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Besicovitch covering property for homogeneous distances on the Heisenberg groups

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Abstract. We prove that the Besicovitch Covering Property (BCP) holds for homogeneous distances on the Heisenberg groups whose unit ball centered at the origin coincides with a Euclidean ball. We thus provide the first examples of homogeneous distances that satisfy BCP on these groups. Indeed, commonly used homogeneous distances, such as (Cygan–)Korányi and Carnot–Carathéodory distances, are known not to satisfy BCP. We also generalize those previous results by giving two geometric criteria that imply the non-validity of BCP and showing that in some sense our examples are sharp. To put our result in another perspective, inspired by an observation of D. Preiss, we prove that in a general metric space with an accumulation point, one can always construct bi-Lipschitz equivalent distances that do not satisfy BCP.

Keywords. Covering theorems, Heisenberg groups, homogeneous distances

1. Introduction

Covering theorems are known to be among the fundamental tools of measure theory. They reflect the geometry of the space and are commonly used to establish connections between local and global behavior of measures. Covering theorems and their applications have been studied for example in [5] and [9]. There are several types of covering results, all with the same purpose: from an arbitrary cover of a set in a metric space, one extracts a subcover as disjoint as possible. More specifically, we will consider the so-called Besicovitch Covering Property (BCP) which originates from the work of Besicovitch ([1], [2], see also [5, 2.8], [20], [21]) in connection with the theory of differentiation of measures. See Section 1.1 for a more detailed presentation of the Besicovitch Covering Property and its applications.

The geometric setting we are interested in is the setting of Carnot groups, and more specifically the Heisenberg groups $\mathbb{H}^n$ (see Section 1.2), equipped with so-called homogeneous distances (see Definition 1.9). Our main result, Theorem 1.14, states that BCP holds for those homogeneous distances on $\mathbb{H}^n$, denoted by d_α in the rest of this paper,

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whose unit ball centered at the origin coincides with a Euclidean ball centered at the origin in exponential coordinates. This gives the first examples of Carnot groups on which one can construct homogeneous distances satisfying BCP. Moreover these distances have simple descriptions, geometrically by means of their unit ball centered at the origin, and also through an explicit algebraic expression (see (1.12)).

Any two homogeneous distances on $\mathbb{H}^n$ are bi-Lipschitz equivalent. Recall that two distances d and $\overline{d}$ are said to be *bi-Lipschitz equivalent* if there exists $C > 1$ such that $C^{-1}d \leq \overline{d} \leq Cd$. Hence, for many purposes, the choice of a specific homogeneous distance doest not matter, and Theorem 1.14 is expected to have several applications. One of them is the extension to $\mathbb{H}^n$ of a result of J. Heinonen and P. Koskela [11, Theorem 1.4] about quasiconformal mappings. This also allows one to replace a "lim sup" by a "lim inf" in the definition of quasiconformal mappings, which suffices for quasisymmetry (see [11, remark after Theorem 1.4] for more details).

Another noticeable consequence of Theorem 1.14 is the validity of the Differentiation Theorem for every locally finite Radon measure on $\mathbb{H}^n$ equipped with some homogeneous distance d_α (see Section 1.1 and in particular Theorem 1.5). In connection with recent developments in geometric measure theory on Carnot groups, this allows one to get a simpler proof of the structure theorem for finite perimeter sets in $\mathbb{H}^n$ (see [7], [8]).

It has already been noticed that two commonly used homogeneous distances on $\mathbb{H}^n$ do not satisfy BCP, namely the Cygan–Korányi distance, usually also called Korányi or gauge distance[1] [13], and the Carnot–Carathéodory distance [22]. It turns out that the validity of BCP strongly depends on the distance the space is endowed with, and more specifically on the geometry of its balls. To give some more evidence for this fact and to put our result in perspective, we also prove two criteria that imply the non-validity of BCP. They give two large families of homogeneous distances on $\mathbb{H}^n$ that do not satisfy BCP, and show that in some sense our example for which BCP holds is sharp (see Section 6, in particular Theorems 6.1 and 6.3).

As a matter of fact, our first criterion applies to the Cygan–Korányi and Carnot–Carathéodory distances, thus also giving new geometric proofs of the failure of BCP for these distances, but the criterion is more general. It also applies to the so-called box-distance (the terminology might not be standard although this distance is a standard homogeneous distance on $\mathbb{H}^n$, see (6.2)), thus proving the non-validity of BCP for the latter homogeneous distance as well.

Going back to the distances considered in the present paper and for which we prove that BCP holds, Hebisch and Sikora showed [10] that in any Carnot group, there are homogeneous distances whose unit ball centered at the origin coincides with a Euclidean ball centered at the origin with a small enough radius. In the specific case of the Heisenberg groups, these distances are related to the Cygan–Korányi distance. They can indeed be expressed in terms of the quadratic mean of the Cygan–Korányi distance (at least for some specific value of the radius of the Euclidean ball which coincides with the unit ball

[1] We adopt here the terminology *Cygan–Korányi* distance, which may not be standard, to emphasize the fact that Cygan [4] was the first to observe that the natural gauge in the Heisenberg group actually induces a distance; also Korányi [12] himself attributes this distance to Cygan.

centered at the origin) together with the pseudo-distance on $\mathbb{H}^n$ given by the Euclidean distance between horizontal components.

These distances have been previously considered in the literature. Lee and Naor [18] proved that these metrics are of negative type on $\mathbb{H}^n$. Recall that a metric space (M, d) is said to be *of negative type* if $(M, \sqrt{d})$ is isometric to a subset of a Hilbert space. Combined with the work of Cheeger and Kleiner [3] about a weak notion of differentiability for maps from $\mathbb{H}^n$ into L^1, which leads in particular to the fact that $\mathbb{H}^n$ equipped with a homogeneous distance does not admit a bi-Lipschitz embedding into L^1, this provides a counterexample to the Goemans–Linial conjecture in theoretical computer science, which was the motivation for those papers. Let us remark that the Cygan–Korányi distance is not of negative type on $\mathbb{H}^n$.

We refer to Section 1.2 for the precise definition of our distances d_α and their connection with the Cygan–Korányi distance and the distances of negative type considered in [18].

1.1. Besicovitch Covering Property

Let (M, d) be a metric space. When speaking of a ball B in M, it will be understood that B is a closed ball and that it comes with a fixed center and radius (although these in general are not uniquely determined by B as a set). Thus $B = B_d(p, r)$ for some $p \in M$ and some $r > 0$ where $B_d(p, r) = \{q \in M;\ d(q, p) \le r\}$.

Definition 1.1 (Besicovitch Covering Property). One says that the *Besicovitch Covering Property* (BCP) holds for the distance d on M if there exists an integer $N \ge 1$ with the following property. Let A be a bounded subset of (M, d) and let $\mathcal{B}$ be a family of balls in (M, d) such that each point of A is the center of some ball of $\mathcal{B}$. Then there is a subfamily $\mathcal{F} \subset \mathcal{B}$ whose balls cover A and which has the property that every point in M belongs to at most N balls of $\mathcal{F}$, that is,

$$\chi_A \le \sum_{B \in \mathcal{F}} \chi_B \le N,$$

where χ_A denotes the characteristic function of the set A.

When equipped with a homogeneous distance, the Heisenberg groups turn out to be doubling metric spaces. Recall that this means that there exists an integer $C \ge 1$ such that each ball of radius $r > 0$ can be covered with less than C balls of radius $r/2$. When (M, d) is a doubling metric space, BCP turns out to be equivalent to a covering property, strictly weaker in general, which we call the Weak Besicovitch Covering Property (w-BCP) (the terminology might not be standard) and with which we shall work in this paper. First, let us fix some more terminology.

Definition 1.2 (Family of Besicovitch balls). We say that a family $\mathcal{B}$ of balls in (M, d) is a *family of Besicovitch balls* if $\mathcal{B} = \{B = B_d(x_B, r_B)\}$ is a finite family of balls such that $x_B \notin B'$ for all distinct $B, B' \in \mathcal{B}$, and $\bigcap_{B \in \mathcal{B}} B \neq \emptyset$.

Definition 1.3 (Weak Besicovitch Covering Property). One says that the *Weak Besicovitch Covering Property* (w-BCP) holds for the distance d on M if there exists an integer $N \geq 1$ such that $\operatorname{Card} \mathcal{B} \leq N$ for every family $\mathcal{B}$ of Besicovitch balls in (M, d).

The validity of BCP implies the validity of w-BCP. We stress that there exist metric spaces for which w-BCP holds although BCP does not. However, when the metric is doubling, both properties are equivalent, as can be proved by following [19, proof of Theorem 2.7] (see also [17]):

Characterization 1.4 (BCP in doubling metric spaces). *Let (M, d) be a doubling metric space. Then BCP holds for the distance d on M if and only if w-BCP does.*

As already said, covering theorems and especially the Besicovitch Covering Property and the Weak Besicovitch Covering Property play an important role in many situations in measure theory, regularity and differentiation of measures, as well as in many problems in harmonic analysis. This is particularly well illustrated by the connection between w-BCP and the so-called Differentiation Theorem. The validity of BCP in the Euclidean space is due to Besicovitch and was a key tool in his proof of the fact that the Differentiation Theorem holds for each locally finite Borel measure on $\mathbb{R}^n$ ([1], [2], see also [5, 2.8], [20]). Moreover, as emphasized in Theorem 1.5, the validity of w-BCP actually turns out to be equivalent to the validity of the Differentiation Theorem for each locally finite Borel measure as shown in [21].

Theorem 1.5 (Preiss [21]). *Let (M, d) be a complete separable metric space. Then the Differentiation Theorem holds for each locally finite Borel measure μ on (M, d), that is,*

$$\lim_{r \to 0^+} \frac{1}{\mu(B_d(p,r))} \int_{B_d(p,r)} f(q)\, d\mu(q) = f(p)$$

for μ-almost every $p \in M$ and for each $f \in L^1(\mu)$ if and only if $M = \bigcup_{n \in \mathbb{N}} M_n$ where, for each $n \in \mathbb{N}$, w-BCP holds for the family of balls centered on M_n with radii less than r_n for some $r_n > 0$.

As already stressed, the fact that BCP holds in a metric space depends strongly on the distance with which the space is endowed. On the one hand, under very mild assumptions on the metric space (namely, as soon as there exists an accumulation point), one can indeed always construct bi-Lipschitz equivalent distances as close as we want to the original distance and for which BCP is not satisfied, as shown in the following result.

Theorem 1.6. *Let (M, d) be a metric space. Assume that there exists an accumulation point in (M, d). Let $0 < c < 1$. Then there exists a distance $\overline{d}$ on M such that $cd \leq \overline{d} \leq d$ and for which w-BCP, and hence BCP, does not hold.*

A slightly different version of this result is stated in [21, Theorem 3]. For the sake of completeness, in Section 8 we give a construction of a distance as stated in Theorem 1.6.

On the other hand, the question whether a metric space can be remetrized so that BCP holds is in general significantly more delicate. As already explained, the main result of the present paper, Theorem 1.14, is a positive answer to this question for the Heisenberg groups equipped with ad-hoc homogeneous distances, namely those whose unit ball at the origin coincides with a Euclidean ball with a small enough radius.

1.2. The Heisenberg group

As a set, we identify the Heisenberg group $\mathbb{H}^n$ with $\mathbb{R}^{2n+1}$ and we equip it with the Euclidean topology. We choose the following convention for the group law:

$$(x, y, z) \cdot (x', y', z') := \big(x + x', y + y', z + z' + \tfrac{1}{2}\langle x, y'\rangle - \tfrac{1}{2}\langle y, x'\rangle\big) \tag{1.7}$$

where x, y, x', y' belong to $\mathbb{R}^n$, z, z' belong to $\mathbb{R}$ and $\langle \cdot, \cdot\rangle$ denotes the usual scalar product in $\mathbb{R}^n$. This corresponds to a choice of exponential and homogeneous coordinates.

The one-parameter family of dilations on $\mathbb{H}^n$ is given by $(\delta_\lambda)_{\lambda>0}$ where

$$\delta_\lambda(x, y, z) := (\lambda x, \lambda y, \lambda^2 z). \tag{1.8}$$

These dilations are group automorphisms.

Definition 1.9 (Homogeneous distance). A distance d on $\mathbb{H}^n$ is said to be *homogeneous* if it is left invariant, that is, $d(p \cdot q, p \cdot q') = d(q, q')$ for all $p, q, q' \in \mathbb{H}^n$, and one-homogeneous with respect to the dilations, that is, $d(\delta_\lambda(p), \delta_\lambda(q)) = \lambda d(p, q)$ for all $p, q \in \mathbb{H}^n$ and all $\lambda > 0$.

We stress that homogeneous distances on $\mathbb{H}^n$ induce the Euclidean topology on $\mathbb{H}^n$. This is a nontrivial fact which follows from the continuity of the dilations with respect to the Euclidean topology together with the homogeneity of the distance as stated in Definition 1.9 (see [15] and [17]).

It turns out that homogeneous distances on $\mathbb{H}^n$ do exist in abundance and make it a doubling metric space. It is also well known that any two homogeneous distances are bi-Lipschitz equivalent. See for example [6] for more details about the Heisenberg groups and more generally Carnot groups.

The (family of) homogeneous distance(s) we consider in this paper can be defined in the following way. For $\alpha > 0$, we denote by B_α the Euclidean ball in $\mathbb{H}^n \simeq \mathbb{R}^{2n+1}$ centered at the origin with radius α, that is,

$$B_\alpha := \{(x, y, z) \in \mathbb{H}^n;\ \|x\|^2_{\mathbb{R}^n} + \|y\|^2_{\mathbb{R}^n} + |z|^2 \le \alpha^2\},$$

where $\|\cdot\|_{\mathbb{R}^n}$ denotes the Euclidean norm in $\mathbb{R}^n$, and we set

$$d_\alpha(p, q) := \inf\{r > 0;\ \delta_{1/r}(p^{-1} \cdot q) \in B_\alpha\}. \tag{1.10}$$

Hebisch and Sikora [10] proved that if $\alpha > 0$ is small enough, then d_α actually defines a distance on $\mathbb{H}^n$. More generally, this holds true in any Carnot group starting from the set B_α given by the Euclidean ball centered at the origin with radius $\alpha > 0$ small enough, where one identifies in the usual way the group with some $\mathbb{R}^m$ where m is the topological dimension of the group.

It then follows from the very definition that d_α is the homogeneous distance on $\mathbb{H}^n$ for which the unit ball centered at the origin coincides with the Euclidean ball of radius α centered at the origin. The geometric description of arbitrary balls that can then be deduced from the unit ball centered at the origin via dilations and left translations is actually of crucial importance for understanding the reasons why BCP eventually holds for these distances.

On the other hand, it is particularly convenient to note that in the specific case of the Heisenberg groups, one also has a fairly simple analytic expression for distances whose unit ball at the origin is given by a Euclidean ball centered at the origin. This will actually be technically extensively used in our proof of Theorem 1.14. This also gives the explicit connection with the Cygan–Korányi distance and the distances of negative type considered by Lee and Naor [18].

Set

$$\rho(p) := \sqrt{\|x\|_{\mathbb{R}^n}^2 + \|y\|_{\mathbb{R}^n}^2} \quad \text{and} \quad \|p\|_{g,\alpha} := (\rho(p)^4 + 4\alpha^2|z|^2)^{1/4} \tag{1.11}$$

for $p = (x, y, z) \in \mathbb{H}^n$. Then, as verified in Section 2, one has

$$d_\alpha(p,q) = \sqrt{\frac{\rho(p^{-1}\cdot q)^2 + \|p^{-1}\cdot q\|_{g,\alpha}^2}{2\alpha^2}}. \tag{1.12}$$

First, note that $d_\rho(p,q) := \rho(p^{-1}\cdot q)$ is a left-invariant pseudo-distance on $\mathbb{H}^n$ that is one-homogeneous with respect to the dilations. Next, when $\alpha = 2$, $\|\cdot\|_{g,2}$ is nothing but the Cygan–Korányi norm which is well known to be a natural gauge in $\mathbb{H}^n$. It can actually be checked by direct computations that $d_{g,\alpha}(p,q) := \|p^{-1}\cdot q\|_{g,\alpha}$ satisfies the triangle inequality for any $0 < \alpha \le 2$, and hence defines a homogeneous distance on $\mathbb{H}^n$. This was first proved by Cygan in [4] when $\alpha = 2$. One then recovers from the analytic expression (1.12) that d_α actually defines a homogeneous distance on $\mathbb{H}^n$ for any $0 < \alpha \le 2$, giving also an explicit range of values of α in $\mathbb{H}^n$ for which this fact holds and was first observed in [10] for general Carnot groups and for small enough values of α.

Theorem 1.13. *For any $0 < \alpha \le 2$, d_α defines a homogeneous distance on $\mathbb{H}^n$.*

There might be other values of $\alpha > 2$ such that d_α defines a homogeneous distance on $\mathbb{H}^n$.

These distances turn out to be those considered by Lee and Naor [18]. They actually proved that d_2 is of negative type in $\mathbb{H}^n$, in order to provide a counterexample to the so-called Goemans–Linial conjecture. Let us mention that it can easily be checked that the proof in [18] extends to the distances d_α for all $0 < \alpha \le 2$.

Let us now state our main result.

Theorem 1.14. *Let $\alpha > 0$ be such that d_α defines a homogeneous distance on $\mathbb{H}^n$. Then BCP holds for the distance d_α.*

For technical and notational simplicity, we will focus our attention on the first Heisenberg group $\mathbb{H} = \mathbb{H}^1$. We shall point out briefly in Section 7 the minor modifications needed to make our arguments work in any Heisenberg group $\mathbb{H}^n$.

The rest of the paper is organized as follows. In Section 2 we fix some conventions about $\mathbb{H}$ and the distance d_α and state three technical lemmas on which the proof of Theorem 1.14 is based. The proofs of these lemmas are given in Sections 4 and 5. Section 3 is devoted to the proof of Theorem 1.14 itself. In Section 6 we prove two criteria, Theorems 6.1 and 6.3, for homogeneous distances on $\mathbb{H}$ that imply that BCP does not hold. Theorem 1.6 is proved in Section 8.

In [17], in part based on some results of the present paper, we prove that on stratified groups of step 2, homogeneous distances satisfying BCP do exist, whereas such homogeneous distances do not exist on stratified groups of step higher than 3.

2. Preliminary results

As already mentioned, in Sections 2 to 6 we will focus on the first Heisenberg group $\mathbb{H} = \mathbb{H}^1$. The modifications needed to handle the case of $\mathbb{H}^n$ for any $n \geq 1$ will be indicated in Section 7.

We first fix some conventions and notation. Next, we state the main lemmas on which the proof of Theorem 1.14 will be based.

Recall that we identify the Heisenberg group $\mathbb{H}$ with $\mathbb{R}^3$ equipped with the group law given in (1.7) and we endow it with the Euclidean topology.

We define the projection $\pi : \mathbb{H} \to \mathbb{R}^2$ by

$$\pi(x, y, z) := (x, y). \tag{2.1}$$

When considering a specific point $p \in \mathbb{H}$, we usually denote its coordinates by (x_p, y_p, z_p) and we set

$$\rho_p := \sqrt{x_p^2 + y_p^2}. \tag{2.2}$$

From now on in this section, as well as in Sections 3, 4 and 5, we fix $\alpha > 0$ such that d_α as given in (1.10) defines a homogeneous distance on $\mathbb{H}$, and all metric notions and properties will be understood relative to the distance d_α. In particular we shall denote the closed balls with center $p \in \mathbb{H}$ and radius $r > 0$ by $B(p, r)$ without explicit reference to the distance d_α.

Remembering (1.10), we have the following properties.

Proposition 2.3. *For $p = (x_p, y_p, z_p) \in \mathbb{H}$, we have*

$$d_\alpha(0, p) \leq r \ \Leftrightarrow\ \frac{\rho_p^2}{r^2} + \frac{z_p^2}{r^4} \leq \alpha^2, \tag{2.4}$$

$$d_\alpha(0, p) = r \ \Leftrightarrow\ \frac{\rho_p^2}{r^2} + \frac{z_p^2}{r^4} = \alpha^2, \tag{2.5}$$

from which we get

$$d_\alpha(0, p) = \sqrt{\frac{\rho_p^2 + \sqrt{\rho_p^4 + 4\alpha^2 z_p^2}}{2\alpha^2}}. \tag{2.6}$$

For a point $p \in \mathbb{H}$, we set

$$r_p := d_\alpha(0, p). \tag{2.7}$$

Using left translations, we have the following properties for any $p, q \in \mathbb{H}$:

$$d_\alpha(p, q) \le r \;\Leftrightarrow\; \frac{\rho^2_{p^{-1}\cdot q}}{r^2} + \frac{z^2_{p^{-1}\cdot q}}{r^4} \le \alpha^2 \tag{2.8}$$

and

$$d_\alpha(p, q) = \sqrt{\frac{\rho^2_{p^{-1}\cdot q} + \sqrt{\rho^4_{p^{-1}\cdot q} + 4\alpha^2 z^2_{p^{-1}\cdot q}}}{2\alpha^2}} \tag{2.9}$$

where

$$\rho_{p^{-1}\cdot q} = \sqrt{(x_q - x_p)^2 + (y_q - y_p)^2}, \qquad z_{p^{-1}\cdot q} = z_q - z_p - \frac{x_p y_q - y_p x_q}{2},$$

by definition of the group law (1.7). Note that if $p = (x_p, y_p, z_p) \in \mathbb{H}$ then $p^{-1} = (-x_p, -y_p, -z_p)$.

Let us point out that balls in $(\mathbb{H}, d_\alpha)$ are convex in the Euclidean sense when identifying $\mathbb{H}$ with $\mathbb{R}^3$ with our chosen coordinates. Indeed, the unit ball centered at the origin is by definition the Euclidean ball of radius α in $\mathbb{H} \simeq \mathbb{R}^3$ and thus is Euclidean convex. Next, dilations (1.8) are linear maps and left translations (see (1.7)) are affine maps, hence

$$B(p, r) = p \cdot \delta_r(B(0, 1))$$

is also a Euclidean convex set in $\mathbb{H} \simeq \mathbb{R}^3$. This will be of crucial use for some of our arguments and we state it below as a proposition for further reference.

Proposition 2.10. *Balls in $(\mathbb{H}, d_\alpha)$ are convex in the Euclidean sense when identifying $\mathbb{H}$ with $\mathbb{R}^3$ with our chosen coordinates.*

We shall also use the following isometries of $(\mathbb{H}, d_\alpha)$. First, *rotations* around the z-axis are defined by

$$\mathrm{R}_\theta : (x, y, z) \mapsto (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z) \tag{2.11}$$

for some angle $\theta \in \mathbb{R}$. Next, the *reflection* R is defined by

$$\mathrm{R}(x, y, z) := (x, -y, -z). \tag{2.12}$$

Using (2.9), one can easily check that these maps are isometries of $(\mathbb{H}, d_\alpha)$.

We now state the main lemmas on which the proof of Theorem 1.14 will be based.

For $\theta \in (0, \pi/2)$ and $a, b > 0$, we set (see Figure 1)

$$\mathcal{P}(a, b, \theta) := \{p \in \mathbb{H};\; x_p > a,\; |z_p| < b,\; |y_p| < x_p \tan\theta\}. \tag{2.13}$$

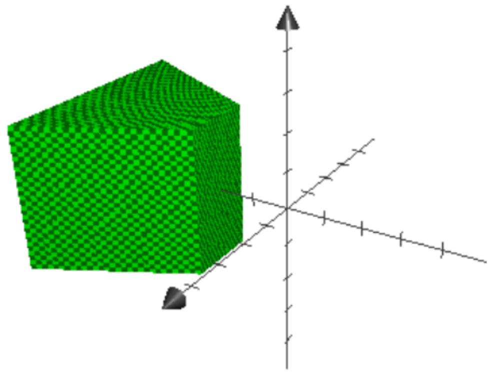
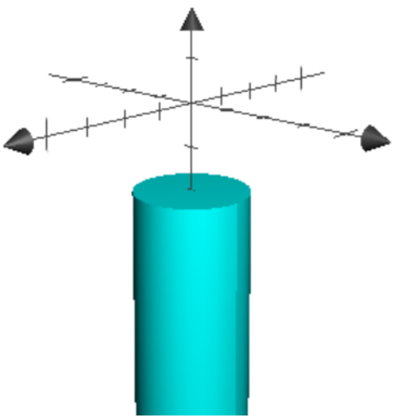


Fig. 1. Two views of the region $\mathcal{P}(a, b, \theta)$.

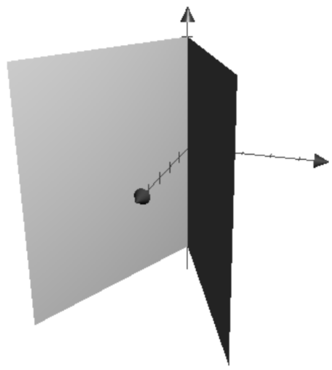
Lemma 2.14. *There exists $\theta_0 \in (0, \pi/4)$, which depends only on α, such that for all $\theta \in (0, \theta_0)$, there exists $a_0(\theta) \geq 1$ such that for all $a > a_0(\theta)$ and all $b \in (0, 1)$, the following holds. Let $p \in \mathbb{H}$ and $q \in \mathbb{H}$ be such that $p \notin B(q, r_q)$ and $q \notin B(p, r_p)$. Then at most one of these two points belongs to $\mathcal{P}(a, b, \theta)$.*

For $a, b > 0$, we set (see Figure 2(a))

$$\mathcal{T}(a, b) := \{p \in \mathbb{H};\ z_p < -a,\ \rho_p < b\}. \tag{2.15}$$



(a) The truncated cylinder $\mathcal{T}(a, b)$



(b) The conic sector $\mathcal{C}(\theta)$.

Fig. 2. The regions $\mathcal{T}(a, b)$ and $\mathcal{C}(\theta)$.

Lemma 2.16. *There exist $a_1 \geq 1$ and $b_1 \in (0, 1)$, depending only on α, such that for all $a > a_1$ and all $b \in (0, b_1)$, the following holds. Let $p, q \in \mathbb{H}$ be such that $p \notin B(q, r_q)$ and $q \notin B(p, r_p)$. Then at most one of these two points belongs to $\mathcal{T}(a, b)$.*

These two lemmas will be proved in Section 4.

For $\theta \in (0, \pi/2)$, we set (see Figure 2(b))

$$\mathcal{C}(\theta) := \{p \in \mathbb{H};\ |y_p| < x_p \tan\theta\}. \tag{2.17}$$

Lemma 2.18. *There exists $\theta_1 \in (0, \pi/8)$, which depends only on α, such that for all $\theta \in (0, \theta_1)$ the following holds. Let $p, q \in \mathbb{H}$ be such that*

$$z_q \leq 0 \quad \textit{and} \quad z_p \leq 0, \tag{2.19}$$
$$\rho_q \leq \rho_p, \tag{2.20}$$
$$q \in \mathcal{C}(\theta) \quad \textit{and} \quad p \in \mathcal{C}(\theta), \tag{2.21}$$
$$q \notin B(p, r_p) \quad \textit{and} \quad p \notin B(q, r_q). \tag{2.22}$$

Then

$$z_q < 2z_p, \tag{2.23}$$
$$\rho_q < \rho_p \cos(2\theta). \tag{2.24}$$

This lemma will be proved in Section 5.

3. Proof of Theorem 1.14

This section is devoted to the proof of Theorem 1.14. Recall that we consider here the case $\mathbb{H} = \mathbb{H}^1$ equipped with the homogeneous distance d_α as defined in (1.10) (see Section 7 for the general case $\mathbb{H}^n$, $n \geq 1$). Recall also that due to Characterization 1.4, Theorem 1.14 will follow if we find an integer $N \geq 1$ such that $\operatorname{Card}\mathcal{B} \leq N$ for every family $\mathcal{B}$ of Besicovitch balls (see Definition 1.2).

We first reduce the proof to the case of some specific families of Besicovitch balls. In what follows, when considering families $\{p_j\}$ of points we shall simplify the notation and set $p_j = (x_j, y_j, z_j)$, $\rho_j = \sqrt{x_j^2 + y_j^2}$ and $r_j = d_\alpha(0, p_j)$. Recall that $\mathcal{C}(\theta)$ is defined in (2.17).

Lemma 3.1. *Let $\theta \in (0, \pi/2)$ and let $\mathcal{B}$ be a family of Besicovitch balls. Then there exists a finite family $\{p_j\}$ of points such that $\mathcal{F} = \{B(p_j, r_j)\}$ is a family of Besicovitch balls with the following properties. For every point p_j in the family, we have*

$$z_j \leq 0, \tag{3.2}$$
$$p_j \in \mathcal{C}(\theta), \tag{3.3}$$
$$\operatorname{Card}\mathcal{B} \leq 2(\pi/\theta + 1)\operatorname{Card}\mathcal{F} + 2. \tag{3.4}$$

Proof. Let $\mathcal{B} = \{B(q_j, t_j)\}_{j=1}^k$ be a family of Besicovitch balls where $k = \operatorname{Card}\mathcal{B}$. Take $q \in \bigcap_{j=1}^k B(q_j, t_j)$. Set $p_j = q^{-1} \cdot q_j$. Remembering that left translations are isometries and that, by convention, we set $r_j = d_\alpha(0, p_j)$, we get $0 \in \bigcap_{j=1}^k B(p_j, r_j)$ and $d_\alpha(p_j, p_i) = d_\alpha(q_j, q_i) > \max(t_j, t_i) \geq \max(r_j, r_i)$, hence $\mathcal{B}' = \{B(p_j, r_j)\}_{j=1}^k$ is a family of Besicovitch balls.

Since balls are Euclidean convex (see Proposition 2.10) and since $0 \in \partial B(p_j, r_j)$ for all $j = 1, \dots, k$, there are at most two balls in $\mathcal{B}'$ with center on the z-axis.

Next, up to replacing the family $\{p_j\}$ by $\{\mathrm{R}(p_j)\}$ (see (2.12) for the definition of the reflection R) and up to reindexing the points, one can find l points $p_1, \dots, p_l$ that satisfy (3.2) such that $\pi(p_1), \dots, \pi(p_l) \neq 0$ (see (2.1) for the definition of the projection π) and $2l \geq k - 2$.

Finally, up to a rotation around the z-axis (see (2.11) for the definition of rotations) and up to reindexing the points, we infer by the pigeonhole principle that there exists an integer k' such that

$$(\pi/\theta + 1)k' \geq l$$

and p_j satisfies (3.3) for all $j = 1, \dots, k'$. Then the family $\mathcal{F} = \{B(p_j, r_j)\}_{j=1}^{k'}$ gives the conclusion. □

We are now ready to conclude the proof of Theorem 1.14 using Lemmas 2.14, 2.16 and 2.18.

Proof of Theorem 1.14. We fix some values of $\theta \in (0, \pi/8)$ and $a, b > 0$ such that the conclusions of Lemmas 2.14, 2.16 and 2.18 hold.

Next, we fix some $R > 0$ large enough so that

$$\{p \in \mathbb{H};\ x_p \in [0, a],\ |z_p| < b,\ |y_p| < x_p \tan\theta\} \subset U(0, R),$$
$$\{p \in \mathbb{H};\ z_p \in [-a, 0],\ \rho_p < b\} \subset U(0, R),$$

where $U(0, R)$ denotes the open ball with center 0 and radius R in $(\mathbb{H}, d_\alpha)$. Such an R exists since in the above two inclusions, the sets on the left are bounded. As a consequence,

$$(\mathbb{H} \setminus U(0, R)) \cap \{p \in \mathbb{H};\ |z_p| < b,\ |y_p| < x_p \tan\theta\} \subset \mathcal{P}(a, b, \theta), \tag{3.5}$$
$$(\mathbb{H} \setminus U(0, R)) \cap \{p \in \mathbb{H};\ z_p \leq 0,\ \rho_p < b\} \subset \mathcal{T}(a, b) \tag{3.6}$$

(see (2.13) for the definition of $\mathcal{P}(a, b, \theta)$ and (2.15) for the definition of $\mathcal{T}(a, b)$).

Let us now consider a family $\mathcal{F} = \{B(p_j, r_j)\}_{j=1}^{k}$ of Besicovitch balls where, by convention, $r_j = d_\alpha(0, p_j)$ and the centers p_j satisfy (3.2) and (3.3). Noting that the family $\{B(\delta_\lambda(p_j), \lambda r_j)\}_{j=1}^{k}$ has the same properties for all $\lambda > 0$, one can assume with no loss of generality that

$$R = \min\{d_\alpha(0, p_j);\ j = 1, \dots, k\}$$

up to a dilation by a factor $\lambda = R/\min\{r_1, \dots, r_k\}$.

Let $m, M > 0$ be defined via

$$-m := \min\{z_p;\ p \in B(0, R)\}, \qquad M := \max\{\rho_p;\ p \in B(0, R)\}.$$

We will bound $k = \operatorname{Card}\mathcal{F}$ in terms of the constants m, M, b and θ.

We reindex the points so that $0 < \rho_1 \leq \dots \leq \rho_k$. Let $l \in \{1, \dots, k\}$ be such that $d_\alpha(0, p_l) = R$. By choice of l and by definition of m and M, we have

$$\rho_l \leq M \quad \text{and} \quad -m \leq z_l.$$

Let $j_0 \geq 1$ be a large enough integer such that $M \cos^{j_0}(2\theta) < b$. Then $l \leq j_0 + 1$. Indeed, otherwise (2.24) would yield

$$0 < \rho_1 < \rho_2 \cos(2\theta) < \cdots < \rho_l \cos^{l-1}(2\theta) \leq M \cos^{j_0+1}(2\theta) < b \cos(2\theta),$$

and hence $\rho_1 < \rho_2 < b$. Then, by choice of R (recall (3.6)), p_1 and p_2 would be distinct points in $\mathcal{T}(a, b)$, which contradicts Lemma 2.16.

Let $j_1 \geq 1$ be a large enough integer such that $2^{-j_1} m < b$. Then $k - l \leq j_1$. Indeed, otherwise (2.23) would give

$$-m \leq z_l < \cdots < 2^{k-l-1} z_{k-1} < 2^{k-l} z_k \leq 0,$$

and hence $|z_k| < |z_{k-1}| < 2^{-(k-l-1)} m \leq 2^{-j_1} m < b$. Then, by choice of R (recall (3.5)), p_{k-1} and p_k would be distinct points in $\mathcal{P}(a, b, \theta)$, which contradicts Lemma 2.14.

Altogether we get the following bound on $\operatorname{Card} \mathcal{F} = k$:

$$\operatorname{Card} \mathcal{F} \leq \log_2(m/b) + \log_{\cos(2\theta)}(b/M) + 3.$$

Combining this with (3.4) in Lemma 3.1, we get a bound on the cardinality of an arbitrary family $\mathcal{B}$ of Besicovitch balls:

$$\operatorname{Card} \mathcal{B} \leq 2(\pi/\theta + 1)\big(\log_2(m/b) + \log_{\cos(2\theta)}(b/M) + 3\big) + 2,$$

which concludes the proof of Theorem 1.14. □

4. Proof of Lemmas 2.14 and 2.16

We begin with a remark that will be technically useful. Given $p, q \in \mathbb{H}$, we set

$$A_p(q) := r_p^2(x_q^2 + y_q^2 - 2x_q x_p - 2y_q y_p) + \left(z_q - \frac{x_p y_q - x_q y_p}{2}\right)^2 - 2z_p\left(z_q - \frac{x_p y_q - x_q y_p}{2}\right).$$

Recall that, following (2.7), we have $r_p = d_\alpha(0, p)$ by convention.

Lemma 4.1. *We have $q \in B(p, r_p)$ if and only if $A_p(q) \leq 0$.*

Proof. Recalling (2.8), we have

$$d_\alpha(p, q) \leq r_p \;\Leftrightarrow\; \frac{(x_q - x_p)^2}{r_p^2} + \frac{(y_q - y_p)^2}{r_p^2} + \frac{\left(z_q - z_p - \frac{x_p y_q - x_q y_p}{2}\right)^2}{r_p^4} \leq \alpha^2.$$

Combining this with (2.5), which gives

$$\frac{x_p^2 + y_p^2}{r_p^2} + \frac{z_p^2}{r_p^4} = \alpha^2,$$

we get the conclusion. □

4.1. Proof of Lemma 2.14

Lemma 4.2. *There exist constants $c_1, c_2 > 0$, depending only on α, such that, for all $\theta \in (0, \pi/4)$ and all $a, b > 0$ with $a^2 \geq b$, we have*

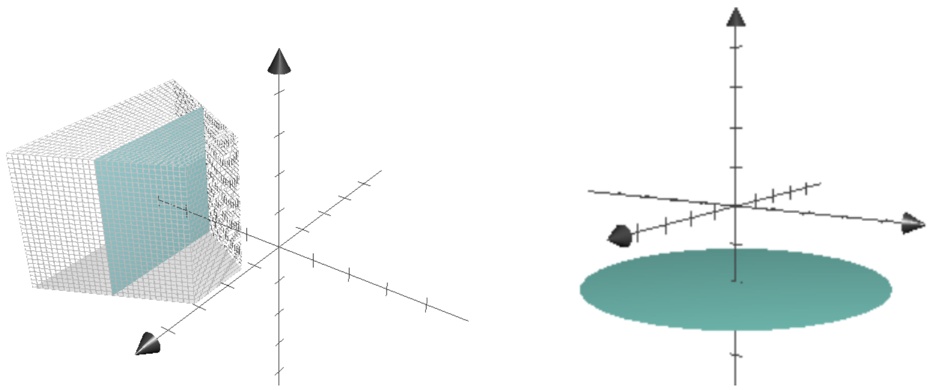
$$c_1 x_p \leq r_p \leq c_2 x_p \quad \text{for all } p \in \mathcal{P}(a, b, \theta).$$

Proof. By (2.6), we always have $r_p^2 \geq x_p^2/(2\alpha^2)$. On the other hand, we can bound r_p^2 from above using $\tan\theta < 1$ (since $\theta < \pi/4$) and $|z_p| < b \leq a^2 \leq x_p^2$ if $p \in \mathcal{P}(a, b, \theta)$ (see (2.13) for the definition of $\mathcal{P}(a, b, \theta)$). Namely, we have

$$\begin{aligned} r_p^2 &= \frac{x_p^2 + y_p^2 + \sqrt{(x_p^2 + y_p^2)^2 + 4\alpha^2 z_p^2}}{2\alpha^2} \\ &\leq \frac{x_p^2(1 + \tan^2\theta) + \sqrt{(x_p^2(1 + \tan^2\theta))^2 + 4\alpha^2 z_p^2}}{2\alpha^2} \\ &\leq \frac{2x_p^2 + \sqrt{4x_p^4 + 4\alpha^2 b^2}}{2\alpha^2} \leq \frac{2 + \sqrt{4 + 4\alpha^2}}{2\alpha^2} x_p^2. \end{aligned}$$ □

For $t \in \mathbb{R}$, $b > 0$ and $\theta \in (0, \pi/2)$, we set (see Figure 3(a))

$$\mathcal{R}(t, b, \theta) := \{p \in \mathbb{H};\ x_p = t,\ |z_p| < b,\ |y_p| < x_p \tan\theta\}.$$



(a) The quadrilateral $\mathcal{R}(t, b, \theta)$ (b) The disc $\mathcal{D}(t, b)$

Fig. 3. The surfaces $\mathcal{R}(t, b, \theta)$ and $\mathcal{D}(t, b)$.

Lemma 4.3. *There exists $\theta_0 \in (0, \pi/4)$, which depends only on α, such that for all $\theta \in (0, \theta_0)$, there exists $a_0(\theta) \geq 1$ such that for all $a > a_0(\theta)$ and all $b \in (0, 1)$, we have*

$$\mathcal{R}(t, b, \theta) \subset B(p, r_p) \quad \text{for all } p \in \mathcal{P}(a, b, \theta) \text{ and all } t \in [1, x_p].$$

Proof. Take $\theta \in (0, \pi/4)$, $a \geq 1 > b$, $p \in \mathcal{P}(a, b, \theta)$, $t > 0$ and consider $q \in \mathcal{R}(t, b, \theta)$. By Lemma 4.1, showing that $q \in B(p, r_p)$ is equivalent to proving that $A_p(q) \leq 0$.

Since $x_q = t$, we have

$$\begin{aligned}
A_p(q) &= r_p^2(t^2 + y_q^2 - 2tx_p - 2y_q y_p) \\
&\quad + \left(z_q - \frac{x_p y_q - t y_p}{2}\right)^2 - 2z_p\left(z_q - \frac{x_p y_q - t y_p}{2}\right) \\
&\le r_p^2(t^2 + y_q^2 - 2tx_p + 2|y_q y_p|) \\
&\quad + \left(|z_q| + \frac{|x_p y_q| + t|y_p|}{2}\right)^2 + 2|z_p|\left(|z_q| + \frac{|x_p y_q| + t|y_p|}{2}\right).
\end{aligned}$$

Note that all terms in the last inequality are positive except $-2tx_p$, since both t and x_p are positive.

We now use the conditions $|y_q| < t\tan\theta$, $|z_q| < b$, $x_p > a$, $|y_p| < x_p \tan\theta$, $|z_p| < b$, $b < 1$ and $\tan\theta < 1$ (since $\theta < \pi/4$) to get

$$\begin{aligned}
A_p(q) &\le r_p^2(t^2 + t^2\tan^2\theta - 2tx_p + 2x_p t\tan^2\theta) + (b + tx_p\tan\theta)^2 + 2b^2 + 2btx_p\tan\theta \\
&\le -2tx_p r_p^2 + r_p^2(t^2 + t^2\tan^2\theta + 2x_p t\tan^2\theta) + (1 + x_p t\tan\theta)^2 + 2(1 + x_p t).
\end{aligned}$$

We now consider separately the cases $t = 1$ and $t = x_p$.

For $t = 1$, we bound, using Lemma 4.2,

$$\begin{aligned}
A_p(q) &\le -2x_p r_p^2 + r_p^2(1 + \tan^2\theta + 2x_p\tan^2\theta) + (1 + x_p\tan\theta)^2 + 2(1 + x_p) \\
&\le -2c_1^2 x_p^3 + c_2^2 x_p^2(1 + \tan^2\theta + 2x_p\tan^2\theta) + (1 + x_p\tan\theta)^2 + 2(1 + x_p) \\
&\le -2(c_1^2 - c_2^2\tan^2\theta)\, x_p^3 + 2c_2^2 x_p^2 + (1 + x_p)^2 + 2(1 + x_p).
\end{aligned}$$

Hence $A_p(q) \le -2(c_1^2 - c_2^2\tan^2\theta)\, x_p^3 + o(x_p^3)$ as $x_p \to \infty$. Thus, choosing θ small enough so that $c_1^2 - c_2^2\tan^2\theta > 0$, we get $A_p(q) \le 0$ provided x_p is large enough.

For $t = x_p$, we use Lemma 4.2 once again to get

$$\begin{aligned}
A_p(q) &\le -2r_p^2 x_p^2 + r_p^2(x_p^2 + 3x_p^2\tan^2\theta) + (1 + x_p^2\tan\theta)^2 + 2(1 + x_p^2) \\
&\le -c_1^2 x_p^4 + 3c_2^2 x_p^4\tan^2\theta + (1 + x_p^2\tan\theta)^2 + 2(1 + x_p^2) \\
&\le -(c_1^2 - 3c_2^2\tan^2\theta - \tan^2\theta)x_p^4 + 1 + 2x_p^2 + 2(1 + x_p^2).
\end{aligned}$$

Hence $A_p(q) \le -(c_1^2 - 3c_2^2\tan^2\theta - \tan^2\theta)x_p^4 + o(x_p^4)$ as $x_p \to \infty$. Thus, choosing θ small enough so that $c_1^2 - 3c_2^2\tan^2\theta - \tan^2\theta > 0$, we get $A_p(q) \le 0$ provided x_p is large enough.

Altogether we have showed that one can find $\theta_0 \in (0, \pi/4)$, depending only on α, and for all $\theta \in (0, \theta_0(\alpha))$, some $a_0(\theta) \ge 1$, such that for $a > a_0(\theta)$ and $b < 1$ and for all $p \in \mathcal{P}(a, b, \theta)$, we have

$$\mathcal{R}(1, b, \theta) \subset B_\alpha(p, r_p) \quad \text{and} \quad \mathcal{R}(x_p, b, \theta) \subset B_\alpha(p, r_p).$$

Since $B_\alpha(p, r_p)$ is Euclidean convex by Proposition 2.10, we conclude the proof by noting that $\mathcal{R}(t, b, \theta)$, for $t \in [1, x_p]$, is in the Euclidean convex hull of $\mathcal{R}(1, b, \theta)$ and $\mathcal{R}(x_p, b, \theta)$. □

Proof of Lemma 2.14. Let $\theta_0 \in (0, \pi/4)$ be given by Lemma 4.3. Let $\theta \in (0, \theta_0)$ and let $a_0(\theta) \geq 1$ be given by Lemma 4.3. Let $a > a_0(\theta)$ and $b \in (0, 1)$. Let $p, q \in \mathbb{H}$ be such that $p \notin B(q, r_q)$ and $q \notin B(p, r_p)$. Assume with no loss of generality that $x_q \leq x_p$. Then, if both p and q were in $\mathcal{P}(a, b, \theta)$, by Lemma 4.3 we would have $q \in \mathcal{R}(x_q, b, \theta) \subset B(p, r_p)$ since $x_q \in [1, x_p]$. But this would contradict the assumptions. □

4.2. *Proof of Lemma 2.16*

Lemma 4.4. *Let $a \geq 1$ and $b > 0$. Then for all $p \in \mathcal{T}(a, b)$, we have*

$$r_p^2 \leq \frac{b^2 + \sqrt{b^4 + 4\alpha^2}}{2\alpha^2} |z_p|.$$

Proof. Let $p \in \mathcal{T}(a, b)$ (see (2.15) for the definition of $\mathcal{T}(a, b)$). Since $1 \leq a < |z_p|$ and $\rho_p < b$, we have (recall (2.6))

$$r_p^2 \leq \frac{|z_p|\rho_p^2 + \sqrt{z_p^2 \rho_p^4 + 4\alpha^2 z_p^2}}{2\alpha^2} = \frac{\rho_p^2 + \sqrt{\rho_p^4 + 4\alpha^2}}{2\alpha^2} |z_p| \leq \frac{b^2 + \sqrt{b^4 + 4\alpha^2}}{2\alpha^2} |z_p|.$$

□

For $t \in \mathbb{R}$ and $b > 0$, we set (see Figure 3(b))

$$\mathcal{D}(t, b) := \{p \in \mathbb{H};\ z_q = t,\ \rho_p < b\}.$$

Lemma 4.5. *There exist $a_1 \geq 1$ and $b_1 \in (0, 1)$, depending only on α, such that for all $a > a_1$ and all $b \in (0, b_1)$, we have*

$$\mathcal{D}(t, b) \subset B(p, r_p) \quad \textit{for all } p \in \mathcal{T}(a, b) \textit{ and all } t \in [z_p, -1].$$

Proof. Take $a \geq 1 > b$, $p \in \mathcal{T}(a, b)$, $t < 0$ and consider $q \in \mathcal{D}(t, b)$. By Lemma 4.1, showing that $q \in B(p, r_p)$ is equivalent to proving that $A_p(q) \leq 0$. Since $z_q = t$, we have

$$\begin{aligned}
A_p(q) &= r_p^2(x_q^2 + y_q^2 - 2x_q x_p - 2y_q y_p) \\
&\quad + \left(t - \frac{x_p y_q - x_q y_p}{2}\right)^2 - 2z_p\left(t - \frac{x_p y_q - x_q y_p}{2}\right) \\
&\leq r_p^2(x_q^2 + y_q^2 + 2|x_q x_p| + 2|y_q y_p|) + \left(|t| + \frac{|x_p y_q| + |x_q y_p|}{2}\right)^2 \\
&\quad - 2t z_p + |z_p|(|x_p y_q| + |x_q y_p|).
\end{aligned}$$

Note that all terms in the last inequality are positive except $-2tz_p$, assuming both t and z_p are negative. Using Lemma 4.4 and the fact that the absolute value of each of the first two components of p and q is smaller than b, we bound

$$\begin{aligned}A_p(q) &\leq 6\frac{b^2+\sqrt{b^4+4\alpha^2}}{2\alpha^2}b^2|z_p| + (|t|+b^2)^2 - 2tz_p + 2b^2|z_p| \\ &\leq -z_p + (|t|+1)^2 - 2tz_p,\end{aligned}$$

where in the last inequality we have assumed that b is small enough, $b < b_1$ for some b_1 which depends only on α.

We now consider separately the cases $t = -1$ and $t = z_p$. For $t = -1$, we need $z_p + 4 \leq 0$, which is true as soon as $z_p \leq -4$. For $t = z_p$, we need $-z_p + (-z_p+1)^2 - 2z_p^2 = -z_p^2 - 3z_p + 1 \leq 0$, which is true as soon as $|z_p|$ is large enough.

Altogether we have shown that one can find $a_1 \geq 1$ and $b_1 \in (0,1)$, depending only on α, such that, for all $a > a_1$ and $b \in (0, b_1)$ and all $p \in \mathcal{T}(a,b)$, we have

$$\mathcal{D}(-1,b) \subset B(p,r_p) \quad \text{and} \quad \mathcal{D}(z_p,b) \subset B(p,r_p).$$

Recall that the set $B(p,r_p)$ is Euclidean convex by Proposition 2.10. This concludes the proof since $\mathcal{D}(t,b)$, for $t \in [z_p,-1]$, is in the Euclidean convex hull of $\mathcal{D}(-1,b)$ and $\mathcal{D}(z_p,b)$. $\square$

Proof of Lemma 2.16. Let $a_1 \geq 1$ and $b_1 \in (0,1)$ be given by Lemma 4.5. Let $a > a_1$ and $b \in (0,b_1)$. Let $p,q \in \mathbb{H}$ be such that $p \notin B(q,r_q)$ and $q \notin B(p,r_p)$. Assume with no loss of generality that $z_p \leq z_q$. Then, if both p and q were in $\mathcal{T}(a,b)$, by Lemma 4.5 we would have $q \in \mathcal{D}(z_q,b) \subset B(p,r_p)$ since $z_q \in [z_p,-1]$. But this would contradict the assumptions. $\square$

5. Proof of Lemma 2.18

We first fix some notation. For $z \in \mathbb{R}$, we set $p_z := (0,0,z)$. For $\theta \in (0,\pi/2)$, $p \in \mathbb{H}$ and $z \in \mathbb{R}$, let $\mathcal{C}(z,\pi(p),\theta)$ denote the two-dimensional Euclidean half-cone in $\mathbb{H} \simeq \mathbb{R}^3$ contained in the plane $\{q \in \mathbb{H};\ z_q = z\}$ with vertex p_z, axis the half-line starting at p_z and passing through (x_p, y_p, z), and aperture 2θ. See Figure 4(a).

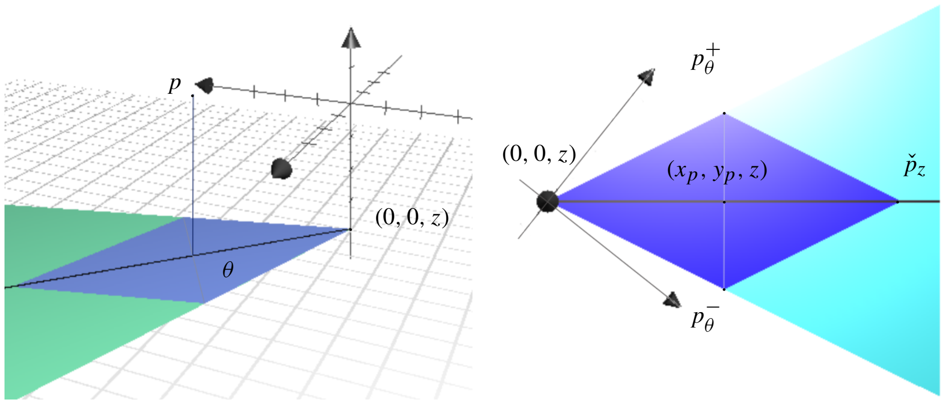
For $\theta \in (0,\pi/2)$, $p \in \mathbb{H}$ and $z \in \mathbb{R}$, let $\mathcal{Q}(z,\pi(p),\theta)$ denote the two-dimensional Euclidean equilateral quadrilateral contained in the plane $\{q \in \mathbb{H};\ z_q = z\}$ with vertices p_z, $p_\theta^+ := (x_p - y_p\tan\theta, y_p + x_p\tan\theta, z)$, $p_\theta^- := (x_p + y_p\tan\theta, y_p - x_p\tan\theta, z)$ and $\check{p}_z := (2x_p, 2y_p, z)$. Note that it is the Euclidean convex hull in $\mathbb{H} \simeq \mathbb{R}^3$ of these four points. See Figure 4(b).

Recall (2.17) for the definition of $\mathcal{C}(\theta)$. Note that $q \in \mathcal{C}(\theta)$ if and only if $(x_q, y_q, 0) \in \mathcal{C}(0,\pi((1,0,0)),\theta)$.

We have the following properties:

$$p,q \in \mathcal{C}(\theta) \Rightarrow q \in \mathcal{C}(z_q,\pi(p),2\theta), \tag{5.1}$$

$$\mathcal{Q}(z,\pi(p),\theta) \subset \mathcal{C}(z,\pi(p),\theta). \tag{5.2}$$



(a) The cone $\mathcal{C}(z, \pi(p), \theta)$ containing the quadrilateral $\mathcal{Q}(z, \pi(p), \theta)$.

(b) The quadrilateral $\mathcal{Q}(z, \pi(p), \theta)$.

Fig. 4. The surfaces $\mathcal{C}(z, \pi(p), \theta)$ and $\mathcal{Q}(z, \pi(p), \theta)$.

For $\theta \in (0, \pi/4)$, we have

$$\mathcal{C}(z, \pi(p), \theta) \cap \{q \in \mathbb{H};\ \rho_q \cos\theta \le \rho_p\} \subset \mathcal{Q}(z, \pi(p), \theta). \tag{5.3}$$

This follows from elementary geometry by noting that the angle between the half-lines starting at p_θ^+ and passing through p_z and $\check{p}_z$ respectively is larger than $\pi/2$.

Lemma 5.4. *There exists $\theta_2 \in (0, \pi/2)$, which depends only on α, such that*

$$\mathcal{Q}(z, \pi(p), \theta) \subset B(p, r_p)$$

for all $0 < \theta \le \theta_2$, all $p \in \mathbb{H} \setminus \{0\}$ and all $z \in \mathbb{R}$ such that $|z - z_p| \le |z_p|$.

Proof. Recalling Proposition 2.10, we only need to prove that the vertices p_z, p_θ^+, p_θ^- and $\check{p}_z$ of $\mathcal{Q}(z, \pi(p), \theta)$ belong to $B(p, r_p)$.

We have $|z - z_p| \le |z_p|$ and, recalling (2.5) and (2.7),

$$\frac{\rho_p^2}{r_p^2} + \frac{|z_p|^2}{r_p^4} = \alpha^2,$$

hence

$$\frac{\rho_p^2}{r_p^2} + \frac{|z - z_p|^2}{r_p^4} \le \frac{\rho_p^2}{r_p^2} + \frac{|z_p|^2}{r_p^4} = \alpha^2,$$

that is, recalling (2.8), $p_z = (0, 0, z) \in B(p, r_p)$.

Similarly we have

$$\frac{(2x_p - x_p)^2}{r_p^2} + \frac{(2y_p - y_p)^2}{r_p^2} + \frac{|z - z_p|^2}{r_p^4} = \frac{\rho_p^2}{r_p^2} + \frac{|z - z_p|^2}{r_p^4} \le \alpha^2,$$

hence $\check{p}_z = (2x_p, 2y_p, z) \in B(p, r_p)$.

Next, let us prove that $p_\theta^+ = (x_p - y_p \tan\theta, y_p + x_p \tan\theta, z) \in B(p, r_p)$. Set

$$\Delta := \frac{(z - z_p - \rho_p^2 \tan\theta/2)^2}{r_p^4} + \frac{\rho_p^2 \tan^2\theta}{r_p^2}.$$

We need to prove that $\Delta \le \alpha^2$. We have

$$\begin{aligned}
\Delta &= \frac{(z - z_p)^2}{r_p^4} + \frac{\rho_p^4 \tan^2\theta}{4r_p^4} - \frac{\rho_p^2 (z - z_p)\tan\theta}{r_p^4} + \frac{\rho_p^2 \tan^2\theta}{r_p^2} \\
&\le \frac{z_p^2}{r_p^4} + \frac{\rho_p^4 \tan^2\theta}{4r_p^4} + \frac{\rho_p^2 |z - z_p| \tan\theta}{r_p^4} + \frac{\rho_p^2 \tan^2\theta}{r_p^2} \\
&\le \alpha^2 - \frac{\rho_p^2}{r_p^2} + \frac{\rho_p^2}{r_p^2}\left(\frac{\alpha^2 \tan^2\theta}{4} + \alpha\tan\theta + \tan^2\theta\right)
\end{aligned}$$

where the last inequality follows from the fact that

$$\frac{\rho_p^2}{r_p^2} + \frac{z_p^2}{r_p^4} = \alpha^2,$$

which implies in particular that $\rho_p^2/r_p^2 \le \alpha^2$ and $|z - z_p|/r_p^2 \le \alpha$. Hence

$$\Delta \le \alpha^2 - \frac{\rho_p^2}{r_p^2}\big(1 - (1 + \alpha^2/4)\tan^2\theta - \alpha\tan\theta\big).$$

Choosing $\theta_2 \in (0, \pi/2)$ small enough so that $1 - (1 + \alpha^2/4)\tan^2\theta - \alpha\tan\theta \ge 0$ for all $0 < \theta \le \theta_2$, we get the conclusion.

The fact that $p_\theta^- \in B(p, r_p)$ is proved in a similar way. □

Proof of Lemma 2.18. Let $\theta_1 = \min(\theta_2/2, \pi/8)$ where θ_2 is given by Lemma 5.4. Let $\theta \in (0, \theta_1)$ and let $p, q \in \mathbb{H}$ satisfy (2.19)–(2.22).

Let us first prove (2.23). Assume for contradiction that $2z_p \le z_q \le 0$. Then $|z_q - z_p| \le |z_p|$. Hence $\mathcal{Q}(z_q, \pi(p), 2\theta) \subset B(p, r_p)$ according to Lemma 5.4. On the other hand, it follows from (2.21), (5.1), (2.20) and (5.3) that $q \in \mathcal{Q}(z_q, \pi(p), 2\theta)$, and hence $q \in B(p, r_p)$, which contradicts (2.22).

Thus $z_q < 2z_p \le z_p \le 0$, and so $|z_p - z_q| \le |z_q|$. It follows from (2.21), (5.1) and (2.22) that $p \in \mathcal{C}(z_p, \pi(q), 2\theta) \setminus B(q, r_q)$. Finally, Lemma 5.4 implies that $p \in \mathcal{C}(z_p, \pi(q), 2\theta) \setminus \mathcal{Q}(z_p, \pi(q), 2\theta)$, and then (2.24) follows from (5.3). □

6. Two criteria for distances for which BCP does not hold

In this section we prove two criteria which imply the non-validity of BCP. This shows that in some sense our example of homogeneous distance d_α for which BCP holds is sharp. Roughly speaking, the first criterion applies to homogeneous distances whose unit sphere centered at the origin either has inward cone-like singularities in the Euclidean sense at the poles (i.e., at the intersection of the sphere with the z-axis), or is flat at the poles with zero curvature in the Euclidean sense. The second criterion applies to homogeneous distances whose unit sphere at the origin has outward cone-like singularities in the Euclidean sense at the poles. Note that the unit sphere centered at the origin of our distance d_α is smooth with positive curvature in the Euclidean sense.

6.1. Distances with ingoing corners or second-order flat at the poles

Let d be a homogeneous distance on $\mathbb{H}$ and let B denote the closed unit ball centered at the origin in $(\mathbb{H}, d)$.

In this subsection, most of the time we shall identify $\mathbb{H}$ with $\mathbb{R}^3$ equipped with its usual differential structure.

For $p \in \mathbb{H}$, $\vec{v} \in \mathbb{R}^3$, $\vec{v} \neq (0,0,0)$, and $\alpha \in (0, \pi/2)$, let $\mathrm{Cone}(p, \vec{v}, \alpha)$ denote the Euclidean half-cone in $\mathbb{H}$, identified with $\mathbb{R}^3$, with vertex p, axis $p+\mathbb{R}^+\vec{v}$ and opening 2α.

We say that $\vec{v} \in \mathbb{R}^3$, $\vec{v} \neq (0,0,0)$, *points out of B at $p \in \partial B$* if there exists an open neighborhood U of p and some $\alpha \in (0, \pi/2)$ such that

$$B \cap \mathrm{Cone}(p, \vec{v}, \alpha) \cap U = \{p\}.$$

Let τ_p denote the left translation defined by $\tau_p(q) := p \cdot q$. We consider it as an affine map from $\mathbb{H}$, identified with $\mathbb{R}^3$, to $\mathbb{R}^3$ whose differential, in the usual Euclidean sense in $\mathbb{R}^3$, is thus a constant linear map and will be denoted by $(\tau_p)_*$. Let $\hat{\pi}$ be defined by $\hat{\pi}(x, y, z) := (x, y, 0)$.

For $\vec{v} \in \mathbb{R}^3$, $\vec{v} \neq (0,0,0)$, and $\epsilon > 0$, let $\Omega(\vec{v})$ denote the set of points $q \in \partial B$ such that $(\tau_{q^{-1}})_*(\vec{v})$ points out of B at q^{-1}, and let $\Omega_\epsilon(\vec{v})$ denote the set of points $q \in \Omega(\vec{v})$ such that $\hat{\pi}(q) \in \mathbb{R}^+\vec{w}$ for some $\vec{w} \in \mathrm{Im}(\hat{\pi})$ such that $\|\vec{w} - \vec{v}\|_{\mathbb{R}^3} \leq \epsilon$ (here $\|\cdot\|_{\mathbb{R}^3}$ denotes the Euclidean norm in $\mathbb{R}^3$).

Theorem 6.1. *Assume that there exists $\vec{v} \in \mathrm{Im}(\hat{\pi})$, $\vec{v} \neq (0,0,0)$, and $\overline{\epsilon} > 0$ such that $\Omega_\epsilon(\vec{v}) \neq \emptyset$ for all $0 < \epsilon \leq \overline{\epsilon}$. Then BCP does not hold in $(\mathbb{H}, d)$.*

Proof. We first construct a sequence $(q_n)_{n\geq 0}$ of points in ∂B such that $q_n \in \Omega(\vec{v})$ for all $n \geq 0$ and $(\tau_{q_k^{-1}})_*(\hat{\pi}(q_n))$ points out of B at q_k^{-1} for all $n \geq 1$ and all $0 \leq k \leq n-1$.

Note that if $q \in \Omega(\vec{v})$ then there exists $\epsilon(q) > 0$ such that $(\tau_{q^{-1}})_*(\vec{v} + \vec{\epsilon})$ points out of B at q^{-1} for all $\vec{\epsilon} \in \mathbb{R}^3$ such that $\|\vec{\epsilon}\|_{\mathbb{R}^3} \leq \epsilon(q)$ (note that the set of vectors that point out of B at some point $p \in \partial B$ is open).

Let us start by choosing some $q_0 \in \Omega(\vec{v})$. By induction assume that $q_0, \ldots, q_n$ have already been chosen. Let $\epsilon = \min(\epsilon(q_0), \ldots, \epsilon(q_n), \overline{\epsilon})$ where each $\epsilon(q_k)$ is associated to $q_k \in \Omega(\vec{v})$ as above. Then we choose $q_{n+1} \in \Omega_\epsilon(\vec{v})$. We have $\hat{\pi}(q_{n+1}) = \lambda(\vec{v} + \vec{\epsilon})$ for some $\lambda > 0$ and some $\vec{\epsilon} \in \mathrm{Im}(\hat{\pi})$ such that $\|\vec{\epsilon}\|_{\mathbb{R}^3} \leq \epsilon$. Hence, by choice of ϵ and of the q_k's, the vector $(\tau_{q_k^{-1}})_*(\hat{\pi}(q_{n+1})) = \lambda\,(\tau_{q_k^{-1}})_*(\vec{v} + \vec{\epsilon})$ points out of B at q_k^{-1} for all $0 \leq k \leq n$ as wanted.

Next, we claim that if $q, q' \in \partial B$ are such that $\hat{\pi}(q') \neq (0,0,0)$ and $(\tau_{q^{-1}})_*(\hat{\pi}(q'))$ points out of B at q^{-1}, then there exists $\overline{\lambda} > 0$ such that $d(q, \delta_\lambda(q')) > 1$ for all

$0 < \lambda \leq \overline{\lambda}$. Indeed, $[0,\infty) \ni \lambda \mapsto q^{-1} \cdot \delta_\lambda(q')$ is a smooth curve starting at q^{-1} and whose tangent vector at $\lambda = 0$ is given by $(\tau_{q^{-1}})_*(\hat{\pi}(q'))$. Since this vector points out of B at q^{-1}, it follows that $q^{-1} \cdot \delta_\lambda(q') \notin B$ for all $\lambda > 0$ small enough, and hence $d(q, \delta_\lambda(q')) = d(0, q^{-1} \cdot \delta_\lambda(q')) > 1$ as desired.

It follows that for all $n \geq 1$, one can find $\lambda_n > 0$ such that for all $0 < \lambda \leq \lambda_n$ and all $0 \leq k < n$, one has

$$d(q_k, \delta_\lambda(q_n)) > 1.$$

Then we set $r_0 = 1$ and by induction we construct a decreasing sequence $(r_n)_{n\geq 0}$ so that

$$d(q_k, \delta_{r_n/r_k}(q_n)) > 1$$

for all $n \geq 1$ and all $0 \leq k < n$. For $n \geq 0$, we set $p_n = \delta_{r_n}(q_n)$. By construction we have

$$d(p_k, p_n) > \max(r_k, r_n)$$

for all $k \geq 0$ and $n \geq 0$ such that $k \neq n$. It follows that $\{B_d(p_n, r_n);\ n \in J\}$ is a family of Besicovitch balls for any finite set $J \subset \mathbb{N}$, and hence BCP does not hold. □

Let us give some examples of homogeneous distances to which the criterion given in Theorem 6.1 applies.

A first class of examples is given by rotationally invariant homogeneous distances d for which there exists $p \in \partial B$ such that $(x_p, y_p) \neq (0,0)$ and

$$z_p = \max\{z > 0;\ (x, y, z) \in \partial B \text{ for some } (x, y) \in \mathbb{R}^2\}.$$

By rotationally invariant distances, we mean distances for which rotations R_θ, $\theta \in \mathbb{R}$, are isometries (see (2.11) for the definition of R_θ).

Indeed, consider $\vec{v} = (1, 0, 0)$, and for $\varepsilon > 0$ set

$$\lambda = \left(\frac{x_p^2 + y_p^2}{1+\varepsilon^2}\right)^{1/2}.$$

Then consider $q = (\lambda, \lambda\varepsilon, -z_p)$. By rotational and left invariance (which implies in particular that $d(0, q) = d(0, q^{-1})$ for all $q \in \mathbb{H}$), one has $q \in \partial B$. On the other hand, since $\{(x, y, z) \in \mathbb{H};\ z > z_p\} \cap B = \emptyset$, any vector with a positive third coordinate points out of B at q^{-1}. In particular $(\tau_{q^{-1}})_*(\vec{v}) = (1, 0, \lambda\varepsilon/2)$ points out of B at q^{-1}. Hence $q \in \Omega_\epsilon(\vec{v})$.

This class of examples includes the so-called *box-distance* d_∞ defined by $d_\infty(p, q) := \|p^{-1} \cdot q\|_\infty$ with

$$\|p\|_\infty := \max((x_p^2 + y_p^2)^{1/2}, 2|z_p|^{1/2}) \tag{6.2}$$

for which the fact that BCP does not hold was not known. It also includes the Carnot–Carathéodory distance, which gives a new proof that BCP fails for this distance. See [22] for a previous different proof.

Other examples of homogeneous distances d to which Theorem 6.1 applies can be obtained in the following way. Assume that B, respectively ∂B, can be described as

$\{q \in \mathbb{H};\ f(q) \leq 0\}$, respectively $\{q \in \mathbb{H};\ f(q) = 0\}$, for some C^1 real valued function f in a neighborhood of a point $p \in \partial B$. Then the outward normal to ∂B at some point $q \in \partial B$ is given in a neighborhood of p by $\nabla f(q)$ (here it is still understood that we identify $\mathbb{H}$ with $\mathbb{R}^3$ and ∇ denotes the usual gradient in $\mathbb{R}^3$). Then Theorem 6.1 applies if one can find a nonzero vector $\vec{v} \in \mathrm{Im}(\hat{\pi})$ such that for all ε small enough, the following holds. There exists $q \in \partial B$ such that $\hat{\pi}(q) \in \mathbb{R}^+\vec{w}$ for some $\vec{w} \in \mathrm{Im}(\hat{\pi})$ with $\|\vec{w} - \vec{v}\|_{\mathbb{R}^3} \leq \epsilon$ and such that q^{-1} lies in a neighborhood of p and

$$\langle \nabla f(q^{-1}), (\tau_{q^{-1}})_*(\vec{v})\rangle > 0$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^3$.

A particular example is when B, respectively ∂B, can be described near the north pole (intersection of ∂B with the positive z-axis) as the subgraph $\{(x, y, z) \in \mathbb{H};\ z \leq \varphi(x, y)\}$, respectively the graph $\{(x, y, z) \in \mathbb{H};\ z = \varphi(x, y)\}$, of a C^2 function φ whose first and second order partial derivatives vanish at the origin. Indeed, in that case one can choose for example $\vec{v} = (1, 0, 0)$ and for a fixed $\epsilon > 0$, one looks for some $q \in \Omega_\epsilon(\vec{v})$ of the form $q = (\lambda, \lambda\epsilon, -\varphi(-\lambda, -\lambda\epsilon))$ for some $\lambda > 0$. Then $q^{-1} = (-\lambda, -\lambda\epsilon, \varphi(-\lambda, -\lambda\epsilon)) \in \partial B$ lies near the north pole for $\lambda > 0$ small and we have

$$\langle \nabla f(q^{-1}), (\tau_{q^{-1}})_*(\vec{v})\rangle = -\partial_x \varphi(-\lambda, -\lambda\epsilon) + \tfrac{1}{2}\lambda\epsilon,$$

which is equivalent to $\lambda\epsilon/2 > 0$ when $\lambda > 0$ is small enough. Hence $\Omega_\epsilon(\vec{v}) \neq \emptyset$.

This argument applies to the Cygan–Korányi distance $d_{g,2}$, and more generally to $d_{g,\alpha}$ for all $\alpha > 0$ such that $d_{g,\alpha}$ defines a distance, thus in particular for all $\alpha \leq 2$. Recall from (1.11) that $d_{g,\alpha}(p, q) := \|p^{-1} \cdot q\|_{g,\alpha}$ where

$$\|p\|_{g,\alpha} := ((x_p^2 + y_p^2)^2 + 4\alpha^2 z_p^2)^{1/4},$$

and that $d_{g,2}$ is the Cygan–Korányi distance. Hence Theorem 6.1 gives in particular a new geometric proof that BCP does not hold for the Cygan–Korányi distance on $\mathbb{H}$ (see [13] and [23] for previous analytic proofs).

6.2. Distances with outgoing corners at the poles

Let d be a homogeneous distance on $\mathbb{H}$ and let B denote the closed unit ball centered at the origin in $(\mathbb{H}, d)$. Set $S^+ := \partial B \cap \{p \in \mathbb{H};\ z_p > 0\}$.

Theorem 6.3. *Assume that there exist two sequences of points $p_n^+, p_n^- \in S^+$ and some $a, \overline{x} > 0$ such that*

$$\begin{aligned} &p_n^- = (x_n^-, 0, z_n^-), \quad p_n^+ = (x_n^+, 0, z_n^+), \quad x_n^- < 0 < x_n^+,\\ &\lim_{n \to 0}(x_n^+ - x_n^-) = 0, \quad z_n^- > z_n^+ > 0, \quad z_n^+ - z_n^- < -a\,(x_n^+ - x_n^-),\\ &\{p \in \mathbb{H};\ x_n^+ \leq x_p \leq \overline{x},\ y_p = 0,\ z_p > z_n^+\} \subset \mathbb{H} \setminus B. \end{aligned}$$

Then BCP does not hold in $(\mathbb{H}, d)$.

The geometric meaning of the above assumptions is the following. In some vertical plane (here we take the xz-plane for simplicity) one can find two sequences of points, p_n^+ and p_n^-, on the unit sphere centered at the origin. The points p_n^+ and p_n^- are on different sides of the z-axis. The two sequences converge to the north pole. The slope between p_n^- and p_n^+ is assumed to be bounded away from zero. We further assume that at the north pole the intersection of the sphere and the xz-plane can be written both as $x = x(z)$ and as $z = z(x)$. See Figure 5.

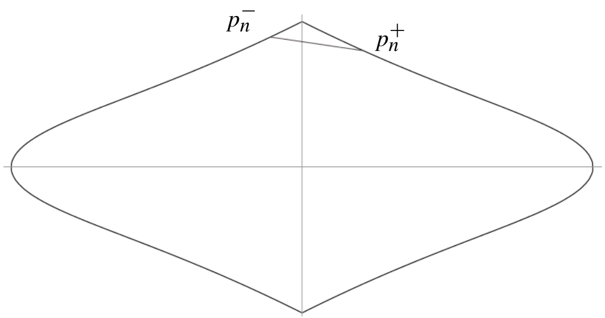


Fig. 5. Intersection of the xz-plane and the unit sphere at the origin of the distance $d_{\kappa,\alpha}$ when $\kappa = 1$ and $\alpha = 2$.

Theorem 6.3 applies in particular if the intersection of B with the xz-plane can be described near the north pole as $\{p \in \mathbb{H};\ -\varepsilon < x_p < \varepsilon,\ y_p = 0,\ 0 < z_p \leq f(x_p)\}$ for some function f of class C^1 on $(-\varepsilon, \varepsilon) \setminus \{0\}$ such that $f'(0^-)$ and $f'(0^+)$ exist and are finite with $f'(0^+) < 0$. This is for instance the case of the following distances built from the Cygan–Korányi distance, and more generally from the distances $d_{g,\alpha}$, and given by

$$d_{\kappa,\alpha}(p, q) := \|p^{-1} \cdot q\|_{\kappa,\alpha} \quad \text{with} \quad \|p\|_{\kappa,\alpha} := \kappa\rho(p) + \|p\|_{g,\alpha}$$

for some $\kappa > 0$. See (1.11) for the definition of $\rho(\cdot)$ and $\|\cdot\|_{g,\alpha}$. Figure 5 is exactly the intersection of the xz-plane and the unit sphere at the origin when $\kappa = 1$ and $\alpha = 2$.

Note that it follows in particular that the l^1-sum of the pseudo-distance d_ρ with the distance $d_{g,\alpha}$ does not satisfy BCP, in contrast with their l^2-sum which is a multiple of the distance d_α.

Proof of Theorem 6.3. By induction, we construct a sequence of points $q_k = (x_k, 0, z_k)$ such that

$$z_{k+1} < z_k < 0 < x_{k+1} < x_k \quad \text{and} \quad r_{k+1} > r_k$$

for all $k \in \mathbb{N}$, where $r_k = d(0, q_k)$, and such that

$$q_l \notin B_d(q_{k+1}, r_{k+1}) \qquad \text{for all } k \in \mathbb{N} \text{ and all } 0 \leq l \leq k.$$

Then we will have $d(q_l, q_k) > \max(r_l, r_k)$ for all $l \in \mathbb{N}$ and $k \in \mathbb{N}$ such that $l \neq k$, so that $\{B_d(q_k, r_k);\ k \in J\}$ is a family of Besicovitch balls for any finite set $J \subset \mathbb{N}$. Hence BCP does not hold.

We start from a point $q_0 = (x_0, 0, z_0)$ with $z_0 < 0 < x_0$. Next assume that $q_0, \ldots, q_k$ have been constructed and choose n large enough so that

$$r_k < \frac{x_k}{x_n^+ - x_n^-}, \tag{6.4}$$

$$-a < \frac{x_n^+ - x_n^-}{x_k^2} z_k < 0, \tag{6.5}$$

$$x_0 \leq \frac{x_k}{x_n^+ - x_n^-}\, \overline{x}. \tag{6.6}$$

We set

$$r_{k+1} := \frac{x_k}{x_n^+ - x_n^-} \quad \text{and} \quad q_{k+1} := \delta_{r_{k+1}}(p_n^-)^{-1}. \tag{6.7}$$

Note that $d(0, q_{k+1}) = r_{k+1}$ since $p_n^- \in \partial B$. We have $r_{k+1} > r_k$ by choice of n (see (6.4)). We also have

$$x_{k+1} = -r_{k+1} x_n^- = \frac{-x_n^-}{x_n^+ - x_n^-}\, x_k < x_k.$$

Hence it remains to check that $z_{k+1} < z_k$ and $q_l \notin B(q_{k+1}, r_{k+1})$ for $0 \leq l \leq k$.

Using dilation, left translation and the assumption $\{p \in \mathbb{H};\ x_n^+ \leq x_p \leq \overline{x},\ y_p = 0,\ z_p > z_n^+\} \subset \mathbb{H} \setminus B$, it follows that

$$\{p \in \mathbb{H};\ x_k \leq x_p \leq r_{k+1}\overline{x} - r_{k+1}x_n^-,\ y_p = 0,\ z_p > z_{k+1} + r_{k+1}^2 z_n^+\} \subset \mathbb{H} \setminus B(q_{k+1}, r_{k+1}).$$

Hence, taking into account the fact that $z_k < \cdots < z_0$ and $x_k < \cdots < x_0$, to prove that $z_{k+1} < z_k$ and that $q_l \notin B(q_{k+1}, r_{k+1})$ for $0 \leq l \leq k$, we only need to check that $x_0 \leq r_{k+1}\overline{x} - r_{k+1}x_n^-$, which follows from (6.6), and that $z_k > z_{k+1} + r_{k+1}^2 z_n^+$. Using the fact that $z_n^+ - z_n^- < -a\,(x_n^+ - x_n^-)$, (6.5) and (6.7), we have

$$\begin{aligned} z_{k+1} + r_{k+1}^2 z_n^+ &= r_{k+1}^2 (z_n^+ - z_n^-) < -a\,(x_n^+ - x_n^-) r_{k+1}^2 \\ &< \frac{(x_n^+ - x_n^-)^2 z_k}{x_k^2} \cdot \frac{x_k^2}{(x_n^+ - x_n^-)^2} = z_k, \end{aligned}$$

which gives the conclusion. □

7. Generalization to any Heisenberg group $\mathbb{H}^n$

The case of $\mathbb{H}^n$ for $n \geq 1$ arbitrary can be easily handled similarly by adopting the following convention. For $p \in \mathbb{H}^n$, we set $p = (x_p, y_p, z_p)$ where $x_p \in \mathbb{R}$, $y_p \in \mathbb{R}^{2n-1}$ and $z_p \in \mathbb{R}$. Note that this is different from the more standard presentation adopted in the introduction (Section 1). To avoid any confusion, the explicit correspondence between theses two conventions is the following. If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and $z \in \mathbb{R}$ denote the exponential and homogeneous coordinates of $p \in \mathbb{H}^n$ as in (1.7), by denoting $p = (x_p, y_p, z_p)$ with $x_p \in \mathbb{R}$, $y_p \in \mathbb{R}^{2n-1}$ and $z_p \in \mathbb{R}$, we mean $x_p = x_1$,

$y_p = (x_2, \ldots, x_n, y_1, \ldots, y_n)$ and $z_p = z$. It follows that y_p^2 should be replaced by $\|y_p\|^2_{\mathbb{R}^{2n-1}}$ and $|y_p|$ by $\|y_p\|_{\mathbb{R}^{2n-1}}$ where $\|\cdot\|_{\mathbb{R}^{2n-1}}$ denotes the Euclidean norm in $\mathbb{R}^{2n-1}$.

In particular, we get

$$\rho_p = \sqrt{x_p^2 + \|y_p\|^2_{\mathbb{R}^{2n-1}}},$$

and setting

$$\begin{aligned} \mathcal{P}(a,b,\theta) &:= \{p \in \mathbb{H}^n;\ x_p > a,\ |z_p| < b,\ \|y_p\|_{\mathbb{R}^{2n-1}} < x_p \tan\theta\}, \\ \mathcal{T}(a,b) &:= \{p \in \mathbb{H}^n;\ z_p < -a,\ \rho_p < b\}, \end{aligned}$$

one can easily check that Lemmas 2.14 and 2.16 hold true in $\mathbb{H}^n$ with essentially the same proofs.

Lemma 2.18 and its proof extend to the case of $\mathbb{H}^n$ by setting

$$\mathcal{C}(\theta) := \{p \in \mathbb{H}^n;\ \|y_p\|_{\mathbb{R}^{2n-1}} < x_p \tan\theta\}$$

and considering the analogue of the sets $\mathcal{C}(z, \pi(p), \theta)$ and $\mathcal{Q}(z, \pi(p), \theta)$ (introduced in Section 5) defined in the following way.

The set $\mathcal{C}(z, \pi(p), \theta)$ is now defined as the $2n$-dimensional Euclidean half-cone contained in the hyperplane $\{q \in \mathbb{H}^n;\ z_q = z\}$ with vertex $p_z = (0, 0, z)$, axis the half-line starting at p_z and passing through (x_p, y_p, z), and aperture 2θ.

The set $\mathcal{Q}(z, \pi(p), \theta)$ is defined as the $2n$-dimensional Euclidean convex hull in the hyperplane $\{q \in \mathbb{H}^n;\ z_q = z\}$ of p_z, $\check{p}_z = (2x_p, 2y_p, z)$ and the $(2n-1)$-dimensional Euclidean ball $\{q \in \mathbb{H}^n;\ z_q = z,\ \langle \pi(q) - \pi(p), \pi(p)\rangle_{\mathbb{R}^{2n}} = 0,\ \|\pi(q) - \pi(p)\|_{\mathbb{R}^{2n}} = \rho_p \tan\theta\}$. Here π denotes the obvious analogue of the map defined in (2.1), $\pi : \mathbb{H}^n \to \mathbb{R}^{2n}$, $\pi(x_p, y_p, z_p) := (x_p, y_p)$.

8. A general construction giving bi-Lipschitz equivalent distances without BCP

This section is devoted to the proof of Theorem 1.6. The construction is inspired by the construction given by the first-named author in [14, Theorem 1.6] where it is proved that there exist translation-invariant distances on $\mathbb{R}$ that are bi-Lipschitz equivalent to the Euclidean distance but that do not satisfy BCP.

Proof of Theorem 1.6. Let (M, d) be a metric space. Assume that $\overline{x}$ is an accumulation point in (M, d) and let $(x_n)_{n\geq 1}$ be a sequence of distinct points in M such that $x_n \neq \overline{x}$ for all $n \geq 1$ and $\lim_{n\to\infty} d(x_n, \overline{x}) = 0$. Set

$$\rho_n := \frac{n}{n+1} d(x_n, \overline{x}).$$

Up to taking a subsequence, one can assume that the sequence $(\rho_n)_{n\geq 1}$ is decreasing.

Fix $0 < c < 1$ and $n_0 \in \mathbb{N}$ large enough so that

$$c(n_0 + 1) < n_0. \tag{8.1}$$

Set

$$\theta(x, y) := \begin{cases} \rho_n & \text{if } \{x, y\} = \{x_n, \overline{x}\} \text{ for some } n \geq n_0, \\ d(x, y) & \text{otherwise,} \end{cases}$$

$$\overline{d}(x, y) := \inf \sum_{i=0}^{N-1} \theta(a_i, a_{i+1}),$$

where the infimum is taken over all $N \in \mathbb{N}^*$ and all chains of points $a_0 = x, \ldots, a_N = y$. Then $\overline{d}$ is a distance on M such that $cd \leq \overline{d} \leq d$: this follows from Lemmas 8.3 and 8.5 below.

Next, we will prove that $\overline{x}$ is an isolated point of $B_{\overline{d}}(x_n, \rho_n)$ for all $n \geq n_0$. More precisely, by definition of $\overline{d}$ we have, for all $n \geq n_0$,

$$\overline{d}(x_n, \overline{x}) \leq \theta(x_n, \overline{x}) = \rho_n,$$

hence $\overline{x} \in B_{\overline{d}}(x_n, \rho_n)$ for all $n \geq n_0$. On the other hand, we will prove in Lemma 8.6 that

$$B_{\overline{d}}(x_n, \rho_n) \cap B_d\left(\overline{x}, \frac{\rho_n}{n(n+1)}\right) = \{\overline{x}\} \quad \text{for all } n \geq n_0. \tag{8.2}$$

Then let us extract a subsequence $(x_{n_k})_{k \geq 0}$ starting at x_{n_0} in such a way that

$$d(\overline{x}, x_{n_k}) < \frac{\rho_{n_j}}{n_j(n_j+1)} \quad \text{for all } k \geq 1 \text{ and } j \in \{0, \ldots, k-1\}.$$

It follows from (8.2) that

$$\overline{d}(x_{n_k}, x_{n_j}) > \rho_{n_j} = \max\{\rho_{n_j}, \rho_{n_k}\} \quad \text{for all } k \geq 1 \text{ and } j \in \{0, \ldots, k-1\}$$

(remember that the sequence $(\rho_h)_{h \geq 1}$ is assumed to be decreasing). Then $\{B_{\overline{d}}(x_{n_k}, \rho_{n_k});\ k \in J\}$ is a family of Besicovitch balls for any finite set $J \subset \mathbb{N}$, which implies that w-BCP, and hence BCP, does not hold in $(M, \overline{d})$. □

Lemma 8.3. *We have $cd \leq \overline{d} \leq d$.*

Proof. By definition of θ, one has $\theta(x, y) \leq d(x, y)$ for all $x, y \in M$. It follows that

$$\overline{d}(x, y) \leq \inf\left\{\sum_{i=0}^{N-1} d(a_i, a_{i+1});\ a_0 = x, \ldots, a_N = y\right\} = d(x, y).$$

Note that since d is a distance, one indeed has

$$d(x, y) = \inf\left\{\sum_{i=0}^{N-1} d(a_i, a_{i+1});\ a_0 = x, \ldots, a_N = y\right\},$$

which follows from the triangle inequality and from the fact that one can consider $N = 1$, $a_0 = x$ and $a_1 = y$, so that $d(x, y) \geq \inf\{\sum_{i=0}^{N-1} d(a_i, a_{i+1});\ a_0 = x, \ldots, a_N = y\}$.

On the other hand, since $s \mapsto s/(s+1)$ is increasing, it follows from the definition of $\theta(x, y)$ and from (8.1) that

$$\theta(x, y) \geq \frac{n_0}{n_0+1} d(x, y) \geq c d(x, y) \quad \text{for all } x, y \in M. \tag{8.4}$$

Hence

$$\overline{d}(x, y) \geq c \inf\left\{\sum_{i=0}^{N-1} d(a_i, a_{i+1}); a_0 = x, \ldots, a_N = y\right\} = cd(xy). \qquad \square$$

Lemma 8.5. *$\overline{d}$ is a distance on M.*

Proof. Lemma 8.3 shows that if $\overline{d}(x, y) = 0$, then $d(x, y) = 0$ and hence $x = y$. Since $\theta(x, y) = \theta(y, x)$, one has $\overline{d}(x, y) = \overline{d}(y, x)$. To prove the triangle inequality, consider x, y and z in M and two arbitrary chains of points $a_0 = x, \ldots, a_N = z$, $b_0 = z, \ldots, b_{N'} = y$. Since $a_0 = x, \ldots, a_N = z = b_0, \ldots, b_{N'} = y$ is a chain of points from x to y, one has

$$\overline{d}(x, y) \leq \sum_{i=0}^{N-1} \theta(a_i, a_{i+1}) + \sum_{i=0}^{N'-1} \theta(b_i, b_{i+1}),$$

and hence

$$\overline{d}(x, y) \leq \overline{d}(x, z) + \overline{d}(z, y). \qquad \square$$

Lemma 8.6. *Let $n \geq n_0$. Assume that $0 < d(\overline{x}, y) < \rho_n/(n(n+1))$. Then $\overline{d}(x_n, y) > \rho_n$.*

Proof. For contradiction, assume that $0 < d(\overline{x}, y) < \rho_n/(n(n+1))$ for some $n \geq n_0$ and $\overline{d}(x_n, y) \leq \rho_n$. Let $\epsilon > 0$ and $a_0 = x_n, \ldots, a_N = y$ be such that

$$\sum_{i=0}^{N-1} \theta(a_i, a_{i+1}) \leq \rho_n + \epsilon. \tag{8.7}$$

First, we claim that $\{a_i, a_{i+1}\} \neq \{x_n, \overline{x}\}$ for all $i \in \{0, \ldots, N-1\}$ provided ϵ is small enough. Indeed, otherwise, with no loss of generality, we would have $a_0 = x_n$ and $a_1 = \overline{x}$, and hence

$$\sum_{i=0}^{N-1} \theta(a_i, a_{i+1}) = \theta(x_n, \overline{x}) + \sum_{i=1}^{N-1} \theta(a_i, a_{i+1}) = \rho_n + \sum_{i=1}^{N-1} \theta(a_i, a_{i+1}) \leq \rho_n + \epsilon,$$

which implies that

$$\sum_{i=1}^{N-1} \theta(a_i, a_{i+1}) \leq \epsilon.$$

On the other hand, (8.4) together with the triangle inequality would give

$$cd(\overline{x}, y) \leq c \sum_{i=1}^{N-1} d(a_i, a_{i+1}) \leq \sum_{i=1}^{N-1} \theta(a_i, a_{i+1}) \leq \epsilon,$$

which is impossible as soon as $\epsilon < cd(\overline{x}, y)$.

Next, we claim that

$$\theta(a_i, a_{i+1}) \geq \frac{n+1}{n+2} d(a_i, a_{i+1}) \tag{8.8}$$

for all $i \in \{0, \ldots, N-1\}$.

Indeed, first, if $\{a_i, a_{i+1}\} = \{\overline{x}, x_m\}$ for some $m \geq n_0$, then we must have $m > n$. Otherwise, since $(\rho_h)_{h\geq 1}$ is decreasing, we would have $\rho_m \geq \rho_{n-1}$. Hence we would get

$$\rho_{n-1} \leq \rho_m = \theta(a_i, a_{i+1}) \leq \sum_{j=0}^{N-1} \theta(a_j, a_{j+1}) \leq \rho_n + \epsilon,$$

which is impossible as soon as $\epsilon < \rho_{n-1} - \rho_n$.

Next, if $\{a_i, a_{i+1}\} = \{\overline{x}, x_m\}$ for some $m > n$, then, by definition of θ and remembering that $s \mapsto s/(s+1)$ is increasing, we have

$$\theta(a_i, a_{i+1}) = \rho_m = \frac{m}{m+1} d(a_i, a_{i+1}) \geq \frac{n+1}{n+2} d(a_i, a_{i+1}),$$

which gives (8.8).

Finally, if $\{a_i, a_{i+1}\} \neq \{\overline{x}, x_m\}$ for all $m \geq n_0$, then it follows from the definition of θ that

$$\theta(a_i, a_{i+1}) = d(a_i, a_{i+1}) \geq \frac{n+1}{n+2} d(a_i, a_{i+1}),$$

which also gives (8.8).

Now, it follows from (8.7) and (8.8) that

$$\rho_n + \epsilon \geq \sum_{i=1}^{N-1} \theta(a_i, a_{i+1}) \geq \frac{n+1}{n+2} \sum_{i=1}^{N-1} d(a_i, a_{i+1}) \geq \frac{n+1}{n+2} d(x_n, y)$$

for all ϵ small enough. Letting $\epsilon \downarrow 0$, we get

$$\rho_n \geq \frac{n+1}{n+2} d(x_n, y) \geq \frac{n+1}{n+2} (d(x_n, \overline{x}) - d(\overline{x}, y)) \geq \frac{n+1}{n+2} \left(\frac{n+1}{n} \rho_n - d(\overline{x}, y) \right),$$

and hence $d(\overline{x}, y) \geq \rho_n/(n(n+1))$, which contradicts the assumptions and concludes the proof. □

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