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Automorphisms of the Lie algebra of vector fields on affine n-space

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Abstract. We study the vector fields $\mathrm{Vec}(\mathbb{A}^n)$ on affine n-space $\mathbb{A}^n$, the subspace $\mathrm{Vec}^c(\mathbb{A}^n)$ of vector fields with constant divergence, and the subspace $\mathrm{Vec}^0(\mathbb{A}^n)$ of vector fields with divergence zero, and we show that their automorphisms, as Lie algebras, are induced by the automorphisms of $\mathbb{A}^n$:

$$\mathrm{Aut}(\mathbb{A}^n) \xrightarrow{\sim} \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}(\mathbb{A}^n)) \xrightarrow{\sim} \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}^c(\mathbb{A}^n)) \xrightarrow{\sim} \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}^0(\mathbb{A}^n)).$$

This generalizes results of the second author obtained in dimension 2 [Reg13]. The case of $\mathrm{Vec}(\mathbb{A}^n)$ goes back to Kulikov [Kul92].

This generalization is crucial in the context of infinite-dimensional algebraic groups, because $\mathrm{Vec}^c(\mathbb{A}^n)$ is canonically isomorphic to the Lie algebra of $\mathrm{Aut}(\mathbb{A}^n)$, and $\mathrm{Vec}^0(\mathbb{A}^n)$ is isomorphic to the Lie algebra of the closed subgroup $\mathrm{SAut}(\mathbb{A}^n) \subset \mathrm{Aut}(\mathbb{A}^n)$ of automorphisms with Jacobian determinant equal to 1.

Keywords. Automorphisms, vector fields, Lie algebras, affine n-space

1. Introduction

Let K be an algebraically closed field of characteristic zero. Denote by $\mathrm{Vec}(\mathbb{A}^n)$ the Lie algebra of polynomial vector fields on affine n-space $\mathbb{A}^n = K^n$:

$$\mathrm{Vec}(\mathbb{A}^n) = \mathrm{Der}(K[x_1, \dots, x_n]) = \left\{ \sum_i f_i \frac{\partial}{\partial x_i} \;\middle|\; f_i \in K[x_1, \dots, x_n] \right\}$$

where we use the standard identification of a derivation δ with $\sum_i \delta(x_i) \frac{\partial}{\partial x_i}$. The group $\mathrm{Aut}(\mathbb{A}^n)$ of polynomial automorphisms of $\mathbb{A}^n$ acts on $\mathrm{Vec}(\mathbb{A}^n)$ in the usual way. For $\varphi \in \mathrm{Aut}(\mathbb{A}^n)$ and $\delta \in \mathrm{Vec}(\mathbb{A}^n) = \mathrm{Der}(K[x_1, \dots, x_n])$ we define

$$\mathrm{Ad}(\varphi)\delta := \varphi^{*-1} \circ \delta \circ \varphi^*$$

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where $\varphi^*\colon K[x_1,\dots,x_n] \to K[x_1,\dots,x_n]$, $f \mapsto f\circ\varphi$, is the comorphism of φ. It is shown in [Kul92] that $\mathrm{Ad}\colon \mathrm{Aut}(\mathbb{A}^n) \to \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}(\mathbb{A}^n))$ is an isomorphism. We will give a short proof in Section 3.

Recall that the *divergence* of a vector field $\delta = \sum_i f_i \frac{\partial}{\partial x_i}$ is defined by $\mathrm{Div}\,\delta := \sum_i \frac{\partial f_i}{\partial x_i}$. This leads to the following subspaces of $\mathrm{Vec}(\mathbb{A}^n)$:

$$\mathrm{Vec}^0(\mathbb{A}^n) := \{\delta \in \mathrm{Vec}(\mathbb{A}^n) \mid \mathrm{Div}\,\delta = 0\} \subset \mathrm{Vec}^c(\mathbb{A}^n) := \{\delta \in \mathrm{Vec}(\mathbb{A}^n) \mid \mathrm{Div}\,\delta \in K\},$$

which are Lie subalgebras, because $\mathrm{Div}[\delta,\eta] = \delta(\mathrm{Div}\,\eta) - \eta(\mathrm{Div}\,\delta)$. We have

$$\mathrm{Vec}^c(\mathbb{A}^n) = \mathrm{Vec}^0(\mathbb{A}^n) \oplus K\,\partial_E \quad \text{where} \quad \partial_E := \sum_i x_i \frac{\partial}{\partial x_i} \text{ is the } \textit{Euler field}.$$

The aim of this note is to prove the following result about the automorphism groups of these Lie algebras.

Main Theorem. *There are canonical isomorphisms*

$$\mathrm{Aut}(\mathbb{A}^n) \xrightarrow{\sim} \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}(\mathbb{A}^n)) \xrightarrow{\sim} \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}^c(\mathbb{A}^n)) \xrightarrow{\sim} \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}^0(\mathbb{A}^n)).$$

Remark 1.1. It is easy to see that the theorem holds for any field K of characteristic zero. In fact, all the homomorphisms are defined over $\mathbb{Q}$, and are equivariant with respect to the obvious actions of the Galois group $\Gamma = \mathrm{Gal}(\bar K/K)$.

As a consequence, we will get the next result (see Corollary 4.4) which goes back to Kulikov [Kul92, Theorem 4].

Corollary. *If every injective endomorphism of the Lie algebra* $\mathrm{Vec}(\mathbb{A}^n)$ *is an automorphism, then the Jacobian Conjecture holds in dimension n.*

Remark 1.2. The Main Theorem has another interesting consequence. The group $\mathrm{Aut}(\mathbb{A}^n)$ is an *infinite-dimensional algebraic group* in the sense of Shafarevich [Sha66, Sha81], briefly an *ind-group* (cf. [Kum02]), and its Lie algebra is canonically isomorphic to $\mathrm{Vec}^c(\mathbb{A}^n)$. It was recently shown by Belov-Kanel and Yu [BKY12] that every automorphism of $\mathrm{Aut}(\mathbb{A}^n)$ as an ind-group is inner. Using the Main Theorem above one can give a new proof of this and extend it to the closed subgroup $\mathrm{SAut}(\mathbb{A}^n) \subset \mathrm{Aut}(\mathbb{A}^n)$ of automorphisms with Jacobian determinant equal to 1. The details can be found in [Kra15] where we also show that the maps in the Main Theorem are isomorphisms of ind-groups.

We add here a lemma which will be used later on.

Lemma 1.3. $\mathrm{Vec}(\mathbb{A}^n)$ *and* $\mathrm{Vec}^0(\mathbb{A}^n)$ *are simple Lie algebras, and*

$$\mathrm{Vec}^0(\mathbb{A}^n) = [\mathrm{Vec}^c(\mathbb{A}^n), \mathrm{Vec}^c(\mathbb{A}^n)].$$

Proof. The formula $\left[\frac{\partial}{\partial x_j}, \sum_i f_i \frac{\partial}{\partial x_i}\right] = \sum_i \frac{\partial f_i}{\partial x_j}\frac{\partial}{\partial x_i}$ shows that every nonzero ideal $\mathfrak{a}$ of $\mathrm{Vec}(\mathbb{A}^n)$ contains a nonzero element from $\sum_i K\frac{\partial}{\partial x_i}$, and $\left[x_\ell \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i}\right] = -\delta_{i\ell}\frac{\partial}{\partial x_j}$ implies that $\sum_i K\frac{\partial}{\partial x_i} \subseteq \mathfrak{a}$. Now we use $\left[f\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i}\right] = -\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j}$ to conclude that $\mathfrak{a} = \mathrm{Vec}(\mathbb{A}^n)$, hence $\mathrm{Vec}(\mathbb{A}^n)$ is simple. (See also [Jor78, Theorem, p. 446].)

The second statement is proved in a similar way and can be found in [Sha81, Lemma 3], and from that the last claim follows immediately. □

2. Group actions and vector fields

If an algebraic group G acts on an affine variety X, we obtain a canonical linear map $\operatorname{Lie} G \to \operatorname{Vec}(X)$ defined in the usual way (cf. [Kra11, II.4.4]). For every $A \in \operatorname{Lie} G$ the associated vector field ξ_A on X is defined by

$$(\xi_A)_x := d\mu_x(A) \quad \text{for } x \in X \tag{2.1}$$

where $\mu_x \colon G \to X$, $g \mapsto gx$, is the orbit map in $x \in X$. It is well-known that the linear map $A \mapsto \xi_A$ is an anti-homomorphism of Lie algebras, and that its kernel is equal to the Lie algebra of the kernel of the action $G \to \operatorname{Aut}(X)$. In particular, for any algebraic subgroup $G \subset \operatorname{Aut}(\mathbb{A}^n)$ we have a canonical injection $\operatorname{Lie} G \hookrightarrow \operatorname{Vec}(\mathbb{A}^n)$; we will denote the image by $L(G)$. Let us point out that a connected $G \subset \operatorname{Aut}(\mathbb{A}^n)$ is determined by $L(G)$, i.e., if $L(G) = L(H)$ for algebraic subgroups $G, H \subset \operatorname{Aut}(\mathbb{A}^n)$, then $G^0 = H^0$.

Recall that the vector field $\delta \in \operatorname{Vec}(\mathbb{A}^n)$ is called *locally nilpotent* if the action of δ on $K[x_1, \ldots, x_n]$ is locally nilpotent, i.e., for any $f \in K[x_1, \ldots, x_n]$ we have $\delta^m(f) = 0$ if m is large enough. Every such δ defines an action of the additive group K^+ on $\mathbb{A}^n$ such that $\delta = \xi_1$ where $1 \in K = \operatorname{Lie} K^+$ (see (2.1) above).

Lemma 2.1. *Let* $\mathbf{u} \subset \operatorname{Vec}(\mathbb{A}^n)$ *be a finite-dimensional commutative Lie subalgebra consisting of locally nilpotent vector fields. Then there is a commutative unipotent algebraic subgroup* $U \subset \operatorname{Aut}(\mathbb{A}^n)$ *such that* $L(U) = \mathbf{u}$*. If* $\mathfrak{cent}_{\operatorname{Vec}(\mathbb{A}^n)}(\mathbf{u}) = \mathbf{u}$*, then* U *acts transitively on* $\mathbb{A}^n$*.*

Proof. It is clear that $\mathbf{u} = L(U)$ for a commutative unipotent subgroup $U \subset \operatorname{Aut}(\mathbb{A}^n)$. In fact, choose a basis $(\delta_1, \ldots, \delta_m)$ of $\mathbf{u}$ and consider the corresponding actions $\rho_i \colon K^+ \to \operatorname{Aut}(\mathbb{A}^n)$. Since the associated vector fields δ_i commute, the same holds for the actions ρ_i, so that we get an action of $(K^+)^m$. It follows that the image $U \subset \operatorname{Aut}(\mathbb{A}^n)$ is a commutative unipotent subgroup with $L(U) = \mathbf{u}$.

Assume that the action of U is not transitive. Then all orbits have dimension $< n$, because orbits of unipotent groups acting on affine varieties are closed (see [Bor91, Chap. I, Proposition 4.10]). But then there is a nonconstant U-invariant function $f \in K[x_1, \ldots, x_n]$. This implies that for every $\delta \in \mathbf{u}$ the vector field $f\delta$ commutes with $\mathbf{u}$ and thus belongs to $\mathfrak{cent}_{\operatorname{Vec}(\mathbb{A}^n)}(\mathbf{u})$, contradicting the assumption. □

Any $\delta \in \operatorname{Vec}(\mathbb{A}^n)$ acts on the functions $K[x_1, \ldots, x_n]$ as a derivation, and on the Lie algebra $\operatorname{Vec}(\mathbb{A}^n)$ by the adjoint action, $\operatorname{ad}(\delta)\mu := [\delta, \mu] = \delta \circ \mu - \mu \circ \delta$. These two actions are related as shown in the following lemma whose proof is obvious.

Lemma 2.2. *Let* $\delta, \mu \in \operatorname{Vec}(\mathbb{A}^n)$ *be commuting vector fields and* $f \in K[x_1, \ldots, x_n]$*. Then*

$$\operatorname{ad}(\delta)(f\mu) = \delta(f)\mu.$$

In particular, if $\operatorname{ad}(\delta)$ *is locally nilpotent on* $\operatorname{Vec}(\mathbb{A}^n)$*, then* δ *is locally nilpotent as a vector field.*

3. Proof of the Main Theorem, part I

We first give a proof of the following result which goes back to Kulikov [Kul92, proof of Theorem 4]; see also [Bav13].

Theorem 3.1. *The canonical map* $\mathrm{Ad}\colon \mathrm{Aut}(\mathbb{A}^n) \to \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}(\mathbb{A}^n))$ *is an isomorphism.*

Denote by $\mathrm{Aff}_n \subset \mathrm{Aut}(\mathbb{A}^n)$ the closed subgroup of affine transformations and by $S = (K^+)^n \subset \mathrm{Aff}_n$ the subgroup of translations. Then

$$L(\mathrm{Aff}_n) = \langle x_i \partial_{x_j}, \partial_{x_k} \mid 1 \leq i, j, k \leq n \rangle \supset L(S) = \langle \partial_{x_1}, \ldots, \partial_{x_n} \rangle \tag{3.1}$$

where $\partial_{x_j} := \partial/\partial x_j$. Set $\mathfrak{aff}_n := \mathrm{Lie}\,\mathrm{Aff}_n$ and $\mathfrak{saff}_n := [\mathfrak{aff}_n, \mathfrak{aff}_n] = \mathrm{Lie}\,\mathrm{SAff}_n$ where $\mathrm{SAff}_n := (\mathrm{Aff}_n, \mathrm{Aff}_n) \subset \mathrm{Aff}_n$ is the commutator subgroup, i.e. the closed subgroup of those affine transformations $x \mapsto gx + b$ where $g \in \mathrm{SL}_n$. The next lemma is certainly known. For the convenience of the reader we indicate a short proof.

Lemma 3.2. *The canonical homomorphisms*

$$\mathrm{Aff}_n \xrightarrow[\simeq]{\mathrm{Ad}} \mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{aff}_n) \xrightarrow[\simeq]{\mathrm{res}} \mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{saff}_n)$$

are isomorphisms.

Proof. It is clear that the homomorphisms

$$\mathrm{Ad}\colon \mathrm{Aff}_n \to \mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{aff}_n) \quad \text{and} \quad \mathrm{res}\colon \mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{aff}_n) \to \mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{saff}_n)$$

are both injective. Thus it suffices to show that the composition $\mathrm{res} \circ \mathrm{Ad}$ is surjective.

We write the elements of Aff_n in the form (v, g) with $v \in S = (K^+)^n$, $g \in \mathrm{GL}_n$ where $(v, g)x = gx + v$ for $x \in \mathbb{A}^n$. It follows that $(v, g)(w, h) = (v + gw, gh)$. Similarly, $(a, A) \in \mathfrak{aff}_n$ means that $a \in \mathfrak{s} := \mathrm{Lie}\, S = K^n$, $A \in \mathfrak{gl}_n$, and $(a, A)x = Ax + a$. For the adjoint representation of $g \in \mathrm{GL}_n$ and of $v \in S$ on $\mathfrak{aff}_n$ we find

$$\mathrm{Ad}(g)(a, A) = (ga, gAg^{-1}) \quad \text{and} \quad \mathrm{Ad}(v)(a, A) = (a - Av, A), \tag{3.2}$$

and thus, for $(b, B) \in \mathfrak{aff}_n$,

$$\mathrm{ad}(B)(a, A) = (Ba, [B, A]) \quad \text{and} \quad \mathrm{ad}(b)(a, A) = (a - Ab, A). \tag{3.3}$$

Now let θ be an automorphism of the Lie algebra $\mathfrak{saff}_n$. Then $\theta(\mathfrak{s}) = \mathfrak{s}$ since $\mathfrak{s}$ is the solvable radical of $\mathfrak{saff}_n$. Since $g := \theta|_{\mathfrak{s}} \in \mathrm{GL}_n$, we can replace θ by $\mathrm{Ad}(g^{-1}) \circ \theta$ and thus assume, by (3.2), that θ is the identity on $\mathfrak{s}$. This implies that $\theta(a, A) = (a + \ell(A), \bar{\theta}(A))$ where $\ell\colon \mathfrak{sl}_n \to \mathfrak{s}$ is a linear map and $\bar{\theta}\colon \mathfrak{sl}_n \xrightarrow{\sim} \mathfrak{sl}_n$ is a Lie algebra automorphism.

From (3.3) we get $\mathrm{ad}(b, B)(a, 0) = \mathrm{ad}(B)(a, 0) = (Ba, 0)$ for all $a \in \mathfrak{s}$, hence

$$\begin{aligned}(Ba, 0) = \theta(Ba, 0) &= \theta(\mathrm{ad}(B)(a, 0)) \\ &= \mathrm{ad}(\theta(B))(a, 0) = \mathrm{ad}(\bar{\theta}(B))(a, 0) = (\bar{\theta}(B)a, 0).\end{aligned}$$

Thus $\bar{\theta}(B) = B$, i.e. $\theta(a, A) = (a + \ell(A), A)$. Now an easy calculation shows that $\ell([A, B]) = A\ell(B) - B\ell(A)$. This means that ℓ is a cocycle of $\mathfrak{sl}_n$. Since $\mathfrak{sl}_n$ is semisimple, ℓ is a coboundary, and thus $\ell(A) = Av$ for a suitable $v \in K^n$. In view of (3.3) this implies that $\theta = \mathrm{Ad}(-v)$, and the claim follows. □

Proof of Theorem 3.1. It is clear that the homomorphism

$$\mathrm{Ad}\colon \mathrm{Aut}(\mathbb{A}^n) \to \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}(\mathbb{A}^n))$$

is injective. So let $\theta \in \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}(\mathbb{A}^n))$ be an arbitrary automorphism.

We have seen above that $L(S) = \langle \partial_{x_1}, \dots, \partial_{x_n} \rangle \subset \mathrm{Vec}(\mathbb{A}^n)$ where $S \subset \mathrm{Aff}_n$ is the subgroup of translations. Clearly, for every $\delta \in L(S)$ the adjoint action $\mathrm{ad}(\delta)$ on $\mathrm{Vec}(\mathbb{A}^n)$ is locally nilpotent, and the same holds for any element from $\mathbf{u} := \theta(L(S))$. It follows from Lemma 2.2 that $\mathbf{u}$ consists of locally nilpotent vector fields. Hence, by Lemma 2.1, $\mathbf{u} = L(U)$ for a commutative unipotent subgroup U of dimension n. Moreover, $\mathfrak{cent}_{\mathrm{Vec}(\mathbb{A}^n)}(L(S)) = L(S)$, and so $\mathfrak{cent}_{\mathrm{Vec}(\mathbb{A}^n)}(\mathbf{u}) = \mathbf{u}$, which implies, again by Lemma 2.1, that U acts transitively on $\mathbb{A}^n$. Thus every orbit map $U \to \mathbb{A}^n$ is an isomorphism. It follows that there is an automorphism $\varphi \in \mathrm{Aut}(\mathbb{A}^n)$ such that $\varphi U \varphi^{-1} = S$. In fact, fix a group isomorphism $\psi\colon U \xrightarrow{\sim} S$ and take the orbit maps $\mu_S\colon S \xrightarrow{\sim} \mathbb{A}^n$ and $\mu_U\colon U \xrightarrow{\sim} \mathbb{A}^n$ at the origin $0 \in \mathbb{A}^n$. Then one easily sees that $\varphi := \mu_S \circ \psi \circ \mu_U^{-1}$ has the property that $\varphi \circ u \circ \varphi^{-1} = \psi(u)$ for all $u \in U$.

It follows that the automorphism $\theta' := \mathrm{Ad}(\varphi) \circ \theta \in \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}(\mathbb{A}^n))$ sends $L(S)$ isomorphically onto itself. Now the relations $[\partial_{x_i}, x_j \partial_{x_k}] = \delta_{ij} \partial_{x_k}$ imply that $\theta'(L(\mathrm{Aff}_n)) = L(\mathrm{Aff}_n)$. By Lemma 3.2, there is an $\alpha \in \mathrm{Aff}_n$ such that $\mathrm{Ad}(\alpha) \circ \theta'$ is the identity on $L(\mathrm{Aff}_n)$. Hence, by the next lemma, $\mathrm{Ad}(\alpha) \circ \theta' = \mathrm{id}$, because $\mathrm{Ad}(\lambda E)$ acts by multiplication with λ on $L(S)$, and so $\theta = \mathrm{Ad}(\varphi^{-1} \circ \alpha^{-1})$. □

Lemma 3.3. *Let θ be an injective endomorphism of one of the Lie algebras $\mathrm{Vec}(\mathbb{A}^n)$, $\mathrm{Vec}^c(\mathbb{A}^n)$ or $\mathrm{Vec}^0(\mathbb{A}^n)$. If θ is the identity on $L(\mathrm{SL}_n)$, then $\theta = \mathrm{Ad}(\lambda E)$ for some $\lambda \in K^*$.*

Proof. We consider the action of GL_n on $\mathrm{Vec}(\mathbb{A}^n)$. Denote by $\mathrm{Vec}(\mathbb{A}^n)_d$ the homogeneous vector fields of degree d, i.e.

$$\mathrm{Vec}(\mathbb{A}^n)_d := \bigoplus_i K[x_1, \dots, x_n]_{d+1}\, \partial_{x_i} \simeq K[x_1, \dots, x_n]_{d+1} \otimes K^n.$$

Note that $\lambda E \in \mathrm{GL}_n$ acts by scalar multiplication with λ^{-d} on $\mathrm{Vec}(\mathbb{A}^n)_d$. We have split exact sequences of GL_n-modules

$$0 \to \mathrm{Vec}^0(\mathbb{A}^n)_d \to \mathrm{Vec}(\mathbb{A}^n)_d \xrightarrow{\mathrm{Div}} K[x_1, \dots, x_n]_d \to 0 \tag{3.4}$$

where $K[x_1, \dots, x_n]_{-1} = (0)$. Moreover, the SL_n-modules $\mathrm{Vec}^0(\mathbb{A}^n)_d$ (for $d \geq -1$) and $K[x_1, \dots, x_n]_d$ (for $d \geq 0$) are simple and pairwise nonisomorphic (see Pieri's formula [Pro07, Chap. 9, Section 10.2]). The splitting of (3.4) is given by $K[x_1, \dots, x_n]_d \partial_E \subset \mathrm{Vec}(\mathbb{A}^n)_d$ where $\partial_E = x_1 \partial_{x_1} + \cdots + x_n \partial_{x_n}$ is the Euler field. In fact, the Euler field is fixed under GL_n and $\mathrm{Div}(f \partial_E) = (d+1) f$ for $f \in K[x_1, \dots, x_n]_d$.

Now let θ be an injective endomorphism of $\mathrm{Vec}(\mathbb{A}^n)$. If θ is the identity on $L(\mathrm{SL}_n)$, then θ is SL_n-equivariant and thus acts as a scalar λ_d on $\mathrm{Vec}^0(\mathbb{A}^n)_d$ and as a scalar μ_d on $K[x_1,\dots,x_n]_d\partial_E$, by Schur's Lemma. The relations

$$[x_j^{e+1}\partial_{x_i}, x_i^{d+1}\partial_{x_j}] = (d+1)x_i^d x_j^{e+1}\partial_{x_j} - (e+1)x_i^{d+1}x_j^e\partial_{x_i}, \qquad i \neq j,$$

show that $\lambda_e\lambda_d = \lambda_{e+d}$, hence $\lambda_d = \lambda^d$ for $\lambda := \lambda_1$. The relations

$$[x_i^e\partial_E, x_i^d\partial_E] = (d-e)x_i^{e+d}\partial_E$$

show that $\mu_e\mu_d = \mu_{e+d}$ for $e \neq d$, which also implies that $\mu_d = \mu^d$ for $\mu := \mu_1$. Finally, from the relation $[\partial_{x_1}, x_2\partial_E] = x_2\partial_{x_1}$, we get $\lambda = \mu$, and so $\theta = \mathrm{Ad}(\lambda^{-1}\,\mathrm{id})$. This proves the claim for $\mathrm{Vec}(\mathbb{A}^n)$. The other two cases follow along the same lines. ◻

4. Étale morphisms and vector fields

In the first section we defined the action of $\mathrm{Aut}(\mathbb{A}^n)$ on the vector fields $\mathrm{Vec}(\mathbb{A}^n)$ by the formula $\mathrm{Ad}(\varphi)\delta := \varphi^{*-1}\circ\delta\circ\varphi^*$. In more geometric terms, considering δ as a section of the tangent bundle $T\mathbb{A}^n = \mathbb{A}^n \times K^n \to \mathbb{A}^n$, one defines the pull-back of δ by

$$\varphi^*(\delta) := (d\varphi)^{-1}\circ\delta\circ\varphi, \quad \text{i.e.,} \quad \varphi^*(\delta)_a = (d\varphi_a)^{-1}(\delta_{\varphi(a)}) \quad \text{for } a \in \mathbb{A}^n.$$

Clearly, $\varphi^*(\delta) = \mathrm{Ad}(\varphi^{-1})\delta$. However, the second formula above shows the well-known fact that the pull-back $\varphi^*(\delta)$ of a vector field δ is also defined for an étale morphism $\varphi\colon \mathbb{A}^n \to \mathbb{A}^n$. In the holomorphic setting this can be understood as lifting the corresponding integral curves.

Proposition 4.1. *Let $\varphi\colon \mathbb{A}^n \to \mathbb{A}^n$ be an étale morphism. For any vector field $\delta \in \mathrm{Vec}(\mathbb{A}^n)$ there is a vector field $\varphi^*(\delta) \in \mathrm{Vec}(\mathbb{A}^n)$ defined by $\varphi^*(\delta)_a := (d\varphi)_a^{-1}\,\delta_{\varphi(a)}$ for $a \in \mathbb{A}^n$. It is uniquely determined by*

$$\varphi^*(\delta)\varphi^*(f) = \varphi^*(\delta f) \quad \textit{for } f \in K[x_1,\dots,x_n]. \tag{4.1}$$

The map $\varphi^*\colon \mathrm{Vec}(\mathbb{A}^n) \to \mathrm{Vec}(\mathbb{A}^n)$ is an injective homomorphism of Lie algebras satisfying $\varphi^*(h\,\delta) = \varphi^*(h)\varphi^*(\delta)$ for $h \in K[x_1,\dots,x_n]$. Moreover, $(\eta\circ\varphi)^* = \varphi^*\circ\eta^*$.

Proof. For a vector field $\delta\colon \mathbb{A}^n \to T\mathbb{A}^n$ and $a \in \mathbb{A}^n$ we have $(d\varphi\circ\delta)_a = d\varphi_a(\delta_a)$. Thus, the equation $(d\varphi)_a(\tilde\delta_a) = (\tilde\delta\circ\varphi)_a = \tilde\delta_{\varphi(a)}$ for the field $\tilde\delta$ has a unique solution, namely

$$\tilde\delta_a := (d\varphi_a)^{-1}(\delta_{\varphi(a)}),$$

which is well-defined since $d\varphi_a$ is invertible. The Jacobian determinant $\det(\mathrm{Jac}(\varphi))$ is a nonzero constant, and so the inverse matrix $\mathrm{Jac}(\varphi)^{-1}$ has entries in $K[x_1,\dots,x_n]$. Therefore, the vector field $\varphi^*(\delta) := \tilde\delta$ defined above is polynomial, and it satisfies (4.1). This proves the first part of the proposition and shows that φ^* is injective. Using (4.1) we find

$$\varphi^*((\delta_1\delta_2)f) = \varphi^*(\delta_1(\delta_2 f)) = \varphi^*(\delta_1)\varphi^*(\delta_2 f) = (\varphi^*(\delta_1)\varphi^*(\delta_2))\varphi^*(f),$$

hence $\varphi^*([\delta_1,\delta_2]f) = [\varphi^*(\delta_1),\varphi^*(\delta_2)]\varphi^*(f)$, and so $\varphi^*([\delta_1,\delta_2]) = [\varphi^*(\delta_1),\varphi^*(\delta_2)]$. Moreover,

$$\varphi^*(h\delta)\varphi^*(f) = \varphi^*((h\delta)f) = \varphi^*(h)\varphi^*(\delta f) = \varphi^*(h)\varphi^*(\delta)\varphi^*(f),$$

hence $\varphi^*(h\delta) = \varphi^*(h)\varphi^*(\delta)$. This proves the second part of the proposition, and the last claim is obvious. □

Remark 4.2. In the notation of the proposition above let $\varphi = (f_1,\dots,f_n)$. Then we get $\varphi^*(\delta x_i) = \varphi^*(\delta)f_i = \sum_j \frac{\partial f_i}{\partial x_j}\varphi^*(\delta)x_j$. Hence, for $\delta = \partial_{x_k}$, we obtain

$$\delta_{ik} = \varphi^*(\partial_{x_k})f_i = \sum_j \frac{\partial f_i}{\partial x_j}\varphi^*(\partial_{x_k})x_j.$$

This shows that the matrix $(\varphi^*(\partial_{x_k})x_j)_{(j,k)}$ is invertible, $(\varphi^*(\partial_{x_k})x_j)^{-1}_{(j,k)} = \mathrm{Jac}(\varphi)$, and

$$\partial_{x_i} = \sum_j \frac{\partial f_i}{\partial x_j}\,\varphi^*(\partial_{x_j}). \tag{4.2}$$

Proposition 4.3. *Let $\varphi\colon \mathbb{A}^n \to \mathbb{A}^n$ be an étale morphism. Then the pull-back map*

$$\varphi^*\colon\ \mathrm{Vec}(\mathbb{A}^n) \to \mathrm{Vec}(\mathbb{A}^n)$$

is an isomorphism if and only if φ is an automorphism.

Proof. Assume that $\varphi^*\colon \mathrm{Vec}(\mathbb{A}^n) \to \mathrm{Vec}(\mathbb{A}^n)$ is an isomorphism. Since φ is étale, the comorphism $\varphi^*\colon K[x_1,\dots,x_n] \to K[x_1,\dots,x_n]$ is injective, and we only have to show that it is surjective. Proposition 4.1 implies that $\varphi^*(\mathrm{Vec}(\mathbb{A}^n)) = \sum_i \varphi^*(K[x_1,\dots,x_n])\varphi^*(\partial_{x_i})$, and from (4.2) we get

$$\mathrm{Vec}(\mathbb{A}^n) = \bigoplus_i K[x_1,\dots,x_n]\partial_{x_i} = \bigoplus_i K[x_1,\dots,x_n]\varphi^*(\partial_{x_i}).$$

Hence $\varphi^*(\mathrm{Vec}(\mathbb{A}^n)) = \mathrm{Vec}(\mathbb{A}^n)$ if and only if $\varphi^*(K[x_1,\dots,x_n]) = K[x_1,\dots,x_n]$. □

As a corollary of the two propositions above, we get the following result due to Kulikov [Kul92, Theorem 4].

Corollary 4.4. *If every injective endomorphism of the Lie algebra* $\mathrm{Vec}(\mathbb{A}^n)$ *is an automorphism, then the Jacobian Conjecture holds in dimension n.*

Remark 4.5. The result of Kulikov is stronger. He proves that every injective endomorphism of $\mathrm{Vec}(\mathbb{A}^n)$ is induced by an étale map φ, which also implies the converse of the statement above: *If the Jacobian Conjecture holds in dimension n, then every injective endomorphism of* $\mathrm{Vec}(\mathbb{A}^n)$ *is an automorphism.*

We finish this section by showing that if the divergence of a vector field is a constant, then the divergence is invariant under an étale morphism. More generally, we have the following result.

Proposition 4.6. *Let $\varphi\colon \mathbb{A}^n \to \mathbb{A}^n$ be an étale morphism, and let δ be a vector field. Then* $\operatorname{Div}\varphi^*(\delta) = \varphi^*(\operatorname{Div}\delta)$. *In particular,* $\delta \in \operatorname{Vec}^c(\mathbb{A}^n)$ *if and only if* $\varphi^*(\delta) \in \operatorname{Vec}^c(\mathbb{A}^n)$*, and in this case we have* $\operatorname{Div}\varphi^*(\delta) = \operatorname{Div}\delta$.

Proof. Set $\varphi = (f_1, \dots, f_n)$, $\delta = \sum_j h_j \partial_{x_j}$ and $\varphi^*(\delta) = \sum_j \tilde{h}_j \partial_{x_j}$. Then, by (4.1),

$$h_k(f_1, \dots, f_n) = \sum_i \tilde{h}_i \frac{\partial f_k}{\partial x_i} \quad \text{for } k = 1, \dots, n.$$

Applying $\frac{\partial}{\partial x_j}$ to the left hand side we get the matrix

$$\left(\sum_i \frac{\partial h_k}{\partial x_i}(f_1, \dots, f_n)\frac{\partial f_i}{\partial x_j}\right)_{(k,j)} = H(f_1, \dots, f_n) \cdot \operatorname{Jac}(\varphi)$$

where $H := \operatorname{Jac}(h_1, \dots, h_n)$. On the right hand side, we obtain similarly

$$\left(\sum_i \frac{\partial \tilde{h}_i}{\partial x_j}\frac{\partial f_k}{\partial x_i} + \sum_i \tilde{h}_i \frac{\partial^2 f_k}{\partial x_i \partial x_j}\right)_{(k,j)} = \tilde{H} \cdot \operatorname{Jac}(\varphi) + \sum_i \tilde{h}_i \frac{\partial}{\partial x_i}\operatorname{Jac}(\varphi).$$

Multiplying this matrix equation on the right by $\operatorname{Jac}(\varphi)^{-1}$ we finally get

$$H(f_1, \dots, f_n) = \tilde{H} + \sum_i \tilde{h}_i \frac{\partial}{\partial x_i}\operatorname{Jac}(\varphi) \cdot \operatorname{Jac}(\varphi)^{-1}.$$

Now we take traces on both sides. Using Lemma 4.7 below and the obvious equalities $\operatorname{Div}\delta = \operatorname{tr} H$ and $\operatorname{Div}\tilde{\delta} = \operatorname{tr}\tilde{H}$, we finally get

$$\operatorname{Div}\tilde{\delta} = (\operatorname{Div}\delta)(f_1, \dots, f_n) = \varphi^*(\operatorname{Div}\delta).$$

The claim follows. □

Lemma 4.7. *Let A be an $n \times n$ matrix whose entries $a_{ij}(t)$ are polynomials in t. Then*

$$\operatorname{tr}\left(\frac{d}{dt}A \cdot \operatorname{Adj}(A)\right) = \frac{d}{dt}\det A$$

where $\operatorname{Adj}(A)$ *is the adjoint matrix of A.*

The proof is a nice exercise in linear algebra which we leave to the reader. It holds for rational entries $a_{ij}(t)$ over any field K, and in case $K = \mathbb{R}$ or $\mathbb{C}$ also for differentiable entries $a_{ij}(t)$.

5. Proof of the Main Theorem, part II

We have seen that the canonical map $\mathrm{Ad}\colon \mathrm{Aut}(\mathbb{A}^n) \to \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}(\mathbb{A}^n))$ is an isomorphism (Theorem 3.1). It follows from Proposition 4.6 that every automorphism of $\mathrm{Vec}(\mathbb{A}^n)$ induces an automorphism of $\mathrm{Vec}^c(\mathbb{A}^n)$. Moreover, since

$$\mathrm{Vec}^0(\mathbb{A}^n) = [\mathrm{Vec}^c(\mathbb{A}^n), \mathrm{Vec}^c(\mathbb{A}^n)]$$

(Lemma 1.3), we get a canonical map $\mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}^c(\mathbb{A}^n)) \to \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}^0(\mathbb{A}^n))$, which is easily seen to be injective. Thus the main theorem follows from the next result.

Theorem 5.1. *The canonical map* $\mathrm{Ad}\colon \mathrm{Aut}(\mathbb{A}^n) \to \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}^0(\mathbb{A}^n))$ *is an isomorphism.*

The proof needs some preparation. The next proposition is a reformulation of some results from [Now86] and [LD12]. For the convenience of the reader we will give a short proof.

Proposition 5.2. *Let* $\delta_1, \dots, \delta_n \in \mathrm{Vec}(\mathbb{A}^n)$ *be pairwise commuting and* K*-linearly independent vector fields. Then the following statements are equivalent:*

(i) *There is an étale morphism* $\varphi\colon \mathbb{A}^n \to \mathbb{A}^n$ *such that* $\varphi^*(\partial_{x_i}) = \delta_i$ *for all* i.
(ii) $\mathrm{Vec}(\mathbb{A}^n) = \bigoplus_i K[x_1, \dots, x_n]\delta_i$.
(iii) *There exist* $f_1, \dots, f_n \in K[x_1, \dots, x_n]$ *such that* $\delta_i(f_j) = \delta_{ij}$.
(iv) $\delta_1, \dots, \delta_n$ *do not have a common Darboux polynomial.*

Recall that a *common Darboux polynomial* of the δ_i is a nonconstant polynomial $f \in K[x_1, \dots, x_n]$ such that $\delta_i(f) = h_i f$ for some $h_i \in K[x_1, \dots, x_n]$, $i = 1, \dots, n$.

Proof. (a) It follows from Remark 4.2 that (i) implies (ii) and (iii). Clearly, (ii) implies (iv) since a common Darboux polynomial for the δ_i is also a common Darboux polynomial for the ∂_{x_i}, which does not exist.

(b) We now show that (ii) implies (i), hence (iii), using the following well-known fact. If $h_1, \dots, h_n \in K[x_1, \dots, x_n]$ satisfy the conditions $\frac{\partial h_i}{\partial x_j} = \frac{\partial h_j}{\partial x_i}$ for all i, j, then there is an $f \in K[x_1, \dots, x_n]$ such that $h_i = \frac{\partial f}{\partial x_i}$ for all i.

By (ii) we have $\partial_{x_i} = \sum_k h_{ik}\delta_k$ for $i = 1, \dots, n$. We claim that $\frac{\partial h_{ik}}{\partial x_j} = \frac{\partial h_{jk}}{\partial x_i}$ for all i, j, k. In fact,

$$\begin{aligned}
0 &= \partial_{x_i}\partial_{x_j} - \partial_{x_j}\partial_{x_i} = \partial_{x_i}\sum_k h_{jk}\delta_k - \partial_{x_j}\sum_k h_{ik}\delta_k \\
&= \sum_k \frac{\partial h_{jk}}{\partial x_i}\delta_k + \sum_k h_{jk}\partial_{x_i}\delta_k - \sum_k \frac{\partial h_{ik}}{\partial x_j}\delta_k - \sum_k h_{ik}\partial_{x_j}\delta_k \\
&= \sum_k \left(\frac{\partial h_{jk}}{\partial x_i} - \frac{\partial h_{ik}}{\partial x_j}\right)\delta_k + \left(\sum_{k,\ell} h_{jk}h_{i\ell}\delta_\ell\delta_k - \sum_{k,\ell} h_{ik}h_{j\ell}\delta_\ell\delta_k\right) \\
&= \sum_k \left(\frac{\partial h_{jk}}{\partial x_i} - \frac{\partial h_{ik}}{\partial x_j}\right)\delta_k + \sum_{k,\ell} h_{ik}h_{j\ell}[\delta_k, \delta_\ell] = \sum_k \left(\frac{\partial h_{jk}}{\partial x_i} - \frac{\partial h_{ik}}{\partial x_j}\right)\delta_k.
\end{aligned}$$

Hence $h_{ik} = \frac{\partial f_k}{\partial x_i}$ for suitable $f_1, \dots, f_n \in K[x_1, \dots, x_n]$. It is clear that the matrix (h_{ik}) is invertible. This implies that the morphism $\varphi := (f_1, \dots, f_n) \colon \mathbb{A}^n \to \mathbb{A}^n$ is étale, and $\partial_{x_i} = \sum_k \frac{\partial f_k}{\partial x_i} \delta_k$, hence $\delta_k = \varphi^*(\partial_{x_k})$, by equation (4.2).

(c) Assume that (iii) holds. Setting $\delta_i = \sum_k h_{ik} \partial_{x_k}$ and applying both sides to f_j, we see that the matrix $(h_{ik}) \in \mathrm{M}_n(K[x_1, \dots, x_n])$ is invertible, hence (ii) holds. Thus the first three statements of the proposition are equivalent, and they imply (iv).

(d) Finally, assume that (iv) holds. Set $\delta_i = \sum_k h_{ik} \partial_{x_k}$. Since $[\delta_i, \delta_j] = 0$ we get $\delta_i(h_{jk}) = \delta_j(h_{ik})$ for all i, j, k. Now an easy calculation shows that $\delta_k(\det(h_{ij})) = \operatorname{Div}(\delta_k) \det(h_{ij})$, and so $\det(h_{ij}) \in K$. If $\det(h_{ij}) \neq 0$, then (ii) follows.

If $\det(h_{ij}) = 0$, then $\operatorname{rank}(\sum_{i=1}^n K[x_1, \dots, x_n]\delta_i) = r < n$, and we can assume that $\operatorname{rank}(\sum_{i=1}^r K[x_1, \dots, x_n]\delta_i) = r$. Choose a nontrivial relation $\sum_{i=1}^{r+1} f_i \delta_i = 0$ where $\gcd(f_1, \dots, f_{r+1}) = 1$. Since $0 = \delta_j(\sum_{i=1}^{r+1} f_i \delta_i) = \sum_{i=1}^{r+1} \delta_j(f_i)\delta_i$ for any j, we see that $\delta_j(f_i) \in K[x_1, \dots, x_n] f_i$, and since the δ_j are K-linearly independent, at least one of the f_i is not a constant, hence a common Darboux polynomial, contradicting (iv). □

The second main ingredient for the proof is the following result.

Lemma 5.3. *Let $\delta_1, \delta_2 \in \operatorname{Vec}^0(\mathbb{A}^n)$ be commuting vector fields. Assume that:*

(a) *δ_1 and δ_2 have a common Darboux polynomial f where $\delta_i f \neq 0$, $i = 1, 2$.*
(b) *Each δ_i acts locally nilpotently on $\operatorname{Vec}^0(\mathbb{A}^n)$.*

Then $K[x_1, \dots, x_n]\delta_1 + K[x_1, \dots, x_n]\delta_2 \subseteq \operatorname{Vec}(\mathbb{A}^n)$ is a $K[x_1, \dots, x_n]$-submodule of rank ≤ 1.

Proof. We will show that there are nonzero polynomials p_1, p_2 such that $p_1\delta_1 = p_2\delta_2$. We have $\delta_i(f) = h_i f$ where $h_1, h_2 \neq 0$. Since δ_1 and δ_2 commute, we get $\delta_1(h_2 f) = \delta_2(h_1 f)$, and so $\delta_1 h_2 = \delta_2 h_1$. In view of the formula $\operatorname{Div}(g\delta) = \delta g + g \operatorname{Div}(\delta)$, this implies that $\delta := h_1\delta_2 - h_2\delta_1 \in \operatorname{Vec}^0(\mathbb{A}^n)$. Moreover, $\delta f = 0$, and so $f\delta \in \operatorname{Vec}^0(\mathbb{A}^n)$. Since

$$[\delta_1, \xi] = [\delta_1, h_1\delta_2] - [\delta_1, h_2\delta_1] = (\delta_1 h_1)\delta_2 - (\delta_1 h_2)\delta_1,$$

we get $(\operatorname{ad} \delta_1)^k \delta = \delta_1^k(h_1)\delta_2 - \delta_1^k(h_2)\delta_1$ and $(\operatorname{ad} \delta_1)^k(f\delta) = \delta_1^k(fh_1)\delta_2 - \delta_1^k(fh_2)\delta_1$. Now, by assumption (b), there is a $k > 0$ such that $(\operatorname{ad} \delta_1)^k \delta = (\operatorname{ad} \delta_1)^k(f\delta) = 0$, hence

$$\delta_1^k(h_1)\delta_2 = \delta_1^k(h_2)\delta_1 \quad \text{and} \quad \delta_1^k(fh_1)\delta_2 = \delta_1^k(fh_2)\delta_1.$$

Thus the claim follows except if $\delta_1^k h_1 = \delta_1^k h_2 = \delta_1^k(fh_1) = \delta_1^k(fh_2) = 0$. We will show that this leads to a contradiction. Since $\delta_1 f = h_1 f$, we get $\delta_1^{k+1} f = 0$. Choose r, s minimal with $\delta_1^r h_1 = 0$ and $\delta_1^s f = 0$. By assumption, $r, s \geq 1$, and we get $\delta_1^{r+s-2}(h_1 f) = \delta_1^{r-1} h_1 \cdot \delta_1^{s-1} f \neq 0$. On the other hand, $\delta_1^{s-1}(h_1 f) = \delta_1^s f = 0$, and we end up with a contradiction, because $s - 1 \leq r + s - 2$. □

Now we can prove the Theorem.

Proof of Theorem 5.1. The case $n = 1$ is handled in Lemma 3.2, so we can assume that $n \geq 2$. Let θ be an automorphism of $\operatorname{Vec}^0(\mathbb{A}^n)$ as a Lie algebra, and set $\delta_i := \theta(\partial_{x_i})$. Then the vector fields $\delta_1, \dots, \delta_n$ are pairwise commuting and K-linearly independent. Since

∂_{x_i} acts locally nilpotently on $\mathrm{Vec}^0(\mathbb{A}^n)$, the same holds for δ_i. Moreover, the centralizer of the δ_i in $\mathrm{Vec}^0(\mathbb{A}^n)$ is the linear span of the δ_i, i.e. $[\delta, \delta_i] = 0$ for all i implies that $\delta \in \bigoplus_i K\delta_i$. In the following we will use vector fields with rational coefficients:

$$\mathrm{Vec}^{\mathrm{rat}}(\mathbb{A}^n) := K(x_1, \dots, x_n) \otimes_{K[x_1,\dots,x_n]} \mathrm{Vec}(\mathbb{A}^n) = \bigoplus_{i=1}^{n} K(x_1, \dots, x_n)\partial_{x_i}.$$

(1) We first claim that the δ_i do not have a common Darboux polynomial. So assume that there exists a nonconstant $f \in K[x_1, \dots, x_n]$ such that $\delta_i f = h_i f$ for all i and some $h_i \in K[x_1, \dots, x_n]$.

First assume that $h_1 = 0$, i.e. $\delta_1 f = 0$. Then $f\delta_1 \in \mathrm{Vec}^0(\mathbb{A}^n)$, and for any $h \in K[x_1, \dots, x_n]$ and every i we have $[\delta_i, hf\delta_1] = \delta_i(hf)\delta_1 = (\delta_i(h) + hh_i)f\delta_1$, and so

$$(\mathrm{ad}\,\delta_i)^k(K[x_1, \dots, x_n]f\delta_1) \subseteq K[x_1, \dots, x_n]f\delta_1 \quad \text{for all } k \geq 0. \tag{5.1}$$

Set $\eta := \theta^{-1}(f\delta_1)$. Then there are $k_1, \dots, k_n \in \mathbb{N}$ such that

$$\eta_0 := (\mathrm{ad}\,\partial_{x_1})^{k_1}(\mathrm{ad}\,\partial_{x_2})^{k_2} \cdots (\mathrm{ad}\,\partial_{x_n})^{k_n}\eta \in K\partial_{x_1} \oplus \cdots \oplus K\partial_{x_n} \setminus \{0\}.$$

Hence, $\theta(\eta_0) = (\mathrm{ad}\,\delta_1)^{k_1}(\mathrm{ad}\,\delta_2)^{k_2} \cdots (\mathrm{ad}\,\delta_n)^{k_n}(f\delta_1) \in K\delta_1 \oplus \cdots \oplus K\delta_n \setminus \{0\}$, which contradicts (5.1), because $f \notin K$.

We are left with the case where no h_i is zero. Then Lemma 5.3 above implies that $\sum_i K[x_1, \dots, x_n]\delta_i \subseteq \mathrm{Vec}(\mathbb{A}^n)$ has rank 1, i.e. there exist $\delta \in \mathrm{Vec}(\mathbb{A}^n)$ and nonzero rational functions $r_i \in K(x_1, \dots, x_n)$ such that $\delta_i = r_i\delta$ for $i = 1, \dots, n$. We can assume that δ is minimal, i.e., not of the form $q\,\delta'$ with a nonconstant polynomial q. For every μ commuting with δ_i, we get $0 = [\mu, \delta_i] = [\mu, r_i\delta] = \mu(r_i)\delta + r_i[\mu, \delta]$, hence $[\mu, \delta] \in K(x_1, \dots, x_n)\delta$. It is easy to see that

$$L := \{\xi \in \mathrm{Vec}(\mathbb{A}^n) \mid [\xi, \delta] \in K(x_1, \dots, x_n)\delta\}$$

is a Lie subalgebra of $\mathrm{Vec}(\mathbb{A}^n)$ which contains all elements commuting with one of the δ_i. Since $\mathrm{Vec}^0(\mathbb{A}^n)$ is generated, as a Lie algebra, by elements commuting with one of the ∂_{x_i} we see that $\theta(\mathrm{Vec}^0(\mathbb{A}^n)) = \mathrm{Vec}^0(\mathbb{A}^n)$ is generated by the elements commuting with one of the δ_i. Thus $\mathrm{Vec}^0(\mathbb{A}^n) \subseteq L$, and so $[\mathrm{Vec}^0(\mathbb{A}^n), \delta] \subseteq K(x_1, \dots, x_n)\delta$. For $\delta = \sum_i p_i\partial_{x_i}$ we get $[\partial_{x_k}, \delta] = \sum_i \frac{\partial p_i}{\partial x_k}\partial_{x_i} = s\delta$ for some $s \in K(x_1, \dots, x_n)$, hence $\frac{\partial p_i}{\partial x_k}p_j = \frac{\partial p_j}{\partial x_k}p_i$ for all pairs i, j. This implies that $\frac{\partial}{\partial x_k}\frac{p_j}{p_i} = 0$ in case $p_i \neq 0$, i.e. $\frac{p_j}{p_i}$ does not depend on x_k. Since this holds for all k, we conclude that $p_j = c_j p_i$ for some $c_j \in K$, hence $\delta = \sum_j c_j\partial_{x_j}$, because δ is minimal. In particular, $[\partial_{x_k}, \delta] = 0$ for all k. Now we get $[x_\ell\partial_{x_k}, \delta] = -c_\ell\partial_{x_k} \in K(x_1, \dots, x_n)\delta$ for all k, ℓ, which implies $\delta = 0$, hence a contradiction.

(2) Now we use the implication (vi)⇒(i) of Proposition 5.2 to see that there is an étale morphism $\varphi\colon \mathbb{A}^n \to \mathbb{A}^n$ with $\delta_i = \varphi^*(\partial_{x_i})$ for all i. Then the composition $\theta' := \theta^{-1} \circ \varphi^*\colon \mathrm{Vec}^0(\mathbb{A}^n) \to \mathrm{Vec}^0(\mathbb{A}^n)$ is an injective homomorphism of Lie algebras (Proposition 4.1) and $\theta'(\partial_{x_i}) = \partial_{x_i}$. Hence, Lemma 5.4 below implies that $\theta' = \mathrm{Ad}(s) = (s^{-1})^*$ where $s \in \mathrm{Aut}(\mathbb{A}^n)$ is a translation, hence $\theta = (\varphi \circ s)^*$. Now Proposition 4.3 implies that $\psi := \varphi \circ s$ is an automorphism of $\mathbb{A}^n$, and so $\theta = \mathrm{Ad}(\psi^{-1})$ as claimed. □

Lemma 5.4. *Let θ be an injective endomorphism of* $\mathrm{Vec}^0(\mathbb{A}^n)$ *such that* $\theta(\partial_{x_i}) = \partial_{x_i}$ *for all* i*. Then* $\theta = \mathrm{Ad}(s)$ *where* $s\colon \mathbb{A}^n \xrightarrow{\sim} \mathbb{A}^n$ *is a translation. In particular,* θ *is an automorphism.*

Proof. We know that $\sum_i K\partial_{x_i} = L(S)$ where $S \subset \mathrm{Aff}_n$ are the translations. Moreover, $L(\mathrm{Aff}_n)$ is the normalizer of $L(S)$ in the Lie algebra $\mathrm{Vec}(\mathbb{A}^n)$. Hence $\theta(L(\mathrm{SAff}_n)) = L(\mathrm{SAff}_n)$, and so there is an affine transformation g such that $\mathrm{Ad}(g)|_{L(\mathrm{SAff}_n)} = \theta|_{L(\mathrm{SAff}_n)}$, by Lemma 3.2. Since $\mathrm{Ad}(g)$ is the identity on $L(S)$, we see that g is a translation. It follows that $\mathrm{Ad}(g^{-1}) \circ \theta$ is the identity on $L(\mathrm{SL}_n)$, hence $\mathrm{Ad}(g^{-1}) \circ \theta = \mathrm{Ad}(\lambda E)$ for some $\lambda \in K^*$, by Lemma 3.3. But $\lambda = 1$, because θ is the identity on $L(S)$, and so $\theta = \mathrm{Ad}(g)$. □

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