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A semi-algebraic version of Zarankiewicz's problem

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Abstract. A bipartite graph G is *semi-algebraic in* $\mathbb{R}^d$ if its vertices are represented by point sets $P, Q \subset \mathbb{R}^d$ and its edges are defined as pairs of points $(p, q) \in P \times Q$ that satisfy a Boolean combination of a fixed number of polynomial equations and inequalities in $2d$ coordinates. We show that for fixed k, the maximum number of edges in a $K_{k,k}$-free semi-algebraic bipartite graph $G = (P, Q, E)$ in $\mathbb{R}^2$ with $|P| = m$ and $|Q| = n$ is at most $O((mn)^{2/3} + m + n)$, and this bound is tight. In dimensions $d \geq 3$, we show that all such semi-algebraic graphs have at most $C((mn)^{d/(d+1)+\varepsilon} + m + n)$ edges, where ε is an arbitrarily small constant and $C = C(d, k, t, \varepsilon)$. This result is a far-reaching generalization of the classical Szemerédi–Trotter incidence theorem. The proof combines tools from several fields: VC-dimension and shatter functions, polynomial partitioning, and Hilbert polynomials.

We also present various applications of our theorem, for example, a general point-variety incidence bound in $\mathbb{R}^d$, an improved bound for a d-dimensional variant of the Erdős unit distances problem, and more.

Keywords. Semi-algebraic graph, extremal graph theory, VC-dimension, polynomial partitioning, incidences

1. Introduction

The problem of Zarankiewicz [46] is a central problem in extremal graph theory. It asks for the maximum number of edges in a bipartite graph which has m vertices in its first class, n vertices in the second class, and neither part contains the complete bipartite graph $K_{k,k}$ with k vertices. In 1954, Kővári, Sós, and Turán [26] proved a general upper bound of the form $c_k(mn^{1-1/k} + n)$ edges, where c_k only depends on k. Well-known constructions of Reiman and Brown show that this bound is tight for $k = 2, 3$ (see [35]).

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However, the Zarankiewicz problem for $k \geq 4$ remains one of the most challenging unsolved problems in extremal graph theory. A recent result of Bohman and Keevash [8] on random graph processes gives the best known lower bound for $k \geq 5$ and $m = n$ of the form $\Omega(n^{2-2/(k+1)}(\log k)^{1/(k^2-1)})$. In this paper, we consider Zarankiewicz's problem for *semi-algebraic*[1] bipartite graphs, that is, bipartite graphs where one vertex set is a collection of points in $\mathbb{R}^{d_1}$, the second vertex set is a collection of points in $\mathbb{R}^{d_2}$, and edges are defined as *pairs* of points that satisfy a Boolean combination of polynomial equations and inequalities in $d_1 + d_2$ coordinates. This framework captures many of the well-studied incidence problems in combinatorial geometry (see, e.g., [38]).

Let $G = (P, Q, E)$ be a semi-algebraic bipartite graph in $(\mathbb{R}^{d_1}, \mathbb{R}^{d_2})$ with $|P| = m$ and $|Q| = n$. Then there are polynomials $f_1, \ldots, f_t \in \mathbb{R}[x_1, \ldots, x_{d_1+d_2}]$ and a Boolean function $\Phi(X_1, \ldots, X_t)$ such that for $(p, q) \in P \times Q \subset \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$,

$$(p, q) \in E \;\Leftrightarrow\; \Phi\big(f_1(p, q) \geq 0, \ldots, f_t(p, q) \geq 0\big) = 1.$$

We say that the edge set E has *description complexity* at most t if E can be described by at most t polynomial equations and inequalities, and each of them has degree at most t. If $G = (P, Q, E)$ is $K_{k,k}$-free, then by the Kővári–Sós–Turán theorem we know that $|E(G)| = O(mn^{1-1/k} + n)$. However, our main result gives a much better bound if G is semi-algebraic of bounded description complexity. In particular, we show that Zarankiewicz's problem for semi-algebraic bipartite graphs primarily depends on the dimension.

Theorem 1.1. *Let $G = (P, Q, E)$ be a semi-algebraic bipartite graph in $(\mathbb{R}^{d_1}, \mathbb{R}^{d_2})$ such that E has description complexity at most t, $|P| = m$, and $|Q| = n$. If G is $K_{k,k}$-free, then*

$$|E(G)| \leq c_1((mn)^{2/3} + m + n) \qquad \text{for } d_1 = d_2 = 2, \tag{1}$$

$$|E(G)| \leq c_2((mn)^{d/(d+1)+\varepsilon} + m + n) \qquad \text{for } d_1 = d_2 = d, \tag{2}$$

and more generally,

$$|E(G)| \leq c_3\big(m^{\frac{d_2(d_1-1)}{d_1d_2-1}+\varepsilon} n^{\frac{d_1(d_2-1)}{d_1d_2-1}} + m + n\big) \qquad \text{for all } d_1, d_2. \tag{3}$$

Here, ε is an arbitrarily small constant and $c_1 = c_1(t, k)$, $c_2 = c_2(d, t, k, \varepsilon)$ and $c_3 = c_3(d_1, d_2, t, k, \varepsilon)$.

To prove the theorem, we combine ideas from the study of VC-dimension with ideas from incidence theory. In the latter, we rely on the concept of polynomial partitioning (as introduced by Guth and Katz [21]) and combine it with a technique that relies on Hilbert polynomials. Recently, similar polynomial partitioning techniques were also studied by Matoušek and Patáková [32] and Basu and Sombra [6]. However, each of the three papers presents different proofs and very different results.

The planar case of Theorem 1.1 (i.e., (1)) is a generalization of the famous Szemerédi–Trotter point-line theorem [42]. Indeed, in the case of $d_1 = d_2 = 2$, by taking P to be the

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[1] A *real semi-algebraic set* in $\mathbb{R}^{d_1+d_2}$ is the locus of all points that satisfy a given finite Boolean combination of polynomial equations and inequalities in the $d_1 + d_2$ coordinates.

point set, Q to be the dual of the lines, and the relationship to be the incidence relationship, we see that G is $K_{2,2}$-free as two distinct lines intersect in at most one point. As we will see below, there are many further applications of Theorem 1.1.

Previous work and lower bounds. Several authors have studied this extremal problem in a more restricted setting: on bounding the number of incidences between an m-element point set P and a set H of n hyperplanes in $\mathbb{R}^d$ where no k points of P lie on k hyperplanes of H. Since each hyperplane $h \subset \mathbb{R}^d$ dualizes[2] to a point in $\mathbb{R}^d$, this problem is equivalent to determining the maximum number of edges in a $K_{k,k}$-free semi-algebraic bipartite graph $G = (P, Q, E)$ in $(\mathbb{R}^d, \mathbb{R}^d)$, where $(p, q) \in E$ if and only if $\langle p, q \rangle = 1$. In this special case, the work of Chazelle [10], Brass and Knauer [9], and Apfelbaum and Sharir [4] implies that $|E(G)| \leq c'((mn)^{d/(d+1)} + m + n)$, where c' depends only on k and d.

On the other hand, Brass and Knauer [9] gave a construction of an m-element point set P and a set H of n hyperplanes in $\mathbb{R}^3$, with no k points from P lying on k hyperplanes of H, with at least $\Omega((mn)^{7/10})$ incidences. For any $d \geq 4$ and $\varepsilon > 0$, Sheffer [39] presented a construction of an m-element point set P and a set H of $n = \Theta(m^{(3-3\varepsilon)/(d+1)})$ hyperplanes in $\mathbb{R}^d$, with no two points from P lying on $(d-1)/\varepsilon$ hyperplanes of H, with $\Omega((mn)^{1-2/(d+4)-\varepsilon})$ incidences. These are the best known lower bounds for Theorem 1.1 that we are aware of. Notice that the gap between these bounds and the upper bound of (2) becomes rather small for large values of d.

Applications. After proving Theorem 1.1, we provide a variety of applications. First, we show how a minor change in our proof leads to the following general incidence bound.

Theorem 1.2. *Let P be a set of m points and let $\mathcal{V}$ be a set of n constant-degree algebraic varieties, both in $\mathbb{R}^d$, such that the incidence graph of $P \times \mathcal{V}$ does not contain a copy of $K_{s,t}$ (here we think of s, t, and d as being fixed constants, and m and n are large). Then for every $\varepsilon > 0$, the number of incidences in $P \times \mathcal{V}$ is*

$$I(P, \mathcal{V}) = O\left(m^{\frac{(d-1)s}{ds-1}+\varepsilon} n^{\frac{d(s-1)}{ds-1}} + m + n\right).$$

Theorem 1.2 subsumes many known incidences results (up to the extra ε in the exponent), and extends them to $\mathbb{R}^d$ (see Section 6). When $s = 2$, the theorem is tight up to subpolynomial factors (see [39]). We also derive an improved bound for a d-dimensional variant of the Erdős unit distances problem, a bound for incidences between points and tubes, and more.

Organization. In Section 2, we give an upper bound on the maximum number of edges in a $K_{k,k}$-free bipartite graph with bounded VC-dimension. In Section 3, we establish the bound (1) from Theorem 1.1. Then in Section 4, we prove the bounds (2) and (3) from Theorem 1.1. The parts of this proof that concern Hilbert polynomials are deferred to Section 5. In Section 6, we discuss applications of Theorem 1.1. Finally, Section 7 concerns the tightness of our results.

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[2] Given a hyperplane $h = \{(x_1, \ldots, x_d) : a_1 x_1 + \cdots + a_d x_d = 1\}$ in $\mathbb{R}^d$, the *dual* of h is the point $h^* = (a_1, \ldots, a_d)$.

2. VC-dimension and shatter functions

Given a bipartite graph $G = (P, Q, E)$ where $E \subset P \times Q$, for any vertex $q \in Q$, let $N_G(q)$ denote the neighborhood of q in G, that is, the set of vertices in P that are connected to q. Then let $\mathcal{F} = \{N_G(q) \subset P : q \in Q\}$ be a set system with ground set P. The dual of $(P, \mathcal{F})$ is the set system obtained by interchanging the roles of P and $\mathcal{F}$, that is, the set system $(\mathcal{F}, \mathcal{F}^*)$, where $\mathcal{F}$ is the ground set and $\mathcal{F}^* = \{\{A \in \mathcal{F} : p \in A\} : p \in P\}$. Obviously, $(\mathcal{F}^*)^* = \mathcal{F}$.

The *Vapnik–Chervonenkis dimension* (for short, VC-dimension) of $(P, \mathcal{F})$ is the *largest* integer d_0 for which there exists a d_0-element set $S \subset P$ such that for every subset $B \subset S$, one can find a member $A \in \mathcal{F}$ with $A \cap S = B$. The *primal shatter function* of $(P, \mathcal{F})$ is defined as

$$\pi_{\mathcal{F}}(z) = \max_{P' \subset P,\, |P'|=z} |\{A \cap P' : A \in \mathcal{F}\}|.$$

In other words, $\pi_{\mathcal{F}}(z)$ is a function whose value at z is the maximum possible number of distinct intersections of the sets of $\mathcal{F}$ with a z-element subset of P. The primal shatter function of $\mathcal{F}^*$ is often called the *dual shatter function* of $\mathcal{F}$.

The VC-dimension of $\mathcal{F}$ is closely related to its shatter functions. A result of Sauer and Shelah states that if $\mathcal{F}$ is a set system with VC-dimension d_0, then

$$\pi_{\mathcal{F}}(z) \leq \sum_{i=0}^{d_0} \binom{z}{i}. \tag{4}$$

On the other hand, suppose that the primal shatter function of $\mathcal{F}$ satisfies $\pi_{\mathcal{F}}(z) \leq cz^d$ for all z. Then, if the VC-dimension of $\mathcal{F}$ is d_0, we have $2^{d_0} \leq cd_0^d$, which implies $d_0 \leq 4d \log(cd)$.

Most of this section is dedicated to proving the following result.

Theorem 2.1. *Let $G = (P, Q, E)$ be a bipartite graph with $|P| = m$ and $|Q| = n$ such that the set system $\mathcal{F}_1 = \{N_G(q) : q \in Q\}$ satisfies $\pi_{\mathcal{F}_1}(z) \leq cz^d$ for all z. Then, if G is $K_{k,k}$-free, we have*

$$|E(G)| \leq c_1(mn^{1-1/d} + n),$$

where $c_1 = c_1(c, d, k)$.

Let $f_1, \ldots, f_\ell$ be d-variate real polynomials with respective zero-sets $V_1, \ldots, V_\ell$, that is, $V_i = \{x \in \mathbb{R}^d : f_i(x) = 0\}$. A vector $\sigma \in \{-1, 0, +1\}^\ell$ is a *sign pattern* of $f_1, \ldots, f_\ell$ if there exists an $x \in \mathbb{R}^d$ such that the sign of $f_j(x)$ is σ_j for all $j = 1, \ldots, \ell$. The Milnor–Thom theorem (see [5, 33, 43]) bounds the number of cells in the arrangement of the zero-sets $V_1, \ldots, V_\ell$, and consequently the number of possible sign patterns.

Theorem 2.2 (Milnor–Thom). *Let $f_1, \ldots, f_\ell$ be d-variate real polynomials of degree at most t, with $\ell \geq d \geq 2$. The number of cells in the arrangement of their zero-sets $V_1, \ldots, V_\ell \subset \mathbb{R}^d$, and consequently the number of sign patterns of $f_1, \ldots, f_\ell$, is at most*

$$(50t\ell/d)^d.$$

We have the following consequence of Theorems 2.1 and 2.2.

Corollary 2.3. *Let* $G = (P, Q, E)$ *be a bipartite semi-algebraic graph in* $(\mathbb{R}^{d_1}, \mathbb{R}^{d_2})$ *with* $|P| = m$ *and* $|Q| = n$ *such that* E *has complexity at most* t*. If* G *is* $K_{k,k}$*-free, then*

$$|E(G)| \leq c'(mn^{1-1/d_2} + n),$$

where $c' = c'(d_1, d_2, t, k)$.

Proof. Let $\mathcal{F}_1 = \{N_G(q) : q \in Q\}$ and $\mathcal{F}_2 = \{N_G(p) : p \in P\}$. By Theorem 2.1, it suffices to show that $\pi_{\mathcal{F}_1}(z) \leq cz^{d_2}$ for all z and a constant $c = c(d_1, d_2, t, k)$.

Since E is semi-algebraic, there are polynomials $f_1, \dots, f_t$ and a Boolean formula Φ such that for $(p, q) \in P \times Q$,

$$(p, q) \in E \;\Leftrightarrow\; \Phi\big(f_1(p, q) \geq 0, \dots, f_t(p, q) \geq 0\big) = 1.$$

Notice that the *dual* of $\mathcal{F}_2$ is isomorphic to the set system $\mathcal{F}_1$. Since any set of z points $p_1, \dots, p_z \in P$ corresponds to z semi-algebraic sets $Z_1, \dots, Z_z \subset \mathbb{R}^{d_2}$ such that $Z_i = \{x \in \mathbb{R}^{d_2} : \Phi(f_1(p_i, x) \geq 0, \dots, f_t(p_i, x) \geq 0) = 1\}$ and $N_G(p_i) = Q \cap Z_i$, by the Milnor–Thom theorem we have

$$\pi_{\mathcal{F}_1}(z) = \pi_{\mathcal{F}_2^*}(z) \leq (50t^2 z/d_2)^{d_2}.$$

This completes the proof of Corollary 2.3. □

The rest of this section is devoted to proving Theorem 2.1, which requires the following lemmas. Let $(P, \mathcal{F})$ be a set system on a ground set P. The *distance* between two sets $A_1, A_2 \in \mathcal{F}$ is $|A_1 \triangle A_2|$, where $A_1 \triangle A_2 = (A_1 \cup A_2) \setminus (A_1 \cap A_2)$ is the symmetric difference of A_1 and A_2. The *unit distance graph* $\mathrm{UD}(\mathcal{F})$ is the graph with vertex set $\mathcal{F}$, and whose edges are pairs of sets (A_1, A_2) that have distance one. We will use the following result of Haussler.

Lemma 2.4 ([22]). *If* $\mathcal{F}$ *is a set system of VC-dimension* d_0 *on a ground set* P*, then the unit distance graph* $\mathrm{UD}(\mathcal{F})$ *has at most* $d_0|\mathcal{F}|$ *edges.*

We say that the set system $\mathcal{F}$ is (k, δ)*-separated* if for any k sets $A_1, \dots, A_k \in \mathcal{F}$ we have

$$|(A_1 \cup \dots \cup A_k) \setminus (A_1 \cap \dots \cap A_k)| \geq \delta.$$

The key tool used to prove Theorem 2.1 is the following *packing lemma*, which was proved by Chazelle for set systems that are $(2, \delta)$-separated. The proof of Lemma 2.5 can be regarded as a modification of Chazelle's argument (see [30]), but we give a self-contained presentation. We note that a weaker result, namely $|\mathcal{F}| \leq O((m/\delta)^d \log^d(m/\delta))$, can be obtained with a simpler proof using epsilon-nets (see [30] or [29]).

Lemma 2.5 (Packing lemma). *Let* $\mathcal{F}$ *be a set system on a ground set* P *such that* $|P| = m$ *and* $\pi_{\mathcal{F}}(z) \leq cz^d$ *for all* z*. If* $\mathcal{F}$ *is* (k, δ)*-separated, then* $|\mathcal{F}| \leq c'(m/\delta)^d$*, where* $c' = c'(c, d, k)$.

Proof. We assume, for contradiction, that $|\mathcal{F}| > c'(m/\delta)^d$ (where the constant c' depends on c, d, k and is set below).

Since the primal shatter function of $\mathcal{F}$ satisfies $\pi_{\mathcal{F}}(z) \le cz^d$ for all z, we know that the VC-dimension of $\mathcal{F}$ is at most $4d\log(cd) =: d_0$. If $\delta \le 4k(k-1)d_0$, then the statement is trivial for sufficiently large c' (by the assumption $|\mathcal{F}| \le cm^d$). Hence, we can assume $\delta > 4k(k-1)d_0$.

Let $S \subset P$ be a random s-element subset, where $s = \lceil 4k(k-1)d_0 m/\delta \rceil$. We set $\mathcal{T} = \{A \cap S : A \in \mathcal{F}\}$, and for each $B \in \mathcal{T}$ we define its *weight* $w(B)$ as the number of sets $A \in \mathcal{F}$ with $A \cap S = B$. Notice that

$$\sum_{B \in \mathcal{T}} w(B) = |\mathcal{F}|.$$

We let E be the edge set of the unit distance graph $\mathrm{UD}(\mathcal{T})$, and define the weight of an edge $e = (B_1, B_2)$ in E as $\min(w(B_1), w(B_2))$. Finally, we set

$$W = \sum_{e \in E} w(e).$$

We will estimate the expectation of W in two ways.

By Lemma 2.4, we know that the unit distance graph $\mathrm{UD}(\mathcal{T})$ has a vertex $B \in \mathcal{T}$ of degree at most $2d_0$. Since the weight of all edges emanating out of B is at most $w(B)$, by removing vertex $B \in \mathcal{T}$, the total edge weight drops by at most $2d_0 w(B)$. By repeating this argument until there are no vertices left, we have

$$W \le 2d_0 \sum_{B \in \mathcal{T}} w(B) = 2d_0 |\mathcal{F}|.$$

Now we bound $\mathbb{E}[W]$ from below. Suppose we first choose a random $(s-1)$-element subset $S' \subset P$, and then choose a single element $p \in P \setminus S'$. Then the set $S = S' \cup \{p\}$ is a random s-element set. Let $E_1 \subset E$ be the edges in the unit distance graph $\mathrm{UD}(\mathcal{T})$ that differ by p, and let

$$W_1 = \sum_{e \in E_1} w(e).$$

By symmetry, we have $\mathbb{E}[W] = s \cdot \mathbb{E}[W_1]$. Hence, we shall bound $\mathbb{E}[W_1]$ from below. To do so, we will estimate $\mathbb{E}[W_1 | S']$ from below, which is the expected value of W_1 when $S' \subset P$ is a fixed $(s-1)$-element subset and we choose p at random from $P \setminus S'$.

Divide $\mathcal{F}$ into equivalence classes $\mathcal{F}_1, \ldots, \mathcal{F}_r$, where $A_1, A_2 \in \mathcal{F}$ are in the same class if and only if $A_1 \cap S' = A_2 \cap S'$. By the assumption $\pi_{\mathcal{F}}(z) \le cz^d$ for all z, we have

$$r \le \pi_{\mathcal{F}}(s-1) \le c_0 (m/\delta)^d,$$

where $c_0 = c_0(c, k, d)$. Let $\mathcal{F}_i$ be one of the equivalence classes such that $|\mathcal{F}_i| = b$. If an element $p \in P \setminus S'$ is chosen such that b_1 sets from $\mathcal{F}_i$ contain p and $b_2 = b - b_1$ sets (from $\mathcal{F}_i$) do not contain p, then $\mathcal{F}_i$ gives rise to an edge in E_1 of weight $\min(b_1, b_2)$. Since $\min(b_1, b_2) \ge b_1 b_2 / b$, we will estimate $\mathbb{E}[b_1 b_2]$ from below when picking p at random. Notice that b_1b_2 is the number of ordered pairs of sets in $\mathcal{F}_i$ that differ by p. Hence,

$$\mathbb{E}[b_1b_2] \geq \sum_{(A_1,A_2)\in\mathcal{F}_i\times\mathcal{F}_i} \mathbb{P}[p \in A_1 \triangle A_2] = \sum_{(A_1,A_2)\in\mathcal{F}_i\times\mathcal{F}_i} \frac{|A_1 \triangle A_2|}{m-s+1}. \tag{5}$$

Now, given any k sets $A_1, \ldots, A_k \in \mathcal{F}_i$, we have

$$\bigcup_{2\leq j\leq k} A_1 \triangle A_j = (A_1 \cup \cdots \cup A_k) \setminus (A_1 \cap \cdots \cap A_k).$$

Since $\mathcal{F}_i$ is (k, δ)-separated, we have

$$\sum_{2\leq j\leq k} |A_1 \triangle A_j| \geq |(A_1 \cup \cdots \cup A_k) \setminus (A_1 \cap \cdots \cap A_k)| \geq \delta.$$

Therefore, any k sets in $\mathcal{F}_i$ contain a pair (A_1, A_j) such that $|A_1 \triangle A_j| \geq \delta/(k-1)$. We define the auxiliary graph $G_i = (\mathcal{F}_i, E_i)$ whose vertices are the members in $\mathcal{F}_i$, and two sets $A_1, A_2 \in \mathcal{F}_i$ are adjacent if and only if $|A_1 \triangle A_2| \geq \delta/(k-1)$. Since G_i does not contain an independent set of size k, by Turán's theorem (see, e.g., [35]) we have $|E_i| \geq b(b-k)/(2k)$. Therefore,

$$\sum_{(A_1,A_2)\in\mathcal{F}_i\times\mathcal{F}_i} |A_1 \triangle A_2| \geq 2\frac{b(b-k)}{2k}\,\frac{\delta}{k-1} = \frac{\delta}{k(k-1)}b(b-k). \tag{6}$$

By combining (5) and (6), we obtain

$$\mathbb{E}[b_1b_2] \geq \frac{\delta}{k(k-1)m}b(b-k).$$

As $\min(b_1, b_2) \geq b_1b_2/b$, the expected contribution of $\mathcal{F}_i$ to W_1 is at least $\frac{\delta}{k(k-1)m}(b-k)$. Summing over all classes, we have

$$\begin{aligned}\mathbb{E}[W_1] &\geq \frac{\delta}{k(k-1)m}\sum_{i=1}^{r}(|\mathcal{F}_i| - k) = \frac{\delta}{k(k-1)m}(|\mathcal{F}| - kr)\\ &\geq \frac{\delta}{k(k-1)m}(|\mathcal{F}| - kc_0(m/\delta)^d).\end{aligned}$$

Recall that $|\mathcal{F}| > c'(m/\delta)^d$. By taking c' to be sufficiently large with respect to k and c_0, and since $2d_0|\mathcal{F}| \geq \mathbb{E}[W] = s \cdot \mathbb{E}[W_1]$, we have

$$2d_0|\mathcal{F}| \geq \frac{s\delta}{k(k-1)m}(|\mathcal{F}| - kc_0(m/\delta)^d) \geq 4d_0|\mathcal{F}| - k4d_0c_0(m/\delta)^d,$$

which implies $|\mathcal{F}| \leq c'(m/\delta)^d$, where $c' = (c, d, k)$. □

Proof of Theorem 2.1. Let $\mathcal{F}_1 = \{N_G(q) : q \in Q\}$ and $\mathcal{F}_2 = \{N_G(p) : p \in P\}$. Notice the dual of $\mathcal{F}_2$ is isomorphic to the set system $\mathcal{F}_1$. Given a set of k points $\{q_1, \ldots, q_k\} \subset Q$, we say that a set $B \in \mathcal{F}_2$ *crosses* $\{q_1, \ldots, q_k\}$ if $\{q_1, \ldots, q_k\} \cap B \neq \emptyset$ and $\{q_1, \ldots, q_k\} \not\subset B$. We make the following observation.

Observation 2.6. *There exist k points* $q_1, \ldots, q_k \in Q$ *such that at most* $2c'm/n^{1/d}$ *sets from* $\mathcal{F}_2$ *cross* $\{q_1, \ldots, q_k\}$, *where* c' *is defined in Lemma 2.5.*

Proof. For the sake of contradiction, suppose that every set of k points has at least $2c'm/n^{1/d}$ sets from $\mathcal{F}_2$ crossing it. Then the dual set system $\mathcal{F}_2^*$ is (k, δ)-separated, where $\delta = 2c'm/n^{1/d}$, and has the property that $\pi_{\mathcal{F}_2^*}(z) = \pi_{\mathcal{F}_1}(z) \leq cz^d$ for all z. By Lemma 2.5, we have

$$n = |\mathcal{F}_2^*| \leq c'(m/\delta)^d.$$

Hence, $\delta \leq (c')^{1/d}m/n^{1/d}$, which is a contradiction. □

Let $q_1, \ldots, q_k$ be the set of k points such that at most $2c'm/n^{1/d}$ sets in $\mathcal{F}_2$ cross it. Since G is $K_{k,k}$-free, there are at most $k-1$ points $p_1, \ldots, p_{k-1} \in P$ with the property that the neighborhood $N_G(p_i)$ contains $\{q_1, \ldots, q_k\}$, for $1 \leq i \leq k-1$. Therefore, the neighborhood of q_1 contains at most $2c'm/n^{1/d} + (k-1)$ points. We remove q_1 and repeat this argument until there are fewer than k vertices remaining in Q, and see that

$$|E(G)| \leq (k-1)m + \sum_{i=k}^{n} \left(2c' \frac{m}{i^{1/d}} + (k-1) \right) \leq c_1(mn^{1-1/d} + n)$$

for sufficiently large $c_1 = c_1(c, d, k)$. □

3. The case where $d_1 = d_2 = 2$

In this section, we shall prove Theorem 1.1 in the case $d_1 = d_2 = 2$, i.e., we shall establish part (1) of Theorem 1.1. Our argument will use the method of "cuttings," which we shall now recall. Let $\Sigma = \{V_1, \ldots, V_n\}$ be a collection of curves of degree at most t in $\mathbb{R}^2$, that is, $V_i = \{x \in \mathbb{R}^2 : f_i(x) = 0\}$ for some bivariate polynomial f_i of degree at most t. We will assume that t is fixed, and n is some number tending to infinity. A *cell* in the arrangement $\mathcal{A}(\Sigma) = \bigcup_i V_i$ is a relatively open connected set defined as follows. Let $\approx$ be an equivalence relation on $\mathbb{R}^2$, where $x \approx y$ if $\{i : x \in V_i\} = \{i : y \in V_i\}$. Then the cells of the arrangement Σ are the connected components of the equivalence classes. The classical Milnor–Thom theorem says that the arrangement $\mathcal{A}(\Sigma)$ subdivides $\mathbb{R}^2$ into at most $O(n^2)$ cells (semi-algebraic sets), but these cells can have very large description complexity. A result of Chazelle et al. [11] shows that these cells can be further subdivided into $O(n^2)$ smaller cells that have constant descriptive complexity. By combining this technique with the standard theory of random sampling [1, 2, 13], one can obtain the following lemma which will be used in the next section. We say that the surface $V_i = \{x \in \mathbb{R}^2 : f_i(x) = 0\}$ *crosses* the cell $\Omega \subset \mathbb{R}^2$ if $V_i \cap \Omega \neq \emptyset$ and V_i does not fully contain Ω.

Lemma 3.1 (Cutting lemma, [11]). *For fixed* $t > 0$, *let* Σ *be a family of n algebraic surfaces in* $\mathbb{R}^2$ *of degree at most t. Then for any* $r > 0$, *there exists a decomposition of* $\mathbb{R}^2$ *into at most* $O(r^2)$ *relatively open connected sets (cells) such that each cell is crossed by at most* n/r *curves from* Σ.

We are now ready to prove the following theorem, which will establish (1).

Theorem 3.2. *Let $G = (P, Q, E)$ be a semi-algebraic bipartite graph in $\mathbb{R}^2$ such that E has description complexity at most t, $|P| = m$, and $|Q| = n$. If G is $K_{k,k}$-free, then*

$$|E(G)| \leq c(m^{2/3}n^{2/3} + m + n),$$

where $c = c(k, t)$.

Proof. If $n > m^2$, then by Corollary 2.3 we have $|E(G)| \leq (c/2)n$ for sufficiently large $c = c(k, t)$, and we are done. Hence, we can assume $n \leq m^2$. Since E is semi-algebraic of description complexity at most t, there are polynomials $f_1, \dots, f_t$ and a Boolean formula Φ such that for $(p, q) \in P \times Q$,

$$(p, q) \in E \;\Leftrightarrow\; \Phi\big(f_1(p, q) \geq 0, \dots, f_t(p, q) \geq 0\big) = 1.$$

For each point $q \in Q$, let $V_{i,q} = \{x \in \mathbb{R}^2 : f_i(x, q) = 0\}$, $1 \leq i \leq t$. Set $\Sigma = \{V_{i,q} : 1 \leq i \leq t,\ q \in Q\}$. Note that $|\Sigma| = tn$.

For $r = m^{2/3}/n^{1/3}$, we apply Lemma 3.1, the cutting lemma, to Σ, which partitions $\mathbb{R}^2$ into at most c_2r^2 cells Ω_i, where $c_2 = c_2(t)$, such that each cell is crossed by at most $|\Sigma|/r$ surfaces from Σ. By the Pigeonhole Principle, there is a cell $\Omega \subset \mathbb{R}^2$ that contains at least

$$\frac{m}{c_2r^2} = \frac{n^{2/3}}{c_2m^{1/3}}$$

points from P. Let $P' \subset P$ be a set of exactly $\lceil n^{2/3}/(c_2m^{1/3})\rceil$ points in $P \cap \Omega$. If $|P'| < k$, we have

$$\frac{n^{2/3}}{c_2m^{1/3}} \leq |P'| < k,$$

which implies $m > n^2/(c_2^3k^3)$. By the dual of Corollary 2.3, we have $|E(G)| \leq (c/2)m$ for sufficiently large $c = c(k, t)$, and we are done. Hence, we can assume $|P'| \geq k$. Let $Q' \subset Q$ be the set of points in Q that gives rise to a surface in Σ that crosses Ω. By the cutting lemma,

$$|Q'| \leq \frac{tn}{r} = t\frac{n^{4/3}}{m^{2/3}} \leq t(c_2)^2|P'|^2.$$

By Corollary 2.3, we have

$$|E(P', Q')| \leq c'(|P'||Q'|^{1/2} + |Q'|) \leq c_3|P'|^2,$$

where c' is defined in Corollary 2.3, and $c_3 = c_3(k, t)$. Hence, there is a point $p \in P'$ such that p has at most $c_3|P'|$ neighbors in Q'. Since G is $K_{k,k}$-free, there are at most $k - 1$ points in $Q \setminus Q'$ that are neighbors to p. Hence,

$$|N_G(p)| \leq c_3|P'| + (k - 1) \leq \frac{c_3}{c_2}\frac{n^{2/3}}{m^{1/3}} + (k - 1).$$

We remove p and repeat this argument until there are no vertices remaining in P and see that

$$|E(G)| \le (c/2)(n+m) + \sum_{i=n^{1/2}}^{m} \left(\frac{c_3}{c_2} \frac{n^{2/3}}{i^{1/3}} + (k-1) \right) \le c(m^{2/3}n^{2/3} + m + n)$$

for sufficiently large $c = c(k,t)$. □

4. The case of general d_1 and d_2

The goal of this section is to prove Theorem 1.1 for all dimensions d_1, d_2 (i.e., parts (2) and (3) of the theorem).

4.1. Preliminaries

We begin by introducing some tools, along with some useful notation. Our proof will use some basic tools from algebraic geometry. A nice introduction to these concepts can be found in [14].

Real varieties. If $V \subset \mathbb{R}^d$ is a real algebraic variety, we define the dimension $\dim V$ of V as in [7, Section 2.8]. Define $V^* \subset \mathbb{C}^d$ to be the *complexification* of V—the smallest complex variety containing V. That is, if $\iota\colon \mathbb{R}^d \to \mathbb{C}^d$ is the usual embedding of $\mathbb{R}^d$ to $\mathbb{C}^d$, then V^* is the Zariski closure (over $\mathbb{C}$) of $\iota(V)$. We define $\deg V = \deg V^*$, where the latter is the usual definition of the degree of a complex variety (i.e., the cardinality of $V^* \cap H$, where $H \subset \mathbb{C}^d$ is a generic flat of codimension $\dim V^*$).

Given a real variety $V \subset \mathbb{R}^d$, we denote by $I(V)$ the ideal of polynomials $f \in \mathbb{R}[x_1, \dots, x_d]$ that vanish on V. We say that a real variety V is *irreducible* if it is irreducible over $\mathbb{R}$ (see e.g. [7, Section 2.8]). In particular, if V is irreducible, then $I(V)$ is a prime ideal. Moreover, for every polynomial $g \in \mathbb{R}[x_1, \dots, x_d]$ such that $g \notin I(V)$, the ideal $(I(V), g)$ strictly contains $I(V)$, and thus $\dim(V \cap Z(g)) < \dim V$.

Polynomial partitioning. Consider a set P of m points in $\mathbb{R}^d$. Given a polynomial $f \in \mathbb{R}[x_1, \dots, x_d]$, we define the *zero-set* of f to be $Z(f) = \{p \in \mathbb{R}^d : f(p) = 0\}$. For $1 < r \le m$, we say that $f \in \mathbb{R}[x_1, \dots, x_d]$ is an *r-partitioning polynomial* for P if no connected component of $\mathbb{R}^d \setminus Z(f)$ contains more than m/r points of P. Notice that there is no restriction on the number of points of P that lie in $Z(f)$.

The following result is due to Guth and Katz [21]. A detailed proof can also be found in [24].

Theorem 4.1 (Polynomial partitioning [21]). *Let P be a set of m points in $\mathbb{R}^d$. Then for every $1 < r \le m$, there exists an r-partitioning polynomial $f \in \mathbb{R}[x_1, \dots, x_d]$ of degree at most $C_{\text{part}} \cdot r^{1/d}$, where C_{part} depends only on d.*

We require the following generalization of Theorem 4.1, which we prove in Section 5.

Theorem 4.2. *Let P be a set of n points in $\mathbb{R}^d$ and let $V \subset \mathbb{R}^n$ be an irreducible variety of degree D and dimension d'. Then there exists an r-partitioning polynomial g for P such that $g \notin I(V)$ and $\deg g \le C_{\text{part}} \cdot r^{1/d'}$, where C_{part} depends only on d and D.*

4.2. *Proof of Theorem 1.1*

We now establish Theorem 1.1 by proving the following more general statement. Theorem 1.1 is immediately implied by Theorem 4.3, by taking V to be $\mathbb{R}^{d_1}$.

Theorem 4.3. *Let $G = (P, Q, E)$ be a bipartite semi-algebraic graph in $(\mathbb{R}^{d_1}, \mathbb{R}^{d_2})$ such that E has complexity at most t, $|P| = m$, and $|Q| = n$. Moreover, let $P \subset V$, where $V \subset \mathbb{R}^{d_1}$ is an irreducible variety of dimension e and degree D. If G is $K_{k,k}$-free, then for any $\varepsilon > 0$,*

$$|E(G)| \le \alpha_{1,e} m^{\frac{d_2(d_1-1)}{d_1d_2-1}+\varepsilon} n^{\frac{d_1(d_2-1)}{d_1d_2-1}} + \alpha_2(m+n), \tag{7}$$

where $\alpha_{1,e}$, α_2 are constants that depend on ε, d_1, d_2, e, t, k, and D.

Proof. As in Section 3, we think of the vertices of P as points in $\mathbb{R}^{d_1}$, and we think of the vertices of Q as semi-algebraic sets in $\mathbb{R}^{d_1}$. That is, every $q \in Q$ is the (semi-algebraic) set of all points $p \in \mathbb{R}^{d_1}$ that satisfy

$$\Phi\big(f_1(p,q) \ge 0, \ldots, f_t(p,q) \ge 0\big) = 1.$$

Abusing notation, we will also refer to the set of incidences in $P \times Q$ as $I(P, Q)$, There is a bijection between the edges of G and the incidences of $I(P, Q)$. Thus, it suffices to prove

$$I(P,Q) \le \alpha_{1,e} m^{\frac{d_2(d_1-1)}{d_1d_2-1}+\varepsilon} n^{\frac{d_1(d_2-1)}{d_1d_2-1}} + \alpha_2(m+n). \tag{8}$$

We prove the theorem by a two-step induction process. First, we induct on e. We can be quite wasteful with each such induction step, since we perform at most d_1 such steps. Within every such step, we perform a second induction on $|P|+|Q| = m+n$. We must be more careful with the steps of the second induction, since we perform many such steps.

By Corollary 2.3, there exists a constant $C_{2.3}$ (depending on d_1, d_2, t, k) such that $|E(G)| \le C_{2.3}(mn^{1-1/d_2} + n)$. When $m \le n^{1/d_2}$ (and when α_2 is sufficiently large) we have $|E(G)| \le \alpha_2 n$. Therefore, in the remainder of the proof we assume that $n < m^{d_2}$, which implies

$$n = n^{\frac{d_1-1}{d_1d_2-1}} n^{\frac{d_1(d_2-1)}{d_1d_2-1}} \le m^{\frac{d_2(d_1-1)}{d_1d_2-1}} n^{\frac{d_1(d_2-1)}{d_1d_2-1}}. \tag{9}$$

Since the conditions in Corollary 2.3 are symmetric with respect to d_1 and d_2, we can replace d_2 with d_1 in the bound of the lemma. Thus, the same argument implies that $m < n^{d_1}$, and hence

$$m \le m^{\frac{d_2(d_1-1)}{d_1d_2-1}} n^{\frac{d_1(d_2-1)}{d_1d_2-1}}. \tag{10}$$

We now consider the base case for the induction. If $m+n$ is sufficiently small, then (8) is immediately implied by choosing sufficiently large values for $\alpha_{1,e}$ and α_2. Similarly, when $e = 0$, we again obtain (8) when $\alpha_{1,e}$ and α_2 are sufficiently large.

Partitioning. Next, we consider the induction step. That is, we assume that (8) holds when $|P| + |Q| < m + n$ or $\dim V < e$. By Theorem 4.2, there exists an r-partitioning polynomial f with respect to V of degree at most $C_{\text{part}} \cdot r^{1/e}$, where r is a large constant that will be set later on. The dependencies between the various constants in the proof are

$$2^{1/\varepsilon}, d_1, d_2, e, t, k, D \ll C_{\text{part}}, C_{\text{cells}}, C_{2.3}, C_{\text{inter}} \ll C_{\text{Höld}} \ll r \ll \gamma_1, \alpha_2 \ll \alpha_1.$$

Denote the cells of the partition as $\Omega_1, \ldots, \Omega_s$. Since we are working over the reals, there exists a polynomial g whose degree depends only on d_1, d_2, and D such that $Z(g) = V$. Thus, by [40, Theorem A.2], there exists a constant C_{cells} such that $s \leq C_{\text{cells}} \cdot r$, where C_{cells} depends on d_1, d_2, e, and D. We partition $I(P, Q)$ into the following three subsets:

- I_1 consists of the incidences $(p, q) \in I(P, Q)$ where p is contained in the variety $V \cap Z(f)$.
- I_2 consists of the incidences $(p, q) \in I(P, Q)$ where p is contained in a cell Ω of the partitioning, and the semi-algebraic set q fully contains Ω.
- $I_3 = I(P, Q) \setminus \{I_1 \cup I_2\}$. This is the set of incidences $(p, q) \in I(P, Q)$ such that p is contained in a cell Ω, and q does not fully contain Ω (i.e., q properly intersects Ω).

Notice that we indeed have

$$I(P, Q) = I_1 + I_2 + I_3. \tag{11}$$

Bounding I_1. The points of $P \subset \mathbb{R}^{d_1}$ that participate in incidences of I_1 are all contained in the variety $V' = V \cap Z(f)$. Set $m_0 = |P \cap V'|$. Since V is an irreducible variety and $f \notin I(V)$, V' is a variety of dimension $e' \leq e - 1$. The intersection $V' = V \cap Z(f)$ can be written as a union of γ_1 irreducible (over $\mathbb{R}$) components, each of dimension at most e' and degree at most γ_2, where γ_1 and γ_2 depend only on D, C_{part}, d, and r (see e.g. [19]). We can now apply the induction hypothesis to each component to obtain

$$I_1 \leq \gamma_1 \alpha_{1,e-1} m_0^{\frac{d_2(d_1-2)}{(d_1-1)d_2-1}+\varepsilon} n^{\frac{(d_1-1)(d_2-1)}{(d_1-1)d_2-1}} + \alpha_2(m_0 + n).$$

Notice that

$$\begin{aligned} m^{\frac{d_2(d_1-2)}{(d_1-1)d_2-1}} n^{\frac{(d_1-1)(d_2-1)}{(d_1-1)d_2-1}} &= m^{\frac{d_2(d_1-2)}{(d_1-1)d_2-1} - \frac{d_2(d_1-1)}{d_1d_2-1} + \frac{d_2(d_1-1)}{d_1d_2-1}} n^{\frac{(d_1-1)(d_2-1)}{(d_1-1)d_2-1} - \frac{d_1(d_2-1)}{d_1d_2-1} + \frac{d_1(d_2-1)}{d_1d_2-1}} \\ &= m^{-\frac{(d_2-1)d_2}{((d_1-1)d_2-1)(d_1d_2-1)} + \frac{d_2(d_1-1)}{d_1d_2-1}} n^{\frac{d_2-1}{((d_1-1)d_2-1)(d_1d_2-1)} + \frac{d_1(d_2-1)}{d_1d_2-1}} \\ &= m^{-\frac{(d_2-1)d_2}{((d_1-1)d_2-1)(d_1d_2-1)}} n^{\frac{d_2-1}{((d_1-1)d_2-1)(d_1d_2-1)}} m^{\frac{d_2(d_1-1)}{d_1d_2-1}} n^{\frac{d_1(d_2-1)}{d_1d_2-1}} \\ &\leq m^{\frac{d_2(d_1-1)}{d_1d_2-1}} n^{\frac{d_1(d_2-1)}{d_1d_2-1}}, \end{aligned} \tag{12}$$

where the last inequality follows from the fact that $m^{-d_2} n \leq 1$. By applying (9) to the $\alpha_2 n$ term and by choosing $\alpha_{1,e}$ to be sufficiently large with respect to $\alpha_{1,e-1}$, γ_1, and α_2, we obtain

$$I_1 \leq \frac{\alpha_{1,e}}{2} m^{\frac{d_2(d_1-1)}{d_1d_2-1}+\varepsilon} n^{\frac{d_1(d_2-1)}{d_1d_2-1}} + \alpha_2 m_0. \tag{13}$$

Bounding I_2. Let $m' = m - m_0$. This is the number of points of P that are not contained in $Z(f)$. A cell of $\Omega_1, \ldots, \Omega_s$ that contains at most $k - 1$ points of P can yield at most

$(k-1)n$ incidences. Since G is $K_{k,k}$-free, a cell that contains at least k points of P can be fully contained in at most $k-1$ of the semi-algebraic sets of Q. Since $s \le C_{\text{cells}} \cdot r$, we obtain

$$I_2 < C_{\text{cells}} \cdot r\big((k-1)n + (k-1)m'\big).$$

By choosing α_2 to be sufficiently large, we have

$$I_2 \le \alpha_2(m' + n). \tag{14}$$

Bounding I_3. We say that a semi-algebraic set $q \in Q$ *properly intersects* a cell Ω if q meets Ω but does not contain Ω. For each $q \in Q$, we now bound the number of cells that q properly intersects. Such a set q is defined by at most t equations, each of degree at most t. For q to properly intersect a cell Ω, at least one of these equations must define a variety that intersects Ω (this condition is necessary but not sufficient). Consider an equation E such that $Z(E)$ does not fully contain V_i (since otherwise it would not properly intersect any cell). Since V_i is irreducible, the dimension of $Z(E) \cap V_i$ is at most $e-1$. Thus, by [40, Theorem A.2], there exists a constant C_{inter} (depending on t, d_1) such that $Z(E)$ intersects at most $C_{\text{inter}}\, r^{(e-1)/e}$ cells of the partition. This in turn implies that every semi-algebraic set $q \in Q$ properly intersects at most $tC_{\text{inter}}\, r^{(e-1)/e}$ cells of the partition.

For $1 \le i \le s$, we denote by Q_i the set of elements of Q that properly intersect the cell Ω_i, and by P_i the set of points of P that are contained in Ω_i. We set $m_i = |P_i|$ and $n_i = |Q_i|$. By the partitioning property, we have $m_i \le m/r$ for every $1 \le i \le s$. By the previous paragraph, we have

$$\sum_{i=1}^{s} n_i \le ntC_{\text{inter}}\, r^{(e-1)/e}.$$

By applying Hölder's inequality, we have

$$\begin{aligned}
\sum_{i=1}^{s} n_i^{\frac{d_1(d_2-1)}{d_1d_2-1}} &\le \Big(\sum_{i=1}^{s} n_i\Big)^{\frac{d_1(d_2-1)}{d_1d_2-1}} \Big(\sum_{i=1}^{s} 1\Big)^{\frac{d_1-1}{d_1d_2-1}}\\
&\le (ntC_{\text{inter}}\, r^{(e-1)/e})^{\frac{d_1(d_2-1)}{d_1d_2-1}} (C_{\text{cells}}\, r)^{\frac{d_1-1}{d_1d_2-1}}\\
&\le C_{\text{Höld}}\, n^{\frac{d_1(d_1-1)}{d_1d_2-1}}\, r^{1-\frac{d_1(d_2-1)}{e(d_1d_2-1)}} \le C_{\text{Höld}}\, n^{\frac{d_1(d_1-1)}{d_1d_2-1}}\, r^{1-\frac{d_2-1}{d_1d_2-1}},
\end{aligned}$$

where $C_{\text{Höld}}$ depends on $t, C_{\text{inter}}, C_{\text{cells}}, d_1, d_2$.

By the induction hypothesis, we have

$$\begin{aligned}
\sum_{i=1}^{s} I(P_i, Q_i) &\le \sum_{i=1}^{s}\Big(\alpha_{1,e} m_i^{\frac{(d_1-1)d_2}{d_1d_2-1}+\varepsilon} n_i^{\frac{d_1(d_2-1)}{d_1d_2-1}} + \alpha_2(m_i + n_i)\Big) &(15)\\
&\le \alpha_{1,e} m^{\frac{(d_1-1)d_2}{d_1d_2-1}+\varepsilon}\Big(r^{\frac{(d_1-1)d_2}{d_1d_2-1}+\varepsilon}\Big)^{-1} \sum_{i=1}^{s} n_i^{\frac{d_1(d_2-1)}{d_1d_2-1}} + \sum_{i=1}^{s} \alpha_2(m_i + n_i) &(16)\\
&= \alpha_{1,e} C_{\text{Höld}}\, r^{-\varepsilon} m^{\frac{(d_1-1)d_2}{d_1d_2-1}+\varepsilon} n^{\frac{d_1(d_2-1)}{d_1d_2-1}} + \alpha_2(m + ntC_{\text{inter}}\, r^{(e-1)/e}). &(17)
\end{aligned}$$

According to (9) and (10), when $\alpha_{1,e}$ is sufficiently large with respect to r, t, C_{inter}, α_2, we have

$$\sum_{i=1}^{s} I(P_i, Q_i) \leq 3\alpha_{1,e} C_{\text{Höld}}\, r^{-\varepsilon} m^{\frac{(d_1-1)d_1}{d_1d_2-1}+\varepsilon} n^{\frac{d_1(d_2-1)}{d_1d_2-1}}.$$

Finally, by choosing r to be sufficiently large with respect to ε, $C_{\text{Höld}}$, we have

$$I_3 = \sum_{i=1}^{s} I(P_i, Q_i) \leq \frac{\alpha_{1,e}}{2} m^{\frac{(d_1-1)d_2}{d_1d_2-1}+\varepsilon} n^{\frac{d_1(d_2-1)}{d_1d_2-1}}. \tag{18}$$

Summing up. By combining (11), (13), (14), and (18), we obtain

$$I(P, Q) \leq \alpha_{1,e} m^{\frac{d_2(d_1-1)}{d_1d_2-1}+\varepsilon} n^{\frac{d_1(d_2-1)}{d_1d_2-1}} + \alpha_2(m+n),$$

which completes the induction step and the proof of the theorem. □

5. Hilbert polynomials and Theorem 4.2

In this section, we will prove Theorem 4.2. Our proof relies on Hilbert polynomials. Before presenting the proof, we begin with some algebraic preliminaries.

5.1. Hilbert polynomials

Let $\mathbb{R}[x_1, \ldots, x_d]_{\leq m}$ be the set of polynomials of degree at most m in $\mathbb{R}[x_1, \ldots, x_d]$. Similarly, if $I \subset \mathbb{R}[x_1, \ldots, x_d]$ is an ideal, let $I_{\leq m} = I \cap \mathbb{R}[x_1, \ldots, x_d]_{\leq m}$ be the set of polynomials in I of degree at most m. It can be easily verified that there are $\binom{d+m}{m}$ monomials in $x_1, \ldots, x_d$ of degree m. Thus, we can consider $\mathbb{R}[x_1, \ldots, x_d]_{\leq m}$ as a vector space of dimension $\binom{d+m}{m}$, and $I_{\leq m}$ as a vector subspace of $\mathbb{R}[x_1, \ldots, x_d]_{\leq m}$. We consider a polynomial $f \in \mathbb{R}[x_1, \ldots, x_d]_{\leq m}$ as equivalent to any of its constant multiples cf (where $c \in \mathbb{R} \setminus \{0\}$), since their zero-sets are identical. Therefore, $\mathbb{R}[x_1, \ldots, x_d]_{\leq m}$ can be identified with the projective space $\mathbb{R}\mathbf{P}^{\binom{d+m}{m}}$, and $I_{\leq m}$ can be identified with a projective variety in $\mathbb{R}\mathbf{P}^{\binom{d+m}{m}}$.

The quotient $\mathbb{R}[x_1, \ldots, x_d]_{\leq m}/I_{\leq m}$ is also a vector space (see, e.g., [14, Section 9.3]). The *Hilbert function* of an ideal $I \subset \mathbb{R}[x_1, \ldots, x_d]$ is defined as

$$h_I(m) = \dim(\mathbb{R}[x_1, \ldots, x_d]_{\leq m}/I_{\leq m}).$$

A nice introduction to Hilbert functions can be found in [14, Chapter 9].

For every ideal $I \subset \mathbb{R}[x_0, \ldots, x_d]$, there exists an integer m_I and a polynomial $H_I(m)$ such that for every $m > m_I$ we have $h_I(m) = H_I(m)$. The polynomial H_I is called the *Hilbert polynomial* of I, and m_I is called the *regularity* of I. We set $t = \deg H_I$, and say that the *dimension* of I is t. Notice that if $I \neq \{0\}$, then $t < d$. Let a_I be the coefficient of the leading monomial of H_I.

If $V \subset \mathbb{R}^d$ is an irreducible variety, then $\dim I(V) = \dim V$, where $\dim V$ is defined in Section 4.1. Furthermore, the leading coefficient $a_I > 0$ is bounded below by a constant $c_{d,\deg V}$ that depends only on d and $\deg V$.

In [20, Theorem B], it is shown that the regularity m_I of I is bounded by a quantity $\tilde{m}$ that depends only on d and $\deg V$.[3]

In particular, there is an integer m' depending only on d and $\deg V$ such that for $m > m'$, we have

$$h_{I(V)}(m) > \frac{c_{d,\deg V}}{2} m^{\dim V}. \tag{19}$$

5.2. *Proof of Theorem 4.2*

We first recall the discrete version of the *ham-sandwich theorem* (see, e.g., [28]). A hyperplane h in $\mathbb{R}^d$ *bisects* a finite point set $S \subset \mathbb{R}^d$ if each of the two open halfspaces bounded by h contains at most $|S|/2$ points of S. The bisecting hyperplane may contain any number of points of S.

Theorem 5.1 (Discrete ham-sandwich theorem). *Any d finite point sets $S_1, \ldots, S_d \subset \mathbb{R}^d$ can be simultaneously bisected by a hyperplane.*

A polynomial $g : \mathbb{R}^d \to \mathbb{R}$ *bisects* a finite point set $S \subset \mathbb{R}^d$ if each of the two sets $\{x \in \mathbb{R}^d : g(x) < 0\}$ and $\{x \in \mathbb{R}^d : g(x) > 0\}$ contains at most $|S|/2$ points of S.

We combine Theorem 5.1 with Hilbert polynomials to obtain a variant of the *polynomial* ham-sandwich theorem (for the original theorem, see for example [21]).

Lemma 5.2. *Let $V \subset \mathbb{R}^d$ be an irreducible variety of dimension d' and degree D, and let $S_1, \ldots, S_k$ be finite sets of points that are contained in V. Then there exist a constant m_0 that depends only on D and d, and a polynomial g, such that $g \notin I(V)$, g bisects each of the sets $S_1, \ldots, S_k$, and*

$$\deg g = \begin{cases} O_{D,d}(1) & \text{if } k < m_0, \\ O_{D,d}(k^{1/d'}) & \text{if } k \geq m_0. \end{cases}$$

Proof. Our proof is a variant of the proof of the polynomial ham-sandwich theorem (as presented, e.g., in [21, 24]). Let $I = I(V)$. As noted in Section 5.1, there exists a constant $\tilde{m}_I$ depending only on d and D such that (19) holds for every $m > \tilde{m}_i$. Thus, the vector space $\mathbb{R}[x_1, \ldots, x_d]_{\leq m}/I_{\leq m}$ has dimension $E_m = \Omega_{d,D}(m^{d'})$. We choose m so that $E_m \geq k$. That is,

$$k = O_{d,D}(m^{d'}), \quad \text{or} \quad m = \Omega_{d,D}(k^{1/d'}).$$

If the resulting m is smaller than $\tilde{m}$, we replace it with $h(\tilde{m}) = O_{D,d}(1)$.

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[3] It is important to note that Giusti's result in [20] applies in any field of characteristic 0. In particular, the field does not need to be algebraically closed. Giusti deals with homogeneous ideals, while we work with affine ideals. However, Giusti's bound also applies in the affine case. Giusti bounds the quantity m_I in terms of the dimension d and the maximum degree of the collection of polynomials needed to generate I. This quantity is in turn bounded by the degree of V.

Let $p_1, \ldots, p_{E_m}$ be a basis for the vector space $\mathbb{R}[x_1, \ldots, x_d]_{\le m}/I_{\le m}$. For each $i = 1, \ldots, E_m$, choose a representative $\tilde{p}_i \in \mathbb{R}[x_1, \ldots, x_d]_{\le m}$ which lies in the equivalence class p_i. We will choose $\tilde{p}_i$ to be of smallest possible degree (note that the choice of $\tilde{p}_i$ need not be unique). Consider the polynomial mapping $\phi : \mathbb{R}^d \to \mathbb{R}\mathbf{P}^{E_m}$ defined by

$$\phi(x) = (\tilde{p}_1(x), \ldots, \tilde{p}_{E_m}(x)).$$

For every $1 \le i \le k$, let $S'_i = \phi(S_i) \subset \mathbb{R}\mathbf{P}^{E_m}$. Note that ϕ is injective on $V = Z(I)$, and thus $|S'_i| = S_i$. By Theorem 5.1, there exists a hyperplane $h \subset \mathbb{R}\mathbf{P}^{E_m}$ that bisects each of the sets $S'_1, \ldots, S'_k$. The hyperplane h can be defined as $Z(a_1 y_1 + \cdots + a_{E_m} y_{E_m})$ for some $a_1, \ldots, a_{E_m} \in \mathbb{R}$. In other words, for each $i = 1, \ldots, k$, we have

$$|\{y \in S'_i : a_1 y_1 + \cdots + a_{E_m} y_{E_m} > 0\}| \le |S'_i|/2,$$
$$|\{y \in S'_i : a_1 y_1 + \cdots + a_{E_m} y_{E_m} < 0\}| \le |S'_i|/2.$$

If $x \in \mathbb{R}^d$, then $a_1 \tilde{p}_1(x) + \cdots + a_{E_m} \tilde{p}_{E_m}(x) = (a_1 \tilde{p}_1 + \cdots + a_{E_m} \tilde{p}_{E_m})(x)$. Thus, if we let $g = a_1 \tilde{p}_1 + \cdots + a_{E_m} \tilde{p}_{E_m}$, then g is a polynomial of degree at most m, $g \notin I$, and for each $i = 1, \ldots, E_m$,

$$|\{y \in S_i : g(y) > 0\}| \le |S_i|/2, \qquad |\{y \in S_i : g(y) < 0\}| \le |S_i|/2,$$

i.e., g bisects each of the sets $S_1, \ldots, S_{E_m}$. □

The standard polynomial partitioning theorem is proved by using the polynomial ham-sandwich theorem. Our variant of the polynomial partitioning theorem is proved by using our variant of the polynomial ham-sandwich theorem (i.e., Lemma 5.2). We now recall the statement of Theorem 4.2, and then prove it.

Theorem 4.2. *Let P be a set of n points in $\mathbb{R}^d$ and let $V \subset \mathbb{R}^d$ be an irreducible variety of dimension d' and degree D. Then there exists an r-partitioning polynomial g for P such that $g \notin I(V)$ and $\deg g = O(r^{1/d'})$. The implicit constant depends only on D and d.*

Proof. In this section, all logarithms will be to base 2. Let m_0 be the constant specified in Lemma 5.2. Let c_D denote the constant in the bound of Lemma 5.2 for the case $k < m_0$ and let c_1 be the constant hidden in the Ω-notation of the case $k \ge m_0$. Finally, let $c_2 = c_1/(1 - 1/2^{1/d'})$.

Let $I = I(V)$. We show that there exists a sequence of polynomials $g_0, g_1, g_2, \ldots$ with the following properties:

- $g_i \notin I$.
- For $0 \le i < \log m_0$, $\deg g_i \le i \cdot c_D$. For $i \ge \log m_0$, $\deg g_i \le c_D \log m_0 + c_2 2^{i/d'}$.
- Every connected component of $\mathbb{R}^d \setminus Z(g_i)$ contains at most $m/2^i$ points of P.

If we can find such a sequence of polynomials, we can complete the proof of the theorem by setting $t = \lceil \log r \rceil$ and taking $g = g_t$.

We prove the existence of $g_0, g_1, g_2, \ldots$ by induction. For the base case, let $g_0 = 1$. For $1 \le i < \log m_0$, by the induction hypothesis there exists a polynomial g_{i-1} of degree at most $(i-1)c_D$ such that every connected component of $\mathbb{R}^d \setminus Z(g_{i-1})$ contains at most $m/2^{i-1}$ points of P. Since $|P| = m$, the number of these connected components that contain more than $m/2^i$ points of P is smaller than 2^i. Let $S_1, \ldots, S_n \subset P$ be the subsets of P that are contained in each of these connected components (that is, $|S_i| > m/2^i$ for each i, and $n < 2^i$). By Lemma 5.2, there is a polynomial $h_{i-1} \notin I$ of degree smaller than c_0 that simultaneously bisects every S_i. We can set $g_i = g_{i-1} \cdot h_{i-1}$, since every connected component of $\mathbb{R}^d \setminus Z(g_{i-1} \cdot h_{i-1})$ contains at most $m/2^i$ points of P and $g_{i-1} \cdot h_{i-1}$ is a polynomial of degree smaller than ic_D. Moreover, since I is a prime ideal that does not contain g_{i-1} and h_{i-1}, it does not contain $g_{i-1} \cdot h_{i-1}$ either.

Next, we consider the case $\log m_0 \le i$, and analyze it similarly. That is, by the induction hypothesis there exists a polynomial $g_{i-1} \notin I$ of degree smaller than $\log m_0 c_D + c_2 2^{(i-1)/d'}$ such that every connected component of $\mathbb{R}^d \setminus Z(g_{i-1})$ contains at most $m/2^{i-1}$ points of P. Since $|P| = m$, the number of these connected components that contain more than $m/2^i$ points of P is smaller than 2^i. Let $S_1, \ldots, S_n \subset P$ be the subsets of P that are contained in each of these connected components (that is, $|S_i| > m/2^i$ for each i, and $n < 2^{i+1}$). By Lemma 5.2, there is a polynomial $h_{i-1} \notin I$ of degree smaller than $c_1 2^{i/d'}$ that simultaneously bisects every S_i. We can set $g_i = g_{i-1} \cdot h_{i-1}$, since every connected component of $\mathbb{R}^d \setminus Z(g_{i-1} \cdot h_{i-1})$ contains at most $m/2^i$ points of P. Moreover, $g_{i-1} \cdot h_{i-1}$ is a polynomial of degree smaller than

$$\begin{aligned} c_D \log m_0 + c_2 2^{(i-1)/d'} + c_1 2^{i/d'} &= c_D \log m_0 + 2^{i/d'} \left(\frac{c_2}{2^{1/d'}} + c_1 \right) \\ &= c_D \log m_0 + c_2 2^{i/d'}. \end{aligned}$$

This completes the induction step, and thus also the proof of the theorem. □

6. Applications

6.1. Incidences with algebraic varieties in $\mathbb{R}^d$

The following theorem is a variant of a well known incidence bound in the plane.

Theorem 6.1 (Pach and Sharir [36, 37]). *Let P be a set of m points and let Γ be a set of n constant-degree algebraic curves, both in $\mathbb{R}^2$, such that the incidence graph of $P \times \Gamma$ does not contain a copy of $K_{s,t}$. Then*

$$I(P, \Gamma) = O\big(m^{s/(2s-1)} n^{(2s-2)/(2s-1)} + m + n\big),$$

where the implicit constant depends on s, t, and the maximum degree of the curves.

While this bound is not tight for many cases, such as incidences with circles or with parabolas, it is the best known *general* incidence bound in $\mathbb{R}^2$. Building on the results in [12], Zahl introduced the following three-dimensional variant of this bound:.

Theorem 6.2 (Zahl [45]). *Let P be a set of m points and let $\mathcal{V}$ be a set of n smooth constant-degree algebraic varieties, both in $\mathbb{R}^3$, such that the incidence graph of $P \times \mathcal{V}$ does not contain a copy of $K_{s,t}$. Then*

$$I(P, \mathcal{V}) = O(m^{2s/(3s-1)} n^{(3s-3)/(3s-1)} + m + n),$$

where the implicit constant depends on s, t, and the maximum degree of the varieties.

Very recently, Basu and Sombra [6] obtained a similar bound in $\mathbb{R}^4$.

Theorem 6.3 (Basu and Sombra [6]). *Let P be a set of points and let $\mathcal{V}$ be a set of constant-degree algebraic varieties, both in $\mathbb{R}^4$, such that the incidence graph of $P \times \mathcal{V}$ does not contain a copy of $K_{s,t}$. Then*

$$I(P, \mathcal{V}) = O(|P|^{3s/(4s-1)} |\mathcal{V}|^{(4s-4)/(4s-1)} + |P| + |\mathcal{V}|),$$

where the implicit constant depends on s, t, and the maximum degree of the varieties.

When we look at the bounds of Theorems 6.1, 6.2, and 6.3, a pattern emerges. An easy variant of our technique in Sections 4 and 5 yields the bound coming from this pattern (up to an extra ε in the exponent).

Theorem 1.2. *Let P be a set of m points and let $\mathcal{V}$ be a set of n constant-degree algebraic varieties, both in $\mathbb{R}^d$, such that the incidence graph of $P \times \mathcal{V}$ does not contain a copy of $K_{s,t}$ (here we think of s, t, and d as being fixed constants, and m and n are large). Then for every $\varepsilon > 0$, we have*

$$I(P, \mathcal{V}) = O\big(m^{\frac{(d-1)s}{ds-1}+\varepsilon} n^{\frac{d(s-1)}{ds-1}} + m + n\big).$$

Theorem 1.2 improves upon a weaker bound that was obtained by Elekes and Szabó [16]. In addition to generalizing Theorems 6.1–6.3, Theorem 1.2 generalizes various other incidence bounds to $\mathbb{R}^d$ (again, up to an extra ε in the exponent). For example, Edelsbrunner, Guibas, and Sharir [15] considered point-plane incidences in $\mathbb{R}^3$, where no three points are collinear. Theorem 1.2 generalizes this result to $\mathbb{R}^d$, where no d points are contained in a common $(d-2)$-flat. A further generalization is to other types of hypersurfaces, such as spheres.

As shown in [39], when $s = 2$ Theorem 1.2 is tight up to subpolynomial factors. Specifically, [39] presents lower bounds for the cases of hyperplanes, hyperspheres, and paraboloids with no $K_{2,t}$ in the incidence graph.

Proof sketch of Theorem 1.2. The proof is very similar to the proof of Theorem 4.3, so here we only explain how to change the original proof. As with Theorem 4.3, Theorem 1.2 comes from the following generalization.

Theorem 6.4. *Let P be a set of points and let $\mathcal{V}$ be a set of constant-degree algebraic varieties, both in $\mathbb{R}^d$, with $|P| = m$, $|\mathcal{V}| = n$, such that the incidence graph of $P \times \mathcal{V}$ does not contain a copy of $K_{s,t}$. Suppose that P is fully contained in an irreducible variety V*

of dimension e and degree D. Suppose furthermore that no surface $S \in \mathcal{V}$ contains V. Then for every $\varepsilon > 0$, we have

$$I(P, \mathcal{V}) \le \alpha_{1,e} m^{\frac{(e-1)s}{es-1}+\varepsilon} n^{\frac{e(s-1)}{es-1}} + \alpha_{2,e}(m+n), \tag{20}$$

where $\alpha_{1,e}$ and $\alpha_{2,e}$ are constants that depend on ε, d, e, s, t, and D.

The proof of Theorem 6.4 parallels that of Theorem 4.3. As in the proof of Theorem 4.3, we will induct both on e (the dimension of the variety V) and on the quantity $m+n$. As before, we can be very wasteful when we induct on e, but we must be more efficient when we induct on $m+n$.

As before, we find an r-partitioning polynomial f. The main difference in the proofs is that we replace Corollary 2.3 with the Kővári–Sós–Turán theorem (see, e.g., [31, Section 4.5]). This allows us to still have (9), but not (10). To overcome this difficulty, we first change the way that the incidences are partitioned into three subsets I_1, I_2, I_3:

- I_1 consists of the incidences $(p, S) \in P \times \mathcal{V}$ such that $p \in Z(f)$ and S properly intersects every irreducible component of $V \cap Z(f)$ that contains p.
- I_2 consists of the incidences $(p, S) \in P \times \mathcal{V}$ such that p is contained in an irreducible component of $V \cap Z(f)$ that is fully contained in S.
- $I_3 = I(P, Q) \setminus \{I_1 \cup I_2\}$. This is the set of incidences $(p, S) \in P \times \mathcal{V}$ such that p is not contained in $V \cap Z(f)$.

Let $V' = V \cap Z(f)$ and let $m_0 = |P \cap V'|$. To bound I_1, we argue as in the proof of Theorem 4.3. This is an incidence problem on the variety $V' = V \cap Z(f)$, which has dimension at most $e-1$. Arguing as in Theorem 4.3, we obtain the bound

$$I_1 \le C\alpha_{1,e-1} m_0^{\frac{(e-2)s}{(e-1)s-1}+\varepsilon} n^{\frac{(e-1)(s-1)}{(e-1)s-1}} + \alpha_{2,e-1}(m_0+n),$$

where the constant C depends on D, d, e, s, t and the degree of f. A computation analogous to (12) shows that this is at most

$$\frac{\alpha_{1,e}}{3} m^{\frac{(e-1)s}{es-1}+\varepsilon} n^{\frac{e(s-1)}{es-1}} + \frac{\alpha_{2,e}}{2} m_0, \tag{21}$$

provided $\alpha_{1,e}$ and $\alpha_{2,e}$ are chosen sufficiently large.

Let $m' = m - m_0$. The proof bounding I_3 proceeds exactly as the proof in Theorem 4.3, and we obtain the bound

$$I_3 \le \frac{\alpha_{1,e}}{3} m^{\frac{(e-1)s}{es-1}+\varepsilon} n^{\frac{e(s-1)}{es-1}} + \alpha_{2,e} m', \tag{22}$$

which is the analogue of (15). Note that (22) should also include the term $\alpha_{2,e} n$, but (9) allows us to combine this with the first term in (22).

It remains to bound I_2. As in the discussion in Theorem 4.3 for bounding the quantity I_1, note that V' can be written as a union of γ_1 irreducible (over $\mathbb{R}$) components, each of dimension at most $e-1$, where γ_1 depends only on D, e, d and the degree of f. Since the incidence graph of $P \times \mathcal{V}$ does not contain a copy of $K_{s,t}$, each of the irreducible components of V' either contains at most s points, or is contained in at most t surfaces from $\mathcal{V}$. The contribution from the first quantity is at most $s\gamma_1 n$, and the contribution from the second quantity is at most tm_0. Combining these bounds with (9), we have

$$I_2 \leq \frac{\alpha_{1,e}}{3} m^{\frac{(e-1)s}{es-1}+\varepsilon} n^{\frac{e(s-1)}{es-1}} + \frac{\alpha_{2,e}}{2} m_0, \tag{23}$$

provided we select $\alpha_{2,e} \geq 2t$. Combining (21)–(23) gives us (20), which establishes Theorem 6.4 and in turn Theorem 1.2. □

6.2. *Unit distances in* $\mathbb{R}^d$

For a finite set $P \subset \mathbb{R}^d$, we define *the number of unit distances that are spanned by* P as the number of pairs $p, q \in P^2$ such that $|p-q| = 1$ (where $|p-q|$ denotes the Euclidean distance between p and q). Let $f_d(n)$ denote the maximum number of unit distances that can be spanned by a set of n points in $\mathbb{R}^d$. The unit distances problem, first posed by Erdős [17, 18], asks for the asymptotic behavior of $f_2(n)$ and $f_3(n)$. Currently, the best known bounds are $f_2(n) = O(n^{4/3})$ [41], $f_2(n) = n^{1+\Omega(1/\log\log n)}$ [17], $f_3(n) = O(n^{3/2})$ [23, 45], and $f_3(n) = \Omega(n^{4/3}\log\log n)$ [18]. For any $d \geq 4$, we have the trivial bound $f_d(n) = \Theta(n^2)$ (see, e.g., [27]). For example, in $\mathbb{R}^4$, let P_1 be a set of $n/2$ points arranged on the circle $x_1^2 + x_2^2 = 1/2$, and let P_2 be a set of $n/2$ points arranged on the circle $x_3^3 + x_4^2 = 1/2$. Then, if $P = P_1 \cup P_2$, the set P has at least $n^2/4$ unit distances.

The problem in $d \geq 4$ becomes non-trivial once we consider only point sets with some restriction on them. Oberlin and Oberlin [34] obtain an improved upper bound under a natural restriction, as follows.

Theorem 6.5 ([34]). *Let $d \geq 2$ and consider an n-point set $P \subset \mathbb{R}^d$ such that no d-element subset of P is contained in a $(d-2)$-flat. Then the number of unit distances that are spanned by P is $O(|P|^{(2d-1)/d})$.*

We now improve Theorem 6.5 by applying Theorem 1.1. First, we show that the configuration described above is essentially the only one that yields $\Theta(n^2)$ unit distances in $\mathbb{R}^4$. Call two circles (C_1, C_2) a *pair of orthogonal circles of radius* $1/\sqrt{2}$ if (after a translation and rotation) they are the two circles $x_1^2 + x_2^2 = 1/2,\ x_3^2 + x_4^2 = 1/2$.

Theorem 6.6 (Unit distances in $\mathbb{R}^4$). *Let P be a set of n points in $\mathbb{R}^4$ such that for any pair of orthogonal circles of radius $1/\sqrt{2}$, one of the circles contains fewer than k points (for some constant k). Then, for any $\varepsilon > 0$, the number of unit distances spanned by P is $O(n^{8/5+\varepsilon})$.*

Proof. Consider the bipartite graph whose vertex set consists of two copies of P, and where an edge (p, q) exists if and only if $|p-q| = 1$. This is a semi-algebraic bipartite graph in $(\mathbb{R}^4, \mathbb{R}^4)$. If we can also show that this graph contains no copy of $K_{k,k}$, then by Theorem 1.1 the number of edges (i.e., the number of unit distances) is as stated in the theorem.

Without loss of generality, we may assume that $k > 2$. We assume, for contradiction, that there exist two collections of points $p_1, \ldots, p_k \subset P$ and $q_1, \ldots, q_k \subset P$ such that

$|p_i - q_j| = 1$ for all indices $1 \le i, j \le k$. That is, if S_i denotes the unit sphere centered at p_i, then $q_1, \dots, q_k \in \bigcap_{i=1}^k S_i$, which implies $|\bigcap_{i=1}^k S_i| > 2$. The intersection of at least three unit hyperspheres cannot be a two-dimensional sphere. Moreover, if such an intersection is zero-dimensional, then it consists of at most two points. Therefore, $\bigcap_{i=1}^k S_i$ must be a circle. Similarly, if S_i' denotes the unit sphere centered at q_i, then $\bigcap_{i=1}^k S_i'$ must be a circle that contains $p_1, \dots, p_k$. Elementary geometry then shows that these two circles must be a pair of orthogonal circles of radius $1/\sqrt{2}$, which contradicts the assumption concerning such circles, and thus completes the proof. □

Notice that Theorem 6.6 implies a better bound than Theorem 6.5, while also relying on a weaker assumption. We now present a general bound for any d, where we have a similar assumption to the one in Theorem 6.6, though with an improved bound.

Theorem 6.7 (Unit distances in $\mathbb{R}^d$). *Let P be a set of n points in $\mathbb{R}^d$ such that every $(d-3)$-dimensional sphere contains fewer than k points (for some constant k). Then, for any $\varepsilon > 0$, the number of unit distances spanned by P is $O(n^{2d/(d+1)+\varepsilon})$.*

Proof. As before, we consider the semi-algebraic bipartite graph whose vertex set consists of two copies of P, and where an edge (p, q) exists if and only if $|p - q| = 1$. This time, this graph is in $(\mathbb{R}^d, \mathbb{R}^d)$. If we can show that this graph contains no copy of $K_{k,k}$, then Theorem 1.1 would imply that the number of edges (i.e., the number of unit distances) is as stated in the theorem.

Without loss of generality, we may assume that $k > 2$. We assume, for contradiction, that there exist two collections of points $p_1, p_2, p_3 \subset P$ and $q_1, \dots, q_k \subset P$ such that $|p_i - q_j| = 1$ for all indices $1 \le i \le 3$ and $1 \le j \le k$. That is, if S_i denotes the unit hypersphere centered at p_i, then $q_1, \dots, q_k \in \bigcap_{i=1}^3 S_i$. The intersection $S_1 \cap S_2$ is fully contained in the perpendicular bisector π_{12} of p_1 and p_2, and the intersection $S_1 \cap S_3$ is fully contained in the perpendicular bisector π_{13} of p_1 and p_3. Since $\pi_{12} \ne \pi_{13}$, the intersection of these two hyperplanes is a $(d-2)$-dimensional flat, and thus $q_1, \dots, q_k \subset \pi_{12} \cap \pi_{13} \cap S_1$. This intersection is a $(d-3)$-dimensional sphere, contradicting the assumption of the theorem. □

6.3. *Incidences between points and tubes*

As an immediate corollary to Theorem 1.1, we establish the following bound on the number of incidences between points and tubes in $\mathbb{R}^d$ (where a *tube* is the set of all points of distance at most δ from a given line). Notice that the set of tubes in $\mathbb{R}^d$ can be parameterized using $2d - 2$ parameters.

Corollary 6.8. *For $\delta > 0$, let P and Σ be a set of m points and n tubes in $\mathbb{R}^d$ respectively, such that each tube has a radius of δ. If the incidence graph contains no copy of $K_{k,k}$, then for any $\varepsilon > 0$ we have*

$$I(P, \Sigma) = O\Big(m^{\frac{(2d-2)(d-1)}{d(2d-2)-1}+\varepsilon} n^{\frac{d(2d-3)}{d(2d-2)-1}} + m + n\Big).$$

In the planar case, we get $O(m^{2/3}n^{2/3} + m + n)$ incidences between points and strips.

6.4. *Incidences with k-dimensional families of varieties*

For each integer $D \geq 0$, let $\mathbb{R}[x_1, \ldots, x_d]_{\leq D}$ be the vector space of polynomials of degree at most D. As in Section 5.1, $\mathbb{R}[x_1, \ldots, x_d]_{\leq D}$ can be identified with $\mathbb{RP}^{\binom{d+D}{D}}$ (here, as in Section 5.1, we identify a polynomial f with all polynomials of the form cf with $c \in \mathbb{R}\backslash\{0\}$). If $\mathcal{M} \subset \mathbb{RP}^{\binom{d+D}{D}}$, we say that the polynomial f is an element of $\mathcal{M}$ if the equivalence class of f (where as above f is identified with cf, $c \in \mathbb{R}\backslash\{0\}$) is an element of $\mathcal{M}$.

Recently, Wang, Yang, and Zhang [44] derived the following result (our formulation is somewhat different from the one in [44]).

Theorem 6.9. *Let $\mathcal{M} \subset \mathbb{RP}^{\binom{D+2}{2}}$ be an algebraic variety of dimension k. Let P be a set of m points in the plane, and let $\Gamma \subset \mathcal{M}$ be a set of n plane curves of degree at most D, each of which is the zero-set of a polynomial that lies in $\mathcal{M}$. Suppose that no two curves of Γ share a common component. Then*

$$I(P, \Gamma) = O(m^{k/(2k-1)} n^{(2k-2)/(2k-1)} + m + n).$$

The implicit constant depends on k and $\mathcal{M}$.

For example, assume that Γ is a set of curves of degree at most D. By Bézout's theorem (see, e.g., [14]), two such curves can intersect in at most D^2 points, so we can apply Theorem 6.1 with $k = D^2 + 1$. However, since a bivariate polynomial of degree D has at most $\binom{D+2}{2} = (D+2)(D+1)/2$ monomials, we obtain an improved bound by applying Theorem 6.9 with $\mathcal{M} = \mathbb{RP}^{\binom{D+2}{2}}$ and $k = (D+2)(D+1)/2$.

Using the techniques developed in this paper, we can extend this result to higher dimensions (though we have an ε loss in the exponents).

Theorem 6.10. *Fix integers d and D. Let $\mathcal{M} \subset \mathbb{RP}^{\binom{D+d}{d}}$ be an algebraic variety of dimension k. Let P be a set of m points in $\mathbb{R}^d$, and let $\mathcal{V}$ be a set of n algebraic varieties in $\mathbb{R}^k$, each of which is the zero-set of some polynomial of degree at most D that lies in $\mathcal{M}$. Suppose that the incidence graph contains no copy of $K_{s,s}$ for some constant s. Then*

$$I(P, \mathcal{V}) = O\big(m^{\frac{k(d-1)}{(dk-1)}+\varepsilon} n^{\frac{d(k-1)}{(dk-1)}} + m + n\big), \tag{24}$$

where the implicit constant depends on $\mathcal{M}$, D, d, and s.

Proof. We need to rephrase this problem in a form that can be addressed by Theorem 4.3. The idea is that we will find a variety $V \subset \mathbb{R}^{\binom{D+d}{d}}$ of dimension k, a collection $Q \subset V$ of n points lying on V, and a collection $\mathcal{Z}$ of m bounded-degree algebraic varieties in $\mathbb{R}^{\binom{D+d}{d}}$ such that the incidence graph of $Q \times \mathcal{Z}$ has no $K_{s,s}$, and $I(Q, \mathcal{Z}) = I(P, \mathcal{V})$.

To each variety $S \in \mathcal{V}$, we can associate an affine polynomial $f_S \in \mathcal{M}$. Without loss of generality, we can assume that none of these polynomials lies on the hyperplane $\{x_0 = 0\}$ (we can always apply a linear change of coordinates to guarantee that this is the case). Let $\tilde{S} \subset \mathbb{R}[x_1, \ldots, x_d]$ be the dehomogenization of S with respect to the coordinate chart $\{x_0 \neq 0\} \subset \mathbb{R}^{\binom{D+d}{d}}$.

For each $p \in \mathbb{R}^d$, let V_p be the dehomogenization of the (homogeneous) variety $\{f \in \mathbb{R}[x_1, \dots, x_d]_{\le D} = \mathbb{R}^{\binom{D+d}{d}} : f(p) = 0\}$ with respect to the coordinate chart $\{x_0 \neq 0\} \subset \mathbb{R}^{\binom{D+d}{d}}$. Then $f_S \in V_p$ if and only if $p \in S$. Furthermore, if $Q = \{f_S : S \in \mathcal{V}\}$ and $\mathcal{Z} = \{V_p : p \in P\}$, then $Q \subset \tilde{\mathcal{M}}$, and the incidence graph of $Q \times \mathcal{Z}$ is the same as the incidence graph of $\mathcal{V} \times P$. In particular, this implies that the former incidence graph contains no $K_{s,s}$, and $I(Q, \mathcal{Z}) = I(P, \mathcal{V})$.

We may now apply Theorem 4.3 to conclude that

$$I(P, \mathcal{V}) = O\Big(m^{\frac{k(d-1)}{(dk-1)}} n^{\frac{d(k-1)}{(dk-1)} + \frac{\varepsilon}{k}} + m + n\Big). \tag{25}$$

Now, if $n > m^k$ then Theorem 6.10 follows immediately from Corollary 2.3. If $n \le m^k$, then (25) implies (24). In either case, Theorem 6.10 is proved. □

7. Discussion

The main open question that arises from this work seems to be whether Theorem 1.1 is tight. The only lower bounds that we are aware of arise from incidence problems with algebraic objects (mainly point-hyperplane incidence problems). These lead to a tight bound only for the case where $d_1 = d_2 = 2$. In any other case, it would be rather interesting to find configurations of semi-algebraic objects that yield an asymptotically larger number of incidences. It is also possible that more sophisticated configurations of algebraic objects (possibly even hyperplanes) suffice to obtain tight bounds for other values of d_1 and d_2. However, for some incidence problems (such as point-circle incidences in $\mathbb{R}^2$, which corresponds to $d_1 = 2$ and $d_2 = 3$) better bounds are known than the ones implied by Theorem 1.1. This might hint that Theorem 1.1 is not tight for various values of d_1 and d_2.

On the other hand, Theorem 2.1 is tight for $m = n$, as implied by the constructions from Kollár, Rónyai, and Szabó [25] and Alon, Rónyai, and Szabó [3]. These works prove, for each fixed d, the existence of bipartite graphs with both parts of size n, $\Omega(n^{2-1/d})$ edges, and no copy of $K_{d,t}$ for $t = (d-1)! + 1$. It is not hard to verify that such graphs satisfy $\pi_{\mathcal{F}}(z) \le cz^d$ for all z. Indeed, for each d-set, there are at most $t - 1$ vertices whose neighborhood is a superset of that d-set, and each vertex with D neighbors in a set of size z gives rise to $\binom{D}{d}$ subsets of size d. This implies that if G is $K_{d,t}$-free, then $\pi_{\mathcal{F}}(z)$ is at most $(t-1)\binom{z}{d} + \binom{z}{d-1} + \cdots + \binom{z}{0} \le cz^d$ for an appropriate choice of $c = c(d, t)$. A more precise bound, showing that the worst case is given by considering the projections being sets of size at most d, with an additional $(t-2)\binom{z}{d}/(d+1)$ projections of size $d+1$, gives an upper bound of $\pi_{\mathcal{F}}(z) \le \frac{d+t-1}{d+1}\binom{z}{d} + \binom{z}{d-1} + \binom{z}{d-2} + \cdots + \binom{z}{0}$.

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