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Rigidity of smooth critical circle maps

Dedicated to Jean-Christophe Yoccoz (1957–2016), in memoriam

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Abstract. We prove that any two C^3 critical circle maps with the same irrational rotation number of bounded type and the same odd criticality are conjugate to each other by a $C^{1+\alpha}$ circle diffeomorphism, for some universal $\alpha > 0$.

Keywords. Critical circle maps, smooth rigidity, renormalization, commuting pairs

1. Introduction

In the theory of real one-dimensional dynamics there exist many levels of equivalence between two systems: combinatorial, topological, quasi-symmetric and smooth equivalence are major examples.

In the circle case, a classical result of Poincaré [45, Chapter 1, Theorem 1.1] states that circle homeomorphisms with the same irrational rotation number are *combinatorially equivalent*: for each $n \in \mathbb{N}$ the first n elements of an orbit are ordered in the same way for any homeomorphism with a given rotation number. This implies that circle homeomorphisms with irrational rotation number are semi-conjugate to the corresponding rigid rotation. According to Denjoy's theorem [7], any two C^2 circle diffeomorphisms with the same irrational rotation number are topologically conjugate to each other. By a fundamental result of Herman [24], improved by Yoccoz [59], any two $C^{2+\varepsilon}$ circle diffeomorphisms whose common rotation number ρ satisfies the *Diophantine* condition

$$\left| \rho - \frac{p}{q} \right| \geq \frac{C}{q^{2+\delta}}, \quad (1.1)$$

for some $\delta \in [0, 1)$ and $C > 0$, and for any positive coprime integers p and q , are conjugate to each other by a smooth circle diffeomorphism (see [29] for precise statements).

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Moreover, any two C^∞ circle diffeomorphisms with the same Diophantine rotation number are C^∞ -conjugate to each other, and real-analytic diffeomorphisms with the same Diophantine rotation number are conjugate to each other by a real-analytic diffeomorphism [45, Chapter I, Section 3].

These are examples of *rigidity* results: lower regularity of conjugacy implies higher regularity under certain conditions.

Since rigidity is totally understood in the setting of circle diffeomorphisms, we continue in this article the study of rigidity problems for critical circle maps developed by de Faria, de Melo, Yampolsky, Khanin and Teplinsky among others.

By a *critical circle map* we mean an orientation-preserving C^3 circle homeomorphism f with exactly one non-flat critical point of odd type (for simplicity, and because this is the generic case, we will assume in this article that the critical point is of cubic type). As usual, a critical point c is called *non-flat* if in a neighbourhood of c the map f can be written as $f(t) = (\phi(t))^d + f(c)$, where ϕ is a C^3 local diffeomorphism with $\phi(c) = 0$, and $d \in \mathbb{N}$ with $d \geq 3$. The *criticality* (or *order*, or *type*, or *exponent*) of the critical point c is d .

Classical examples of critical circle maps are obtained from the two-parameter family $\tilde{f}_{a,b} : \mathbb{C} \rightarrow \mathbb{C}$ of entire maps in the complex plane

$$\tilde{f}_{a,b}(z) = z + a - \frac{b}{2\pi} \sin(2\pi z) \quad \text{for } a \in [0, 1) \text{ and } b \geq 0. \quad (1.2)$$

Since each $\tilde{f}_{a,b}$ commutes with unitary horizontal translation, it is the lift of a holomorphic map $f_{a,b} : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C} \setminus \{0\}$ of the punctured plane via the universal cover $z \mapsto e^{2\pi iz}$. Since $\tilde{f}_{a,b}$ preserves the real axis, $f_{a,b}$ preserves the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and therefore induces a two-parameter family of real-analytic circle maps. This classical family was introduced by Arnold [3], and is called the *Arnold family*.

For $b = 0$ the family $f_{a,b} : S^1 \rightarrow S^1$ is just the family of rigid rotations $z \mapsto e^{2\pi ia} z$, and for $b \in (0, 1)$ the family is still contained in the space of real-analytic circle diffeomorphisms.

For $b = 1$ each $\tilde{f}_{a,b}$ still restricts to an increasing real-analytic homeomorphism of the real line that projects to an orientation-preserving real-analytic circle homeomorphism, having one critical point of cubic type at 1, the projection of the integers. Denote by $\rho(a)$ the rotation number of the circle homeomorphism $f_{a,1}$. It is well-known that $a \mapsto \rho(a)$ is continuous, non-decreasing, maps $[0, 1)$ onto itself and is such that the interval $\rho^{-1}(\theta) \subset [0, 1)$ degenerates to a point whenever $\theta \in [0, 1) \setminus \mathbb{Q}$ (see [24]). Furthermore, the set $\{a \in [0, 1) : \rho(a) \in \mathbb{R} \setminus \mathbb{Q}\}$ has zero Lebesgue measure (see [54]). For $0 \leq p < q$ coprime integers we know that $\rho^{-1}(\{p/q\})$ is always a non-degenerate closed interval. In the interior of this interval we find critical circle maps with two periodic orbits (of period q), one attracting and one repelling, which collapse to a single parabolic orbit on the boundary of the interval (see [9]).

For $b > 1$ the maps $f_{a,b} : S^1 \rightarrow S^1$ are not invertible any more (they have two critical points of even degree). These examples show how critical circle maps arise as bifurcations from circle diffeomorphisms to endomorphisms, and in particular, from zero to positive topological entropy (compare with infinitely renormalizable unimodal maps

[45, Chapter VI]). This is one of the main reasons why critical circle maps attracted the attention of physicists and mathematicians interested in understanding the *boundary of chaos* [8], [15], [26], [32], [33], [37], [38], [46], [49]–[52].

For more examples of critical circle maps see the introduction of [19] and the expository notes [20, Section 5.2]. For the ergodic aspects see [27], [17], [12] and the references therein. For applications to local connectivity and Lebesgue measure of Julia sets (associated to generic quadratic polynomials with Siegel disks) see [47], [41], [55], [48].

Since our goal is to study smoothness of conjugacies, we will focus on critical circle maps without periodic orbits, that is, the ones with irrational rotation number. In [60], Yoccoz proved that the rotation number is the unique invariant of the topological classes. More precisely, any C^3 orientation-preserving circle homeomorphism with only non-flat critical points (maybe more than one) and with irrational rotation number is topologically conjugate to the corresponding rigid rotation. We remark that the condition of non-flatness of the critical points cannot be removed: in [23] Hall was able to construct C^∞ homeomorphisms of the circle with no periodic points and no dense orbits.

Recall that an irrational number is of *bounded type* if it satisfies the Diophantine condition (1.1) for $\delta = 0$, that is, ρ in $[0, 1]$ is of bounded type if there exists $C > 0$ such that

$$\left| \rho - \frac{p}{q} \right| \geq \frac{C}{q^2}$$

for any integers p and $q \neq 0$. On the one hand, this is a respectable class: the set of numbers of bounded type is dense in $[0, 1]$, with Hausdorff dimension equal to one. On the other hand, from the metrical viewpoint, this is a rather restricted class: while the Diophantine numbers have full Lebesgue measure in $[0, 1]$, the set of numbers of bounded type has zero Lebesgue measure. More important for our purposes, bounded type numbers are characterized by the property that their *continued fraction expansion* have bounded partial quotients (see Section 3 for precise definitions).

Since a critical circle map cannot be smoothly conjugate to a rigid rotation, in order to study smooth-rigidity problems we must restrict ourselves to the class of critical circle maps. Numerical observations [15], [46], [52] suggested in the early eighties that smooth critical circle maps with rotation number of bounded type are geometrically rigid. This was posed as a conjecture in several works by Lanford [32], [33], Rand [49]–[51] (see also [46]) and Shenker [52] (see also [15]) among others:

Rigidity Conjecture. *Any two C^3 critical circle maps with the same irrational rotation number of bounded type and the same odd criticality are conjugate to each other by a $C^{1+\alpha}$ circle diffeomorphism for some $\alpha > 0$.*

The conjecture has been proved by de Faria and de Melo for real-analytic critical circle maps [14] and nowadays (after the work of Yampolsky, Khanin and Teplinsky) it is understood without any assumption on the irrational rotation number: inside each topological class of real-analytic critical circle maps the degree of the critical point is the unique invariant of the C^1 -conjugacy classes. In the following result we summarize many contributions of the authors quoted above:

Theorem A (de Faria–de Melo, Khmelev–Yampolsky, Khanin–Teplinsky). *Let f and g be real-analytic circle homeomorphisms with the same irrational rotation number and with a unique critical point of the same odd type. Let h be the conjugacy between f and g (given by Yoccoz’s result) that maps the critical point of f to the critical point of g (note that this determines h). Then:*

- (1) h is a C^1 diffeomorphism;
- (2) h is $C^{1+\alpha}$ at the critical point of f for a universal $\alpha > 0$;
- (3) for a full Lebesgue measure set of rotation numbers (that contains all bounded type numbers), h is globally $C^{1+\alpha}$.

See Section 8 for the definition of the full measure set of rotation numbers considered in conclusion (3) of Theorem A. Let us point out that, by a result of Avila [5], there exist two real analytic critical circle maps with the same irrational rotation number and the same criticality that are not $C^{1+\beta}$ conjugate for any $\beta > 0$. See Section 8 for further comments.

Item (1) of Theorem A was proved by Khanin and Teplinsky [28], building on earlier work of de Faria, de Melo and Yampolsky [10], [11], [13], [14], [55]–[58]. Item (2) was proved in [31], and item (3) is obtained by combining [13] with [58]. The proof of Theorem A relies on methods coming from complex analysis and complex dynamics (see [39], [40] and the references therein), and that is why rigidity is well understood for real-analytic critical circle maps, but nothing has been known yet for smooth ones (even in the C^∞ setting). In this article we take the final step and solve positively the Rigidity Conjecture:

Theorem B (Main result). *Any two C^3 critical circle maps with the same irrational rotation number of bounded type and the same odd criticality are conjugate to each other by a $C^{1+\alpha}$ circle diffeomorphism, for some universal $\alpha > 0$.*

The novelties of this article in order to transfer rigidity from real-analytic dynamics to (finitely) smooth ones are two: the first one is a *bidimensional* version of the *glueing procedure* (first introduced by Lanford [32], [33]) developed in Section 7, and the second one is the notion of *asymptotically holomorphic maps*, to be defined in Section 6 (Definition 6.3). Asymptotically holomorphic maps were already used in one-dimensional dynamics by Lyubich in the early nineties [35], and more recently by Graczyk, Sands and Świątek in [16], but as far as we know never for critical circle maps.

Remark 1.1. We report here recent progress in the rigidity theory of smooth critical circle maps, since the preprint version [22] of this paper first appeared: in [21], written in collaboration with Marco Martens, we were able to extend Theorem A to the C^4 category. More precisely, we proved that any two C^4 critical circle maps with the same irrational rotation number and the same odd criticality are conjugate to each other by a C^1 circle diffeomorphism, which is $C^{1+\alpha}$ at the critical point, and, for some universal $\alpha > 0$, this conjugacy is a $C^{1+\alpha}$ diffeomorphism for Lebesgue almost every rotation number (see [21, Theorem A]). See Section 8 for further comments.

Let us discuss the main ideas of the proof of Theorem B. A C^3 critical circle map f with irrational rotation number generates a sequence $\{\mathcal{R}^n(f) = (\eta_n, \xi_n)\}_{n \in \mathbb{N}}$ of com-

muting pairs of interval maps, each being the renormalization of the previous one (see Definition 3.4). To prove Theorem B we need to prove the exponential convergence of the orbits generated by two critical circle maps with a given combinatorics of bounded type (see Theorem 2.1).

Our main task (see Theorem D in Section 4) is to show the existence of a sequence $\{f_n = (\tilde{\eta}_n, \tilde{\xi}_n)\}_{n \in \mathbb{N}}$ that belongs to a universal C^ω -compact set of real-analytic critical commuting pairs, such that $\mathcal{R}^n(f)$ is C^0 -exponentially close to f_n at a universal rate, and both have the same rotation number. In Section 4, using the exponential contraction of the renormalization operator on the space of real-analytic critical commuting pairs (see Theorem 2.3), we deduce the exponential contraction of the renormalization operator in the space of C^3 critical commuting pairs with bounded combinatorics (see Theorem C in Section 2), and therefore the $C^{1+\alpha}$ rigidity as stated in Theorem B.

To realize the main task we extend the initial commuting pair to a pair of C^3 maps in an open complex neighbourhood of each original interval (the so-called *extended lift*, see Definition 6.5), which are asymptotically holomorphic (see Definition 6.3), each having a unique cubic critical point at the origin.

Using the real bounds (see Theorem 3.1), the almost Schwarz inclusion (see Proposition 6.7) and the asymptotic holomorphic property we prove that for all $n \in \mathbb{N}$ greater than or equal to some n_0 , both η_n and ξ_n extend to a definite neighbourhood of their interval domains in the complex plane, giving rise to maps with a unique cubic critical point at the origin, and with exponentially small conformal distortion (see Theorem 6.1). Theorem 6.1 also gives us some geometric control that will imply the desired compactness (we will not study the dynamics of these extensions, just their geometric behaviour).

Using the Ahlfors–Bers theorem (see Proposition 5.5) we construct for each $n \geq n_0$ a C^3 diffeomorphism Φ_n , exponentially close to the identity in definite domains around the dynamical intervals, that conjugates (η_n, ξ_n) to a C^3 critical commuting pair $(\widehat{\eta}_n, \widehat{\xi}_n)$ exponentially close to (η_n, ξ_n) , and such that $\widehat{\eta}_n^{-1} \circ \widehat{\xi}_n$ is a holomorphic diffeomorphism between complex neighbourhoods of the endpoints of the union of the dynamical intervals (see Subsection 7.1). Using this holomorphic diffeomorphism to glue the ends of a band around the union of the dynamical intervals we obtain a Riemann surface conformally equivalent to a round annulus A_{R_n} around the unit circle. This identification gives rise to a holomorphic local diffeomorphism P_n mapping the band onto the annulus and such that, via P_n , the pair $(\widehat{\eta}_n, \widehat{\xi}_n)$ induces a C^3 map G_n from an annulus in A_{R_n} to A_{R_n} , having exponentially small conformal distortion, which restricts to a critical circle map on S^1 (see Proposition 7.7). The commuting condition of each pair $(\widehat{\eta}_n, \widehat{\xi}_n)$ is equivalent to the continuity of the corresponding G_n , and that is why we project to the annulus A_{R_n} .

Using again the Ahlfors–Bers theorem we construct a holomorphic map H_n , on a smaller but definite annulus around the unit circle, that is exponentially close to G_n and restricts to a real-analytic critical circle map with the same combinatorics as the restriction of G_n to S^1 (see Proposition 7.8 for much more properties).

Finally, using the projection P_n , we lift each H_n to a real-analytic critical commuting pair $f_n = (\tilde{\eta}_n, \tilde{\xi}_n)$ exponentially close to $(\widehat{\eta}_n, \widehat{\xi}_n)$, having the same combinatorics and with complex extensions C^0 -exponentially close to the ones of $\mathcal{R}^n(f)$ produced in Theo-

rem 6.1 (see Proposition 7.17). Compactness then follows from the geometric properties obtained in Theorem 6.1 (see Lemma 7.18).

The organization of this article is the following: in Section 2 we reduce Theorem B to Theorem C, which states the exponential convergence of the renormalization orbits of C^3 critical circle maps with the same bounded combinatorics. In Section 3 we introduce the renormalization operator in the space of critical commuting pairs, and review its basic properties. In Section 4 we reduce Theorem C to Theorem D, which states the existence of a C^ω -compact piece of real-analytic critical commuting pairs such that for a given C^3 critical circle map f , with any irrational rotation number, there exists a sequence $\{f_n\}$, contained in that compact piece, such that $\mathcal{R}^n(f)$ is C^0 -exponentially close to f_n at a universal rate, and both have the same rotation number. In Section 5 we state a corollary of the Ahlfors–Bers theorem (Proposition 5.5) that will be fundamental in Section 7 (its proof will be given in Appendix B). In Section 6 we construct the *extended lift* of a C^3 critical circle map (see Definition 6.5), and then we state and prove Theorem 6.1 as described above. In Section 7 we develop a bidimensional glueing procedure in order to prove Theorem D. Finally, in Section 8, we review some further questions.

2. A first reduction of the main result

Just as in the case of unimodal maps, the main tool in order to obtain a smooth conjugacy between critical circle maps is the use of *renormalization* group methods (see [4], [42] and [43]). As was already clear in the early eighties [15], [46], it is convenient to construct a renormalization operator \mathcal{R} (see Definition 3.4) acting not on the space of critical circle maps but on a suitable space of *critical commuting pairs* (see Definition 3.2).

Just as in the case of unimodal maps (see for instance [45, Chapter VI, Theorem 9.4]), the principle that *exponential convergence of the renormalization operator is equivalent to smooth conjugacy* also holds for critical circle maps. The following result is due to de Faria and de Melo [13, First Main Theorem, p. 341]. For any $0 \leq r < \infty$ denote by d_r the C^r metric in the space of critical commuting pairs (see Definition 3.3):

Theorem 2.1 (de Faria–de Melo 1999). *There exists a set \mathbb{A} in $[0, 1]$, having full Lebesgue measure and containing all irrational numbers of bounded type, for which the following holds. Let f and g be C^3 critical circle maps with the same irrational rotation number in the set \mathbb{A} and with the same odd type at the critical point. If $d_0(\mathcal{R}^n(f), \mathcal{R}^n(g))$ converges to zero exponentially fast as $n \rightarrow \infty$, then f and g are $C^{1+\alpha}$ -conjugate to each other for some $\alpha > 0$.*

Roughly speaking, the full Lebesgue measure set \mathbb{A} is composed of irrational numbers in $[0, 1]$ whose coefficients in the continued fraction expansion may be unbounded, but their growth is less than quadratic (see Section 8 or [13, Appendix C] for the precise definition). Let us point out that, in sharp contrast with the case of diffeomorphisms, \mathbb{A} does not contain all Diophantine numbers, and contains some Liouville numbers (again, see Section 8). The remaining cases were more recently solved by Khanin and Teplinsky [28, Theorem 2, p. 198]:

Theorem 2.2 (Khanin–Teplinsky 2007). *Let f and g be C^3 critical circle maps with the same irrational rotation number and the same odd type at the critical point. If $d_2(\mathcal{R}^n(f), \mathcal{R}^n(g))$ converges to zero exponentially fast as $n \rightarrow \infty$, then f and g are C^1 -conjugate to each other.*

To obtain the smooth conjugacy (item (1) of Theorem A), Khanin and Teplinsky combined Theorem 2.2 with the following fundamental result:

Theorem 2.3 (de Faria–de Melo 2000, Yampolsky 2003). *There exists a universal constant λ in $(0, 1)$ with the following property. If ζ_1 and ζ_2 are real-analytic critical commuting pairs with the same irrational rotation number and the same odd type at the critical point, then there exists a constant $C > 0$ such that*

$$d_r(\mathcal{R}^n(\zeta_1), \mathcal{R}^n(\zeta_2)) \leq C\lambda^n$$

for all $n \in \mathbb{N}$ and for any $0 \leq r < \infty$. Moreover, given a C^ω -compact set \mathcal{K} of real-analytic critical commuting pairs, the constant C can be chosen the same for any ζ_1 and ζ_2 in \mathcal{K} .

Theorem 2.3 was proved by de Faria and de Melo [14] for rotation numbers of bounded type, and extended by Yampolsky [58] to cover all irrational rotation numbers.

With Theorem 2.1 at hand, our main result (Theorem B) reduces to the following one:

Theorem C. *There exists $\lambda \in (0, 1)$ such that if f and g are C^3 critical circle maps with the same irrational rotation number of bounded type and the same criticality, then there exists $C > 0$ such that for all $n \in \mathbb{N}$,*

$$d_0(\mathcal{R}^n(f), \mathcal{R}^n(g)) \leq C\lambda^n,$$

where d_0 is the C^0 distance in the space of critical commuting pairs.

This article is devoted to proving Theorem C. Let us fix some notation that we will use along this article. \mathbb{N} , \mathbb{Z} , \mathbb{Q} , \mathbb{R} and \mathbb{C} denote respectively the set of natural, integer, rational, real and complex numbers. The real part of a complex number z will be denoted by $\Re(z)$, and its imaginary part by $\Im(z)$. $B(z, r)$ denotes the Euclidean open ball of radius $r > 0$ around a complex number z . \mathbb{H} and $\widehat{\mathbb{C}}$ denote respectively the upper half-plane and the Riemann sphere. $\mathbb{D} = B(0, 1)$ denotes the unit disk in the complex plane, and $S^1 = \partial \mathbb{D}$ denotes its boundary, that is, the unit circle. $\text{Diff}_+^3(S^1)$ denotes the group (under composition) of orientation-preserving C^3 diffeomorphisms of the unit circle. $\text{Leb}(A)$ is the Lebesgue measure of a Borel set A in the plane, and $\text{diam}(A)$ its Euclidean diameter. Given a bounded interval I in the real line we denote its Euclidean length by $|I|$. Moreover, for any $\alpha > 0$, let

$$N_\alpha(I) = \{z \in \mathbb{C} : d(z, I) < \alpha|I|\},$$

where d denotes the Euclidean distance in the complex plane.

3. Renormalization of critical commuting pairs

In this section we define the space of C^3 critical commuting pairs (Definition 3.2), and we endow it with the C^3 metric (Definition 3.3). This metric space, which is neither compact nor locally compact, contains the phase space of the renormalization operator (Definition 3.4). Each C^3 critical circle map with irrational rotation number gives rise to an infinite renormalization orbit in this phase space, and the asymptotic behaviour of these orbits is the subject of this article.

As we said in the introduction, a *critical circle map* is an orientation-preserving C^3 circle homeomorphism f with exactly one critical point $c \in S^1$ of odd type. For simplicity, and since this is the generic case, we will assume that the critical point is of cubic type. Suppose that the rotation number $\rho(f) = \theta$ in $[0, 1)$ is irrational, and let $[a_0, a_1, \dots, a_n, \dots]$ be its continued fraction expansion:

$$\theta = \lim_{n \rightarrow \infty} \frac{1}{a_0 + \frac{1}{a_1 + \frac{1}{\ddots + \frac{1}{a_n}}}}$$

We define recursively the *return times* of θ by

$$q_0 = 1, \quad q_1 = a_0, \quad q_{n+1} = a_n q_n + q_{n-1} \quad \text{for } n \geq 1.$$

Recall that the numbers q_n are also obtained as the denominators of the truncated expansion of order n of θ :

$$\frac{p_n}{q_n} = [a_0, a_1, \dots, a_{n-1}] = \frac{1}{a_0 + \frac{1}{a_1 + \frac{1}{\ddots + \frac{1}{a_{n-1}}}}}$$

Let R_θ be the rigid rotation of angle $2\pi\theta$ of the unit circle. The arithmetical properties of the continued fraction expansion imply that the iterates $\{R_\theta^{q_n}(c)\}_{n \in \mathbb{N}}$ are the closest returns of the orbit of c under the rotation R_θ :

$$d(c, R_\theta^{q_n}(c)) < d(c, R_\theta^j(c)) \quad \text{for any } j \in \{1, \dots, q_n - 1\},$$

where d denote the standard distance in S^1 . The sequence $\{q_n\}$ of return times increases at least exponentially fast as $n \rightarrow \infty$, and the sequence $\{d(c, R_\theta^{q_n}(c))\}$ of return distances decreases to zero at least exponentially fast as $n \rightarrow \infty$. Moreover the sequence $\{R_\theta^{q_n}(c)\}_{n \in \mathbb{N}}$ approaches the point c from alternating sides:

$$R_\theta^{q_1}(c) < R_\theta^{q_3}(c) < \dots < c < R_\theta^{q_2}(c) < R_\theta^{q_0}(c).$$

By Poincaré’s result quoted at the beginning of the introduction, this information remains true at the combinatorial level for f : for any $n \in \mathbb{N}$ the interval $[c, f^{q_n}(c)]$ contains no other iterates $f^j(c)$ for $j \in \{1, \dots, q_n - 1\}$, and if we denote by μ the unique

invariant Borel probability of f , we can say that $\mu([c, f^{q_n}(c)]) < \mu([c, f^j(c)])$ for any $j \in \{1, \dots, q_n - 1\}$.

We say that $\rho(f)$ is of *bounded type* if there exists a constant $M \in \mathbb{N}$ such that $a_n < M$ for any $n \in \mathbb{N}$ (it is not difficult to see that this definition is equivalent to the one given in the introduction, see [30, Chapter II, Theorem 23]). As we said in the introduction, the set of numbers of bounded type has zero Lebesgue measure in $[0, 1]$.

3.1. Dynamical partitions

Denote by I_n the interval $[c, f^{q_n}(c)]$ and define

$$\mathcal{P}_n = \{I_n, f(I_n), \dots, f^{q_{n+1}-1}(I_n)\} \cup \{I_{n+1}, f(I_{n+1}), \dots, f^{q_n-1}(I_{n+1})\}.$$

A crucial combinatorial fact is that \mathcal{P}_n is a partition (modulo boundary points) of the circle for every $n \in \mathbb{N}$. We call it the n -th *dynamical partition* of f associated with the point c . Note that the partition \mathcal{P}_n is determined by the piece of orbit

$$\{f^j(c) : 0 \leq j \leq q_n + q_{n+1} - 1\}.$$

The transitions from \mathcal{P}_n to \mathcal{P}_{n+1} can be described in the following easy way. The interval $I_n = [c, f^{q_n}(c)]$ is subdivided by the points $f^{jq_{n+1}+q_n}(c)$ with $1 \leq j \leq a_{n+1}$ into $a_{n+1} + 1$ subintervals. This subpartition is spread by the iterates of f to all the $f^j(I_n) = f^j([c, f^{q_n}(c)])$ with $0 \leq j < q_{n+1}$. The other elements of the partition \mathcal{P}_n , which are the $f^j(I_{n+1})$ with $0 \leq j < q_n$, remain unchanged.

As we are working with critical circle maps, our partitions in this article are always determined by the critical orbit. A major result for critical circle maps is the following:

Theorem 3.1 (Real bounds). *There exists $K > 1$ such that given a C^3 critical circle map f with irrational rotation number, there exists $n_0 = n_0(f)$ such that for all $n \geq n_0$ and for every pair I, J of adjacent atoms of \mathcal{P}_n we have*

$$K^{-1}|I| \leq |J| \leq K|I|.$$

Moreover, if Df denotes the first derivative of f , then

$$\frac{1}{K} \leq \frac{|Df^{q_n-1}(x)|}{|Df^{q_n-1}(y)|} \leq K \quad \text{for all } x, y \in f(I_{n+1}) \text{ and all } n \geq n_0,$$

$$\frac{1}{K} \leq \frac{|Df^{q_{n+1}-1}(x)|}{|Df^{q_{n+1}-1}(y)|} \leq K \quad \text{for all } x, y \in f(I_n) \text{ and all } n \geq n_0.$$

Theorem 3.1 was proved by Świątek and Herman (see [25], [54], [18] and [13]). The control on the distortion of the return maps follows from the Koebe distortion principle (see [13, Section 3]). Note that for a rigid rotation we have $|I_n| = a_{n+1}|I_{n+1}| + |I_{n+2}|$. If a_{n+1} is large, then I_n is much larger than I_{n+1} . Thus, even for rigid rotations, real bounds do not hold in general.

3.2. Critical commuting pairs

The first return map of the union $I_n \cup I_{n+1}$ of adjacent intervals is given respectively by $f^{q_{n+1}}$ and f^{q_n} . This pair of interval maps

$$(f^{q_{n+1}}|_{I_n}, f^{q_n}|_{I_{n+1}})$$

motivates the following definition:

Definition 3.2. A critical commuting pair $\zeta = (\eta, \xi)$ consists of two smooth orientation-preserving interval homeomorphisms $\eta : I_\eta \rightarrow \eta(I_\eta)$ and $\xi : I_\xi \rightarrow \xi(I_\xi)$ where:

- (1) $I_\eta = [0, \xi(0)]$ and $I_\xi = [\eta(0), 0]$;
- (2) there exists a neighbourhood of the origin where both η and ξ have homeomorphic extensions (with the same degree of smoothness) which commute, that is, $\eta \circ \xi = \xi \circ \eta$;
- (3) $(\eta \circ \xi)(0) = (\xi \circ \eta)(0) \neq 0$;
- (4) $\eta'(0) = \xi'(0) = 0$;
- (5) $\eta'(x) \neq 0$ for all $x \in I_\eta \setminus \{0\}$ and $\xi'(x) \neq 0$ for all $x \in I_\xi \setminus \{0\}$.

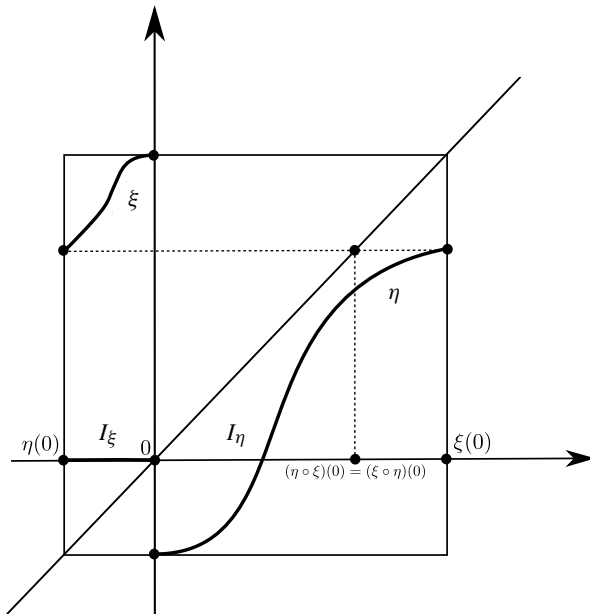


Fig. 1. A commuting pair.

Any critical circle map f with irrational rotation number θ induces a sequence of critical commuting pairs in a natural way. Let \tilde{f} be the lift of f to the real line (for the canonical covering $t \mapsto e^{2\pi i t}$) satisfying $\tilde{f}'(0) = 0$ and $0 < \tilde{f}(0) < 1$. For each $n \geq 1$ let \tilde{I}_n be the closed interval in the real line, adjacent to the origin, that projects to I_n . Let $T : \mathbb{R} \rightarrow \mathbb{R}$ be the translation $x \mapsto x + 1$ and define $\eta : \tilde{I}_n \rightarrow \mathbb{R}$ and $\xi : \tilde{I}_{n+1} \rightarrow \mathbb{R}$ as

$$\eta = T^{-p_{n+1}} \circ \tilde{f}^{q_{n+1}} \quad \text{and} \quad \xi = T^{-p_n} \circ \tilde{f}^{q_n}.$$

Then $(\eta|_{I_n}, \xi|_{I_{n+1}})$ is a critical commuting pair that we denote by $(f^{q_{n+1}}|_{I_n}, f^{q_n}|_{I_{n+1}})$ to simplify notation.

A converse of this construction was introduced by Lanford [32], [33] and it is known as the *glueing procedure*: the map $\eta^{-1} \circ \xi$ is a diffeomorphism from a small neighbourhood of $\eta(0)$ onto a neighbourhood of $\xi(0)$. Identifying $\eta(0)$ and $\xi(0)$ in this way we obtain from the interval $[\eta(0), \xi(0)]$ a smooth compact boundaryless one-dimensional manifold M . The discontinuous piecewise smooth map

$$f_\zeta(t) = \begin{cases} \xi(t) & \text{for } t \in [\eta(0), 0), \\ \eta(t) & \text{for } t \in [0, \xi(0)], \end{cases}$$

projects to a smooth homeomorphism on the quotient manifold M . By choosing a diffeomorphism $\psi : M \rightarrow S^1$ we obtain a critical circle map on S^1 , just by conjugating with ψ . Although there is no canonical choice for the diffeomorphism ψ , any two different choices give rise to smoothly-conjugate critical circle maps in S^1 . Therefore any critical commuting pair induces a whole *smooth conjugacy class* of critical circle maps. In Section 7 we propose a bidimensional extension of this procedure, in order to prove our main result (Theorem B).

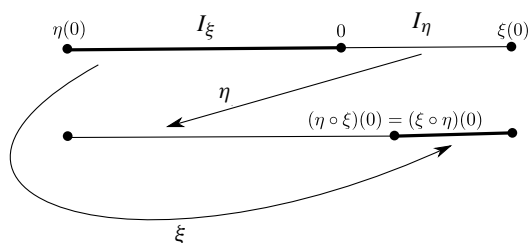


Fig. 2. Scheme of a commuting pair.

3.3. The C^r metric

We endow the space of C^3 critical commuting pairs with the C^3 metric. Given two critical commuting pairs $\zeta_1 = (\eta_1, \xi_1)$ and $\zeta_2 = (\eta_2, \xi_2)$ let A_1 and A_2 be the Möbius transformations such that for $i = 1, 2$,

$$A_i(\eta_i(0)) = -1, \quad A_i(0) = 0, \quad A_i(\xi_i(0)) = 1.$$

Definition 3.3. For any $0 \leq r < \infty$ define the C^r metric on the space of C^r critical commuting pairs in the following way:

$$d_r(\zeta_1, \zeta_2) = \max \left\{ \left| \frac{\xi_1(0)}{\eta_1(0)} - \frac{\xi_2(0)}{\eta_2(0)} \right|, \|A_1 \circ \zeta_1 \circ A_1^{-1} - A_2 \circ \zeta_2 \circ A_2^{-1}\|_r \right\},$$

where $\|\cdot\|_r$ is the C^r norm for maps in $[-1, 1]$ with one discontinuity at the origin, and ζ_i is the piecewise map defined by η_i and ξ_i :

$$\zeta_i : I_{\xi_i} \cup I_{\eta_i} \rightarrow I_{\xi_i} \cup I_{\eta_i} \quad \text{such that} \quad \zeta_i|_{I_{\xi_i}} = \xi_i \quad \text{and} \quad \zeta_i|_{I_{\eta_i}} = \eta_i.$$

Note that d_r is a pseudo-metric since it is invariant under conjugacy by homotheties: if α is a positive real number, and $H_\alpha(t) = \alpha t$ and $\zeta_1 = H_\alpha \circ \zeta_2 \circ H_\alpha^{-1}$, then $d_r(\zeta_1, \zeta_2) = 0$. In order to have a metric we need to restrict to *normalized* critical commuting pairs: for a commuting pair $\zeta = (\eta, \xi)$ denote by $\tilde{\zeta}$ the pair $(\tilde{\eta}|_{\tilde{I}_\eta}, \tilde{\xi}|_{\tilde{I}_\xi})$ where tilde means rescaling by the linear factor $\lambda = 1/|I_\xi|$. Note that $|\tilde{I}_\xi| = 1$ and \tilde{I}_η has length equal to the ratio between the lengths of I_η and I_ξ . Equivalently $\tilde{\eta}(0) = -1$ and $\tilde{\xi}(0) = |I_\eta|/|I_\xi| = \xi(0)/|\eta(0)|$.

When we are dealing with real-analytic critical commuting pairs, we consider the C^ω -topology defined in the usual way: we say that $(\eta_n, \xi_n) \rightarrow (\eta, \xi)$ if there exist open sets $U_\eta \supset I_\eta$ and $U_\xi \supset I_\xi$ in the complex plane and $n_0 \in \mathbb{N}$ such that η and η_n for $n \geq n_0$ extend continuously to $\overline{U_\eta}$, are holomorphic in U_η and satisfy $\|\eta_n - \eta\|_{C^0(\overline{U_\eta})} \rightarrow 0$, and ξ and ξ_n for $n \geq n_0$ extend continuously to $\overline{U_\xi}$, are holomorphic in U_ξ and satisfy $\|\xi_n - \xi\|_{C^0(\overline{U_\xi})} \rightarrow 0$. We say that a set \mathcal{C} of real-analytic critical commuting pairs is *closed* if whenever $\{\zeta_n\} \subset \mathcal{C}$ and $\zeta_n \rightarrow \zeta$, we have $\zeta \in \mathcal{C}$. This defines a Hausdorff topology, stronger than the C^r -topology for any $0 \leq r \leq \infty$.

3.4. The renormalization operator

Let $\zeta = (\eta, \xi)$ be a C^3 critical commuting pair according to Definition 3.2, and recall that $(\eta \circ \xi)(0) = (\xi \circ \eta)(0) \neq 0$. Suppose that $(\xi \circ \eta)(0) \in I_\eta$ (just as in Figures 1 and 2 above), and define the *height* $\chi(\zeta)$ of the commuting pair $\zeta = (\eta, \xi)$ as r if

$$\eta^{r+1}(\xi(0)) \leq 0 \leq \eta^r(\xi(0)),$$

and $\chi(\zeta) = \infty$ if no such r exists (note that in the latter case the map $\eta|_{I_\eta}$ has a fixed point, so when we are dealing with commuting pairs induced by critical circle maps with irrational rotation number, we have finite height). Note also that the height of the pair $(f^{q_{n+1}}|_{I_n}, f^{q_n}|_{I_{n+1}})$ induced by a critical circle map f is exactly a_{n+1} where $\rho(f) = [a_0, a_1, a_2, \dots]$ (because the combinatorics of f are the same as for the rigid rotation $R_{\rho(f)}$).

For a pair $\zeta = (\eta, \xi)$ with $(\xi \circ \eta)(0) \in I_\eta$ and $\chi(\zeta) = r < \infty$ we see that

$$(\eta|_{[0, \eta^r(\xi(0))]}, \eta^r \circ \xi|_{I_\xi})$$

is again a commuting pair, and if $\zeta = (\eta, \xi)$ is induced by a critical circle map

$$\zeta = (\eta, \xi) = (f^{q_{n+1}}|_{I_n}, f^{q_n}|_{I_{n+1}}),$$

we have

$$(\eta|_{[0, \eta^r(\xi(0))]}, \eta^r \circ \xi|_{I_\xi}) = (f^{q_{n+1}}|_{I_{n+2}}, f^{q_{n+2}}|_{I_{n+1}}).$$

This motivates the following definition (the definition in the case $(\xi \circ \eta)(0) \in I_\xi$ is analogous):

Definition 3.4. Let $\zeta = (\eta, \xi)$ be a critical commuting pair with $(\xi \circ \eta)(0) \in I_\eta$. We say that ζ is *renormalizable* if $\chi(\zeta) = r < \infty$. In this case we define the *renormalization* of ζ as the critical commuting pair

$$\mathcal{R}(\zeta) = (\tilde{\eta}|_{[0, \widetilde{\eta^r(\xi(0))}]}, \widetilde{\eta^r \circ \xi}|_{\tilde{I}_\xi}).$$

A critical commuting pair is a special case of a *generalized interval exchange map* of two intervals, and the renormalization operator defined above is just the restriction of the *Zorich accelerated version* of the *Rauzy–Veech renormalization* for interval exchange maps (see for instance [61]). However, in this article we will keep the classical terminology for critical commuting pairs.

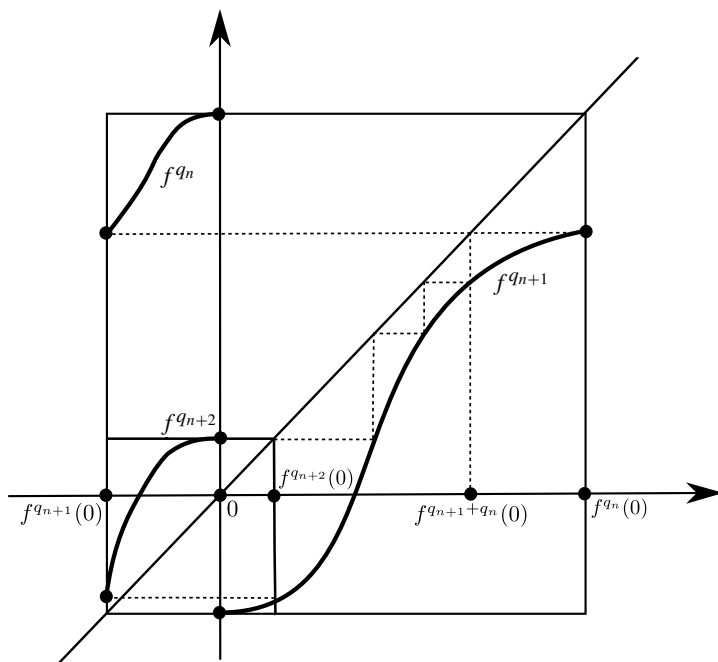


Fig. 3. Two consecutive renormalizations of f , without rescaling (recall that f^{q_n} means $T^{-p_n} \circ \tilde{f}^{q_n}$). In this example $a_{n+1} = 4$.

Definition 3.5. Let ζ be a critical commuting pair. If $\chi(\mathcal{R}^j(\zeta)) < \infty$ for $j \in \{0, 1, \dots, n - 1\}$ we say that ζ is *n-times renormalizable*, and if $\chi(\mathcal{R}^j(\zeta)) < \infty$ for all $j \in \mathbb{N}$ we say that ζ is *infinitely renormalizable*. In this case the irrational number θ whose continued fraction expansion is equal to

$$[\chi(\zeta), \chi(\mathcal{R}(\zeta)), \dots, \chi(\mathcal{R}^n(\zeta)), \dots]$$

is called the *rotation number* of the critical commuting pair ζ , and denoted by $\rho(\zeta) = \theta$.

The rotation number of a critical commuting pair can also be defined with the help of the glueing procedure described above, just as the rotation number of any representative of the conjugacy class obtained after glueing and uniformizing.

An immediate but important remark is that when ζ is induced by a critical circle map with irrational rotation number, the pair ζ is automatically infinitely renormalizable (and both notions of rotation number coincide): any C^3 critical circle map f with irrational

rotation number gives rise to a well defined orbit $\{\mathcal{R}^n(f)\}_{n \geq 1}$ of infinitely renormalizable C^3 critical commuting pairs defined by

$$\mathcal{R}^n(f) = (\widetilde{f}^{q_n}|_{\widetilde{I}_{n-1}}, \widetilde{f}^{q_{n-1}}|_{\widetilde{I}_n}) \quad \text{for all } n \geq 1.$$

For any positive number θ denote by $\lfloor \theta \rfloor$ the integer part of θ , that is, $\lfloor \theta \rfloor \in \mathbb{N}$ and $\lfloor \theta \rfloor \leq \theta < \lfloor \theta \rfloor + 1$. Recall that the Gauss map $G : [0, 1] \rightarrow [0, 1]$ is defined by

$$G(\theta) = 1/\theta - \lfloor 1/\theta \rfloor \quad \text{for } \theta \neq 0 \quad \text{and} \quad G(0) = 0,$$

and note that ρ semi-conjugates the renormalization operator to the Gauss map:

$$\rho(\mathcal{R}^n(\zeta)) = G^n(\rho(f))$$

for any ζ at least n -times renormalizable. In particular the renormalization operator acts as a left shift on the continued fraction expansion of the rotation number: if $\rho(\zeta) = [a_0, a_1, \dots]$ then $\rho(\mathcal{R}^n(\zeta)) = [a_n, a_{n+1}, \dots]$.

3.5. Lipschitz continuity along the orbits

For $K > 1$ and $r \in \{0, 1, \dots, \infty, \omega\}$ denote by $\mathcal{P}^r(K)$ the space of C^r critical commuting pairs $\zeta = (\eta, \xi)$ such that $\eta(0) = -1$ (they are normalized) and $\xi(0) \in [K^{-1}, K]$. Recall also that T denotes the translation $t \mapsto t + 1$ on the real line. Let $K_0 > 1$ be the universal constant given by the real bounds. In the next section we will use the following:

Lemma 3.6. *Given $M > 0$ and $K > K_0$ there exists $L > 1$ with the following property. Let f be a C^3 critical circle map with irrational rotation number $\rho(f) = [a_0, a_1, \dots]$ satisfying $a_n < M$ for all $n \in \mathbb{N}$. There exists $n_0 = n_0(f) \in \mathbb{N}$ such that for any $n \geq n_0$ and any renormalizable critical commuting pair $\zeta = (\eta, \xi)$ satisfying:*

- (1) $\zeta, \mathcal{R}(\zeta) \in \mathcal{P}^3(K)$,
- (2) $\lfloor 1/\rho(\zeta) \rfloor = a_n$,
- (3) if $(T^{-p_{n+1}} \circ \widetilde{f}^{q_{n+1}})(0) < 0 < (T^{-p_n} \circ \widetilde{f}^{q_n})(0)$ then

$$\left| \left| \frac{(T^{-p_n} \circ \widetilde{f}^{q_n})(0)}{(T^{-p_{n+1}} \circ \widetilde{f}^{q_{n+1}})(0)} \right| - \xi(0) \right| < \frac{1}{K^2} \cdot \frac{K+1}{K-1};$$

otherwise, if $(T^{-p_n} \circ \widetilde{f}^{q_n})(0) < 0 < (T^{-p_{n+1}} \circ \widetilde{f}^{q_{n+1}})(0)$, then

$$\left| \left| \frac{(T^{-p_{n+1}} \circ \widetilde{f}^{q_{n+1}})(0)}{(T^{-p_n} \circ \widetilde{f}^{q_n})(0)} \right| - \xi(0) \right| < \frac{1}{K^2} \cdot \frac{K+1}{K-1};$$

- (4) $(\eta \circ \xi)(0)$ and $(T^{-p_{n+1}-p_n} \circ \widetilde{f}^{q_{n+1}+q_n})(0)$ have the same sign,

we have

$$d_0(\mathcal{R}^{n+1}(f), \mathcal{R}(\zeta)) \leq L \cdot d_0(\mathcal{R}^n(f), \zeta),$$

where d_0 is the C^0 distance in the space of critical commuting pairs.

We postpone the proof of Lemma 3.6 until Appendix A. We remark that to obtain Lipschitz estimates for the renormalization operator acting on unbounded combinatorics is a much more difficult problem (see [21, Lemma 4.1]).

4. Reduction of Theorem C

In this section we reduce Theorem C to the following:

Theorem D. *There exist a C^ω -compact set \mathcal{K} of real-analytic critical commuting pairs and a constant $\lambda \in (0, 1)$ with the following property. Given a C^3 critical circle map f with any irrational rotation number there exist $C > 0$ and a sequence $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{K}$ such that*

$$d_0(\mathcal{R}^n(f), f_n) \leq C\lambda^n \quad \text{for all } n \in \mathbb{N},$$

and f_n has the same rotation number as $\mathcal{R}^n(f)$ for all $n \in \mathbb{N}$.

Note that \mathcal{K} is C^r -compact for any $0 \leq r \leq \infty$ (see Section 3.3). Note also that Theorem D is true for any combinatorics. The following argument was inspired by [44].

Proof that Theorem D implies Theorem C. Let \mathcal{K} be the C^ω -compact set of real-analytic critical commuting pairs given by Theorem D. By the real bounds there exists a uniform constant $n_0 \in \mathbb{N}$ such that $\mathcal{R}^n(\zeta) \in \mathcal{P}^\omega(K_0)$ for all $\zeta \in \mathcal{K}$ and all $n \geq n_0$. Therefore there exists $K > K_0$ such that $\mathcal{R}^n(\zeta) \in \mathcal{P}^\omega(K)$ for all $\zeta \in \mathcal{K}$ and all $n \geq 1$. Let $M > \max_{n \in \mathbb{N}} a_n$ where $\rho(f) = \rho(g) = [a_0, a_1, \dots]$, and let $L > 1$ be given by Lemma 3.6.

By Theorem D there exist constants $\lambda_1 \in (0, 1)$ and $C_1(f), C_1(g) > 0$ and sequences $\{f_n\}_{n \in \mathbb{N}}, \{g_n\}_{n \in \mathbb{N}} \subset \mathcal{K}$ such that for all $n \in \mathbb{N}$ we have $\rho(f_n) = \rho(g_n) = [a_n, a_{n+1}, \dots]$ and

$$d_0(\mathcal{R}^n(f), f_n) \leq C_1(f)\lambda_1^n \quad \text{and} \quad d_0(\mathcal{R}^n(g), g_n) \leq C_1(g)\lambda_1^n. \tag{4.1}$$

Let $n_0(f), n_0(g) \in \mathbb{N}$ be given by Lemma 3.6, and consider $n_0 = \max\{n_0(f), n_0(g)\}$ and $C_1 = \max\{C_1(f), C_1(g)\}$. Fix $\alpha \in (0, 1)$ such that $\alpha > \log L / (\log L - \log \lambda_1)$, and for all $n > (1/\alpha)n_0$ let $m = \lfloor \alpha n \rfloor$. By the choice of $K > K_0$, and since $f_m, g_m \in \mathcal{K}$ for all $m \in \mathbb{N}$, we have $\mathcal{R}^j(f_m) \in \mathcal{P}^3(K)$ for all $j \in \mathbb{N}$. By the real bounds,

$$\left| \frac{(T^{-p_{n+1}-p_n} \circ \tilde{f}^{q_{n+1}+q_n})(0)}{(T^{-p_{n+1}} \circ \tilde{f}^{q_{n+1}})(0)} \right| \in [1/K, K] \quad \text{for all } n \geq n_0,$$

and by (4.1) we have items (3) and (4) of Lemma 3.6 for $\zeta = f_n$, on taking n_0 large enough. Applying Lemma 3.6 we obtain

$$d_0(\mathcal{R}^n(f), \mathcal{R}^{n-m}(f_m)) \leq L^{n-m} \cdot d_0(\mathcal{R}^m(f), f_m) \leq C_1 L^{n-m} \lambda_1^m,$$

and for the same reasons

$$d_0(\mathcal{R}^n(g), \mathcal{R}^{n-m}(g_m)) \leq L^{n-m} \cdot d_0(\mathcal{R}^m(g), g_m) \leq C_1 L^{n-m} \lambda_1^m.$$

Let $\lambda_2 = L^{1-\alpha} \lambda_1^\alpha$; note that $\lambda_2 \in (0, 1)$ by the choice of α . Consider also $C_2 = 2C_1(1/\lambda_1)L > 0$. Since f_m and g_m are real-analytic and have the same combinatorics, we know by Yampolsky’s result (Theorem 2.3) that there exist constants $\lambda_3 \in (0, 1)$ and $C_3 > 0$ (uniform in \mathcal{K}) such that

$$d_0(\mathcal{R}^{n-m}(f_m), \mathcal{R}^{n-m}(g_m)) \leq C_3 \lambda_3^{n-m} \quad \text{for all } n \in \mathbb{N}.$$

Finally, consider $C = C_2 + C_3 > 0$ and $\lambda = \max\{\lambda_2, \lambda_3^{1-\alpha}\} \in (0, 1)$. By the triangle inequality,

$$d_0(\mathcal{R}^n(f), \mathcal{R}^n(g)) \leq C\lambda^n \quad \text{for all } n \in \mathbb{N}. \quad \square$$

5. Approximation by holomorphic maps

5.1. The Beltrami equation

Until now we have been working on the real line; now we start to work on the complex plane. We assume that the reader is familiar with the notion of *quasiconformality* (the books of Ahlfors [1] and of Lehto and Virtanen [34] are classical references). Recall the two basic differential operators of complex calculus:

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Instead of $\frac{\partial F}{\partial z}$ and $\frac{\partial F}{\partial \bar{z}}$ we will use the more compact notation ∂F and $\bar{\partial} F$ respectively. To be more explicit, if $F : \Omega \subset \mathbb{C} \rightarrow \mathbb{C}$ is differentiable at $w \in \Omega$ then $(DF(w))(z) = \partial F(w)z + \bar{\partial} F(w)\bar{z}$ for any $z \in \mathbb{C}$.

Recall that a continuous real function $h : \mathbb{R} \rightarrow \mathbb{R}$ is *absolutely continuous* if it has a derivative at Lebesgue almost every point, the derivative is integrable and $h(b) - h(a) = \int_a^b h'(t) dt$. A continuous function $F : \Omega \subset \mathbb{C} \rightarrow \mathbb{C}$ is *absolutely continuous on lines* in Ω if its real and imaginary parts are absolutely continuous on Lebesgue almost every horizontal line, and on Lebesgue almost every vertical line.

Definition 5.1. Let $\Omega \subset \mathbb{C}$ be a domain and let $K \geq 1$. An orientation-preserving homeomorphism $F : \Omega \rightarrow F(\Omega)$ is *K-quasiconformal* (*K-q.c.* from now on) if F is absolutely continuous on lines and

$$|\bar{\partial} F(z)| \leq \frac{K - 1}{K + 1} |\partial F(z)| \quad \text{a.e. in } \Omega.$$

Given a *K-q.c.* homeomorphism $F : \Omega \rightarrow F(\Omega)$ we define its *Beltrami coefficient* as the measurable function $\mu_F : \Omega \rightarrow \mathbb{D}$ given by

$$\mu_F(z) = \frac{\bar{\partial} F(z)}{\partial F(z)} \quad \text{for a.e. } z \in \Omega.$$

Note that μ_F belongs to $L^\infty(\Omega)$ and satisfies $\|\mu_F\|_\infty \leq (K - 1)/(K + 1) < 1$. Conversely, any measurable function from Ω to \mathbb{C} with L^∞ norm less than one is the Beltrami coefficient of a quasiconformal homeomorphism:

Theorem 5.2 (Morrey 1938). *Given any measurable function $\mu : \Omega \rightarrow \mathbb{D}$ such that $|\mu(z)| \leq (K - 1)/(K + 1) < 1$ almost everywhere in Ω for some $K \geq 1$, there exists a *K-q.c.* homeomorphism $F : \Omega \rightarrow F(\Omega)$ which is a solution of the Beltrami equation*

$$\partial F(z) \mu(z) = \bar{\partial} F(z) \quad \text{a.e.}$$

The solution is unique up to post-composition with conformal diffeomorphisms. In particular, if Ω is the entire Riemann sphere, there is a unique solution (called the normalized solution) that fixes 0, 1 and ∞ .

See [1, Chapter V, Section B] or [34, Chapter V] for the proof. Note that Theorem 5.2 not only states the existence of a solution of the Beltrami equation, but also the fact that any solution is a homeomorphism. Moreover we have:

Proposition 5.3. *If $\mu_n \rightarrow 0$ in the unit ball of L^∞ , the normalized quasiconformal homeomorphisms f^{μ_n} converge to the identity uniformly on compact sets of \mathbb{C} . In general if $\mu_n \rightarrow \mu$ almost everywhere in \mathbb{C} and $\|\mu_n\|_\infty \leq k < 1$ for all $n \in \mathbb{N}$, then the normalized quasiconformal homeomorphisms f^{μ_n} converge to f^μ uniformly on compact sets of \mathbb{C} .*

See [1, Chapter V, Section C].

5.2. Ahlfors–Bers theorem

The Beltrami equation induces therefore a one-to-one correspondence between the space of quasiconformal homeomorphisms of $\widehat{\mathbb{C}}$ that fix 0, 1 and ∞ , and the space of Borel measurable complex-valued functions μ on $\widehat{\mathbb{C}}$ for which $\|\mu\|_\infty < 1$. The following classical result expresses the analytic dependence of the solution of the Beltrami equation with respect to μ :

Theorem 5.4 (Ahlfors–Bers 1960). *Let Λ be an open subset of some complex Banach space and consider a map $\Lambda \times \mathbb{C} \rightarrow \mathbb{D}$, denoted by $(\lambda, z) \mapsto \mu_\lambda(z)$, satisfying the following properties:*

- (1) *for every λ the function $\mathbb{C} \rightarrow \mathbb{D}$ given by $z \mapsto \mu_\lambda(z)$ is measurable, and $\|\mu_\lambda\|_\infty \leq k$ for some fixed $k < 1$;*
- (2) *for Lebesgue almost every $z \in \mathbb{C}$, the function $\Lambda \rightarrow \mathbb{D}$ given by $\lambda \mapsto \mu_\lambda(z)$ is holomorphic.*

For each λ let F_λ be the unique quasiconformal homeomorphism of the Riemann sphere that fixes 0, 1 and ∞ , and whose Beltrami coefficient is μ_λ (F_λ is given by Theorem 5.2). Then $\lambda \mapsto F_\lambda(z)$ is holomorphic for all $z \in \mathbb{C}$.

See [2] or [1, Chapter V, Section C] for the proof. In Section 7 we will make repeated use of the following corollary of the Ahlfors–Bers theorem:

Proposition 5.5. *For any bounded domain U in the complex plane there exists a number $C(U) > 0$, with $C(U) \leq C(W)$ if $U \subseteq W$, such that the following holds. Let $\{G_n : U \rightarrow G_n(U)\}_{n \in \mathbb{N}}$ be a sequence of quasiconformal homeomorphisms such that:*

- *the domains $G_n(U)$ are uniformly bounded: there exists $R > 0$ such that $G_n(U) \subset B(0, R)$ for all $n \in \mathbb{N}$;*
- *$\mu_n \rightarrow 0$ in the unit ball of L^∞ , where μ_n is the Beltrami coefficient of G_n in U .*

Then for any domain V such that $\overline{V} \subset U$ there exist $n_0 \in \mathbb{N}$ and a sequence $\{H_n : V \rightarrow H_n(V)\}_{n \geq n_0}$ of biholomorphisms such that

$$\|H_n - G_n\|_{C^0(V)} \leq C(U) \frac{R}{d(\partial V, \partial U)} \|\mu_n\|_\infty \quad \text{for all } n \geq n_0,$$

where $d(\partial V, \partial U)$ denotes the Euclidean distance between the boundaries of U and V (which are disjoint compact sets in the complex plane, since V is compactly contained in the bounded domain U).

We postpone the proof of Proposition 5.5 until Appendix B. In the next section we will also use the following extension of Koebe’s classical one-quarter theorem [6, Theorem 1.3]:

Proposition 5.6. *Given $\varepsilon > 0$ there exists $K > 1$ for which the following holds. Let $f : \mathbb{D} \rightarrow f(\mathbb{D}) \subset \mathbb{C}$ be a K -quasiconformal homeomorphism such that $f(0) = 0$, $f((-1, 1)) \subset \mathbb{R}$ and $f(\mathbb{D}) \subset B(0, 1/\varepsilon)$. Suppose that $f|_{(-1/2, 1/2)}$ is differentiable and $|f'(t)| > \varepsilon$ for all $t \in (-1/2, 1/2)$, where f' denotes the real one-dimensional derivative of the restriction of f to the interval $(-1/2, 1/2)$. Then*

$$B(0, \varepsilon/16) \subset f(\mathbb{D}).$$

Proof. Suppose, for contradiction, that there exist $\varepsilon > 0$ and a sequence $\{f_n : \mathbb{D} \rightarrow f_n(\mathbb{D}) \subset \mathbb{C}\}_{n \in \mathbb{N}}$ of quasiconformal homeomorphisms with the following properties:

- (1) each f_n is K_n -q.c. with $K_n \rightarrow 1$ as $n \rightarrow \infty \rightarrow \infty$;
- (2) $f_n(0) = 0$ and $f_n((-1, 1)) \subset \mathbb{R}$ for all $n \in \mathbb{N}$;
- (3) $f_n(\mathbb{D}) \subset B(0, 1/\varepsilon)$ for all $n \in \mathbb{N}$;
- (4) $f_n|_{(-1/2, 1/2)}$ is differentiable and $|f'_n(t)| > \varepsilon$ for all $t \in (-1/2, 1/2)$ and $n \in \mathbb{N}$;
- (5) $B(0, \varepsilon/16)$ is not contained in $f_n(\mathbb{D})$ for any $n \in \mathbb{N}$.

By compactness, since $K_n \rightarrow 1$ and $f_n(0) = 0$ for all $n \in \mathbb{N}$, we can assume by taking a subsequence that there exists $f : \mathbb{D} \rightarrow \mathbb{C}$ holomorphic such that $f_n \rightarrow f$ uniformly on compact sets of \mathbb{D} as $n \rightarrow \infty$ (see for instance [34, Chapter II, Section 5]). Of course $f(0) = 0$ and $f((-1, 1)) \subset \mathbb{R}$. We claim that $|Df(0)| > \varepsilon/2$, where Df denotes the complex derivative of the holomorphic map f . Indeed, note that item (3) implies that

$$(-\varepsilon/m, \varepsilon/m) \subset f_n([-1/m, 1/m]) \quad \text{for all } n, m \in \mathbb{N},$$

and then by uniform convergence we have

$$(-\varepsilon/m, \varepsilon/m) \subset f([-1/m, 1/m]) \quad \text{for all } m \in \mathbb{N}.$$

Since f is holomorphic, this implies the claim. From the claim we see that f is univalent in \mathbb{D} , since the uniform limit of quasiconformal homeomorphisms is either constant or a quasiconformal homeomorphism (again see [34, Chapter II, Section 5]). Finally, by Koebe’s one-quarter theorem we have $B(0, \varepsilon/8) \subset f(\mathbb{D})$, but this contradicts that $B(0, \varepsilon/16)$ is not contained in $f_n(\mathbb{D})$ for any $n \in \mathbb{N}$. \square

6. Complex extensions of $\mathcal{R}^n(f)$

For every C^3 critical circle map, with any irrational rotation number, we will construct in this section a suitable extension to an annulus around the unit circle in the complex plane with the property that, after a finite number of renormalizations, this extension has good

geometric bounds and exponentially small Beltrami coefficient. In the next section we will perturb this extension in order to get a holomorphic map with the same combinatorics and also good bounds.

Recall that given a bounded interval I of the real line we denote its Euclidean length by $|I|$, and for any $\alpha > 0$ we denote by $N_\alpha(I)$ the \mathbb{R} -symmetric topological disk

$$N_\alpha(I) = \{z \in \mathbb{C} : d(z, I) < \alpha|I|\},$$

where d denotes the Euclidean distance in the complex plane. The goal of this section is the following:

Theorem 6.1. *There exist universal constants $\lambda \in (0, 1)$ and $\alpha > 0$ and $\beta > 0$ with the following property. Let f be a C^3 critical circle map with any irrational rotation number. For all $n \geq 1$ denote by (η_n, ξ_n) the components of the critical commuting pair $\mathcal{R}^n(f)$. Then there exist constants $n_0 \in \mathbb{N}$ and $C > 0$ such that for each $n \geq n_0$ both ξ_n and η_n extend (after normalization) to \mathbb{R} -symmetric orientation-preserving C^3 maps defined in $N_\alpha([-1, 0])$ and $N_\alpha([0, \xi_n(0)])$ respectively, where we have the following seven properties:*

- (1) ξ_n and η_n each have a unique critical point at the origin, which is of cubic type;
- (2) the extensions η_n and ξ_n commute in $B(0, \lambda)$, that is, both compositions $\eta_n \circ \xi_n$ and $\xi_n \circ \eta_n$ are well defined in $B(0, \lambda)$, and they coincide;
- (3) $N_\beta(\xi_n([-1, 0])) \subset \xi_n(N_\alpha([-1, 0]))$;
- (4) $N_\beta([-1, (\eta_n \circ \xi_n)(0)]) \subset \eta_n(N_\alpha([0, \xi_n(0)]))$;
- (5) $\eta_n(N_\alpha([0, \xi_n(0)])) \cup \xi_n(N_\alpha([-1, 0])) \subset B(0, \lambda^{-1})$;

$$(6) \quad \max_{z \in N_\alpha([-1, 0]) \setminus \{0\}} \frac{|\bar{\partial}\xi_n(z)|}{|\partial\xi_n(z)|} \leq C\lambda^n;$$

$$(7) \quad \max_{z \in N_\alpha([0, \xi_n(0)]) \setminus \{0\}} \frac{|\bar{\partial}\eta_n(z)|}{|\partial\eta_n(z)|} \leq C\lambda^n.$$

In this section we prove Theorem 6.1 (see Subsection 6.3), and in Section 7 we prove Theorem D.

6.1. Extended lifts of critical circle maps

In this subsection we lift a critical circle map to the real line, and then we extend this lift in a suitable way to a neighbourhood of the real line in the complex plane (see Definition 6.5 below).

Let f and g be C^3 critical circle maps with cubic critical points c_f and c_g , and critical values v_f and v_g respectively. Recall that $\text{Diff}_+^3(S^1)$ denotes the group (under composition) of orientation-preserving C^3 diffeomorphisms of the unit circle, endowed with the C^3 topology. Let $\mathcal{A}, \mathcal{B} \subset \text{Diff}_+^3(S^1)$ be defined by

$$\mathcal{A} = \{\psi \in \text{Diff}_+^3(S^1) : \psi(c_f) = c_g\} \quad \text{and} \quad \mathcal{B} = \{\phi \in \text{Diff}_+^3(S^1) : \phi(v_g) = v_f\}.$$

There is a canonical bijection between \mathcal{A} and \mathcal{B} :

$$\psi \mapsto R_{\theta_1} \circ \psi \circ R_{\theta_2},$$

where R_{θ_1} is the rigid rotation that takes c_g to v_f , and R_{θ_2} is the rigid rotation that takes v_g to c_f . We will be interested, however, in another identification between \mathcal{A} and \mathcal{B} :

Lemma 6.2. *There exists a bijection $T : \mathcal{A} \rightarrow \mathcal{B}$ such that for any $\psi \in \mathcal{A}$ we have*

$$\begin{array}{ccc} S^1 & \xrightarrow{f} & S^1 \\ \psi \downarrow & & \uparrow T(\psi) \\ S^1 & \xrightarrow{g} & S^1 \end{array} \quad f = T(\psi) \circ g \circ \psi.$$

The lemma is true precisely because the maps f and g have the same type at their respective critical points.

Proof of Lemma 6.2. Let ψ in $\text{Diff}_+^3(S^1)$ be such that $\psi(c_f) = c_g$, and consider the orientation-preserving circle homeomorphism

$$T(\psi) = f \circ \psi^{-1} \circ g^{-1}$$

that maps the critical value of g to the critical value of f . To see that $T(\psi)$ is in $\text{Diff}_+^3(S^1)$ note that if $z \neq v_g$ then $T(\psi)$ is smooth at z , with non-vanishing derivative equal to

$$(DT(\psi))(z) = D\psi^{-1}(g^{-1}(z)) \left(\frac{Df((\psi^{-1} \circ g^{-1})(z))}{Dg(g^{-1}(z))} \right).$$

In the limit we have

$$\lim_{z \rightarrow v_g} D\psi^{-1}(g^{-1}(z)) \left(\frac{Df((\psi^{-1} \circ g^{-1})(z))}{Dg(g^{-1}(z))} \right) = D\psi^{-1}(c_g) \left(\frac{(D^3 f)(c_f)}{(D^3 g)(c_g)} \right),$$

a well-defined number in $(0, \infty)$. This proves that $T(\psi)$ is in \mathcal{B} for every $\psi \in \mathcal{A}$. Moreover, T is invertible with inverse $T^{-1} : \mathcal{B} \rightarrow \mathcal{A}$ given by $T^{-1}(\phi) = g^{-1} \circ \phi^{-1} \circ f$. \square

Let $A : S^1 \rightarrow S^1$ be the map corresponding to the parameters $a = 0$ and $b = 1$ in the Arnold family (1.2), defined in the introduction, and recall that the lift of A to the real line, under the covering $\pi : \mathbb{R} \rightarrow S^1 : \pi(t) = \exp(2\pi i t)$, fixing the origin is given by

$$\tilde{A}(t) = t - \frac{1}{2\pi} \sin(2\pi t).$$

The critical point of A in the unit circle is at 1, and it is of cubic type (the critical point is also a fixed point for A). Now let f be a C^3 critical circle map with a unique cubic critical point at 1, and let \tilde{f} be the unique lift of f to the real line under the covering π satisfying $\tilde{f}'(0) = 0$ and $0 < \tilde{f}(0) < 1$. By Lemma 6.2 we can consider

two C^3 orientation-preserving circle diffeomorphisms h_1 and h_2 , with $h_1(1) = 1$ and $h_2(1) = f(1)$, such that $h_2 \circ A \circ h_1 = f$, that is, the following diagram commutes:

$$\begin{array}{ccc} S^1 & \xrightarrow{f} & S^1 \\ h_1 \downarrow & & \uparrow h_2 \\ S^1 & \xrightarrow{A} & S^1 \end{array}$$

For each $i \in \{1, 2\}$ let \tilde{h}_i be the lift of h_i to the real line under the covering π , determined by $\tilde{h}_i(0) \in [0, 1)$. In Proposition 6.4 below we will extend both \tilde{h}_1 and \tilde{h}_2 to complex neighbourhoods of the real line in a suitable way. For that purpose we recall the definition of asymptotically holomorphic maps:

Definition 6.3. Let I be a compact interval of the real line, let U be a neighbourhood of I in \mathbb{R}^2 and let $H : U \rightarrow \mathbb{C}$ be a C^1 map (not necessarily a diffeomorphism). We say that H is *asymptotically holomorphic* of order $r \geq 1$ in I if for every $x \in I$,

$$\bar{\partial}H(x, 0) = 0 \quad \text{and} \quad \frac{\bar{\partial}H(x, y)}{(d((x, y), I))^{r-1}} \rightarrow 0$$

uniformly as $(x, y) \in U \setminus I$ converges to I . We say that H is \mathbb{R} -*asymptotically holomorphic* of order r if it is asymptotically holomorphic of order r in each compact interval of \mathbb{R} .

The sum or product of \mathbb{R} -asymptotically holomorphic maps is also \mathbb{R} -asymptotically holomorphic. The inverse of an asymptotically holomorphic diffeomorphism of order r is asymptotically holomorphic of order r . Composition of asymptotically holomorphic maps is asymptotically holomorphic.

In the following proposition we suppose $r \geq 1$ even though we will apply it for $r \geq 3$. In the proof we follow the exposition of Graczyk, Sands and Świątek [16, Lemma 2.1, p. 623].

Proposition 6.4. For $i = 1, 2$ there exists $H_i : \mathbb{C} \rightarrow \mathbb{C}$ of class C^r such that:

- (1) H_i is an extension of \tilde{h}_i : $H_i|_{\mathbb{R}} = \tilde{h}_i$;
- (2) H_i commutes with the unitary horizontal translation: $H_i \circ T = T \circ H_i$;
- (3) H_i is \mathbb{R} -asymptotically holomorphic of order r ;
- (4) H_i is \mathbb{R} -symmetric: $H_i(\bar{z}) = \overline{H_i(z)}$.

Moreover there exist $R > 0$ and four domains B_R, U_R, V_R and W_R in \mathbb{C} , symmetric about the real line and such that:

- $B_R = \{z \in \mathbb{C} : -R < \Im(z) < R\}$;
- H_1 is an orientation-preserving diffeomorphism between B_R and U_R ;
- $\tilde{A}(U_R) = V_R$;
- H_2 is an orientation-preserving diffeomorphism between V_R and W_R ;
- both $\inf_{z \in B_R} |\partial H_1(z)|$ and $\inf_{z \in V_R} |\partial H_2(z)|$ are positive numbers.

Proof. For $z = x + iy \in \mathbb{C}$, with $y \neq 0$, let $P_{x,y}$ be the degree r polynomial map that coincides with \tilde{h}_i at the $r + 1$ real numbers

$$\left\{ x + \frac{j}{r}y \right\}_{j \in \{0,1,\dots,r\}}.$$

Recall that $P_{x,y}$ can be given by the following linear combination (the so-called *Lagrange's form of the interpolation polynomial*):

$$\begin{aligned} P_{x,y}(z) &= \sum_{j=0}^r \tilde{h}_i(x + (j/r)y) \prod_{\substack{l=0 \\ l \neq j}}^r \frac{z - (x + (l/r)y)}{(x + (j/r)y) - (x + (l/r)y)} \\ &= \sum_{j=0}^r \tilde{h}_i(x + (j/r)y) \prod_{\substack{l=0 \\ l \neq j}}^r \frac{z - x - (l/r)y}{((j - l)/r)y}. \end{aligned}$$

We define $H_i(x + iy) = P_{x,y}(x + iy)$, that is,

$$H_i(x + iy) = P_{x,y}(x + iy) = \sum_{j=0}^r \tilde{h}_i(x + (j/r)y) \prod_{\substack{l=0 \\ l \neq j}}^r \frac{ir - l}{j - l}.$$

After computation we obtain

$$H_i(x + iy) = P_{x,y}(x + iy) = \frac{1}{N} \sum_{j=0}^r \frac{(-1)^j \binom{r}{j}}{1 + i(j/r)} \tilde{h}_i(x + (j/r)y)$$

where

$$N = \sum_{j=0}^r \frac{(-1)^j \binom{r}{j}}{1 + i(j/r)} \neq 0.$$

Note that H_i is as smooth as \tilde{h}_i , and $H_i(x) = \tilde{h}_i(x)$ for any real number x (item (1)). As \tilde{h}_i is a lift, for any $j \in \{0, 1, \dots, r\}$ we have $\tilde{h}_i(x + 1 + (j/r)y) = \tilde{h}_i(x + (j/r)y) + 1$, but then $P_{x+1,y}(x + 1 + (j/r)y) = P_{x,y}(x + (j/r)y) + 1$ for any $j \in \{0, 1, \dots, r\}$, and this implies $P_{x+1,y} \circ T = T \circ P_{x,y}$ in the whole complex plane. This proves item (2).

To prove that H_i is \mathbb{R} -asymptotically holomorphic of order r note that

$$\bar{\partial} H_i(x + iy) = \frac{1}{2N} \sum_{j=0}^r (-1)^j \binom{r}{j} \tilde{h}'_i(x + (j/r)y),$$

and for any $k \in \{0, \dots, r\}$,

$$\frac{\partial^k}{\partial y^k} \bar{\partial} H_i(x + iy) = \frac{1}{2N} \cdot \frac{1}{r^k} \sum_{j=0}^r (-1)^j j^k \binom{r}{j} \tilde{h}_i^{(k+1)}(x + (j/r)y).$$

By using the identity $\sum_{j=0}^r (-1)^j j^k \binom{r}{j} = 0$ for any $k \in \{0, \dots, r - 1\}$, we obtain, for any $x \in \mathbb{R}$,

$$\bar{\partial} H_i(x) = \frac{1}{2N} \tilde{h}'_i(x) \sum_{j=0}^r (-1)^j \binom{r}{j} = 0,$$

and for any $k \in \{0, \dots, r - 1\}$,

$$\frac{\partial^k}{\partial y^k} \bar{\partial} H_i(x) = \frac{1}{2N} \cdot \frac{\tilde{h}_i^{(k+1)}(x)}{r^k} \sum_{j=0}^r (-1)^j j^k \binom{r}{j} = 0.$$

By Taylor’s theorem

$$\lim_{y \rightarrow 0} \frac{\bar{\partial} H_i(x + iy)}{y^{r-1}} = 0$$

uniformly on compact subsets of the real line, and hence H_i is \mathbb{R} -asymptotically holomorphic of order r (item (3)). To obtain the symmetry as in item (4) we can take $z \mapsto H_i(z) + \overline{H_i(\bar{z})}/2$, preserving all the other properties.

Finally, note that the Jacobian of H_i at a point x in \mathbb{R} is equal to $|\tilde{h}'_i(x)|^2 \neq 0$. This gives us a complex neighbourhood of the real line where H_i is an orientation-preserving diffeomorphism, and the positive constant R . Since we also have $|\partial H_i| = |\tilde{h}'_i|$ on the real line, and each \tilde{h}_i is the lift of a circle diffeomorphism, we obtain the last item of Proposition 6.4. \square

These are the extensions that we will consider:

Definition 6.5. The map $F : B_R \rightarrow W_R$ defined by $F = H_2 \circ \tilde{A} \circ H_1$ is called the *extended lift* of the critical circle map f .

$$\begin{array}{ccc} B_R & \xrightarrow{F} & W_R \\ H_1 \downarrow & & \uparrow H_2 \\ U_R & \xrightarrow{\tilde{A}} & V_R \end{array}$$

We have the following properties:

- F is C^r in the horizontal band B_R ;
- $T \circ F = F \circ T$ in B_R ;
- F is \mathbb{R} -symmetric (in particular F preserves the real line), and F restricted to the real line is f ;
- F is asymptotically holomorphic in \mathbb{R} of order r ;
- the critical points of F in B_R are integers (the same as for \tilde{A}), and they are of cubic type.

We remark that the extended lift of a real-analytic critical circle map will be C^∞ in the corresponding horizontal strip, but not necessarily holomorphic.

The preimage of the real axis under F consists of \mathbb{R} itself together with two families of C^r curves, $\{\gamma_1(k)\}_{k \in \mathbb{Z}}$ and $\{\gamma_2(k)\}_{k \in \mathbb{Z}}$, arising as solutions of $\Im(F(x + iy)) = 0$. Note that $\gamma_1(k)$ and $\gamma_2(k)$ meet in the critical point $c_k = k$.

Let $\gamma_i^+(k) = \gamma_i(k) \cap \mathbb{H}$ and $\gamma_i^-(k) = \gamma_i(k) \cap \mathbb{H}^-$ for $i = 1, 2$. We also denote $\gamma_i^+(0)$ just by γ_i^+ .

Lemma 6.6. *We can choose R small enough to have γ_1^+ contained in $T = \{\arg(z) \in (\pi/6, \pi/2)\} \cap B_R$ (that is, the open triangle with vertices $0, iR$ and $(\sqrt{3} + i)R$), γ_2^+ contained in $-\bar{T}$, γ_1^- contained in $-T$, and γ_2^- is contained in \bar{T} .*

Proof. The derivative of H_1 at real points is conformal, so the angle between both γ_1 and γ_2 and the real line at zero is $\pi/3$. □

6.2. Poincaré disks

Besides the notion of asymptotically holomorphic maps, the main tool in order to prove Theorem 6.1 is the notion of Poincaré disk, introduced into the subject by Sullivan in his seminal article [53].

Given an open interval $I = (a, b) \subset \mathbb{R}$ let $\mathbb{C}_I = (\mathbb{C} \setminus \mathbb{R}) \cup I = \mathbb{C} \setminus (\mathbb{R} \setminus I)$. For $\theta \in (0, \pi)$ let D be the open disk in the plane intersecting the real line along I and for which the angle from \mathbb{R} to ∂D at the point b (measured anticlockwise) is θ . Let $D^+ = D \cap \{z : \Im(z) > 0\}$ and let D^- be the image of D^+ under complex conjugation.

Define the *Poincaré disk* of angle θ based on I as $D_\theta(a, b) = D^+ \cup I \cup D^-$, that is, $D_\theta(a, b)$ is the set of points in the complex plane that view I at an angle $\geq \theta$ (see Figure 4). Note that for $\theta = \pi/2$ the Poincaré disk $D_\theta(a, b)$ is the Euclidean disk whose diameter is the interval (a, b) .

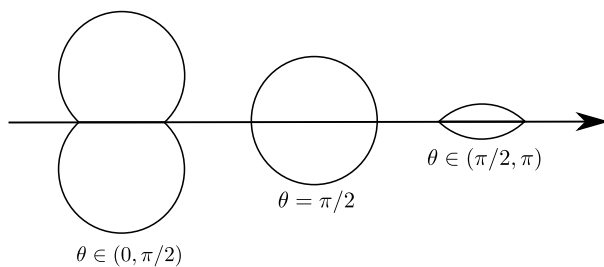


Fig. 4. Poincaré disks.

Poincaré disks have a geometrical meaning: \mathbb{C}_I is an open, connected and simply connected set which is not the whole plane. By the Riemann mapping theorem we can endow \mathbb{C}_I with a complete and conformal Riemannian metric of constant curvature equal to -1 , just by pulling back the Poincaré metric of \mathbb{D} by any conformal uniformization. Note that I is always a hyperbolic geodesic by symmetry.

For $\theta \in (0, \pi)$ consider $\varepsilon(\theta) = \log \tan(\pi/2 - \theta/4)$, which is an orientation-reversing real-analytic diffeomorphism between $(0, \pi)$ and $(0, \infty)$. An elementary computation shows that the set of points in \mathbb{C}_I whose hyperbolic distance to I is less than ε is precisely $D_\theta(a, b)$.

In particular we can state the Schwarz lemma in the following way. Let I and J be two intervals in the real line and let $\phi : \mathbb{C}_I \rightarrow \mathbb{C}_J$ be a holomorphic map such that $\phi(I) \subset J$. Then for any $\theta \in (0, \pi)$ we have $\phi(D_\theta(I)) \subset D_\theta(J)$.

With this at hand (and a very clever inductive argument, see also [36]), Yampolsky was able to obtain *complex bounds* for critical circle maps in the *Epstein class* [55,

Theorem 1.1]. The reason why we chose asymptotically holomorphic maps to extend our (finitely smooth) one-dimensional dynamics (see Proposition 6.4 and Definition 6.5 above) is the following asymptotic Schwarz lemma, obtained by Graczyk, Sands and Świątek [16, Proposition 2, p. 629] for asymptotically holomorphic maps (as explained at the end of the introduction, we denote by $\text{diam}(D_\theta(a, b))$ the Euclidean diameter of the Poincaré disk $D_\theta(a, b)$).

Proposition 6.7 (Almost Schwarz inclusion). *Let $h : I \rightarrow \mathbb{R}$ be a C^3 diffeomorphism from a compact interval I with non-empty interior into the real line. Let H be any C^3 extension of h to a complex neighbourhood of I , which is asymptotically holomorphic of order 3 on I . Then there exist $M, \delta > 0$ such that if $a, c \in I$ are different, $\theta \in (0, \pi)$ and $\text{diam}(D_\theta(a, c)) < \delta$ then*

$$H(D_\theta(a, c)) \subseteq D_{\tilde{\theta}}(h(a), h(c)),$$

where $\tilde{\theta} = \theta - M|c - a| \text{diam}(D_\theta(a, c))$. Moreover, $\tilde{\theta} > 0$.

Let us point out that a predecessor of this almost Schwarz inclusion, for real-analytic maps, already appeared in the work of de Faria and de Melo [14, Lemma 3.3, p. 350].

6.3. Proof of Theorem 6.1

With Proposition 6.7 at hand, we are ready to start the proof of Theorem 6.1. We will work with $\tilde{f}^{q_{n+1}}|_{I_n}$, the proof for $\tilde{f}^{q_n}|_{I_{n+1}}$ being the same.

Proposition 6.8. *Let f be a C^3 critical circle map with irrational rotation number, and let F be its extended lift (according to Definition 6.5). There exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ there exist $K_n \geq 1$ and $\theta_n > 0$ satisfying $K_n \rightarrow 1$ and $\theta_n \rightarrow 0$ as $n \rightarrow \infty$, and*

$$\lim_{n \rightarrow \infty} \left| \frac{\text{diam}(D_{\theta_n/K_n}(\tilde{f}(I_n)))}{|\tilde{f}(I_n)|} - \frac{\text{diam}(D_{\theta_n}(\tilde{f}^{q_{n+1}}(I_n)))}{|\tilde{f}^{q_{n+1}}(I_n)|} \right| = 0$$

with the following property. Let $\theta \geq \theta_n$, $1 \leq j \leq q_{n+1}$ and let J be an open interval such that

$$I_n \subseteq \bar{J} \subseteq (\tilde{f}^{q_{n-1}-q_{n+1}}(0), \tilde{f}^{q_n-q_{n+1}}(0)).$$

Then the inverse branch F^{-j+1} mapping $\tilde{f}^j(J)$ back to $\tilde{f}(J)$ is well-defined over $D_\theta(\tilde{f}^j(J))$, and maps this neighbourhood diffeomorphically onto an open set contained in $D_{\theta/K_n}(\tilde{f}(J))$.

To simplify notation we will prove Proposition 6.8 for the case $J = I_n$ and $j = q_{n+1}$.

Proof of Proposition 6.8. For each $n \in \mathbb{N}$ and $j \in \{1, \dots, q_{n+1} - 1\}$ we know by combinatorics that \tilde{f} is a C^3 diffeomorphism from $\tilde{f}^j(I_n)$ to $\tilde{f}^{j+1}(I_n)$. Let $M_{j,n}, \delta_{j,n} > 0$ be given by Proposition 6.7 applied to the corresponding inverse branch of the extended lift F . Moreover, let $M_n = \max_{j \in \{1, \dots, q_{n+1}-1\}} M_{j,n}$ and $\delta_n = \min_{j \in \{1, \dots, q_{n+1}-1\}} \delta_{j,n}$. For each $n \in \mathbb{N}$ let A_n and B_n be the affine maps given by

$$A_n(t) = (1/|\tilde{f}^{q_{n+1}}(I_n)|)(t - \tilde{f}^{q_{n+1}}(0)) \quad \text{and} \quad B_n(t) = (1/|\tilde{f}(I_n)|)(t - \tilde{f}(0)).$$

By the real bounds, the C^3 diffeomorphism $T_n : [0, 1] \rightarrow [0, 1]$ given by

$$T_n = B_n \circ \tilde{f}^{-q_{n+1}+1} \circ A_n^{-1}$$

has universally bounded distortion, and therefore

$$\inf_{\substack{t \in [0,1] \\ n \geq n_0}} |T'_n(t)| > 0.$$

In particular $M = \sup_{n \geq n_0} M_n$ is finite, and $\delta = \inf_{n \geq n_0} \delta_n$ is positive. Let $d_n = \max_{1 \leq j \leq q_{n+1}} |\tilde{f}^j(I_n)|$, and recall that by the real bounds the sequence $\{d_n\}_{n \geq 1}$ goes to zero exponentially fast as $n \rightarrow \infty$. In particular we can choose a sequence $\{\alpha_n\}_{n \geq 1} \subset (0, \pi/2)$ also convergent to zero but such that

$$\lim_{n \rightarrow \infty} \frac{d_n}{(\alpha_n)^3} = 0.$$

Let $\psi : (0, \pi) \rightarrow [1, \infty)$ be defined by

$$\psi(\theta) = \max \left\{ 1, \frac{1 + \cos \theta}{\sin \theta} \right\} = \begin{cases} \frac{1 + \cos \theta}{\sin \theta} & \text{for } \theta \in (0, \pi/2), \\ 1 & \text{for } \theta \in [\pi/2, \pi). \end{cases}$$

Note that ψ is an orientation-reversing real-analytic diffeomorphism between $(0, \pi/2)$ and $(1, \infty)$. A straightforward computation shows that for any $\theta \in (0, \pi)$ and any real numbers $a < b$ we have $\text{diam}(D_\theta(a, b)) = \psi(\theta)|b - a|$. Consider

$$\begin{aligned} \theta_n &= \alpha_n + \psi(\alpha_n) \delta M \sum_{j=0}^{q_{n+1}-1} |\tilde{f}^{j+1}(I_n)|^2 > \alpha_n > 0, \\ K_n &= \frac{\theta_n}{\alpha_n} = 1 + \frac{\psi(\alpha_n)}{\alpha_n} \delta M \sum_{j=0}^{q_{n+1}-1} |\tilde{f}^{j+1}(I_n)|^2 > 1. \end{aligned}$$

By the choice of α_n we have $\lim_{n \rightarrow \infty} (\psi(\alpha_n)/\alpha_n)d_n = 0$, and since $\sum_{j=0}^{q_{n+1}-1} |\tilde{f}^{j+1}(I_n)|^2 \leq d_n$, we see that $\theta_n \rightarrow 0$ and $K_n \rightarrow 1$ as $n \rightarrow \infty$. Moreover,

$$\begin{aligned} |\psi(\theta_n/K_n) - \psi(\theta_n)| &\leq \left(\max_{\theta \in [\theta_n/K_n, \theta_n]} |\psi'(\theta)| \right) |\theta_n - \theta_n/K_n| = |\psi'(\theta_n/K_n)| |\theta_n - \theta_n/K_n| \\ &= \frac{\psi(\theta_n/K_n)}{\sin(\theta_n/K_n)} |\theta_n - \theta_n/K_n| = \frac{\psi(\alpha_n)}{\sin \alpha_n} |\theta_n - \alpha_n| \\ &= \delta M \frac{(\psi(\alpha_n))^2}{\sin \alpha_n} \sum_{j=0}^{q_{n+1}-1} |\tilde{f}^{j+1}(I_n)|^2 \leq \delta M \frac{(\psi(\alpha_n))^2}{\sin \alpha_n} d_n, \end{aligned}$$

and this goes to zero by the choice of α_n . In particular

$$\lim_{n \rightarrow \infty} \left| \frac{\text{diam}(D_{\theta_n/K_n}(\tilde{f}(I_n)))}{|\tilde{f}(I_n)|} - \frac{\text{diam}(D_{\theta_n}(\tilde{f}^{q_{n+1}}(I_n)))}{|\tilde{f}^{q_{n+1}}(I_n)|} \right| = 0$$

as stated. We choose $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $\psi(\alpha_n)d_n < \delta$.

Define inductively $\{\theta_j\}_{j=1}^{q_{n+1}}$ by $\theta_{q_{n+1}} = \theta_n$ and for $1 \leq j \leq q_{n+1} - 1$,

$$\theta_j = \theta_{j+1} - M|\tilde{f}^{j+1}(I_n)| \operatorname{diam}(D_{\theta_{j+1}}(\tilde{f}^{j+1}(I_n))) = \theta_{j+1} - M\psi(\theta_{j+1})|\tilde{f}^{j+1}(I_n)|^2.$$

We want to show that $\theta_j > \alpha_n = \theta_n/K_n$ for all $1 \leq j \leq q_{n+1}$. With this purpose, we claim that for any $1 \leq j \leq q_{n+1}$ we have

$$\theta_j \geq \alpha_n + \psi(\alpha_n)\delta M \sum_{k=0}^{j-1} |\tilde{f}^{k+1}(I_n)|^2 > \alpha_n.$$

The claim follows by (reverse) induction on j (the case $j = q_{n+1}$ holds by definition). If the claim is true for $j + 1$ we have $\psi(\theta_{j+1}) < \psi(\alpha_n)$, which implies $\theta_j > \theta_{j+1} - \psi(\alpha_n)\delta M|\tilde{f}^{j+1}(I_n)|^2$, and hence the claim is true for j . It follows that

$$\operatorname{diam}(D_{\theta_j}(\tilde{f}^j(I_n))) = \psi(\theta_j)|\tilde{f}^j(I_n)| < \psi(\alpha_n)d_n < \delta \leq \delta_j \quad \text{for all } 1 \leq j \leq q_{n+1}.$$

By Proposition 5.6 the inverse branch F^{-1} mapping $\tilde{f}^{j+1}(I_n)$ back to $\tilde{f}^j(I_n)$ is a well-defined diffeomorphism from the Poincaré disk $D_{\theta_{j+1}}(\tilde{f}^{j+1}(I_n))$ onto its image, and by Proposition 6.7 we know that

$$F^{-1}(D_{\theta_{j+1}}(\tilde{f}^{j+1}(I_n))) \subseteq D_{\theta_j}(\tilde{f}^j(I_n)).$$

The claim also gives

$$\theta_1 \geq \alpha_n + \psi(\alpha_n)\delta M|\tilde{f}(I_n)|^2 > \alpha_n = \theta_n/K_n,$$

and this finishes the proof. □

Corollary 6.9. *There exist constants $\alpha, C_1, C_2 > 0$ and $\lambda \in (0, 1)$ with the following property. Let f be a C^3 critical circle map with irrational rotation number, and let F be its extended lift. There exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ there exists an \mathbb{R} -symmetric topological disk Y_n with*

$$N_\alpha(\tilde{f}(I_n)) \subset Y_n$$

such that the composition $F^{q_{n+1}-1} : Y_n \rightarrow F^{q_{n+1}-1}(Y_n)$ is a well-defined C^3 diffeomorphism and we have:

$$(1) \quad C_1 < \frac{\operatorname{diam}(F^{q_{n+1}-1}(Y_n))}{|\tilde{f}^{q_{n+1}}(I_n)|} < C_2,$$

$$(2) \quad \sup_{z \in Y_n} \frac{|\bar{\partial} F^{q_{n+1}-1}(z)|}{|\partial F^{q_{n+1}-1}(z)|} \leq C_2 \lambda^n.$$

Proof. For each $n \in \mathbb{N}$ let:

- I_n be the closed interval whose endpoints are 0 and $(T^{-p_n} \circ \tilde{f}^{q_n})(0)$,
- J_n be the open interval containing the origin that projects to

$$(f^{q_{n+1}}(1), f^{q_n - q_{n+3}}(1))$$

under the covering $\pi(t) = e^{2\pi it}$, and

- K_n be the open interval containing the origin that projects to

$$(f^{q_{n-1}-q_{n+1}}(1), f^{q_n-q_{n+1}}(1))$$

under the covering π .

Note that $I_n \cup I_{n+1} \subset J_n \subset \overline{J_n} \subset K_n$ (see Figure 5). By combinatorics, the map $\tilde{f} : \tilde{f}^j(K_n) \rightarrow \tilde{f}^{j+1}(K_n)$ is a diffeomorphism for all $j \in \{1, \dots, q_{n+1} - 1\}$, and therefore all restrictions $\tilde{f} : \tilde{f}^j(J_n) \rightarrow \tilde{f}^{j+1}(J_n)$ are diffeomorphisms for any $j \in \{1, \dots, q_{n+1} - 1\}$ (just as in the proof of Proposition 6.8).

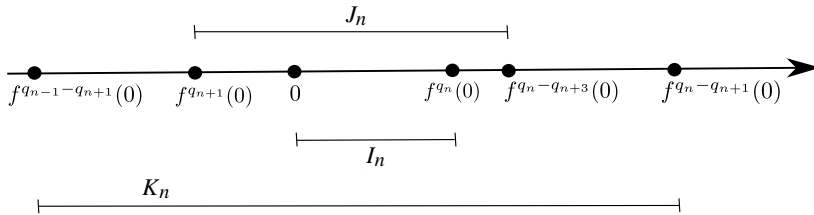


Fig. 5. Relative positions of the relevant points in the proof of Corollary 6.9.

Recall that the extended lift $F : B_R \rightarrow W_R$ is given by the composition $F = H_2 \circ \tilde{A} \circ H_1$ (see Definition 6.5). Let $n_0 \in \mathbb{N}$ be given by Proposition 6.8, and for each $n \geq n_0$ let $K_n \geq 1$ and $\theta_n > 0$ be also given by Proposition 6.8. Fix $\theta \in (0, \pi)$ such that $\theta > \theta_n$ for all $n \geq n_0$ and

$$|\mu_{H_i}(z)| < \frac{1}{2}(d(z, \tilde{f}^j(J_n)))^2$$

for any $z \in D_{\theta/K_n}(\tilde{f}^j(J_n))$, any $j \in \{1, \dots, q_{n+1} - 1\}$ and any $i \in \{1, 2\}$ (as before, μ_{H_i} denotes the Beltrami coefficient of the quasiconformal homeomorphism H_i , and d denotes the Euclidean distance in the complex plane). The existence of such a θ is guaranteed by Proposition 6.8, the fact that both H_i are \mathbb{R} -asymptotically holomorphic of order 3, and the last item in Proposition 6.4.

Let $Y_n \subset F^{-q_{n+1}+1}(D_\theta(\tilde{f}^{q_{n+1}}(J_n)))$ be the preimage of $D_\theta(\tilde{f}^{q_{n+1}}(J_n))$ under $F^{q_{n+1}-1}$ given by Proposition 6.8, and note that:

- Y_n is an \mathbb{R} -symmetric topological disk;
- $\tilde{f}(I_n) \subset Y_n$;
- $\tilde{f}(I_{n+1}) \subset Y_n$;
- by Proposition 6.8, $F^j(Y_n) \subset D_{\theta/K_n}(\tilde{f}^{j+1}(J_n))$ for all $j \in \{0, 1, \dots, q_{n+1} - 1\}$.

Moreover,

$$\text{diam}(F^{q_{n+1}-1}(Y_n)) = \text{diam}(D_\theta(\tilde{f}^{q_{n+1}}(J_n))) = \psi(\theta)|\tilde{f}^{q_{n+1}}(J_n)|,$$

and by the real bounds, $|\tilde{f}^{q_{n+1}}(J_n)|$ and $|\tilde{f}^{q_{n+1}}(I_n)|$ are comparable (with universal constants independent of $n \geq n_0$). Again the map ψ is the same as in the proof of Proposition 6.8. This gives us item (1), and now we prove (2). For each $n \geq n_0$ let $k_n \in [0, 1)$ be

the conformal distortion of $F^{q_{n+1}-1}$ at Y_n , that is,

$$k_n = \sup_{z \in Y_n} \frac{|\bar{\partial} F^{q_{n+1}-1}(z)|}{|\partial F^{q_{n+1}-1}(z)|}.$$

Moreover, for each $j \in \{1, \dots, q_{n+1} - 1\}$ let $K_{n,j}$, $K_{n,j}(1)$ and $K_{n,j}(2)$ in $[1, \infty)$ be respectively the quasiconformality of F at $F^{j-1}(Y_n)$, of H_1 also at $F^{j-1}(Y_n)$, and of H_2 at $(\tilde{A} \circ H_1)(F^{j-1}(Y_n))$. Since \tilde{A} is conformal, we have

$$\begin{aligned} k_n &\leq \log \left(\prod_{j=1}^{q_{n+1}-1} K_{n,j} \right) = \sum_{j=1}^{q_{n+1}-1} \log K_{n,j} \\ &= \sum_{j=1}^{q_{n+1}-1} (\log K_{n,j}(1) + \log K_{n,j}(2)) \\ &\leq \sum_{j=1}^{q_{n+1}-1} M_0 (\text{diam}(F^{j-1}(Y_n)))^2 \quad (\text{for some } M_0 > 1) \\ &\leq \sum_{j=1}^{q_{n+1}-1} M_0 (\text{diam}(D_{\theta/K_n}(\tilde{f}^j(J_n))))^2 \\ &= \sum_{j=1}^{q_{n+1}-1} M_0 (\psi(\theta/K_n))^2 |\tilde{f}^j(J_n)|^2 < M_1 \sum_{j=1}^{q_{n+1}-1} |\tilde{f}^j(J_n)|^2. \end{aligned}$$

The last inequality follows from the fact that $K_n \rightarrow 1$ as $n \rightarrow \infty$. By combinatorics the projection of the family $\{\tilde{f}^j(J_n)\}_{j=1}^{q_{n+1}-1}$ to the unit circle has finite multiplicity of intersection (independent of $n \geq n_0$), and therefore

$$\sum_{j=1}^{q_{n+1}-1} |\tilde{f}^j(J_n)|^2 < M_2 \left(\max_{j \in \{1, \dots, q_{n+1}-1\}} |\tilde{f}^j(J_n)| \right), \tag{6.1}$$

where the constant $M_2 > 0$ only depends on the multiplicity of intersection of the projection of the family $\{\tilde{f}^j(J_n)\}_{j=1}^{q_{n+1}-1}$ to the unit circle. By the real bounds, the right-hand side of (6.1) goes to zero exponentially fast at a universal rate (independent of f), and therefore we obtain constants $\lambda \in (0, 1)$ and $C > 0$ such that

$$k_n = \sup_{z \in Y_n} \frac{|\bar{\partial} F^{q_{n+1}-1}(z)|}{|\partial F^{q_{n+1}-1}(z)|} \leq C\lambda^n \quad \text{for all } n \geq n_0.$$

To finish the proof of Corollary 6.9 we need to obtain definite domains around $\tilde{f}(I_n)$ contained in Y_n . As in the proof of Proposition 6.8, for each $n \geq n_0$ let A_n and B_n be the affine maps given by

$$A_n(z) = (1/|\tilde{f}^{q_{n+1}}(I_n)|)(z - \tilde{f}^{q_{n+1}}(0)) \quad \text{and} \quad B_n(z) = (1/|\tilde{f}(I_n)|)(z - \tilde{f}(0)),$$

and also let $Z_n = A_n(D_{\theta}(\tilde{f}^{q_{n+1}}(J_n)))$. By the real bounds there exists a universal constant $\alpha_0 > 0$ such that

$$N_{\alpha_0}([0, 1]) \subset Z_n \quad \text{for all } n \geq n_0.$$

The \mathbb{R} -symmetric orientation-preserving C^3 diffeomorphism $T_n : Z_n \rightarrow T_n(Z_n)$ given by

$$T_n = B_n \circ F^{-q_{n+1}+1} \circ A_n^{-1}$$

induces a diffeomorphism in $[0, 1]$ which, again by the real bounds, has universally bounded distortion. In particular there exists $\varepsilon > 0$ such that $|T'_n(t)| > \varepsilon$ for all $t \in [0, 1]$ and all $n \geq n_0$. By Proposition 5.6 there exists $\alpha > 0$ (only depending on α_0 and ε) such that (for n_0 large enough)

$$N_\alpha([0, 1]) \subset T_n(Z_n) \quad \text{for all } n \geq n_0,$$

and therefore $N_\alpha(\tilde{f}(I_n)) \subset Y_n$ for all $n \geq n_0$. □

Proposition 6.10. *There exist constants $\alpha, C_1, C_2 > 0$ and $\lambda \in (0, 1)$ with the following property. Let f be a C^3 critical circle map with irrational rotation number, and let F be its extended lift. There exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ there exists an \mathbb{R} -symmetric topological disk X_n with*

$$N_\alpha(I_n) \subset X_n, \quad \text{where } I_n = [0, (T^{-p_n} \circ \tilde{f}^{q_n})(0)],$$

such that the composition $F^{q_{n+1}}$ is well-defined in X_n , has a unique critical point at the origin, and:

- (1) $C_1 < \frac{\text{diam}(F^{q_{n+1}}(X_n))}{|\tilde{f}^{q_{n+1}}(I_n)|} < C_2,$
- (2) $\sup_{z \in X_n \setminus \{0\}} \frac{|\bar{\partial} F^{q_{n+1}}(z)|}{|\partial F^{q_{n+1}}(z)|} \leq C_2 \lambda^n.$

Proof. From the construction of the extended lift F in Subsection 6.1 (see also Lemma 6.6) there exists a complex neighbourhood Ω of the origin such that the restriction $F : \Omega \rightarrow F(\Omega)$ is of the form $Q \circ \psi$, where $Q(z) = z^3 + \tilde{f}(0)$, and $\psi : \Omega \rightarrow Q^{-1}(F(\Omega))$ is an \mathbb{R} -symmetric orientation-preserving C^3 diffeomorphism fixing the origin. In particular there exist $\varepsilon, \delta > 0$ such that if $t \in (-\delta, \delta)$ then $|(\psi^{-1})'(t)| > \varepsilon$, where $(\psi^{-1})'$ denotes the one-dimensional derivative of the restriction of ψ^{-1} to $Q^{-1}(F(\Omega)) \cap \mathbb{R}$. Let $K > 1$ be given by Proposition 5.6 applied to $\varepsilon > 0$. Since ψ is asymptotically holomorphic of order 3 in Ω , we can choose Ω so small that ψ is K -quasiconformal. By taking $n_0 \in \mathbb{N}$ large enough we can assume that $|\psi(I_n)| < \delta$ and $Y_n \subset F(\Omega)$ for all $n \geq n_0$, where the topological disk Y_n is the one given by Corollary 6.9. By Corollary 6.9 and elementary properties of the cube root map (see for instance [55, Lemma 2.2]) there exists a universal constant $\alpha_0 > 0$ such that for all $n \geq n_0$ we have

$$N_{\alpha_0}(\psi(I_n)) \subset Q^{-1}(Y_n). \tag{6.2}$$

Define $X_n \subset \Omega$ by $X_n = F^{-1}(Y_n) = \psi^{-1}(Q^{-1}(Y_n))$. Item (1) follows directly from Corollary 6.9(1) since $F^{q_{n+1}}(X_n) = F^{q_{n+1}-1}(Y_n)$. By (6.2) and Proposition 5.6 there exists a universal constant $\alpha > 0$ such that

$$N_\alpha(I_n) \subset X_n \subset \Omega \quad \text{for all } n \geq n_0.$$

To obtain item (2) recall that by Corollary 6.9(2) we have

$$\sup_{z \in Y_n} \frac{|\bar{\partial} F^{q_{n+1}-1}(z)|}{|\partial F^{q_{n+1}-1}(z)|} \leq C\lambda^n.$$

Since Q is a polynomial, it is conformal at its regular points, and since $\|\mu_\psi\|_\infty \leq (K-1)/(K+1) < 1$ in Ω we have

$$\sup_{z \in X_n \setminus \{0\}} \frac{|\bar{\partial} F^{q_{n+1}}(z)|}{|\partial F^{q_{n+1}}(z)|} \leq C\lambda^n. \quad \square$$

Theorem 6.1 follows directly from Proposition 6.10 and its analogue for $\tilde{f}^{q_n}|_{I_{n+1}}$.

7. Proof of Theorem D

As its title indicates, this section is entirely devoted to the proof of Theorem D, and recall that Theorem D implies our main result (Theorem B) as we saw in Section 4.

First let us fix some notation and terminology. By a *topological disk* we mean an open, connected and simply connected set properly contained in the complex plane. Let $\pi : \mathbb{C} \rightarrow \mathbb{C} \setminus \{0\}$ be the holomorphic covering $z \mapsto \exp(2\pi iz)$, and let $T : \mathbb{C} \rightarrow \mathbb{C}$ be the horizontal translation $z \mapsto z + 1$ (which is a generator of the group of automorphisms of the covering). For any $R > 1$ consider the *band*

$$B_R = \{z \in \mathbb{C} : -\log R < 2\pi \Im(z) < \log R\},$$

which is the universal cover of the *round annulus*

$$A_R = \{z \in \mathbb{C} : 1/R < |z| < R\}$$

via the holomorphic covering π . Since B_R is T -invariant, the translation generates the group of automorphisms of the covering. The restriction $\pi : \mathbb{R} \rightarrow S^1 = \partial\mathbb{D}$ is also a covering map, the automorphism T preserves the real line, and again generates the group of automorphisms of the covering.

More generally, an *annulus* is an open and connected set A in the complex plane whose fundamental group is isomorphic to \mathbb{Z} . By the Uniformization Theorem such an annulus is conformally equivalent either to the punctured disk $\mathbb{D} \setminus \{0\}$, to the punctured plane $\mathbb{C} \setminus \{0\}$, or to some round annulus $A_R = \{z \in \mathbb{C} : 1/R < |z| < R\}$. In the last case the value of $R > 1$ is unique, and there exists a holomorphic covering from \mathbb{D} to A whose group of deck transformations is infinite cyclic, and such that any generator is a Möbius transformation that has exactly two fixed points on the boundary of the unit disk.

Since the deck transformations are Möbius transformations, they are isometries of the Poincaré metric on \mathbb{D} , and therefore there exists a unique Riemannian metric on A such that the covering map provided by the Uniformization Theorem is a local isometry. This metric is complete, and in particular any two points can be joined by a minimizing geodesic. There exists a unique simple closed geodesic in A whose hyperbolic length is equal to $\pi^2/\log R$. The length of this closed geodesic is therefore a conformal invariant.

We denote by Θ the antiholomorphic involution $z \mapsto 1/\bar{z}$ of the punctured plane $\mathbb{C} \setminus \{0\}$, and we say that a map is S^1 -symmetric if it commutes with Θ . An annulus is S^1 -symmetric if it is invariant under Θ (for instance, the round annulus A_R described above is S^1 -symmetric). In this case, the unit circle is the *core curve* (the unique simple closed geodesic) for the hyperbolic metric in A . In this section we will deal only with S^1 -symmetric annuli. In particular any time that some annulus A_0 is contained in some other annulus A_1 , the annulus A_0 separates the boundary components of A_1 (more technically, the inclusion is *essential* in the sense that the fundamental group $\pi_1(A_0)$ injects into $\pi_1(A_1)$).

Besides Theorem 6.1, the main tool in order to prove Theorem D is Proposition 5.5. The proof will be divided into three subsections. Along the proof, C will denote a positive constant (independent of $n \in \mathbb{N}$) and n_0 will denote a positive (large enough) natural number. First, let $n_0 \in \mathbb{N}$ be given by Theorem 6.1. Moreover, set $W_1 = N_\alpha([-1, 0])$, $W_2 = W_2(n) = N_\alpha([0, \xi_n(0)])$, $W_0 = B(0, \lambda)$ and $\mathcal{V} = B(0, \lambda^{-1})$, where $\alpha > 0$ and $\lambda \in (0, 1)$ are the universal constants given by Theorem 6.1. Recall that $\eta_n(0) = -1$ for all $n \geq 1$ after normalization.

7.1. *A first perturbation and a bidimensional glueing procedure*

From Theorem 6.1 we have:

Lemma 7.1. *There exists an \mathbb{R} -symmetric topological disk U with $-1 \in U \subset W_1 \setminus W_0$ such that for all $n \geq n_0$ the composition*

$$\eta_n^{-1} \circ \xi_n : U \rightarrow (\eta_n^{-1} \circ \xi_n)(U)$$

is an \mathbb{R} -symmetric orientation-preserving C^3 diffeomorphism.

For each $n \geq n_0$ denote by A_n the diffeomorphism $\eta_n^{-1} \circ \xi_n$. Note that $\|\mu_{A_n}\|_\infty \leq C\lambda^n$ in U for all $n \geq n_0$, and that the domains $\{A_n(U)\}_{n \geq n_0}$ are uniformly bounded since they are contained in $\bigcup_j W_2^j$. Fix $\varepsilon, \delta > 0$ such that the rectangle

$$V = (-1 - \varepsilon, -1 + \varepsilon) \times (-i\delta, i\delta)$$

is compactly contained in U , and apply Proposition 5.5 to the sequence

$$\{A_n : U \rightarrow A_n(U)\}_{n \geq n_0}$$

of \mathbb{R} -symmetric orientation-preserving C^3 diffeomorphisms to obtain a sequence

$$\{B_n : V \rightarrow B_n(V)\}_{n \geq n_0}$$

of \mathbb{R} -symmetric biholomorphisms such that

$$\|A_n - B_n\|_{C^0(V)} \leq C\lambda^n \quad \text{for all } n \geq n_0.$$

By combining Theorem 6.1 with the commuting condition, we obtain the following configuration, which somehow resembles the notion of *holomorphic commuting pairs* introduced by de Faria in his PhD thesis [10] (see also [11, Section 2] and [14, Section 2]).

Lemma 7.2. For each $n \geq n_0$ there exist \mathbb{R} -symmetric topological disks $V_i(n)$ for $i \in \{1, 2, 3\}$ with the following properties:

- $0 \in V_1(n) \subset W_0$;
- $(\eta_n \circ \xi_n)(0) = (\xi_n \circ \eta_n)(0) = \xi_n(-1) \in V_2(n) \subset W_2$;
- $\xi_n(0) \in V_3(n) \subset W_2$;
- when restricted to $V_1(n)$, both η_n and ξ_n are orientation-preserving three-fold C^3 branched coverings onto V and $V_3(n)$ respectively, with a unique critical point at the origin;
- both $\xi_n|_V$ and $\eta_n|_{V_3(n)}$ are orientation-preserving C^3 diffeomorphisms onto $V_2(n)$.

In particular $\eta_n^{-1} \circ \xi_n$ is an orientation-preserving C^3 diffeomorphism from V onto $V_3(n)$ for all $n \geq n_0$.

For each $n \geq n_0$ let $U_1(n)$, $U_2(n)$ and $U_3(n)$ be \mathbb{R} -symmetric topological disks such that:

- $\overline{U_1(n)}$, $\overline{U_2(n)}$ and $\overline{U_3(n)}$ are pairwise disjoint;
- $V \cap U_j(n) = \emptyset$ and $V_i(n) \cap U_j(n) = \emptyset$ for $i, j \in \{1, 2, 3\}$;
- $\overline{U_1(n)} \subset W_1$ and $\overline{U_2(n)} \cup \overline{U_3(n)} \subset W_2$,

and such that

$$\mathcal{U}_n = \text{int} \left[V \cup \left(\bigcup_{i=1}^3 V_i(n) \right) \cup \left(\bigcup_{j=1}^3 \overline{U_j(n)} \right) \right]$$

is an \mathbb{R} -symmetric topological disk (see Figure 6). Note that

$$\overline{I_{\xi_n} \cup I_{\eta_n}} \subset \mathcal{U}_n \subset W_1 \cup W_2 \quad \text{for all } n \geq n_0,$$

and $\mathcal{U}_n \setminus \overline{V \cup V_1(n) \cup V_2(n) \cup V_3(n)}$ has three connected components, which are precisely $U_1(n)$, $U_2(n)$ and $U_3(n)$. By Theorem 6.1 we can choose $U_1(n)$, $U_2(n)$ and $U_3(n)$ so as

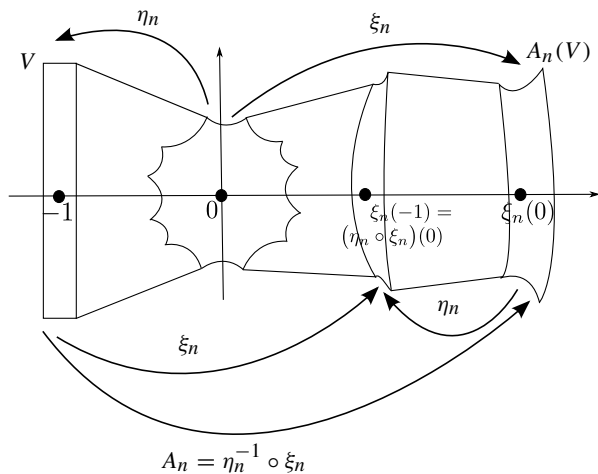


Fig. 6. The domain \mathcal{U}_n .

to also have

$$\overline{N_\delta([-1, 0]) \cup N_\delta([0, \xi_n(0)])} \subset \mathcal{U}_n \quad \text{for all } n \geq n_0,$$

for some universal constant $\delta > 0$ independent of $n \geq n_0$. Note also that each \mathcal{U}_n is uniformly bounded since it is contained in $N_\alpha([-1, K])$, where $\alpha > 0$ is given by Theorem 6.1, and $K > 1$ is the universal constant given by the real bounds.

For each $n \geq n_0$ let \mathcal{T}_n be an \mathbb{R} -symmetric topological disk such that:

- $V, V_1(n), V_2(n)$ and $B_n(V)$ are contained in \mathcal{T}_n ;
- $\mathcal{T}_n \setminus (V \cup B_n(V))$ is connected and simply connected;
- the Hausdorff distance between $\overline{\mathcal{T}_n}$ and $\overline{\mathcal{U}_n}$ is no more than

$$\|A_n - B_n\|_{C^0(V)} \leq C\lambda^n.$$

Lemma 7.3. *For each $n \geq n_0$ there exists an orientation-preserving \mathbb{R} -symmetric C^3 diffeomorphism $\Phi_n : \mathcal{U}_n \rightarrow \mathcal{T}_n$ such that:*

- $\Phi_n \equiv \text{Id}$ in the interior of $V \cup \overline{U_1(n)} \cup V_1(n)$, in particular $\Phi_n(0) = 0$;
- $B_n = \Phi_n \circ (\eta_n^{-1} \circ \xi_n) \circ \Phi_n^{-1}$ in V , that is, $\Phi_n \circ A_n = B_n \circ \Phi_n$ in V ;
- $\|\Phi_n - \text{Id}\|_{C^0(\mathcal{U}_n)} \leq C\lambda^n$;
- $\|\mu_{\Phi_n}\|_\infty \leq C\lambda^n$ in \mathcal{U}_n .

Proof. For each $n \geq n_0$ we have $\|A_n - B_n\|_{C^0(V)} \leq C\lambda^n$, and therefore

$$\|\text{Id} - (B_n \circ A_n^{-1})\|_{C^0(V_3(n))} \leq C\lambda^n.$$

If we define $\Phi_n|_{V_3(n)} = B_n \circ A_n^{-1}$ we also have $\|\mu_{\Phi_n}\|_\infty = \|\mu_{A_n^{-1}}\|_\infty$ in $V_3(n)$, which is equal to $\|\mu_{A_n}\|_\infty$ in V . In particular $\|\mu_{\Phi_n}\|_\infty \leq C\lambda^n$ in $V_3(n)$, and then we define Φ_n in the whole \mathcal{U}_n by interpolating $B_n \circ A_n^{-1}$ in $V_3(n)$ with the identity in the interior of $V \cup \overline{U_1(n)} \cup V_1(n)$. □

Consider the seven topological disks:

$$\begin{aligned} X_1(n) &= \text{int}(V \cup \overline{U_1(n)} \cup V_1(n)) \subset W_1 \cap \mathcal{U}_n, \\ X_2(n) &= \text{int}(V_1(n) \cup \overline{U_2(n)} \cup V_2(n) \cup \overline{U_3(n)} \cup V_3(n)) \subset W_2 \cap \mathcal{U}_n, \\ \widehat{X}_1(n) &= \{z \in X_1(n) : \xi_n(z) \in \mathcal{U}_n\}, \quad \widehat{X}_2(n) = \{z \in X_2(n) : \eta_n(z) \in \mathcal{U}_n\}, \\ \widehat{\mathcal{T}}_n &= \Phi_n(\widehat{X}_1(n)) \cup \Phi_n(\widehat{X}_2(n)) \subset \mathcal{T}_n, \\ Y_1(n) &= X_1(n) \cap \Phi_n(\widehat{X}_1(n)), \quad Y_2(n) = X_2(n) \cap \Phi_n(\widehat{X}_2(n)). \end{aligned}$$

Note that $V, V_1(n)$ and $B_n(V)$ are contained in $\widehat{\mathcal{T}}_n$ for all $n \geq n_0$. Moreover, we have the following two corollaries of Theorem 6.1:

Lemma 7.4. *There exists $\delta > 0$ such that for all $n \geq n_0$ we have*

$$N_\delta([-1, 0]) \subset Y_1(n) \quad \text{and} \quad N_\delta([0, \xi_n(0)]) \subset Y_2(n).$$

Lemma 7.5. *Both*

$$\sup_{n \geq n_0} \sup_{z \in Y_1(n)} \det(D\xi_n(z)) \quad \text{and} \quad \sup_{n \geq n_0} \sup_{z \in Y_2(n)} \det(D\eta_n(z))$$

are finite, where $\det(\cdot)$ denotes the determinant of a square matrix.

Let

$$\begin{aligned} \widehat{\xi}_n &: \Phi_n(\widehat{X}_1(n)) \rightarrow (\Phi_n \circ \xi_n)(\widehat{X}_1(n)), & \widehat{\xi}_n &= \Phi_n \circ \xi_n \circ \Phi_n^{-1}, \\ \widehat{\eta}_n &: \Phi_n(\widehat{X}_2(n)) \rightarrow (\Phi_n \circ \eta_n)(\widehat{X}_2(n)) & \widehat{\eta}_n &= \Phi_n \circ \eta_n \circ \Phi_n^{-1}. \end{aligned}$$

Since each Φ_n is an \mathbb{R} -symmetric C^3 diffeomorphism, the pair $(\widehat{\eta}_n, \widehat{\xi}_n)$ restricts to a critical commuting pair with the same rotation number as (η_n, ξ_n) , and the same criticality (which we assume to be cubic, for simplicity). Note also that $\widehat{\eta}_n(0) = -1$ for all $n \geq n_0$. Moreover, from Lemma 7.5 and $\|\Phi_n - \text{Id}\|_{C^0(\mathcal{U}_n)} \leq C\lambda^n$ we have

$$\|\xi_n - \widehat{\xi}_n\|_{C^0(Y_1(n))} \leq C\lambda^n \quad \text{and} \quad \|\eta_n - \widehat{\eta}_n\|_{C^0(Y_2(n))} \leq C\lambda^n \quad \text{for all } n \geq n_0.$$

Therefore it is enough to shadow the sequence $(\widehat{\eta}_n, \widehat{\xi}_n)$ in the domains $Y_1(n)$ and $Y_2(n)$, instead of (η_n, ξ_n) (the shadowing sequence will be constructed in Subsection 7.3 below). The main advantage of working with the sequence $(\widehat{\eta}_n, \widehat{\xi}_n)$ is precisely that $\widehat{\eta}_n^{-1} \circ \widehat{\xi}_n$ is univalent in V for all $n \geq n_0$ (since it coincides with B_n). In particular we can choose each topological disk \mathcal{U}_n and \mathcal{T}_n defined above with the additional property that, identifying V with $B_n(V)$ via the biholomorphism B_n , we obtain from \mathcal{T}_n an abstract annular Riemann surface \mathcal{S}_n (with the complex structure induced by the quotient).

Denote by $p_n : \mathcal{T}_n \rightarrow \mathcal{S}_n$ the canonical projection (note that p_n is not a covering map, just a surjective local diffeomorphism). The projection of the real line, $p_n(\mathbb{R} \cap \mathcal{T}_n)$, is real-analytic diffeomorphic to the unit circle S^1 . We call it the *equator* of \mathcal{S}_n .

Since complex conjugation leaves \mathcal{T}_n invariant and commutes with B_n , it induces an antiholomorphic involution $F_n : \mathcal{S}_n \rightarrow \mathcal{S}_n$ acting as the identity on the equator $p_n(\mathbb{R} \cap \mathcal{T}_n)$. Note that F_n has a continuous extension to $\partial\mathcal{S}_n$ that switches the boundary components.

Since \mathcal{S}_n is obviously not biholomorphic to $\mathbb{D} \setminus \{0\}$ or to $\mathbb{C} \setminus \{0\}$, we have $\text{mod}(\mathcal{S}_n) < \infty$ for all $n \geq n_0$, where $\text{mod}(\cdot)$ denotes the conformal modulus of an annular Riemann surface. For each $n \geq n_0$ define a constant R_n in $(1, \infty)$ by

$$R_n = \exp(\text{mod}(\mathcal{S}_n)/2),$$

that is, \mathcal{S}_n is conformally equivalent to $A_{R_n} = \{z \in \mathbb{C} : R_n^{-1} < |z| < R_n\}$. Any biholomorphism between \mathcal{S}_n and A_{R_n} must send the equator $p_n(\mathbb{R} \cap \mathcal{T}_n)$ onto the unit circle S^1 (because the equator is invariant under the antiholomorphic involution F_n , and the unit circle is invariant under the antiholomorphic involution $z \mapsto 1/\bar{z}$ in A_{R_n}). Let $\Psi_n : \mathcal{S}_n \rightarrow A_{R_n}$ be the conformal uniformization determined by $\Psi_n(p_n(0)) = 1$, and let $P_n : \mathcal{T}_n \rightarrow A_{R_n}$ be the holomorphic surjective local diffeomorphism

$$P_n = \Psi_n \circ p_n$$

(see Figure 7). Note that $P_n(0) = 1$ and $P_n(\mathcal{T}_n \cap \mathbb{R}) = S^1$ for all $n \geq n_0$. Moreover $P_n(z)P_n(\bar{z}) = 1$ for all $z \in \mathcal{T}_n$ and all $n \geq n_0$. From now on we forget about the abstract cylinder \mathcal{S}_n .

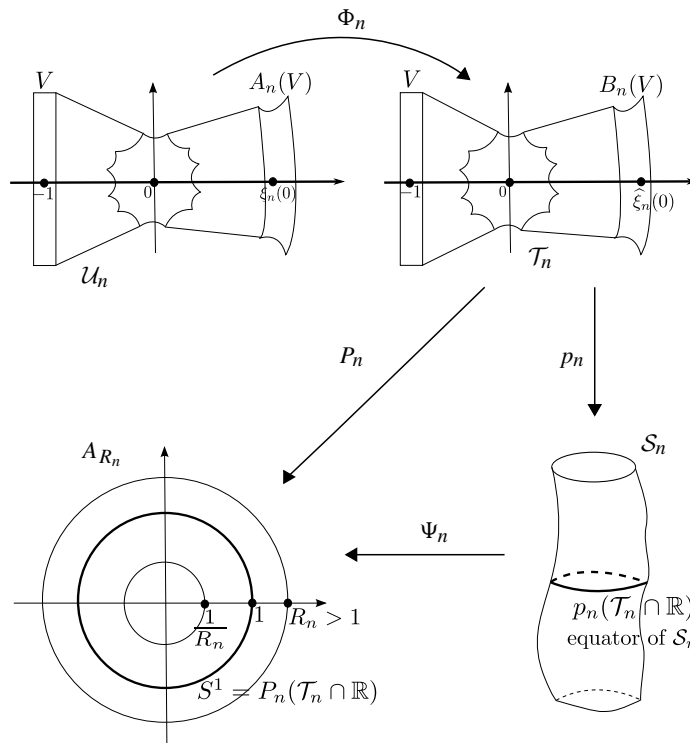


Fig. 7. Bidimensional glueing procedure.

Lemma 7.6. *There exist constants $\delta > 0$ and $C > 1$ such that for all $n \geq n_0$ and for all $z \in N_\delta([-1, \tilde{\xi}_n(0)])$ we have $z \in \widehat{\mathcal{T}}_n \subset \mathcal{T}_n$ and*

$$1/C < |P'_n(z)| < C.$$

Proof. By the real bounds there exists a universal constant $C_0 > 1$ such that for each $n \geq n_0$ there exists $w_n \in [-1, \tilde{\xi}_n(0)]$ such that

$$1/C_0 < |P'_n(w_n)| < C_0.$$

We need to construct a definite complex domain around $[-1, \tilde{\xi}_n(0)]$ where P_n has universally bounded distortion. Again by the real bounds there exist $\delta > 0$ and $l \in \mathbb{N}$ with the following properties. For each $n \geq n_0$ there exist $z_1, \dots, z_{k_n} \in [-1, \tilde{\xi}_n(0)]$ with $k_n < l$ for all $n \geq n_0$ such that:

- $[-1, \tilde{\xi}_n(0)] \subset \bigcup_{i=1}^{k_n} B(z_i, \delta)$;
- $B(z_i, 2\delta) \subset \widehat{\mathcal{T}}_n \subset \mathcal{T}_n$ for all $i \in \{1, \dots, k_n\}$;
- $P_n|_{B(z_i, 2\delta)}$ is univalent for all $i \in \{1, \dots, k_n\}$.

By convexity, for all $n \geq n_0$ and all $i \in \{1, \dots, k_n\}$ we have

$$\sup_{v, w \in B(z_i, \delta)} \frac{|P'_n(v)|}{|P'_n(w)|} \leq \exp\left(\sup_{w \in B(z_i, \delta)} \frac{|P''_n(w)|}{|P'_n(w)|}\right),$$

and by the Koebe distortion theorem (see for instance [6, Section I.1, Theorem 1.6]),

$$\sup_{w \in B(z_i, \delta)} \frac{|P''_n(w)|}{|P'_n(w)|} \leq \frac{2}{\delta} \quad \text{for all } n \geq n_0 \text{ and } i \in \{1, \dots, k_n\}. \quad \square$$

Now we project each commuting pair $(\tilde{\eta}_n, \tilde{\xi}_n)$ from $\widehat{\mathcal{T}}_n$ to the round annulus A_{R_n} .

Proposition 7.7 (Glueing procedure). *The pair*

$$\widehat{\xi}_n : \Phi_n(\widehat{X}_1(n)) \rightarrow \mathcal{T}_n \quad \text{and} \quad \widehat{\eta}_n : \Phi_n(\widehat{X}_2(n)) \rightarrow \mathcal{T}_n$$

projects under P_n to a well-defined orientation-preserving C^3 map

$$G_n : P_n(\widehat{\mathcal{T}}_n) \subset A_{R_n} \rightarrow A_{R_n}.$$

For each $n \geq n_0$, $P_n(\widehat{\mathcal{T}}_n)$ is a Θ -invariant annulus with positive and finite modulus. Each G_n is S^1 -symmetric, in particular G_n preserves the unit circle.

When restricted to the unit circle, G_n produces a C^3 critical circle map $g_n : S^1 \rightarrow S^1$ with cubic critical point at $P_n(0) = 1$, and with rotation number $\rho(g_n) = \rho(\mathcal{R}^n(f)) \in \mathbb{R} \setminus \mathbb{Q}$.

$$\begin{array}{ccc} \widehat{\mathcal{T}}_n \subset \mathcal{T}_n & \xrightarrow{(\widehat{\eta}_n, \widehat{\xi}_n)} & \mathcal{T}_n \\ P_n \downarrow & & \downarrow P_n \\ P_n(\widehat{\mathcal{T}}_n) \subset A_{R_n} & \xrightarrow{G_n} & A_{R_n} \end{array}$$

Moreover the unique critical point of G_n in $P_n(\widehat{\mathcal{T}}_n)$ is the one on the unit circle (at the point 1) and

$$|\bar{\partial}G_n(z)| \leq C\lambda^n |\partial G_n(z)| \quad \text{for all } z \in P_n(\widehat{\mathcal{T}}_n) \setminus \{1\},$$

that is, $\|\mu_{G_n}\|_\infty \leq C\lambda^n$ in $P_n(\widehat{\mathcal{T}}_n)$.

Proof. This follows from:

- the construction of \mathcal{U}_n and \mathcal{T}_n ;
- the property $B_n = \Phi_n \circ (\eta_n^{-1} \circ \xi_n) \circ \Phi_n^{-1}$ in V ;
- the commuting condition in $V_1(n)$;
- the symmetry $P_n(z)\overline{P_n(\bar{z})} = 1$ for all $z \in \mathcal{T}_n$ and all $n \geq n_0$;
- the fact that $P_n : \mathcal{T}_n \rightarrow A_{R_n}$ is holomorphic, $P_n(0) = 1$ and $P_n(\mathcal{T}_n \cap \mathbb{R}) = S^1$ for all $n \geq n_0$. □

Note that each g_n belongs to the smooth conjugacy class obtained with the glueing procedure (described in Section 3.2) applied to the C^3 critical commuting pair $(\widehat{\eta}_n, \widehat{\xi}_n)$. In the next subsection we will construct a sequence of real-analytic critical circle maps, with the desired combinatorics, that extend to holomorphic maps exponentially close to G_n in a definite annulus around the unit circle (see Proposition 7.8 below).

7.2. Main perturbation

The goal of this subsection is to construct the following sequence of perturbations:

Proposition 7.8 (Main perturbation). *There exist a constant $r > 1$ and a sequence*

$$\{H_n : A_r \rightarrow \mathbb{C}\}_{n \geq n_0}$$

of holomorphic maps defined in the annulus A_r such that for all $n \geq n_0$ the following holds:

- $A_r \subset P_n(\widehat{\mathcal{T}}_n) \subset P_n(\mathcal{T}_n) = A_{R_n}$;
- $\|H_n - G_n\|_{C^0(A_r)} \leq C\lambda^n$;
- $H_n(A_r) \subset (G_n \circ P_n)(\widehat{\mathcal{T}}_n) \subset P_n(\mathcal{T}_n) = A_{R_n}$;
- H_n preserves the unit circle and, when restricted to the unit circle, produces a real-analytic critical circle map $h_n : S^1 \rightarrow S^1$ such that:
 - the unique critical point of h_n is at $P_n(0) = 1$, and is of cubic type;
 - the critical value of h_n coincides with the one of g_n , that is, $h_n(1) = g_n(1) \in P_n(V \cap \mathbb{R})$;
 - $\rho(h_n) = \rho(g_n) = \rho(\mathcal{R}^n(f)) \in \mathbb{R} \setminus \mathbb{Q}$;
- the unique critical point of H_n in A_r is the one on the unit circle.

The remainder of this subsection is devoted to proving Proposition 7.8. We will not perturb the maps G_n directly (basically because they are non-invertible). Instead, we will decompose them (see Lemma 7.9 below), and then we will perturb their coefficients (see the definition after the statement of Lemma 7.9). Those perturbations will be done, again, with the help of Proposition 5.5.

Let $A : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C} \setminus \{0\}$ be the map corresponding to the parameters $a = 0$ and $b = 1$ in the Arnold family (1.2), defined in the introduction. The lift of A to the complex plane by the holomorphic covering $z \mapsto \exp(2\pi iz)$ is the entire map $\tilde{A} : \mathbb{C} \rightarrow \mathbb{C}$ given by

$$\tilde{A}(z) = z - \frac{1}{2\pi} \sin(2\pi z).$$

Then A preserves the unit circle, and its restriction $A : S^1 \rightarrow S^1$ is a real-analytic critical circle map. The critical point of A on the unit circle is at 1, and is of cubic type (the critical point is also a fixed point for A). The following is a bidimensional version of Lemma 6.2:

Lemma 7.9. *For each $n \geq n_0$ there exist:*

- $S_n > 1$;
- an S^1 -symmetric orientation-preserving C^3 diffeomorphism $\psi_n : P_n(\widehat{\mathcal{T}}_n) \rightarrow A_{S_n}$;
- an S^1 -symmetric biholomorphism $\phi_n : A(A_{S_n}) \rightarrow (G_n \circ P_n)(\widehat{\mathcal{T}}_n)$ such that

$$G_n = \phi_n \circ A \circ \psi_n \quad \text{in } P_n(\widehat{\mathcal{T}}_n).$$

The diffeomorphisms ψ_n and ϕ_n are called the *coefficients* of G_n in $P_n(\widehat{\mathcal{T}}_n)$.

$$\begin{array}{ccc}
 P_n(\widehat{\mathcal{T}}_n) & \xrightarrow{G_n} & (G_n \circ P_n)(\widehat{\mathcal{T}}_n) \\
 \psi_n \downarrow & & \uparrow \phi_n \\
 A_{S_n} & \xrightarrow{A} & A(A_{S_n})
 \end{array}$$

Proof of Lemma 7.9. For each $n \geq n_0$ let $S_n > 1$ be such that $A(A_{S_n})$ is a Θ -invariant annulus with

$$\text{mod}(A(A_{S_n})) = \text{mod}((G_n \circ P_n)(\widehat{\mathcal{T}}_n)).$$

In particular there exists a biholomorphism $\phi_n : A(A_{S_n}) \rightarrow (G_n \circ P_n)(\widehat{\mathcal{T}}_n)$ that commutes with Θ . Each ϕ_n preserves the unit circle and we can choose it so that $\phi_n(1) = G_n(1)$, that is, ϕ_n takes the critical value of A to the critical value of G_n .

Since both G_n and A are three-fold branched coverings around their critical points and local diffeomorphisms away from them, the equation $G_n = \phi_n \circ A \circ \psi_n$ induces an orientation-preserving C^3 diffeomorphism $\psi_n : P_n(\widehat{\mathcal{T}}_n) \rightarrow A_{S_n}$ that commutes with Θ and has $\psi_n(1) = 1$, that is, ψ_n takes the critical point of G_n to the one of A . The fact that ψ_n is smooth at 1 with non-vanishing derivative follows from the fact that the critical points of G_n and A have the same degree (see Lemma 6.2). \square

Note that, at the beginning of the proof of Lemma 7.9, we have used the fact that the image under the Arnold map A of a small round annulus around the unit circle is also an annulus. This is true, even though A has a critical point on the unit circle (placed at 1, and being also a fixed point of A). Even more is true: the conformal modulus of the annulus $A(A_s)$ depends continuously on $s > 1$ (and we have also used this fact in the proof).

As we said, the idea of the proof of Proposition 7.8 is to perturb each diffeomorphism ψ_n by means of Proposition 5.5. In order to control the C^0 size of those perturbations we will need some geometric control, which we state in four lemmas, before entering into the proof of Proposition 7.8. From Lemma 7.6 we have:

Lemma 7.10.

$$1 < \inf_{n \geq n_0} R_n \quad \text{and} \quad \sup_{n \geq n_0} R_n < \infty.$$

Lemma 7.11. *For all $n \geq n_0$ both $P_n(\widehat{\mathcal{T}}_n)$ and $(G_n \circ P_n)(\widehat{\mathcal{T}}_n)$ are Θ -invariant annuli with finite modulus. Moreover, there exists a universal constant $K > 1$ such that*

$$1/K < \text{mod}(P_n(\widehat{\mathcal{T}}_n)) < K \quad \text{for all } n \geq n_0.$$

Proof. By Lemma 7.10 we know that $R = \sup_{n \geq n_0} R_n$ is finite, and since for all $n \geq n_0$ both $P_n(\widehat{\mathcal{T}}_n)$ and $(G_n \circ P_n)(\widehat{\mathcal{T}}_n)$ are contained in the corresponding A_{R_n} , we see at once that both have finite modulus, and also $\sup_{n \geq n_0} \text{mod}(P_n(\widehat{\mathcal{T}}_n))$ is finite. Just as in Lemma 7.10, the fact that $\inf_{n \geq n_0} \text{mod}(P_n(\widehat{\mathcal{T}}_n))$ is positive follows from Lemmas 7.4 and 7.6. \square

Lemma 7.12. *There exists a constant $r_0 > 1$ such that $\bar{A}_{r_0} \subset P_n(\widehat{T}_n)$ for all $n \geq n_0$.*

Proof. By the invariance with respect to the antiholomorphic involution $z \mapsto 1/\bar{z}$, the unit circle is the core curve (the unique closed geodesic for the hyperbolic metric) of each annulus $P_n(\widehat{T}_n)$. Since $\inf_{n \geq n_0} \text{mod}(P_n(\widehat{T}_n)) > 0$, the statement is well-known: see for instance [39, Chapter 2, Theorem 2.5]. \square

Lemma 7.13. *We have*

$$s = \inf_{n \geq n_0} S_n > 1 \quad \text{and} \quad S = \sup_{n \geq n_0} S_n < \infty.$$

Proof. Since $\mu_{\psi_n} = \mu_{G_n}$ in $P_n(\widehat{T}_n)$, we have $\|\mu_{\psi_n}\|_\infty \leq C\lambda^n$ in $P_n(\widehat{T}_n)$ for all $n \geq n_0$. By the geometric definition of quasiconformal homeomorphisms (see for instance [34, Chapter I, Section 7]) we have

$$\frac{1 - C\lambda^n}{1 + C\lambda^n} \text{mod}(P_n(\widehat{T}_n)) \leq 2 \log(S_n) \leq \frac{1 + C\lambda^n}{1 - C\lambda^n} \text{mod}(P_n(\widehat{T}_n))$$

for all $n \geq n_0$, and we are done by Lemma 7.11. \square

With this geometric control at hand, we are ready to prove Proposition 7.8:

Proof of Proposition 7.8. Let $r_0 > 1$ be given by Lemma 7.12 (recall that $\bar{A}_{r_0} \subset P_n(\widehat{T}_n)$ for all $n \geq n_0$), and fix $r \in (1, (1+r_0)/2)$. How small $r - 1$ must be will be determined in the course of the argument (see Lemma 7.14 below). For any $r \in (1, (1+r_0)/2)$ consider $r_- = r_0 - (r - 1) \in ((1+r_0)/2, r_0)$.

The sequence

$$\{\psi_n : A_{r_0} \rightarrow \psi_n(A_{r_0})\}_{n \geq n_0}$$

of S^1 -symmetric C^3 diffeomorphisms satisfies the hypothesis of Proposition 5.5 since:

- $\mu_{\psi_n} = \mu_{G_n}$ in $P_n(\widehat{T}_n)$, and therefore $\|\mu_{\psi_n}\|_\infty \leq C\lambda^n$ for all $n \geq n_0$;
- $\psi_n(A_{r_0}) \subset A_{S_n} \subset A_S$ for all $n \geq n_0$ (see Lemma 7.13).

Apply Proposition 5.5 to the bounded domain A_{r_-} , compactly contained in A_{r_0} , to obtain a sequence of S^1 -symmetric biholomorphisms

$$\{\widehat{\psi}_n : A_{r_-} \rightarrow \widehat{\psi}_n(A_{r_-})\}_{n \geq n_0}$$

such that

$$\|\widehat{\psi}_n - \psi_n\|_{C^0(A_{r_-})} \leq C\lambda^n \quad \text{for all } n \geq n_0.$$

Fix n_0 large enough to have $\widehat{\psi}_n(A_{r_-}) \subset A_{S_n}$, and note that we can suppose that each $\widehat{\psi}_n$ fixes the point 1 (just as ψ_n) by considering

$$z \mapsto \frac{1}{\widehat{\psi}_n(1)} \widehat{\psi}_n(z).$$

Since $|\widehat{\psi}_n(z)| \leq S$ for all $z \in A_{r_-}$ and all $n \geq n_0$ (where $S \in (1, \infty)$ is given by Lemma 7.13), and since $|\widehat{\psi}_n(1) - 1| \leq C\lambda^n$ for all $n \geq n_0$, we know that this new map (which we will still denote by $\widehat{\psi}_n$ for simplicity) satisfies all the properties that we want for $\widehat{\psi}_n$, and also fixes the point $z = 1$.

For each $n \geq n_0$ consider the holomorphic map $H_n : A_r \rightarrow \mathbb{C}$ defined by $H_n = \phi_n \circ A \circ \widehat{\psi}_n$. We have:

- $H_n(A_r) \subset (G_n \circ P_n)(\widehat{\mathcal{T}}_n) \subset A_{R_n}$;
- H_n is S^1 -symmetric and therefore it preserves the unit circle;
- when restricted to the unit circle, H_n produces a real-analytic critical circle map $h_n : S^1 \rightarrow S^1$;
- the unique critical point of H_n in A_r is on the unit circle, at $P_n(0) = 1$, and is of cubic type;
- the critical value of H_n coincides with the one of G_n , that is, $H_n(1) = G_n(1) \in P_n(V \cap \mathbb{R})$.

We divide the rest of the proof of Proposition 7.8 into three lemmas. We need to prove first that, for a suitable $r > 1$, H_n is C^0 -exponentially close to G_n in the annulus A_r (Lemma 7.14 below), and then that we can choose each H_n with the desired combinatorics for its restriction h_n to the unit circle (Lemma 7.15). This last perturbation will change the critical value of each H_n (it will not coincide with the one of G_n any more). We will finish the proof of Proposition 7.8 with Lemma 7.16, which allows us to keep the critical point of H_n at $P_n(0) = 1$, and to place the critical value of H_n at $g_n(1)$ for all $n \geq n_0$. This will be important in the following subsection, the last one of this section.

Lemma 7.14. *There exists $r \in (1, (1 + r_0)/2)$ such that in the annulus A_r we have*

$$\|H_n - G_n\|_{C^0(A_r)} \leq C\lambda^n \quad \text{for all } n \geq n_0.$$

Proof. The proof is divided into three claims:

Claim 1. *There exists $\beta > 1$ such that $\overline{A}_\beta \subset A(A_{S_n})$ for all $n \geq n_0$.*

Indeed, by Lemma 7.13 the round annulus $A_{(1+s)/2}$ is compactly contained in A_{S_n} for all $n \geq n_0$, and therefore the annulus $A(A_{(1+s)/2})$ is contained in $A(A_{S_n})$ for all $n \geq n_0$. Thus we just take $\beta > 1$ such that $\overline{A}_\beta \subset A(A_{(1+s)/2})$, and Claim 1 is proved.

From now on we fix $\alpha \in (1, \beta)$.

Claim 2. *There exists $r \in (1, (1 + r_0)/2)$ so close to 1 that $(A \circ \widehat{\psi}_n)(A_r) \subset A_\alpha$ and $(A \circ \psi_n)(A_r) \subset A_\alpha$ for all $n \geq n_0$.*

Indeed, since $\overline{A}_r \subset A_r$, $\widehat{\psi}_n$ is holomorphic, and $\widehat{\psi}_n(A_r) \subset A_{S_n} \subset A_S$ for all $n \geq n_0$ (where $S \in (1, \infty)$ is given by Lemma 7.13), by the Cauchy derivative estimate we have $\sup_{n \geq n_0} \{|\widehat{\psi}'_n(z)| : z \in A_r\} < \infty$. Since each $\widehat{\psi}_n$ preserves the unit circle, and since $\|\widehat{\psi}_n - \psi_n\|_{C^0(A_r)} \leq C\lambda^n$ for all $n \geq n_0$, Claim 2 is proved.

Another way to prove the second claim is by noting that since $\overline{A}_\alpha \subset A_\beta \subset \overline{A}_\beta \subset A(A_{S_n})$ for all $n \geq n_0$, the hyperbolic metric on any annulus $A(A_{S_n})$ and the Euclidean metric are comparable in A_α with universal parameters, that is, there exists a constant $K > 1$ such that

$$\frac{1}{K}|z - w| \leq d_{A(A_{S_n})}(z, w) \leq K|z - w|$$

for all $z, w \in A_\alpha$ and all $n \geq n_0$, where $d_{A(A_{S_n})}$ denote the hyperbolic distance in the annulus $A(A_{S_n})$ (this is well-known, see for instance [6, Section 1.4, Theorem 4.3]). Since each $A \circ \widehat{\psi}_n : A_r \rightarrow A(A_{S_n})$ is holomorphic and preserves the unit circle, we know by the Schwarz lemma that for all $z \in A_r$ and all $n \geq n_0$,

$$d_{A(A_{S_n})}((A \circ \widehat{\psi}_n)(z), S^1) \leq d_{A_r}(z, S^1),$$

where d_{A_r} denote the hyperbolic distance in the annulus A_r . Since all distances $d_{A(A_{S_n})}$ are comparable with the Euclidean distance in A_δ with universal parameters, for all $z \in A_r$ and all $n \geq n_0$ we have

$$d((A \circ \widehat{\psi}_n)(z), S^1) \leq K d_{A_r}(z, S^1),$$

where d is just the Euclidean distance in the plane. Fix $r \in (1, (1 + r_0)/2)$ so close to 1 that $z \in A_r$ implies $d_{A_r}(z, S^1) < (\alpha - 1)/(K\alpha)$ (and therefore $(A \circ \widehat{\psi}_n)(z) \in A_\alpha$ for all $n \geq n_0$). Again since $\|\widehat{\psi}_n - \psi_n\|_{C^0(A_r)} \leq C\lambda^n$ for all $n \geq n_0$, Claim 2 is proved.

Claim 3. *There exists a positive number M such that $|\phi'_n(z)| < M$ for all $z \in A_\alpha$ and all $n \geq n_0$.*

Indeed, recall that $\phi_n(A(A_{S_n})) = (G_n \circ P_n)(\widehat{T}_n) \subset A_{R_n}$ for all $n \geq n_0$. By Lemma 7.10 there exists a (finite) number Δ such that $\phi_n(A(A_{S_n})) \subset B(0, \Delta)$ for all $n \geq n_0$. Since $\overline{A_\alpha} \subset A_\beta \subset \overline{A_\beta} \subset A(A_{S_n})$ for all $n \geq n_0$, Claim 3 follows from the Cauchy derivative estimate.

With the three claims at hand, Lemma 7.14 follows. □

To control the combinatorics after perturbation we use the *monotonicity* of the rotation number:

Lemma 7.15. *Let f be a C^3 critical circle map and let g be a real-analytic critical circle map that extends holomorphically to the annulus*

$$A_R = \{z \in \mathbb{C} : 1/R < |z| < R\} \quad \text{for some } R > 1.$$

There exists a real-analytic critical circle map h , with $\rho(h) = \rho(f)$, also extending holomorphically to A_R , such that

$$\|h - g\|_{C^0(A_R)} \leq d_{C^0(S^1)}(f, g).$$

In particular

$$d_{C^r(S^1)}(h, g) \leq d_{C^0(S^1)}(f, g) \quad \text{for any } 0 \leq r \leq \infty.$$

Proof. Let F and G be the respective lifts of f and g to the real line satisfying

$$\rho(f) = \lim_{n \rightarrow \infty} F^n(0)/n \quad \text{and} \quad \rho(g) = \lim_{n \rightarrow \infty} G^n(0)/n.$$

Consider the band $B_R = \{z \in \mathbb{C} : -\log R < 2\pi \Im(z) < \log R\}$, which is the universal cover of the annulus A_R via the holomorphic covering $z \mapsto e^{2\pi iz}$. Let $\delta = \|F - G\|_{C^0(\mathbb{R})}$,

and for any t in $[-1, 1]$ let $G_t : B_R \rightarrow \mathbb{C}$ be defined as $G_t = G + t\delta$. Each G_t preserves the real line, and its restriction is the lift of a real-analytic critical circle map. Moreover, each G_t commutes with the unitary horizontal translation in B_R .

Note that $\|G_t - G\|_{C^0(B_R)} = |t|\delta \leq \|F - G\|_{C^0(\mathbb{R})}$ for any $t \in [-1, 1]$. Moreover, for any $x \in \mathbb{R}$ the family $\{G_t(x)\}_{t \in [-1, 1]}$ is monotone in t , and we have $G_{-1}(x) \leq F(x) \leq G_1(x)$. In particular there exists $t_0 \in [-1, 1]$ such that

$$\lim_{n \rightarrow \infty} G_{t_0}^n(0)/n = \rho(F),$$

and we define h as the projection of G_{t_0} to the annulus A_R . □

After the perturbation given by Lemma 7.15 we still have the critical point of h_n placed at 1, but its critical value is no longer $g_n(1)$ (however, they are exponentially close). To finish the proof of Proposition 7.8 we need to fix this, without changing the combinatorics of h_n in S^1 . Until now each H_n is S^1 -symmetric, in the sense that it commutes with $z \mapsto 1/\bar{z}$ in the annulus A_r . We will lose this property in the following perturbation, which turns out to be the last one.

Lemma 7.16. *For each $n \geq n_0$ consider the (unique) Möbius transformation M_n which maps the unit disk \mathbb{D} onto itself fixing the basepoint $z = 1$, and which maps $H_n(1)$ to $G_n(1)$. Then there exists $\rho \in (1, r)$ such that $\bar{A}_\rho \subset M_n(A_r)$ for all $n \geq n_0$. Moreover for each $n \geq n_0$ we have*

$$\|M_n \circ H_n \circ M_n^{-1} - G_n\|_{C^0(A_\rho)} \leq C\lambda^n.$$

Note that, when restricted to the unit circle, each M_n gives rise to an orientation-preserving real-analytic diffeomorphism which is, as Lemma 7.16 indicates, C^∞ -exponentially close to the identity.

Proof of Lemma 7.16. Consider the biholomorphism $\psi : \mathbb{H} \rightarrow \mathbb{D}$ given by $\psi(z) = \frac{z-i}{z+i}$, whose inverse $\psi^{-1} : \mathbb{D} \rightarrow \mathbb{H}$ is given by $\psi^{-1}(z) = i\left(\frac{1+z}{1-z}\right)$. Note that ψ maps the vertical geodesic $\{z \in \mathbb{H} : \Re(z) = 0\}$ onto the interval $(-1, 1)$ in \mathbb{D} . Since ψ and ψ^{-1} are Möbius transformations, both extend uniquely to corresponding biholomorphisms of the entire Riemann sphere. The extension of ψ is a real-analytic diffeomorphism between the compactification of the real line and the unit circle, which maps the point at infinity to the point $z = 1$. For each $n \geq n_0$ consider the real number t_n defined by

$$t_n = \psi^{-1}(G_n(1)) - \psi^{-1}(H_n(1)) = 2i \frac{G_n(1) - H_n(1)}{(1 - G_n(1))(1 - H_n(1))}.$$

Each t_n is finite since for all $n \geq n_0$ neither $G_n(1)$ nor $H_n(1)$ is equal to one. Moreover, we claim that

$$\inf_{n \geq n_0} |G_n(1) - 1| > 0 \quad \text{and} \quad \inf_{n \geq n_0} |H_n(1) - 1| > 0.$$

Indeed, since $|H_n(1) - G_n(1)| \leq C\lambda^n$ for all $n \geq n_0$, it is enough to prove that $\inf_{n \geq n_0} |G_n(1) - 1| > 0$, and this follows from Lemma 7.6 since $1 = P_n(0)$ and

$G_n(1) = P_n(-1)$ for all $n \geq n_0$. In particular, again using $|H_n(1) - G_n(1)| \leq C\lambda^n$ for all $n \geq n_0$, we see that $|t_n| \leq C\lambda^n$ for all $n \geq n_0$. From the explicit formula

$$M_n(z) = \frac{(2i - t_n)z + t_n}{(2i + t_n) - t_n z} = \frac{z - \frac{t_n}{t_n - 2i}}{1 - \frac{t_n}{t_n + 2i}z} \cdot \frac{2i - t_n}{2i + t_n} \quad \text{for all } n \geq n_0,$$

we see that the pole of each M_n is at the point $z_n = 1 + i(2/t_n)$, and since $|t_n| \leq C\lambda^n$ for all $n \geq n_0$, we can take n_0 large enough to have $z_n \in \mathbb{C} \setminus \overline{B(0, 2R)}$, where $R = \sup_{n \geq n_0} R_n < \infty$ is given by Lemma 7.10. A straightforward computation gives

$$(M_n - \text{Id})(z) = \frac{t_n(z - 1)^2}{(2i + t_n) - t_n z} \quad \text{for all } n \geq n_0,$$

and therefore

$$\|M_n - \text{Id}\|_{C^0(A_R)} \leq C\lambda^n \quad \text{for all } n \geq n_0.$$

In particular for any fixed $\rho \in (1, r)$ we can choose n_0 large enough to have $\overline{A_\rho} \subset M_n(A_r)$ for all $n \geq n_0$. Moreover, given any $z \in A_\rho$ we have

$$\begin{aligned} (M_n \circ H_n \circ M_n^{-1} - G_n)(z) &= (M_n - \text{Id})((H_n \circ M_n^{-1})(z)) + (H_n - G_n)(z) \\ &\quad + (H_n(M_n^{-1}(z)) - H_n(z)). \end{aligned}$$

In particular

$$\begin{aligned} \|M_n \circ H_n \circ M_n^{-1} - G_n\|_{C^0(A_\rho)} &\leq \|M_n - \text{Id}\|_{C^0(H_n(A_r))} + \|H_n - G_n\|_{C^0(A_\rho)} \\ &\quad + \|H_n\|_{C^1(A_r)} \|M_n^{-1} - \text{Id}\|_{C^0(A_\rho)}. \end{aligned}$$

Since $H_n(A_r) \subset A_R$ and $A_\rho \subset A_r \subset A_R$, the terms $\|M_n - \text{Id}\|_{C^0(H_n(A_r))}$, $\|H_n - G_n\|_{C^0(A_\rho)}$ and $\|M_n^{-1} - \text{Id}\|_{C^0(A_\rho)}$ are each less than or equal to $C\lambda^n$ for all $n \geq n_0$.

Finally, since each H_n is holomorphic and $\overline{A_r} \subset A_r$ and $H_n(A_r) \subset (G_n \circ P_n)(\widehat{T}_n) \subset A_{R_n} \subset A_R$ for all $n \geq n_0$, from the Cauchy derivative estimate we obtain

$$\sup_{n \geq n_0} \|H_n\|_{C^1(A_r)} < \infty,$$

and therefore $\|M_n \circ H_n \circ M_n^{-1} - G_n\|_{C^0(A_\rho)} \leq C\lambda^n$ for all $n \geq n_0$. □

With Lemma 7.16 at hand we are done, since $(M_n \circ H_n \circ M_n^{-1})(1) = G_n(1)$. We have finished the proof of Proposition 7.8. □

7.3. The shadowing sequence

This is the final subsection of Section 7, which is devoted to proving Theorem D. Let us recall what we have done: in Subsection 7.1 we constructed a suitable sequence $\{G_n\}_{n \geq n_0}$ of S^1 -symmetric C^3 extensions of C^3 critical circle maps g_n to some annulus $P_n(\widehat{T}_n)$.

When lifted via the corresponding projection P_n (also constructed in Subsection 7.1), each g_n gives rise to a C^3 critical commuting pair $(\widehat{\eta}_n, \widehat{\xi}_n)$ exponentially close to $\mathcal{R}^n(f)$ and having the same combinatorics at each step (moreover, with complex extensions C^0 -exponentially close to the ones of $\mathcal{R}^n(f)$ produced in Theorem 6.1; see Proposition 7.7 above for more properties).

In Subsection 7.2 we perturbed each G_n in a definite annulus A_r , in order to obtain a sequence of real-analytic critical circle maps, each of them having the same combinatorics as the corresponding $\mathcal{R}^n(f)$, that extend to holomorphic maps H_n exponentially close to G_n in A_r (see Proposition 7.8 above for more properties). Both the critical point and the critical value of each H_n coincide with the ones of the corresponding G_n , more precisely, the critical point of each H_n is $P_n(0) = 1 \in P_n(V_1(n)) \cap S^1$, and its critical value is $H_n(1) = G_n(1) \in P_n(V) \cap S^1 = P_n(B_n(V)) \cap S^1$. Recall also that $H_n(A_r) \subset P_n(\mathcal{T}_n)$ for all $n \geq n_0$.

In this subsection we lift each $H_n : A_r \rightarrow A_{R_n}$ via the holomorphic projection $P_n : \mathcal{T}_n \rightarrow A_{R_n}$ in the canonical way: Let $\alpha > 0$ be such that for all $n \geq n_0$ we have

$$\overline{N_\alpha([-1, 0]) \cup N_\alpha([0, \widehat{\xi}_n(0)])} \subset \widehat{\mathcal{T}}_n,$$

and $P_n(N_\alpha([-1, 0]) \cup N_\alpha([0, \widehat{\xi}_n(0)]))$ is an annulus contained in A_r and containing the unit circle (the existence of such an α is guaranteed by Lemmas 7.4 and 7.6). Set $Z_1(n) = N_\alpha([-1, 0])$ and $Z_2(n) = N_\alpha([0, \widehat{\xi}_n(0)])$. For each $n \geq n_0$ let $\widetilde{\eta}_n : Z_2(n) \rightarrow \mathcal{T}_n$ be the \mathbb{R} -preserving holomorphic map defined by

$$H_n \circ P_n = P_n \circ \widetilde{\eta}_n \quad \text{in } Z_2(n), \quad \text{and} \quad \widetilde{\eta}_n(0) = -1.$$

In the same way let $\widetilde{\xi}_n : Z_1(n) \rightarrow \mathcal{T}_n$ be the \mathbb{R} -preserving holomorphic map defined by the two conditions:

$$H_n \circ P_n = P_n \circ \widetilde{\xi}_n \quad \text{in } Z_1(n), \quad \text{and} \quad \widetilde{\xi}_n(0) = \widehat{\xi}_n(0).$$

$$\begin{array}{ccc} Z_1(n) \cup Z_2(n) \subset \mathcal{T}_n & \xrightarrow{(\widetilde{\eta}_n, \widetilde{\xi}_n)} & \mathcal{T}_n \\ \downarrow P_n & & \downarrow P_n \\ A_r \subset A_{R_n} & \xrightarrow{H_n} & A_{R_n} \end{array}$$

In the next proposition we summarize the main properties of this lift, which are all straightforward:

Proposition 7.17 (The shadowing sequence). *For each $n \geq n_0$ the pair $f_n = (\widetilde{\eta}_n, \widetilde{\xi}_n)$ restricts to a real-analytic critical commuting pair with domains $I(\widetilde{\xi}_n) = [\widetilde{\eta}_n(0), 0] = [-1, 0]$ and $I(\widetilde{\eta}_n) = [0, \widetilde{\xi}_n(0)] = [0, \widehat{\xi}_n(0)]$, and such that $\rho(f_n) = \rho(\widehat{\eta}_n, \widehat{\xi}_n) = \rho(\mathcal{R}^n(f)) \in \mathbb{R} \setminus \mathbb{Q}$. Moreover, $\widetilde{\xi}_n$ and $\widetilde{\eta}_n$ extend to holomorphic maps in $Z_1(n)$ and $Z_2(n)$ respectively where:*

- $\widetilde{\xi}_n$ has a unique critical point in $Z_1(n)$, which is at the origin and of cubic type;

- $\tilde{\eta}_n$ has a unique critical point in $Z_2(n)$, which is at the origin and of cubic type;
- $\|\tilde{\xi}_n - \hat{\xi}_n\|_{C^0(Z_1(n) \cap \Phi_n(\hat{X}_1(n)))} \leq C\lambda^n$;
- $\|\tilde{\eta}_n - \hat{\eta}_n\|_{C^0(Z_2(n) \cap \Phi_n(\hat{X}_2(n)))} \leq C\lambda^n$.

With Proposition 7.17 at hand, Theorem D follows directly from the following consequence of Montel's theorem:

Lemma 7.18. *Let α be a constant in $(0, 1)$ and let \mathcal{V} be an \mathbb{R} -symmetric bounded topological disk such that $[-1, \alpha^{-1}] \subset \mathcal{V}$. Let W_1 and W_2 be topological disks whose closures are contained in \mathcal{V} and $[-1, 0] \subset W_1$ and $[0, \alpha^{-1}] \subset W_2$. Denote by \mathcal{K} the set of all normalized real-analytic critical commuting pairs $\zeta = (\eta, \xi)$ such that:*

- $\eta(0) = -1$ and $\xi(0) \in [\alpha, \alpha^{-1}]$;
- $\alpha|\eta([0, \xi(0)])| \leq |\xi([-1, 0])| \leq \alpha^{-1}|\eta([0, \xi(0)])|$;
- both ξ and η extend to holomorphic maps (with a unique cubic critical point at the origin) defined in W_1 and W_2 respectively, and satisfy:

- (1) $N_\alpha(\xi([-1, 0])) \subset \xi(W_1)$;
- (2) $N_\alpha(\eta([0, \xi(0)])) \subset \eta(W_2)$;
- (3) $\xi(W_1) \cup \eta(W_2) \subset \mathcal{V}$.

Then \mathcal{K} is C^ω -compact.

8. Concluding remarks

As was already mentioned in the introduction (see Remark 1.1 after the statement of Theorem B), since the preprint version [22] of this paper first appeared, we were able to prove (in collaboration with Marco Martens) that any two C^4 critical circle maps with the same irrational rotation number and the same odd criticality are conjugate to each other by a C^1 circle diffeomorphism, which is $C^{1+\alpha}$ at the critical point, and that, for some universal $\alpha > 0$, this conjugacy is a $C^{1+\alpha}$ diffeomorphism for Lebesgue almost every rotation number [21]. In other words, the precise statement of Theorem A holds in the C^4 category (see [21, Theorem A]).

To the best of our knowledge, the possibility of generalizing these results to C^3 dynamics with unbounded combinatorics remains open (nor do we know whether rigidity holds for less regularity, for instance $C^{2+\alpha}$ smoothness, even for bounded combinatorics). Moreover, we do not know how to deal with critical points of non-integer criticality (not even with fractional criticality).

The question of “how smooth the conjugacy is” is a delicate problem. On the one hand, the presence of the critical point gives us more rigidity than in the case of diffeomorphisms: smooth conjugacy is obtained in the C^4 category for *any* irrational rotation number, with no Diophantine conditions (again, see [21] and the references therein). On the other hand, there exist examples [5], [13] showing that this smooth diffeomorphism may not be globally $C^{1+\alpha}$ in general, even for real-analytic dynamics. It seems possible, although difficult, to obtain some arithmetical condition on the rotation numbers that would decide whether the conjugacy may or may not be “better than C^1 ”.

Let us be more precise. The set $\mathbb{A} \subset [0, 1]$ of de Faria and de Melo (see Theorem 2.1) is the set of rotation numbers $\rho = [a_0, a_1, \dots]$ satisfying the following conditions:

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \log a_j &< \infty, \\ \lim_{n \rightarrow \infty} \frac{1}{n} \log a_n &= 0, \\ \frac{1}{n} \sum_{j=k+1}^{k+n} \log a_j &\leq \omega_\rho\left(\frac{n}{k}\right) \quad \text{for all } 0 < n \leq k, \end{aligned}$$

where $\omega_\rho(t)$ is a positive function (that depends on the rotation number) defined for $t > 0$ such that $t\omega_\rho(t) \rightarrow 0$ as $t \rightarrow 0$ (for instance, we can take $\omega_\rho(t) = C_\rho(1 - \log t)$ where $C_\rho > 0$ depends on the number).

The set \mathbb{A} obviously contains all rotation numbers of bounded type, and it has full Lebesgue measure in $[0, 1]$ (see [13, Appendix C] or [20, Appendix, p. 63]).

Is there a condition on the rotation number equivalent to $C^{1+\alpha}$ rigidity? This is not clear even in the real-analytic setting (see [5]). We remark that $C^{1+\alpha}$ rigidity fails for some Diophantine rotation numbers (see [13, Section 5]).

Another difficult problem is the following: what can be said, in terms of smooth rigidity, for maps with more than one critical point? More precisely, let f and g be orientation-preserving C^3 circle homeomorphisms with the same irrational rotation number, and with $N \geq 1$ non-flat critical points of odd type. Denote by $S_f = \{c_1, \dots, c_N\}$ the ordered critical set of f , by $S_g = \{c'_1, \dots, c'_N\}$ the ordered critical set of g , and suppose that the criticalities of c_i and c'_i are the same for all $i \in \{1, \dots, N\}$ (the cubic case is the generic one). Finally, denote by μ_f and μ_g the corresponding unique invariant measures of f and g .

By Yoccoz's result [60], f and g are topologically conjugate to each other. For elementary reasons, the condition $\mu_f([c_i, c_{i+1}]) = \mu_g([c'_i, c'_{i+1}])$ for all $i \in \{1, \dots, N-1\}$ is necessary (and sufficient) in order to have a topological conjugacy between f and g that sends the critical points of f to the critical points of g .

Is this conjugacy a smooth diffeomorphism? If yes, what is its degree of smoothness? To the best of our knowledge, these questions remain completely open.

Appendix A. Proof of Lemma 3.6

In this appendix we prove Lemma 3.6, stated at the end of Section 3 and used in Section 4. For this purpose, we need the following fact:

Lemma A.1. *Let f_1, \dots, f_n be C^1 maps with C^1 norm bounded by some constant $B > 0$, and let g_1, \dots, g_n be C^0 maps. Then*

$$\|f_n \circ \dots \circ f_1 - g_n \circ \dots \circ g_1\|_{C^0} \leq \left(\sum_{j=0}^{n-1} B^j \right) \max_{i \in \{1, \dots, n\}} \|f_i - g_i\|_{C^0}$$

whenever the compositions make sense.

Proof. The proof goes by induction on n (when $n = 1$ we have nothing to prove). Suppose that

$$\|f_{n-1} \circ \dots \circ f_1 - g_{n-1} \circ \dots \circ g_1\|_{C^0} \leq \left(\sum_{j=0}^{n-2} B^j\right) \max_{i \in \{1, \dots, n-1\}} \|f_i - g_i\|_{C^0}.$$

Then for any t ,

$$\begin{aligned} |(f_n \circ \dots \circ f_1 - g_n \circ \dots \circ g_1)(t)| &\leq |f_n((f_{n-1} \circ \dots \circ f_1)(t)) - f_n((g_{n-1} \circ \dots \circ g_1)(t))| \\ &\quad + |f_n((g_{n-1} \circ \dots \circ g_1)(t)) - g_n((g_{n-1} \circ \dots \circ g_1)(t))| \\ &\leq B|(f_{n-1} \circ \dots \circ f_1 - g_{n-1} \circ \dots \circ g_1)(t)| + \|f_n - g_n\|_{C^0} \\ &\leq B\left(\sum_{j=0}^{n-2} B^j\right) \max_{i \in \{1, \dots, n-1\}} \|f_i - g_i\|_{C^0} + \|f_n - g_n\|_{C^0} \\ &\leq \left(\sum_{j=0}^{n-1} B^j\right) \max_{i \in \{1, \dots, n\}} \|f_i - g_i\|_{C^0}. \quad \square \end{aligned}$$

For $K > 1$ and $r \in \{0, 1, \dots, \infty, \omega\}$ recall from Section 3 that we denote by $\mathcal{P}^r(K)$ the space of C^r critical commuting pairs $\zeta = (\eta, \xi)$ such that $\eta(0) = -1$ (they are normalized) and $\xi(0) \in [K^{-1}, K]$.

Lemma A.2. *Given $M \in \mathbb{N}$, $B > 0$ and $K > 1$ there exists $L(M, B, K) > 1$ with the following property. Let $\zeta_1 = (\eta_1, \xi_1)$ and $\zeta_2 = (\eta_2, \xi_2)$ be renormalizable C^3 critical commuting pairs satisfying the following conditions:*

- (1) $\zeta_1, \mathcal{R}(\zeta_1), \zeta_2$ and $\mathcal{R}(\zeta_2)$ belong to $\mathcal{P}^3(K)$;
- (2) the continued fraction expansion of both rotation numbers $\rho(\zeta_1)$ and $\rho(\zeta_2)$ have the same first term, say a_0 , with $a_0 \leq M$. More precisely,

$$\lfloor 1/\rho(\zeta_1) \rfloor = \lfloor 1/\rho(\zeta_2) \rfloor = a_0 \in \{1, \dots, M\};$$

- (3) $\max\{\|\eta_1\|_{C^1}, \|\xi_1\|_{C^1}\} < B$;
- (4) $(\eta_1 \circ \xi_1)(0)$ and $(\eta_2 \circ \xi_2)(0)$ have the same sign;
- (5) $|\xi_1(0) - \xi_2(0)| < \frac{1}{K^2} \cdot \frac{K+1}{K-1}$.

Then

$$d_0(\mathcal{R}(\zeta_1), \mathcal{R}(\zeta_2)) \leq L \cdot d_0(\zeta_1, \zeta_2),$$

where d_0 is the C^0 distance in the space of critical commuting pairs (see Section 3.3).

In the proof of Lemma A.2 we will use the following notation. For $\alpha > 0$ denote by T_α the (unique) Möbius transformation that fixes -1 and 0 , and maps α to 1 . Moreover, for each $i \in \{1, 2\}$ let

$$x_i = \xi_i(0), \quad y_i = \eta_i^{a_0}(x_i) = \eta_i^{a_0}(\xi_i(0)), \quad z_i = T_{x_i}(y_i) = T_{\xi_i(0)}(\eta_i^{a_0}(\xi_i(0))).$$

Proof of Lemma A.2. Suppose that both $(\eta_1 \circ \xi_1)(0)$ and $(\eta_2 \circ \xi_2)(0)$ are positive, and let $V \subset \mathbb{R}$ be the interval $[0, \max\{(\eta_1 \circ \xi_1)(0), (\eta_2 \circ \xi_2)(0)\}]$. Note that (in the notation

described before the proof) $p_\alpha = \alpha + \alpha \frac{1+\alpha}{1-\alpha}$ is the pole of T_α . If $\alpha > K/(K + 2)$ then $p_\alpha \notin [1/K, K]$, and if $\alpha \in [1/K, K/(K + 2)]$ then $p_\alpha - \alpha \geq \frac{1}{K} \cdot \frac{K+1}{K-1}$. By item (5) in the hypothesis, and since ζ_1 and ζ_2 belong to $\mathcal{P}^3(K)$ by (1), there exists $L_0(K) > 1$ such that

$$\begin{aligned} \|T_{x_1}\|_{C^1(V)} &\leq L_0, & \|T_{x_1} - T_{x_2}\|_{C^0(V)} &\leq L_0|x_1 - x_2| \leq L_0 \cdot d_0(\zeta_1, \zeta_2), \\ |y_1 - y_2| &\leq L_0|\tilde{\eta}_1^{a_0}(1) - \tilde{\eta}_2^{a_0}(1)|, \\ \|\tilde{\eta}_1\|_{C^1([0,1])} &\leq L_0\|\eta_1\|_{C^1([0,x_1])} \leq L_0B, \end{aligned}$$

where $\tilde{\eta}_i = T_{x_i} \circ \eta_i \circ T_{x_i}^{-1}$ for $i \in \{1, 2\}$. By Lemma A.1,

$$|y_1 - y_2| \leq L_0^2 \left(\sum_{j=0}^{a_0-1} B^j \right) \|\tilde{\eta}_1 - \tilde{\eta}_2\|_{C^0([0,1])}.$$

Defining $L_1(M, B, K) = L_0^2 \sum_{j=0}^{M-1} B^j$ we obtain

$$|y_1 - y_2| \leq L_1 \cdot d_0(\zeta_1, \zeta_2).$$

Therefore

$$\begin{aligned} |z_1 - z_2| &\leq |z_1 - T_{x_1}(y_2)| + |T_{x_1}(y_2) - z_2| \\ &\leq L_0|y_1 - y_2| + L_0 \cdot d_0(\zeta_1, \zeta_2) \leq (L_0L_1 + L_0) \cdot d_0(\zeta_1, \zeta_2). \end{aligned}$$

Defining $L_2(M, B, K) = L_0L_1 + L_0$ we obtain

$$|z_1 - z_2| \leq L_2 \cdot d_0(\zeta_1, \zeta_2). \tag{A.1}$$

Moreover, there exists $L_3(M, B, K) \geq L_2$ with the following properties:

- from (1), both Möbius transformations T_{z_i} , and also their inverses, have C^1 norm bounded by L_3 in $W = [0, \max\{z_1, z_2\}]$;
- both Möbius transformations T_{z_i} are at C^0 distance less than or equal to $L_3 \cdot d_0(\zeta_1, \zeta_2)$ in W (this follows from (A.1) and (1));
- the same with their inverses, that is, both Möbius transformations $T_{z_i}^{-1}$ are at C^0 distance less than or equal to $L_3 \cdot d_0(\zeta_1, \zeta_2)$ in $[0, 1]$ (again from (A.1) and (1));
- the maps

$$T_{x_1} \circ \eta_1^{a_0} \circ \xi_1 \circ T_{x_1}^{-1} \quad \text{and} \quad T_{x_1} \circ \eta_1 \circ T_{x_1}^{-1}$$

have C^1 norm bounded by L_3 in $[-1, 0]$ and $[0, 1]$ respectively (follows from (1)–(3)).

Note that for $i = \{1, 2\}$ we have

$$T_{y_i} \circ \eta_i^{a_0} \circ \xi_i \circ T_{y_i}^{-1} = T_{z_i} \circ (T_{x_i} \circ \eta_i^{a_0} \circ \xi_i \circ T_{x_i}^{-1}) \circ T_{z_i}^{-1}$$

in $[-1, 0]$, and

$$T_{y_i} \circ \eta_i \circ T_{y_i}^{-1} = T_{z_i} \circ (T_{x_i} \circ \eta_i \circ T_{x_i}^{-1}) \circ T_{z_i}^{-1}$$

in $[0, 1]$. By Lemma A.1 and the four properties listed above there exists $L_4 = L_4(M, B, K) \geq L_3$ such that

$$\begin{aligned} & \|T_{y_1} \circ \eta_1^{a_0} \circ \xi_1 \circ T_{y_1}^{-1} - T_{y_2} \circ \eta_2^{a_0} \circ \xi_2 \circ T_{y_2}^{-1}\|_{C^0} \\ & \leq L_4 \max\{\|T_{z_1} - T_{z_2}\|_{C^0}, d_0(\zeta_1, \zeta_2), \|T_{z_1}^{-1} - T_{z_2}^{-1}\|_{C^0}\} \leq L_3 L_4 \cdot d_0(\zeta_1, \zeta_2) \end{aligned}$$

in $[-1, 0]$, and

$$\begin{aligned} & \|T_{y_1} \circ \eta_1 \circ T_{y_1}^{-1} - T_{y_2} \circ \eta_2 \circ T_{y_2}^{-1}\|_{C^0} \\ & \leq L_4 \max\{\|T_{z_1} - T_{z_2}\|_{C^0}, d_0(\zeta_1, \zeta_2), \|T_{z_1}^{-1} - T_{z_2}^{-1}\|_{C^0}\} \leq L_3 L_4 \cdot d_0(\zeta_1, \zeta_2) \end{aligned}$$

in $[0, 1]$. Therefore we are done by taking $L \geq L_3 L_4$. □

With Lemma A.2 at hand, we are ready to prove Lemma 3.6.

Proof of Lemma 3.6. Let f be a C^3 critical circle map with irrational rotation number $\rho(f) = [a_0, a_1, \dots]$, and recall that we are assuming that $a_n < M$ for all $n \in \mathbb{N}$. Let $n_0(f) \in \mathbb{N}$ be given by the real bounds, and note that $\mathcal{R}^n(f) \in \mathcal{P}^3(K)$ for all $n \geq n_0$ (since $K > K_0$ by hypothesis, and so $\mathcal{P}^3(K) \supset \mathcal{P}^3(K_0)$). As a well-known corollary of the real bounds (see for instance [13, Theorem 3.1]), there exists a constant $B > 0$ such that the sequence $\{\mathcal{R}^n(f)\}_{n \in \mathbb{N}}$ is bounded by B in the C^1 metric, and we are done by taking $L > 1$ given by Lemma A.2. □

Appendix B. Proof of Proposition 5.5

Assume that each μ_n is defined in the whole complex plane, just by extending it by zero in the complement of the domain U , that is,

$$\mu_n(z) \partial G_n(z) = \overline{\partial} G_n(z) \quad \text{for a.e. } z \in U, \quad \text{and} \quad \mu_n(z) = 0 \quad \text{for all } z \in \mathbb{C} \setminus U.$$

Fix $n \in \mathbb{N}$. If $\mu_n \equiv 0$ we take $H_n = G_n|_U$, so assume that $\|\mu_n\|_\infty > 0$ and fix some small $\varepsilon \in (0, 1 - \|\mu_n\|_\infty)$. Denote by Λ the open disk $B(0, (1 - \varepsilon)/\|\mu_n\|_\infty)$ in the complex plane (note that $\overline{\mathbb{D}} \subset \Lambda$). Consider the one-parameter family $\{\mu_n(t)\}_{t \in \Lambda}$ of Beltrami coefficients defined by

$$\mu_n(t) = t\mu_n.$$

Note that for all $t \in \Lambda$ we have $\|\mu_n(t)\|_\infty < 1 - \varepsilon < 1$. Denote by $f^{\mu_n(t)}$ the solution of the Beltrami equation with coefficient $\mu_n(t)$, given by Theorem 5.2, normalized to fix 0, 1 and ∞ . Note that $f^{\mu_n(0)}$ is the identity and, by uniqueness, there exists a biholomorphism $H_n : f^{\mu_n(1)}(U) \rightarrow G_n(U)$ such that

$$G_n = H_n \circ f^{\mu_n(1)} \quad \text{in } U.$$

By the Ahlfors–Bers theorem (Theorem 5.4), for any $z \in \mathbb{C}$ the curve $\{f^{\mu_n(t)}(z) : t \in [0, 1]\}$ is smooth, that is, the derivative of $f^{\mu_n(t)}$ with respect to the parameter t exists at any $z \in \mathbb{C}$. Following Ahlfors [1, Chapter V, Section C], we use the notation

$$\dot{f}_n(z, s) = \lim_{t \rightarrow 0} \frac{f^{\mu_n(s+t)}(z) - f^{\mu_n(s)}(z)}{t}.$$

The limit exists for every $z \in \mathbb{C}$ and every $s \in [0, 1]$ (actually for every $s \in \Lambda$), and the convergence is uniform on compact subsets of \mathbb{C} . Then

$$\|f^{\mu_n(1)} - \text{Id}\|_{C^0(U)} = \sup_{z \in U} |f^{\mu_n(1)}(z) - z| \leq \sup_{z \in U} \int_0^1 |\dot{f}_n(z, s)| ds.$$

Moreover, \dot{f}_n has the following integral representation (see [1, Chapter V, Section C, Theorem 5] for the explicit computation):

$$\dot{f}_n(z, s) = -\frac{1}{\pi} \iint_{\mathbb{C}} \mu_n(w) S(f^{\mu_n(s)}(w), f^{\mu_n(s)}(z)) (\partial f^{\mu_n(s)}(w))^2 dx dy$$

for every $z \in \mathbb{C}$ and every $s \in [0, 1]$, where $w = x + iy$ and

$$S(w, z) = \frac{1}{w - z} - \frac{z}{w - 1} + \frac{z - 1}{w} = \frac{z(z - 1)}{w(w - 1)(w - z)}.$$

Since each μ_n is supported in U , we have

$$\dot{f}_n(z, s) = -\frac{1}{\pi} \iint_U \mu_n(w) S(f^{\mu_n(s)}(w), f^{\mu_n(s)}(z)) (\partial f^{\mu_n(s)}(w))^2 dx dy.$$

From the formula

$$|\partial f^{\mu_n(s)}(w)|^2 = \left(\frac{1}{1 - |s|^2 |\mu_n(w)|^2} \right) \det(Df^{\mu_n(s)}(w))$$

we obtain

$$\begin{aligned} |\dot{f}_n(z, s)| &\leq \frac{1}{\pi} \iint_U \frac{|\mu_n(w)|}{1 - |s|^2 |\mu_n(w)|^2} \det(Df^{\mu_n(s)}(w)) |S(f^{\mu_n(s)}(w), f^{\mu_n(s)}(z))| dx dy \\ &\leq \frac{1}{\pi} \cdot \frac{\|\mu_n\|_\infty}{1 - |s|^2 \|\mu_n\|_\infty^2} \iint_U \det(Df^{\mu_n(s)}(w)) |S(f^{\mu_n(s)}(w), f^{\mu_n(s)}(z))| dx dy \\ &= \frac{1}{\pi} \cdot \frac{\|\mu_n\|_\infty}{1 - |s|^2 \|\mu_n\|_\infty^2} \iint_{f^{\mu_n(s)}(U)} |S(w, f^{\mu_n(s)}(z))| dx dy. \end{aligned}$$

Therefore the length of the curve $\{f^{\mu_n(t)}(z) : t \in [0, 1]\}$ is at most

$$\begin{aligned} \frac{1}{\pi} \int_0^1 \left[\frac{\|\mu_n\|_\infty}{1 - |s|^2 \|\mu_n\|_\infty^2} \iint_{f^{\mu_n(s)}(U)} |S(w, f^{\mu_n(s)}(z))| dx dy \right] ds \\ \leq \frac{1}{\pi} \cdot \frac{\|\mu_n\|_\infty}{1 - \|\mu_n\|_\infty^2} \int_0^1 \left[\iint_{f^{\mu_n(s)}(U)} |S(w, f^{\mu_n(s)}(z))| dx dy \right] ds. \end{aligned}$$

If we define

$$M_n(U) = \frac{1}{\pi} \sup_{z \in U} \int_0^1 \left[\iint_{f^{\mu_n(s)}(U)} |S(w, f^{\mu_n(s)}(z))| dx dy \right] ds,$$

we get

$$\|f^{\mu_n(1)} - \text{Id}\|_{C^0(U)} \leq \frac{\|\mu_n\|_\infty}{1 - \|\mu_n\|_\infty^2} M_n(U).$$

We have two remarks:

First remark: Since $\mu_n \rightarrow 0$ in the unit ball of L^∞ , we know by Proposition 5.3 that for any $s \in [0, 1]$ the normalized quasiconformal homeomorphisms $f^{\mu_n(s)}$ converge to the identity uniformly on compact subsets of \mathbb{C} , in particular on \bar{U} . Therefore the sequence $M_n(U)$ converges to

$$\frac{1}{\pi} \sup_{z \in U} \iint_U |S(w, z)| dx dy < \frac{1}{\pi} \sup_{z \in U} \iint_{\mathbb{C}} |S(w, z)| dx dy < \infty.$$

For fixed $z \in \mathbb{C}$, $S(w, z)$ is in $L^1(\mathbb{C})$ since it has simple poles at $0, 1$ and z , and is $O(|w|^{-3})$ near ∞ . The finiteness then follows from the compactness of \bar{U} .

Second remark: $x \mapsto x/(1-x^2)$ is an orientation-preserving real-analytic diffeomorphism between $(-1, 1)$ and the real line, which is tangent to the identity at the origin. In fact $x/(1-x^2) = x + o(x^2)$ in $(-1, 1)$.

With these two remarks we obtain $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$ we have

$$\|f^{\mu_n(1)} - \text{Id}\|_{C^0(U)} \leq M(U)\|\mu_n\|_\infty, \quad \text{where} \quad M(U) = \frac{2}{\pi} \sup_{z \in U} \iint_U |S(w, z)| dx dy.$$

Since V is compactly contained in the bounded domain U , the boundaries ∂V and ∂U are disjoint compact sets. Let $\delta > 0$ be their Euclidean distance, that is, $\delta = d(\partial V, \partial U) = \min\{|z - w| : z \in \partial V, w \in \partial U\}$. Again by Proposition 5.3, since $\mu_n \rightarrow 0$, there exists $n_0 \geq n_1$ in \mathbb{N} such that for all $n \geq n_0$ we have $V \subset f^{\mu_n(1)}(U)$ and

$$f^{\mu_n(1)}(U) \supseteq B(z, \delta/2) \quad \text{for all } z \in V.$$

If we consider the restriction of H_n to the domain V we have

$$\|H_n - G_n\|_{C^0(V)} \leq \|H'_n\|_{C^0(V)} \|f^{\mu_n(1)} - \text{Id}\|_{C^0(U)} \leq \|H'_n\|_{C^0(V)} M(U) \|\mu_n\|_\infty.$$

By Cauchy's derivative estimate we know that for all $z \in V$,

$$\begin{aligned} |H'_n(z)| &= \left| \frac{1}{2\pi i} \int_{\partial B(z, \delta/2)} \frac{H_n(w)}{(w-z)^2} dw \right| \leq 2 \|H_n\|_{C^0(f^{\mu_n(1)}(U))} / \delta = 2 \|G_n\|_{C^0(U)} / \delta \\ &\leq 2R/\delta \quad \text{for all } n \geq n_0. \end{aligned}$$

That is,

$$\|H'_n\|_{C^0(V)} \leq \frac{2R}{d(\partial V, \partial U)} \quad \text{for all } n \geq n_0,$$

and we deduce that for all $n \geq n_0$,

$$\frac{\|H_n - G_n\|_{C^0(V)}}{\|\mu_n\|_\infty} \leq \frac{R}{d(\partial V, \partial U)} \cdot \frac{4}{\pi} \sup_{z \in U} \iint_U |S(w, z)| dx dy.$$

Therefore it is enough to set $C(U) = (4/\pi) \sup_{z \in U} \iint_U |S(w, z)| dx dy$. □

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References

- [1] Ahlfors, L.: Lectures on Quasiconformal Mappings. Van Nostrand (1966) [Zbl 0138.06002](#) [MR 0200442](#)
- [2] Ahlfors, L., Bers, L.: Riemann's mapping theorem for variable metrics. *Ann. of Math.* **72**, 385–404 (1960) [Zbl 0104.29902](#) [MR 0115006](#)
- [3] Arnol'd, V. I.: Small denominators I. Mappings of the circle onto itself. *Izv. Akad. Nauk Ser. Mat.* **25**, 21–86 (1961) (in Russian); English transl.: *Amer. Math. Soc. Transl. (2)* **46**, 213–284 (1965) [Zbl 0152.41905](#) [MR 0140699](#)
- [4] Avila, A.: Dynamics of renormalization operators. In: *Proc. Int. Congress of Mathematicians (Hyderabad, 2010)*, 154–175 [Zbl 1252.37030](#) [MR 2827889](#)
- [5] Avila, A.: On rigidity of critical circle maps. *Bull. Braz. Math. Soc.* **44**, 611–619 (2013) [Zbl 1314.37028](#) [MR 3167125](#)
- [6] Carleson, L., Gamelin, T.: *Complex Dynamics*. Springer (1993) [Zbl 0782.30022](#) [MR 1230383](#)
- [7] Denjoy, A.: Sur les courbes définies par les équations différentielles à la surface du tore. *J. Math. Pures Appl.* **11**, 333–375 (1932) [Zbl 0006.30501](#)
- [8] Dixon, T. W., Gherghetta, T., Kenny, B. G.: Universality in the quasiperiodic route to chaos. *Chaos* **6**, 32–42 (1996) [Zbl 1055.37517](#) [MR 1376893](#)
- [9] Epstein, A., Keen, L., Tresser, C.: The set of maps $F_{a,b} : x \mapsto x + a + \frac{b}{2\pi} \sin(2\pi x)$ with any given rotation interval is contractible. *Comm. Math. Phys.* **173**, 313–333 (1995) [Zbl 0839.58021](#) [MR 1355627](#)
- [10] de Faria, E.: Proof of universality for critical circle mappings. PhD thesis, CUNY (1992) [MR 2688469](#)
- [11] de Faria, E.: Asymptotic rigidity of scaling ratios for critical circle mappings. *Ergodic Theory Dynam. Systems* **19**, 995–1035 (1999) [Zbl 0996.37045](#) [MR 1709428](#)
- [12] de Faria, E., Guarino, P.: Real bounds and Lyapunov exponents. *Discrete Contin. Dynam. Systems A* **36**, 1957–1982 (2016) [Zbl 06498599](#) [MR 3411549](#)
- [13] de Faria, E., de Melo, W.: Rigidity of critical circle mappings I. *J. Eur. Math. Soc.* **1**, 339–392 (1999) [Zbl 0988.37047](#) [MR 1728375](#)
- [14] de Faria, E., de Melo, W.: Rigidity of critical circle mappings II. *J. Amer. Math. Soc.* **13**, 343–370 (2000) [Zbl 0988.37048](#) [MR 1711394](#)
- [15] Feigenbaum, M., Kadanoff, L., Shenker, S.: Quasi-periodicity in dissipative systems. A renormalization group analysis. *Phys. D* **5**, 370–386 (1982) [MR 0680571](#)
- [16] Graczyk, J., Sands, D., Świątek, G.: Decay of geometry for unimodal maps: negative Schwarzian case. *Ann. of Math.* **161**, 613–677 (2005) [Zbl 1091.37015](#) [MR 2153397](#)
- [17] Graczyk, J., Świątek, G.: Singular measures in circle dynamics. *Comm. Math. Phys.* **157**, 213–230 (1993) [Zbl 0792.58025](#) [MR 1244865](#)
- [18] Graczyk, J., Świątek, G.: Critical circle maps near bifurcation. *Comm. Math. Phys.* **176**, 227–260 (1996) [Zbl 0857.58034](#) [MR 1374412](#)

- [19] Guarino, P.: Rigidity conjecture for C^3 critical circle maps. PhD thesis, IMPA (2012) (available at www.preprint.impa.br)
- [20] Guarino, P.: Some topological and geometrical aspects of circle dynamics. *Cadernos de Matemática* **16**, 39–69 (2015) (available at www.icmc.usp.br)
- [21] Guarino, P., Martens, M., de Melo, W.: Rigidity of critical circle maps. [arXiv:1511.02792](https://arxiv.org/abs/1511.02792) (2015)
- [22] Guarino, P., de Melo, W.: Rigidity of smooth critical circle maps. [arXiv:1303.3470](https://arxiv.org/abs/1303.3470) (2013)
- [23] Hall, G. R.: A C^∞ Denjoy counterexample. *Ergodic Theory Dynam. Systems* **1**, 261–272 (1981) [Zbl 0501.58035](#) [MR 0662469](#)
- [24] Herman, M.: Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations. *Publ. Math. IHES* **49**, 5–234 (1979) [Zbl 0448.58019](#) [MR 0538680](#)
- [25] Herman, M.: Conjugaison quasi-symétrique des homéomorphismes du cercle à des rotations. Manuscript (1988)
- [26] Kadanoff, L., Shenker, S.: Critical behavior of a KAM surface. I. Empirical results. *J. Statist. Phys.* **27**, 631–656 (1982) [Zbl 0925.58021](#) [MR 0661681](#)
- [27] Khanin, K.: Universal estimates for critical circle mappings. *Chaos* **1**, 181–186 (1991) [Zbl 0899.58051](#) [MR 1135906](#)
- [28] Khanin, K., Teplinsky, A.: Robust rigidity for circle diffeomorphisms with singularities. *Invent. Math.* **169**, 193–218 (2007) [Zbl 1151.37035](#) [MR 2308853](#)
- [29] Khanin, K., Teplinsky, A.: Herman’s theory revisited, *Invent. Math.* **178**, 333–344 (2009) [Zbl 1179.37059](#) [MR 2545684](#)
- [30] Khinchin, A. Ya.: *Continued Fractions*. Reprint of the 1964 translation, Dover Publ. (1997) [Zbl 0117.28601](#) [MR 1451873](#)
- [31] Khmelev, D., Yampolsky, M.: The rigidity problem for analytic critical circle maps. *Moscow Math. J.* **6**, 317–351 (2006) [Zbl 1124.37024](#) [MR 2270617](#)
- [32] Lanford, O. E.: Renormalization group methods for critical circle mappings with general rotation number. In: *VIIIth Int. Congress on Mathematical Physics (Marseille, 1986)*, World Sci., Singapore, 532–536 (1987) [MR 0915597](#)
- [33] Lanford, O. E.: Renormalization group methods for circle mappings. In: *Nonlinear Evolution and Chaotic Phenomena*, NATO Adv. Sci. Inst. Ser. B Phys. 176, Plenum, New York, 25–36 (1988) [Zbl 0707.58043](#) [MR 1109968](#)
- [34] Lehto, O., Virtanen, K. I.: *Quasiconformal Mappings in the Plane*. Springer (1973) [Zbl 0267.30016](#) [MR 0344463](#)
- [35] Lyubich, M.: Teichmüller space of Fibonacci maps. [arXiv:9311213](https://arxiv.org/abs/9311213) (1993)
- [36] Lyubich, M., Yampolsky, M.: Dynamics of quadratic polynomials: complex bounds for real maps. *Ann. Inst. Fourier (Grenoble)* **47**, 1219–1255 (1997) [Zbl 0881.58053](#) [MR 1488251](#)
- [37] MacKay, R. S.: A renormalisation approach to invariant circles in area-preserving maps. *Phys. D* **7**, 283–300 (1983) [Zbl 0568.70023](#) [MR 0719057](#)
- [38] MacKay, R. S.: *Renormalisation in Area-Preserving Maps*. World Sci. (1993) [Zbl 0791.58002](#) [MR 1336593](#)
- [39] McMullen, C. T.: *Complex Dynamics and Renormalization*. Ann. of Math. Stud. 135, Princeton Univ. Press (1994) [Zbl 0822.30002](#) [MR 1312365](#)
- [40] McMullen, C. T.: Renormalization and 3-manifolds which Fiber over the Circle. Ann. of Math. Stud. 142, Princeton Univ. Press (1996) [Zbl 0860.58002](#) [MR 1401347](#)
- [41] McMullen, C. T.: Self-similarity of Siegel disks and Hausdorff dimension of Julia sets. *Acta Math.* **180**, 247–292 (1998) [Zbl 0930.37022](#) [MR 1638776](#)
- [42] McMullen, C. T.: Rigidity and inflexibility in conformal dynamics. In: *Proc. Int. Congress of Mathematicians (Berlin, 1998)*, Doc. Math. Extra Vol. II, 841–855 (1998) [Zbl 0914.58015](#) [MR 1648131](#)

- [43] de Melo, W.: Rigidity and renormalization in one-dimensional dynamics. In: Proc. Int. Congress of Mathematicians (Berlin, 1998), Doc. Math. Extra Vol. II, 765–778 (1998) [Zbl 0914.58016](#) [MR 1648124](#)
- [44] de Melo, W., Pinto, A.: Rigidity of C^2 infinitely renormalizable unimodal maps. *Comm. Math. Phys.* **208**, 91–105 (1999) [Zbl 0982.37026](#) [MR 1729879](#)
- [45] de Melo, W., van Strien, S.: *One-Dimensional Dynamics*. Springer (1993) [Zbl 0791.58003](#) [MR 1239171](#)
- [46] Ostlund, S., Rand, D., Sethna, J., Siggia, E.: Universal properties of the transition from quasi-periodicity to chaos in dissipative systems. *Phys. D* **8**, 303–342 (1983) [Zbl 0538.58025](#) [MR 0719630](#)
- [47] Petersen, C. L.: Local connectivity of some Julia sets containing a circle with an irrational rotation. *Acta Math.* **177**, 163–224 (1996) [Zbl 0884.30020](#) [MR 1440932](#)
- [48] Petersen, C. L., Zakeri, S.: On the Julia set of a typical quadratic polynomial with a Siegel disk. *Ann. of Math.* **159**, 1–52 (2004) [Zbl 1069.37038](#) [MR 2051390](#)
- [49] Rand, D.: Universality and renormalisation in dynamical systems. In: *New Directions in Dynamical Systems*, T. Bedford and J. W. Swift (eds.), Cambridge Univ. Press, Cambridge, 1–56 (1987) [Zbl 0651.58022](#) [MR 0953969](#)
- [50] Rand, D.: Global phase space universality, smooth conjugacies and renormalisation: I. The $C^{1+\alpha}$ case. *Nonlinearity* **1**, 181–202 (1988) [Zbl 0697.58029](#) [MR 0928952](#)
- [51] Rand, D.: Existence, non-existence and universal breakdown of dissipative golden invariant tori: I. Golden critical circle maps. *Nonlinearity* **5**, 639–662 (1992) [Zbl 0759.58040](#) [MR 1166539](#)
- [52] Shenker, S.: Scaling behavior in a map of a circle onto itself: empirical results. *Phys. D* **5**, 405–411 (1982) [MR 0680574](#)
- [53] Sullivan, D.: Bounds, quadratic differentials, and renormalization conjectures. In: *AMS Centennial Publications*, Vol. II, 417–466 (1992) [Zbl 0936.37016](#) [MR 1184622](#)
- [54] Świątek, G.: Rational rotation numbers for maps of the circle. *Comm. Math. Phys.* **119**, 109–128 (1988) [Zbl 0656.58017](#) [MR 0968483](#)
- [55] Yampolsky, M.: Complex bounds for renormalization of critical circle maps. *Ergodic Theory Dynam. Systems* **19**, 227–257 (1999) [Zbl 0918.58049](#) [MR 1677153](#)
- [56] Yampolsky, M.: The attractor of renormalization and rigidity of towers of critical circle maps. *Comm. Math. Phys.* **218**, 537–568 (2001) [Zbl 0978.37033](#) [MR 1828852](#)
- [57] Yampolsky, M.: Hyperbolicity of renormalization of critical circle maps. *Publ. Math. IHES* **96**, 1–41 (2002) [Zbl 1030.37027](#) [MR 1985030](#)
- [58] Yampolsky, M.: Renormalization horseshoe for critical circle maps. *Comm. Math. Phys.* **240**, 75–96 (2003) [Zbl 1078.37034](#) [MR 2004980](#)
- [59] Yoccoz, J.-C.: Conjugaison différentiable des difféomorphismes du cercle dont le nombre de rotation vérifie une condition diophantienne. *Ann. Sci. École Norm. Sup.* **17**, 333–359 (1984) [Zbl 0595.57027](#) [MR 0777374](#)
- [60] Yoccoz, J.-C.: Il n’y a pas de contre-exemple de Denjoy analytique. *C. R. Acad. Sci. Paris* **298**, 141–144 (1984) [Zbl 0573.58023](#) [MR 0741080](#)
- [61] Yoccoz, J.-C.: Continued fraction algorithms for interval exchange maps: an introduction. In: *Frontiers in Number Theory, Physics and Geometry*, Vol. 1: On random matrices, zeta functions and dynamical systems, Springer, 401–435 (2006) [Zbl 1127.28011](#) [MR 2261103](#)