_{12}\varphi\right| = \left|cx_3 \int_A \frac{(y_1 - h)y_2}{((y_1 - h)^2 + y_2^2 + x_3^2)^{7/2}}\varphi(y_1, y_2)\, dy_1\, dy_2\right|.$$

By the substitution $z_1 = y_1 - h$, $z_2 = y_2$ this is equal to

$$\left|cx_3 \int_{B(0,h/2)} \frac{z_1 z_2}{(z_1^2 + z_2^2 + x_3^2)^{7/2}}\varphi(z_1 + h, z_2)\, dz_1\, dz_2\right| = \left|cx_3 \int_W \frac{z_1 z_2 g(z_1, z_2)}{(z_1^2 + z_2^2 + x_3^2)^{7/2}}\, dz_1\, dz_2\right|, \qquad (35)$$

where $g(z_1, z_2) = \varphi(z_1 + h, z_2) - \varphi(-z_1 + h, z_2) - \varphi(z_1 + h, -z_2) + \varphi(-z_1 + h, -z_2)$ and $W = \{z \in B(0, h/2) : z_1, z_2 \ge 0\}$. Note that for $z \in W$ we have $g(z_1, z_2) = 4z_1 z_2 \varphi_{12}(\xi_1 + h, \xi_2)$ for some $\xi_1 \in (-z_1, z_1)$, $\xi_2 \in (-z_2, z_2)$. By Lemma 3.11, for $z \in W$ and ξ_1, ξ_2 as above we have

$$|\varphi_{12}(\xi_1 + h, \xi_2)| \le ch^{-1/2}|\log h| + cz_2 h^{-3/2}.$$

It follows that (35) is bounded from above by

$$cx_3 \int_W \frac{z_1^2 z_2^2 (h^{-1/2}|\log h| + z_2 h^{-3/2})}{(z_1^2 + z_2^2 + x_3^2)^{7/2}}\, dz_1\, dz_2. \qquad (36)$$

Set $W_1 = \{z : z_1, z_2 \in [0, x_3]\}$ and $W_2 = \{z \in B(0, h/2) \setminus B(0, x_3) : z_1, z_2 \ge 0\}$. We have $W \subset W_1 \cup W_2$. Thus (36) is bounded from above by

$$cx_3 \int_{W_1} \frac{z_1^2 z_2^2 (h^{-1/2}|\log h| + z_2 h^{-3/2})}{x_3^7}\, dz_1\, dz_2 + cx_3 \int_{W_2} \frac{z_1^2 z_2^2 (h^{-1/2}|\log h| + z_2 h^{-3/2})}{(z_1^2 + z_2^2)^{7/2}}\, dz_1\, dz_2 \le ch^{-1/2}|\log h|. \qquad \square$$

Proposition 4.6. *There exists $h_0 = h_0(\Lambda) \in (0, r/8]$ such that for any $h \in (0, h_0]$ we have $|u_{23}(x)| \le ch^{-3/4}|\log h|$ for $x \in S_4(h)$.*

Proof. Let $h \in (0, r/8]$. Set $p = (-r, 0)$; recall that $z = (r, 0)$. We have

$$\begin{aligned} u_{23}(x) &= \int_{\mathbb{R}^2} K_2(x_1 - y_1, -y_2, x_3) u_3(y_1, y_2, 0)\, dy_1\, dy_2 \\ &= \int_{B(0,r/4)\cap B(p,r)} K_2 u_3 + \int_{(D\cap B(0,r/4))\setminus(B(p,r)\cup B(z,r))} K_2 u_3 \\ &\quad + \int_{(D^c\cap B(0,r/4))\setminus(B(p,r)\cup B(z,r))} K_2 u_3 + \int_{B(0,r/4)\cap B(z,r)} K_2 u_3 \\ &\quad + \int_{B(0,r/4)^c} K_2 u_3 = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V}. \end{aligned}$$

Note that $u_3(y_1, y_2, 0) = -(-\Delta)^{1/2}\varphi(y_1, y_2)$ for $(y_1, y_2) \in \mathbb{R}^2 \setminus \partial D$.

Set $A = B(0, r/4) \cap B(p, r)$. For $y \in A$ by Corollary 3.13 we get $|(-\Delta)^{1/2}\varphi(y)| \le c\delta_D^{-1/2}(y) \le c|y_1|^{-1/2}$. It follows that

$$\begin{aligned} |\mathrm{I}| &\le cx_3 \int_A \frac{y_2|y_1|^{-1/2}}{((h - y_1)^2 + y_2^2 + x_3^2)^{5/2}}\, dy_1\, dy_2 \\ &\le cx_3 \int_0^h dy_2 \int_{-r/4}^{-f_1(y_2)} dy_1\, \frac{y_2|y_1|^{-1/2}}{h^5} + cx_3 \int_h^{r/4} dy_2 \int_{-r/2}^{-f_1(y_2)} dy_1\, \frac{y_2|y_1|^{-1/2}}{y_2^5} \\ &\le cx_3 h^{-3}. \end{aligned}$$

We also have

$$|\mathrm{II}| \le cx_3 \int_0^h dy_2 \int_0^{f_1(y_2)} dy_1\, y_2 h^{-5} + cx_3 \int_h^{r/2} dy_2 \int_0^{f_1(y_2)} dy_1\, y_2 y_2^{-5} \le cx_3 h^{-1}.$$

For $y \in (D^c \cap B(0, r/4)) \setminus (B(p, r) \cup B(z, r))$ by Corollary 3.13 we get $|(-\Delta)^{1/2}\varphi(y)| \le c\delta_D^{-1/2}(y) \approx (f(y_2) - y_1)^{-1/2}$. Hence

$$|\mathrm{III}| \le cx_3 \int_0^{r/4} dy_2 \int_{-f_1(y_2)}^{f(y_2)} dy_1\, (f(y_2) - y_1)^{-1/2} \frac{y_2}{h^5 \vee y_2^5}.$$

For $y_2 \in (0, r/4)$ we have

$$\int_{-f_1(y_2)}^{f(y_2)} (f(y_2) - y_1)^{-1/2}\, dy_1 = \int_0^{f_1(y_2)+f(y_2)} z^{-1/2}\, dz \le cy_2.$$

It follows that

$$|\mathrm{III}| \le cx_3 \int_0^h \frac{y_2^2}{h^5}\, dy_2 + cx_3 \int_h^{r/4} \frac{y_2^2}{y_2^5}\, dy_2 \le \frac{cx_3}{h^2}.$$

Clearly

$$\mathrm{IV} = \int_{B(0,r/4)\cap B(z,r)} \frac{-cx_3 y_2}{((h - y_1)^2 + y_2^2 + x_3^2)^{5/2}}\, dy_1\, dy_2 = 0.$$

Using Corollary 3.13 we get

$$|\mathrm{V}| \leq cx_3 \int_D dy + cx_3 \int_{D^c} \frac{\delta_D^{-1/2}(y)}{(1+|y|)^{5/2}}\, dy \leq cx_3.$$

It follows that for $x \in S_4(h)$ we have

$$|u_{23}(x)| \leq |\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}| \leq cx_3/h^3. \tag{37}$$

On the other hand, for $x \in S_4(h)$ we have

$$u_{23}(x) = \int_D K_{23}(x_1-y_1,-y_2,x_3)\varphi(y_1,y_2)\, dy_1\, dy_2.$$

Set $W = B((h,0),h/2)$ and $W_+ = \{y \in W : y_2 > 0\}$. For $x \in S_4(h)$ one may show $|\int_{D\setminus W} K_{23}\varphi| \leq ch^{-1/2}|\log h|$. The proof of this inequality is omitted; it is very similar to the proof of $|\int_{D\setminus W} K_{12}\varphi| \leq cx_3 h^{-3/2}|\log h|$ (see the proof of Proposition 4.5).

We have

$$\begin{aligned}\int_W K_{23}\varphi &= -c\int_W \frac{12x_3^2-3(y_1-h)^2-3y_2^2}{((y_1-h)^2+y_2^2+x_3^2)^{7/2}} y_2\varphi(y_1,y_2)\, dy_1\, dy_2 \\ &= -c\int_{W_+} \frac{12x_3^2-3(y_1-h)^2-3y_2^2}{((y_1-h)^2+y_2^2+x_3^2)^{7/2}} y_2(\varphi(y_1,y_2)-\varphi(y_1,-y_2))\, dy_1\, dy_2. \end{aligned} \tag{38}$$

For $y \in W_+$ we have $\varphi(y_1,y_2)-\varphi(y_1,-y_2) = 2y_2\varphi_2(y_1,\xi_2)$ for some $\xi_2 \in (-y_2,y_2)$, and $\varphi_2(y_1,\xi_2) = \varphi_2(h,0) + (y_1-h,\xi_2)\circ\nabla\varphi_2(\xi')$, where ξ' is a point between $(h,0)$ and (y_1,ξ_2). It follows that (38) equals

$$\begin{aligned}&-c\varphi_2(h,0)\int_{W_+} \frac{12x_3^2-3(y_1-h)^2-3y_2^2}{((y_1-h)^2+y_2^2+x_3^2)^{7/2}} 2y_2^2\, dy_1\, dy_2 \\ &\quad - c\int_{W_+} \frac{12x_3^2-3(y_1-h)^2-3y_2^2}{((y_1-h)^2+y_2^2+x_3^2)^{7/2}} 2y_2^2(y_1-h,\xi_2)\circ\nabla\varphi_2(\xi')\, dy_1\, dy_2 = \mathrm{I}+\mathrm{II}.\end{aligned}$$

Set $V = B(0,h/2)$ and $V_+ = \{z \in V : z_2 > 0\}$. By the substitution $z_1 = y_1 - h$, $z_2 = y_2$ we obtain

$$\begin{aligned}\mathrm{I} &= -c\varphi_2(h,0)\int_{V_+} \frac{12x_3^2-3z_1^2-3z_2^2}{(z_1^2+z_2^2+x_3^2)^{7/2}} 2z_2^2\, dy_1\, dy_2 \\ &= -c\varphi_2(h,0)\int_{V} \frac{12x_3^2-3z_1^2-3z_2^2}{(z_1^2+z_2^2+x_3^2)^{7/2}} z_2^2\, dy_1\, dy_2.\end{aligned}$$

By symmetry of z_1, z_2 the above integral equals

$$\frac{1}{2}\int_V \frac{12x_3^2-3z_1^2-3z_2^2}{(z_1^2+z_2^2+x_3^2)^{7/2}}(z_1^2+z_2^2)\, dy_1\, dy_2.$$

Let us introduce polar coordinates $z_1 = \rho\cos\theta$, $z_2 = \rho\sin\theta$. Then the above expression equals $\pi\int_0^{h/2} \frac{12x_3^2 - 3\rho^2}{(\rho^2+x_3^2)^{7/2}}\rho^3\,d\rho$. By elementary calculation this is equal to $(3\pi/16)h^4 \times (x_3^2 + h^2/4)^{-5/2}$. By Lemma 3.11, $\varphi_2(h, 0) \le ch^{1/2}|\log h|$. Hence $|\mathrm{I}| \le ch^{-1/2}|\log h|$.

Now we estimate II. For $y \in W_+$ and ξ_2, ξ' as above we have

$$(y_1 - h, \xi_2) \circ \nabla\varphi_2(\xi') = (y_1 - h)\varphi_{12}(\xi') + \xi_2\varphi_{22}(\xi'). \tag{39}$$

For any $w \in W$ by Lemma 3.11 we get $|\varphi_{12}(w)| \le ch^{-1/2}|\log h|$, $|\varphi_{22}(w)| \le ch^{-1/2}$, so (39) is bounded from above by $c|y_1 - h|h^{-1/2}|\log h| + c|y_2|h^{-1/2}$. Set $B_+((h, 0), x_3) = \{y \in B((h, 0), x_3) : y_2 > 0\}$. It follows that

$$\begin{aligned}|\mathrm{II}| &\le \frac{c}{x_3^5}\int_{B_+((h,0),x_3)} |y - (h, 0)|^3 h^{-1/2}|\log h|\,dy \\ &\quad + c\int_{W_+\setminus B_+((h,0),x_3)} |y - (h, 0)|^{-2}h^{-1/2}|\log h|\,dy \le ch^{-1/2}|\log h|\,|\log x_3|.\end{aligned}$$

Hence for $x \in S_4(h)$ we have

$$|u_{23}(x)| \le \left|\int_{D\setminus W} K_{23}\varphi\right| + |\mathrm{I}| + |\mathrm{II}| \le ch^{-1/2}|\log h|\,|\log x_3|. \tag{40}$$

For any $\beta > 0$ and $x \in S_4(h)$ we get $|u_{23}(x)|^\beta \le c_1^\beta x_3^\beta h^{-3\beta}$ by (37). Using this and (40) we get $|u_{23}(x)|^{1+\beta} \le cc_1^\beta x_3^\beta|\log x_3|h^{-3\beta-1/2}|\log h|$. Setting $\beta = 1/9$ we obtain $|u_{23}(x)| \le ch^{-3/4}|\log h|^{9/10} \le ch^{-3/4}|\log h|$. □

Lemma 4.7. *For any* $(x_1, x_2) \in D$ *we have* $u_{13}(x_1, x_2, 0) = u_{23}(x_1, x_2, 0) = 0$ *and* $u_{33}(x_1, x_2, 0) > 0$.

Proof. The equalities $u_{13}(x_1, x_2, 0) = u_{23}(x_1, x_2, 0) = 0$ for $(x_1, x_2) \in D$ follow easily from (8). For $(x_1, x_2) \in \mathrm{int}(D^c)$ we have

$$u_3(x_1, x_2, 0) = -(-\Delta)^{1/2}\varphi(x) = \frac{1}{2\pi}\int_D \frac{\varphi(y)}{|y - x|^3}\,dy > 0.$$

By Corollary 3.13 we have $f(x_1, x_2) = u_3(x_1, x_2, 0) \in L^1(\mathbb{R}^2)$. By the normal derivative lemma [15, Lemma 2.33] we get $u_{33}(x_1, x_2, 0) > 0$ for $(x_1, x_2) \in D$. □

5. Harmonic extension for a ball

The aim of this section is to prove the following result.

Proposition 5.1. *Let* φ *be the solution of* (1)–(2) *for the ball* $B(0, 1) \subset \mathbb{R}^2$ *and* u *be the harmonic extension of* φ *given by* (6)–(10). *We have*

$$H(u)(x) > 0, \quad x \in \mathbb{R}^3 \setminus (B(0, 1)^c \times \{0\}). \tag{41}$$

Recall that $H(u)(x)$ is the determinant of the Hessian matrix of u at x. Recall also that the solution of (1)–(2) for the ball $B(0,1)$ is given by the explicit formula $\varphi(x) = C_B(1-|x|)^{1/2}$, $C_B = 2/\pi$. Hence for $x = (x_1, x_2, x_3)$ where $x_3 > 0$, the function u is given by the explicit formula $u(x) = \int_{B(0,1)} K(x_1 - y_1, x_2 - y_2, x_3)\varphi(y_1, y_2)\, dy_1\, dy_2$. Applying this it is easy to check numerically that (41) holds (e.g. using Mathematica). Unfortunately, it seems hard to formally prove (41) directly using the explicit formula for u.

Instead, to show (41) we use a trick: we add an auxiliary function w to u and we use Lewy's Theorem 1.6. First, we briefly present the idea of the proof. We define

$$\Psi^{(b)}(x) = (1-b)u(x) + bw(x), \quad b \in [0,1],$$

where w is an appropriately chosen auxiliary function, namely

$$w(x) = K(x_1, x_2, x_3 + \sqrt{3/2}). \tag{42}$$

Note that for any $q \geq 0$ we have $\{(x_1, x_2, x_3) : K_{33}(x_1, x_2, x_3 + q) = 0, x_3 > -q\} = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 = (2/3)(x_3 + q)^2,\ x_3 > -q\}$. The function w is chosen so that $w_{33}(x) = 0$ for $x \in \partial B(0,1) \times \{0\}$, i.e. for $x = (x_1, x_2, 0)$ with $x_1^2 + x_2^2 = 1$. Such a choice helps to control $H(\Psi^{(b)})(x)$ near $\partial B(0,1) \times \{0\}$. One can directly check that $\Psi^{(1)} = w$ satisfies $H(\Psi^{(1)})(x) > 0$ for $x \in \mathbb{R}^3_+ \cup B(0,1) \times \{0\}$ (recall that $\mathbb{R}^3_+ = \{(x_1, x_2, x_3) : x_3 > 0\}$). If $\Psi^{(0)} = u$ does not satisfy $H(\Psi^{(0)})(x) > 0$ for $x \in \mathbb{R}^3_+ \cup B(0,1) \times \{0\}$, one can show that there exists $b \in [0,1)$ such that $H(\Psi^{(b)})(x) \geq 0$ for $x \in \mathbb{R}^3_+ \cup B(0,1) \times \{0\}$ and there exists $x_0 \in \mathbb{R}^3_+$ for which $H(\Psi^{(b)})(x_0) = 0$. This contradicts Theorem 1.6. If $\Psi^{(0)} = u$ does not satisfy $H(\Psi^{(0)})(x) > 0$ for $x \in \mathbb{R}^3_-$, one can use Lemma 2.7 and again obtain a contradiction. This finishes the presentation of the idea of the proof.

Lemma 5.2. *Let w be given by (42) and $v = u + aw$ with $a \geq 0$. There exist $M_1 \geq 10$ and $h_1 \in (0, 1/2]$ such that for any $a \geq 0$ we have*

$$H(v)(x) > 0, \qquad x \in A_1 \cup A_2 \cup A_3 \cup A_4,$$

where

$$\begin{aligned}
A_1 &= \{(x_1, x_2, x_3) : x_1^2 + x_2^2 \in [(1-h_1)^2, (1+h_1)^2],\ x_3 \in (0, h_1]\},\\
A_2 &= \{(x_1, x_2, x_3) : x_1^2 + x_2^2 \in [(1+h_1)^2, M_1^2],\ x_3 \in (0, h_1]\},\\
A_3 &= \{(x_1, x_2, 0) : x_1^2 + x_2^2 < 1\},\\
A_4 &= \{(x_1, x_2, x_3) \in \mathbb{R}^3_+ : x_1^2 + x_2^2 \geq M_1^2 \text{ or } x_3 \geq M_1\}.
\end{aligned}$$

Proof. First note that for any fixed $x_3 > 0$ the function $(x_1, x_2) \mapsto v(x_1, x_2, x_3)$ is radial, so it is enough to show the assertion for $x \in (A_1 \cup A_2 \cup A_3 \cup A_4) \cap L$, where $L = \{(x_1, x_2, x_3) : x_2 = 0,\ x_1 \leq 0\}$. Set $A_i' = A_i \cap L$, $i = 1, 2, 3, 4$. For $x \in A_1' \cup A_2' \cup A_3' \cup A_4'$ we have $v_{12}(x) = v_{23}(x) = 0$ and $v_{22}(x) < 0$. Hence $H(v)(x) = v_{22}(x) f(a, x)$, where

$$f(a,x) = \begin{vmatrix} v_{11} & v_{13} \\ v_{13} & v_{33} \end{vmatrix} = \begin{vmatrix} u_{11} + aw_{11} & u_{13} + aw_{13} \\ u_{13} + aw_{13} & u_{33} + aw_{33} \end{vmatrix}, \tag{43}$$

and it is enough to show $f(a, x) < 0$ for $x \in A_1' \cup A_2' \cup A_3' \cup A_4'$.

We will consider four cases: $x \in A'_1$, $x \in A'_2$, $x \in A'_3$, $x \in A'_4$.

Case 1: $x \in A'_1$. Set $q_0 = \sqrt{3/2}$ and $z_0 = (-1, 0, 0)$. Note that $w_{33}(z_0) = 0$, $w_{11}(z_0) = C_K q_0(12 - 3q_0^2)(1 + q_0^2)^{-7/2} \approx 9.185 C_K(1 + q_0^2)^{-7/2}$ and $w_{13}(z_0) = -C_K(12q_0^2 - 3)(1 + q_0^2)^{-7/2} = -15C_K(1 + q_0^2)^{-7/2}$. Denote $w_{11}(x) = p_1(x)$, $w_{13}(x) = p_2(x)$. It is clear that for sufficiently small h_1 and $x \in A'_1$ we have

$$\sqrt{9/10}\,|p_2(x)| > |p_1(x)|. \tag{44}$$

Let h_0 denote the minimum of the constants h_0 from Propositions 4.1–4.6. For any $h \in (0, h_0]$ denote

$$T_1(h) = \{(-1 + h, 0, x_3) : x_3 \in (0, h/4]\},$$
$$T_2(h) = \{(-1 + h, 0, x_3) : x_3 \in (h/4, h]\} \cup \{(x_1, 0, h) : x_1 \in [-1, -1 + h)\},$$
$$T_3(h) = \{(x_1, 0, h) : x_1 \in [-\sqrt{2/3}\,h - 1, -1]\},$$
$$T_4(h) = \{(x_1, 0, h) : x_1 \in [-1 - h, -\sqrt{2/3}\,h - 1)\} \cup \{(-1 - h, 0, x_3) : x_3 \in (0, h)\}.$$

Note that the value $-\sqrt{2/3}\,h - 1$ in the definition of $T_3(h)$, $T_4(h)$ is chosen so that $w_{33}(-\sqrt{2/3}\,h - 1, 0, h) = 0$. Note also that $w_{33}(x) \ge 0$ for $x \in T_1(h) \cup T_2(h) \cup T_3(h)$ and $w_{33}(x) < 0$ for $x \in T_4(h)$.

We will consider four subcases: $x \in T_1(h)$, $x \in T_2(h)$, $x \in T_3(h)$, $x \in T_4(h)$.

Subcase 1a: $x \in T_1(h)$. By (43), Propositions 4.1, 4.4 and definition of w we have

$$f(a, x) = \begin{vmatrix} -b_1(x)h^{-3/2} + p_1(x)a & -b_2(x)h^{-3/2} - p_2(x)a \\ -b_2(x)h^{-3/2} - p_2(x)a & \varepsilon(x)a + b_1(x)h^{-3/2} + b_3(x)h^{-1/2} \end{vmatrix},$$

where $0 < B'_1 \le b_1(x) \le B_1$, $0 \le b_2(x) \le B_2$, $0 < B'_3 \le b_3(x) \le B_3$, $0 < P'_1 \le p_1(x) \le P_1$, $0 < P'_2 \le p_2(x) \le P_2$, and $0 \le \varepsilon(x) \le E(h) \le E(h_0)$ with $\lim_{h\to 0^+} E(h) = 0$. More precisely, the estimates of $b_1(x)$, $b_2(x)$ follow from the estimates of $u_{11}(x)$, $u_{13}(x)$ on $S_4(h)$ in Proposition 4.4, while the estimates of $b_3(x)$ follow from $u_{33}(x) = -u_{11}(x) - u_{22}(x)$ and the estimates of $u_{11}(x)$, $u_{22}(x)$ on $S_4(h)$ in Propositions 4.1 and 4.4. The estimates of $p_1(x)$, $p_2(x)$ follow from the formulas for $w_{11}(z_0)$, $w_{13}(z_0)$ and continuity of $w_{11}(x)$, $w_{13}(x)$ near z_0. The estimates of $\varepsilon(x)$ and $\lim_{h\to 0^+} E(h) = 0$ follow from $w_{33}(z_0) = 0$ and continuity of $w_{33}(x)$ near z_0. Hence

$$f(a, x) = -\varepsilon(x)b_1(x)ah^{-3/2} - b_1^2(x)h^{-3} - b_1(x)b_3(x)h^{-2} + \varepsilon(x)p_1(x)a^2$$
$$+ b_1(x)p_1(x)ah^{-3/2} + p_1(x)b_3(x)ah^{-1/2} - b_2^2(x)h^{-3} - p_2^2(x)a^2 - 2b_2(x)p_2(x)ah^{-3/2}.$$

Note that for sufficiently small h we have

$$p_1(x)b_3(x)ah^{-1/2} < p_1(x)b_1(x)ah^{-3/2}.$$

For sufficiently small h, using this and (44) we get

$$(9/10)p_2^2(x)a^2 + b_1^2(x)h^{-3} > p_1^2(x)a^2 + b_1^2(x)h^{-3} \ge 2b_1(x)p_1(x)ah^{-3/2}$$
$$> b_1(x)p_1(x)ah^{-3/2} + b_3(x)p_1(x)ah^{-1/2}.$$

For sufficiently small h we also have $p_1(x)\varepsilon(x)a^2 < (1/10)p_2^2(x)a^2$. It follows that for sufficiently small $h_1 > 0$ and for all $0 < h \le h_1$, $a \ge 0$, $x \in T_1(h)$ we have $f(a,x) < 0$.

Subcase 1b: $x \in T_2(h)$. By (43), Propositions 4.1, 4.2, 4.4 and definition of w we have

$$f(a,x) = \begin{vmatrix} b_1(x)h^{-3/2} + p_1(x)a & -b_2(x)h^{-3/2} - p_2(x)a \\ -b_2(x)h^{-3/2} - p_2(x)a & \varepsilon(x)a - b_1(x)h^{-3/2} + b_3(x)h^{-1/2} \end{vmatrix},$$

where $-B_1 \le b_1(x) \le B_1$, $0 < B_2' \le b_2(x) \le B_2$, $0 < B_3' \le b_3(x) \le B_3$, $0 < P_1' \le p_1(x) \le P_1$, $0 < P_2' \le p_2(x) \le P_2$, and $0 \le \varepsilon(x) \le E(h) \le E(h_0)$ with $\lim_{h\to 0^+} E(h) = 0$. More precisely, the estimates of $b_1(x)$, $b_2(x)$ follow from the estimates of $u_{11}(x)$, $u_{13}(x)$ on $S_3(h)$ in Propositions 4.2 and 4.4, while the estimates of $b_3(x)$ follow from $u_{33}(x) = -u_{11}(x) - u_{22}(x)$ and the estimates of $u_{11}(x)$, $u_{22}(x)$ on $S_3(h)$ in Propositions 4.1 and 4.2. The estimates of $p_1(x)$, $p_2(x)$, $\varepsilon(x)$, and $\lim_{h\to 0^+} E(h) = 0$, follow by the same arguments as in Subcase 1a. Hence

$$\begin{aligned} f(a,x) &= \varepsilon(x)b_1(x)ah^{-3/2} - b_1^2(x)h^{-3} + b_1(x)b_3(x)h^{-2} + \varepsilon(x)p_1(x)a^2 \\ &- b_1(x)p_1(x)ah^{-3/2} + p_1(x)b_3(x)ah^{-1/2} - b_2^2(x)h^{-3} - p_2^2(x)a^2 - 2b_2(x)p_2(x)ah^{-3/2}. \end{aligned}$$

First assume that $b_1(x) \ge 0$. Then for sufficiently small h we have

$$\begin{aligned} \varepsilon(x)b_1(x)ah^{-3/2} &< b_2(x)p_2(x)ah^{-3/2}, \\ p_1(x)b_3(x)ah^{-1/2} &< b_2(x)p_2(x)ah^{-3/2}, \\ b_1(x)b_3(x)h^{-2} &< b_2^2(x)h^{-3}, \\ \varepsilon(x)p_1(x)a^2 &< p_2^2(x)a^2, \end{aligned}$$

which implies $f(a,x) < 0$.

Now assume that $b_1(x) < 0$. By (44) for sufficiently small h we get

$$\begin{aligned} &(9/10)p_2^2(x)a^2 + b_1^2(x)h^{-3} > p_1^2(x)a^2 + b_1^2(x)h^{-3} \ge |2b_1(x)p_1(x)ah^{-3/2}|, \\ &p_1(x)\varepsilon(x)a^2 < (1/10)p_2^2(x)a^2, \\ &p_1(x)b_3(x)ah^{-1/2} < 2b_2(x)p_2(x)ah^{-3/2}, \end{aligned}$$

which implies $f(a,x) < 0$.

It follows that for sufficiently small $h_1 > 0$ and for all $0 < h \le h_1$, $a \ge 0$, $x \in T_2(h)$ we have $f(a,x) < 0$.

Subcase 1c: $x \in T_3(h)$. By (43), Propositions 4.1–4.3 and definition of w we have

$$f(a,x) = \begin{vmatrix} b_1(x)h^{-3/2} + p_1(x)a & -b_2(x)h^{-3/2} - p_2(x)a \\ -b_2(x)h^{-3/2} - p_2(x)a & \varepsilon(x)a - b_1(x)h^{-3/2} + b_3(x)h^{-1/2} \end{vmatrix},$$

where $0 < B_1' \le b_1(x) \le B_1$, $-B_2 \le b_2(x) \le B_2$, $0 < B_3' \le b_3(x) \le B_3$, $0 < P_1' \le p_1(x) \le P_1$, $0 < P_2' \le p_2(x) \le P_2$, and $0 \le \varepsilon(x) \le E(h) \le E(h_0)$ with $\lim_{h\to 0^+} E(h) = 0$. More precisely, the estimates of $b_1(x)$, $b_2(x)$ follow from the estimates of $u_{11}(x)$, $u_{13}(x)$ on $S_2(h)$ in Propositions 4.2 and 4.3, while the estimates of $b_3(x)$

follow from $u_{33}(x) = -u_{11}(x) - u_{22}(x)$ and the estimates of $u_{11}(x)$, $u_{22}(x)$ on $S_2(h)$ in Propositions 4.1–4.3. The estimates of $p_1(x)$, $p_2(x)$, $\varepsilon(x)$, and $\lim_{h\to 0^+} E(h) = 0$, follow by the same arguments as in Subcase 1a.

For sufficiently small h we have

$$b_3(x)h^{-1/2} < b_1(x)h^{-3/2}/2, \tag{45}$$

$$\frac{2B_2}{B_1'}\varepsilon(x) < \frac{P_2'}{2}, \tag{46}$$

$$\varepsilon(x)(p_1(x) + 2\varepsilon(x)) < p_2^2(x)/4. \tag{47}$$

If $\varepsilon(x)a - b_1(x)h^{-3/2} + b_3(x)h^{-1/2} < 0$ then clearly $f(a,x) < 0$. So we may assume $\varepsilon(x)a - b_1(x)h^{-3/2} + b_3(x)h^{-1/2} \geq 0$, which implies (see (45))

$$\varepsilon(x)a \geq b_1(x)h^{-3/2} - b_3(x)h^{-1/2} > (b_1(x)h^{-3/2})/2, \tag{48}$$

$$\varepsilon(x)a > \varepsilon(x)a - b_1(x)h^{-3/2} + b_3(x)h^{-1/2} \geq 0. \tag{49}$$

By (46) and (48) we get

$$|b_2(x)|h^{-3/2} = \frac{2|b_2(x)|}{b_1(x)}\,\frac{b_1(x)h^{-3/2}}{2} < \frac{2B_2}{B_1'}\varepsilon(x)a < \frac{P_2'a}{2} < \frac{p_2(x)a}{2}. \tag{50}$$

By (47)–(50) we get

$$\begin{aligned} f(a,x) &\leq (p_1(x)a + b_1(x)h^{-3/2})\varepsilon(x)a - (p_2(x)a/2)^2 \\ &\leq (p_1(x)a + 2\varepsilon(x)a)\varepsilon(x)a - p_2^2(x)a^2/4 < 0. \end{aligned}$$

It follows that for sufficiently small $h_1 > 0$ and for all $0 < h \leq h_1$, $a \geq 0$, $x \in T_3(h)$ we have $f(a,x) < 0$.

Subcase 1d: $x \in T_4(h)$. Note that for $x = (x_1, 0, x_3) \in T_4(h)$ we have $w_{33}(x) < 0$. Moreover,

$$u_{33}(x) = \int_{B(0,1)} K_{33}(x_1 - y_1, x_2 - y_2, x_3)\varphi(y_1, y_2)\,dy_1\,dy_2.$$

Recall that $K_{33}(x_1 - y_1, x_2 - y_2, x_3) = C_K x_3((x_1-y_1)^2 + (x_2-y_2)^2 + x_3^2)^{-7/2} \times (6x_3^2 - 9(x_1 - y_1)^2 - 9(x_2 - y_2)^2)$. Hence to have $K_{33}(x_1 - y_1, -y_2, x_3) < 0$ for all $(y_1, y_2) \in B(0,1)$ and $x_1 \leq -1$ it is sufficient to prove $6x_3^2 - 9(x_1 + 1)^2 < 0$. Note that for $x = (x_1, 0, x_3) \in T_4(h)$ we have $0 < x_3 < -\sqrt{3/2}\,(x_1 + 1)$, $x_1 < -1$. It follows that $6x_3^2 - 9(x_1+1)^2 < 0$ and $u_{33}(x) < 0$. Hence $u_{33}(x) + aw_{33}(x) < 0$. Note that $u_{22}(x) + aw_{22}(x) < 0$, so $u_{11}(x) + aw_{11}(x) = -u_{22}(x) - aw_{22}(x) - u_{33}(x) - aw_{33}(x) > 0$. Together with (43) this implies that $f(a,x) < 0$ for any $a \geq 0$ and $x \in T_4(h)$.

Case 2: $x \in A_2'$. This case follows by the same arguments as in Subcase 1d.

Case 3: $x \in A_3'$. Note that $w_{33}(x) > 0$ for $x \in A_3'$. Set $\overline{x}_3 = x_3 + \sqrt{3/2}$. We have

$$w_{11}(x) = C_K \overline{x}_3 (x_1^2 + \overline{x}_3^2)^{-7/2}(12x_1^2 - 3\overline{x}_3^2).$$

Note that

$$\{(x_1, 0, x_3) : w_{11}(x_1, 0, x_3) = 0,\ x_1 \le 0,\ x_3 > -\sqrt{3/2}\} \\ = \{(x_1, 0, x_3) : x_3 + \sqrt{3/2} = -2x_1\}.$$

Set

$$T_1 = \left\{(x_1, 0, 0) : x_1 \in \left[\frac{-\sqrt{3}}{2\sqrt{2}}, 0\right]\right\}, \qquad T_2 = \left\{(x_1, 0, 0) : x_1 \in \left(-1, \frac{-\sqrt{3}}{2\sqrt{2}}\right)\right\}.$$

Then we have $A_2' = T_1 \cup T_2$. Note that $w_{11}(-\sqrt{3}/(2\sqrt{2}), 0, 0)) = 0$, $w_{11}(x) \le 0$ for $x \in T_1$ and $w_{11}(x) > 0$ for $x \in T_2$. Moreover for $x = (x_1, 0, 0) \in A_3'$ we have $u(x) = \varphi(x_1, 0) = C_B(1 - x_1^2)^{1/2}$, so $u_{11}(x) < 0$.

We will consider two subcases: $x \in T_1$, $x \in T_2$.

Subcase 3a: $x \in T_1$. Note that $w_{11}(x) \le 0$ and $u_{11}(x) < 0$, so $u_{11}(x) + aw_{11}(x) < 0$ for $a \ge 0$. It follows that $u_{33}(x) + aw_{33}(x) > 0$ (because $u_{33} + aw_{33} = -(u_{11} + aw_{11} + u_{22} + aw_{22})$). Hence $f(a, x) < 0$.

Subcase 3b: $x \in T_2$. For $(y_1, y_2) \in B(0, 1)$ and $y = (y_1, y_2, 0)$ we have $u(y) = \varphi(y_1, y_2) = C_B(1 - y_1^2 - y_2^2)^{1/2}$. Therefore for $x \in T_2$ we obtain $u_{11}(x) = \varphi_{11}(x_1, 0) = -C_B(1 - x_1^2)^{-3/2}$, $u_{33}(x) = -\varphi_{11}(x_1, 0) - \varphi_{22}(x_1, 0) = C_B(1 - x_1^2)^{-3/2}(2 - x_1^2)$. Hence

$$u_{33}(x) < 2|u_{11}(x)|. \tag{51}$$

For $x \in T_2$ we also have $-w_{22}(x) - w_{11}(x) = w_{33}(x) > 0$, so

$$|w_{22}(x)| > |w_{11}(x)|. \tag{52}$$

Note that for $x = (x_1, x_2, x_3) = (x_1, 0, 0) \in T_2$ we have $\overline{x}_3/|x_1| = \sqrt{3/2}\,/|x_1|$ and $\overline{x}_3/|x_1| \in (\sqrt{3/2}, 2)$.

For $x \in T_2$ we have

$$\frac{|w_{13}(x)|}{|w_{22}(x)|} = \frac{|x_1|}{\overline{x}_3} \frac{12\overline{x}_3^2 - 3x_1^2}{(3x_1^2 + 3\overline{x}_3^2)} = \frac{|x_1|}{\overline{x}_3}\left(4 - \frac{5}{(\overline{x}_3/|x_1|)^2 + 1}\right) > \frac{2|x_1|}{\overline{x}_3} > 1,$$

so

$$|w_{13}(x)| > |w_{22}(x)|. \tag{53}$$

If $a = 0$ then by the explicit formulas, $f(a, x) < 0$. If $a > 0$ and $u_{11}(x) + aw_{11}(x) \le 0$ then $u_{33}(x) + aw_{33}(x) = -(u_{11}(x) + aw_{11}(x) + u_{22}(x) + aw_{22}(x)) > 0$ and $u_{13}(x) + aw_{13}(x) = aw_{13}(x) \ne 0$ (see (53)), so $f(a, x) < 0$. So we may assume $a > 0$ and $u_{11}(x) + aw_{11}(x) > 0$.

Again by (43) and (51), (53) we get

$$f(a, x) < \begin{vmatrix} u_{11}(x) + aw_{11}(x) & a|w_{22}(x)| \\ a|w_{22}(x)| & 2|u_{11}(x)| - aw_{11}(x) - aw_{22}(x) \end{vmatrix}.$$

Hence

$$f(a,x) < -2|u_{11}(x)|^2 + 3|u_{11}(x)|w_{11}(x)a - |u_{11}(x)|\,|w_{22}(x)|a \\ -w_{11}^2(x)a^2 + w_{11}(x)|w_{22}(x)|a^2 - |w_{22}(x)|^2a^2.$$

By (52) this is bounded from above by

$$-2|u_{11}(x)|^2 + 2|u_{11}(x)|\,|w_{11}(x)|a - w_{11}^2(x)a^2 + w_{11}(x)|w_{22}(x)|a^2 - |w_{22}(x)|^2a^2 \\ = -\left(\sqrt{2}|u_{11}(x)| - \frac{w_{11}(x)a}{\sqrt{2}}\right)^2 - \left(\frac{w_{11}(x)a}{\sqrt{2}} - \frac{|w_{22}(x)|a}{\sqrt{2}}\right)^2 - \left(\frac{|w_{22}(x)|a}{\sqrt{2}}\right)^2 < 0.$$

Case 4: $x \in A_4'$. Recall that $\overline{x}_3 = x_3 + \sqrt{3/2}$ and write $\overline{x} = (x_1, x_2, \overline{x}_3)$. Recall also that $w(x) = K(\overline{x})$. We have

$$K_{11}(\overline{x}) = C_K\overline{x}_3(x_1^2 + x_2^2 + \overline{x}_3^2)^{-7/2}(12x_1^2 - 3x_2^2 - 3\overline{x}_3^2),$$
$$K_{13}(\overline{x}) = C_K x_1(x_1^2 + x_2^2 + \overline{x}_3^2)^{-7/2}(12\overline{x}_3^2 - 3x_1^2 - 3x_2^2),$$
$$K_{33}(\overline{x}) = C_K\overline{x}_3(x_1^2 + x_2^2 + \overline{x}_3^2)^{-7/2}(6\overline{x}_3^2 - 9x_1^2 - 9x_2^2).$$

For any $M \geq 10$ denote

$$T_1(M) = \{(x_1, 0, x_3) : \overline{x}_3 = M,\ x_1 \leq 0,\ \overline{x}_3 \geq 3|x_1|\},$$
$$T_2(M) = \{(x_1, 0, x_3) : \overline{x}_3 = M,\ x_1 \leq 0,\ \sqrt{3/2}\,|x_1| \leq \overline{x}_3 < 3|x_1|\},$$
$$T_3(M) = \{(x_1, 0, x_3) : \overline{x}_3 = M,\ x_1 \leq 0,\ |x_1| \leq \overline{x}_3 < \sqrt{3/2}\,|x_1|\} \\ \cup \{(x_1, 0, x_3) : x_1 = -M,\ 0 < \overline{x}_3 < M\}.$$

We will consider three subcases: $x \in T_1(M)$, $x \in T_2(M)$, $x \in T_3(M)$.

Subcase 4a: $x \in T_1(M)$. Set $B = B(0,1) \subset \mathbb{R}^2$. We have

$$u_{11}(x) = \int_B (K_{11}(x_1 - y_1, -y_2, x_3) - K_{11}(\overline{x}))\varphi(y_1, y_2)\,dy_1\,dy_2 \\ + K_{11}(\overline{x}) \int_B \varphi(y_1, y_2)\,dy_1\,dy_2,$$

$$K_{11}(\overline{x}) = \frac{C_K\overline{x}_3(12x_1^2 - 3\overline{x}_3^2)}{(x_1^2 + \overline{x}_3^2)^{7/2}} < \frac{C_K\overline{x}_3^3(12/9 - 3)}{(x_1^2 + \overline{x}_3^2)^{7/2}} < \frac{-c}{\overline{x}_3^4}. \tag{54}$$

For $(y_1, y_2) \in B$ we also have

$$|K_{11}(x_1 - y_1, -y_2, x_3) - K_{11}(\overline{x})| \leq (|y_1| + |y_2| + |x_3 - \overline{x}_3|)|\nabla K_{11}(\xi)| \leq 4|\nabla K_{11}(\xi)|,$$

where ξ is a point between $(x_1 - y_1, -y_2, x_3)$ and $\overline{x} = (x_1, 0, \overline{x}_3)$. For such ξ we have

$$|\nabla K_{11}(\xi)| \leq c/x_3^5. \tag{55}$$

By (54), (55) for sufficiently large M and all $x \in T_1(M)$ we have $u_{11}(x) < 0$. We also have $aw_{11}(x) = aK_{11}(\overline{x}) < 0$ for $a \geq 0$, $x \in T_1(M)$. Hence $u_{11}(x) + aw_{11}(x) < 0$, which implies $f(a,x) < 0$. It follows that for sufficiently large $M_1 \geq 10$ and for all $M \geq M_1$, $a \geq 0$, $x \in T_1(M)$ we have $f(a,x) < 0$.

Subcase 4b: $x \in T_2(M)$. First we need the following auxiliary lemma.

Lemma 5.3. *Let* $f(y_1, y_3) = -6y_1^3 - 3y_1^2 y_3 + 24 y_1 y_3^2 - 3y_3^3$. *For any* $y_3 > 0$ *and* $y_1 \in [y_3/3, y_3]$ *we have* $f(y_1, y_3) > 4y_3^3$.

Proof. The proof is elementary. Fix $y_3 > 0$ and set $g(y_1) = f(y_1, y_3)$. We have $g'(y_1) = -18y_1^2 - 6y_1 y_3 + 24y_3^2$, $g'(y_1) = 0$ for $y_1 = (-8/6)y_3$ and $y_1 = y_3$, so g is increasing for $y_1 \in [(-8/6)y_3, y_3]$. We also have $g(y_3/3) = (40/9)y_3^3$, so for any $y_1 \in [y_3/3, y_3]$ we have $g(y_1) > 4y_3^3$. □

Set $b = \int_B \varphi(y_1, y_2)\, dy_1\, dy_2$. For $x \in T_2(M)$ we have

$$f(a, x) = \begin{vmatrix} K_{11}(\overline{x})(a+b) + \varepsilon_{11}(x) & K_{13}(\overline{x})(a+b) + \varepsilon_{13}(x) \\ K_{13}(\overline{x})(a+b) + \varepsilon_{13}(x) & K_{33}(\overline{x})(a+b) + \varepsilon_{33}(x) \end{vmatrix},$$

where

$$\varepsilon_{ij}(x) = \int_B (K_{ij}(x_1 - y_1, -y_2, x_3) - K_{ij}(\overline{x}))\varphi(y_1, y_2)\, dy_1\, dy_2$$

for $(i, j) = (1, 1)$ or $(1, 3)$ or $(3, 3)$. For $(y_1, y_2) \in B$ we have

$$|K_{ij}(x_1 - y_1, -y_2, x_3) - K_{ij}(\overline{x})| \le (|y_1| + |y_2| + |x_3 - \overline{x}_3|)|\nabla K_{ij}(\xi)| \le 4|\nabla K_{ij}(\xi)|,$$

where ξ is a point between $(x_1 - y_1, -y_2, x_3)$ and $\overline{x} = (x_1, 0, \overline{x}_3)$. We have $|\nabla K_{ij}(\xi)| \le c x_3^{-5}$, so

$$|\varepsilon_{ij}(x)| \le cb/x_3^5. \tag{56}$$

Write

$$f_1(a, x) = \begin{vmatrix} K_{11}(\overline{x})(a+b) & K_{13}(\overline{x})(a+b) \\ K_{13}(\overline{x})(a+b) & K_{33}(\overline{x})(a+b) \end{vmatrix}.$$

We have $|K_{ij}(\overline{x})| \le c x_3^{-4}$, so by (56) we obtain

$$|f(a, x) - f_1(a, x)| \le c(a+b)b x_3^{-9}. \tag{57}$$

On the other hand,

$$\begin{aligned} |f_1(a, x)| &\ge (a+b)^2 (K_{13}^2(\overline{x}) - K_{11}(\overline{x})K_{33}(\overline{x})) \\ &\ge (a+b)^2 \left(K_{13}^2(\overline{x}) - \left(\frac{K_{11}(\overline{x}) + K_{33}(\overline{x})}{2} \right)^2 \right) \\ &= (a+b)^2 \left(|K_{13}(\overline{x})|^2 - \left(\frac{|K_{22}(\overline{x})|}{2} \right)^2 \right). \end{aligned} \tag{58}$$

We have

$$|K_{13}(\overline{x})| - |K_{22}(\overline{x})|/2 = \tfrac{1}{2} C_K (|x_1|^2 + \overline{x}_3^2)^{-7/2} (-6|x_1|^3 - 3|x_1|^2 \overline{x}_3 + 24|x_1|\overline{x}_3^2 - 3\overline{x}_3^3).$$

By Lemma 5.3 we obtain

$$|K_{13}(\overline{x})| - |K_{22}(\overline{x})|/2 \ge \tfrac{1}{2} C_K (|x_1|^2 + \overline{x}_3^2)^{-7/2} 4\overline{x}_3^3 \ge c x_3^{-4}.$$

Using this and (58) yields

$$|f_1(a,x)| \geq (a+b)^2(|K_{13}(\overline{x})| - |K_{22}(\overline{x})|/2)^2 \geq c(a+b)^2 x_3^{-8}.$$

It follows that $f_1(a,x) < -c(a+b)^2 x_3^{-8}$. Using this and (57) we find that for sufficiently large $M_1 \geq 10$ and for all $M \geq M_1$, $a \geq 0$, $x \in T_2(M)$ we have $f(a,x) < 0$.

Subcase 4c: $x \in T_3(M)$. This subcase follows from the same arguments as in Subcase 1d. □

Proof of Proposition 5.1. Assume on the contrary that there exists $z = (z_1, z_2, z_3) \in \mathbb{R}^3 \setminus (B(0,1)^c \times \{0\})$ such that $H(u)(z) \leq 0$. By Lemma 2.7 we may assume that $z_1 \geq 0$. By the explicit formula for φ and Lemma 4.7 we may assume that $z_1 > 0$. Define

$$\Psi^{(b)}(x) = (1-b)u(x) + bw(x), \qquad b \in [0,1],$$

where w is given by (42). By direct computation for any $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ with $x_3 > -\sqrt{3/2}$ we have

$$H(w)(x) = C_K^3 \frac{27(x_3 + \sqrt{3/2})(x_1^2 + x_2^2 + 2(x_3 + \sqrt{3/2})^2)}{(x_1^2 + x_2^2 + (x_3 + \sqrt{3/2})^2)^{15/2}} > 0.$$

Recall that $\mathbb{R}^3_+ = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_3 > 0\}$ and set $\Omega = \mathbb{R}^3_+ \setminus (A_1 \cup A_2 \cup A_4)$, where A_1, A_2, A_4 are sets from Lemma 5.2. By that lemma we find that $z \in \Omega$ and $H(\Psi^{(b)})(x) > 0$ for all $b \in [0,1]$ and $x \in \partial\Omega$. Note that $\Psi^{(0)} = u$, $\Psi^{(1)} = w$, $H(\Psi^{(0)})(z) < 0$ and $H(\Psi^{(1)})(x) > 0$ for all $x \in \overline{\Omega}$. Clearly, all second order partial derivatives of $\Psi^{(b)}$ are uniformly Lipschitz continuous on $\overline{\Omega}$, that is,

$$\exists c\ \forall b \in [0,1]\ \forall x, y \in \overline{\Omega}\ \forall i, j \in \{1,2,3\} \qquad |\Psi_{ij}^{(b)}(x) - \Psi_{ij}^{(b)}(x)| \leq c|x-y|.$$

It follows that there exists $b_0 \in [0,1)$ such that $H(\Psi^{(b_0)})(z_0) = 0$ for some $z_0 \in \Omega$ and $H(\Psi^{(b_0)})(x) \geq 0$ for all $x \in \overline{\Omega}$. This contradicts Theorem 1.6. □

6. Concavity of φ

In this section we prove the main result of this paper, Theorem 1.1. This is done by using the method of continuity, Lewy's Theorem 1.6 and results from Sections 3–5.

For any $\varepsilon \geq 0$ we define

$$v^{(\varepsilon)}(x) = u(x) + \varepsilon(-x_1^2/2 - x_2^2/2 + x_3^2), \qquad x \in \mathbb{R}^3 \setminus (D^c \times \{0\}), \tag{59}$$

where u is the harmonic extension of φ given by (6)–(10) and φ is the solution of (1)–(2) for an open bounded set $D \subset \mathbb{R}^2$. When D is not fixed, we will sometimes write $v^{(\varepsilon,D)}$ instead of $v^{(\varepsilon)}$.

Lemma 6.1. *Let $C_1, R_1 > 0$, $\kappa_2 \geq \kappa_1 > 0$, $D \in F(C_1, R_1, \kappa_1, \kappa_2)$, let φ be the solution of (1)–(2) for D and u the harmonic extension of φ given by (6)–(10). For any $\varepsilon \geq 0$ let $v^{(\varepsilon)}$ be given by (59). For any $(x_1, x_2, x_3) \in \mathbb{R}^3_+$ we have $H(v^{(\varepsilon)})(x_1, x_2, -x_3) = H(v^{(\varepsilon)})(x_1, x_2, x_3)$.*

The proof of this lemma is similar to the proof of Lemma 2.7 and is omitted.

Proposition 6.2. *Fix* $C_1, R_1 > 0$, $\kappa_2 \geq \kappa_1 > 0$ *and* $D \in F(C_1, R_1, \kappa_1, \kappa_2)$. *Denote* $\Lambda = \{C_1, R_1, \kappa_1, \kappa_1\}$. *Let* φ *be the solution of* (1)–(2) *for* D, u *the harmonic extension of* φ *and* $v^{(\varepsilon)}$ *given by* (59). *For* $M \geq 10$, $h \in (0, 1/2]$, $\eta \in (0, 1/2]$ *define* (*see Figure* 7)

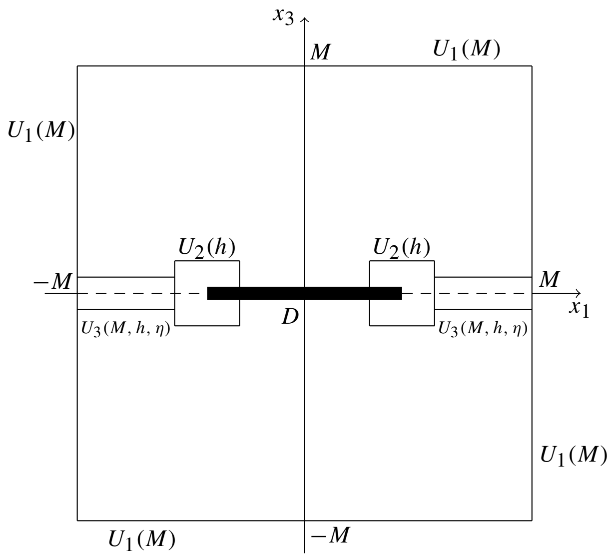
$$\begin{aligned} U_1(M) &= \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq M^2,\ x_3 = M \text{ or } x_3 = -M\} \\ &\quad \cup \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 = M^2,\ x_3 \in [-M, M] \setminus \{0\}\}, \\ U_2(h) &= \{x \in \mathbb{R}^3 :\ (x_1, x_2) \in D,\ \delta_D((x_1, x_2)) \leq h,\ x_3 \in [-h, h]\} \\ &\quad \cup \{x \in \mathbb{R}^3 :\ (x_1, x_2) \notin D,\ \delta_D((x_1, x_2)) \leq h,\ x_3 \in [-h, h] \setminus \{0\}\}, \\ U_3(M, h, \eta) &= \{x \in \mathbb{R}^3 : (x_1, x_2) \notin D,\ \delta_D((x_1, x_2)) \geq h,\ x_1^2 + x_2^2 \leq M^2, \\ &\qquad\qquad x_3 \in [-\eta, \eta] \setminus \{0\}\}, \\ U_4(h) &= \{x \in \mathbb{R}^3 :\ (x_1, x_2) \in D,\ \delta_D((x_1, x_2)) \leq h,\ x_3 = 0\}. \end{aligned}$$

Then

$$\exists c_1 = c_1(\Lambda) \in (0, 1]\ \exists M_0 \geq 10\ \exists h_1 = h_1(\Lambda) \in (0, 1/2]\ \forall M \geq M_0\ \forall \varepsilon \in (0, c_1 M^{-7}]$$
$$\exists \eta = \eta(\Lambda, M, \varepsilon) \in (0, 1/2]\ \exists C = C(\Lambda, M, \varepsilon) > 0\ \forall x \in U_1(M) \cup U_2(h_1) \cup U_3(M, h_1, \eta)$$
$$H(v^{(\varepsilon)})(x) \geq C.$$

Moreover

$$\exists \tilde{h} = \tilde{h}(\Lambda) \in (0, 1/2]\ \exists \tilde{C} = \tilde{C}(\Lambda) > 0\ \forall x \in U_4(\tilde{h}) \quad H(u)(x) \geq \tilde{C}. \tag{60}$$



A cross section parallel to the $x_1 x_3$ plane

Fig. 7

Proof. In the whole proof we use the convention stated in Remark 2.9. We have $H(v^{(\varepsilon)})(x) = W_1(x) + W_2(x) + W_3(x)$, where

$$W_1(x) = v_{12}^{(\varepsilon)}(x)\big(v_{13}^{(\varepsilon)}(x)v_{23}^{(\varepsilon)}(x) - v_{12}^{(\varepsilon)}(x)v_{33}^{(\varepsilon)}(x)\big),$$
$$W_2(x) = -v_{23}^{(\varepsilon)}(x)\big(v_{11}^{(\varepsilon)}(x)v_{23}^{(\varepsilon)}(x) - v_{13}^{(\varepsilon)}(x)v_{12}^{(\varepsilon)}(x)\big),$$
$$W_3(x) = v_{22}^{(\varepsilon)}(x)f(\varepsilon,x),$$
$$f(\varepsilon,x) = v_{11}^{(\varepsilon)}(x)v_{33}^{(\varepsilon)}(x) - (v_{13}^{(\varepsilon)}(x))^2.$$

The proof consists of three parts.

Part 1: *Estimates on* $U_1(M)$. We may assume in this part that $x_2 = 0$, $x_3 > 0$, $x_1 \le 0$. By the formulas $u_{ij}(x) = \int_D K_{ij}(x_1 - y_1, x_2 - y_2, x_3)\varphi(y_1,y_2)\,dy_1\,dy_2$ and the explicit formulas for K_{ij} (see Section 2), there exist $M_1 \ge 10$ and c such that for any $M \ge M_1$ and $x \in U_1(M)$ we have $|u_{11}(x)| \le cx_3M^{-5}$, $u_{22}(x) \approx -x_3M^{-5}$, $|u_{33}(x)| \le cx_3M^{-5}$, $|u_{13}(x)| \le cM^{-4}$, $|u_{23}(x)| \le cM^{-5}$ and $|u_{12}(x)| \le cx_3M^{-6}$.

Fix $M \ge M_1$.

Let $x \in U_1(M)$ (recall that we assume that $x_2 = 0$, $x_3 > 0$, $x_1 \le 0$). We have

$$|W_1(x)| \le cx_3M^{-6}(M^{-4}M^{-5} + x_3M^{-6}(x_3M^{-5} + 2\varepsilon)) \le cx_3M^{-15} + c\varepsilon M^{-10}, \quad (61)$$
$$|W_2(x)| \le cM^{-5}((x_3M^{-5} + \varepsilon)M^{-5} + M^{-4}x_3M^{-6}) \le cx_3M^{-15} + c\varepsilon M^{-10}. \quad (62)$$

Now we estimate $W_3(x)$. We have

$$v_{22}^{(\varepsilon)}(x) = u_{22}(x) - \varepsilon \approx -cx_3M^{-5} - \varepsilon. \quad (63)$$

The most important is the estimate of $f(\varepsilon,x)$. To obtain this estimate we will consider six cases.

Case 1.1: $x_3 = M$, $|x_1| < x_3/3$. Set $m(x) = C_K(x_1^2 + x_3^2)^{-7/2}$. We have

$$u_{11}(x) \approx K_{11}(x) = m(x)x_3(12x_1^2 - 3x_3^2) < cM^{-7}x_3(12(x_3/3)^2 - 3x_3^2),$$

so $u_{11}(x) \le -cM^{-4}$. Moreover,

$$u_{33}(x) \approx K_{33}(x) = m(x)x_3(6x_3^2 - 9x_1^2) \ge cM^{-7}x_3(6x_3^2 - 9(x_3/3)^2),$$

so $u_{33}(x) \ge cM^{-4}$. Therefore for any $\varepsilon \ge 0$ we have $v_{11}^{(\varepsilon)}(x) \le -cM^{-4}$ and $v_{33}^{(\varepsilon)}(x) \ge cM^{-4}$. Hence $f(\varepsilon,x) \le -cM^{-8}$.

Case 1.2: $x_3 = M$, $|x_1| \in [x_3/3, x_3/\sqrt{3/2}\,]$. By the arguments of Subcase 4b in the proof of Lemma 5.2 we have $u_{11}(x)u_{33}(x) - (u_{13}(x))^2 < -cM^{-8}$ for sufficiently large M. For any $\varepsilon \ge 0$ we have

$$\big|f(\varepsilon,x) - \big(u_{11}(x)u_{33}(x) - (u_{13}(x))^2\big)\big| \le 2\varepsilon^2 + 2\varepsilon|u_{11}(x)| + \varepsilon|u_{33}(x)|.$$

For any $c_1 \in (0,1]$ and all $\varepsilon \in (0, c_1M^{-7}]$ this is bounded from above by cc_1M^{-11}. It follows that for sufficiently small $c_1 \in (0,1]$, for sufficiently large M and all $\varepsilon \in (0, c_1M^{-7}]$ we have $f(\varepsilon,x) < -cM^{-8}$.

Case 1.3: $x_3 = M$, $|x_1| \in [x_3/\sqrt{3/2}, x_3]$. We have

$$u_{11}(x) \approx K_{11}(x) = m(x)x_3(12x_1^2 - 3x_3^2) \approx M^{-7}x_3\left(12\frac{x_3^2}{3/2} - 3x_3^2\right) \approx M^{-4}.$$

Moreover, for $y \in D \subset B(0, 1)$,

$$\begin{aligned} K_{33}(x_1 - y_1, -y_2, x_3) &\le C_K x_3((x_1 - y_1)^2 + y_2^2 + x_3^2)^{-7/2}(6x_3^2 - 9(x_1 - y_1)^2) \\ &= C_K x_3((x_1 - y_1)^2 + y_2^2 + x_3^2)^{-7/2}(6x_3^2 - 9x_1^2 + 18x_1y_1 - 9y_1^2) \le cM^{-5}, \end{aligned}$$

so $u_{33}(x) \le cM^{-5}$. For sufficiently small $c_1 \in (0, 1]$ and all $\varepsilon \in (0, c_1M^{-7}]$ we obtain $v_{11}^{(\varepsilon)}(x) \approx M^{-4}$ and $v_{33}^{(\varepsilon)}(x) \le cM^{-5}$. We also have $u_{13}(x) \approx K_{13}(x) = m(x)x_1(12x_3^2 - 3x_1^2) \ge cM^{-4}$. It follows that for sufficiently small c_1, for sufficiently large M and all $\varepsilon \in (0, c_1M^{-7}]$ we have $f(\varepsilon, x) < -cM^{-8}$.

Case 1.4: $x_3 \in [M/4, M]$, $x_1 = -M$. We have

$$u_{11}(x) \approx K_{11}(x) = m(x)x_3(12x_1^2 - 3x_3^2),$$

so $u_{11}(x) \ge cM^{-4}$. Moreover,

$$u_{33}(x) \approx K_{33}(x) = m(x)x_3(6x_3^2 - 9x_1^2),$$

so $u_{33}(x) \le -cM^{-4}$. Therefore for sufficiently small $c_1 \in (0, 1]$ and all $\varepsilon \in (0, c_1M^{-7}]$ we have $v_{11}^{(\varepsilon)}(x) \ge cM^{-4}$ and $v_{33}^{(\varepsilon)}(x) \le -cM^{-4}$. Hence $f(\varepsilon, x) \le -cM^{-8}$.

Case 1.5: $x_3 \in [1, M/4]$, $x_1 = -M$. We have

$$u_{13}(x) \approx K_{13}(x) = m(x)x_1(12x_3^2 - 3x_1^2),$$

so $u_{13}(x) \le -cM^{-4}$. Moreover,

$$\begin{aligned} u_{11}(x) &\approx K_{11}(x) = m(x)x_3(12x_1^2 - 3x_3^2), \\ u_{33}(x) &\approx K_{33}(x) = m(x)x_3(6x_3^2 - 9x_1^2), \end{aligned}$$

so $u_{11}(x) \ge cM^{-5}$ and $u_{33}(x) \le -cM^{-5}$. Therefore for sufficiently small $c_1 \in (0, 1]$ and all $\varepsilon \in (0, c_1M^{-7}]$ we have $v_{11}^{(\varepsilon)}(x) \ge cM^{-5}$ and $v_{33}^{(\varepsilon)}(x) \le -cM^{-5}$. Hence $f(\varepsilon, x) \le -cM^{-8}$.

Case 1.6: $x_3 \in (0, 1]$, $x_1 = -M$. By similar arguments to Case 1.5 we get $u_{13}(x) \le -cM^{-4}$, $|u_{11}(x)| \le cM^{-5}$ and $|u_{33}(x)| \le cM^{-5}$. Therefore for sufficiently small $c_1 \in (0, 1]$ and all $\varepsilon \in (0, c_1M^{-7}]$ we have $|v_{11}^{(\varepsilon)}(x)| \le cM^{-5}$ and $|v_{33}^{(\varepsilon)}(x)| \le cM^{-5}$. Hence for sufficiently small $c_1 \in (0, 1]$, for sufficiently large M and all $\varepsilon \in (0, c_1M^{-7}]$ we have $f(\varepsilon, x) \le -cM^{-8}$.

Finally, in all six cases, for sufficiently small $c_1 \in (0, 1]$, for sufficiently large M and all $\varepsilon \in (0, c_1M^{-7}]$ we have $f(\varepsilon, x) \le -cM^{-8}$. By (63) we get $W_3(x) = v_{22}^{(\varepsilon)}(x)f(\varepsilon, x) \ge cx_3M^{-13} + c\varepsilon M^{-8}$. By (61), (62) we have $|W_1(x) + W_2(x)| \le cx_3M^{-15} + c\varepsilon M^{-10}$.

Recall that $H(v^{(\varepsilon)})(x) = W_1(x) + W_2(x) + W_3(x)$. It follows that there exist sufficiently small $c_1' = c_1'(\Lambda) \in (0, 1]$ and sufficiently large $M_0 \geq M_1 \geq 10$ such that for any $M \geq M_0$ and $\varepsilon \in (0, c_1' M^{-7}]$ and all $x \in U_1(M)$ we have $H(v^{(\varepsilon)})(x) \geq c\varepsilon M^{-8}$.

Let us fix the above M_0 and $M \geq M_0$ in the rest of the proof of the proposition.

Part 2: *Estimates on $U_2(h)$.* We will use the notation and results from Section 4 (Propositions 4.1–4.6). In particular we choose a point on ∂D and a Cartesian coordinate system with origin at that point in the same way as in Section 4 (see Figures 1 and 4). Let $h \in (0, h_0]$, where h_0 denotes the minimum of the constants h_0 from Propositions 4.1–4.6. By Lemma 6.1 we may assume $x_3 \geq 0$, and by continuity we may assume $x_3 > 0$. Hence it is enough to estimate $H(v^{(\varepsilon)})(x)$ for $x \in S_1(h) \cup S_2(h) \cup S_3(h) \cup S_4(h)$. We will consider two cases. Assume that $\varepsilon \in (0, 1]$.

Case 2.1: $x \in S_1(h) \cup S_2(h) \cup S_3(h)$. If $x \in S_1(h) \cup S_3(h)$ we have $(v_{13}^{(\varepsilon)}(x))^2 = u_{13}^2(x) \geq ch^{-3}$, $v_{11}^{(\varepsilon)}(x)v_{33}^{(\varepsilon)}(x) = u_{11}(x)u_{33}(x) + 2\varepsilon u_{11}(x) - \varepsilon u_{33}(x) - 2\varepsilon^2$, $|2\varepsilon u_{11}(x)| \leq c\varepsilon h^{-3/2}$ and $|-\varepsilon u_{33}(x)| \leq c\varepsilon h^{-3/2}$.

If $u_{11}(x) \leq 0$ or $u_{33}(x) \leq 0$ then $u_{11}(x)u_{33}(x) \leq 0$ (recall that $u_{11}(x) + u_{33}(x) = -u_{22}(x) > 0$). If $u_{11}(x) > 0$ and $u_{33}(x) > 0$ then

$$u_{11}(x)u_{33}(x) \leq \left(\frac{u_{11}(x) + u_{33}(x)}{2}\right)^2 = \left(\frac{u_{22}(x)}{2}\right)^2 \leq ch^{-1}.$$

Hence $f(\varepsilon, x) = -(v_{13}^{(\varepsilon)}(x))^2 + v_{11}^{(\varepsilon)}(x)v_{33}^{(\varepsilon)}(x) \leq -ch^{-3}$ for sufficiently small h and all $\varepsilon \in (0, 1]$.

If $x \in S_2(h)$ we have $u_{11}(x) \approx h^{-3/2}$ and $u_{33}(x) \approx -h^{-3/2}$. Hence for sufficiently small h and all $\varepsilon \in (0, 1]$ we have $v_{11}^{(\varepsilon)}(x) \approx h^{-3/2}$, $v_{33}^{(\varepsilon)}(x) \approx -h^{-3/2}$ and $f(\varepsilon, x) \leq -ch^{-3}$.

Hence for any $x \in S_1(h) \cup S_2(h) \cup S_3(h)$ for sufficiently small h and all $\varepsilon \in (0, 1]$ we obtain $f(\varepsilon, x) \leq -ch^{-3}$. We have $v_{22}^{(\varepsilon)}(x) \approx -x_3 h^{-3/2} - \varepsilon$. It follows that $W_3(x) = v_{22}^{(\varepsilon)}(x) f(\varepsilon, x) \geq cx_3 h^{-9/2} + c\varepsilon h^{-3}$. Moreover,

$$\begin{aligned}
|W_1(x)| &\leq cx_3 h^{-3/2}|\log h|\big(h^{-3/2}h^{-1/2}|\log h| + (2\varepsilon + x_3 h^{-5/2})x_3 h^{-3/2}|\log h|\big) \\
&\leq cx_3 h^{-7/2}|\log h|^2 + c\varepsilon h^{-1}|\log h|^2, \\
|W_2(x)| &\leq ch^{-1/2}|\log h|\big((\varepsilon + x_3 h^{-5/2})h^{-1/2}|\log h| + h^{-3/2}x_3 h^{-3/2}|\log h|\big) \\
&\leq cx_3 h^{-7/2}|\log h|^2 + c\varepsilon h^{-1}|\log h|^2.
\end{aligned}$$

Hence there exists a sufficiently small h_1' such that for all $h \in (0, h_1']$ and $\varepsilon \in (0, 1]$ we have $H(v^{(\varepsilon)})(x) \geq cx_3 h^{-9/2} + c\varepsilon h^{-3}$.

Case 2.2: $x \in S_4(h)$. For sufficiently small h and all $\varepsilon \in [0, 1]$ we have $W_3(x) \geq ch^{-1/2}h^{-3} = ch^{-14/4}$ and

$$\begin{aligned}
|W_1(x)| &\leq ch^{-1/2}|\log h|(h^{-3/2}h^{-3/4}|\log h| + h^{-3/2}h^{-1/2}|\log h|) \\
&\leq ch^{-11/4}|\log h|^2,
\end{aligned}$$

$$|W_2(x)| \le ch^{-3/4}|\log h|(h^{-3/2}h^{-3/4}|\log h| + h^{-1/2}|\log h|h^{-3/2}) \le ch^{-12/4}|\log h|^2.$$

So there exists a sufficiently small h_1'' such that for all $h \in (0, h_1'']$ and $\varepsilon \in [0, 1]$ we have $H(v^{(\varepsilon)})(x) \ge ch^{-14/4}$.

Since $u = v^{(0)}$ is continuous in a neighbourhood of any $x \in D \times \{0\}$, we obtain (60).

Fix $h_1 = h_1' \wedge h_1''$ in the rest of the proof of the proposition.

Part 3: *Estimates on* $U_3(M, h_1, \eta)$. Choose a point on ∂D and a Cartesian coordinate system as in Part 2. Note that it is enough to estimate $H(v^{(\varepsilon)})(x)$ for $x \in U_3'(M, h_1, \eta) = \{(x_1, x_2, x_3) : x_2 = 0, x_1 \in [-M, -h_1], x_3 \in (0, \eta]\}$ and sufficiently small $\eta = \eta(\Lambda, M, \varepsilon)$.

Let $x \in U_3'(M, h_1, 1/2)$. Note that $\mathrm{dist}(x, \partial D) \ge h_1$. By the formulas $u_{ij}(x) = \int_D K_{ij}(x_1 - y_1, x_2 - y_2, x_3)\varphi(y_1, y_2)\,dy_1\,dy_2$ and the explicit formulas for K_{ij} (see Section 2) we have $|u_{11}(x)| \le cx_3h_1^{-5}$, $|u_{22}(x)| \le cx_3h_1^{-5}$, $|u_{33}(x)| \le cx_3h_1^{-5}$, $|u_{13}(x)| \le ch_1^{-4}$, $|u_{23}(x)| \le ch_1^{-4}$ and $|u_{12}(x)| \le cx_3h_1^{-5}$. Note also that by our choice of coordinate system for any $y = (y_1, y_2) \in D$ we have $y_1 > 0$. From now on we assume additionally that $x = (x_1, x_2, x_3) \in U_3'(M, h_1, 1/2)$ with $x_3 \le |x_1|/\sqrt{6}$ (this condition implies $12x_3^2 \le 2x_1^2$). For such $x = (x_1, x_2, x_3)$ and any $y = (y_1, y_2) \in D$ we have $12x_3^2 - 3(x_1 - y_1)^2 - 3(x_2 - y_2)^2 \le -(x_1 - y_1)^2 \le -x_1^2 \le -h_1^2$.

It follows that

$$|u_{13}(x)| = \left| C_K \int_D \frac{(x_1 - y_1)(12x_3^2 - 3(x_1 - y_1)^2 - 3(x_2 - y_2)^2)}{((x_1 - y_1)^2 + (x_2 - y_2)^2 + x_3^2)^{7/2}} \varphi(y_1, y_2)\,dy_1\,dy_2 \right| \ge \frac{\tilde{C}h_1^3}{M^7}. \tag{64}$$

The constant $\tilde{C}$ will play an important role in the rest of the proof, and this is why it is not denoted by c as usual. Clearly, $\tilde{C}$ depends only on Λ.

Recall that in Parts 1 and 2 of this proof we have fixed constants M_0, $M \ge M_0$, h_1. At the end of Part 1 we have chosen a constant $c_1' \in (0, 1]$. Set

$$c_1 = c_1' \wedge \tfrac{1}{4}\tilde{C}h_1^3, \tag{65}$$

where $\tilde{C}$ is the constant from (64). In the rest of the proof we fix this constant c_1 and $\varepsilon \in (0, c_1M^{-7}]$. The reason for defining c_1 by (65) is that $2\varepsilon^2 \le 2c_1^2M^{-14} \le \frac{1}{8}\tilde{C}^2h_1^6M^{-14}$, which implies

$$2\varepsilon^3 \le \frac{1}{4}\,\frac{\varepsilon}{2}\tilde{C}^2h_1^6M^{-14}, \tag{66}$$

which will be crucial in the following.

Note that for sufficiently small $\eta = \eta(\Lambda, M, \varepsilon)$ and $x \in U_3'(M, h_1, \eta)$ we have $x_3 \le |x_1|/\sqrt{6}$ and

$$v_{22}^{(\varepsilon)}(x) = -\varepsilon + u_{22}(x) \le -\varepsilon + cx_3h_1^{-5} \le -\varepsilon/2,$$
$$v_{11}^{(\varepsilon)}(x) = -\varepsilon + u_{11}(x) \le -\varepsilon + cx_3h_1^{-5} \le -\varepsilon/2.$$

We have

$$
\begin{aligned}
&H(v^{(\varepsilon)})(x) = v_{11}^{(\varepsilon)}(x)v_{22}^{(\varepsilon)}(x)v_{33}^{(\varepsilon)}(x) + 2v_{12}^{(\varepsilon)}(x)v_{23}^{(\varepsilon)}(x)v_{13}^{(\varepsilon)}(x)\\
&\qquad - v_{22}^{(\varepsilon)}(x)(v_{13}^{(\varepsilon)}(x))^2 - v_{11}^{(\varepsilon)}(x)(v_{23}^{(\varepsilon)}(x))^2 - v_{33}^{(\varepsilon)}(x)(v_{12}^{(\varepsilon)}(x))^2,\\
&-v_{22}^{(\varepsilon)}(x)(v_{13}^{(\varepsilon)}(x))^2 \geq \frac{\varepsilon}{2}\frac{\tilde{C}^2 h_1^6}{M^{14}}, \quad (67)\\
&-v_{11}^{(\varepsilon)}(x)(v_{23}^{(\varepsilon)}(x))^2 \geq 0,\\
&|v_{33}^{(\varepsilon)}(x)(v_{12}^{(\varepsilon)}(x))^2| \leq (cx_3h_1^{-5})^2(2\varepsilon + cx_3h_1^{-5}), \quad (68)\\
&|v_{12}^{(\varepsilon)}(x)v_{23}^{(\varepsilon)}(x)v_{13}^{(\varepsilon)}(x)| \leq cx_3h_1^{-5}h_1^{-4}h_1^{-4}, \quad (69)\\
&|v_{11}^{(\varepsilon)}(x)v_{22}^{(\varepsilon)}(x)v_{33}^{(\varepsilon)}(x)| \leq (\varepsilon + cx_3h_1^{-5})^2(2\varepsilon + cx_3h_1^{-5}). \quad (70)
\end{aligned}
$$

Note that the right hand sides of (68)–(70) are bounded by $2\varepsilon^3 + x_3C(\Lambda, h_1)$ (note that h_1 depends only on Λ, so $C(\Lambda, h_1) = C(\Lambda)$). By (66) and (67) we have $2\varepsilon^3 \leq -\frac{1}{4}v_{22}^{(\varepsilon)}(x)(v_{13}^{(\varepsilon)}(x))^2$. Moreover, $x_3C(\Lambda, h_1) < -\frac{1}{4}v_{22}^{(\varepsilon)}(x)(v_{13}^{(\varepsilon)}(x))^2$ for sufficiently small $\eta = \eta(\Lambda, M, \varepsilon)$ and $x \in U_3'(M, h_1, \eta)$. For such η and x we have

$$
H(v^{(\varepsilon)})(x) \geq -\frac{1}{2}v_{22}^{(\varepsilon)}(x)(v_{13}^{(\varepsilon)}(x))^2 \geq \frac{\varepsilon}{4}\frac{\tilde{C}^2 h_1^6}{M^{14}}. \qquad \square
$$

Lemma 6.3. *Let φ be the solution of (1)–(2) for $B(0, 1)$, u the harmonic extension of φ and $v^{(\varepsilon)}$ given by (59). For $M \geq 10$, $h \in (0, 1/2]$, $\eta \in (0, 1/2]$ we define*

$$
\begin{aligned}
U_1(M) &= \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq M^2,\ x_3 = M \text{ or } x_3 = -M\}\\
&\quad \cup \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 = M^2,\ x_3 \in [-M, M] \setminus \{0\}\},\\
U_2(h) &= \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 \in [(1-h)^2, 1),\ x_3 \in [-h, h]\}\\
&\quad \cup \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 \in [1, (1+h)^2],\ x_3 \in [-h, h] \setminus \{0\}\},\\
U_3(M, h, \eta) &= \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 \in [(1+h)^2, M^2],\ x_1^2 + x_2^2 \leq M^2,\ x_3 \in [-\eta, \eta] \setminus \{0\}\}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\exists c_1 \in (0, 1]\ \exists M_0 \geq 10\ \exists h_1 \in (0, 1/2]\ \forall M \geq M_0\ \exists \eta = \eta(M) \in (0, 1/2]\\
&\forall \varepsilon \in (0, c_1M^{-7}]\ \forall x \in U_1(M) \cup U_2(h_1) \cup U_3(M, h_1, \eta) \qquad H(v^{(\varepsilon)})(x) > 0.
\end{aligned}
$$

Remark 6.4. It is important here that η does not depend on ε.

Proof of Lemma 6.3. Existence of c_1, M_0, h_1 and the estimate $H(v^{(\varepsilon)})(x) > 0$ for $x \in U_1(M) \cup U_2(h_1)$ (where $M \geq M_0$ and $\varepsilon \in (0, c_1M^{-7}]$) follow from the arguments in the proof of Proposition 6.2.

Let $\varepsilon \in (0, 1]$. Fix $M \geq M_0$ and let $x \in U_3(M, h_1, 1/2)$. We may assume that $x_2 = 0$, $x_3 > 0$, $x_1 < 0$. We have $H(v^{(\varepsilon)})(x) = v_{22}^{(\varepsilon)}(x)f(\varepsilon, x)$, where $f(\varepsilon, x) =$

$v_{11}^{(\varepsilon)}(x)v_{33}^{(\varepsilon)}(x) - (v_{13}^{(\varepsilon)}(x))^2$. We have $u_{22}(x) < 0$, so $v_{22}^{(\varepsilon)}(x) = u_{22}(x) - \varepsilon < 0$. Moreover, $|u_{11}(x)| \le cx_3h_1^{-5}$ and $|u_{33}(x)| \le cx_3h_1^{-5}$, which gives

$$v_{11}^{(\varepsilon)}(x)v_{33}^{(\varepsilon)}(x) = (u_{11}(x) - \varepsilon)(u_{33}(x) + 2\varepsilon) < cx_3h_1^{-10} + cx_3h_1^{-5}.$$

Let us additionally assume that x_3 is so small that $x_3 \le (|x_1| - 1)/\sqrt{6}$. For such x by the arguments from the proof of Proposition 6.2 we have $|u_{13}(x)| \ge ch_1^3M^{-7}$, so $|v_{13}^{(\varepsilon)}(x)|^2 = |u_{13}(x)|^2 \ge ch_1^6M^{-14}$. Hence for sufficiently small $\eta = \eta(M)$ and $x \in U_3(M, h_1, \eta)$ we have $f(\varepsilon, x) < 0$, which implies $H(v^{(\varepsilon)})(x) > 0$. □

Proposition 6.5. *Let φ be the solution of (1)–(2) for $B(0, 1)$, u the harmonic extension of φ, and $v^{(\varepsilon)}$ given by (59). For $M \ge 10$ define*

$$\Omega_M = \{x \in \mathbb{R}^3 : x_1^2+x_2^2 \le M^2, x_3 \in [-M, M]\} \setminus \{x \in \mathbb{R}^3 : x_1^2+x_2^2 \in [1, M^2], x_3 = 0\}.$$

Let c_1 and M_0 be the constants from Lemma 6.3. Then

$$\forall M \ge M_0 \, \forall \varepsilon \in (0, c_1M^{-7}] \, \forall x \in \Omega_M \quad H(v^{(\varepsilon)})(x) > 0.$$

Proof. Assume on the contrary that there exist $M_1 \ge M_0$, $\varepsilon_1 \in (0, c_1M_1^{-7}]$ and $z \in \Omega_{M_1}$ such that $H(v^{(\varepsilon_1)})(z) \le 0$. By Lemma 6.3 there exist $h_1 \in (0, 1/2]$ and $\eta_1 = \eta_1(M_1) \in (0, 1/2]$ such that $H(v^{(\varepsilon)})(x) > 0$ for all $\varepsilon \in (0, c_1M_1^{-7}]$ and $x \in U_1(M_1) \cup U_2(h_1) \cup U_3(M_1, h_1, \eta_1)$.

Note that from $v^{(0)} = u$ and Proposition 5.1 we have $H(v^{(0)})(x) > 0$ for all $x \in \Omega_{M_1}$. It follows that there exist $\varepsilon_2 \in (0, \varepsilon_1]$ and $\tilde{z} \in \Omega_{M_1} \setminus (U_1(M_1) \cup U_2(h_1) \cup U_3(M_1, h_1, \eta_1))$ such that $H(v^{(\varepsilon_2)})(\tilde{z}) = 0$ and $H(v^{(\varepsilon_2)})(x) \ge 0$ for all $x \in \Omega_{M_1}$. This contradicts Theorem 1.6. □

As a direct consequence of Propositions 6.2 and 6.5 we obtain

Corollary 6.6. *Fix $C_1, R_1 > 0$, $\kappa_2 \ge \kappa_1 > 0$ and $D \in F(C_1, R_1, \kappa_1, \kappa_2)$. Denote $\Lambda = \{C_1, R_1, \kappa_1, \kappa_1\}$. Let $\varphi^{(D)}$ be the solution of (1)–(2) for D, $u^{(D)}$ the harmonic extension of $\varphi^{(D)}$ given by (6)–(10) and $v^{(\varepsilon,D)}$ given by (59). Then*

$$\exists c_1 = c_1(\Lambda) \in (0, 1] \, \exists c_2 = c_2(\Lambda) > 0 \, \exists M_0 \ge 10 \, \exists h_1 = h_1(\Lambda) \in (0, 1/2] \, \forall M \ge M_0$$
$$\forall \varepsilon \in (0, c_1M^{-7}] \, \exists \eta = \eta(\Lambda, M, \varepsilon) \in (0, (1/2) \wedge \varepsilon] \, \exists c_3 = c_3(\Lambda, M, \varepsilon) > 0$$

$$\forall x \in Q(M, D, \varepsilon) \qquad H(v^{(\varepsilon,D)})(x) \ge c_3,$$
$$\forall x \in \Omega(M, B(0, 1)) \qquad H(v^{(\varepsilon,B(0,1))})(x) \ge c_3,$$
$$\forall x \in Q_4(D) \qquad H(u^{(D)})(x) \ge c_2,$$

where (see Figure 8) $Q(M, D, \varepsilon) = Q_1(M) \cup Q_2(M, D, \varepsilon) \cup Q_3(M, D, \varepsilon)$,

$$Q_1(M) = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 \le M^2, \ x_3 = M \text{ or } x_3 = -M\}$$
$$\cup \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 = M^2, \ x_3 \in [-M, M] \setminus \{0\}\},$$
$$Q_2(M, D, \varepsilon) = \{x \in \mathbb{R}^3 : \ (x_1, x_2) \in D, \ \delta_D((x_1, x_2)) \le h_1, \ x_3 \in [-\eta, \eta]\},$$
$$Q_3(M, D, \varepsilon) = \{x \in \mathbb{R}^3 : \ (x_1, x_2) \in D^c, \ x_1^2 + x_2^2 \le M^2, \ x_3 \in [-\eta, \eta] \setminus \{0\}\},$$
$$\Omega(M, D) = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 < M^2, \ x_3 \in (-M, M)\} \setminus (D^c \times \{0\}),$$
$$Q_4(D) = \{x \in \mathbb{R}^3 : \ (x_1, x_2) \in D, \ \delta_D((x_1, x_2)) \le h, \ x_3 = 0\}.$$

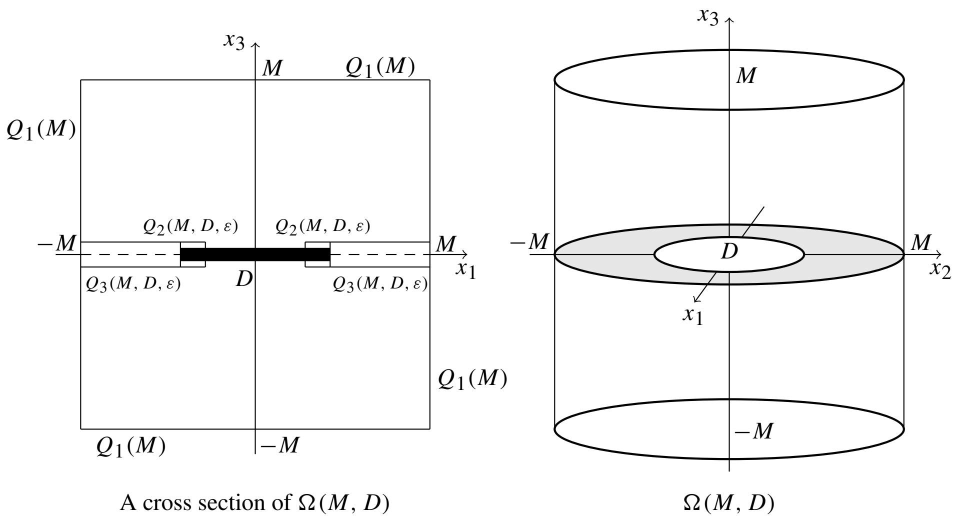


Fig. 8

Proof of Theorem 1.1.

Step 1. In this step we will use the notation from Corollary 6.6. We will show that for any $\Lambda = \{C_1, R_1, \kappa_1, \kappa_2\}$, $D \in F(\Lambda)$ and $x \in \mathbb{R}^3 \setminus (D^c \times \{0\})$ we have $H(u^{(D)})(x) > 0$.

Fix $\Lambda = \{C_1, R_1, \kappa_1, \kappa_2\}$ where $C_1, R_1 > 0$, $\kappa_2 \geq \kappa_1 > 0$ and fix $D_0 \in F(\Lambda)$. Let $\{D(t)\}_{t\in[0,1]}$ with $D(0) = D_0$ and $D(1) = B(0, 1)$ be the family of domains defined by (16). By Lemma 2.4 there exists $\Lambda' = \{C_1', R_1', \kappa_1', \kappa_2'\}$ where $C_1', R_1' > 0$, $\kappa_2' \geq \kappa_1' > 0$ such that $D(t) \in F(\Lambda')$ for all $t \in [0, 1]$. Set $v^{(\varepsilon,t)} = v^{(\varepsilon,D(t))}$.

We will apply Corollary 6.6 to $\Lambda' = \{C_1', R_1', \kappa_1', \kappa_1'\}$. Fix $M \geq M_0 \geq 10$ and $\varepsilon \in (0, c_1 M^{-7}]$. Let

$$T = \{t \in [0, 1] : H(v^{(\varepsilon,t)})(x) > 0 \text{ for all } x \in \Omega(M, D(t))\}.$$

Let $\Omega_+(M) = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 < M^2, x_3 \in (0, M)\}$ and $\Omega_-(M) = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 < M^2, x_3 \in (-M, 0)\}$. Observe that $H(v^{(\varepsilon,t)})(x) > 0$ for all $x \in \Omega(M, D(t))$ if and only if $H(v^{(\varepsilon,t)})(x) > 0$ for all $x \in \Omega_+(M)$. Indeed, if the latter inequality holds then $H(v^{(\varepsilon,t)})(x) > 0$ for all $x \in \Omega_-(M)$ by Lemma 6.1 and $H(v^{(\varepsilon,t)})(x) > 0$ for all $x \in D(t) \times \{0\}$ by Lewy's theorem. It follows that

$$T = \{t \in [0, 1] : H(v^{(\varepsilon,t)})(x) > 0 \text{ for all } x \in \Omega_+(M)\}.$$

The reason to consider $\Omega_+(M)$ instead of $\Omega(M, D(t))$ is that $\Omega_+(M)$ does not depend on t. By Corollary 6.6 we have $1 \in T$, so T is nonempty. We will show that T is both open and closed (relatively in $[0, 1]$), which implies that $T = [0, 1]$.

By Lemma 2.5 and standard arguments, $v^{(\varepsilon,t)}(x) \to v^{(\varepsilon,s)}(x)$ for $x \in \Omega_+(M)$ as $[0, 1] \ni t \to s$.

Assume that $\{t_n : n = 1, 2, \ldots\} \subset T$ and $t_n \to t_0$ as $n \to \infty$. Then $H(v^{(\varepsilon,t_0)})(x) \geq 0$ for all $x \in \Omega_+(M)$. By Corollary 6.6, $H(v^{(\varepsilon,t_0)})(x)$ does not vanish identically in $\Omega_+(M)$.

By Lewy's theorem $H(v^{(\varepsilon,t)})(x) > 0$ for all $x \in \Omega_+(M)$. Hence $t_0 \in T$, which implies that T is closed.

Now, assume on the contrary that T is not open. Then there exists $t_0 \in T$ and a sequence $\{t_n\}$ such that $[0, 1] \ni t_n \to t_0$ as $n \to \infty$ and $t_n \notin T$ for any $n = 1, 2, \dots$. Hence there exist $x_n \in \Omega_+(M)$ such that $H(v^{(\varepsilon,t_n)})(x_n) \le 0$. Taking a subsequence if necessary, we may assume that $x_n \to x_0 \in \overline{\Omega_+(M)}$ as $n \to \infty$. If $x_0 \in D(t_0)^c \times \{0\}$ then for sufficiently large n we get $x_n \in Q_2(M, D(t_n), \varepsilon) \cup Q_3(M, D(t_n), \varepsilon)$, contrary to Corollary 6.6. If $x_0 \in \Omega_+(M) \cup Q_1(M) \cup (D(t_0) \times \{0\})$ then by standard arguments $H(v^{(\varepsilon,t_n)})(x_n) \to H(v^{(\varepsilon,t_0)})(x_0) \le 0$ as $n \to \infty$. If $x_0 \in \Omega_+(M) \cup (D(t_0) \times \{0\})$ then we get a contradiction with our assumption that $t_0 \in T$. If $x_0 \in \Omega_1(M)$ we get a contradiction to Corollary 6.6. So T is open.

It follows that for any fixed $M \ge M_0 \ge 10$ and $\varepsilon \in (0, c_1 M^{-7})$ we have $H(v^{(\varepsilon,D_0)})(x) > 0$ for all $x \in \Omega(M, D_0)$. By letting $\varepsilon \to 0$ we obtain $H(u^{(D_0)})(x) \ge 0$ for all $x \in \Omega(M, D_0)$. By the estimates of $H(u^{(D_0)})$ on $Q_4(D_0)$ from Corollary 6.6 we deduce that $H(u^{(D_0)})(x)$ does not vanish near $\partial D_0 \times \{0\}$. Hence Lewy's theorem implies that $H(u^{(D_0)})(x) > 0$ for all $x \in \Omega(M, D_0)$. Since $M \ge M_0 \ge 10$ was arbitrary, we get $H(u^{(D_0)})(x) > 0$ for all $x \in \mathbb{R}^3 \setminus (D_0^c \times \{0\})$.

Step 2. We denote by sign(Hess($u(y)$)) the signature of the Hessian matrix of $u(y)$. In this step we will show that for all $\Lambda = \{C_1, R_1, \kappa_1, \kappa_1\}$, $D \in F(\Lambda)$ and $y \in \mathbb{R}^3 \setminus (D^c \times \{0\})$ we have $\operatorname{sign}(\operatorname{Hess}(u(y))) = (1, 2)$ and φ is strictly concave on D.

Fix $\Lambda = \{C_1, R_1, \kappa_1, \kappa_1\}$ where $C_1, R_1 > 0$, $\kappa_2 \ge \kappa_1 > 0$ and fix $D \in F(\Lambda)$. Let φ be the solution of (1)–(2) for D, and u the harmonic extension of φ. Let $(x_1, x_2) \in D$ and $x = (x_1, x_2, 0)$. Denote $f(x) = u_{11}(x)u_{22}(x) - u_{12}^2(x)$. By Lemma 4.7, $u_{13}(x) = u_{23}(x) = 0$ and $u_{33}(x) > 0$. By Step 1, $H(u)(x) > 0$. Hence $f(x) > 0$. We have $u_{11}(x) + u_{22}(x) + u_{33}(x) = 0$, so $u_{11}(x) + u_{22}(x) < 0$. Therefore $f(x) > 0$ implies that $u_{11}(x) < 0$ and $u_{22}(x) < 0$. Hence $\operatorname{sign}(\operatorname{Hess}(u(x))) = (1, 2)$. Since $H(u)(y) > 0$ for any $y \in \mathbb{R}^3 \setminus (D^c \times \{0\})$, we get $\operatorname{sign}(\operatorname{Hess}(u(y))) = (1, 2)$.

The inequalities $f(x) > 0$, $u_{11}(x) < 0$ and $u_{22}(x) < 0$ show that $\varphi(x_1, x_2) = u(x_1, x_2, 0)$ is strictly concave on D.

Step 3. In this step we will show that for any open bounded convex set $D \subset \mathbb{R}^2$, φ is concave on D.

Fix an open bounded convex set $D \subset B(0, 1) \subset \mathbb{R}^2$. It is well known (see e.g. [9, p. 451]) that there exists a sequence of sets D_n such that $D_n \in F(\Lambda_n)$ for some $\Lambda_n = \{C_{1,n}, R_{1,n}, \kappa_{1,n}, \kappa_{2,n}\}$ and $\bigcup_{n=1}^{\infty} D_n = D$, $D_n \subset D_{n+1}$, $n \in \mathbb{N}$, and $d(D_n, D) \to 0$ as $n \to \infty$ (where $C_{1,n}, R_{1,n} > 0$, $\kappa_{2,n} \ge \kappa_{1,n} > 0$). Let $\varphi^{(n)}$, φ denote solutions of (1)–(2) for D_n and D. By Step 2, $\varphi^{(n)}$ are concave on D_n. By Lemma 2.5 we have $\lim_{n\to\infty} \varphi^{(n)}(x) = \varphi(x)$ for $x \in D$. So φ is concave on D.

By scaling we may relax the assumption $D \subset B(0, 1)$. □

7. Extensions and conjectures

Proof of Theorem 1.5. (a) It is well known that if $\psi_r(x) = \psi(rx)$ for some $r > 0$ and all $x \in \mathbb{R}^d$ then $(-\Delta)^{\alpha/2}\psi_r(x) = r^\alpha(-\Delta)^{\alpha/2}\psi(rx)$ (see e.g. [4, p. 9]). Fix $x_0 \in \partial D$ and

$\lambda \in (0,1)$. Set $f(x) = \varphi(\lambda x + (1-\lambda)x_0) - \lambda^\alpha \varphi(x)$. We have $(-\Delta)^{\alpha/2} f(x) = 0$ for $x \in D$ and $f(x) \ge 0$ for $x \in D^c$. Hence $f(x) \ge 0$ for $x \in D$.

(b) Fix $x, y \in D$ and $\lambda \in (0,1)$. Set $z = \lambda x + (1-\lambda)y$. Let l be the line through x and y. Let $x_0 \in \partial D$ be the point on l which is closer to x than to y, and $y_0 \in \partial D$ be the point on l which is closer to y than to x. We have

$$z = y\frac{|z-x_0|}{|y-x_0|} + x_0\left(1 - \frac{|z-x_0|}{|y-x_0|}\right).$$

By (a) we get

$$\varphi(z) \ge \left(\frac{|z-x_0|}{|y-x_0|}\right)^\alpha \varphi(y) \ge \left(\frac{|z-x|}{|y-x|}\right)^\alpha \varphi(y) = (1-\lambda)^\alpha \varphi(y).$$

Moreover,

$$z = x\frac{|z-y_0|}{|x-y_0|} + y_0\left(1 - \frac{|z-y_0|}{|x-y_0|}\right).$$

Again by (a) we get

$$\varphi(z) \ge \left(\frac{|z-y_0|}{|x-y_0|}\right)^\alpha \varphi(x) \ge \left(\frac{|z-y|}{|x-y|}\right)^\alpha \varphi(x) = \lambda^\alpha \varphi(x). \qquad \square$$

Now we present some conjectures concerning solutions of (3)–(4).

Conjecture 7.1. *Let $\alpha = 1$ and $d \ge 3$. If $D \subset \mathbb{R}^d$ is a bounded convex set then the solution of* (3)–(4) *is concave on D.*

It seems that one can show this conjecture by using the generalization of H. Lewy's result obtained by S. Gleason and T. Wolff [20, Theorem 1]. Let $\alpha = 1$, $d \ge 3$ and $D \subset \mathbb{R}^d$ be a sufficiently smooth bounded convex set such that ∂D has a strictly positive curvature, φ the solution of (3)–(4) and u its harmonic extension in $\mathbb{R}^{d+1}$. It seems that using the method of continuity, as in this paper, one can show that the Hessian matrix of u has constant signature $(1, d-1)$. This implies concavity of φ on D. Anyway, Conjecture 7.1 remains an open challenging problem.

Conjecture 7.2. *Let $d \ge 2$, $D \subset \mathbb{R}^d$ be a bounded convex set and φ be the solution of* (3)–(4).

(a) *If $\alpha \in (1,2)$ then φ is $1/\alpha$-concave on D.*
(b) *If $\alpha \in (0,1)$ then φ is concave on D.*

Remark 7.3. For any $\alpha \in (1,2)$, $\eta \in (0, 1-1/\alpha)$ and $d \ge 2$ there exists a bounded convex set $D \subset \mathbb{R}^d$ (a sufficiently narrow bounded cone) such that the solution of (3)–(4) is not $1/\alpha + \eta$-concave on D.

Justification of Remarks 1.4 and 7.3. It is clear that it is sufficient to prove Remark 7.3. For any $\theta \in (0, \pi/2)$ and $d \ge 2$ let

$$D(\theta) = \{(x_1, \dots, x_d) : \sqrt{x_2^2 + \dots + x_d^2} < x_1 \tan\theta,\ |x| < 1\}.$$

Let $\alpha \in (0,2)$ and φ be the solution of (3)–(4) for $D(\theta)$.

By [29, Theorem 3.13, Lemma 3.7] for any $\varepsilon > 0$ there exist $\theta \in (0, \pi/2)$ and $c > 0$ such that

$$\varphi(x) \le c|x|^{\alpha-\varepsilon}, \qquad x \in D(\theta). \tag{71}$$

Theorem 3.13 and Lemma 3.7 in [29] are formulated only for $d \ge 3$, but small modifications of the proofs in [29] give these results also for $d = 2$. (71) for any $d \ge 2$ also follows from the recent paper [7].

Fix $d \ge 2$, $\alpha \in (1, 2)$, $\eta \in (0, 1 - 1/\alpha)$ and $\varepsilon \in \left(0, \frac{\alpha^2\eta}{1+\eta\alpha}\right)$. There exist $\theta \in (0, \pi/2)$ and $c > 0$ such that the solution φ of (3)–(4) for $D(\theta)$ satisfies $\varphi(x) \le c|x|^{\alpha-\varepsilon}$. Fix $x_0 = (a, 0, \dots, 0) \in D(\theta)$. If φ is $1/\alpha + \eta$-concave on $D(\theta)$ then for any $\lambda \in (0, 1)$ we have

$$\varphi(\lambda x_0) \ge \lambda^{\frac{\alpha}{1+\eta\alpha}} \varphi(x_0) = \lambda^{\alpha - \frac{\alpha^2\eta}{1+\eta\alpha}} \varphi(x_0).$$

On the other hand $\varphi(\lambda x_0) \le c\lambda^{\alpha-\varepsilon}|x_0|^{\alpha-\varepsilon}$, so

$$c\lambda^{\alpha-\varepsilon}|x_0|^{\alpha-\varepsilon} \ge \lambda^{\alpha - \frac{\alpha^2\eta}{1+\eta\alpha}} \varphi(x_0),$$

which gives

$$\lambda^{\frac{\alpha^2\eta}{1+\eta\alpha} - \varepsilon} \ge \varphi(x_0) c^{-1} |x_0|^{\varepsilon-\alpha}$$

for any $\lambda \in (0, 1)$, a contradiction. □

We finish this section with an open problem concerning p-concavity of the first eigenfunction for the fractional Laplacian with Dirichlet boundary condition.

Let $\alpha \in (0, 2)$, $d \ge 1$, $D \subset \mathbb{R}^d$ be a bounded open set and consider the following Dirichlet eigenvalue problem for $(-\Delta)^{\alpha/2}$:

$$(-\Delta)^{\alpha/2}\varphi_n(x) = \lambda_n \varphi_n(x), \qquad x \in D, \tag{72}$$

$$\varphi_n(x) = 0, \qquad x \in D^c. \tag{73}$$

It is well known (see e.g. [13], [27]) that there exists a sequence of eigenvalues $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$, $\lambda_n \to \infty$ and corresponding eigenfunctions $\varphi_n \in L^2(D)$. The φ_n form an orthonormal basis in $L^2(D)$, they are continuous and bounded on D, and one may assume that $\varphi_1 > 0$ on D.

Open problem. For any $\alpha \in (0, 2)$ and $d \ge 2$ find $p = p(d, \alpha) \in [-\infty, 1]$ such that for every open bounded convex set $D \subset \mathbb{R}^d$ the first eigenfunction of (72)–(73) is p-concave on D. It is not clear whether such a p exists.

Any results, even numerical, concerning this problem would be very interesting.

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