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Classical solutions and higher regularity for nonlinear fractional diffusion equations

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Abstract. We study the regularity properties of the solutions to the nonlinear equation with fractional diffusion,

$$\partial_t u + (-\Delta)^{\sigma/2}\varphi(u) = 0,$$

for $x \in \mathbb{R}^N$ and $t > 0$, with $0 < \sigma < 2$ and $N \geq 1$. If the nonlinearity satisfies some not very restrictive conditions: $\varphi \in C^{1,\gamma}(\mathbb{R})$ with $1 + \gamma > \sigma$, and $\varphi'(u) > 0$ for every $u \in \mathbb{R}$, we prove that bounded weak solutions are classical solutions for all positive times. We also explore sufficient conditions on the nonlinearity to obtain higher regularity for the solutions, even C^∞ regularity. Degenerate and singular cases, including the power nonlinearity $\varphi(u) = |u|^{m-1}u$, $m > 0$, are also considered, and the existence of positive classical solutions until the possible extinction time if $m < \frac{N-\sigma}{N}$, $N > \sigma$, and for all times otherwise, is proved.

Keywords. Nonlinear fractional diffusion, nonlocal diffusion operators, classical solutions, optimal regularity

1. Introduction

This paper is devoted to establishing the regularity of bounded weak solutions for the nonlinear parabolic equation involving fractional diffusion,

$$\partial_t u + (-\Delta)^{\sigma/2}\varphi(u) = 0 \quad \text{in } Q = \mathbb{R}^N \times (0, \infty). \tag{1.1}$$

Here $(-\Delta)^{\sigma/2} = \mathcal{F}^{-1}(|\cdot|^\sigma \mathcal{F})$, where $\mathcal{F}$ denotes Fourier transform, is the usual fractional Laplacian with $0 < \sigma < 2$ and $N \geq 1$. The constitutive function φ is assumed to be at least continuous and nondecreasing. Further conditions will be introduced as needed.

The existence of a unique weak solution to the Cauchy problem for equation (1.1) has been fully investigated in [18, 19] for the case where φ is a positive power. The solution

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in that case is in fact bounded for positive times even if the initial data are not, provided they are in a suitable integrability space. The theory can be easily extended to the case of more general functions φ; see Section 8 at the end of the paper for a brief survey on the existence and uniqueness theory under the present assumptions.

When $\varphi(u) = u$ the equation is the so-called fractional heat equation, which has been studied in a number of papers, mainly coming from probability. Explicit representation with a kernel allows us to show in this case that solutions are C^∞ and bounded for every $t > 0$, $x \in \mathbb{R}^N$, under the assumption that the initial data are integrable. In the nonlinear case such a representation is not available. Nevertheless, we will still be able to deduce that bounded weak solutions are smooth if the equation is "uniformly parabolic", $0 < c \le \varphi'(u) \le C < \infty$.

Classical solutions. Our first result establishes that if the nonlinearity is smooth enough, compared to the order of the equation, $\max\{1, \sigma\}$, then bounded weak solutions are classical solutions, in the sense that all terms appearing in the statement of the equation are continuous functions.

Theorem 1.1. *Let u be a bounded weak solution to (1.1), and assume $\varphi \in C^{1,\gamma}(\mathbb{R})$ for some $0 < \gamma < 1$, and $\varphi'(s) > 0$ for every $s \in \mathbb{R}$. If $1+\gamma > \sigma$, then $\partial_t u$ and $(-\Delta)^{\sigma/2}\varphi(u)$ are Hölder continuous functions and (1.1) is satisfied everywhere.*

The precise regularity of the solution is determined by the regularity of the nonlinearity φ; see Section 5 for the details. Notice that the condition $\varphi' > 0$ together with the boundedness of u implies that the equation is uniformly parabolic.

The idea of the proof is as follows: thanks to the results of Athanasopoulos and Caffarelli [2], we already know that bounded weak solutions are C^α regular for some $\alpha \in (0, 1)$. In order to improve this regularity we write (1.1) as a fractional linear heat equation with a source term. This term is in principle not very smooth, but it has some good properties. To be precise, given $(x_0, t_0) \in Q$, we have

$$\partial_t u + (-\Delta)^{\sigma/2} u = (-\Delta)^{\sigma/2} f, \tag{1.2}$$

where

$$f(x,t) := u(x,t) - \frac{\varphi(u(x,t))}{\varphi'(u(x_0,t_0))},$$

after the time rescaling $t \to t/\varphi'(u(x_0, t_0))$. It turns out, as we will prove in Sections 4 and 5, that solutions to the linear equation (1.2) have the same regularity as f.

Next, using the nonlinearity we observe that f in the actual right-hand side is more regular than u near (x_0, t_0) (see (4.1)). Once we have arrived at this point, we are in a situation that is somewhat similar to the one considered by Caffarelli and Vasseur in [8, Appendix B]. There, they prove $C^{1,\alpha}$ regularity for an equation, motivated by the study of geostrophic equations, of the form

$$\partial_t u + (-\Delta)^{1/2} u = \operatorname{div}(\mathbf{v}u),$$

where $\mathbf{v}$ is a divergence free vector. Comparing with (1.2), we see two differences: in their case $\sigma = 1$, and the source term is local. These two differences will significantly complicate our analysis. We want to stress that the main difficulty in [8] was proving that the solution is C^α. In our case this initial regularity is already available, and the demanding part is to gain higher regularity.

In order to obtain the above-mentioned regularity for the solutions u to (1.2), we will use the fact that they are given by the representation formula

$$u(x,t) = \int_{\mathbb{R}^N} P_\sigma(x-\overline{x},t)u(\overline{x},0)\,d\overline{x} + \int_0^t \int_{\mathbb{R}^N} (-\Delta)^{\sigma/2} P_\sigma(x-\overline{x},t-\overline{t}) f(\overline{x},\overline{t})\,d\overline{x}\,d\overline{t}, \qquad (x,t)\in Q,$$

where P_σ is the kernel of the σ-fractional linear heat equation; see Section 3 for a proof of this fact, which falls into the linear theory. Therefore, we are led to study the singular kernel $A_\sigma(x,t) := (-\Delta)^{\sigma/2}P_\sigma(x,t)$. Unfortunately, P_σ, and hence A_σ, is only explicit when $\sigma = 1$. However, using the self-similar structure of P_σ, we will be able to obtain the required estimates and cancellation properties for A_σ (see Section 2).

Singular and degenerate equations. The hypotheses made in Theorem 1.1 exclude all the powers $\varphi(u) = |u|^{m-1}u$ for $m > 0$, $m \neq 1$, since they are degenerate ($m > 1$) or singular ($m < 1$) at the level $u = 0$. Nevertheless, a close look at our proof shows that we may in fact get a result that is local in some sense: weak solutions defined in the whole space that are globally bounded and C^α are indeed classical solutions (in the sense mentioned above) in any set where the equation is uniformly parabolic. This is carefully stated in Theorem 7.1 which applies to more general nonlinearities φ. To be clear, in the case of the power nonlinearities we may even consider signed solutions and the result ensures that they are indeed classical solutions in the open sets where they are strictly positive (or strictly negative). It is to be noted that local regularity results are not to be expected for nonlocal equations unless appropriate assumptions are made on the far-away behaviour. Our mild hypotheses do precisely that.

Higher regularity. If φ is C^∞ we prove that solutions are C^∞. The result will be a consequence of the regularity already provided by Theorem 1.1 plus a result for linear equations with variable coefficients, Theorem 6.1, which is of independent interest. The case $\sigma < 1$ is a little more involved since we first have to raise the regularity in space exponent from σ to 1. See more in Section 6, where higher regularity results depending on the smoothness of φ are given.

Theorem 1.2. *Let u be a bounded weak solution to equation* (1.1). *If $\varphi \in C^\infty(\mathbb{R})$ and $\varphi' > 0$ in $\mathbb{R}$, then $u \in C^\infty(Q)$.*

As a direct precedent of the present work, let us mention the paper [20], where we consider the nonlinearity $\varphi(u) = \log(1+u)$ in the case $\sigma = 1 = N$, and prove that solutions with initial data in some $L \log L$ space become instantaneously bounded and C^∞. Notice that in this case $\varphi'(u) = 1/(1+u)$, and hence the equation is uniformly parabolic.

We expect some of these ideas to have wider applicability. We point out several possible extensions, together with some comments and applications of (1.1), in Section 9.

2. Kernel properties

In this section we consider two issues for the kernel $A_\sigma = (-\Delta)^{\sigma/2} P_\sigma$, which play an important role in the study of regularity, namely some estimates and a cancellation property. Before doing this, it will be convenient to introduce a certain Hölder space adapted to equation (1.1), together with appropriate notation. For simplicity, we will omit the subscript σ in what follows when no confusion arises. It will also be convenient to use the notation $Y = (x, t) \in \mathbb{R}^{N+1}$.

The kernel P has as Fourier transform $\widehat{P}(\xi, t) = e^{-|\xi|^\sigma t}$. Therefore, it has the self-similar form

$$P(x,t) = t^{-N/\sigma}\Phi(z), \qquad z = xt^{-1/\sigma} \in \mathbb{R}^N,\ t > 0. \tag{2.1}$$

Moreover, the profile Φ is a C^∞ positive, radially decreasing function $\Phi(z) = \widetilde{\Phi}(|z|)$, satisfying $\widetilde{\Phi}(s) \sim s^{-N-\sigma}$ for $s \to \infty$ (see [4]). We will exploit all these properties in what follows.

The σ-distance and the associated Hölder space. The self-similar structure of P motivates the use of the *σ-parabolic "distance"* $|Y_1 - Y_2|_\sigma$, where

$$|Y|_\sigma := (|x|^2 + |t|^{2/\sigma})^{1/2} = t^{1/\sigma}(|z|^2 + 1)^{1/2}.$$

This is not really a distance unless $\sigma \ge 1$, since the triangle inequality does not hold if $\sigma < 1$. However it is a quasimetric, with relaxed triangle inequality

$$|Y_1 - Y_3|_\sigma \le 2^{(1-\sigma)_+/\sigma}(|Y_1 - Y_2|_\sigma + |Y_2 - Y_3|_\sigma). \tag{2.2}$$

This will be enough for our purposes.

Observe the relation between the standard Euclidean distance and this σ-parabolic distance:

$$|Y| \le c|Y|_\sigma^\nu \quad \text{for every } |Y| \le 1, \quad \nu := \min\{1, \sigma\}. \tag{2.3}$$

The σ-parabolic ball is defined as $B_R := \{Y \in \mathbb{R}^{N+1} : |Y|_\sigma < R\}$. Performing the change of variables

$$s = |x|\,|t|^{-1/\sigma}, \qquad r = (|x|^2 + |t|^{2/\sigma})^{1/2}, \tag{2.4}$$

we get, for all $\delta > -N - \sigma$,

$$\int_{B_R} |Y|_\sigma^\delta \, dY = 2\sigma N\omega_N \int_0^R r^{\delta+N+\sigma-1}\, dr \int_0^\infty \frac{s^{N-1}}{(1+s^2)^{(N+\sigma)/2}}\, ds = cR^{\delta+N+\sigma}.$$

In particular, the volume of the ball B_R is proportional to $R^{N+\sigma}$. In the same way, $\int_{B_R^c} |Y|_\sigma^{-\delta}\, dY = cR^{-\delta+N+\sigma}$ for every $\delta > N + \sigma$.

The Hölder space $C_\sigma^\alpha(Q)$, $\alpha \in (0, \nu)$, consists of all functions u defined in Q such that for some constant $c > 0$,

$$|u(Y_1) - u(Y_2)| \le c|Y_1 - Y_2|_\sigma^\alpha \quad \text{for every } Y_1, Y_2 \in Q.$$

The estimates. Using formula (2.1) we deduce that $A = (-\Delta)^{\sigma/2} P$ has the self-similar expression

$$A(x,t) = t^{-1-N/\sigma}\Psi(z),$$

where $z = xt^{-1/\sigma}$ and $\Psi(z) = (-\Delta)^{\sigma/2}\Phi(z)$. This is the basis for the estimates.

Proposition 2.1. *For every $Y \in Q$ the kernel A satisfies*

$$|A(Y)| \le \frac{c}{|Y|_\sigma^{N+\sigma}}, \qquad |\partial_t A(Y)| \le \frac{c}{|Y|_\sigma^{N+2\sigma}}, \qquad |\nabla_x A(Y)| \le \frac{c}{|Y|_\sigma^{N+\sigma+1}}. \tag{2.5}$$

Proof. We observe that $\widehat{\Phi}(\xi) = e^{-|\xi|^\sigma}$, hence $\widehat{\Psi}(\xi) = |\xi|^\sigma e^{-|\xi|^\sigma}$. Using the expression of the inverse Fourier transform of a radial function we get

$$\Psi(z) = \widetilde{\Psi}(|z|) = c_N |z|^{1-N/2}\int_0^\infty e^{-r^\sigma} r^{N/2+\sigma} J_{(N-2)/2}(r|z|)\,dr,$$

where J_α denotes the Bessel function of the first kind of order α. This yields the decay $|\widetilde{\Psi}(s)| = O(s^{-N-\sigma})$ for s large [21, Lemma 1]. Since $\widetilde{\Psi}$ is bounded, we have

$$|\widetilde{\Psi}(|z|)| = |(-\Delta)^{\sigma/2}\Phi(z)| \le c(1+|z|^2)^{-(N+\sigma)/2}, \tag{2.6}$$

which implies

$$|A(Y)| \le \frac{c}{t^{1+N/\sigma}(1+|xt^{-1/\sigma}|^2)^{(N+\sigma)/2}} = \frac{c}{|Y|_\sigma^{N+\sigma}}.$$

The estimate for the time derivative is a consequence of

$$\partial_t A(Y) = -(-\Delta)^\sigma P(x,t) = -t^{-N/\sigma-2}(-\Delta)^\sigma\Phi(z), \tag{2.7}$$

which follows immediately from the equation satisfied by P, and (2.6). Indeed,

$$|\partial_t A(Y)| \le \frac{c}{t^{N/\sigma+2}(1+|xt^{-1/\sigma}|^2)^{(N+2\sigma)/2}} = \frac{c}{|Y|_\sigma^{N+2\sigma}}.$$

In order to estimate the spatial derivative $\nabla_x A(Y)$, we consider the equation relating the profiles Φ and Ψ,

$$\sigma(-\Delta)^\sigma\Phi(z) - (N+\sigma)\Psi(z) - z\cdot\nabla\Psi(z) = 0,$$

which follows from (2.7). It implies that

$$|\nabla\Psi(z)| \le \frac{c}{|z|}\big(|\Psi(z)| + |(-\Delta)^\sigma\Phi(z)|\big).$$

Since $\nabla\Psi$ is bounded, we deduce that $|\widetilde{\Psi}'(s)| \le c(1+s^2)^{-(N+\sigma+1)/2}$. Finally,

$$|\nabla_x A(Y)| = t^{-1-(N+1)/\sigma}|\widetilde{\Psi}'(s)| \le \frac{c}{t^{1+(N+1)/\sigma}(1+|xt^{-1/\sigma}|^2)^{(N+\sigma+1)/2}} = \frac{c}{|Y|_\sigma^{N+\sigma+1}}. \qquad \square$$

Let us point out that further derivatives may be estimated in a similar way.

Cancellation. We now show that the function A has zero integral in the sense of principal value adapted to the self-similar variables: we take out a small σ-ball and integrate, and then pass to the limit.

Proposition 2.2. *For every $R > \varepsilon > 0$,*

$$\int_{B_R^+ \setminus B_\varepsilon} A(x,t)\,dx\,dt = \int_{B_R^- \setminus B_\varepsilon} A(x,t)\,dx\,dt = 0 \tag{2.8}$$

where $B_R^+ = B_R \cap \{t > 0\}$ and $B_R^- = B_R \cap \{t < 0\}$.

Proof. From the equation for the profile Φ,

$$\sigma(-\Delta)^{\sigma/2}\Phi(z) - N\Phi(z) - z \cdot \nabla\Phi(z) = 0,$$

we get an alternative expression for the profile of A,

$$\widetilde{\Psi}(s) = \frac{1}{\sigma}(N\widetilde{\Phi}(s) + s\widetilde{\Phi}'(s)) = \frac{s^{1-N}}{\sigma}(s^N\widetilde{\Phi}(s))'.$$

Hence, using the change of variables (2.4) and the behaviour of $\widetilde{\Phi}$ at infinity, we get

$$\begin{aligned}\int_{B_R^\pm \setminus B_\varepsilon} A(x,t)\,dx\,dt &= N\omega_N\sigma \int_\varepsilon^R \int_0^\infty \frac{s^{N-1}\widetilde{\Psi}(s)}{r}\,ds\,dr \\ &= N\omega_N \log(R/\varepsilon)(s^N\widetilde{\Phi}(s))\big|_{s=0}^\infty = 0. \qquad \square\end{aligned}$$

3. The linear problem

As we have said in the Introduction, the solution u to equation (1.1) will be analyzed by writing it as a solution of a linear problem with a particular right-hand side. This leads to the representation of u by means of a variation of constants formula. We give a proof of this independent fact and then proceed to establish the regularity of the linear problem.

3.1. A representation formula

Let us consider the Cauchy problem associated to the fractional linear heat equation with a source term,

$$\begin{cases} \partial_t u + (-\Delta)^{\sigma/2}u = (-\Delta)^{\sigma/2} f & \text{in } Q, \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^N. \end{cases} \tag{3.1}$$

We assume that $f \in L^\infty_{\mathrm{loc}}((0,\infty) : L^1(\mathbb{R}^N, \rho\,dx))$ with $\rho(x) = (1+|x|)^{-(N+\sigma)}$. We define a *very weak solution* to problem (3.1) as a function $u \in C([0,\infty) : L^1(\mathbb{R}^N, \rho\,dx))$ such that

$$\iint_Q u(x,t)\partial_t\zeta(x,t)\,dx\,dt = \int_Q (u-f)(x,t)(-\Delta)^{\sigma/2}\zeta(x,t)\,dx\,dt$$

for all $\zeta \in C_c^\infty(Q)$, and $u(\cdot,0) = u_0$ almost everywhere.

Theorem 3.1. *If f belongs to $L^\infty_{loc}([0,\infty) : L^1(\mathbb{R}^N, \rho\, dx))$ and is locally Hölder, and $u_0 \in L^p(\mathbb{R}^N)$ for some $1 \le p \le \infty$, then there is a unique very weak solution of problem (3.1), which is given by Duhamel's formula:*

$$u(x,t) = \int_{\mathbb{R}^N} P(x-\overline{x},t)u_0(\overline{x})\,d\overline{x} + \int_0^t \int_{\mathbb{R}^N} (-\Delta)^{\sigma/2} P(x-\overline{x}, t-\overline{t}) f(\overline{x},\overline{t})\,d\overline{x}\,d\overline{t}, \qquad (x,t) \in Q. \tag{3.2}$$

Proof. Step 1: Uniqueness. We may assume $u_0 = f = 0$ and then apply the results of [3] where a wider class of data and solutions is treated.

Step 2: u is well defined. We have only to take care of the last term in (3.2), which can be written as

$$\int_Q A(Y-\overline{Y})\chi_{\{\overline{t}<t\}} f(\overline{Y})\,d\overline{Y}, \qquad Y \in Q \subset \mathbb{R}^{N+1}.$$

In order to prove that this integral is well defined we decompose $Q = B_r^- \cup (Q \setminus B_r^-)$, with $r > 0$ small, where $B_r^- = \{\overline{Y} = (\overline{x},\overline{t}) : |Y-\overline{Y}|_\sigma < r,\ \overline{t} \le t\}$. The cancellation property (2.8) combined with the local Hölder regularity allow us to estimate the inner integral:

$$\begin{aligned}\left|\int_{B_r^-} A(Y-\overline{Y})\chi_{\{\overline{t}<t\}} f(\overline{Y})\,d\overline{Y}\right| &= \left|\int_{B_r^-} A(Y-\overline{Y})\chi_{\{\overline{t}<t\}}\big(f(\overline{Y})-f(Y)\big)\,d\overline{Y}\right| \\ &\le \int_{B_r^-} |A(Y-\overline{Y})|\,|f(\overline{Y})-f(Y)|\,d\overline{Y} \\ &\le c\int_{B_r^-} \frac{d\overline{Y}}{|\overline{Y}-Y|_\sigma^{N+\sigma-\alpha}} \le c.\end{aligned}$$

The outer integral is bounded by using estimate (2.5).

Step 3: u is a very weak solution. In order to justify the representation formula we proceed by approximation. Let $t > 0$ and take $f \in C_c^\infty(Q)$ with $f(x,\overline{t}) = 0$ for $\overline{t} \ge t-r$, r small, thus avoiding the singularity. In that case the integral in the ball B_r^- vanishes identically. Since moreover u given by (3.2) is a bounded classical solution, hence a very weak solution, the assertion holds.

Next, for any $f \in C^\infty(Q)$ and compactly supported in space (in a uniform way), we use approximation with functions f_n as before by modifying f in the time interval $t - r_n \le \overline{t} \le t$. Using the fact that the fractional Laplacian can be applied to f_n instead of P, it is easy to see that we can pass to the limit u of the solutions u_n, which is still a bounded classical solution. Moreover, the formula as it is written holds for functions f of this class by integrating by parts and the integrability estimate from Step 2.

Finally, for general f as in the statement, we use approximation of f in a compact set by functions $f_n \in C^\infty(Q)$ compactly supported in space. Passing to the limit in the very weak formulation, which is again justified thanks to Step 2, we conclude that $u = \lim u_n$ is a very weak solution. □

Remark. Any bounded, locally Hölder function f satisfies the conditions required by Theorem 3.1.

3.2. Regularity of the linear problem

The first term on the right-hand side of the representation formula (3.2) is regular. Hence, u has the same regularity as

$$g(Y) = \int_{\mathbb{R}_+^{N+1}} A(Y - \overline{Y})\chi_{\{\bar{t}<t\}} f(\overline{Y})\, d\overline{Y}. \tag{3.3}$$

We start by proving that g has the same σ-Hölder regularity as f.

Lemma 3.1. *Let $f \in C^\alpha_\sigma(Q) \cap L^\infty(Q)$ for some $0 < \alpha < \nu$, and let g be given by (3.3). Then $g \in C^\alpha_\sigma(Q) \cap L^\infty(Q)$.*

Proof. Let $Y_1 = (x_1, t_1)$, $Y_2 = (x_2, t_2) \in Q$ be two points with $|Y_1 - Y_2|_\sigma = h > 0$ small. We assume without loss of generality that $f(Y_1) = 0$. We have to estimate the difference

$$g(Y_1) - g(Y_2) = \int_{\mathbb{R}_+^{N+1}} \big(A(Y_1 - \overline{Y})\chi_{\{\bar{t}<t_1\}} - A(Y_2 - \overline{Y})\chi_{\{\bar{t}<t_2\}}\big) f(\overline{Y})\, d\overline{Y} \tag{3.4}$$

and see if it is $O(h^\alpha)$. We decompose Q into four regions, depending on the sizes of $|\bar{x} - x_1|$ and $|\bar{t} - t_1|$ (see Figure 1).

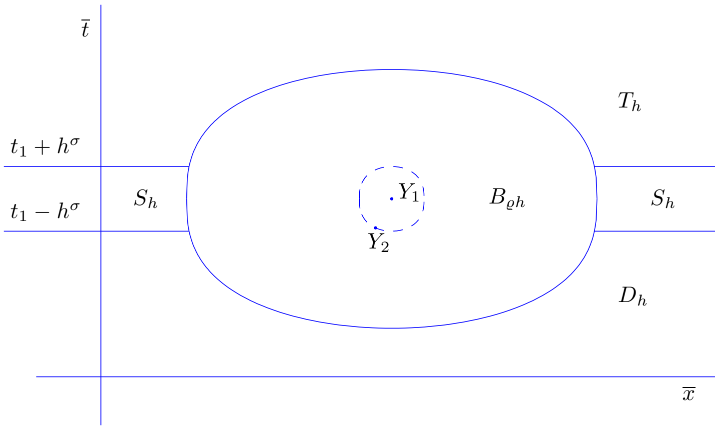

Fig. 1. Integration regions.

(i) *The small "ball"* $B_{\varrho h}(Y_1)$, where $\varrho > 1$ is a constant to be fixed later. We take h small enough ($\varrho h < \min\{t_1, 1\}$) so that on the one hand $B_{\varrho h} \subset Q$, and on the other hand we can use the relation (2.3). The difficulty in this region is the nonintegrable singularity of $A(Y)$ at $Y = 0$. Integrability will be gained thanks to the regularity of f. We first have, repeating the computations in Step 2 of the proof of Theorem 3.1,

$$\left| \int_{B_{\varrho h(Y_1)}} A(Y_1 - \overline{Y})\chi_{\{\bar{t}<t_1\}} f(\overline{Y})\, d\overline{Y} \right| \le ch^\alpha.$$

To estimate the second term in (3.4), we consider the ball $B_h(Y_2)$. To be sure that the distance from $\partial B_{\varrho h}(Y_1)$ to $B_h(Y_2)$ is positive, we take $\varrho = \max\{4, 2^{2/\sigma}\}$ (see (2.2)). Using again the cancellation property (2.8), we get

$$\begin{aligned}\int_{B_{\varrho h}(Y_1)} & A(Y_2-\overline{Y})\chi_{\{\overline{t}<t_2\}} f(\overline{Y})\,d\overline{Y}\\ &= \underbrace{\int_{B_{\varrho h}(Y_1)} A(Y_2-\overline{Y})\chi_{\{\overline{t}<t_2\}}\big(f(\overline{Y})-f(Y_2)\big)\,d\overline{Y}}_{I_1}\\ &\quad+ \underbrace{f(Y_2)\int_{B_{\varrho h}(Y_1)\setminus B_h(Y_2)} A(Y_2-\overline{Y})\chi_{\{\overline{t}<t_2\}}\,d\overline{Y}}_{I_2}. \qquad (3.5)\end{aligned}$$

The first integral satisfies $|I_1| \le ch^\alpha$. As to I_2, since we are far from the singularity of A,

$$|I_2| \le ch^\alpha \int_{B_{\varrho h}(Y_1)\setminus B_h(Y_2)} \frac{d\overline{Y}}{h^{N+\sigma}} \le ch^\alpha.$$

(ii) *The narrow strip* $S_h = \{\overline{Y} \in \overline{B}^c_{\varrho h}(Y_1),\ |\overline{t}-t_1| < h^\sigma\}$. In this region we have $|\overline{Y}-Y_1|_\sigma \le \varrho_1|\overline{Y}-Y_2|_\sigma$ and $|\overline{x}-x_1| > \varrho_2 h$, for some positive constants ϱ_1, ϱ_2 depending only on σ. Therefore,

$$\begin{aligned}&\left|\int_{S_h}\big(A(Y_1-\overline{Y})\chi_{\{\overline{t}<t_1\}} - A(Y_2-\overline{Y})\chi_{\{\overline{t}<t_2\}}\big) f(\overline{Y})\,d\overline{Y}\right|\\ &\qquad\le \int_{S_h}\big(|A(Y_1-\overline{Y})| + |A(Y_2-\overline{Y})|\big)|f(\overline{Y})|\,d\overline{Y} \le \int_{S_h}\frac{d\overline{Y}}{|\overline{Y}-Y_1|_\sigma^{N+\sigma-\alpha}}\\ &\qquad\le c\int_{t_1-h^\sigma}^{t_1+h^\sigma}\int_{\{|\overline{x}-x_1|>\varrho_2 h\}}\frac{d\overline{x}\,d\overline{t}}{|\overline{x}-x_1|^{N+\sigma-\alpha}} \le ch^\alpha.\end{aligned}$$

Notice that $\alpha < \sigma$, so that the last integral is convergent.

(iii) *The complement of the ball* $B_{\varrho h}(Y_1)$ *for large times,* $T_h = \{\overline{Y} \in \overline{B}^c_{\varrho h}(Y_1), \overline{t} > t_1 + h^\sigma\}$. The integral in this region is 0, since here we have

$$A(Y_1-\overline{Y})\chi_{\{\overline{t}<t_1\}} = A(Y_2-\overline{Y})\chi_{\{\overline{t}<t_2\}} = 0.$$

(iv) *The complement of the ball* $B_{\varrho h}(Y_1)$ *for small times,* $D_h = \{\overline{Y} \in \overline{B}^c_{\varrho h}(Y_1),\ \overline{t} < t_1 - h^\sigma\}$. The required estimate is obtained here using the fact that

$$A(Y_1-\overline{Y})\chi_{\{\overline{t}<t_1\}} - A(Y_2-\overline{Y})\chi_{\{\overline{t}<t_2\}} = A(Y_1-\overline{Y}) - A(Y_2-\overline{Y}).$$

Thus we are integrating a difference of A's, so there will be some cancellation. Indeed, by the Mean Value Theorem,

$$|A(Y_1-\overline{Y}) - A(Y_2-\overline{Y})| \le |Y_1-Y_2|\max\{|\nabla_x A(\theta)|, |\partial_t A(\theta)|\},$$

where $\theta = s(Y_1-\overline{Y}) + (1-s)(Y_2-\overline{Y})$ for some $s \in (0,1)$.

Since $|Y_1 - Y_2| \leq |Y_1 - Y_2|_\sigma^\nu = h^\nu$, and in D_h we have $|Y_1 - \overline{Y}|_\sigma \leq 2|\theta|_\sigma$, Proposition 2.1 yields

$$|A(Y_1 - \overline{Y}) - A(Y_2 - \overline{Y})| \leq \frac{ch^\nu}{|\theta|_\sigma^{N+\sigma+\nu}} \leq \frac{ch^\nu}{|\overline{Y} - Y_1|_\sigma^{N+\sigma+\nu}}.$$

Therefore we conclude that

$$\left| \int_{D_h} \big(A(Y_1 - \overline{Y}) \chi_{\{\overline{t} < t_1\}} - A(Y_2 - \overline{Y}) \chi_{\{\overline{t} < t_2\}} \big) f(\overline{Y}) \, d\overline{Y} \right|$$
$$\leq ch^\nu \int_{D_h} \frac{d\overline{Y}}{|Y_1 - \overline{Y}|_\sigma^{N+\sigma+\nu-\alpha}} \leq ch^\nu \int_{\{|\overline{Y}|_\sigma > \varrho h\}} \frac{d\overline{Y}}{|\overline{Y}|_\sigma^{N+\sigma+\nu-\alpha}} = ch^\alpha,$$

assuming $\alpha < \nu$ so that the last integral is convergent. □

Remark. Notice that if some derivative (even a fractional one) of f belongs to $C_\sigma^\alpha(Q) \cap L^\infty(Q)$, then a computation analogous to that in the above lemma shows that the convolution of this derivative against the kernel A also belongs to $C_\sigma^\alpha(Q) \cap L^\infty(Q)$. We conclude that g has the same regularity as f.

As a corollary of Lemma 3.1, we obtain a maximal regularity result for the linear equation with a standard right-hand side that has independent interest.

Corollary 3.1. *Let* $f \in C_\sigma^\alpha(Q) \cap L^\infty(Q)$ *for some* $0 < \alpha < \min\{1, \sigma\}$. *If* w *is a very weak solution to*

$$\partial_t w + (-\Delta)^{\sigma/2} w = f, \tag{3.6}$$

then $\partial_t w, (-\Delta)^{\sigma/2} w \in C_\sigma^\alpha(Q) \cap L^\infty(Q)$.

Proof. The function $u = (-\Delta)^{\sigma/2} w$ solves (3.1) in the distributional sense, hence it is in $C_\sigma^\alpha(Q) \cap L^\infty(Q)$. The result now follows by noticing that $\partial_t w = f - u$. □

See also [6, Appendix A] for a related regularity result for the linear parabolic equation (3.6).

4. Improving σ-Hölder regularity

We now return to the nonlinear equation (1.1). For bounded weak energy solutions the equation is neither degenerate nor singular. Hence, the results from [2] guarantee that they are C_σ^α for some $\alpha \in (0, \nu)$. The aim of this section is to improve this regularity showing that the solutions belong to C_σ^α for *all* $\alpha < \nu$. Further regularity, showing that the solution is classical, will be obtained in Section 5.

The idea is to use the fact that the solution u to the nonlinear equation (1.1) is, after a scaling in time, a solution to the linear equation

$$\partial_t u + (-\Delta)^{\sigma/2} u = (-\Delta)^{\sigma/2} f, \qquad f(Y) = u(Y) - \frac{\varphi(u(Y))}{\varphi'(u(Y_0))}.$$

The function f satisfies the hypotheses of Theorem 3.1. Hence, u is given by the representation formula (3.2). Moreover, since $u \in C^\alpha_\sigma(Q)$ and φ is uniformly parabolic, and since $\varphi' \in C^\gamma(\mathbb{R})$, applying the Mean Value Theorem we get

$$\begin{aligned}
|f(Y_1) - f(Y_2)| &= \left|1 - \frac{\varphi'(\theta)}{\varphi'(u(Y_0))}\right| |u(Y_1) - u(Y_2)| \le \frac{|u(Y_0) - \theta|^\gamma}{\varphi'(u(Y_0))} |Y_1 - Y_2|^\alpha_\sigma \\
&\le c \max\{|u(Y_1) - u(Y_0)|^\gamma, |u(Y_2) - u(Y_0)|^\gamma\} |Y_1 - Y_2|^\alpha_\sigma \\
&\le c \max\{|Y_1 - Y_0|^{\alpha\gamma}_\sigma, |Y_2 - Y_0|^{\alpha\gamma}_\sigma\} |Y_1 - Y_2|^\alpha_\sigma, \qquad (4.1)
\end{aligned}$$

where θ is some value between $u(Y_1)$ and $u(Y_2)$. This shows not only that f has the same regularity as u, namely $f \in C^\alpha_\sigma(Q)$, but a bit more that will be enough to improve the σ-Hölder regularity of u by a constant factor.

Lemma 4.1. *Let f be bounded and locally Hölder continuous, so that g in (3.3) is well defined. Assume that there exist $c, \delta_0, \epsilon > 0$ with $\alpha + \epsilon < \nu$ such that*

$$|f(Y) - f(Y_0)| \le c|Y - Y_0|^{\alpha+\epsilon}_\sigma, \qquad (4.2)$$

$$|f(Y) - f(\overline{Y})| \le c\delta^\epsilon |Y - \overline{Y}|^\alpha_\sigma, \qquad (4.3)$$

for all $0 < \delta < \delta_0$ and $Y, \overline{Y} \in B_\delta(Y_0)$. Then

$$|g(Y) - g(Y_0)| \le c'|Y - Y_0|^{\alpha+\epsilon}_\sigma$$

for all $Y \in B_{\delta_0/2}(Y_0)$, where c' depends on c.

Proof. Since f is bounded, condition (4.2) holds for every $Y \in Q$. This is enough to make all the estimates used to prove Lemma 3.1 work, yielding terms which are $O(h^{\alpha+\epsilon})$, except for the integral I_1 in (3.5). To estimate this term, take $\varrho h < \delta_0$ and observe that (4.3) gives

$$\begin{aligned}
|I_1| &= \left| \int_{B_{\varrho h}(Y_0)} A(Y - \overline{Y}) \chi_{\{\bar{t} < t\}} \big(f(\overline{Y}) - f(Y)\big)\, d\overline{Y} \right| \\
&\le ch^\epsilon \int_{B_{\varrho h}(Y_0)} \frac{1}{|Y - \overline{Y}|^{N+\sigma}_\sigma} |Y - \overline{Y}|^\alpha_\sigma \, d\overline{Y} \le ch^{\alpha+\epsilon}. \qquad \square
\end{aligned}$$

Applying this lemma a finite number of times we obtain the desired regularity for the nonlinear problem.

Theorem 4.1. *Let u be a bounded weak solution to equation (1.1). Assume $\varphi \in C^{1,\gamma}(\mathbb{R})$ and $\varphi'(s) > 0$ for every $s \in \mathbb{R}$. Then $u \in C^\alpha_\sigma(Q)$ for all $\alpha \in (0, \nu)$.*

We must remark that the restriction $\alpha + \epsilon < \nu$ in Lemma 4.1 is only needed to make the outer integrals convergent; the estimate in the ball $B_{\varrho h}(Y_0)$ is true for any $\alpha \in (0, \nu)$ and $\epsilon \in (0, 1]$. This observation turns out to be of great importance in obtaining further regularity in the next section.

5. Classical solutions

Our next aim is to go beyond the C^ν_σ threshold of regularity. Here we encounter an additional difficulty, stemming from the nonlocal character of the fractional Laplacian operator, which is not present in the work [8], namely that $(-\Delta)^{\sigma/2}P(Y)\chi_{\{t>0\}}$ is not smooth through $t=0$ outside $x=0$. For that reason we must treat the second order estimates in the time and space variables separately. We begin by improving regularity in space, to obtain $u(\cdot,t)\in C^\alpha(\mathbb{R}^N)$ uniformly in t for some $\alpha>\nu$ depending on the regularity of the nonlinearity φ. We then use equation (1.1) to get Lipschitz regularity in time, which is later improved to $u(x,\cdot)\in C^{\nu(1+\gamma)/\sigma}(\mathbb{R}_+)$ uniformly in x. The last step is to reach the desired smoothness in space, $u(\cdot,t)\in C^{\nu(1+\gamma)}(\mathbb{R}^N)$ uniformly in t.

Notation. By writing $u\in C^\alpha$ with $\alpha\in[1,2)$ we mean $u\in C^{1,\alpha-1}$ if $\alpha\in(1,2)$, and $u\in C^{0,1}$ if $\alpha=1$.

Lemma 5.1. *Let f be bounded and locally Hölder continuous, and g as in (3.3). If f satisfies in addition (4.2) and (4.3) with some $0<\alpha<\nu$ and $0<\epsilon<1$, then, for every $e\in\mathbb{R}^N$ with $|e|=1$,*

$$|g(x_0+he,t_0)-2g(x_0,t_0)+g(x_0-he,t_0)|\le ch^{\alpha+\epsilon}\quad \text{for every } h>0 \text{ small.} \tag{5.1}$$

Proof. Set $Y=Y_0+(he,0)$ and let $Y^*=2Y_0-Y$ be its symmetric point with respect to Y_0. We have to estimate the second difference

$$g(Y)-2g(Y_0)+g(Y^*)=\int_{\mathbb{R}^{N+1}_+}\mathcal{A}(Y,Y_0,\overline{Y})f(\overline{Y})\,d\overline{Y},$$

where

$$\begin{aligned}\mathcal{A}(Y,Y_0,\overline{Y})&=A(Y-\overline{Y})\chi_{\{\bar t<t\}}-2A(Y_0-\overline{Y})\chi_{\{\bar t<t_0\}}+A(Y^*-\overline{Y})\chi_{\{\bar t<t^*\}}\\&=\big(A(Y-\overline{Y})-2A(Y_0-\overline{Y})+A(Y^*-\overline{Y})\big)\chi_{\{\bar t<t_0\}}.\end{aligned}$$

As in the proof of Lemma 3.1, we consider separately the contributions to the integral of several regions, though here we only need to consider the ball $B_{\varrho h}(Y_0)$ and its complement $B^c_{\varrho h}(Y_0)$. The contribution of the integral in $B_{\varrho h}(Y_0)$ is decomposed as $J_1-2J_2+J_3$, where

$$\begin{aligned}J_1&=\int_{B_{\varrho h}(Y_0)}A(Y-\overline{Y})\chi_{\{\bar t<t_0\}}f(\overline{Y})\,d\overline{Y},\\J_2&=\int_{B_{\varrho h}(Y_0)}A(Y_0-\overline{Y})\chi_{\{\bar t<t_0\}}f(\overline{Y})\,d\overline{Y},\\J_3&=\int_{B_{\varrho h}(Y_0)}A(Y^*-\overline{Y})\chi_{\{\bar t<t_0\}}f(\overline{Y})\,d\overline{Y}.\end{aligned}$$

Thus $|\int_{B_{\varrho h}(Y_0)}\mathcal{A}f|\le|J_1|+2|J_2|+|J_3|$. The integrals J_1 and J_2 were already estimated in the course of the proof of Lemma 3.1, modified as in Lemma 4.1 (see the comment after that lemma), thus giving $O(h^{\alpha+\epsilon})$. Since $Y^*-Y_0=Y_0-Y$, the integral J_3 is estimated in just the same way.

To estimate the contribution in the complement of the ball we use Taylor's formula. We have, by Proposition 2.1,

$$|\mathcal{A}(Y, Y_0, \overline{Y})| = |A(Y - \overline{Y}) - 2A(Y_0 - \overline{Y}) + A(2Y_0 - Y - \overline{Y})|$$
$$\leq ch^2 |D_x^2 A(\theta)| \leq \frac{ch^2}{|Y_0 - \overline{Y}|_\sigma^{N+\sigma+2}},$$

where θ is, as before, some intermediate point. This gives

$$\int_{B^c_{\varrho h}(Y_0)} |\mathcal{A}(Y, Y_0, \overline{Y})| \, |f(\overline{Y})| \, d\overline{Y} \leq ch^2 \int_{B^c_{\varrho h}(Y_0)} \frac{d\overline{Y}}{|Y_0 - \overline{Y}|_\sigma^{N+\sigma+2-\alpha-\epsilon}} = ch^{\alpha+\epsilon}.$$

We have used the fact that $\alpha + \epsilon < 2$, and so the integral converges. This completes the desired estimate. □

Lemma 5.2. *Under the hypotheses of Theorem 1.1, bounded weak solutions u to equation (1.1) satisfy $u(\cdot, t) \in C^{\alpha(1+\gamma)}(\mathbb{R}^N)$ for every $\alpha \in (0, \nu)$ uniformly in $t \geq \tau > 0$ for every $\tau > 0$.*

Proof. For each given $Y_0 \in Q$ we define a function g and, as in the proof of Lemma 4.1, we deduce estimate (5.1) with $\epsilon = \alpha\gamma$ at that point, which is translated into the same estimate for the solution u. Since the constants do not depend on the particular point chosen, we find that u satisfies

$$|u(x + he, t) - 2u(x, t) + u(x - he, t)| \leq ch^{\alpha(1+\gamma)}, \tag{5.2}$$

with constant uniform in Q. We can thus prove that $(-\Delta)^{\delta/2} u$ is bounded in $\mathbb{R}^N$ for every $t > \tau > 0$ and every $\delta \in (0, \alpha(1 + \gamma))$. Indeed,

$$|(-\Delta)^{\delta/2} u(x, t)| = \left| c \int_{\mathbb{R}^{N+1}} \frac{u(x + z, t) - 2u(x, t) + u(x - z, t)}{|z|^{N+\delta}} \, dz \right|$$
$$\leq c \int_{\{|z|<\tau\}} \frac{|z|^{\alpha(1+\gamma)}}{|z|^{N+\delta}} \, dz + c \int_{\{|z|>\tau\}} \frac{dz}{|z|^{N+\delta}} \leq c. \tag{5.3}$$

The result now follows from [22, Proposition 2.9]. □

Lemma 5.3. *Under the hypotheses of Theorem 1.1, $u(x, \cdot) \in C^{\nu(1+\gamma)/\sigma}(\mathbb{R}_+)$ uniformly in $x \in \mathbb{R}^N$.*

Proof. We first show that $|(-\Delta)^{\sigma/2} \varphi(u)|$ is bounded in Q. For that purpose we estimate the second differences in x of $\varphi(u)$ in terms of second differences in x of u and use the previous result. If $Z = (he, 0)$, $e \in \mathbb{R}^N$, $|e| = 1$, then

$$\begin{aligned}
&|\varphi(u(Y_0 + Z)) - 2\varphi(u(Y_0)) + \varphi(u(Y_0 - Z))| \\
&\quad \leq |\varphi(u(Y_0 + Z)) - 2\varphi(u(Y_0)) + \varphi(2u(Y_0) - u(Y_0 + Z))| \\
&\qquad + |\varphi(u(Y_0 - Z)) - \varphi(2u(Y_0) - u(Y_0 + Z))| \\
&\quad \leq [\varphi]_{C^{1,\gamma}} |u(Y_0 + Z) - u(Y_0)|^{1+\gamma} \\
&\qquad + \|\varphi'(u)\|_{L^\infty(Q)} |u(Y_0 + Z) - 2u(Y_0) + u(Y_0 - Z)| \leq ch^{\alpha(1+\gamma)}
\end{aligned}$$

for every $\alpha < \nu$. Since $\nu(1+\gamma) > \sigma$ we find, analogously to how we obtained (5.3), that $|(-\Delta)^{\sigma/2}\varphi(u)| \le c$ in Q. Now, using the equation we get $|\partial_t u| \le c$ in Q, that is, u is Lipschitz continuous in time, uniformly in space. This means $u \in C^{\sigma}_{\sigma}(Q)$. With this information we now try to repeat the above calculations of Lemma 5.2 with x_0 fixed and varying t. To this end we consider the point $Y = Y_0 + (0, h)$, $h > 0$ (for simplicity), and we replace h by $h^{1/\sigma}$ in the regions of integration (see the proof of Theorem 3.1).

First, the integral in the ball $B_{\varrho h^{1/\sigma}}(Y_0)$ is estimated as in Lemma 3.1, taking note of (4.1), which holds with $\alpha = \nu$. Thus $|\int_{B_{\varrho h^{1/\sigma}}(Y_0)} \mathcal{A}f| = O(h^{\nu(1+\gamma)/\sigma})$.

Now consider the region $D_{h^{1/\sigma}} = \{\overline{Y} \in \overline{B}^c_{\varrho h^{1/\sigma}}(Y_0),\ \bar{t} < t_0 - h\}$. The idea here is that the characteristic functions take all the value one. Thus, by using Taylor's expansion, since only t varies, we have

$$
\begin{aligned}
|\mathcal{A}(Y, Y_0, \overline{Y})| &= |A(Y-\overline{Y}) - 2A(Y_0-\overline{Y}) + A(2Y_0 - Y - \overline{Y})| \\
&\le ch^2|\partial_t^2 A(\theta)| \le \frac{ch^2}{|Y_0-\overline{Y}|_\sigma^{N+3\sigma}},
\end{aligned}
$$

where θ is some intermediate point. This gives

$$
\int_{D_{h^{1/\sigma}}} |\mathcal{A}(Y, Y_0, \overline{Y})|\,|f(\overline{Y})|\,d\overline{Y} \le ch^2 \int_{D_{h^{1/\sigma}}} \frac{|Y_0-\overline{Y}|_\sigma^{\nu(1+\gamma)}}{|Y_0-\overline{Y}|_\sigma^{N+3\sigma}}\,d\overline{Y} \le ch^{\nu(1+\gamma)/\sigma}.
$$

We now turn our attention to the difficult part, the small slice $S_{h^{1/\sigma}} = \{\overline{Y} \in \overline{B}^c_{\varrho h^{1/\sigma}}(Y_0),\ |\bar{t} - t_0| < h\}$, where we have to look more carefully at the possible cancellations. We have

$$
\begin{aligned}
&\int_{S_{h^{1/\sigma}}} \mathcal{A}(Y, Y_0, \overline{Y}) f(\overline{Y})\,d\overline{Y} \\
&\quad= \int_{S_{h^{1/\sigma}}} \big(A(Y-\overline{Y})\chi_{\{\bar{t}<t\}} - 2A(Y_0-\overline{Y})\chi_{\{\bar{t}<t_0\}} + A(Y^*-\overline{Y})\chi_{\{\bar{t}<t^*\}}\big) f(\overline{Y})\,d\overline{Y} \\
&\quad= \int_{S_{h^{1/\sigma}}} \big(A(Y-\overline{Y}) - 2A(Y_0-\overline{Y})\chi_{\{\bar{t}<t_0\}}\big) f(\overline{Y})\,d\overline{Y} \\
&\quad= \underbrace{\int_{S_{h^{1/\sigma}}} \big(A(Y-\overline{Y}) - A(Y_0-\overline{Y})\big) f(\overline{Y})\,d\overline{Y}}_{J_1} \\
&\qquad+ \underbrace{\int_{S_{h^{1/\sigma}}} \big(A(Y_0-\overline{Y})\chi_{\{\bar{t}>t_0\}} - A(Y_0-\overline{Y})\chi_{\{\bar{t}<t_0\}}\big) f(\overline{Y})\,d\overline{Y}}_{J_2}.
\end{aligned}
$$

First, by the Mean Value Theorem applied to A in the time variable, together with the C^{ν}_{σ} regularity of u and Lemma 3.1, we have

$$|J_1| \le \int_{t_0-h}^{t_0+h} \int_{\{|\overline{x}-x_0|>\varrho h^{1/\sigma}\}} \frac{ch|\overline{Y}-Y_0|_{\sigma}^{\nu(1+\gamma)}}{|\overline{Y}-Y_0|_{\sigma}^{N+2\sigma}}\, d\overline{x}\, d\overline{t}$$
$$\le ch^2 \int_{\{|\overline{x}-x_0|>\varrho h^{1/\sigma}\}} \frac{d\overline{x}}{|\overline{x}-x_0|^{N+2\sigma-\nu(1+\gamma)}} = ch^{\nu(1+\gamma)/\sigma}.$$

As to the second integral J_2, performing the change of variables $\overline{Y} \to Z_1 = \overline{Y}^* \equiv (\overline{x}, 2t_0 - \overline{t})$, symmetric in time, in the second term (and writing again $\overline{Y}$ instead of Z_1), we have

$$J_2 = \int_{S_{h^{1/\sigma}}} A(Y_0 - \overline{Y})\chi_{\{\overline{t}>t_0\}} f(\overline{Y})\, d\overline{Y} - \int_{S_{h^{1/\sigma}}} A(\overline{Y} - Y_0)\chi_{\{\overline{t}>t_0\}} f(\overline{Y}^*)\, d\overline{Y}$$
$$= \int_{S_{h^{1/\sigma}}} A(Y_0 - \overline{Y})\chi_{\{\overline{t}>t_0\}} \big(f(\overline{Y}) - f(\overline{Y}^*)\big)\, d\overline{Y}.$$

Now we observe that

$$|f(\overline{Y}) - f(\overline{Y}^*)| \le c|u(\overline{Y}) - u(Y_0)|^{\gamma} |u(\overline{Y}) - u(\overline{Y}^*)| \le ch|\overline{Y} - Y_0|_{\sigma}^{\nu\gamma}$$

(see (4.1)). Thus

$$|J_2| \le ch \int_{t_0}^{t_0+h} \int_{\{|\overline{x}-x_0|>\varrho h^{1/\sigma}\}} \frac{|\overline{Y}-Y_0|_{\sigma}^{\nu\gamma}}{|\overline{Y}-Y_0|_{\sigma}^{N+\sigma}}\, d\overline{x}\, d\overline{t} \le ch^{1+\nu\gamma/\sigma}.$$

We conclude by noting that $1 + \nu\gamma/\sigma \ge \nu(1+\gamma)/\sigma$. □

Lemma 5.4. *Under the hypotheses of Theorem 1.1, $u(\cdot, t) \in C^{\nu(1+\gamma)}(\mathbb{R}^N)$ uniformly in t.*

Proof. Once we know that $u(x, \cdot) \in C^{\nu(1+\gamma)/\sigma}(0, \infty)$ uniformly in $x \in \mathbb{R}^N$, we can repeat the calculations in the proof of Lemma 5.2 with α replaced by ν. □

Using the worst case we can write the joint regularity in the form

$$u \in \begin{cases} C^{(1+\gamma)/\sigma}(Q) & \text{if } \sigma \ge 1, \\ C^{\sigma(1+\gamma)}(Q) & \text{if } \sigma \le 1, \end{cases}$$

with both variables playing the same role. We also see that the solution is classical since it has continuous derivatives in the sense required in the equation.

Corollary 5.1. *Under the hypotheses of Theorem 1.1, the function $z := \partial_t u = -(-\Delta)^{\sigma/2}\varphi(u)$ satisfies $z \in C_{x,t}^{\nu(1+\gamma)-\sigma,(\nu(1+\gamma)-\sigma)/\sigma}(Q)$.*

Proof. We point out that both sides of the equation are bounded functions and equal almost everywhere. We also know that $\partial_t u$ is Hölder continuous as a function of t for a.e. x, and the Hölder continuity is locally uniform. On the other hand, we easily conclude that $(-\Delta)^{\sigma/2}\varphi(u)$ is Hölder continuous as a function of x for a.e. t, and this happens again locally uniformly. Hölder continuity everywhere in both variables follows. □

Let us recall that under our assumptions, $\sigma < \nu(1+\gamma)$, so that we are getting Hölder regularity for $\partial_t u$ in all cases.

6. Higher regularity

We have already proved that solutions of (1.1) are differentiable in time. However, in view of Lemma 5.4 at this stage they are only known to be differentiable in space if $\sigma(1+\gamma) > 1$, where γ is the Hölder exponent of φ'. This assumption can be weakened.

Proposition 6.1. *Under the assumptions of Theorem 1.1, if $\sigma < 1$ and $\gamma + \sigma > 1$, then $u \in C^{1,\alpha}(Q)$ for some $\alpha \in (0,1)$.*

Proof. We only need to study the case $\sigma(1+\gamma) \le 1$, when necessarily $\sigma < 1$.

We consider the function $z = \partial_t u$, which belongs to $C^{\alpha}_{\sigma}(Q)$ for all $\alpha < \sigma$. Let $Y_0 = (x_0, t_0) \in Q$ be fixed and denote $a(Y) = \varphi'(u(Y))$, $z_0 = z(Y_0)$, $a_0 = a(Y_0)$. Then z is a distributional solution to the inhomogeneous fractional heat equation

$$\partial_t z + a_0(-\Delta)^{\sigma/2} z = (-\Delta)^{\sigma/2} F_1 + (-\Delta)^{\sigma/2} F_2,$$

where

$$F_1 = -(a-a_0)(z-z_0), \qquad F_2 = -z_0 a.$$

We decompose z as $z_1 + z_2$, where z_i is a solution to

$$\partial_t z_i + a_0(-\Delta)^{\sigma/2} z_i = (-\Delta)^{\sigma/2} F_i, \qquad i = 1, 2.$$

The term z_2 inherits the regularity of F_2, that is, the regularity of $a(Y)$. As to z_1, we use the fact that the function $F_1 = (a-a_0)(v-v_0)$ satisfies conditions (4.2) and (4.3), which implies, thanks to Lemma 4.1, that z_1 is more smooth than a, hence more smooth than z_2. Therefore, we concentrate on the "bad" term, z_2.

The regularity of F_2, that is, the regularity of $\varphi'(u)$, coincides with the minimum between the regularities of φ' and u. Therefore, $F_2(x,\cdot)$ belongs to $C^{\gamma}(\mathbb{R})$ uniformly in x. As for spatial regularity, at this stage we know that $F_2(\cdot,t)$ is $C^{\alpha}(\mathbb{R}^N)$ uniformly in t for all $\alpha < \min\{\sigma(1+\gamma), \gamma\}$.

If $\sigma(1+\gamma) \ge \gamma$, we get $z_2(\cdot,t) \in C^{\gamma}(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$ uniformly in t, and so does $z = \partial_t u$. Using the equation we conclude that $u(\cdot,t) \in C^{\gamma+\sigma}(\mathbb{R}^N)$ uniformly in time. Since we have assumed that $\gamma + \sigma > 1$, this means that u is differentiable also in x.

If $\sigma(1+\gamma) < \gamma$, we get $z_2(\cdot,t) \in C^{\sigma(1+\gamma)}(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$ uniformly in t. Since $w = (-\Delta)^{\sigma/2} z_2$ satisfies

$$\partial_t w + a_0(-\Delta)^{\sigma/2} w = (-\Delta)^{\sigma/2}(-\Delta)^{\sigma/2} F_2,$$

the regularity of w is given by the regularity of $(-\Delta)^{\sigma/2}F_2$. This implies that $z_2(\cdot,t) \in C^\alpha(\mathbb{R}^N)$ uniformly in t for all $\alpha < \min\{\sigma(2+\gamma),\gamma\}$. Repeating this argument as many times as needed, we finally deduce that $z(\cdot,t) \in C^\alpha(\mathbb{R}^N)$ for all $\alpha < \gamma$ uniformly in t. Using the equation, we conclude that $u \in C^{\alpha,1+\gamma}_{x,t}(Q)$ for all $\alpha < \sigma+\gamma$. □

A similar argument allows us to prove a regularity result for linear equations with variable coefficients that is of independent interest.

Theorem 6.1. *Let u be a bounded very weak solution to $\partial_t u + (-\Delta)^{\sigma/2}(au+b) = 0$, where the coefficients satisfy $a, b \in C^{1,\alpha}(Q) \cap L^\infty(Q)$ and $a(x,t) \geq \delta > 0$. If $u \in C^\alpha(Q)$ then $\partial_t u, \partial_{x_i} u \in C^\alpha(Q \cap \{t > \tau\}) \cap L^\infty(Q \cap \{t > \tau\})$, $i = 1, \dots, N$, for every $\tau > 0$.*

Proof. The proof of $C^{1,\alpha}$ regularity is done by considering the linear equations satisfied by the derivatives. Boundedness for the derivatives then immediately follows, since u is in $C^{1,\alpha}(Q \cap \{t > \tau\}) \cap L^\infty(Q \cap \{t > \tau\})$. □

This linear result is now used to obtain further regularity for the nonlinear problem, which covers in particular Theorem 1.2.

Theorem 6.2. *If in addition to the hypotheses of Theorem 1.1, $\varphi \in C^{k,\gamma}(\mathbb{R})$ for some $k \geq 2$ and $0 < \gamma < 1$, then $u \in C^{k,\alpha}(Q)$ for some $\alpha \in (0,1)$.*

Proof. We proceed by induction on the differentiability order. Since $\varphi \in C^{1,\theta}$ for all $\theta \in (0,1)$, Proposition 6.1 yields $\partial_t u, \partial_{x_i} u \in C^\alpha(Q) \cap L^\infty(Q)$ for some $\alpha \in (0,1)$.

Assume that the result is true for derivatives of order $j \leq k-1$. Let

$$v_\beta = \partial_t^{\beta_0}\partial_{x_1}^{\beta_1}\cdots\partial_{x_N}^{\beta_N} u, \qquad \sum_{i=0}^N \beta_i = j.$$

It is easily checked that v_β satisfies an equation of the form

$$\partial_t v_\beta + (-\Delta)^{\sigma/2}(\varphi'(u)v_\beta + b_\beta) = 0,$$

where b_β is a polynomial in $v_{\beta'}$, $\beta'_i \leq \beta_i$, $i = 0, \dots, N$, $\sum_{i=0}^N \beta'_i \leq j-1$, with coefficients involving the derivatives $\varphi^{(l)}(u)$, $0 < l \leq j$. By hypothesis, $b_\beta \in C^{1,\alpha}(Q) \cap L^\infty(Q)$ for some $\alpha \in (0,1)$. Since u is bounded, we have $a = \varphi'(u) \geq \delta > 0$. Hence we may apply Theorem 6.1 to conclude the proof. □

7. Nonlinear degenerate and singular equations

A careful inspection of the proof of Theorem 1.1 shows that the result has a certain "local" nature. This will now be exploited to treat more general equations, which may be degenerate or singular at some points.

Theorem 7.1. *Let u be a bounded, locally Hölder continuous weak solution of equation (1.1) in Q. Let φ be continuous and nondecreasing, and such that $\varphi \in C^{1,\gamma}$ and $\varphi'(u) \geq c > 0$ in an interval of values $u \in (a,b)$. Then $\partial_t u$ and $(-\Delta)^{\sigma/2}\varphi(u)$ are Hölder continuous in $\mathcal{O} = \{(x,t) : a < u(x,t) < b\}$. Hence, u is a classical solution of (1.1) in $\mathcal{O}$.*

Proof. The first time where a modification in our original proof is needed is in Lemma 5.2. Indeed, the estimate of the second order difference (5.2) holds uniformly in compact subsets of $\mathcal{O}$, but this is not enough to obtain the desired spatial regularity. To avoid this difficulty, we take a smooth cut-off function ϕ with support contained in $\mathcal{O}$, with $\phi \equiv 1$ in a subset $\mathcal{O}' \subset \mathcal{O}$. The second order difference of $\psi = g\phi$ satisfies

$$\begin{aligned}\psi(Y_0+Z) - 2\psi(Y_0) + \psi(Y_0 - Z) &= \big(g(Y_0+Z) - 2g(Y_0) + g(Y_0-Z)\big)\phi(Y_0)\\ &\quad + \big(\phi(Y_0+Z) - 2\phi(Y_0) + \phi(Y_0-Z)\big)g(Y_0+Z)\\ &\quad + \big(\phi(Y_0) - \phi(Y_0-Z)\big)\big(g(Y_0+Z) - g(Y_0-Z)\big) = O(h^{\alpha(1+\gamma)}),\end{aligned}$$

uniformly in Q, $Z = (he, 0)$, $e \in \mathbb{R}^N$, $|e| = 1$, $0 < \alpha < \nu$. Then $(-\Delta)^{\delta/2}\psi$ is bounded for every $\delta < \alpha(1+\gamma)$, which implies that ψ is $C^{\alpha(1+\gamma)}$, and thus $g \in C_x^{\alpha(1+\gamma)}(\mathcal{O}')$, for every $0 < \alpha < \nu$. The same trick allows one to fix the proofs of Lemmas 5.3 and 5.4. □

Remark. This local type result about classical regularity for bounded weak solutions of a nonlocal equation is remarkable, since it is not expected to hold under general conditions for this type of equations. Even for the linear fractional heat equation a counterexample to time regularity is shown in [9] for a problem posed in a bounded domain, by giving discontinuous Dirichlet conditions outside the domain. Our result applies to the more difficult degenerate parabolic equation, and the precise wording of the necessary conditions is important. This difficulty explains why we are not attempting to prove higher regularity in the setting of this section.

Application to the fractional porous medium equation. In order to apply Theorem 7.1 we need to make sure that the solution is locally Hölder continuous. In the fractional porous medium case, $\varphi(u) = |u|^{m-1}u$, $m > 1$, this follows from [2]. However, the equation degenerates when $u = 0$. Hence the application of Theorem 7.1 only yields the regularity stated there in the set $\{u \neq 0\}$.

In the fast diffusion case, $m < 1$, the required local Hölder regularity has been recently obtained in [17]. Again, further regularity is only guaranteed where the solution is different from 0.

On the other hand, in our paper [19] we prove, for all $m \geq (N-\sigma)_+/N$, that when the initial value is nonnegative, the solution is positive everywhere for positive times. The application of the results of the present paper then implies that the solution is classical. This positivity property is in sharp contrast with the nonlinear theory for the standard Laplacian and $m > 1$, where the existence of free boundaries is well-known [24]. For $0 < m < (N-\sigma)/N$ with $N > \sigma$, solutions may vanish identically after a finite time [19, Theorem 9.5]. However, if the initial value is nonnegative, the solution is positive, and hence classical, up to the extinction time.

8. Existence and basic properties

As a complement to the previous regularity theory, we devote this section to a survey of the main facts of the existence and uniqueness theory for the Cauchy problem for equation (1.1),

$$\begin{cases} \partial_t u + (-\Delta)^{\sigma/2}\varphi(u) = 0 & \text{in } Q, \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^N. \end{cases} \tag{CP}$$

Such a theory has been developed in great detail in [19] for φ a power function. As in the case of the standard (local) porous medium equation, many of the basic features of the theory can be extended to more general nonlinearities φ, as long as they are continuous and nondecreasing [11]. Therefore, we will now outline how such extension can be done in the fractional case $\sigma \in (0,2)$, with special attention to the points where the arguments differ.

Let us recall the concept of *weak solution* to the Cauchy problem (CP): it is a function $u \in C([0,\infty) : L^1(\mathbb{R}^N))$ such that (i) $\varphi(u) \in L^2_{\text{loc}}((0,\infty) : \dot{H}^{\sigma/2}(\mathbb{R}^N))$; (ii) the identity

$$\int_0^\infty \int_{\mathbb{R}^N} u\partial_t\zeta \, dx\, dt - \int_0^\infty \int_{\mathbb{R}^N} (-\Delta)^{\sigma/4}\varphi(u)(-\Delta)^{\sigma/4}\zeta \, dx\, dt = 0$$

holds for every $\zeta \in C_c^\infty(Q)$; and (iii) $u(\cdot,0) = u_0$ almost everywhere. The (homogeneous) fractional Sobolev space $\dot{H}^{\sigma/2}(\mathbb{R}^N)$ is the space of locally integrable functions ζ such that $(-\Delta)^{\sigma/4}\zeta \in L^2(\mathbb{R}^N)$. We point out that this is a convenient choice among other possible notions of weak solution, and it can be described as a weak L^1-energy solution to be specific.

8.1. Solutions with bounded initial data

We will start by considering the theory for initial data

$$u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N).$$

Existence and uniqueness are proved by using the definition of the fractional Laplace operator based on the extension technique developed by Caffarelli and Silvestre [7], which is a generalization of the well-known Dirichlet to Neumann operator corresponding to $\sigma = 1$. Thus, if $g = g(x)$ is a smooth bounded function defined in $\mathbb{R}^N$, its σ-harmonic extension to the upper half-space, $v = \mathrm{E}(g)$, is the unique smooth bounded solution $v = v(x,y)$ to

$$\begin{cases} \nabla\cdot(y^{1-\sigma}\nabla v) = 0 & \text{in } \mathbb{R}^{N+1}_+ \equiv \{(x,y)\in\mathbb{R}^{N+1} : x\in\mathbb{R}^N,\ y>0\}, \\ v(\cdot,0) = g & \text{in } \mathbb{R}^N. \end{cases} \tag{8.1}$$

Then it turns out (see [7]) that

$$-\frac{\partial v}{\partial y^\sigma} \equiv -\mu_\sigma \lim_{y\to 0^+} y^{1-\sigma}\frac{\partial v}{\partial y} = (-\Delta)^{\sigma/2} g, \tag{8.2}$$

where $\mu_\sigma = 2^{\sigma-1}\Gamma(\sigma/2)/\Gamma(1-\sigma/2)$. In (8.1) the operator ∇ acts in all (x,y) variables, while in (8.2), $(-\Delta)^{\sigma/2}$ acts only on the $x = (x_1,\ldots,x_N)$ variables.

In this approach, problem (CP) can be written in an equivalent local form. If u is a solution, then $w = \mathrm{E}(\varphi(u))$ solves

$$\begin{cases} \nabla\cdot(y^{1-\sigma}\nabla w) = 0, & (x,y)\in\mathbb{R}^{N+1}_+,\ t>0,\\ \dfrac{\partial w}{\partial y^\sigma} - \dfrac{\partial \beta(w)}{\partial t} = 0, & x\in\mathbb{R}^N,\ y=0,\ t>0,\\ \beta(w) = u_0, & x\in\mathbb{R}^N,\ y=0,\ t=0, \end{cases} \tag{8.3}$$

where $\beta = \varphi^{-1}$. Conversely, if we obtain a solution w to (8.3), then $u = \beta(w)|_{y=0}$ is a solution to (CP).

We use the concept of weak solution for problem (8.3) obtained by multiplying by a test function ζ,

$$\int_0^\infty\int_{\mathbb{R}^N}\beta(w)\frac{\partial\zeta}{\partial t}\,dx\,dt - \mu_\sigma\int_0^\infty\int_{\mathbb{R}^{N+1}_+} y^{1-\sigma}\langle\nabla w,\nabla\zeta\rangle\,dx\,dy\,dt = 0.$$

We then introduce the energy space $X^\sigma(\mathbb{R}^{N+1}_+)$, the completion of $C_c^\infty(\mathbb{R}^{N+1}_+)$ with the norm

$$\|v\|_{X^\sigma} = \left(\mu_\sigma\int_{\mathbb{R}^{N+1}_+} y^{1-\sigma}|\nabla v|^2\,dx\,dy\right)^{1/2}.$$

In order to solve the evolution problem, which is our concern, we use the Nonlinear Semigroup Generation Theorem due to Crandall and Liggett [10]. We are thus reduced to deal with the related elliptic problem

$$\begin{cases} \nabla\cdot(y^{1-\sigma}\nabla w) = 0, & x\in\mathbb{R}^N,\ y>0,\\ -\dfrac{\partial w}{\partial y^\sigma} + \beta(w) = g, & x\in\mathbb{R}^N,\ y=0, \end{cases} \tag{8.4}$$

with $g\in L^1_+(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$. As in the case treated in [19], in order to get a solution by variational techniques, it is convenient to replace the half-space $\mathbb{R}^{N+1}_+$ by a half-ball $B_R^+ = \{|x|^2+y^2<R^2 : x\in\mathbb{R}^N,\ y>0\}$. We impose zero Dirichlet data on the "new part" of the boundary. Thus we are led to study the problem

$$\begin{cases} \nabla\cdot(y^{1-\sigma}\nabla w) = 0 & \text{in } B_R^+,\\ w = 0 & \text{on } \partial B_R^+\cap\{y>0\},\\ -\dfrac{\partial w}{\partial y^\sigma} + \beta(w) = g & \text{on } D_R := \{|x|<R,\ y=0\}, \end{cases} \tag{8.5}$$

with $g\in L^\infty(D_R)$ given. Minimizing the functional

$$J(w) = \frac{\mu_\sigma}{2}\int_{B_R^+} y^{1-\sigma}|\nabla w|^2 + \int_{D_R} B(w) - \int_{D_R} wg,$$

with $B' = \beta$, in the admissible set $\mathcal{A} = \{w\in X_0^\sigma(B_R^+) : 0\le\beta(w)\le\|g\|_\infty\}$, where

$$X_0^\sigma(B_R^+) = \left\{f : B_R^+\to\mathbb{R} : \int_{B_R^+} y^{1-\sigma}|\nabla f|^2\,dx\,dy<\infty,\ f|_{\{\partial B_R^+,\,y>0\}} = 0\right\},$$

we obtain a unique solution $w = w_R$ to problem (8.5). Moreover, if g_1 and g_2 are two admissible data, then the corresponding weak solutions satisfy the L^1-contraction property

$$\int_{D_R} \big(\beta(w_1(x,0)) - \beta(w_2(x,0))\big)_+ \, dx \le \int_{\mathbb{R}} (g_1(x) - g_2(x))_+ \, dx.$$

In order to pass to the limit $R \to \infty$ we assume $g \ge 0$; thus, by monotonicity in R of the approximate solutions w_R we obtain a function $w_\infty = \lim_{R\to\infty} w_R$ which is a weak solution to problem (8.4). The above contractivity property also holds in the limit. Moreover, $\|\beta(w_\infty(\cdot,0))\|_{L^\infty(\mathbb{R})} \le \|g\|_{L^\infty(\mathbb{R})}$, and $w_\infty \ge 0$. In the general case where g changes sign we use comparison with the solutions corresponding to g^+ and g^- to show that the family w_R is uniformly bounded, and then use compactness to pass to the limit and obtain a weak solution.

Now, using the Crandall–Liggett Theorem we obtain the existence of a unique mild solution $\overline{w}$ to the evolution problem (8.3). To prove that $\overline{w}$ is moreover a weak solution to (8.3), one needs to show that it lies in the right energy space. This is done by using the same technique as in [18], which yields the energy estimate

$$\mu_\sigma \int_0^T \int_0^\infty \int_{\mathbb{R}^N} y^{1-\sigma} |\nabla \overline{w}(x,y,t)|^2 \, dx\, dy\, dt \le \int_{\mathbb{R}^N} \widetilde{B}(\varphi(u_0(x)))\, dx \quad \text{for every } T > 0,$$

where $\widetilde{B}'(w) = \beta'(w)w$; that is, $\widetilde{B}(\varphi(u_0)) = \psi(u_0)$ with $\psi' = \varphi$. Hence the function $u = \beta(\overline{w}(\cdot,0))$ is a weak solution to problem (CP). In addition we have $\|\beta(\overline{w}(\cdot,0))\|_{L^\infty(\mathbb{R}\times(0,\infty))} \le \|u_0\|_{L^\infty(\mathbb{R})}$. Recalling the isometry between $\dot{H}^{\sigma/2}(\mathbb{R}^N)$ and $X^\sigma(\mathbb{R}^{N+1}_+)$, we obtain

$$\int_0^T \int_{\mathbb{R}^N} |(-\Delta)^{\sigma/4} \varphi(u)(x,t)|^2 \, dx\, dt \le \int_{\mathbb{R}^N} \psi(u_0(x))\, dx \quad \text{for every } T > 0.$$

The semigroup theory also guarantees that the solutions we constructed satisfy the L^1-contraction property $\|u(\cdot,t) - \widetilde{u}(\cdot,t)\|_1 \le \|u_0 - \widetilde{u}_0\|_1$.

Uniqueness follows by the standard argument due to Oleĭnik et al. [16], using here the test function

$$\zeta(x,t) = \begin{cases} \int_t^T (\varphi(u) - \varphi(\widetilde{u}))(x,s)\, ds, & 0 \le t \le T, \\ 0, & t \ge T, \end{cases}$$

in the weak formulation for the difference of two solutions u and $\widetilde{u}$.

Summarizing, we have proved the following existence and uniqueness result.

Theorem 8.1. *Let $\varphi \in C(\mathbb{R})$ be nondecreasing. Given $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ there exists a unique bounded weak L^1-energy solution to problem* (CP).

8.2. Solutions with unbounded data. Boundedness and decay

If the (nondecreasing) nonlinearity φ satisfies $\varphi'(u) \geq C|u|^{m-1}$ for some $m > 0$, then weak solutions with initial data in $L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, where $p \geq 1$ satisfies $p > p(m) = (1-m)N/\sigma$, become immediately bounded, hence, thanks to our results, classical.

The idea is to take as test function in the weak formulation $\zeta = \zeta(u) = |u|^{p-2}u$. Though u is not differentiable in time a.e. for a general φ, this is not needed for the proof, since a regularization procedure, using some Steklov averages, allows one to bypass this difficulty; see for example the classical paper [1] for the case of local operators. More precisely, up to the above mentioned regularization, the formal argument proceeds as follows: we multiply the equation by $\partial_t\varphi(u)$ to show that $\partial_t \ell(u) \in L^2_{\text{loc}}((0,\infty) : L^2(\mathbb{R}^N))$, where $(\ell')^2 = \varphi'$; observing that $|\partial_t |u|^p| \leq C|u|^{(2p-m-1)/2}|\partial_t \ell(u)|$, we deduce that $\partial_t |u|^p \in L^2_{\text{loc}}((0,\infty) : L^2(\mathbb{R}^N))$ for every $p \geq (m+1)/2$; now we use the results of [1] which allow us to write the identity

$$\int_{\mathbb{R}^N} |u(x,t_2)|^p\,dx - \int_{\mathbb{R}^N} |u(x,t_1)|^p\,dx + \int_{t_1}^{t_2}\int_{\mathbb{R}^N} (-\Delta)^{\sigma/2}\varphi(u)|u|^{p-2}u\,dx\,dt = 0 \tag{8.6}$$

provided $|u|^{p-2}u \in L^2_{\text{loc}}((0,\infty) : \dot{H}^{\sigma/2}(\mathbb{R}^N))$. It can be seen that this last property holds for every $p \geq 2$ by using the following result applied to $v = \varphi(u)$.

Proposition 8.1. *If $v \in \dot{H}^{\sigma/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $f' \in L^\infty_{\text{loc}}(\mathbb{R}^N)$, then $f(v) \in \dot{H}^{\sigma/2}(\mathbb{R}^N)$.*

Proof. Using the extension technique we have

$$\begin{aligned} \|f(v)\|^2_{\dot{H}^{\sigma/2}(\mathbb{R}^N)} &= \mu_\sigma \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma}|\nabla \operatorname{E}(f(v))|^2\,dx\,dy \\ &\leq \mu_\sigma \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma}|\nabla f(\operatorname{E}(v))|^2\,dx\,dy \\ &\leq \mu_\sigma \|f'(\operatorname{E}(v))\|^2_\infty \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma}|\nabla \operatorname{E}(v)|^2\,dx\,dy \leq c\|v\|^2_{\dot{H}^{\sigma/2}(\mathbb{R}^N)}. \end{aligned}$$

□

Remark. If moreover f is convex then, noting that $\|\operatorname{E}(v)\|_\infty \leq \|v\|_\infty$, we deduce the estimate $\|f(v)\|_{\dot{H}^{\sigma/2}(\mathbb{R}^N)} \leq \|f'(v)\|_\infty \|v\|_{\dot{H}^{\sigma/2}(\mathbb{R}^N)}$.

The smoothing effect argument now starts from identity (8.6) for $p \geq \max\{2, (m+1)/2\}$. Applying the generalized Stroock–Varopoulos inequality, proved in [19, Lemma 5.2], we obtain

$$\int_{\mathbb{R}^N} |u(x,t_1)|^p\,dx \geq c\int_{t_1}^{t_2}\int_{\mathbb{R}^N} |(-\Delta)^{\sigma/4}G(u)|^2\,dx\,dt,$$

where

$$|G'(u)|^2 = \varphi'(u)|u|^{p-2} \geq c|u|^{m+p-3}.$$

Now we assume $N > \sigma$, which is always the case if $N \geq 2$. Using the Hardy–Littlewood–Sobolev inequality [14, 23], we get

$$\int_{\mathbb{R}^N} |u(x,t_1)|^p\,dx \geq c\int_{t_1}^{t_2}\left(\int_{\mathbb{R}^N} |u(x,t)|^{\frac{(m+p-1)N}{N-\sigma}}\,dx\right)^{\frac{N-\sigma}{N}}\,dt \tag{8.7}$$

for every $p \geq \max\{2, (m+1)/2\}$. If in addition $p > (1-m)N/\sigma$, we have $\frac{(m+p-1)N}{N-\sigma} > p$. Inequality (8.7) is then enough to apply Moser's standard iteration technique to obtain an L^p-L^∞ smoothing effect. We can weaken the restriction on p using interpolation (see [19]).

Let us state precisely the smoothing result thus obtained.

Theorem 8.2. *Let $\varphi \in C^1(\mathbb{R})$ be such that $\varphi'(u) \geq C|u|^{m-1}$ for some $m > 0$. Let $u_0 \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, where $p \geq 1$ satisfies $p > (1-m)N/\sigma$, and assume $N > \sigma$. Then there exists a unique weak L^1-energy solution to the Cauchy problem (CP) which is bounded in $\mathbb{R}^N \times (\tau, \infty)$ for all $\tau > 0$. This solution moreover satisfies*

$$\sup_{x \in \mathbb{R}^N} |u(x,t)| \leq C\, t^{-\gamma_p} \|u_0\|_p^{\delta_p} \tag{8.8}$$

with $\gamma_p = N/(N(m-1)+\sigma p)$ and $\delta_p = \sigma p \gamma_p / N$, the constant C depending on m, p, σ, and N.

Remark. The result is also valid if $\varphi'(u) \geq C|u|^{m-1}$ for $|u| \geq A$, for some $m \geq 1$ and $A > 0$.

In the case $N = 1 \leq \sigma < 2$ we must replace the Hardy–Littlewood–Sobolev inequality by a Nash–Gagliardo–Nirenberg inequality (see [19, Lemma 5.3]). To get an inequality like (8.7) we must also assume the upper estimate $\varphi'(u) \leq C|u|^{m-1}$. We omit further details.

Existence for data which are unbounded is proved by approximation; see [19] for the details in the case where the nonlinearity is a pure power. As for uniqueness, continuity in L^1 guarantees that two solutions with the same initial data do not differ by more than ε in L^1 norm for some small enough time. Since for positive times solutions are assumed to be bounded, we may use the L^1 contraction property to prove that the distance between the two solutions stays smaller than ε for any later time. Since ε is arbitrary, uniqueness follows.

9. Extensions and comments

Some applications. Equation (1.1) appears in the study of hydrodynamic limits of interacting particle systems with long range dynamics. Thus, in [13], Jara and co-authors study the nonequilibrium functional central limit theorem for the position of a tagged particle in a mean-zero one-dimensional zero-range process. The asymptotic behaviour of the particle is described by a stochastic differential equation governed by the solution of (1.1).

In several space dimensions, equations like (1.1) occur in boundary heat control, as already mentioned by Athanasopoulos and Caffarelli [2], who refer to the model formulated in the book by Duvaut and Lions [12], and use the extension technique of Caffarelli and Silvestre.

For a more thorough discussion on applications see [5].

Regularity for unbounded solutions. In our proofs we are requiring the solutions to be bounded in order to make the integrals on unbounded sets convergent. However, this requirement can perhaps be weakened. It may be enough that the solutions belong to $C([0,T] : L^1(\mathbb{R}^N, \rho\, dx))$. It would be interesting to explore this possibility, since this may be helpful in the study of higher regularity.

Higher regularity for the fractional porous medium equation. The main difficulty in obtaining further regularity in this case is that, since the equation is not uniformly parabolic at infinity (it is not true that $0 \le c \le \varphi'(u) \le C < \infty$), we do not know the derivatives to be bounded. Hence, we cannot apply Theorem 6.1 directly. However, as mentioned in the previous paragraph, this might be circumvented by substituting the boundedness requirement by some less restrictive condition. A precise quantitative statement of the positivity property obtained in [5] might be helpful.

The fractional porous medium equation with sign changes. Our results only show that the equation is satisfied in the classical sense where the solution is different from 0. It remains to determine what is the optimal regularity for sign changing solutions. A first step would be to study whether solutions are strong, i.e., whether $\partial_t u$ (and hence $(-\Delta)^{\sigma/2} u$) are functions, and not only distributions.

The very fast fractional porous medium equation. The nonlinearities $\varphi(u) = ((1+u)^m - 1)/m$, $m \neq 0$, are uniformly parabolic if we restrict ourselves to nonnegative solutions. Moreover, they fall within the hypotheses of Theorem 8.2 if we modify the nonlinearity suitably for $u < 0$, which does not matter if we only consider nonnegative solutions. Therefore, we obtain existence of C^∞ solutions for all nonnegative initial data in $L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ with p large enough. If $\sigma > 1 - m$ and $N = 1$ we can even take $p = 1$.

The nonlinearity $\varphi(u) = \log(1+u)$ is also uniformly parabolic if we restrict to nonnegative solutions. In addition, after a suitable modification for $u < 0$, it satisfies the hypotheses of Theorem 8.2 with $m = 0$. Thus, if $N = 1$ and $\sigma = 1$ we are in the critical case where we need a bit more than integrability to have existence. In [20] we proved that it is enough for u_0 to belong to some $L \log L$ space. The solution is then guaranteed to be C^∞.

The singular nonlinearities $\varphi(u) = u^m/m$, $m < 0$, and $\varphi(u) = \log u$ (with $u > 0$) cannot be treated in the same way, and require new ideas.

The fractional Stefan problem. For the Stefan nonlinearity $\varphi(s) = (s-1)_+$, the function $\varphi(u)$ is continuous [2], but it is not known to be Hölder continuous. Though u is continuous in the set $\{u > 1\}$, we cannot proceed further, even if we restrict to that set.

Problems in fluid mechanics. We now explore an interesting and enlightening connection, in the case $N = \sigma = 1$, between equation (1.1) and Morlet's family of one-dimensional nonlocal transport equations with viscosity [15],

$$\partial_t v - \delta \partial_y (H(v)v) - (1-\delta) H(v) \partial_y v = -(-\Delta)^{1/2} v, \quad 0 \le \delta \le 1. \tag{9.1}$$

For a nonnegative solution u to equation (1.1), we consider the change of variables $(x,t,u) \mapsto (y,\tau,v)$ given by the Bäcklund type transform

$$y = \int_0^x (1+u(s,t))\,ds - c(t), \qquad \tau = t, \qquad v(y,\tau) = \varphi(u(x,t)),$$

with $c'(t) = H(\varphi(u))(0,t)$. We denote $(y,\tau) = J(x,t)$. Notice that the Jacobian of the transformation J is $\frac{\partial(y,\tau)}{\partial(x,t)} = 1+u \neq 0$, since $u \geq 0$. Then we may write the inverse

$$x = \int_0^y \frac{d\theta}{1+\varphi^{-1}(v(\theta,\tau))} - \bar{c}(\tau),$$

with $\bar{c}\,'(\tau) = -H(\varphi(u))(0,t)/(1+u(0,t))$.

We recall that if the operators act on smooth enough functions, then the half-Laplacian $(-\Delta)^{1/2}$ can be written in terms of the Hilbert transform

$$Hf(x) = \frac{1}{\pi}\text{P.V.}\int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy$$

as $(-\Delta)^{1/2} = H\partial_x = \partial_x H$. Therefore,

$$\partial_x y = 1+u, \qquad \partial_t y = -H(\varphi(u)) = -\widetilde{H}(v),$$

where $\widetilde{H}(v) = H(v \circ J) \circ J^{-1}$ is the conjugate of the Hilbert transform H by the transformation J. Specifically,

$$\begin{aligned}\widetilde{H}(v(y,\tau)) = H(\varphi(u(x,t))) &= \frac{1}{\pi}\,\text{P.V.}\int_{\mathbb{R}} \frac{\varphi(u(x',t))}{x-x'}\,dx' \\ &= \frac{1}{\pi}\,\text{P.V.}\int_{\mathbb{R}} \frac{v(y',\tau)}{(1+\varphi^{-1}(v(y',\tau)))\int_{y'}^{y}\frac{d\theta}{1+\varphi^{-1}(v(\theta,\tau))}}\,dy'.\end{aligned}$$

If $\varphi(u) = ((1+u)^m - 1)/m$, $m \in [-1,0)$, then equation (1.1) becomes

$$\partial_t v - \widetilde{H}(v)\partial_y v = -(1+mv)\partial_y \widetilde{H}(v).$$

If instead of $\widetilde{H}$ we had the standard Hilbert transform H, and we took $m = -\delta$, we would have an equation in Morlet's family (9.1). The connection also works for $m = 0$ if we take $\varphi(u) = \log(1+u)$ (see [20]).

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