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# The tracial Hahn–Banach theorem, polar duals, matrix convex sets, and projections of free spectrahedra

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**Abstract.** This article investigates matrix convex sets and introduces their tracial analogs which we call contractively tracial convex sets. In both contexts completely positive (cp) maps play a central role: unital cp maps in the case of matrix convex sets and trace preserving cp (CPTP) maps in the case of contractively tracial convex sets. CPTP maps, also known as quantum channels, are fundamental objects in quantum information theory.

Free convexity is intimately connected with Linear Matrix Inequalities (LMIs)  $L(x) = A_0 + A_1x_1 + \dots + A_gx_g \geq 0$  and their matrix convex solution sets  $\{X : L(X) \geq 0\}$ , called *free spectrahedra*. The Effros–Winkler Hahn–Banach Separation Theorem for matrix convex sets states that matrix convex sets are solution sets of LMIs with operator coefficients. Motivated in part by cp interpolation problems, we develop the foundations of convex analysis and duality in the tracial setting, including tracial analogs of the Effros–Winkler Theorem.

The projection of a free spectrahedron in  $g + h$  variables to  $g$  variables is a matrix convex set called a *free spectrahedron*. As a class, free spectrahedrons are more general than free spectrahedra, but at the same time more tractable than general matrix convex sets. Moreover, many matrix convex sets can be approximated from above by free spectrahedrons. Here a number of fundamental results for spectrahedrons and their polar duals are established. For example, the free polar dual of a free spectrahedron is again a free spectrahedron. We also give a Positivstellensatz for free polynomials that are positive on a free spectrahedron.

**Keywords.** Linear matrix inequality (LMI), polar dual, LMI domain, spectrahedron, spectrahedron, convex hull, free real algebraic geometry, noncommutative polynomial, cp interpolation, quantum channel, tracial hull, tracial Hahn–Banach theorem

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## 1. Introduction

This article investigates matrix convex sets from the perspective of the emerging areas of free real algebraic geometry and free analysis [Voi04, Voi10, KVV14, MS11, Pop10, AM15, BB07, dOHMP09, HKM13b, PNA10]. It also introduces *contractively tracial convex sets*, the tracial analogs of matrix convex sets appropriate for the quantum channel and quantum operation interpolation problems. Matrix convex sets arise naturally in a number of contexts, including engineering systems theory, operator spaces, systems and algebras, and are inextricably linked to unital completely positive (ucp) maps [SIG97, Arv72, Pau02, Far00, HMPV09]. On the other hand, completely positive trace preserving (CPTP) maps are central to quantum information theory [NC10, JKPP11]. Hence there is an inherent similarity between matrix convex sets and structures naturally occurring in quantum information theory.

Given positive integers  $g$  and  $n$ , let  $\mathbb{S}_n^g$  denote the set of  $g$ -tuples  $X = (X_1, \dots, X_g)$  of complex  $n \times n$  hermitian matrices and let  $\mathbb{S}^g$  denote the sequence  $(\mathbb{S}_n^g)_n$ . We use  $M_n$  to

denote the algebra of  $n \times n$  complex matrices. A subset  $\Gamma \subseteq \mathbb{S}^g$  is a sequence  $\Gamma = (\Gamma(n))_n$  such that  $\Gamma(n) \subseteq \mathbb{S}_n^g$  for each  $n$ . A *matrix convex set* is a subset  $\Gamma \subseteq \mathbb{S}^g$  that is closed with respect to direct sums and (simultaneous) conjugation by isometries. *Closed under direct sums* means that if  $X \in \Gamma(n)$  and  $Y \in \Gamma(m)$ , then

$$X \oplus Y := \left( \begin{pmatrix} X_1 & 0 \\ 0 & Y_1 \end{pmatrix}, \dots, \begin{pmatrix} X_g & 0 \\ 0 & Y_g \end{pmatrix} \right) \in \Gamma(n+m). \quad (1.1)$$

Likewise, *closed under conjugation by isometries* means that if  $X \in \Gamma(n)$  and  $V$  is an  $n \times m$  isometry, then

$$V^* X V := (V^* X_1 V, \dots, V^* X_g V) \in \Gamma(m).$$

The simplest examples of matrix convex sets arise as solution sets of linear matrix inequalities (LMIs). The use of LMIs is a major advance in systems engineering in the past two decades [SIG97]. Furthermore, LMIs underlie the theory of semidefinite programming [BPR13, BN02], itself a recent major innovation in convex optimization [Nem06].

Matrix convex sets determined by LMIs are based on a free analog of an affine linear functional, often called a *linear pencil*. Given a positive integer  $d$  and  $g$  hermitian  $d \times d$  matrices  $A_j$ , let

$$L(x) = A_0 + \sum_{j=1}^g A_j x_j. \quad (1.2)$$

This linear pencil is often denoted by  $L_A$  to emphasize the dependence on  $A$ . In the case that  $A_0 = I_d$ , we call  $L$  *monic*. Replacing  $x \in \mathbb{R}^g$  with a tuple  $X = (X_1, \dots, X_g)$  of  $n \times n$  hermitian matrices and letting  $W \otimes Z$  denote the Kronecker product of matrices leads to the evaluation of the free affine linear functional,

$$L(X) = A_0 \otimes I_n + \sum A_j \otimes X_j. \quad (1.3)$$

The inequality  $L(X) \succeq 0$  is a *free linear matrix inequality (free LMI)*. The solution set  $\Gamma$  of this free LMI is the sequence of sets

$$\Gamma(n) = \{X \in \mathbb{S}_n^g : L(X) \succeq 0\},$$

known as a *free spectrahedron* (or *free LMI domain*). It is easy to see that  $\Gamma$  is a matrix convex set.

By the Effros–Winkler matricial Hahn–Banach Separation Theorem [EW97], (up to a technical hypothesis) every matrix convex set is the solution set of  $L(X) \succeq 0$ , as in equation (1.3), for some monic linear pencil *provided* the  $A_j$  are allowed to be hermitian operators on a (common) Hilbert space. More precisely, every matrix convex set is a (perhaps infinite) intersection of free spectrahedra. Thus, being a spectrahedron imposes a strict finiteness condition on a matrix convex set.

In between (closed) matrix convex sets and spectrahedra lie the class of domains we call *spectrahedrops*, namely coordinate projections of free spectrahedra. A subset  $\Delta \subseteq \mathbb{S}^g$  is a *free spectrahedrop* if there exists a pencil

$$L(x, y) = A_0 + \sum_{j=1}^g A_j x_j + \sum_{k=1}^h B_k y_k$$

in  $g + h$  variables such that

$$\Delta(n) = \{X \in \mathbb{S}_n^g : \exists Y \in \mathbb{S}_n^h, L(X, Y) \geq 0\}. \quad (1.4)$$

In applications, presented with a convex set, one would like, for optimization purposes say, to know if it is a spectrahedron or a spectrahedron. Alternatively, presented with an algebraically defined set  $\Gamma \subseteq \mathbb{S}^g$  that is not necessarily convex, it is natural to consider the relaxation obtained by replacing  $\Gamma$  with its matrix convex hull or an approximation thereof. Thus, it is of interest to know when the convex hull of a set is a spectrahedron or perhaps a spectrahedron. An approach to these problems via approximating from above by spectrahedrons was pursued in [HKM16]. Here we develop the duality approach. Typically the second polar dual of a set is its closed matrix convex hull.

### 1.1. Results on polar duals and free spectrahedrons

We list here our main results on free spectrahedrons and polar duals. For the reader unfamiliar with the terminology, the definitions not already introduced can be found in Section 2 with the exception of *free polar dual* whose definition appears in Subsection 4.2.

- (1) A perfect free Positivstellensatz (Theorem 5.1) for any symmetric free polynomial  $p$  on a free spectrahedron  $\Delta$  as in (1.4) says that  $p(X)$  is positive semidefinite for all  $X \in \Delta$  if and only if  $p$  has the form

$$p(x) = f(x)^* f(x) + \sum_{\ell} q_{\ell}(x)^* L(x, y) q_{\ell}(x)$$

where  $f$  and  $q_{\ell}$  are vectors with polynomial entries. If the degree of  $p$  is less than or equal to  $2r + 1$ , then  $f$  and  $q_{\ell}$  have degree no greater than  $r$ .

- (2) The free polar dual of a free spectrahedron is a free spectrahedron (Theorem 4.11 and Corollary 4.17).
- (3) The matrix convex hull of a union of finitely many bounded free spectrahedrons is a bounded free spectrahedron (Proposition 4.18).
- (4) A matrix convex set is, in a canonical sense, generated by a finite set (equivalently a single point) if and only if it is the polar dual of a free spectrahedron (Theorem 4.6).

### 1.2. Results on interpolation of cp maps and quantum channels

A completely positive (cp) map  $M_n \rightarrow M_m$  that is trace preserving is called a *quantum channel*, and a cp map that is trace nonincreasing for positive semidefinite arguments is a *quantum operation*. These maps figure prominently in quantum information theory [NC10].

The cp interpolation problem is formulated as follows. Given  $A \in \mathbb{S}_n^g$  and  $B \in \mathbb{S}_m^g$ , does there exist a cp map  $\Phi : M_n \rightarrow M_m$  such that, for  $1 \leq \ell \leq g$ ,

$$\Phi(A_{\ell}) = B_{\ell}?$$

One can require further that  $\Phi$  be unital, a quantum channel or a quantum operation. Imposing either of the latter two constraints pertains to quantum information theory [Hal1, Kle07, NCSB98], where one is interested in quantum channels (resp., quantum operations) that send a prescribed set of quantum states into another set of quantum states.

*1.2.1. Algorithmic aspects.* A byproduct of the methods used in this paper and in [HKM13a] produces solutions to these cp interpolation problems in the form of an algorithm, Theorem 3.4 in Subsection 3.2. Ambrozie and Gheondea [AG15] solved these interpolation problems with LMI algorithms. While equivalent to theirs, our solutions are formulated as concrete LMIs that can be solved with a standard semidefinite programming (SDP) solver. These interpolation results are a basis for proofs of the results outlined in Section 1.1.

*1.3. Free tracial Hahn–Banach Theorem*

Matrix convex sets are closely connected with ranges of unital cp maps. Indeed, given a tuple  $A \in \mathbb{S}_m^g$ , the matrix convex hull of the set  $\{A\}$  is the sequence of sets

$$\left( \{B \in \mathbb{S}_n^g : B_j = \Phi(A_j) \text{ for some ucp map } \Phi : M_m \rightarrow M_n\} \right)_n.$$

From the point of view of quantum information theory it is natural to consider hulls of ranges of quantum operations. We say that  $\mathcal{Y} \subseteq \mathbb{S}^g$  is *contractively tracial* if for all positive integers  $m, n$  and finite collections  $\{C_\ell\}$  of  $n \times m$  matrices such that

$$\sum C_\ell^* C_\ell \preceq I_m,$$

we have  $\sum C_j Y C_j^* \in \mathcal{Y}(n)$  for all  $Y \in \mathcal{Y}(m)$ . It is clear that an intersection of contractively tracial sets is again contractively tracial, giving rise, in the usual way, to the notion of the *contractive tracial hull*, denoted  $\text{cthull}$ . For a tuple  $A$ ,

$$\text{cthull}(A) = \{B : \Phi(A) = B \text{ for some quantum operation } \Phi\}.$$

While the unital and quantum interpolation problems have very similar formulations, contractive tracial hulls possess far less structure than matrix convex hulls. A subset  $\mathcal{Y} \subseteq \mathbb{S}^g$  is *levelwise convex* if each  $\mathcal{Y}(m)$  is convex (as a subset of  $\mathbb{S}_m^g$ ). (Generally *levelwise* refers to a property holding for each  $\mathcal{Y}(m) \subseteq \mathbb{S}_m^g$ .) As is easily seen, contractive tracial hulls need not be levelwise convex nor closed with respect to direct sums. However, they do have a few good properties. These we develop in Section 6.

Section 7 contains notions of free spectrahedra and corresponding Hahn–Banach type separation theorems tailored to the tracial setting. To understand *convex* contractively tracial sets, given  $B \in \mathbb{S}_k^g$ , let  $\mathfrak{H}_B = (\mathfrak{H}_B(m))_m$  denote the sequence of sets

$$\mathfrak{H}_B(m) = \left\{ Y \in \mathbb{S}_m^g : \exists T \succeq 0, \text{tr}(T) \leq 1, I \otimes T - \sum B_j \otimes Y_j \succeq 0 \right\}.$$

We call  $\mathfrak{H}_B$  a *tracial spectrahedron*. (Note that  $\mathfrak{H}_B$  is not closed under direct sums, and thus it is not a matrix convex set.) These  $\mathfrak{H}_B$  are all contractively tracial and levelwise

convex. Indeed for such structural reasons, and in view of the tracial Hahn–Banach separation theorem immediately below, we believe these to be the natural analogs of free spectrahedra in the tracial context.

**Theorem 1.1** (cf. Theorem 7.6). *If  $\mathcal{Y} \subseteq \mathbb{S}^g$  is contractively tracial, levelwise convex and closed, and if  $Z \in \mathbb{S}_m^g$  is not in  $\mathcal{Y}(m)$ , then there exists a  $B \in \mathbb{S}_m^g$  such that  $\mathcal{Y} \subseteq \mathfrak{H}_B$ , but  $Z \notin \mathfrak{H}_B$ .*

Because of the asymmetry between  $B$  and  $Y$  in the definition of  $\mathfrak{H}_B$ , there is a second type of tracial spectrahedron. Given  $Y \in \mathbb{S}_k^g$ , we define the *opp-tracial spectrahedron* as the sequence  $\mathfrak{H}_Y^{\text{opp}} = (\mathfrak{H}_Y^{\text{opp}}(m))_m$  where

$$\mathfrak{H}_Y^{\text{opp}}(m) = \left\{ B \in \mathbb{S}_m^g : \exists T \geq 0, \text{tr}(T) \leq 1, I \otimes T - \sum B_j \otimes Y_j \geq 0 \right\}. \tag{1.5}$$

Proposition 7.11 computes the hulls resulting from the two different double duals determined by the two notions of tracial spectrahedron.

### 1.4. Reader’s guide

The paper is organized as follows. Section 2 introduces terminology and notation used throughout the paper. Section 3 solves the cp interpolation problems, and includes a background section on cp maps. Section 4 contains our main results on polar duals, free spectrahedra and free spectrahedrops. It uses the results of Section 3. In particular, we show that a matrix convex set is finitely generated if and only if it is the polar dual of a free spectrahedron (Theorem 4.6). Furthermore, we prove that the polar dual of a free spectrahedron is again a free spectrahedron (Theorem 4.11). Section 5 contains the “perfect” Convex Positivstellensatz for free polynomials positive semidefinite on free spectrahedrops. The proof depends upon the results of Section 4. In Section 6 we introduce tracial sets and hulls and discuss their connections with the quantum interpolation problems from Section 3. Finally, Section 7 introduces tracial spectrahedra and proves a Hahn–Banach separation theorem in the tracial context (Theorem 7.6). This theorem is then used to suggest corresponding notions of duality. Section 8 contains examples.

## 2. Preliminaries

This section introduces terminology and presents preliminaries on free polynomials, free sets and free convexity that we need.

### 2.1. Free sets

A set  $\Gamma \subseteq \mathbb{S}^g$  is closed with respect to (*simultaneous*) *unitary conjugation* if for each  $n$ , each  $A \in \Gamma(n)$  and each  $n \times n$  unitary matrix  $U$ ,

$$U^*AU = (U^*A_1U, \dots, U^*A_gU) \in \Gamma(n).$$

The set  $\Gamma$  is a *free set* if it is closed with respect to direct sums (see (1.1)) and simultaneous unitary conjugation. In particular, a matrix convex set is a free set. We refer the reader to [Voi04, Voi10, KVV14, MS11, Pop10, AM15, BB07] for a systematic study of free sets and free function theory. The set  $\Gamma$  is (*uniformly*) *bounded* if there is a  $C \in \mathbb{R}_{>0}$  such that  $C - \sum X_j^2 \geq 0$  for all  $X \in \Gamma$ .

### 2.2. Free polynomials

One natural way free sets arise is as the nonnegativity set of a free polynomial. Given positive integers  $\ell$  and  $\nu$ , let  $\mathbb{C}^{\ell \times \nu}$  denote the collection of  $\ell \times \nu$  complex matrices. An expression of the form

$$P = \sum_w B_w w \in \mathbb{C}^{\ell \times \nu} \langle x \rangle,$$

where  $B_w \in \mathbb{C}^{\ell \times \nu}$ , and the sum is a finite sum over the words in the variables  $x$ , is a *free (noncommutative) matrix-valued polynomial*. The collection of all  $\ell \times \nu$ -valued free polynomials is denoted  $\mathbb{C}^{\ell \times \nu} \langle x \rangle$ , and  $\mathbb{C} \langle x \rangle$  denotes the set of scalar-valued free polynomials. We use  $\mathbb{C}^{\ell \times \nu} \langle x \rangle_k$  to denote free polynomials of degree  $\leq k$ . Here the *degree* of a word is its length. The free polynomial  $P$  is evaluated at an  $X \in \mathbb{S}_n^g$  by

$$P(X) = \sum_{w \in \langle x \rangle} B_w \otimes w(X) \in \mathbb{C}^{\ell n \times \nu n},$$

where  $\otimes$  denotes the (Kronecker) tensor product.

There is a natural involution  $*$  on words that reverses the order. This involution extends to  $\mathbb{C}^{\ell \times \nu} \langle x \rangle$  by

$$P^* = \sum_w B_w^* w^* \in \mathbb{C}^{\nu \times \ell} \langle x \rangle.$$

If  $\mu = \ell$  and  $P^* = P$ , then  $P$  is *symmetric*. Note that if  $P \in \mathbb{C}^{\ell \times \ell} \langle x \rangle$  is symmetric, and  $X \in \mathbb{S}_n^g$ , then  $P(X) \in \mathbb{C}^{\ell n \times \ell n}$  is a hermitian matrix.

### 2.3. Free semialgebraic sets

The *nonnegativity set*  $\mathcal{D}_P \subseteq \mathbb{S}^g$  of a symmetric free polynomial is the sequence of sets

$$\mathcal{D}_P(n) = \{X \in \mathbb{S}_n^g : P(X) \geq 0\}.$$

It is readily checked that  $\mathcal{D}_P$  is a free set. By analogy with (commutative) real algebraic geometry, we call  $\mathcal{D}_P$  a *basic free semialgebraic set*. Often it is assumed that  $P(0) \succ 0$ . The free set  $\mathcal{D}_P$  has the additional property that it is *closed with respect to restriction to reducing subspaces*, that is, if  $X \in \mathcal{D}_P(n)$  and  $\mathcal{H} \subseteq \mathbb{C}^n$  is a reducing (equivalently invariant) subspace for  $X$  of dimension  $m$ , then  $X$  restricted to  $\mathcal{H}$  is in  $\mathcal{D}_P(m)$ .

2.4. Free convexity

If  $\Gamma$  is matrix convex, it is easy to show that  $\Gamma$  is levelwise convex. More generally, if  $A^\ell = (A_1^\ell, \dots, A_g^\ell)$  are in  $\Gamma(n_\ell)$  for  $1 \leq \ell \leq k$ , then  $A = \bigoplus_{\ell=1}^k A^\ell \in \Gamma(n)$ , where  $n = \sum n_\ell$ . Hence, if  $V_\ell$  are  $n_\ell \times m$  matrices (for some  $m$ ) such that  $V = (V_1^* \dots V_k^*)^*$  is an isometry (equivalently  $\sum_{\ell=1}^k V_\ell^* V_\ell = I_m$ ), then

$$V^*AV = \sum_{\ell=1}^k V_\ell^* A^\ell V_\ell \in \Gamma(m). \tag{2.1}$$

A sum as in (2.1) is a matrix (free) convex combination of the  $g$ -tuples  $\{A^\ell : \ell = 1, \dots, k\}$ .

**Lemma 2.1** ([HKM16, Lemma 2.3]). *Suppose  $\Gamma$  is a free subset of  $\mathbb{S}^g$ .*

- (1) *If  $\Gamma$  is closed with respect to restriction to reducing subspaces, then the following are equivalent:*
  - (i)  *$\Gamma$  is matrix convex;*
  - (ii)  *$\Gamma$  is levelwise convex.*
- (2) *If  $\Gamma$  is (nonempty and) matrix convex, then  $0 \in \Gamma(1)$  if and only if  $\Gamma$  is closed with respect to (simultaneous) conjugation by contractions.*

Convex subsets of  $\mathbb{R}^g$  are defined as intersections of half-spaces and are thus described by linear functionals. Analogously, matrix convex subsets of  $\mathbb{S}^g$  are defined by linear pencils [EW97, HM12]. We next present basic facts about linear pencils and their associated matrix convex sets.

2.4.1. *Linear pencils.* Recall the definition (see (1.2)) of the (affine) linear pencil  $L_A(x)$  associated to a tuple  $A = (A_0, \dots, A_g) \in \mathbb{S}_k^{g+1}$ . If  $A_0 = 0$ , i.e.,  $A = (A_1, \dots, A_g) \in \mathbb{S}_k^g$ , let

$$\Lambda_A(x) = \sum_{j=1}^g A_j x_j$$

denote the corresponding homogeneous (truly) linear pencil, and

$$\mathfrak{L}_A = I - \Lambda_A$$

the associated monic linear pencil.

The pencil  $L_A$  (and also  $\mathfrak{L}_A$ ) is a free polynomial with matrix coefficients, so it is naturally evaluated on  $X \in \mathbb{S}_n^g$  using (Kronecker’s) tensor product, yielding (1.3). The free semialgebraic set  $\mathcal{D}_{L_A}$  is easily seen to be matrix convex. We will refer to  $\mathcal{D}_{L_A}$  as a free spectrahedron or free LMI domain, and say that a free set  $\Gamma$  is freely LMI representable if there is a linear pencil  $L$  such that  $\Gamma = \mathcal{D}_L$ . In particular, if  $\Gamma$  is freely LMI representable with a monic  $\mathfrak{L}_A$ , then  $0$  is in the interior of  $\Gamma(1)$ .

The following is a special case of a theorem due to Effros and Winkler [EW97] (see also [HKM13a, Theorem 3.1]). Given a free set  $\Gamma$ , if  $0 \in \Gamma(1)$ , then  $0 \in \Gamma(n)$  for each  $n$ . In this case we will write  $0 \in \Gamma$ .



**Theorem 2.2.** *If  $\mathcal{C} = (\mathcal{C}(n))_{n \in \mathbb{N}} \subseteq \mathbb{S}^{\mathbb{S}}$  is a closed matrix convex set containing 0 and  $Y \in \mathbb{S}_m^{\mathbb{S}}$  is not in  $\mathcal{C}(m)$ , then there is a monic linear pencil  $\mathcal{L}$  of size  $m$  such that  $\mathcal{L}(X) \succeq 0$  for all  $X \in \mathcal{C}$ , but  $\mathcal{L}(Y) \not\succeq 0$ .*

By the following result from [HM12], linear matrix inequalities account for matrix convexity of free semialgebraic sets.

**Theorem 2.3.** *Fix a symmetric matrix polynomial  $p$ . If  $p(0) \succ 0$  and the strict positivity set  $\mathfrak{P}_p = \{X : p(X) \succ 0\}$  of  $p$  is bounded, then  $\mathfrak{P}_p$  is matrix convex if and only if it is freely LMI representable with a monic pencil.*

### 2.5. A Convex Positivstellensatz and LMI domination

Positivstellensätze are pillars of real algebraic geometry [BCR98]. We next recall the Positivstellensatz for a free polynomial  $p$ . It is the algebraic certificate for nonnegativity of  $p$  on the free spectrahedron  $\mathcal{D}_L$  from [HKM12]. It is “perfect” in the sense that  $p$  is only assumed to be nonnegative on  $\mathcal{D}_L$ , and we obtain degree bounds on the scale of  $\deg(p)/2$  for the polynomials involved in the positivity certificate. In Section 5, we will extend this Positivstellensatz to free spectrahedrops (i.e., projections of free spectrahedra)—see Theorem 5.1.

**Theorem 2.4.** *Suppose  $\mathcal{L}$  is a monic linear pencil. A matrix polynomial  $p$  is positive semidefinite on  $\mathcal{D}_{\mathcal{L}}$  if and only if it has a weighted sum of squares representation with optimal degree bounds:*

$$p = s^*s + \sum_j^{\text{finite}} f_j^* \mathcal{L} f_j,$$

where  $s, f_j$  are matrix polynomials of degree no greater than  $\deg(p)/2$ .

In particular, if  $\mathcal{L}_A, \mathcal{L}_B$  are monic linear pencils, then  $\mathcal{D}_{\mathcal{L}_B} \subseteq \mathcal{D}_{\mathcal{L}_A}$  if and only if there exists a positive integer  $\mu$  and a contraction  $V$  such that

$$A = V^*(I_{\mu} \otimes B)V. \tag{2.2}$$

If  $\mathcal{D}_{\mathcal{L}_B}$  is bounded, then  $V$  can be chosen to be an isometry.

*Proof.* The first statement is [HKM12, Theorem 1.1]. Applying this result to the LMI domination problem  $\mathcal{D}_{\mathcal{L}_B} \subseteq \mathcal{D}_{\mathcal{L}_A}$ , we see that  $\mathcal{D}_{\mathcal{L}_B} \subseteq \mathcal{D}_{\mathcal{L}_A}$  is equivalent to

$$\mathcal{L}_A(x) = S^*S + \sum_{j=1}^{\mu} V_j^* \mathcal{L}_B(x) V_j \tag{2.3}$$

for some matrices  $S, V_j$ , i.e.,

$$I = S^*S + \sum_j V_j^* V_j = S^*S + V^*V, \tag{2.4}$$

$$A = \sum_{j=1}^{\mu} V_j^* B V_j = V^*(I_{\mu} \otimes B)V, \tag{2.5}$$

where  $V$  is the block column matrix of the  $V_j$ . Equation (2.4) simply says that  $V$  is a contraction, and (2.5) is (2.2). The last statement is proved in [HKM13a]. Alternatively, as is shown in [HKM12, Proposition 4.2], if  $\mathcal{D}_{\mathcal{L}_B}$  is bounded, then there are finitely many matrices  $W_j$  such that

$$I = \sum W_j^* \mathcal{L}_B(x) W_j.$$

Writing  $S^*S = \sum (W_j S)^* \mathcal{L}_B(x) (W_j S)$  and inserting into (2.3) completes the proof.  $\square$

Example 8.1 shows that it is not necessarily possible to choose  $V$  in (2.5) to be an isometry without additional hypothesis on the tuple  $B$ .

### 3. Completely positive interpolation

Theorem 3.4 below provides a solution to three cp interpolation problems in terms of concrete LMIs that can be solved with a standard semidefinite programming (SDP) solver. The unital cp interpolation problem comes from efforts to understand matrix convex sets that arise in convex optimization. Its solution plays an important role in the proof of the main result on the polar dual of a free spectrahedron (Theorem 4.11) via its appearance in the proof of Proposition 4.14.

The trace preserving and trace nonincreasing cp interpolation problems arise in quantum information theory and the study of quantum channels, where one is interested in sending one (finite) set of prescribed quantum states to another.

#### 3.1. Basics of completely positive maps

This subsection collects basic facts about completely positive (cp) maps  $\phi : \mathcal{S} \rightarrow M_d$ , where  $\mathcal{S}$  is a subspace of  $M_n$  closed under conjugate transpose (see for instance [Pau02]) containing a positive definite matrix. Thus  $\mathcal{S}$  is an *operator system*.

Suppose  $\mathcal{S}$  is a subspace of  $M_n$  closed under conjugate transpose,  $\phi : \mathcal{S} \rightarrow M_d$  is a linear map and  $\ell$  is a positive integer. The ( $\ell$ -th) *ampliation*  $\phi_\ell : M_\ell(\mathcal{S}) \rightarrow M_\ell(M_d)$  of  $\phi$  is defined by applying  $\phi$  entrywise,

$$\phi_\ell(S_{j,k}) = (\phi(S_{j,k})).$$

The map  $\phi$  is *symmetric* if  $\phi(S^*) = \phi(S)^*$ , and it is *completely positive* if each  $\phi_\ell$  is positive in the sense that if  $S \in M_\ell(\mathcal{S})$  is positive semidefinite, then so is  $\phi_\ell(S) \in M_\ell(M_d)$ . In what follows,  $\mathcal{S}$  is often a subspace of  $\mathbb{S}_n$  (and is thus automatically closed under the conjugate transpose operation).

The *Choi matrix* of a mapping  $\phi : M_n \rightarrow M_d$  is the  $n \times n$  block matrix with  $d \times d$  matrix entries given by

$$(C_\phi)_{i,j} = (\phi(E_{i,j}))_{i,j}.$$

On the other hand, a matrix  $C = (C_{i,j}) \in M_n(M_d)$  determines a mapping  $\phi_C : M_n \rightarrow M_d$  by  $\phi_C(E_{i,j}) = C_{i,j} \in M_d$ . A matrix  $C$  is a *Choi matrix for  $\phi : \mathcal{S} \rightarrow M_d$*  if the mapping  $\phi_C$  agrees with  $\phi$  on  $\mathcal{S}$ .

**Theorem 3.1.** For  $\phi : M_n \rightarrow M_d$ , the following are equivalent:

- (a)  $\phi$  is completely positive;
- (b) the Choi matrix  $C_\phi$  is positive semidefinite.

Suppose  $\mathcal{S} \subseteq M_n$  is an operator system. For a symmetric  $\phi : \mathcal{S} \rightarrow M_d$ , the following are equivalent:

- (i)  $\phi$  is completely positive;
- (ii)  $\phi_d$  is positive;
- (iii) there exists a completely positive mapping  $\Phi : M_n \rightarrow M_d$  extending  $\phi$ ;
- (iv) there is a positive semidefinite Choi matrix for  $\phi$ ;
- (v) there exist  $n \times d$  matrices  $V_1, \dots, V_{nd}$  such that

$$\phi(A) = \sum V_j^* A V_j. \quad (3.1)$$

Finally, for a subspace  $\mathcal{S}$  of  $M_n$ , a mapping  $\phi : \mathcal{S} \rightarrow M_d$  has a completely positive extension  $\Phi : M_n \rightarrow M_d$  if and only if  $\phi$  has a positive semidefinite Choi matrix.

**Lemma 3.2.** The cp mapping  $\phi : M_n \rightarrow M_d$  as in (3.1) is

- (a) unital (that is,  $\phi(I_n) = I_d$ ) if and only if

$$\sum_j V_j^* V_j = I;$$

- (b) trace preserving if and only if

$$\sum_j V_j V_j^* = I;$$

- (c) trace nonincreasing for positive semidefinite matrices (i.e.,  $\text{tr}(\phi(P)) \leq \text{tr}(P)$  for all positive semidefinite  $P$ ) if and only if

$$\sum_j V_j V_j^* \leq I.$$

*Proof.* We prove (c), and leave (a) and (b) as an easy exercise for the reader. For  $A \in M_n$ ,

$$\text{tr}(\phi(A)) = \sum_j \text{tr}(V_j^* A V_j) = \text{tr}\left(A \sum_j V_j V_j^*\right).$$

Hence the trace nonincreasing property for  $\phi$  is equivalent to

$$\text{tr}\left(P\left(I - \sum_j V_j V_j^*\right)\right) \geq 0$$

for all positive semidefinite  $P$ , i.e.,  $I - \sum_j V_j V_j^* \geq 0$ . □

**Proposition 3.3.** *The linear mapping  $\phi : M_n \rightarrow M_d$  is*

(a) *unital (that is,  $\phi(I_n) = I_d$ ) if and only if its Choi matrix  $C$  satisfies*

$$\sum_{j=1}^n C_{j,j} = I;$$

(b) *trace preserving if and only if its Choi matrix  $C$  satisfies*

$$(\operatorname{tr}(C_{i,j}))_{i,j=1}^n = I_n;$$

(c) *trace nonincreasing for positive semidefinite matrices (i.e.,  $\operatorname{tr}(\phi(P)) \leq \operatorname{tr}(P)$  for all positive semidefinite  $P$ ) if and only if its Choi matrix  $C$  satisfies*

$$(\operatorname{tr}(C_{i,j}))_{i,j} \preceq I_n,$$

*Proof.* Statement (a) follows from

$$\phi(I_n) = \phi\left(\sum_{j=1}^n E_{j,j}\right) = \sum_{j=1}^n C_{j,j},$$

where  $C$  is the Choi matrix for  $\phi$ . Here  $E_{i,j}$  denote the matrix units.

For (b), let  $X = \sum_{i,j=1}^n \alpha_{i,j} E_{i,j}$ . Then

$$\operatorname{tr}(X) = \sum_{i=1}^n \alpha_{i,i}, \quad \operatorname{tr}(\phi(X)) = \sum_{i,j=1}^n \alpha_{i,j} \operatorname{tr}(C_{i,j}).$$

Since  $\operatorname{tr}(\phi(X)) = \operatorname{tr}(X)$  for all  $X$ , this linear system yields  $\operatorname{tr}(C_{i,j}) = \delta_{i,j}$  for all  $i, j$ .

Finally, for (c), if  $\phi$  is trace nonincreasing, choosing  $X = xx^*$  a rank one matrix,  $X = (x_i x_j)$ , we find that

$$\sum x_i x_j \operatorname{tr}(C_{i,j}) = \operatorname{tr}(\phi(X)) \leq \operatorname{tr}(X) = \sum x_i^2.$$

Hence  $I - (\operatorname{tr}(C_{i,j})) \succeq 0$ . Conversely, if  $I - (\operatorname{tr}(C_{i,j})) \succeq 0$ , then for any positive semidefinite rank one matrix  $X$ , the computation above shows that  $\operatorname{tr}(\phi(X)) \leq \operatorname{tr}(X)$ . Finally, use the fact that any positive semidefinite matrix is a sum of rank one positive semidefinite matrices to complete the proof.  $\square$

The Arveson extension theorem [Arv69] says that any cp (resp. ucp) map on an operator system extends to a cp (resp. ucp) map on the full algebra. Example 8.2 shows that a CPTP map need not extend to a CPTP map on the full algebra.

### 3.2. Quantum interpolation problems and semidefinite programming

The cp interpolation problem is formulated as follows. Given  $A^1 \in \mathbb{S}_n^g$  and  $A^2 \in \mathbb{S}_m^g$ , does there exist a cp map  $\Phi : M_n \rightarrow M_m$  such that

$$A_\ell^2 = \Phi(A_\ell^1) \quad \text{for } \ell = 1, \dots, g?$$

One can require further that

- (1)  $\Phi$  be *unital*, or
- (2)  $\Phi$  be *trace preserving*, or
- (3)  $\Phi$  be *trace nonincreasing* in the sense that  $\text{tr}(\Phi(P)) \leq \text{tr}(P)$  for all positive semidefinite  $P$ .

Our solutions to these interpolation problems are formulated as concrete LMIs that can be solved with a standard semidefinite programming (SDP) solver. They are equivalent to, but stated quite differently than, the earlier results in [AG15].

**Theorem 3.4.** *Suppose for  $\ell = 1, \dots, g$  the matrices  $A_\ell^1 \in \mathbb{S}_n$  and  $A_\ell^2 \in \mathbb{S}_m$  are given. Let  $\alpha_{p,q}^\ell$  denote the  $(p, q)$ -entry of  $A_\ell^1$ . There exists a cp map  $\Phi : M_n \rightarrow M_m$  that solves the interpolation problem*

$$\Phi(A_\ell^1) = A_\ell^2, \quad \ell = 1, \dots, g,$$

if and only if the following feasibility semidefinite programming problem has a solution:

$$(C_{p,q})_{p,q=1}^n =: C \succeq 0, \quad \forall \ell = 1, \dots, g, \quad \sum_{p,q} \alpha_{p,q}^\ell C_{p,q} = A_\ell^2, \quad (3.2)$$

for the unknown  $mn \times mn$  symmetric matrix  $C = (C_{p,q})_{p,q=1}^n$  consisting of  $m \times m$  blocks  $C_{p,q}$ . Furthermore,

- (1) the map  $\Phi$  is unital if and only if, in addition to (3.2),

$$\sum_{p=1}^n C_{p,p} = I_m; \quad (3.3)$$

- (2) the map  $\Phi$  is a quantum channel if and only if, in addition to (3.2),

$$(\text{tr}(C_{p,q}))_{p,q} = I_n; \quad (3.4)$$

- (3) the map  $\Phi$  is a quantum operation if and only if, in addition to (3.2),

$$(\text{tr}(C_{p,q}))_{p,q} \preceq I_n. \quad (3.5)$$

In each case the constraints on  $C$  are LMIs, and the set of solutions  $C$  constitutes a bounded spectrahedron.

**Remark 3.5.** In the unital case the resulting spectrahedron is free: for fixed  $A^1 \in \mathbb{S}_n^g$ , the sequence of solution sets to (3.2) and (3.3) parametrized over  $m$  is a free spectrahedron (see Proposition 4.14 for details). In the two quantum cases, for each  $m$ , the solutions  $\mathcal{D}(m)$  at level  $m$  form a spectrahedron, but the sequence  $\mathcal{D} = (\mathcal{D}(m))_m$  is in general not a free spectrahedron since it fails to respect direct sums.

*Proof of Theorem 3.4.* This interpolation result is a consequence of Theorem 3.1. Let  $\mathcal{S}$  denote the span of  $\{A_\ell^1\}$  and  $\phi$  the mapping from  $\mathcal{S}$  to  $M_m$  defined by  $\phi(A_\ell^1) = A_\ell^2$ . This mapping has a completely positive extension  $\Phi : M_n \rightarrow M_m$  if and only if it has a positive semidefinite Choi matrix. The conditions on  $C$  evidently are exactly those needed to say that  $C$  is a positive semidefinite Choi matrix for  $\phi$ .

The additional conditions in (3.3) and (3.4) (i.e.,  $\phi(I_n) = I_m$  and trace preservation) are clearly linear, so produce a spectrahedron in  $\mathbb{S}_{mn}$ . Both spectrahedra are bounded. Indeed, in each case  $C_{p,p} \leq I_m$ , so  $C \leq I_{mn}$ . Likewise, the additional condition in (3.5) is an LMI constraint, producing a bounded spectrahedron.  $\square$

We note that cp maps between subspaces of matrix algebras in the absence of positive definite elements were treated in [HKN14, Section 8] (see also [KS13, KTT13]).

#### 4. Free spectrahedra and polar duals

This section starts by recalling the definition of a free spectrahedron as the coordinate projection of a spectrahedron. It then continues with a review of free polar duals [EW97] and their basic properties, before turning to two main results, stated now without technical hypotheses. Firstly, a free convex set is, in a canonical sense, generated by a finite set (equivalently a single point) if and only if it is the polar dual of a free spectrahedron (Theorem 4.6). Secondly, the polar dual of a free spectrahedron is again a free spectrahedron (Theorem 4.11).

##### 4.1. Projections of free spectrahedra: free spectrahedra

Let  $L$  be a linear pencil in the variables  $(x_1, \dots, x_g; y_1, \dots, y_h)$ . Thus, for some  $d$  and  $d \times d$  hermitian matrices  $D, \Omega_1, \dots, \Omega_g, \Gamma_1, \dots, \Gamma_h$ ,

$$L(x, y) = D + \sum_{j=1}^g \Omega_j x_j + \sum_{\ell=1}^h \Gamma_\ell y_\ell.$$

The set

$$\text{proj}_x \mathcal{D}_L(1) = \{x \in \mathbb{R}^g : \exists y \in \mathbb{R}^h, L(x, y) \succeq 0\}$$

is known as a *spectrahedral shadow* or a *semidefinite programming (SDP) representable set* [BPR13] and the representation afforded by  $L$  is an *SDP representation*. SDP representable sets are evidently convex and lie in a middle ground between LMI representable sets and general convex sets. They play an important role in convex optimization [Nem06]. If  $S \subseteq \mathbb{R}^g$  is closed semialgebraic and satisfies some mild additional hypothesis, it is proved in [HN10] based upon the Lasserre–Parrilo construction [Las09, Par06] that the convex hull of  $S$  is SDP representable.

Given a linear pencil  $L$ , let  $\text{proj}_x \mathcal{D}_L = (\text{proj}_x \mathcal{D}_L(n))_n$  denote the free set

$$\text{proj}_x \mathcal{D}_L(n) = \{X \in \mathbb{S}_n^g : \exists Y \in \mathbb{S}_n^h, L(X, Y) \succeq 0\}.$$

We call a set of the form  $\text{proj}_x \mathcal{D}_L$  a *free spectrahedron* and  $\mathcal{D}_L$  an *LMI lift* of  $\text{proj}_x \mathcal{D}_L$ . Thus a free spectrahedron is a coordinate projection of a free spectrahedron. Clearly, free spectrahedrons are matrix convex. In particular, they are closed with respect to restrictions to reducing subspaces.

**Lemma 4.1** ([HKM16, §4.1]). *If  $\mathcal{K} = \text{proj}_x \mathcal{D}_L$  is a free spectrahedron containing  $0 \in \mathbb{R}^g$  in the interior of  $\mathcal{K}(1)$ , then there exists a monic linear pencil  $\mathfrak{L}(x, y)$  such that*

$$\mathcal{K} = \text{proj}_x \mathcal{D}_{\mathfrak{L}} = \{X \in \mathbb{S}^g : \exists Y \in \mathbb{S}^h, \mathfrak{L}(X, Y) \succeq 0\}.$$

*If, in addition,  $\mathcal{D}_L$  is bounded, then we may further ensure  $\mathcal{D}_{\mathfrak{L}}$  is bounded.*

If the free spectrahedron  $\mathcal{K}$  is closed and bounded, and contains 0 in its interior, then there is a monic linear pencil  $\mathfrak{L}$  such that  $\mathcal{D}_{\mathfrak{L}}$  is bounded and  $\mathcal{K} = \text{proj}_x \mathcal{D}_{\mathfrak{L}}$  (see Theorem 4.11).

Let  $p = 1 - x_1^2 - x_2^4$ . It is well known that  $\mathcal{D}_p(1) = \{(x, y) \in \mathbb{R}^2 : 1 - x_1^2 - x_2^4 \geq 0\}$  is a spectrahedral shadow. On the other hand,  $\mathcal{D}_p(2)$  is not convex (in the usual sense), and hence  $\mathcal{D}_p$  is not a spectrahedron. Further details can be found in Example 8.3.

#### 4.2. Basics of polar duals

By precise analogy with the classical  $\mathbb{R}^g$  notion, the *free polar dual*  $\mathcal{K}^\circ = (\mathcal{K}^\circ(n))_n$  of a free set  $\mathcal{K} \subseteq \mathbb{S}^g$  is

$$\mathcal{K}^\circ(n) := \left\{ A \in \mathbb{S}_n^g : \mathfrak{L}_A(X) = I \otimes I - \sum_{j=1}^g A_j \otimes X_j \succeq 0 \text{ for all } X \in \mathcal{K} \right\}.$$

Given  $\varepsilon > 0$ , consider the free  $\varepsilon$ -ball centered at 0,

$$\mathcal{N}_\varepsilon := \{X \in \mathbb{S}^g : \|X\| \leq \varepsilon\} = \left\{ X : \varepsilon^2 I \succeq \sum_j X_j^2 \right\}.$$

It is easy to see that its polar dual is bounded. In fact,

$$\mathcal{N}_{1/(g\varepsilon)} \subseteq \mathcal{N}_\varepsilon^\circ \subseteq \mathcal{N}_{\sqrt{g}/\varepsilon}.$$

We say that 0 is in the interior of the subset  $\Gamma \subseteq \mathbb{S}^g$  if  $\Gamma$  contains some free  $\varepsilon$ -ball centered at 0.

**Lemma 4.2.** *Suppose  $\mathcal{K} \subseteq \mathbb{S}^g$  is matrix convex. The following are equivalent:*

- (i)  $0 \in \mathbb{R}^g$  is in the interior of  $\mathcal{K}(1)$ ;
- (ii)  $0 \in \mathbb{S}_n^g$  is in the interior of  $\mathcal{K}(n)$  for some  $n$ ;
- (iii)  $0 \in \mathbb{S}_n^g$  is in the interior of  $\mathcal{K}(n)$  for all  $n$ ;
- (iv) 0 is in the interior of  $\mathcal{K}$ .

*Proof.* It is clear that (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii). Assume (ii) holds. There is an  $\varepsilon > 0$  with  $\mathcal{N}_\varepsilon(n) \subseteq \mathcal{K}(n)$ . Since  $\mathcal{K}$  is closed with respect to restriction to reducing subspaces, and

$$\mathcal{N}_\varepsilon(1) \oplus \cdots \oplus \mathcal{N}_\varepsilon(1) \subseteq \mathcal{N}_\varepsilon(n),$$

we see  $\mathcal{N}_\varepsilon(1) \subseteq \mathcal{K}(1)$ , i.e., (i) holds.

Now suppose (i) holds, i.e.,  $\mathcal{N}_\varepsilon(1) \subseteq \mathcal{K}(1)$  for some  $\varepsilon > 0$ . We claim that  $\mathcal{N}_{\varepsilon/g^2} \subseteq \mathcal{K}$ . Let  $X \in \mathcal{N}_{\varepsilon/g^2}$ . It is clear that  $[-\varepsilon/g, \varepsilon/g]^g \subseteq \mathcal{K}(1)$ , hence  $[-\varepsilon/g, \varepsilon/g]^g \otimes I_n \subseteq \mathcal{K}(n)$ . Since each  $X_j$  has norm  $\leq \varepsilon/g^2$ , matrix convexity of  $\mathcal{K}$  implies that

$$(0, \dots, 0, gX_j, 0, \dots, 0) \in \mathcal{K},$$

and thus

$$X = \frac{1}{g}((gX_1, 0, \dots, 0) + \cdots + (0, \dots, 0, gX_g)) \in \mathcal{K}. \quad \square$$

For the readers' convenience, the following proposition lists some properties of  $\mathcal{K}^\circ$ . The bipolar result of item (6) is due to [EW97]. Given a collection  $\Gamma_\alpha$  of matrix convex sets, it is readily verified that  $\Gamma = (\Gamma(n))_n$  defined by  $\Gamma(n) = \bigcap_\alpha \Gamma_\alpha(n)$  is again matrix convex. Likewise, if  $\Gamma$  is matrix convex, then so is its closure  $\overline{\Gamma} = (\overline{\Gamma(n)})_n$ . Given a subset  $\mathcal{K}$  of  $\mathbb{S}^g$ , let  $\text{co}^{\text{mat}} \mathcal{K}$  denote the intersection of all matrix convex sets containing  $\mathcal{K}$ . Thus,  $\text{co}^{\text{mat}} \mathcal{K}$  is the smallest matrix convex set containing  $\mathcal{K}$ . Likewise,  $\overline{\text{co}}^{\text{mat}} \mathcal{K} = \overline{\text{co}^{\text{mat}} \mathcal{K}}$  is the smallest closed matrix convex set containing  $\mathcal{K}$ . Details, and an alternative characterization of the matrix convex hull of a free set  $\mathcal{K}$ , can be found in [HKM16].

**Proposition 4.3.** *Suppose  $\mathcal{K} \subseteq \mathbb{S}^g$ .*

- (1)  $\mathcal{K}^\circ$  is a closed matrix convex set containing 0;
- (2) if 0 is in the interior of  $\mathcal{K}$ , then  $\mathcal{K}^\circ$  is bounded;
- (3)  $\mathcal{K}(n) \subseteq \mathcal{K}^{\circ\circ}(n)$  for all  $n$ , that is,  $\mathcal{K} \subseteq \mathcal{K}^{\circ\circ}$ ;
- (4)  $\mathcal{K}$  is bounded if and only if 0 is in the interior of  $\mathcal{K}^\circ$ ;
- (5) if there is an  $m$  such that  $0 \in \mathcal{K}(m)$ , then  $\mathcal{K}^{\circ\circ} = \overline{\text{co}}^{\text{mat}} \mathcal{K}$ ;
- (6) if  $\mathcal{K}$  is a closed matrix convex set containing 0, then  $\mathcal{K} = \mathcal{K}^{\circ\circ}$ ;
- (7) if  $\mathcal{K}$  is matrix convex, then  $\mathcal{K}(1)^\circ = \mathcal{K}^\circ(1)$ .

*Proof.* Matrix convexity in (1) is straightforward.

If  $\mathcal{K}$  has 0 in its interior, then there is a small free neighborhood  $\mathcal{N}_\varepsilon$  of 0 inside  $\mathcal{K}$ . Hence  $\mathcal{K}^\circ \subseteq \mathcal{N}_\varepsilon^\circ = \mathcal{N}_{1/\varepsilon}$  is bounded.

Item (3) is a tautology. Indeed, if  $X \in \mathcal{K}(n)$ , then we want to show  $\mathfrak{L}_X(A) \geq 0$  whenever  $\mathfrak{L}_A(Y) \geq 0$  for all  $Y$  in  $\mathcal{K}$ . But this follows simply from the fact that  $\mathfrak{L}_X(A)$  and  $\mathfrak{L}_A(X)$  are unitarily equivalent.

If  $\mathcal{K}$  is bounded, then it is evident that 0 is in the interior of  $\mathcal{K}^\circ$ . If 0 is in the interior of  $\mathcal{K}^\circ$ , then, by (2),  $\mathcal{K}^{\circ\circ}$  is bounded. By (3),  $\mathcal{K} \subseteq \mathcal{K}^{\circ\circ}$ , and thus  $\mathcal{K}$  is bounded.

To prove (5), first note that  $0 \in \overline{\text{co}}^{\text{mat}} \mathcal{K}(m)$  and since  $\overline{\text{co}}^{\text{mat}} \mathcal{K}(m)$  is matrix convex,  $0 \in \overline{\text{co}}^{\text{mat}} \mathcal{K}(1)$ . Now suppose  $W \notin \overline{\text{co}}^{\text{mat}} \mathcal{K}$ . The Effros–Winkler matricial Hahn–Banach Theorem 2.2 produces a monic linear pencil  $\mathfrak{L}_A$  (with the size of  $A$  no larger than the size of  $W$ ) separating  $W$  from  $\overline{\text{co}}^{\text{mat}} \mathcal{K}$ , that is,  $\mathfrak{L}_A(W) \not\geq 0$  and  $\mathfrak{L}_A(X) \geq 0$  for



$X \in \text{co}^{\text{mat}} \mathcal{K}$ . Hence  $A \in \mathcal{K}^\circ$ . Using the unitary equivalence of  $\mathfrak{L}_W(A)$  and  $\mathfrak{L}_A(W)$  we see that  $\mathfrak{L}_W(A) \not\geq 0$ , and thus  $W \notin \mathcal{K}^{\circ\circ}$ . Thus,  $\mathcal{K}^{\circ\circ} \subseteq \overline{\text{co}}^{\text{mat}} \mathcal{K}$ . The reverse inclusion follows from (3).

Finally, suppose  $\mathcal{K}$  is matrix convex and  $y \in \mathcal{K}(1)^\circ$ . Thus,  $\sum y_j x_j = \langle y, x \rangle \leq 1$  for all  $x \in \mathcal{K}(1)$ . Given  $X \in \mathcal{K}(m)$  and a unit vector  $v \in \mathbb{C}^m$ , since  $v^* X v \in \mathcal{K}(1)$ , we have

$$1 \geq \sum y_j v^* X_j v.$$

Hence,

$$v^* \left( I - \sum y_j X_j \right) v \geq 0$$

for all unit vectors  $v$ . So  $y \in \mathcal{K}^\circ(1)$ . The reverse inclusion is immediate. □

**Corollary 4.4.** *If  $\mathcal{K} \subseteq \mathbb{S}^g$ , then  $\mathcal{K}^{\circ\circ} = \overline{\text{co}}^{\text{mat}}(\mathcal{K} \cup \{0\})$ . Here  $0 \in \mathbb{R}^g$ .*

*Proof.* Note that  $\mathcal{K}^\circ = (\mathcal{K} \cup \{0\})^\circ$ , and hence  $\mathcal{K}^{\circ\circ} = (\mathcal{K} \cup \{0\})^{\circ\circ}$ . By Proposition 4.3(5),

$$\overline{\text{co}}^{\text{mat}}(\mathcal{K} \cup \{0\}) = (\mathcal{K} \cup \{0\})^{\circ\circ}. \quad \square$$

**Lemma 4.5.** *Suppose  $\mathcal{K} \subseteq \mathbb{S}^{g+h}$ , and consider its image  $\text{proj } \mathcal{K} \subseteq \mathbb{S}^g$  under the projection  $\text{proj} : \mathbb{S}^{g+h} \rightarrow \mathbb{S}^g$ . A tuple  $A \in \mathbb{S}^g$  is in  $(\text{proj } \mathcal{K})^\circ$  if and only if  $(A, 0) \in \mathcal{K}^\circ$ .*

*Proof.* Note that  $A \in (\text{proj } \mathcal{K})^\circ$  if and only if for all  $X \in \text{proj } \mathcal{K}$  we have  $\mathfrak{L}_A(X) \geq 0$ , if and only if  $\mathfrak{L}_{(A,0)}(X, Y) \geq 0$  for all  $X \in \text{proj } \mathcal{K}$  and all  $Y \in \mathbb{S}^h$ , if and only if  $\mathfrak{L}_{(A,0)}(X, Y) \geq 0$  for all  $(X, Y) \in \mathcal{K}$ , if and only if  $(A, 0) \in \mathcal{K}^\circ$ . □

The polar dual of the set  $\{(x_1, x_2) \in \mathbb{R}^2 : 1 - x_1^2 - x_2^4 \geq 0\}$  is computed and seen not to be a spectrahedron in Example 8.4.

### 4.3. Polar duals of free spectrahedra

The next theorem completely characterizes finitely generated matrix convex sets  $\mathcal{K}$  containing 0 in their interior. Namely, such sets are exactly polar duals of bounded free spectrahedra.

**Theorem 4.6.** *Suppose  $\mathcal{K}$  is a closed matrix convex set with 0 in its interior. If there is an  $\Omega \in \mathcal{K}$  such that for each  $X \in \mathcal{K}$  there is a  $\mu \in \mathbb{N}$  and an isometry  $V$  such that*

$$X_j = V^*(I_\mu \otimes \Omega_j)V, \tag{4.1}$$

then

$$\mathcal{K}^\circ = \mathcal{D}_{\mathfrak{L}_\Omega}, \tag{4.2}$$

where  $\mathfrak{L}_\Omega$  is the monic linear pencil  $\mathfrak{L}_\Omega(x) = I - \sum \Omega_j x_j$ .

Conversely, if there is an  $\Omega$  such that (4.2) holds, then  $\Omega \in \mathcal{K}$  and for each  $X \in \mathcal{K}$  there is an isometry  $V$  such that (4.1) holds.

A variant of Theorem 4.6 in which the condition that 0 is in the interior of  $\mathcal{K}$  is replaced by the weaker hypothesis that 0 is merely in  $\mathcal{K}$ , and of course with a slightly weaker conclusion, is separately stated in Proposition 4.9 below.

**Lemma 4.7.** Suppose  $\Omega \in \mathbb{S}_d^g$  and consider the monic linear pencil  $\mathfrak{L}_\Omega = I - \sum \Omega_j x_j$ .

(1) Let  $\Omega' = \Omega \oplus 0$  where  $0 \in \mathbb{S}_d^g$ . A tuple  $X \in \mathbb{S}^g$  is in  $\mathcal{D}_{\mathfrak{L}_\Omega}^\circ$  if and only if there is an isometry  $V$  such that

$$X_j = V^*(I \otimes \Omega'_j)V.$$

(2) If  $\mathcal{D}_{\mathfrak{L}_\Omega}$  is bounded, then  $X \in \mathbb{S}^g$  is in  $\mathcal{D}_{\mathfrak{L}_\Omega}^\circ$  if and only if there is an isometry  $V$  such that (4.1) holds.

**Remark 4.8.** As an alternative of (2),  $X \in \mathcal{D}_{\mathfrak{L}_\Omega}^\circ$  if and only if there exists a contraction  $V$  such that (4.1) holds.

*Proof of Lemma 4.7.* Note that  $X \in \mathcal{D}_{\mathfrak{L}_\Omega}^\circ$  if and only if  $\mathcal{D}_{\mathfrak{L}_\Omega} \subseteq \mathcal{D}_{\mathfrak{L}_X}$ . Thus if  $\mathcal{D}_{\mathfrak{L}_\Omega}$  is bounded, then the result follows directly from the last part of Theorem 2.4. On the other hand, if  $X$  has a representation as in (4.1), then evidently  $X \in \mathcal{D}_{\mathfrak{L}_\Omega}^\circ$ .

If  $\mathcal{D}_{\mathfrak{L}_\Omega}$  is not necessarily bounded and  $X \in \mathcal{D}_{\mathfrak{L}_\Omega}^\circ(m)$ , then, by Theorem 2.4,

$$X = \sum_{j=1}^{\mu} V_j^* \Omega_j V_j$$

for some  $\mu$  and operators  $V_j : \mathbb{C}^m \rightarrow \mathbb{C}^n$  such that

$$I - \sum V_j^* V_j \geq 0.$$

There is a  $\nu > \mu$  and  $m \times n$  matrices  $V_{\mu+1}, \dots, V_\nu$  such that

$$\sum_{j=1}^{\nu} V_j^* V_j = I.$$

Let

$$W_j = \begin{cases} \begin{pmatrix} V_j \\ 0 \end{pmatrix} & \text{for } 1 \leq j \leq \mu, \\ (0 \ V_j^*)^* & \text{for } \mu < j \leq \nu. \end{cases}$$

With this choice of  $W$ , note that  $\sum W_j^* W_j = I_m$  and

$$\sum W_j^* \Omega'_j W_j = \sum W_j^* (\Omega_j \oplus 0) W_j = \sum_{j=1}^{\nu} V_j^* \Omega_j V_j = X_j. \tag{4.3}$$

If  $X$  has a representation as in (4.3) and  $\mathfrak{L}_\Omega(Y) \geq 0$ , then

$$\mathfrak{L}_X(Y) = \sum_j (W_j \otimes I)^* \mathfrak{L}_{\Omega'}(Y) (W_j \otimes I).$$

On the other hand,  $\mathfrak{L}_{\Omega'}(Y) = \mathfrak{L}_\Omega(Y) \oplus I \geq 0$ . Hence  $X \in \mathcal{D}_{\mathfrak{L}_\Omega}^\circ$ . □

*Proof of Theorem 4.6.* Suppose first (4.2) holds for some  $\Omega \in \mathbb{S}_n^g$ . Since  $\mathcal{D}_{\mathfrak{L}_\Omega}^\circ = \mathcal{K}$  and evidently  $\Omega \in \mathcal{D}_{\mathfrak{L}_\Omega}^\circ$ , it follows that  $\Omega \in \mathcal{K}$ . Since  $0$  is assumed to be in the interior of  $\mathcal{K}$ ,

its polar dual  $\mathcal{K}^\circ = \mathcal{D}_{\mathfrak{L}_\Omega}$  is bounded by Proposition 4.3. Thus, if  $X \in \mathcal{K} = \mathcal{D}_{\mathfrak{L}_\Omega}^\circ$ , then by Lemma 4.7,  $X$  has a representation as in (4.1).

Conversely, assume that  $\Omega \in \mathcal{K}$  has the property that any  $X \in \mathcal{K}$  can be represented as in (4.1). Consider the matrix convex set

$$\Gamma = \{V^*(I_\mu \otimes \Omega)V : \mu \in \mathbb{N}, V^*V = I\}.$$

Since  $\Omega \in \mathcal{K}$ , it follows that  $\Gamma \subseteq \mathcal{K}$ . On the other hand, the hypothesis is that  $\mathcal{K} \subseteq \Gamma$ . Hence  $\mathcal{K} = \Gamma$ . Now, for  $\mathfrak{L}_X$  a monic linear pencil,  $\mathfrak{L}_X(\Omega) \succeq 0$  if and only if

$$\mathfrak{L}_X(V^*(I_\mu \otimes \Omega)V) = (V \otimes I)^* \mathfrak{L}_X(I_\mu \otimes \Omega) (V \otimes I) \succeq 0$$

for all choices of  $\mu$  and isometries  $V$ . Thus,  $X \in \mathcal{K}^\circ$  if and only if  $\mathfrak{L}_X(\Omega) \succeq 0$ . On the other hand,  $\mathfrak{L}_X(\Omega)$  is unitarily equivalent to  $\mathfrak{L}_\Omega(X)$ . Thus  $X \in \mathcal{K}^\circ$  if and only if  $X \in \mathcal{D}_{\mathfrak{L}_\Omega}$ .  $\square$

**Proposition 4.9.** *Suppose  $\mathcal{K}$  is a closed matrix convex set containing 0. If there is an  $\Omega \in \mathcal{K}$  such that for each  $X \in \mathcal{K}$  there is a  $\mu \in \mathbb{N}$  and an isometry  $V$  such that*

$$X_j = V^*(I_\mu \otimes \Omega'_j)V, \tag{4.4}$$

then

$$\mathcal{K}^\circ = \mathcal{D}_{\mathfrak{L}_\Omega}, \tag{4.5}$$

where  $\mathfrak{L}_\Omega$  is the monic linear pencil  $\mathfrak{L}_\Omega(x) = I - \sum \Omega_j x_j$ . Here  $\Omega' = \Omega \oplus 0$  as in Lemma 4.7.

Conversely, if there is an  $\Omega$  such that (4.5) holds, then  $\Omega \in \mathcal{K}$  and for each  $X \in \mathcal{K}$  there is an isometry  $V$  such that (4.4) holds.

*Proof.* Suppose first (4.5) holds for some  $\Omega \in \mathbb{S}_n^g$ . Since  $\mathcal{D}_{\mathfrak{L}_\Omega}^\circ = \mathcal{K}$  and evidently  $\Omega \in \mathcal{D}_{\mathfrak{L}_\Omega}^\circ$ , it follows that  $\Omega \in \mathcal{K}$ . By Lemma 4.7, if  $X \in \mathcal{K} = \mathcal{D}_{\mathfrak{L}_\Omega}^\circ$ , then  $X$  has a representation as in (4.4).

Conversely, assume that  $\Omega$  has the property that any  $X \in \mathcal{K}$  can be represented as in (4.4). Consider the matrix convex set

$$\Gamma = \{V^*(I_\mu \otimes \Omega')V : \mu \in \mathbb{N}, V^*V = I\}.$$

Since  $0, \Omega \in \mathcal{K}$ , it follows that  $\Omega' = \Omega \oplus 0 \in \mathcal{K}$ , and thus  $\Gamma \subseteq \mathcal{K}$ . On the other hand, the hypothesis is that  $\mathcal{K} \subseteq \Gamma$ . Hence  $\mathcal{K} = \Gamma$ . Now, for  $\mathfrak{L}_X$  a monic linear pencil,  $\mathfrak{L}_X(\Omega) \succeq 0$  if and only if  $\mathfrak{L}_X(\Omega') \succeq 0$ , if and only if

$$\mathfrak{L}_X(V^*(I_\mu \otimes \Omega')V) = (V \otimes I)^* \mathfrak{L}_X(I_\mu \otimes \Omega') (V \otimes I) \succeq 0$$

for all choices of  $\mu$  and isometries  $V$ . Thus,  $X \in \mathcal{K}^\circ$  if and only if  $\mathfrak{L}_X(\Omega) \succeq 0$ . On the other hand,  $\mathfrak{L}_X(\Omega)$  is unitarily equivalent to  $\mathfrak{L}_\Omega(X)$ . Thus  $X \in \mathcal{K}^\circ$  if and only if  $X \in \mathcal{D}_{\mathfrak{L}_\Omega}$ .  $\square$

**Remark 4.10.** (1) For perspective, in the classical (not free) situation, when  $g = 2$  it is known that  $K \subseteq \mathbb{R}^2$  has an LMI representation if and only if  $K^\circ$  is a numerical range [Hen10, HS12]. It is well known that the polar dual of a spectrahedron is not necessarily a spectrahedron. This is the case even in  $\mathbb{R}^s$  (cf. [BPR13, Section 5] or Example 8.4).

(2) In the commutative case the polar dual of a spectrahedron (more generally, of a spectrahedral shadow) is a spectrahedral shadow (see [GN11] or [BPR13, Chapter 5]).

(3) It turns out that the  $\Omega$  in Theorem 4.6 can be taken to be an extreme point of  $\mathcal{K}$  in a very strong free sense. We refer to [Far00, Kls14, WW99] for more on matrix extreme points.

#### 4.4. The polar dual of a free spectrahedron is a free spectrahedron

This subsection contains a duality result for free spectrahedrons (Theorem 4.11) and several of its corollaries.

It can happen that  $\mathcal{D}_{\mathcal{L}}$  is not bounded, but the projection  $\mathcal{K} = \text{proj}_x \mathcal{D}_{\mathcal{L}}$  is. Corollary 4.13 below says that a free spectrahedron is closed and bounded if and only if it is the projection of some bounded free spectrahedron. For expositional purposes, it is convenient to introduce the following terminology. A free spectrahedron  $\mathcal{K}$  is called *stratospherically bounded* if there is a linear pencil  $\mathcal{L}$  such that  $\mathcal{K} = \text{proj}_x \mathcal{D}_{\mathcal{L}}$  and  $\mathcal{D}_{\mathcal{L}}$  is bounded.

**Theorem 4.11.** *Suppose  $\mathcal{K}$  is a closed matrix convex set containing 0.*

- (1) *If  $\mathcal{K}$  is a free spectrahedron and 0 is in the interior of  $\mathcal{K}$ , then  $\mathcal{K}^\circ$  is a stratospherically bounded free spectrahedron.*
- (2) *If  $\mathcal{K}^\circ$  is a free spectrahedron containing 0 in its interior, then  $\mathcal{K}$  is a stratospherically bounded free spectrahedron.*

*In particular, if  $\mathcal{K}$  is a bounded free spectrahedron with 0 in its interior, then both  $\mathcal{K}$  and  $\mathcal{K}^\circ$  are stratospherically bounded free spectrahedrons (with 0 in their interiors).*

Before presenting the proof of the theorem we state a few corollaries and Proposition 4.14 needed in the proof.

**Corollary 4.12.** *Given  $\Omega \in \mathbb{S}_d^s$ , let  $\mathcal{L}_\Omega$  denote the corresponding monic linear pencil. The free set  $\mathcal{D}_{\mathcal{L}_\Omega}^\circ$  is a stratospherically bounded free spectrahedron.*

*Proof.* The set  $\mathcal{D}_{\mathcal{L}_\Omega}$  is (trivially) a free spectrahedron with 0 in its interior. Thus, by Theorem 4.11,  $\mathcal{D}_{\mathcal{L}_\Omega}^\circ$  is a stratospherically bounded free spectrahedron.  $\square$

**Corollary 4.13.** *A free spectrahedron  $\mathcal{K} \subseteq \mathbb{S}^s$  is closed and bounded if and only if it is stratospherically bounded.*

*Proof.* Implication  $(\Leftarrow)$  is obvious.  $(\Rightarrow)$  Let us first reduce to the case where  $\mathcal{K}(1)$  has nonempty interior. If  $\mathcal{K}(1)$  has empty interior, then it is contained in a proper affine hyperplane  $\{\ell = 0\}$  of  $\mathbb{R}^s$ . Here  $\ell$  is an affine linear functional. In this case we can solve for one of the variables, thereby reducing the codimension of  $\mathcal{K}(1)$ . (Note that  $\ell = 0$  on  $\mathcal{K}(1)$  implies  $\ell = 0$  on  $\mathcal{K}$  [HKM16, Lemma 3.3].)

Now let  $\hat{x} \in \mathbb{R}^s$  be an interior point of  $\mathcal{K}(1)$ . Consider the translation

$$\tilde{\mathcal{K}} = \mathcal{K} - \hat{x} = \bigcup_{n \in \mathbb{N}} \{X - \hat{x}I_n : X \in \mathcal{K}(n)\}. \tag{4.6}$$

Clearly,  $\tilde{\mathcal{K}}$  is a bounded free spectrahedron with 0 in its interior. Hence by Theorem 4.11, it is stratospherically bounded. Translating back, we see  $\mathcal{K}$  is a stratospherically bounded free spectrahedron.  $\square$

Each stratospherically bounded free spectrahedron is closed, since it is the projection of a (levelwise) compact spectrahedron. Hence a bounded free spectrahedron  $\mathcal{K}$  will not be stratospherically bounded if it is not closed. For a concrete example, consider the linear pencil

$$L(x, y) = \begin{pmatrix} 2 - x & 1 \\ 1 & 2 - y \end{pmatrix} \oplus (2 + x),$$

and let  $\mathcal{K} = \text{proj}_x \mathcal{D}_L$ . Then  $\mathcal{K} = \{X \in \mathbb{S} : -2 \leq X < 2\}$  is bounded but not closed.

**Proposition 4.14.** *Given  $\Omega \in \mathbb{S}_d^g$  and  $\Gamma \in \mathbb{S}_d^h$ , the sequence  $\mathcal{K} = (\mathcal{K}(n))_n$ ,*

$$\begin{aligned} \mathcal{K}(n) = \{A \in \mathbb{S}_n^g : A = V^*(I_\mu \otimes \Omega)V, \\ 0 = V^*(I_\mu \otimes \Gamma)V \text{ for some isometry } V \text{ and } \mu \leq nd\}, \end{aligned}$$

*is a stratospherically bounded free spectrahedron.*

*Let  $\mathfrak{L}_{(\Omega, \Gamma)}$  denote the monic linear pencil corresponding to  $(\Omega, \Gamma)$ . The free set*

$$\mathcal{C} = \{A : (A, 0) \in \mathcal{D}_{\mathfrak{L}_{(\Omega, \Gamma)}}^\circ\}$$

*is a stratospherically bounded free spectrahedron.*

*Proof.* Let  $\mathcal{S}$  denote the span of  $\{I, \Omega_1, \dots, \Omega_g, \Gamma_1, \dots, \Gamma_h\}$ . Thus  $\mathcal{S}$  is an operator system in  $M_d$  (the fact that  $I \in \mathcal{S}$  implies  $\mathcal{S}$  contains a positive definite element). Let  $\phi : \mathcal{S} \rightarrow M_n$  denote the linear mapping determined by

$$I \mapsto I, \quad \Omega_j \mapsto A_j, \quad \Gamma_\ell \mapsto 0.$$

Observe that, by Theorem 3.1,  $A \in \mathcal{K}(n)$  if and only if  $\phi$  has a completely positive extension  $\Phi : M_d \rightarrow M_n$ . Theorem 3.4 expresses existence of such a  $\Phi$  as a (unital) cp interpolation problem in terms of a free spectrahedron. For the reader’s convenience we write out this critical LMI explicitly. Let  $\omega_{pq}^j$  denote the  $(p, q)$ -entry of  $\Omega_j$  and  $\gamma_{pq}^\ell$  the  $(p, q)$ -entry of  $\Gamma_\ell$ . For a complex matrix (or scalar)  $Q$  we use  $\hat{Q}$  to denote its real part and  $i\check{Q}$  for its imaginary part. Thus  $\hat{Q} = \frac{1}{2}(Q + Q^*)$  and  $\check{Q} = -\frac{i}{2}(Q - Q^*)$ .

Now  $A$  is in  $\mathcal{K}(n)$  if and only if there exist  $n \times n$  matrices  $C_{p,q}$  satisfying

- (i)  $\sum_{p,q=1}^d E_{p,q} \otimes C_{p,q} \succeq 0$ ;
- (ii)  $\sum_{p=1}^d C_{p,p} = I_n$ ;
- (iii)  $\sum_{p,q=1}^d \omega_{pq}^\ell C_{p,q} = A_\ell$  for  $\ell = 1, \dots, g$ ; and
- (iv)  $\sum_{p,q=1}^d \gamma_{pq}^\ell C_{p,q} = 0$  for  $\ell = 1, \dots, h$ .

Since the  $C_{p,q}$  for  $p \neq q$  are not hermitian matrices, we rewrite the system (i)–(iv) into one with hermitian unknowns  $\hat{C}_{p,q}$  and  $\check{C}_{p,q}$ . Property (i) transforms into

$$\sum_{p,q} (\hat{E}_{p,q} \otimes \hat{C}_{p,q} - \check{E}_{p,q} \otimes \check{C}_{p,q}) + i(\hat{E}_{p,q} \otimes \check{C}_{p,q} + \check{E}_{p,q} \otimes \hat{C}_{p,q}) \geq 0,$$

i.e.,

$$\begin{aligned} \sum_{p,q} \hat{E}_{p,q} \otimes \hat{C}_{p,q} - \check{E}_{p,q} \otimes \check{C}_{p,q} &\geq 0, \\ \sum_{p,q} \hat{E}_{p,q} \otimes \check{C}_{p,q} + \check{E}_{p,q} \otimes \hat{C}_{p,q} &= 0. \end{aligned} \tag{4.7}$$

In (ii) we simply replace  $C_{p,p}$  with  $\hat{C}_{p,p}$ ,

$$\sum_{p=1}^d \hat{C}_{p,p} = I_n. \tag{4.8}$$

Properties (iii) and (iv) are handled similarly to (i). Thus

$$\begin{aligned} \sum_{p,q} \hat{\omega}_{pq}^\ell \hat{C}_{p,q} - \check{\omega}_{pq}^\ell \check{C}_{p,q} &= A_\ell, \\ \sum_{p,q} \hat{\omega}_{pq}^\ell \check{C}_{p,q} + \check{\omega}_{pq}^\ell \hat{C}_{p,q} &= 0, \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} \sum_{p,q} \hat{\gamma}_{pq}^\ell \hat{C}_{p,q} - \check{\gamma}_{pq}^\ell \check{C}_{p,q} &= A_\ell, \\ \sum_{p,q} \hat{\gamma}_{pq}^\ell \check{C}_{p,q} + \check{\gamma}_{pq}^\ell \hat{C}_{p,q} &= 0. \end{aligned} \tag{4.10}$$

Hence,  $\mathcal{K}$  is the linear image of the explicitly constructed free spectrahedron (in the variables  $\hat{C}_{p,q}$  and  $\check{C}_{p,q}$ ) given by (4.7)–(4.10). Moreover, (i) and (ii) together imply  $0 \leq C_{p,p} \leq I$ . It now follows that  $\|\hat{C}_{p,q}\|, \|\check{C}_{p,q}\| \leq 1$  for all  $p, q$ . Thus, this free spectrahedron is bounded. It is now routine to verify that  $\mathcal{K}$  is a projection of a bounded free spectrahedron and is thus a stratospherically bounded free spectrahedron.

Let  $\Omega' = \Omega \oplus 0$  and  $\Gamma' = \Gamma \oplus 0$  where  $0 \in \mathbb{S}_d^g$ . Note that

$$\mathcal{D}_{\mathcal{L}(\Omega, \Gamma)} = \mathcal{D}_{\mathcal{L}(\Omega', \Gamma')}.$$

By Lemma 4.7,  $(A, 0) \in \mathcal{D}_{\mathcal{L}(\Omega, \Gamma)}^\circ$  if and only if

$$A \in \{B : \exists \mu \in \mathbb{N} \text{ and an isometry } V \text{ such that } B = V^*(I_\mu \otimes \Omega')V, 0 = V^*(I_\mu \otimes \Gamma')V\}.$$

By the first part of the proposition (applied to the tuple  $(\Omega', \Gamma')$ ), it follows that  $\mathcal{C}$  is a stratospherically bounded free spectrahedron.  $\square$

*Proof of Theorem 4.11.* Suppose  $\mathcal{K}$  is a free spectrahedron with 0 in its interior. By Lemma 4.1, there exists  $(\Omega, \Gamma)$ , a pair of tuples of matrices, such that

$$\mathcal{K} = \{X : \exists Y, (X, Y) \in \mathcal{D}_{\mathfrak{L}(\Omega, \Gamma)}\} = \text{proj}_x \mathcal{D}_{\mathfrak{L}(\Omega, \Gamma)},$$

where  $\mathfrak{L}_{(\Omega, \Gamma)}(x, y)$  is the monic linear pencil associated to  $(\Omega, \Gamma)$ .

Observe that  $A \in \mathcal{K}^\circ$  if and only if

$$\mathfrak{L}_A(X) \geq 0 \quad \text{for each } X \in \mathcal{K}.$$

Thus,  $A \in \mathcal{K}^\circ$  if and only if

$$\mathfrak{L}_{(A, 0)}(X, Y) \geq 0 \quad \text{for all } (X, Y) \in \mathcal{D}_{\mathfrak{L}(\Omega, \Gamma)},$$

if and only if

$$(A, 0) \in \mathcal{D}_{\mathfrak{L}(\Omega, \Gamma)}^\circ.$$

Summarizing,  $A \in \mathcal{K}^\circ$  if and only if  $(A, 0) \in \mathcal{D}_{\mathfrak{L}(\Omega, \Gamma)}^\circ$ . Thus, by the second part of Proposition 4.14,  $\mathcal{K}^\circ$  is a stratospherically bounded free spectrahedron.

Because  $\mathcal{K}$  contains 0 and is a closed matrix convex set,  $\mathcal{K}^{\circ\circ} = \mathcal{K}$  by Proposition 4.3. Thus, if  $\mathcal{K}^\circ$  is a free spectrahedron with 0 in its interior, then, by what has already been proved,  $\mathcal{K}^{\circ\circ} = \mathcal{K}$  is a stratospherically bounded free spectrahedron.

Finally, if  $\mathcal{K}$  is a bounded free spectrahedron with 0 in its interior, then  $\mathcal{K}^\circ$  contains 0 in its interior and is a stratospherically bounded free spectrahedron. Hence,  $\mathcal{K} = \mathcal{K}^{\circ\circ}$  is also a stratospherically bounded free spectrahedron.  $\square$

Note that the polar dual of a free spectrahedron is a matrix convex set generated by a singleton (Theorem 4.6) and is a free spectrahedron by the above corollary.

**Corollary 4.15.** *Let  $\mathfrak{L}$  denote the monic linear pencil associated with  $(\Omega, \Gamma)$ . If  $\mathcal{K} = \text{proj}_x \mathcal{D}_{\mathfrak{L}}$  is bounded, then its polar dual is the free set given by*

$$\begin{aligned} \mathcal{K}^\circ(n) &= \{A \in \mathbb{S}_n^g : (A, 0) \in \mathcal{D}_{\mathfrak{L}}^\circ\} \\ &= \{A \in \mathbb{S}_n^g : \exists \mu \in \mathbb{N} \text{ and an isometry } V \text{ with } A = V^*(I_\mu \otimes \Omega)V, 0 = V^*(I_\mu \otimes \Gamma)V\}. \end{aligned}$$

Whether or not  $\mathcal{K}$  is bounded, its polar dual is the free set

$$\begin{aligned} \mathcal{K}^\circ(n) &= \{A \in \mathbb{S}_n^g : (A, 0) \in \mathcal{D}_{\mathfrak{L}}^\circ\} \\ &= \{A \in \mathbb{S}_n^g : \exists \mu \in \mathbb{N} \text{ and an isometry } V \text{ with } A = V^*(I_\mu \otimes \Omega')V, 0 = V^*(I_\mu \otimes \Gamma')V\}, \end{aligned}$$

where  $\Omega' = \Omega \oplus 0$  and  $\Gamma' = \Gamma \oplus 0$ , as in Lemma 4.7.

*Proof.* From the proof of Theorem 4.11,  $\mathcal{K}^\circ = \{A : (A, 0) \in \mathcal{D}_{\mathfrak{L}}^\circ\}$ . Writing  $\mathfrak{L} = \mathfrak{L}_\Delta$ , by Lemma 4.7 (whether or not  $\mathcal{D}_{\mathfrak{L}}$  is bounded),

$$\mathcal{D}_{\mathfrak{L}}^\circ = \{X : \exists \mu \in \mathbb{N} \text{ and an isometry } V \text{ such that } X = V^*(I_\mu \otimes \Delta')V\}.$$

To obtain the stronger conclusion under the assumption that  $\mathcal{K}$  is bounded, an additional argument along the lines of [HKM13a, §3.1] is needed; see also [Zal17, Theorem 2.12]. Let  $(A, 0) \in \mathcal{D}_{\mathcal{L}}^{\circ}$ . We need to show that the unital linear map

$$\begin{aligned} \tau : \text{span}\{I, \Omega_1, \dots, \Omega_g, \Gamma_1, \dots, \Gamma_h\} &\rightarrow \text{span}\{I, A_1, \dots, A_g\}, \\ \Gamma_j &\mapsto A_j, \quad \Gamma_k \mapsto 0, \end{aligned}$$

is completely positive. Assume

$$I \otimes X_0 + \sum_j \Omega_j \otimes X_j + \sum_k \Gamma_k \otimes Y_k \geq 0 \tag{4.11}$$

for some hermitian  $X_0, \dots, X_g, Y_1, \dots, Y_h$ . In particular,  $X_0 = X_0^*$ . We claim that  $X_0 \geq 0$ . Suppose  $X_0 \not\geq 0$ . By compressing we may reduce to  $X_0 < 0$ . From (4.11) it now follows that

$$I \otimes tX_0 + \sum_j \Omega_j \otimes tX_j + \sum_k \Gamma_k \otimes tY_k \geq 0$$

for every  $t > 0$ . Since  $tX_0 < 0$ , this implies

$$I \otimes I + \sum_j \Omega_j \otimes tX_j + \sum_k \Gamma_k \otimes tY_k \geq 0,$$

whence  $(tX_1, \dots, tX_g) \in \mathcal{K}$  for every  $t > 0$ . If  $(X_1, \dots, X_g) \neq 0$ , this contradicts the boundedness of  $\mathcal{K}$ . Otherwise  $(X_1, \dots, X_g) = 0$ , and

$$\sum_k \Gamma_k \otimes Y_k \succ -I \otimes X_0 \succ 0.$$

Hence for any tuple  $(X_1, \dots, X_g)$  of hermitian matrices of the same size as the  $Y_k$ ,

$$I \otimes I + \sum_j \Omega_j \otimes X_j + \sum_k \Gamma_k \otimes tY_k \geq 0$$

for some  $t > 0$ . This again contradicts the boundedness of  $\mathcal{K}$ . Thus  $X_0 \geq 0$ .

By adding a small multiple of the identity to  $X_0$  there is no harm in assuming  $X_0 \succ 0$ . Hence multiplying (4.11) by  $X_0^{-1/2}$  on the left and right yields the tuple  $X_0^{-1/2}(X_1, \dots, X_g, Y_1, \dots, Y_h)X_0^{-1/2} \in \mathcal{D}_{\mathcal{L}}$ . Since  $(A, 0) \in \mathcal{D}_{\mathcal{L}}^{\circ}$ ,

$$\mathcal{L}_{(A,0)}(X_0^{-1/2}(X_1, \dots, X_g, Y_1, \dots, Y_h)X_0^{-1/2}) = I \otimes I + \sum_k A_k \otimes X_0^{-1/2} X_k X_0^{-1/2} \geq 0.$$

Multiplying by  $X_0^{1/2}$  on the left and right gives

$$I \otimes X_0 + \sum_k A_k \otimes X_k \geq 0,$$

as required. □

To each subset  $\Gamma \subseteq \mathbb{S}^g$  we associate its interior  $\text{int } \Gamma = (\text{int } \Gamma(n))_{n \in \mathbb{N}}$ , where  $\text{int } \Gamma(n)$  denotes the interior of  $\Gamma(n)$  in the Euclidean space  $\mathbb{S}_n^g$ . We say  $\Gamma$  has nonempty interior if there is  $n$  with  $\text{int } \Gamma(n) \neq \emptyset$ .



**Corollary 4.16.** *If  $\mathcal{K} \subseteq \mathbb{S}^g$  is a bounded free spectrahedron, then  $\overline{\mathcal{K}}$  is a free spectrahedron.*

*Proof.* As in the proof of Corollary 4.13, we may assume the interior of  $\mathcal{K}$  is nonempty. This implies there is an  $\hat{x} \in \mathbb{R}^g$  in the interior of  $\mathcal{K}(1)$ . Consider the translation  $\tilde{\mathcal{K}} = \mathcal{K} - \hat{x}$  as in (4.6). This is a free spectrahedron containing 0 in its interior. Hence its closure  $\overline{\tilde{\mathcal{K}}} = \tilde{\mathcal{K}}^{\circ\circ}$  is a free spectrahedron by Theorem 4.11. Thus so is  $\overline{\mathcal{K}} = \overline{\tilde{\mathcal{K}}} + \hat{x}$ .  $\square$

**Corollary 4.17.** *If  $\mathcal{K} \subseteq \mathbb{S}^g$  is a free spectrahedron with nonempty interior, then  $\mathcal{K}^\circ$  is a free spectrahedron.*

*Proof.* Assume  $\mathcal{K} = \text{proj } \mathcal{D}_{L_A}$ . Applying Lemma 4.5 gives

$$\mathcal{K}^\circ = \{B \in \mathbb{S}^g : (B, 0) \in \mathcal{D}_{L_A}^\circ\}.$$

So if we prove  $\mathcal{D}_{L_A}^\circ$  is a free spectrahedron, then

$$\mathcal{K}^\circ = \text{proj}(\mathcal{D}_{L_A}^\circ \cap (\mathbb{S}^g \otimes \{0\}^h))$$

is the intersection of two free spectrahedrons, so a free spectrahedron. Thus without loss of generality we may take  $\mathcal{K} = \mathcal{D}_{L_A}$  and proceed. We will demonstrate that the corollary in this case is a consequence of the Convex Positivstellensatz, Theorem 2.4.

Suppose  $\hat{x} \in \mathbb{R}^g$  is in the interior of  $\mathcal{D}_{L_A}$ . Without loss of generality we may assume

$$L_0 = L_A(\hat{x}) \succ 0$$

(cf. [HV07]). Define the monic linear pencil

$$\mathfrak{L}(y) = L_0^{-1/2} L(y + \hat{x}) L_0^{-1/2} = I + \sum_{j=1}^g L_0^{-1/2} A_j L_0^{-1/2} y_j.$$

By definition, a tuple  $\Omega \in \mathbb{S}^g$  is in  $\mathcal{D}_{L_A}^\circ$  if and only if  $\mathcal{D}_{L_A} \subseteq \mathcal{D}_{\mathfrak{L}\Omega}$ . Equivalently, with

$$L(y) = \left( I + \sum_{j=1}^g \Omega_j \hat{x}_j \right) + \sum_{j=1}^g \Omega_j y_j,$$

we have  $\mathcal{D}_{\mathfrak{L}} \subseteq \mathcal{D}_L$ . By Theorem 2.4, there is an  $S \succeq 0$  and matrices  $V_k$  with

$$L(y) = S + \sum_k V_k^* \mathfrak{L}(y) V_k.$$

That is,

$$I + \sum_{j=1}^g \Omega_j \hat{x}_j \succeq \sum_k V_k^* V_k, \quad \text{and} \quad \Omega_j = \sum_k V_k^* L_0^{-1/2} A_j L_0^{-1/2} V_k, \quad j = 1, \dots, g.$$

Equivalently, there is a completely positive mapping  $\Phi$  satisfying

$$\Phi(L_0^{-1/2}A_jL_0^{-1/2}) = \Omega_j, \quad j = 1, \dots, g, \tag{4.12}$$

$$\Phi(I) \preceq I + \sum_j \Omega_j \hat{x}_j. \tag{4.13}$$

As in Theorem 3.4 we now employ the Choi matrix  $C$ . Conditions (4.12) translate into linear constraints on the block entries  $C_{ij}$  of  $C$ . Similarly, (4.13) transforms into an LMI constraint on the entries of  $C$ . Thus  $C$  provides a free spectrahedral lift of  $\mathcal{D}_{L_A}^\circ$ .  $\square$

#### 4.5. The free convex hull of a union

In this subsection we prove that the convex hull of a union of free spectrahedra is again a free spectrahedron.

**Proposition 4.18.** *If  $\mathcal{S}_1, \dots, \mathcal{S}_t \subseteq \mathbb{S}^g$  are stratospherically bounded free spectrahedra and each contains 0 in its interior, then  $\overline{\text{co}}^{\text{mat}}(\mathcal{S}_1 \cup \dots \cup \mathcal{S}_t)$  is a stratospherically bounded free spectrahedron with 0 in its interior.*

*Proof.* Let  $\mathcal{K} = \mathcal{S}_1 \cup \dots \cup \mathcal{S}_t$ . Then

$$\mathcal{K}^\circ = \mathcal{S}_1^\circ \cap \dots \cap \mathcal{S}_t^\circ.$$

Since each  $\mathcal{S}_j$  is a stratospherically bounded free spectrahedron with 0 in its interior, the same holds true for  $\mathcal{S}_j^\circ$  by Theorem 4.11. It is clear that these properties are preserved under a finite intersection, so  $\mathcal{K}^\circ$  is again a stratospherically bounded free spectrahedron with 0 in its interior. Hence, by Proposition 4.3,

$$(\mathcal{K}^\circ)^\circ = \overline{\text{co}}^{\text{mat}} \mathcal{K}$$

is a stratospherically bounded free spectrahedron with 0 in its interior by Theorem 4.11.  $\square$

### 5. Positivstellensatz for free spectrahedra

This section focuses on polynomials positive on a free spectrahedron, extending our Positivstellensatz for free polynomials positive on free spectrahedra, Theorem 2.4, to a Convex Positivstellensatz for free spectrahedra, Theorem 5.1.

Let  $\mathfrak{L}$  denote a monic linear pencil of size  $d$ ,

$$\mathfrak{L}(x, y) = I + \sum_{j=1}^g \Omega_j x_j + \sum_{k=1}^h \Gamma_k y_k, \tag{5.1}$$

and let  $\mathcal{K} = \text{proj}_x \mathcal{D}_{\mathfrak{L}}$ . If  $\mu$  is a positive integer and  $q_\ell \in \mathbb{C}^{d \times \mu} \langle x \rangle$  (and so are polynomials in the  $x$  variables only), and if  $\sum_\ell q_\ell(x)^* \Gamma_k q_\ell(x) = 0$  for each  $k$ , then

$$\sum_\ell q_\ell^*(x) \mathfrak{L}(x, y) q_\ell(x)$$

is a polynomial in the  $x$  variables and is thus in  $\mathbb{C}^{\mu \times \mu} \langle x \rangle$ . For positive integers  $\mu$  and  $r$  we define the *truncated quadratic module* in  $\mathbb{C}^{\mu \times \mu} \langle x \rangle$  associated to  $\mathcal{L}$  and  $\mathcal{K}$  by

$$M_x^\mu(\mathcal{L})_r = \left\{ \sum_{\ell} q_\ell^* \mathcal{L} q_\ell + \sigma : q_\ell \in \mathbb{C}^{d \times \mu} \langle x \rangle_r, \sigma \in \Sigma_r^\mu \langle x \rangle, \sum_{\ell} q_\ell^* \Gamma_k q_\ell = 0 \text{ for all } k \right\}.$$

Here  $\Sigma_r^\mu = \Sigma_r^\mu \langle x \rangle$  denotes the set of all sums of hermitian squares  $h^*h$  for  $h \in \mathbb{C}^{\mu \times \mu} \langle x \rangle_r$ . It is easy to see that  $M_x^\mu(\mathcal{L}) = \bigcup_{r \in \mathbb{N}} M_x^\mu(\mathcal{L})_r$  is a quadratic module in  $\mathbb{C}^{\mu \times \mu} \langle x \rangle$ .

The main result of this section is the following Positivstellensatz:

**Theorem 5.1.** *A symmetric polynomial  $p \in \mathbb{C}^{\mu \times \mu} \langle x \rangle_{2r+1}$  is positive semidefinite on  $\mathcal{K}$  if and only if  $p \in M_x^\mu(\mathcal{L})_r$ .*

**Remark 5.2.** (1) In case there are no  $y$  variables in  $\mathcal{L}$ , Theorem 5.1 reduces to the Convex Positivstellensatz of [HKM12].

(2) If  $r = 0$ , i.e.,  $p$  is linear, then Theorem 5.1 reduces to Corollary 4.15.

(3) A Positivstellensatz for commutative polynomials *strictly positive* on spectrahedrops was established by Gouveia and Netzer [GN11]. A major distinction is that the degrees of the  $q_i$  and  $\sigma$  in the commutative theorem behave very badly.

(4) Observe that  $\mathcal{K}$  is in general not closed. Thus Theorem 5.1 yields a “perfect” Positivstellensatz for certain nonclosed sets.

### 5.1. Proof of Theorem 5.1

We begin with some auxiliary results.

**Proposition 5.3.** *With  $\mathcal{L}$  a monic linear pencil as in (5.1),  $M_x^\mu(\mathcal{L})_r$  is a closed convex cone in the set of all symmetric polynomials in  $\mathbb{C}^{\mu \times \mu} \langle x \rangle_{2r+1}$ .*

The convex cone property is obvious. For the proof that this cone is closed, it is convenient to introduce a norm compatible with  $\mathcal{L}$ .

Given  $\varepsilon > 0$ , let

$$\mathcal{B}_\varepsilon^g(n) := \{X \in \mathbb{S}_n^g : \|X\| \leq \varepsilon\}.$$

There is an  $\varepsilon > 0$  such that for all  $n \in \mathbb{N}$ , if  $(X, Y) \in \mathbb{S}_n^{g+h}$  and  $\|(X, Y)\| \leq \varepsilon$ , then  $\mathcal{L}(X, Y) \succeq 1/2$ . In particular,  $\mathcal{B}_\varepsilon^{g+h} \subseteq \mathcal{D}_\mathcal{L}$ . Using this  $\varepsilon$  we norm matrix polynomials in  $g + h$  variables by

$$\| \| p(x, y) \| \| := \max\{\|p(X, Y)\| : (X, Y) \in \mathcal{B}_\varepsilon^{g+h}\}. \tag{5.2}$$

(Note that by the nonexistence of polynomial identities for matrices of all sizes,  $\| \| p(x, y) \| \| = 0$  iff  $p(x, y) = 0$  [Row80, §2.5, §1.4]. Furthermore, on the right-hand side of (5.2) the maximum is attained because the bounded free semialgebraic set  $\mathcal{B}_\varepsilon^{g+h}$  is levelwise compact and matrix convex; see [HM04, Section 2.3] for details.) Note that if  $f \in \mathbb{C}^{d \times \mu} \langle x \rangle_\beta$  and  $\| \| f(x)^* \mathcal{L}(x, y) f(x) \| \| \leq N^2$ , then  $\| \| f^* f \| \| \leq 2N^2$ .

*Proof of Proposition 5.3.* Suppose  $(p_n)$  is a sequence from  $M_x^\mu(\mathcal{L})_r$  that converges to some symmetric  $p \in \mathbb{C}^{\mu \times \mu} \langle x \rangle$  of degree at most  $2r + 1$ . By Carathéodory’s convex hull

theorem (see e.g. [Bar02, Theorem I.2.3]), there is an  $M$  such that for each  $n$  there exist matrix-valued polynomials  $r_{n,i} \in \mathbb{C}^{\mu \times \mu} \langle x \rangle_r$  and  $t_{n,i} \in \mathbb{C}^{d \times \mu} \langle x \rangle_r$  such that

$$p_n = \sum_{i=1}^M r_{n,i}^* r_{n,i} + \sum_{i=1}^M t_{n,i}^* \mathfrak{L}(x, y) t_{n,i}.$$

Since  $\|p_n\| \leq N^2$ , it follows that  $\|r_{n,i}\| \leq N$  and likewise  $\|t_{n,i}^* \mathfrak{L}(x, y) t_{n,i}\| \leq N^2$ . In view of the remarks preceding the proof, we obtain  $\|t_{n,i}\| \leq \sqrt{2} N$  for all  $i, n$ . Hence for each  $i$ , the sequences  $(r_{n,i})$  and  $(t_{n,i})$  are bounded in  $n$ . They thus have convergent subsequences. Passing to one of these subsequential limits finishes the proof.  $\square$

Next is a variant of the Gelfand–Naimark–Segal (GNS) construction.

**Proposition 5.4.** *If  $\lambda : \mathbb{C}^{\nu \times \nu} \langle x \rangle_{2k+2} \rightarrow \mathbb{C}$  is a linear functional that is nonnegative on  $\Sigma_{k+1}^\nu$  and positive on  $\Sigma_k^\nu \setminus \{0\}$ , then there exists a tuple  $X = (X_1, \dots, X_g)$  of hermitian operators on a Hilbert space  $\mathcal{X}$  of dimension at most  $\nu \dim \mathbb{C} \langle x \rangle_k$  and a vector  $\gamma \in \mathcal{X}^{\oplus \nu}$  such that*

$$\lambda(f) = \langle f(X)\gamma, \gamma \rangle \tag{5.3}$$

for all  $f \in \mathbb{C}^{\nu \times \nu} \langle x \rangle_{2k+1}$ , where  $\langle \cdot, \cdot \rangle$  is the inner product on  $\mathcal{X}$ . Further, if  $\lambda$  is nonnegative on  $M_x^\nu(\mathfrak{L})_k$ , then  $X$  is in the closure  $\overline{\mathcal{K}}$  of the free spectrahedron  $\mathcal{K}$  coming from  $\mathfrak{L}$ .

Conversely, if  $X = (X_1, \dots, X_g)$  is a tuple of symmetric operators on a Hilbert space  $\mathcal{X}$  of dimension  $N$ , the vector  $\gamma \in \mathcal{X}^{\oplus \nu}$ , and  $k$  is a positive integer, then the linear functional  $\lambda : \mathbb{C}^{\nu \times \nu} \langle x \rangle_{2k+2} \rightarrow \mathbb{C}$  defined by

$$\lambda(f) = \langle f(X)\gamma, \gamma \rangle$$

is nonnegative on  $\Sigma_{k+1}^\nu$ . Further, if  $X \in \overline{\mathcal{K}}$ , then  $\lambda$  is nonnegative also on  $M_x^\nu(\mathfrak{L})_k$ .

*Proof.* The first part of the forward direction is standard: see e.g. [HKM12, Proposition 2.5]. In the course of the proof one constructs  $X_j$  as the operators of multiplication by  $x_j$  on a Hilbert space  $\mathcal{X}$ , that, as a set, is  $\mathbb{C} \langle x \rangle_k^{1 \times \nu}$  (the set of row vectors of length  $\nu$  whose entries are polynomials of degree at most  $k$ ). The vector space  $\mathcal{X}^{\oplus \nu}$  in which  $\gamma$  lies is  $\mathbb{C} \langle x \rangle_k^{\nu \times \nu}$  and  $\gamma$  can be thought of as the identity matrix in  $\mathbb{C} \langle x \rangle_k^{\nu \times \nu}$ . Indeed, the (column) vector  $\gamma$  has  $j$ -th entry the row vector with  $j$ -th entry the empty set (which plays the role of multiplicative identity) and zeros elsewhere.

In particular, for  $p \in \mathcal{X} = \mathbb{C} \langle x \rangle_k^{1 \times \nu}$ , we have  $p = p(X)\gamma$ . Let  $\sigma$  denote the dimension of  $\mathcal{X}$  (which turns out to be  $\nu$  times the dimension of  $\mathbb{C} \langle x \rangle_k$ ).

We next assume that  $\lambda$  is nonnegative on  $M_x^\nu(\mathfrak{L})_k$  and claim that then  $X \in \overline{\mathcal{K}}$ . Assume otherwise. Then, as  $\overline{\mathcal{K}}$  is closed matrix convex (and  $\mathcal{K}$  contains 0 since  $\mathfrak{L}$  is monic), the matricial Hahn–Banach Theorem 2.2 applies: there is a monic linear pencil  $\mathfrak{L}_\Lambda$  of size  $\sigma$  such that  $\mathfrak{L}_\Lambda|_{\mathcal{K}} \geq 0$  and  $\mathfrak{L}_\Lambda(X) \not\geq 0$ . In particular,  $\mathcal{D}_{\mathfrak{L}_\Lambda} \supseteq \mathcal{K}$ , whence

$$\mathcal{D}_{\mathfrak{L}_\Lambda}^\circ \subseteq \mathcal{K}^\circ.$$

By Corollary 4.15,

$$\mathcal{K}^\circ(n) = \left\{ A \in \mathbb{S}_n^g : \exists \mu \in \mathbb{N} \exists \text{isometry } V, \sum_{j=1}^\mu V_j^* \Gamma V_j = 0, \sum_{j=1}^\mu V_j^* \Omega V_j = A \right\}. \quad (5.4)$$

Since  $\Lambda \in \mathcal{K}^\circ$ , there is an isometry  $W$  with

$$\sum_{j=1}^\eta W_j^* \Gamma W_j = 0, \quad \sum_{j=1}^\eta W_j^* \Omega W_j = \Lambda.$$

Here,  $W = \text{col}(W_1, \dots, W_\eta)$  for some  $\eta$ , and  $W_j \in \mathbb{C}^{d \times \sigma}$ .

Since  $\mathcal{L}_\Lambda(X) \not\geq 0$ , there is  $u \in \mathbb{C}^\sigma \otimes \mathcal{X}$  with

$$u^* \mathcal{L}_\Lambda(X) u < 0. \quad (5.5)$$

Let

$$u = \sum_i e_i \otimes v_i,$$

where  $e_i \in \mathbb{C}^\sigma$  are the standard basis vectors, and  $v_i \in \mathcal{X}$ . By the construction of  $X$  and  $\gamma$ , there is a polynomial  $p_i \in \mathbb{C}\langle x \rangle_k^{1 \times \nu}$  with  $v_i = p_i(X)\gamma$ . Now (5.5) can be written as follows:

$$\begin{aligned} 0 > u^* \mathcal{L}_\Lambda(X) u &= \left( \sum_i e_i \otimes v_i \right)^* \mathcal{L}_\Lambda(X) \left( \sum_j e_j \otimes v_j \right) \\ &= \sum_{i,j,\ell} (e_i \otimes v_i)^* (W_\ell \otimes I)^* \mathcal{L}(X, Y) (W_\ell \otimes I) (e_j \otimes v_j) \\ &= \sum_{i,j,\ell} (W_\ell e_i \otimes p_i(X)\gamma)^* \mathcal{L}(X, Y) (W_\ell e_j \otimes p_j(X)\gamma). \end{aligned} \quad (5.6)$$

If we let  $\vec{p}_\ell(x) = \sum_j W_\ell e_j \otimes p_j(x) \in \mathbb{C}^{d \times \nu} \langle x \rangle_k$ , then (5.6) is further equivalent to

$$0 > \sum_\ell (\vec{p}_\ell(X)\gamma)^* \mathcal{L}(X, Y) (\vec{p}_\ell(X)\gamma) = \lambda(q), \quad (5.7)$$

where  $q = \sum_\ell \vec{p}_\ell(x)^* \mathcal{L}(x, y) \vec{p}_\ell(x)$  is a matrix polynomial only in  $x$  by (5.4), and thus  $q \in M_x^\mu(\mathcal{L})_k$ . But now (5.7) contradicts the nonnegativity of  $\lambda$  on  $M_x^\mu(\mathcal{L})_k$ .

The converse is obvious. □

*Proof of Theorem 5.1.* Let  $\text{Sym } \mathbb{C}^{\mu \times \mu} \langle x \rangle_{2r+1}$  denote the symmetric elements of  $\mathbb{C}^{\mu \times \mu} \langle x \rangle_{2r+1}$ . Arguing towards a contradiction, suppose  $p \in \text{Sym } \mathbb{C}^{\mu \times \mu} \langle x \rangle_{2r+1}$  and  $p|_{\mathcal{K}} \geq 0$ , but  $p \notin M_x^\mu(\mathcal{L})_r$ . By the scalar Hahn–Banach theorem and Proposition 5.3, there is a strictly separating positive (real) linear functional  $\lambda : \text{Sym } \mathbb{C}^{\mu \times \mu} \langle x \rangle_{2r+1} \rightarrow \mathbb{R}$  nonnegative on  $M_x^\mu(\mathcal{L})_r$ . We first extend  $\lambda$  to a (complex) linear functional on the whole

$\mathbb{C}^{\mu \times \mu} \langle x \rangle_{2r+1}$  by sending  $q + is$  to  $\lambda(q) + i\lambda(s)$  for symmetric  $q, s$ . We then extend  $\lambda$  to a linear functional (still called  $\lambda$ ) on  $\mathbb{C}^{\mu \times \mu} \langle x \rangle_{2r+2}$  by

$$E_{i,j} \otimes u^*v \mapsto \begin{cases} 0 & \text{if } i \neq j \text{ or } u \neq v, \\ C & \text{otherwise,} \end{cases}$$

where  $i, j = 1, \dots, \mu$ , and  $u, v \in \langle x \rangle$  are of length  $r + 1$ . For  $C > 0$  large enough, this  $\lambda$  will be nonnegative on  $\Sigma_{r+1}^\mu$ . Perturbing  $\lambda$  if necessary, we may further assume  $\lambda$  is strictly positive on  $\Sigma_r^\mu \setminus \{0\}$ . Now applying Proposition 5.4 yields a matrix tuple  $X \in \overline{K}$  and a vector  $\gamma$  satisfying (5.3) (with  $k = r$ ). But then

$$0 > \lambda(p) = \langle p(X)\gamma, \gamma \rangle \geq 0,$$

a contradiction. □

### 6. Tracial sets

While this paper’s original motivation arose from considerations of free optimization as it appears in linear systems theory, determining the matrix convex hull of a free set has an analog in quantum information theory [LP11]. In free optimization, the relevant maps are completely positive and *unital* (ucp). In quantum information theory, the relevant maps are completely positive and *trace preserving* (CPTP) or *trace nonincreasing*. This section begins by recalling the two *quantum interpolation problems* from Subsection 3.2 before reformulating these problems in terms of *tracial hulls*. Corresponding duality results are the topic of the next section.

Recall a quantum channel is a cp map  $\Phi$  from  $M_n$  to  $M_k$  that is trace preserving,

$$\text{tr}(\Phi(X)) = \text{tr}(X).$$

The dual  $\Phi'$  of  $\Phi$  is the mapping from  $M_k$  to  $M_n$  defined by

$$\text{tr}(\Phi(X)Y^*) = \text{tr}(X\Phi'(Y)^*).$$

**Lemma 6.1** ([LP11, Proposition 1.2]).  *$\Phi'$  is a quantum channel cp if and only if  $\Phi$  is unital cp.*

Recall the cp interpolation problem from Subsection 3.2. It asks, given  $A \in \mathbb{S}_n^g$  and  $B \in \mathbb{S}_m^g$ : does there exist a unital cp map  $\Phi : M_n \rightarrow M_m$  such that  $B_j = \Phi(A_j)$  for  $j = 1, \dots, g$ ? The set of solutions  $B$  for a given  $A$  is the matrix convex hull of  $A$ . The versions of the interpolation problem arising in quantum information theory [Hal1, Kle07, NCSB98] replace unital with trace preserving or trace nonincreasing. Namely, does  $B_j$  equal  $\Phi(A_j)$  for  $j = 1, \dots, g$  for some trace preserving (resp. trace nonincreasing) cp map  $\Phi : M_n \rightarrow M_m$ ? The set of all solutions  $B$  for a given  $A$  is the *tracial hull* of  $A$ . Thus,

$$\text{thull}(A) = \{B : \Phi(A) = B \text{ for some trace preserving cp map } \Phi\}. \tag{6.1}$$

We define the *contractive tracial hull* of a tuple  $A$  by

$$\text{cthull}(A) = \{B : \Phi(A) = B \text{ for some cp trace nonincreasing map } \Phi\}.$$

The article [LP11] determines when  $B \in \text{thull}(A)$  for  $g = 1$  (see Section 3.2). For any  $g \geq 0$  the paper [AG15, Section 3] converts this problem to an LMI suitable for semidefinite programming; see Theorem 3.4 here for a similar result.

While the unital and trace preserving (or trace nonincreasing) interpolation problems have very similar formulations, tracial hulls possess far less structure than matrix convex hulls. Indeed, as is easily seen, tracial hulls need not be convex (levelwise), and contractive tracial hulls need not be closed with respect to direct sums. Tracial hulls are studied in Subsection 6.1, and contractive tracial hulls in Subsection 6.2. Section 7 contains “tracial” notions of half-space and corresponding Hahn–Banach type separation theorems.

### 6.1. Tracial sets and hulls

A set  $\mathcal{Y} \subseteq \mathbb{S}^s$  is *tracial* if whenever  $Y \in \mathcal{Y}(n)$  and  $C_\ell$  are  $m \times n$  matrices such that

$$\sum C_\ell^* C_\ell = I_n,$$

then  $\sum C_j Y C_j^* \in \mathcal{Y}(m)$ . The *tracial hull* of a subset  $\mathcal{S} \subseteq \mathbb{S}^s$  is the smallest tracial set containing  $\mathcal{S}$ , denoted  $\text{thull}(\mathcal{S})$ . Note that if  $\mathcal{S}$  is a singleton, this definition is consistent with the definition afforded by (6.1).

The following lemma is an easy consequence of a theorem of Choi, stated in [Pau02, Proposition 4.7]. It caps the number of terms needed in a convex combination to represent a given matrix tuple  $Z$  in the tracial hull of  $T$ . Hence it is an analog of Carathéodory’s convex hull theorem (see e.g. [Bar02, Theorem I.2.3]).

**Lemma 6.2.** *Suppose  $T \in \mathbb{S}_n^s$  and  $C_1, \dots, C_N$  are  $m \times n$  matrices with  $\sum C_\ell^* C_\ell = I_n$ . If  $Z = \sum_{\ell=1}^N C_\ell T C_\ell^*$ , then there exist  $m \times n$  matrices  $V_1, \dots, V_{mn}$  such that  $\sum V_\ell^* V_\ell = I_n$  and*

$$Z = \sum_{\ell=1}^{mn} V_\ell T V_\ell^*.$$

*Proof.* The mapping  $\Phi : M_n \rightarrow M_m$  defined by

$$\Phi(X) = \sum C_\ell X C_\ell^*$$

is completely positive. Hence, by [Pau02, Proposition 4.7], there exist (at most)  $nm$  matrices  $V_j : \mathbb{C}^m \rightarrow \mathbb{C}^n$  such that

$$\Phi(X) = \sum_{\ell=1}^{mn} V_\ell X V_\ell^*.$$

In particular,

$$Z = \Phi(T) = \sum V_\ell T V_\ell^*.$$

Further, for all  $m \times m$  matrices  $X$ ,

$$\begin{aligned} \operatorname{tr}(X) &= \operatorname{tr}\left(X \sum C_\ell^* C_\ell\right) = \operatorname{tr}\left(\sum C_\ell X C_\ell^*\right) = \operatorname{tr}(\Phi(X)) \\ &= \operatorname{tr}\left(\sum V_\ell X V_\ell^*\right) = \operatorname{tr}\left(X \sum V_\ell^* V_\ell\right). \end{aligned}$$

It follows that  $\sum V_\ell^* V_\ell = I$ . □

**Lemma 6.3.** For  $\mathcal{S} = \{T\}$  a singleton,

$$\operatorname{thull}(\{T\}) = \left\{ \sum C_\ell T C_\ell^* : \sum C_\ell^* C_\ell = I \right\}.$$

Moreover, this set is closed (levelwise).

The tracial hull of a subset  $\mathcal{S} \subseteq \mathbb{S}^g$  is

$$\operatorname{thull}(\mathcal{S}) = \left\{ \sum C_\ell T C_\ell^* : \sum C_\ell^* C_\ell = I, T \in \mathcal{S} \right\} = \bigcup_{T \in \mathcal{S}} \operatorname{thull}(\{T\}).$$

If  $\mathcal{S}$  is a finite set, then the tracial hull of  $\mathcal{S}$  is closed.

*Proof.* The first statement follows from the observation that  $\{\sum C_\ell T C_\ell^* : \sum C_\ell^* C_\ell = I\}$  is tracial.

To prove the “moreover” part, suppose  $T$  has size  $n$  and suppose  $Z^k$  is a sequence from  $\mathcal{Y}(m)$ . By Lemma 6.2 for each  $k$  there exist  $nm$  matrices  $V_{k,\ell}$  of size  $n \times m$  such that

$$Z^k = \sum_\ell V_{k,\ell} T V_{k,\ell}^*$$

and each  $V_{k,\ell}$  is a contraction. Hence, by passing to a subsequence, we can assume that for each fixed  $\ell$ , the sequence  $(V_{k,\ell})_k$  converges to some  $W_\ell$ . Hence  $Z^k$  converges to  $Z = \sum_\ell W_\ell T W_\ell^*$ . Also, since  $\sum_\ell V_{k,\ell}^* V_{k,\ell} = I$  for each  $k$ , we have  $\sum_\ell W_\ell^* W_\ell = I$ , whence  $Z \in \mathcal{Y}(m)$ .

To prove the second statement, let  $\mathcal{S} \subseteq \mathbb{S}^g$ . Evidently,

$$\mathcal{S} \subseteq \bigcup_{T \in \mathcal{S}} \operatorname{thull}(\{T\}) \subseteq \operatorname{thull}(\mathcal{S}).$$

Hence it suffices to show that  $\bigcup_{T \in \mathcal{S}} \operatorname{thull}(\{T\})$  is itself tracially convex. To this end, suppose  $X \in \bigcup_{T \in \mathcal{S}} \operatorname{thull}(\{T\})$  and  $C_1, \dots, C_N$  with  $\sum C_\ell^* C_\ell = I$  are given (and of the appropriate sizes). There is an  $S \in \mathcal{S}$  such that  $X \in \operatorname{thull}(\{S\})$ . Hence, by the first part of the lemma,  $\sum C_\ell X C_\ell^* \in \operatorname{thull}(\{S\}) \subseteq \bigcup_{T \in \mathcal{S}} \operatorname{thull}(\{T\})$ , and the desired conclusion follows.

The final statement of the lemma follows by combining its first two assertions and using the fact that the closure of a finite union is the finite union of the closures. □



6.2. Contractively tracial sets and hulls

A set  $\mathcal{Y} \subseteq \mathbb{S}^g$  is *contractively tracial* if whenever  $Y \in \mathcal{Y}(m)$  and  $C_\ell$  are  $n \times m$  matrices such that

$$\sum C_\ell^* C_\ell \preceq I_m, \tag{6.2}$$

then  $\sum C_j Y C_j^* \in \mathcal{Y}(n)$ . Note that, in this case,  $\mathcal{Y}$  is closed under unitary conjugation and compression to subspaces, but not necessarily direct sums. It is clear that intersections of contractively tracial sets are again contractively tracial.

If  $\mathcal{S}$  is a singleton, the *contractive tracial hull* of a set  $\mathcal{S}$ , defined as the smallest contractively tracial set containing  $\mathcal{S}$ , is consistent with our earlier definition in terms of cp maps.

**Lemma 6.4.** *The contractive tracial hull of a subset  $\mathcal{S} \subseteq \mathbb{S}^g$  is*

$$\text{cthull}(\mathcal{S}) = \left\{ \sum C_\ell T C_\ell^* : \sum C_\ell^* C_\ell \preceq I, T \in \mathcal{S} \right\} = \bigcup_{T \in \mathcal{S}} \text{cthull}(\{T\}).$$

*If  $\mathcal{S}$  is a finite set, then the contractive tracial hull of  $\mathcal{S}$  is closed.*

*Proof.* The proof is the same as for Lemma 6.3, so is omitted. □

Tracial and contractively tracial sets are not necessarily convex, as Example 8.6 illustrates, and they are not necessarily free sets because they may not respect direct sums. Lemma 6.5 below explains the relation between these two failings. Recall that a subset  $\mathcal{Y}$  of  $\mathbb{S}^g$  is *levelwise convex* if each  $\mathcal{Y}(n)$  is convex (in the usual sense as a subset of  $\mathbb{S}_n^g$ ). Say that  $\mathcal{Y}$  is *closed with respect to convex direct sums* if given  $\ell$  and  $Y^1, \dots, Y^\ell \in \mathcal{Y}$  and given  $\lambda_1, \dots, \lambda_\ell \geq 0$  with  $\sum \lambda_j \leq 1$ , we have

$$\bigoplus_j \lambda_j Y^j \in \mathcal{Y}.$$

**Lemma 6.5.** *If  $\mathcal{Y}$  is contractively tracial, then  $\mathcal{Y}$  is levelwise convex if and only if  $\mathcal{Y}$  is closed with respect to convex direct sums.*

*Proof.* Suppose each  $\mathcal{Y}(m)$  is convex. Given  $Y^j \in \mathcal{Y}(m_j)$  for  $1 \leq j \leq \ell$ , let  $m = \sum m_j$ . Consider the block operator column  $W_j$  embedding  $\mathbb{C}^{m_j}$  into  $\mathbb{C}^m = \bigoplus_j \mathbb{C}^{m_j}$ . Note that  $W_j^* W_j = I_{m_j}$  and thus contractively tracial implies  $W_j Y^j W_j^* \in \mathcal{Y}(m)$ . Hence, given  $\lambda_j \geq 0$  with  $\sum \lambda_j = 1$ , convexity of  $\mathcal{Y}(m)$  (in the ordinary sense) implies

$$\bigoplus_j \lambda_j Y^j = \sum \lambda_j W_j Y^j W_j^* \in \mathcal{Y}(m).$$

To prove the converse, suppose  $Y^j \in \mathcal{Y}(n)$  and  $m = \ell n$ . In this case,  $\sum W_j W_j^* = I_n$ , and hence tracial implies

$$\sum W_j^* \left( \bigoplus_j \lambda_j Y^j \right) W_j = \sum \lambda_j Y^j \in \mathcal{Y}(n). \tag{□}$$

6.3. Classical duals of free convex hulls and of tracial hulls

This subsection gives properties of the classical polar dual of matrix convex hulls and tracial hulls. Real linear functionals  $\lambda : \mathbb{S}_n^g \rightarrow \mathbb{R}$  are in one-one correspondence with elements  $B \in \mathbb{S}_n^g$  via the pairing

$$\lambda(X) = \text{tr}\left(\sum B_j X_j\right), \quad X = (X_1, \dots, X_g).$$

Write  $\lambda_B$  for this  $\lambda$ . To avoid confusion with the free polar duals appearing earlier in this article, let  $\mathcal{U}^{oc}$  denote the *conventional polar dual* of a subset  $\mathcal{U} \subseteq \mathbb{S}_n^g$ ,

$$\mathcal{U}^{oc} = \{B \in \mathbb{S}_n^g : \lambda_B(X) \leq 1 \text{ for all } X \in \mathcal{U}\}.$$

**Lemma 6.6.** *Suppose  $A \in \mathbb{S}_n^g$ .*

- (i)  $\text{co}^{\text{mat}}(A)^{oc} = \{Y : \text{thull}(Y) \subseteq \{A\}^{oc}\}$ ;
- (ii)  $\text{thull}(A)^{oc} = \{Y : \{A\}^{oc} \supseteq \text{co}^{\text{mat}}(Y)\}$ ; and
- (iii)  $\text{thull}(B) \subseteq \text{thull}(A)$  if and only if  $\{A\}^{oc} \supseteq \text{co}^{\text{mat}}(Y)$  implies  $\{B\}^{oc} \supseteq \text{co}^{\text{mat}}(Y)$ .

*Proof.* The first formula:

$$\begin{aligned} \text{co}^{\text{mat}}(A)^{oc} &= \left\{ Y : 1 - \text{tr}\left(\sum_j V_j^* A V_j Y\right) \geq 0, \sum_j V_j^* V_j = I \right\} \\ &= \left\{ Y : 1 - \text{tr}\left(A \sum_j V_j Y V_j^*\right) \geq 0, \sum_j V_j^* V_j = I \right\} \\ &= \{Y : 1 - \text{tr}(AG) \geq 0, G \in \text{thull}(Y)\} \\ &= \{Y : \{A\}^{oc} \supseteq \text{thull}(Y)\}. \end{aligned}$$

The second formula:

$$\begin{aligned} \text{thull}(A)^{oc} &= \left\{ Y : 1 - \text{tr}\left(\sum_j V_j^* A V_j Y\right) \geq 0, \sum_j V_j V_j^* = I \right\} \\ &= \left\{ Y : 1 - \text{tr}\left(A \sum_j V_j Y V_j^*\right) \geq 0, \sum_j V_j V_j^* = I \right\} \\ &= \{Y : \{A\}^{oc} \supseteq \text{co}^{\text{mat}}(Y)\}. \end{aligned}$$

The third formula:  $\text{thull}(B) \subseteq \text{thull}(A)$  if and only if  $\text{thull}(B)^{oc} \supseteq \text{thull}(A)^{oc}$ , if and only if

$$\{Y : \{B\}^{oc} \supseteq \text{co}^{\text{mat}}(Y)\} \supseteq \{Y : \{A\}^{oc} \supseteq \text{co}^{\text{mat}}(Y)\},$$

if and only if  $\{A\}^{oc} \supseteq \text{co}^{\text{mat}}(Y)$  and  $\{B\}^{oc} \supseteq \text{co}^{\text{mat}}(Y)$ . □

### 7. Tracial spectrahedra and an Effros–Winkler separation theorem

Classically, convex sets are delineated by half-spaces. In this section a notion of half-space suitable in the tracial context—we call these *tracial spectrahedra*—is introduced. Subsection 7.3 contains a free Hahn–Banach separation theorem for tracial spectrahedra. The section concludes with applications of this Hahn–Banach theorem. Subsection 7.4 suggests several notions of duality based on the tracial separation theorem from Subsection 7.3. Subsection 7.5 studies free (convex) cones.

#### 7.1. Tracial spectrahedra

Polar duality considerations in the trace nonincreasing context lead naturally to inequalities of the type

$$I \otimes T - \sum_{j=1}^g B_j \otimes Y_j \succeq 0$$

for tuples  $B, Y \in \mathbb{S}^g$  and a positive semidefinite matrix  $T$  with trace at most one. Two notions, in a sense dual to one another, of half-space are obtained by fixing either  $B$  or  $Y$ .

Given  $B \in \mathbb{S}_k^g$ , let

$$\begin{aligned} \mathfrak{H}_B &= \bigcup_{m \in \mathbb{N}} \left\{ Y \in \mathbb{S}_m^g : \exists T \succeq 0, \operatorname{tr}(T) \leq 1, I \otimes T - \sum B_j \otimes Y_j \succeq 0 \right\} \\ &= \bigcup_{m \in \mathbb{N}} \left\{ Y \in \mathbb{S}_m^g : \exists T \succeq 0, \operatorname{tr}(T) = 1, I \otimes T - \sum B_j \otimes Y_j \succeq 0 \right\}. \end{aligned}$$

We call sets of the form  $\mathfrak{H}_B$  *tracial spectrahedra*. Tracial spectrahedra obtained by fixing  $Y$ , and parametrizing over  $B$ , appear in Subsubsection 7.4.2.

**Proposition 7.1.** *Let  $B \in \mathbb{S}_k^g$ . Then*

- (a) *the set  $\mathfrak{H}_B$  is contractively tracial;*
- (b) *for each  $m$ , the set  $\mathfrak{H}_B(m)$  is convex; and*
- (c) *for each  $m$ , the set  $\mathfrak{H}_B(m)$  is closed.*

*In summary,  $\mathfrak{H}_B$  is levelwise compact and closed, and it is contractively tracial.*

**Remark 7.2.** Of course  $\mathfrak{H}_B$  is not a free set since, in particular, it is not closed with respect to direct sums.

*Proof of Proposition 7.1.* Suppose  $Y \in \mathfrak{H}_B(m)$  and  $C_\ell$  satisfy (6.2). There is an  $m \times m$  positive semidefinite matrix  $T$  with trace at most one such that

$$I \otimes T - \sum B_j \otimes Y_j \succeq 0.$$

It follows that

$$0 \preceq I \otimes \sum_{\ell} C_{\ell} T C_{\ell}^* - \sum_j B_j \otimes \sum_{\ell} C_{\ell} Y_j C_{\ell}^*.$$

Note that  $T' = \sum_{\ell} C_{\ell} T C_{\ell}^* \geq 0$  and

$$\text{tr}(T') = \text{tr}\left(T \sum C_{\ell}^* C_{\ell}\right) = \text{tr}(T^{1/2} C_{\ell}^* C_{\ell} T^{1/2}) \leq \text{tr}(T) \leq 1.$$

Hence  $\sum C_{\ell} Y C_{\ell}^* \in \mathcal{Y}(n)$ , and (a) is proved.

To prove (b), suppose both  $Y^1$  and  $Y^2$  are in  $\mathcal{Y}_B$ . To each there is an associated positive semidefinite matrix of trace at most one, say  $T_1$  and  $T_2$ . If  $0 \leq s_1, s_2 \leq 1$  and  $s_1 + s_2 = 1$ , then  $T = \sum s_{\ell} T_{\ell}$  is positive semidefinite and has trace at most one. Moreover, with  $Y = \sum s_j Y^j$ ,

$$I \otimes T - \sum_j B_j \otimes \left(\sum_{\ell} s_{\ell} Y_j^{\ell}\right) = \sum_{\ell} s_{\ell} \left(I \otimes T_{\ell} - \sum_j B_j \otimes Y_j^{\ell}\right) \geq 0.$$

To prove (c), suppose the sequence  $(Y^k)_k$  from  $\mathfrak{H}_B(m)$  converges to  $Y \in \mathbb{S}_m^g$ . For each  $k$  there is a positive semidefinite matrix  $T_k$  of trace at most one such that

$$I \otimes T_k - \Lambda_B(Y^k) \geq 0.$$

Choose a convergent subsequence of the  $T_k$  with limit  $T$ . This  $T$  witnesses  $Y \in \mathfrak{H}_B(m)$ . □

To proceed toward the separation theorem we start with some preliminaries.

### 7.2. An auxiliary result

Given a positive integer  $n$ , let  $\mathcal{T}_n$  denote the positive semidefinite  $n \times n$  matrices of trace one. Each  $T \in \mathcal{T}_n$  corresponds to a state on  $M_n$  via the trace

$$M_n \ni A \mapsto \text{tr}(AT). \tag{7.1}$$

Conversely, to each state  $\varphi$  on  $M_n$  we can assign a matrix  $T$  such that  $\varphi$  is the map (7.1). Note that  $\mathcal{T}_n$  is a convex, compact subset of  $\mathbb{S}_n$ , the symmetric  $n \times n$  matrices.

The following lemma is a version of [EW97, Lemma 5.2]. An affine (real) linear mapping  $f : \mathbb{S}_n \rightarrow \mathbb{R}$  is a function of the form  $f(x) = a_f + \lambda_f(x)$ , where  $\lambda_f$  is (real) linear and  $a_f \in \mathbb{R}$ .

**Lemma 7.3.** *Suppose  $\mathcal{F}$  is a convex set of affine linear mappings  $f : \mathbb{S}_n \rightarrow \mathbb{R}$ . If for each  $f \in \mathcal{F}$  there is a  $T \in \mathcal{T}_n$  such that  $f(T) \geq 0$ , then there is a  $\mathfrak{T} \in \mathcal{T}_n$  such that  $f(\mathfrak{T}) \geq 0$  for every  $f \in \mathcal{F}$ .*

*Proof.* For  $f \in \mathcal{F}$ , let

$$B_f = \{T \in \mathcal{T}_n : f(T) \geq 0\} \subseteq \mathcal{T}_n.$$

By hypothesis each  $B_f$  is nonempty and it suffices to prove that

$$\bigcap_{f \in \mathcal{F}} B_f \neq \emptyset.$$

Since each  $B_f$  is compact, it suffices to prove that the collection  $\{B_f : f \in \mathcal{F}\}$  has the finite intersection property. Accordingly, let  $f_1, \dots, f_m \in \mathcal{F}$  and suppose

$$\bigcap_{j=1}^m B_{f_j} = \emptyset.$$

Define  $F : \mathbb{S}_n \rightarrow \mathbb{R}^m$  by

$$F(T) = (f_1(T), \dots, f_m(T)).$$

Then  $F(\mathcal{T}_n)$  is both convex and compact because  $\mathcal{T}_n$  is both convex and compact and each  $f_j$ , and hence  $F$ , is affine linear. Moreover,  $F(\mathcal{T}_n)$  does not intersect

$$\mathbb{R}_{\geq 0}^m = \{x = (x_1, \dots, x_m) \in \mathbb{R}^m : x_j \geq 0 \text{ for each } j\}.$$

Hence there is a linear functional  $\lambda : \mathbb{R}^m \rightarrow \mathbb{R}$  such that

$$\lambda(F(\mathcal{T}_n)) < 0 \quad \text{and} \quad \lambda(\mathbb{R}_{\geq 0}^m) \geq 0.$$

There exists  $\lambda_j \in \mathbb{R}$  such that  $\lambda(x) = \sum \lambda_j x_j$ . Since  $\lambda(\mathbb{R}_{\geq 0}^m) \geq 0$  it follows that  $\lambda_j \geq 0$  for all  $j$ , and since  $\lambda \neq 0$ , for at least one  $k$  we have  $\lambda_k > 0$ . Without loss of generality, it may be assumed that  $\sum \lambda_j = 1$ . Let

$$f = \sum \lambda_j f_j.$$

Since  $\mathcal{F}$  is convex, it follows that  $f \in \mathcal{F}$ . On the other hand,  $f(T) = \lambda(F(T))$ . Hence if  $T \in \mathcal{T}_n$ , then  $f(T) < 0$ . Thus, for this  $f$  there does not exist a  $T \in \mathcal{T}_n$  such that  $f(T) \geq 0$ , a contradiction which completes the proof.  $\square$

### 7.3. A tracial spectrahedron separating theorem

The following lemma is proved by a variant of the Effros–Winkler construction of separating LMIs (i.e., the matricial Hahn–Banach theorem) in the theory of matrix convex sets.

**Lemma 7.4.** *Fix positive integers  $m, n$ , and suppose that  $\mathcal{S}$  is a nonempty subset of  $\mathbb{S}_n^g$ . Let  $\mathcal{U}$  denote the subset of  $\mathbb{S}_n^g$  consisting of all tuples of the form*

$$\sum_{\ell=1}^{\mu} C_{\ell} Y^{\ell} C_{\ell}^*,$$

where each  $C_{\ell}$  is  $n \times m$ , each  $Y^{\ell}$  is in  $\mathcal{S}$  and  $\sum C_{\ell}^* C_{\ell} \preceq I$ . If  $B \in \mathbb{S}_n^g$  is in the conventional polar dual of  $\mathcal{U}$ , then there exists a positive semidefinite  $m \times m$  matrix  $T$  with trace at most one such that

$$I \otimes T - \sum B_j \otimes Y_j \succeq 0 \quad \text{for every } Y \in \mathcal{S}.$$

*Proof.* Recall the definition of  $\lambda_B$  from Subsection 6.3. Given  $C_\ell$  and  $Y^\ell$  as in the statement of the lemma, define  $f_{C,Y} : \mathbb{S}_m^g \rightarrow \mathbb{R}$  by

$$f_{C,Y}(X) = \text{tr}\left(\sum C_\ell X C_\ell^*\right) - \lambda_B\left(\sum C_\ell Y C_\ell^*\right).$$

Let  $\mathcal{F} = \{f_{C,Y} : C, Y\}$ . Thus  $\mathcal{F}$  is a set of affine (real) linear mappings from  $\mathbb{S}_m^g$  to  $\mathbb{R}$ . To show that  $\mathcal{F}$  is convex, suppose, for  $1 \leq s \leq N$ ,  $C^s = (C_1^s, \dots, C_{\mu_s}^s)$  is a tuple of  $n \times m$  matrices, for  $1 \leq s \leq N$  and  $1 \leq j \leq \mu_s$  the matrices  $Y^{s,j}$  are in  $\mathcal{S}$ , and  $\lambda_1, \dots, \lambda_N$  are positive numbers with  $\sum \lambda_s = 1$ . In this case,

$$\sum \lambda_s f_{C^s, Y^{s,\cdot}} = f_{C,Y}$$

for

$$C = \left(\frac{1}{\sqrt{\lambda_s}} C_\ell^s\right)_{s,\ell}, \quad Y = (Y^{s,\ell})_{s,\ell}.$$

Hence  $\mathcal{F}$  is convex.

Given  $n \times m$  matrices  $C_1, \dots, C_\mu$  and  $Y^1, \dots, Y^\mu \in \mathcal{S}$ , let  $D = \sum C_\ell^* C_\ell$ . Assuming  $D$  has norm one, there is a unit vector  $\gamma$  such that  $\|D\gamma\| = \|D\| = 1$ . Choose  $T = \gamma\gamma^*$ . Thus  $T \in \mathcal{T}_m$ . Moreover,

$$\text{tr}\left(\sum C_\ell T C_\ell^*\right) = \text{tr}(TD) = \langle D\gamma, \gamma \rangle = 1.$$

Thus, using the assumption that  $B$  is in  $\mathcal{U}^{oc}$ , we see that

$$f_{C,Y}(T) = 1 - \lambda_B\left(\sum C_\ell Y^\ell C_\ell^*\right) \geq 0.$$

If  $D$  is not of norm one, a simple scaling argument gives the same conclusion,

$$f_{C,Y}(T) \geq 0.$$

Thus, for each  $f \in \mathcal{F}$  there exists a  $T \in \mathcal{T}_m$  such that  $f(T) \geq 0$ . By Lemma 7.3, it follows that there is a  $\mathfrak{T} \in \mathcal{T}_m$  such that  $f_C(\mathfrak{T}) \geq 0$  for all  $C$  and  $Y$ , i.e.,

$$\text{tr}\left(\sum C_\ell \mathfrak{T} C_\ell^*\right) - \lambda_B\left(\sum C_\ell Y^\ell C_\ell^*\right) \geq 0, \tag{7.2}$$

regardless of the norm of  $\sum C_\ell^* C_\ell$ .

Now the aim is to show that

$$\Delta := I \otimes \mathfrak{T} - \sum_j B_j \otimes Y_j \succeq 0 \quad \text{for every } Y \in \mathcal{S}.$$

Accordingly, let  $Y \in \mathcal{S}$  and  $\gamma = \sum e_s \otimes \gamma_s \in \mathbb{R}^n \otimes \mathbb{R}^m$ . Compute

$$\langle \Delta \gamma, \gamma \rangle = \sum_s \langle \mathfrak{T} \gamma_s, \gamma_s \rangle - \sum_j \sum_{s,t} (B_j)_{s,t} \langle Y_j \gamma_s, \gamma_t \rangle.$$

Now let  $\Gamma^*$  denote the matrix with  $s$ -th column  $\gamma_s$ . Hence  $\Gamma$  is  $n \times m$  and

$$\lambda_B(\Gamma Y \Gamma^*) = \text{tr}\left(\sum_j B_j(\Gamma Y_j \Gamma^*)\right) = \sum_j \sum_{s,t} (B_j)_{s,t} \langle Y_j \gamma_s, \gamma_t \rangle.$$

Similarly,

$$\text{tr}(\Gamma \mathfrak{T} \Gamma^*) = \sum_s \langle \mathfrak{T} \gamma_s, \gamma_s \rangle.$$

Thus, by (7.2),

$$\langle \Delta \gamma, \gamma \rangle = \text{tr}(\Gamma \mathfrak{T} \Gamma^*) - \lambda_B(\Gamma Y \Gamma^*) \geq 0.$$

It is in this last step that the contractively tracial, not just tracial is needed, so that it is not necessary for  $\Gamma^* \Gamma$  to be a multiple of the identity.  $\square$

**Proposition 7.5.** *If  $\mathcal{Y} \subseteq \mathbb{S}^s$  is contractively tracial and if  $B \in \mathbb{S}_n^g$  is in the conventional polar dual  $\mathcal{Y}(n)^{oc}$  of  $\mathcal{Y}(n)$ , then  $\mathcal{Y} \subseteq \mathfrak{H}_B$ .*

*Proof.* Suppose  $\mathcal{Y}$  is contractively tracial and  $Y \in \mathcal{Y}(m)$ . Letting  $\mathcal{S} = \{Y\}$  in Lemma 7.4, it follows that there is a  $T$  such that

$$I \otimes T - \sum B_j \otimes Y_j \geq 0.$$

Thus,  $Y \in \mathfrak{H}_B$ , and the proof is complete.  $\square$

We are now ready to state the separation result for closed levelwise convex tracial sets.

**Theorem 7.6.** (i) *If  $\mathcal{Y} \subseteq \mathbb{S}^s$  is contractively tracial, levelwise convex, and if  $Z \in \mathbb{S}_m^g$  is not in the closure of  $\mathcal{Y}(m)$ , then there exists a  $B \in \mathbb{S}_m^g$  such that  $\mathcal{Y} \subseteq \mathfrak{H}_B$ , but  $Z \notin \mathfrak{H}_B$ . Hence,*

$$\overline{\mathcal{Y}} = \bigcap \{\mathfrak{H}_B : \mathfrak{H}_B \supseteq \mathcal{Y}\} = \bigcap_{n \in \mathbb{N}} \bigcap_{B \in \mathcal{Y}(n)^{oc}} \mathfrak{H}_B.$$

(ii) *The levelwise closed convex contractively tracial hull of a subset  $\mathcal{Y}$  of  $\mathbb{S}^s$  is*

$$\bigcap \{\mathfrak{H}_B : \mathfrak{H}_B \supseteq \mathcal{Y}\}.$$

*Proof.* To prove (i), suppose  $Z \in \mathbb{S}_m^g$  but  $Z \notin \overline{\mathcal{Y}(m)}$ . Since  $\mathcal{Y}$  is levelwise convex, there is  $\lambda_B$  such that  $\lambda_B(Y) \leq 1$  for all  $Y \in \mathcal{Y}(m)$ , but  $\lambda_B(Z) > 1$  by the usual Hahn–Banach separation theorem for closed convex sets. Thus  $B$  is in the conventional polar dual of  $\mathcal{Y}(m)^{oc}$ . From Proposition 7.5,  $\mathcal{Y} \subseteq \mathfrak{H}_B$ .

On the other hand, if  $T \in \mathcal{T}_m$  and  $\{e_1, \dots, e_m\}$  is an orthonormal basis for  $\mathbb{R}^m$ , then, with  $e = \sum e_s \otimes e_s \in \mathbb{R}^m \otimes \mathbb{R}^m$ ,

$$\left\langle \left( I \otimes T - \sum B_j \otimes Z_j \right) e, e \right\rangle = \text{tr}(T) - \text{tr}\left(\sum B_j Z_j\right) = 1 - \lambda_B(Z) < 0.$$

Hence  $Z \notin \mathfrak{H}_B$  and the conclusion follows.

To prove (ii), first note that if  $\mathcal{I}$  denotes the intersection of the  $\mathfrak{H}_B$  that contain  $\mathcal{Y}$ , then  $\mathcal{Y} \subseteq \mathcal{I}$ . Since the intersection of tracial spectrahedra is levelwise closed and convex, and contractively tracial, the levelwise closed convex tracial hull  $\mathcal{H}$  of  $\mathcal{Y}$  is also contained in  $\mathcal{I}$ . On the other hand, from (i),

$$\mathcal{H} = \bigcap \{\mathfrak{H}_B : \mathfrak{H}_B \supseteq \mathcal{H}\} \supseteq \mathcal{I} \supseteq \mathcal{H}. \quad \square$$

**Remark 7.7** (The contractive tracial hull of a point). Fix a  $Y \in \mathbb{S}_n^g$  and let  $\mathcal{Y}$  denote its contractive tracial hull,

$$\mathcal{Y} = \left\{ \sum V_j Y V_j^* : \sum V_j^* V_j \preceq I \right\}.$$

Evidently each  $\mathcal{Y}(m)$  (taking  $V_j : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ) is a convex set. From Lemma 6.4,  $\mathcal{Y}$  is closed. Hence Theorem 7.6 applies and gives a duality description of  $\mathcal{Y}$ . Namely,  $\tilde{Y}$  is in the contractive tracial hull  $\mathcal{Y}$  if and only if for each  $B$  for which there exists a positive semidefinite  $T$  of trace at most one such that

$$I \otimes T - \sum B_j \otimes Y_j \succeq 0,$$

there exists a positive semidefinite  $\tilde{T}$  of trace at most one such that

$$I \otimes \tilde{T} - \sum B_j \otimes \tilde{Y}_j \succeq 0.$$

7.4. Tracial polar duals

We now introduce two natural notions of polar duals based on the tracial spectrahedra. Rather than exhaustively studying these duals, we list a few properties to illustrate the possibilities.

7.4.1. *Ex situ tracial dual.* Suppose  $\mathcal{K} \subseteq \mathbb{S}^g$ . Let  $\hat{\mathcal{K}}$  denote its *ex situ tracial dual* defined by

$$\hat{\mathcal{K}} = \bigcap_{B \in \mathcal{K}} \mathfrak{H}_B.$$

Thus,

$$\hat{\mathcal{K}}(n) = \left\{ Y \in \mathbb{S}_n^g : \forall B \in \mathcal{K} \exists T \succeq 0, \text{tr}(T) \leq 1, I \otimes T - \sum B_j \otimes Y_j \succeq 0 \right\}.$$

**Proposition 7.8.** *If  $\mathcal{K}$  is matrix convex and each  $\mathcal{K}^\circ(n)$  is bounded (equivalently,  $\mathcal{K}(1)$  contains 0 in its interior), then*

- (i)  $\hat{\mathcal{K}}(n) = \{Y \in \mathbb{S}_n^g : \exists T \succeq 0, \text{tr}(T) \leq 1 \forall B \in \mathcal{K}, I \otimes T - \sum B_j \otimes Y_j \succeq 0\}$ ;
- (ii)  $\hat{\mathcal{K}}(n) = \{SMS : M \in \mathcal{K}^\circ(n), S \succeq 0, \text{tr}(S^2) \leq 1\}$ .

*Proof.* Suppose  $\mathcal{K}$  is matrix convex. To prove (i), let  $Y \in \hat{\mathcal{K}}(n)$ . For each  $B$ , let  $\mathcal{T}_B = \{T \in \mathcal{T}_n : I \otimes T - \sum B_j \otimes Y_j \succeq 0\}$ . Thus, the hypothesis that  $Y \in \hat{\mathcal{K}}(n)$  is equivalent to assuming that for every  $B$  in  $\mathcal{K}$ , the set  $\mathcal{T}_B$  is nonempty.

That  $\mathcal{T}_B$  is compact will be verified by showing it has the finite intersection property. Now given  $B^1, \dots, B^\ell \in \mathcal{K}$ , let  $B = \bigoplus_k B^k \in \mathcal{K}$ . Since  $B \in \mathcal{K}$ , there is a  $T$  such that

$$\bigoplus_k \left( I \otimes T - \sum B_j^k \otimes Y_j \right) = I \otimes T - \sum B_j \otimes Y_j \succeq 0.$$



Hence  $T \in \bigcap_{k=1}^{\ell} \mathcal{T}_{B^k}$ . It follows that the collection  $\{\mathcal{T}_B : B \in \mathcal{K}\}$  has the finite intersection property and hence there is a  $T \in \bigcap_{B \in \mathcal{K}} \mathcal{T}_B$ , and the forward inclusion in (i) follows. The reverse inclusion holds whether or not  $\mathcal{K}$  is matrix convex.

To prove (ii), suppose  $Y \in \hat{\mathcal{K}}(n)$ . Thus, by what has already been proved, there is a positive semidefinite matrix  $S$  such that  $\text{tr}(S^2) \leq 1$  and

$$I \otimes S^2 - \sum B_j \otimes Y_j \succeq 0 \tag{7.3}$$

for all  $B \in \mathcal{K}$ . For positive integers  $k$ , let  $S_k^+$  denote the inverse of  $S + 1/k$ . Multiplying (7.3) on the left and on the right by  $I \otimes S_k^+$  yields

$$I \otimes P - \sum B_j \otimes S_k^+ Y_j S_k^+ \succeq 0,$$

where  $P$  is the projection onto the range of  $S$ . It follows that  $M_k = S_k^+ Y S_k^+ \in \mathcal{K}^\circ(n)$ . Since  $\mathcal{K}^\circ(n)$  is bounded (by assumption) and closed, it is compact, and consequently a subsequence of  $(M_k)_k$  converges to some  $M \in \mathcal{K}^\circ(n)$ . Hence,  $Y = SMS$ .

Reversing the argument above shows that if  $M \in \mathcal{K}^\circ(n)$  and  $S$  is positive semidefinite with  $\text{tr}(S^2) \leq 1$ , then  $Y = SMS \in \hat{\mathcal{K}}(n)$ , and the proof is complete.  $\square$

**Proposition 7.9.** *The ex situ tracial dual  $\hat{\mathcal{K}}$  of a free spectrahedron  $\mathcal{K} = \mathcal{D}_{\Omega}$  is exactly the set*

$$\left\{ \sum_{\ell} C_{\ell}^* \Omega C_{\ell} : \text{tr} \left( \sum_{\ell} C_{\ell}^* C_{\ell} \right) \leq 1 \right\}.$$

*Proof.* Suppose  $Y$  is in the ex situ tracial dual. By Proposition 7.8, there is a positive semidefinite matrix  $S$  with  $\text{tr}(S^2) \leq 1$  and an  $M \in \mathcal{K}^\circ$  such that  $Y = SMS$ . Since  $M \in \mathcal{K}^\circ$ , by Remark 4.8 there is a positive integer  $\mu$  and a contraction  $V$  such that

$$M = V^* (I_{\mu} \otimes \Omega) V = \sum_{k=1}^{\mu} V_k^* \Omega V_k.$$

Hence,

$$Y = \sum_k S V_k^* \Omega V_k S.$$

Finally,

$$\text{tr} \left( \sum S V_k^* V_k S \right) \leq \text{tr}(S^2) \leq 1.$$

Conversely, suppose  $\text{tr}(\sum C_{\ell}^* C_{\ell}) \leq 1$  and  $Y = \sum C_{\ell}^* \Omega C_{\ell}$ . Let  $T = \sum C_{\ell}^* C_{\ell}$  and note that for  $B \in \mathcal{K}$ ,

$$I \otimes T - \sum B_j \otimes Y_j = \sum_{\ell} C_{\ell}^* \left( I \otimes I - \sum_j B_j \otimes \Omega_j \right) C_{\ell} \succeq 0. \quad \square$$

7.4.2. *In situ tracial dual.* Given a free set  $\mathcal{K} \subseteq \mathbb{S}^g$ , we can define another dual set we call the *in situ*  $\mathcal{K}^\triangleright = (\mathcal{K}^\triangleright(m))_m$  by

$$\mathcal{K}^\triangleright(m) = \{B \in \mathbb{S}_m^g : \mathcal{K} \subseteq \mathfrak{H}_B\}.$$

Equivalently,

$$\mathcal{K}^\triangleright(m) = \left\{ B \in \mathbb{S}_m^g : \forall Y \in \mathcal{K} \exists T \succeq 0, \operatorname{tr}(T) \leq 1, I \otimes T - \sum B_j \otimes Y_j \succeq 0 \right\}.$$

Each  $\mathcal{K}^\triangleright(m)$  is levelwise convex. Moreover, if  $B \in \mathcal{K}^\triangleright$  and  $V^*V \preceq I$ , then  $V^*BV \in \mathcal{K}^\triangleright$ . On the other hand, there is no reason to expect that  $\mathcal{K}^\triangleright$  is closed with respect to direct sums. Hence it need not be matrix convex.

A subset  $\mathcal{Y}$  of  $\mathbb{S}^g$  is *contractively stable* if  $\sum C_j^*YC_j \in \mathcal{Y}$  for all  $Y \in \mathcal{Y}$  whenever  $\sum C_j^*C_j \preceq I$ . In general, contractively stable sets need not be levelwise convex, as Example 8.7 shows.

**Proposition 7.10.** *The set  $\mathcal{K}^\triangleright$  is contractively stable.*

*Proof.* Suppose  $B \in \mathcal{K}^\triangleright(m)$ . Let  $n \times m$  matrices  $C_1, \dots, C_\ell$  be such that  $\sum C_k^*C_k \preceq I$  and consider the  $n \times n$  matrix  $D = \sum C_k B C_k^*$ .

Given  $Y \in \mathcal{K}(p)$ , there exists a positive semidefinite  $p \times p$  matrix  $T$  of trace at most one such that

$$I \otimes T - \sum B_j \otimes Y_j \succeq 0.$$

Thus,

$$\begin{aligned} I \otimes T - \sum_{j=1}^g D_j \otimes Y_j &= I \otimes T - \sum_{j=1}^g \sum_k C_k^* B_j C_k \otimes Y_j \\ &= \left( I - \sum C_k^* C_k \right) \otimes T + \sum_k (C_k \otimes I)^* \left( I \otimes T - \sum_j B_j \otimes Y_j \right) (C_k \otimes I) \succeq 0. \end{aligned}$$

Hence  $D \in \mathcal{K}^\triangleright$ , and the proof is complete. □

The *contractive convex hull* of  $\mathcal{Y}$  is the smallest levelwise closed set containing  $\mathcal{Y}$  that is contractively stable. The following proposition finds the two hulls defined by applying the two notions of tracial polar duals introduced above.

**Proposition 7.11.** *For  $\mathcal{K} \subseteq \mathbb{S}^g$ , the set  $\widehat{\mathcal{K}}^\triangleright$  is the levelwise closed convex contractively tracial hull of  $\mathcal{K}$ . Similarly,  $(\widehat{\mathcal{K}})^\triangleright$  is the levelwise closed contractively stable hull of  $\mathcal{K}$ .*

The proof of the second statement rests on the following companion to Lemma 7.4. Recall from (1.5) the opp-tracial spectrahedron,

$$\mathfrak{H}_Y^{\text{opp}} = \left\{ B : \exists T \succeq 0, \operatorname{tr}(T) \leq 1, I \otimes T - \sum B_j \otimes Y_j \succeq 0 \right\}.$$

**Lemma 7.12.** Fix positive integers  $m, n$ , and suppose that  $\mathcal{S}$  is a nonempty subset of  $\mathbb{S}_n^g$ . Let  $\mathcal{U}$  denote the subset of  $\mathbb{S}_m^g$  consisting of all tuples of the form

$$\sum_{\ell=1}^{\mu} C_{\ell}^* B^{\ell} C_{\ell},$$

where each  $C_{\ell}$  is  $n \times m$ , each  $B^{\ell}$  is in  $\mathcal{S}$  and  $\sum C_{\ell}^* C_{\ell} \leq I$ .

(1) If  $Y \in \mathbb{S}_m^g$  is in the conventional polar dual of  $\mathcal{U}$ , then there exists a positive semidefinite  $m \times m$  matrix  $T$  with trace at most one such that

$$I \otimes T - \sum B_j \otimes Y_j \geq 0 \quad \text{for every } B \in \mathcal{S}.$$

(2) The tracial spectrahedra  $\mathfrak{H}_Y^{\text{opp}}$  are closed and contractively stable.

(3) If  $\mathcal{K} \subseteq \mathbb{S}^g$  is contractively stable and  $Y \in \mathbb{S}_m^g$  is in the conventional polar dual  $\mathcal{K}(m)^{\text{oc}}$  of  $\mathcal{K}(m)$ , then  $\mathcal{K} \subseteq \mathfrak{H}_Y^{\text{opp}}$ .

(4) If  $\mathcal{K} \subseteq \mathbb{S}^g$  is levelwise closed and convex, and contractively stable, then

$$\mathcal{K} = \bigcap \{ \mathfrak{H}_Y^{\text{opp}} : \mathfrak{H}_Y^{\text{opp}} \supseteq \mathcal{K} \} = \bigcap_n \bigcap \{ \mathfrak{H}_Y^{\text{opp}} : Y \in \mathcal{K}(n)^{\text{oc}} \}.$$

(5) The levelwise closed and convex contractively stable hull of  $\mathcal{K} \subseteq \mathbb{S}^g$  is

$$\bigcap \{ \mathfrak{H}_Y^{\text{opp}} : \mathfrak{H}_Y^{\text{opp}} \supseteq \mathcal{K} \}.$$

(6) For  $\mathcal{K} \subseteq \mathbb{S}^g$ , we have  $Y \in \hat{\mathcal{K}}(n)$  if and only if  $\mathcal{K} \subseteq \mathfrak{H}_Y^{\text{opp}}$ .

*Proof.* The proof of (1) is similar to the proof of Lemma 7.4 and is omitted. Likewise, the proof of (2) follows an argument given in the proof of Proposition 7.10.

To prove (3), suppose that  $Y \in \mathcal{K}(m)^{\text{oc}}$ . Given  $B \in \mathcal{K}(n)$ , an application of (1) with  $\mathcal{S} = \{B\}$  produces an  $m \times m$  positive semidefinite matrix  $T$  with  $\text{tr}(T) \leq 1$  such that  $I \otimes T - \sum B_j \otimes Y_j \geq 0$ . Hence,  $B \in \mathfrak{H}_Y^{\text{opp}}$ .

We now move on to (4). From (3), if  $Y \in \mathcal{K}(m)^{\text{oc}}$ , then  $\mathcal{K} \subseteq \mathfrak{H}_Y^{\text{opp}}$ . On the other hand, if  $Y \in \mathbb{S}_m^g$  and  $\mathcal{K} \subseteq \mathfrak{H}_Y^{\text{opp}}$ , then, for  $B \in \mathcal{K}(m)$ ,

$$I \otimes T - \sum B_j \otimes Y_j \geq 0$$

for some positive semidefinite  $T$  with trace at most one. In particular, if we set  $e = \sum_{s=1}^m e_s \otimes e_s \in \mathbb{R}^m \otimes \mathbb{R}^m$ , then

$$0 \leq \left\langle I \otimes T - \sum B_j \otimes Y_j e, e \right\rangle = \text{tr}(T) - \lambda_Y(B).$$

Hence  $Y \in \mathcal{K}(m)^{\text{oc}}$ . Continuing with the proof of (4), from (3) we obtain

$$\mathcal{K} \subseteq \bigcap_n \bigcap \{ \mathfrak{H}_Y^{\text{opp}} : Y \in \mathcal{K}(n)^{\text{oc}} \}.$$

To establish the reverse inclusion, suppose that  $C$  is not in  $\mathcal{K}(m)$ . Since  $\mathcal{K}(m)$  is assumed to be closed and convex, there exists a  $Y \in \mathcal{K}(m)^{\text{oc}}$ , the conventional polar dual of  $\mathcal{K}(m)$

(so that  $\lambda_Y(\mathcal{K}(m)) \leq 1$ ) with  $\lambda_Y(C) > 1$ . In particular,  $\mathcal{K} \subseteq \mathfrak{H}_Y^{\text{opp}}$ . On the other hand, if  $T$  is  $m \times m$  and positive semidefinite with trace at most one, then with  $e = \sum e_s \otimes e_s$ ,

$$\left\langle \left( I \otimes T - \sum C_j \otimes Y_j \right) e, e \right\rangle = \text{tr}(T) - \sum_j \text{tr}(C_j Y_j) = \text{tr}(T) - \lambda_Y(C) < 0.$$

Hence,  $C \notin \mathfrak{H}_Y^{\text{opp}}$ .

To prove (5), let  $\mathcal{H}$  denote the contractively stable hull of  $\mathcal{K}$ . Let also  $\mathcal{I}$  denote the intersection of all the tracial spectrahedra  $\mathfrak{H}_Y^{\text{opp}}$  such that  $\mathcal{K} \subseteq \mathfrak{H}_Y^{\text{opp}}$ . Evidently  $\mathcal{H} \subseteq \mathcal{I}$ . On the other hand, by (4),

$$\mathcal{K} \subseteq \mathcal{H} \subseteq \mathcal{I} \subseteq \bigcap \{ \mathfrak{H}_Y^{\text{opp}} : \mathfrak{H}_Y^{\text{opp}} \supseteq \mathcal{H} \} = \mathcal{H}.$$

Finally, for (6), first suppose  $Y \in \hat{\mathcal{K}}(n)$ . By definition, for each  $B \in \mathcal{K}$  there is a positive semidefinite  $T$  of trace at most one such that  $I \otimes T - \sum B_j \otimes Y_j \geq 0$ . Hence  $\mathcal{K} \subseteq \mathfrak{H}_Y^{\text{opp}}$ . Conversely, if  $B \in \mathfrak{H}_Y^{\text{opp}}$ , then  $I \otimes T - \sum B_j \otimes Y_j \geq 0$  for some positive semidefinite  $T$  of trace at most one depending on  $B$ . Thus, if  $\mathcal{K} \subseteq \mathfrak{H}_Y^{\text{opp}}$ , then  $Y \in \hat{\mathcal{K}}(n)$ . □

*Proof of Proposition 7.11.* Since

$$\widehat{\mathcal{K}}^\triangleright = \bigcap_{B \in \mathcal{K}^\triangleright} \mathfrak{H}_B = \bigcap_{\mathcal{K} \subseteq \mathfrak{H}_B} \mathfrak{H}_B,$$

Theorem 7.6(ii) gives the first conclusion of the proposition.

Likewise,

$$(\hat{\mathcal{K}})^\triangleright = \{ B : \hat{\mathcal{K}} \subseteq \mathfrak{H}_B \} = \bigcap_{Y \in \hat{\mathcal{K}}} \mathfrak{H}_Y^{\text{opp}} = \bigcap \{ \mathfrak{H}_Y^{\text{opp}} : \mathfrak{H}_Y^{\text{opp}} \supseteq \mathcal{K} \},$$

and the term on the right-hand side is, by Lemma 7.12, the closed contractive convex hull of  $\mathcal{K}$ . □

### 7.5. Matrix convex tracial sets and free cones

In this subsection we introduce and study properties of free (convex) cones.

A subset  $\mathcal{S}$  of  $\mathbb{S}^g$  is a *free cone* if for all positive integers  $m, n, \ell$ , tuples  $T \in \mathcal{S}(n)$  and  $n \times m$  matrices  $C_1, \dots, C_\ell$ , the tuple  $\sum C_i^* T C_i$  is in  $\mathcal{S}(m)$ . The set  $\mathcal{S}$  is a *free convex cone* if for all positive integers  $m, n, \ell$ , tuples  $T^1, \dots, T^\ell \in \mathcal{S}(n)$  and  $n \times m$  matrices  $C_1, \dots, C_\ell$ , the tuple  $\sum_i C_i^* T^i C_i$  lies in  $\mathcal{S}(m)$ . Finally, a subset  $\mathcal{Y}$  of  $\mathbb{S}^g$  is a *contractively tracial convex set* if  $\mathcal{Y}$  is contractively tracial and given positive integers  $m, n, \mu$  and  $Y^1, \dots, Y^\mu \in \mathcal{Y}(m)$  and  $n \times m$  matrices  $C_1, \dots, C_\mu$  with  $\sum C_j^* C_j \leq I$ , the tuple

$$\sum C_j Y^j C_j^*$$

lies in  $\mathcal{Y}(n)$ . This condition is an analog to matrix convexity of a set containing 0 which we studied earlier in this paper. Surprisingly:

**Proposition 7.13.** *Every contractively tracial convex set is a free convex cone.*

For the proof of this proposition we introduce an auxiliary notion and then give a lemma. A subset  $\mathcal{Y}$  of  $\mathbb{S}^g$  is *closed with respect to identical direct sums* if for each  $Y \in \mathcal{Y}$  and positive integer  $\ell$ , the tuple  $I_\ell \otimes Y$  is in  $\mathcal{Y}$ .

**Lemma 7.14.** *Suppose  $\mathcal{Y} \subseteq \mathbb{S}^g$ .*

- (1) *If  $\mathcal{Y}$  is contractively tracial and closed with respect to identical direct sums, then  $\mathcal{Y}$  is a free cone.*
- (2) *If  $\mathcal{Y}$  is contractively tracial and closed with respect to direct sums, then  $\mathcal{Y}$  is a free convex cone.*
- (3) *If  $\mathcal{Y}$  is a tracial set containing 0 which is levelwise convex and closed with respect to identical direct sums, then each  $\mathcal{Y}(m)$  is a cone in the ordinary sense.*

*Proof.* To prove (1), let  $Y \in \mathcal{Y}(n)$  and let  $\ell$  be a positive integer. Let  $V_k$  denote the block  $1 \times \ell$  row matrices with  $m \times n$  matrix entries with  $I_n$  in the  $k$ -th position and 0 elsewhere, for  $k = 1, \dots, \ell$ . It follows that  $\sum V_k^* V_k = I$ . Since also  $I_\ell \otimes Y$  is in  $\mathcal{Y}$  and  $\mathcal{Y}$  is tracial,

$$\sum V_k(Y \otimes I_\ell)V_k^* = kY \in \mathcal{Y}(n).$$

Now let  $m$  and  $\ell$  be positive integers and consider  $m \times n$  matrices  $C_1, \dots, C_\ell$  and  $Y^1, \dots, Y^\ell \in \mathcal{Y}(n)$ . Choose a positive integer  $k$  such that each  $D_j = C_j/\sqrt{k}$  has norm at most one. Let  $M_j$  be the block  $1 \times \ell$  row matrix with  $m \times n$  entries with  $D_j$  in the  $j$ -th position and 0 elsewhere, for  $j = 1, \dots, \ell$ . It follows that

$$\sum_j M_j^* M_j = \text{diag}(D_1^* D_1, \dots, D_\ell^* D_\ell) \preceq I.$$

Since  $\mathcal{Y}$  is tracial, either assuming  $Y^j = Y^k$  for all  $j, k$  and  $\mathcal{Y}$  is closed under identical direct sums, or assuming that  $\mathcal{Y}$  is closed under direct sums  $\bigoplus_{j=1}^\ell Y^j$  is in  $\mathcal{Y}$ , and hence

$$\sum M_j \left( \bigoplus_{j=1}^\ell Y^j \right) M_j^* = k \sum_{j=1}^\ell D_j Y^j D_j^* = \sum C_j Y^j C_j^* \in \mathcal{Y}(n).$$

Thus, in the first case  $\mathcal{Y}$  is a free cone and in the second a free convex cone.

To prove the third statement, note that the argument used to prove the first part of the lemma shows that if  $\mathcal{Y}$  is a tracial set that is closed with respect to identical direct sums and if each  $C_j$  is  $I$ , then  $\ell Y = \sum C_j(Y \otimes I_\ell)C_j^*$  is in  $\mathcal{Y}(n)$ . If  $\mathcal{Y}$  is levelwise convex, since also  $0 \in \mathcal{Y}(n)$ , it follows that  $\mathcal{Y}(n)$  is a convex cone.  $\square$

*Proof of Proposition 7.13.* Fix positive integers  $n$  and  $\nu$ . Let  $Y^1, \dots, Y^\nu \in \mathcal{Y}(n)$ . Let  $C_\ell$  denote the inclusion of  $\mathbb{R}^n$  as the  $\ell$ -th coordinate in  $\mathbb{R}^{n\nu} = \bigoplus_{i=1}^\nu \mathbb{R}^n$ . In particular,  $C_\ell^* C_\ell = I_n$ , and hence  $Z^\ell = C_\ell Y^\ell C_\ell^* \in \mathcal{Y}(n\nu)$  (based only on  $\mathcal{Y}$  being a tracial set). Now let  $V_\ell$  denote the block  $\nu \times \nu$  matrix with  $n \times n$  entries with  $I_n$  in the  $(\ell, \ell)$  position and zeros ( $n \times n$  matrices) elsewhere. Note that  $\sum V_\ell^* V_\ell = I_{n\nu}$ . Hence,

$$\sum V_\ell Z^\ell V_\ell^* = \text{diag}(Y^1 \dots Y^\nu) \in \mathcal{Y}(n\nu).$$

Thus  $\mathcal{Y}$  is closed with respect to identical direct sums. By Lemma 7.14(2),  $\mathcal{Y}$  is a free convex cone.  $\square$

**Remark 7.15.** If  $\mathcal{Y} \subseteq \mathbb{S}^g$  is a cone and if  $B \in \mathbb{S}_n^g$  is in the polar dual of the set  $\mathcal{U}$  consisting of all tuples  $\sum C_j Y^j C_j^*$  for  $Y^j \in \mathcal{Y}$  and  $C_j$  such that  $\sum C_j^* C_j \preceq I$ , then

$$\sum B_j \otimes Y_j \preceq 0$$

for all  $Y \in \mathcal{Y}(m)$ . In particular, the polar dual  $\mathcal{B} = \mathcal{Y}^\circ$  of a cone  $\mathcal{Y}$  is a free convex cone.

To prove this assertion, pick  $B \in \mathbb{S}_n^g$  in the polar dual of  $\mathcal{U}$ . Fix a positive integer  $m$ . By Lemma 7.4, there exists a positive semidefinite  $T$  with trace at most one such that

$$I \otimes T - \sum B_j \otimes Y_j \succeq 0$$

for all  $Y \in \mathcal{Y}(m)$ . Since  $\mathcal{Y}(m)$  is a cone,  $I \otimes T - \sum B_j \otimes t^2 Y_j \succeq 0$  for all real  $t$ , and hence

$$-\sum B_j \otimes Y_j \succeq 0.$$

It follows that

$$-\sum C^* B_j C \otimes Y_j \succeq 0 \tag{7.4}$$

for any  $C$ . The conventional polar dual of a set is convex, which implies convex combinations with various  $C_j$  in (7.4) are in  $\mathcal{B}$ . Hence  $\mathcal{B}$  is a free convex cone.

## 8. Examples

The examples referenced in the body of the paper are gathered together in this section. Some of the examples consider the scalar level  $\Gamma(1) \subseteq \mathbb{R}^g$  of a free set  $\Gamma \subseteq \mathbb{S}^g$ .

**Example 8.1.** This example shows that it is not necessarily possible to choose  $V$  to be an isometry in equation (2.5) of Theorem 2.4 if the boundedness assumption on  $\mathcal{D}_{\mathfrak{L}_B}$  is omitted. Let  $g = 1$ , and consider  $\mathfrak{L}_A(x) = 1 + x$ ,  $\mathfrak{L}_B(x) = 1 + 2x$ . In this case,

$$\mathcal{D}_{\mathfrak{L}_B} = \{X : X \succeq -1/2\} \subseteq \mathcal{D}_{\mathfrak{L}_A} = \{X : X \succeq -1\}.$$

It is clear that there does not exist a  $\mu$  and an isometry  $V$  such that  $A = V^*(I_\mu \otimes B)V$ . This example is in fact representative in the sense that if  $\mathfrak{L}_B$  is a monic linear pencil and  $\mathcal{D}_{\mathfrak{L}_B}$  is unbounded, then there is a monic linear pencil  $\mathfrak{L}_A$  with  $\mathcal{D}_{\mathfrak{L}_B} \subseteq \mathcal{D}_{\mathfrak{L}_A}$  for which there does not exist a  $\mu$  and an isometry  $V$  such that  $A = V^*(I_\mu \otimes B)V$ .

**Example 8.2.** Here is an example of a trace preserving cp map  $\phi : \mathcal{S} \rightarrow M_2$  with domain an operator system  $\mathcal{S}$  that does not admit an extension to a trace nonincreasing cp map  $\phi : M_2 \rightarrow M_2$ . This phenomenon contrasts with the classical Arveson extension theorem [Arv69] which says that any ucp map extends to the full algebra.

Let

$$\mathcal{S} = \text{span}\{I_2, E_{1,2}, E_{2,1}\}, \quad V = \begin{pmatrix} \sqrt{1/2} & 0 \\ 0 & \sqrt{3/2} \end{pmatrix},$$

and consider the cp map  $\phi : \mathcal{S} \rightarrow M_2$ ,

$$\phi(A) = V^*AV \quad \text{for } A \in \mathcal{S}.$$

We have

$$\phi(I_2) = V^*V = \begin{pmatrix} 1/2 & 0 \\ 0 & 3/2 \end{pmatrix}, \quad \phi(E_{1,2}) = \frac{\sqrt{3}}{2}E_{1,2}, \quad \phi(E_{2,1}) = \frac{\sqrt{3}}{2}E_{2,1},$$

so  $\phi$  is trace preserving on  $\mathcal{S}$ .

Now let us consider a cp extension (still denoted by  $\phi$ ) of  $\phi$  to  $M_2$ . Let

$$\phi(E_{1,1}) = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Then the Choi matrix for  $\phi$  is

$$C = \begin{pmatrix} a & b & 0 & \sqrt{3}/2 \\ b & c & 0 & 0 \\ 0 & 0 & 1/2 - a & -b \\ \sqrt{3}/2 & 0 & -b & 3/2 - c \end{pmatrix} \succeq 0.$$

Suppose  $\phi : M_2 \rightarrow M_2$  is trace nonincreasing. Then

$$1 = \text{tr}(E_{1,1}) \geq \text{tr}(\phi(E_{1,1})) = a + c,$$

$$1 = \text{tr}(E_{2,2}) \geq \text{tr}(\phi(E_{2,2})) = 2 - a - c,$$

whence  $a + c = 1$ . Since  $C$  is positive semidefinite, the nonnegativity of the diagonal of  $C$  now gives us  $0 \leq a \leq 1/2$ . But then the  $2 \times 2$  minor

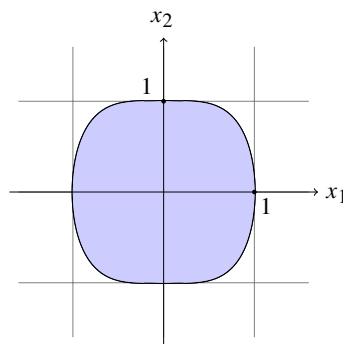
$$\begin{pmatrix} a & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 + a \end{pmatrix}$$

is not positive semidefinite, a contradiction.

**Example 8.3.** Consider

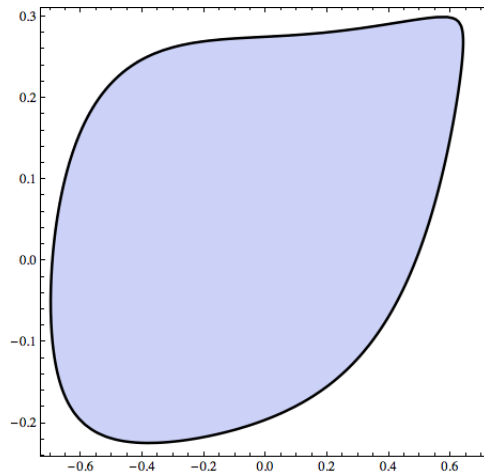
$$p = 1 - x_1^2 - x_2^4.$$

In this case  $p$  is symmetric with  $p(0) = 1 > 0$ .



$$\text{Bent TV screen } \mathcal{D}_p(1) = \{(x_1, x_2) \in \mathbb{R}^2 : 1 - x_1^2 - x_2^4 \geq 0\}.$$

The free semialgebraic set  $\mathcal{D}_p$  is called the real *bent free TV screen*, or (bent) TV screen for short. While  $\mathcal{D}_p(1)$  is convex, it is known that  $\mathcal{D}_p$  is not matrix convex (see [DHM07] or [BPR13, Chapter 8]). Indeed, already  $\mathcal{D}_p(2)$  is not a convex set.



A nonconvex 2-dimensional slice of  $\mathcal{D}_p(2)$ .

That the set  $\mathcal{D}_p(1)$  is a spectrahedral shadow is well known. Indeed, letting

$$L(x_1, x_2, y) = \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & y \\ x_1 & y & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & x_2 \\ x_2 & y \end{pmatrix},$$

we readily check that  $\text{proj}_x \mathcal{D}_L(1) = \mathcal{D}_p(1)$ . Further, Lemma 4.1 implies that  $L$  can be replaced by a monic linear pencil  $\mathcal{L}$ . An explicit construction of such an  $\mathcal{L}$  can be found in [HKM16, §7.1]. We remark that  $\text{proj}_x \mathcal{D}_L$  strictly contains the matrix convex hull of  $\mathcal{D}_p$  [HKM16, §7.1].

The next example is one in a classical commutative situation. We refer the reader to [BPR13] for background on classical convex algebraic geometry.

**Example 8.4** (The polar dual of the bent TV screen  $\mathcal{D}_p = \{(X, Y) : 1 - X^2 - Y^4 \geq 0\}$ . We note that  $\mathcal{D}_p^\circ(1)$  coincides with the classical polar dual of  $\mathcal{D}_p(1)$  by Proposition 4.3 (cf. [HKM16, Example 4.7]).

We first find the boundary  $\partial \mathcal{D}_p^\circ(1)$  using Lagrange multipliers. Consider a linear function  $1 - (c_1x + c_2y)$  that is nonnegative but not strictly positive on  $\mathcal{D}_p^\circ(1)$ , and its values on (the boundary of)  $\mathcal{D}_p(1)$ . The KarushKuhnTucker (KKT) conditions for first order optimality give

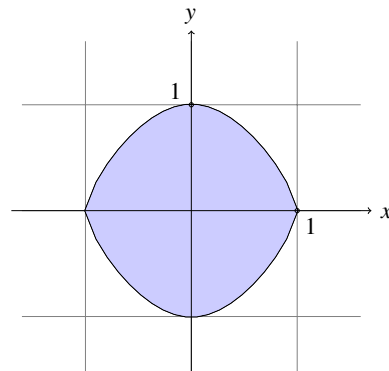
$$1 - x^2 - y^4 = 0, \quad c_1 = 2\lambda x, \quad c_2 = 4\lambda y^3, \quad 1 = c_1x + c_2y.$$

Eliminating  $x, y, \lambda$  leads to the following formula relating  $c_1, c_2$ :

$$q(c_1, c_2) := -16c_1^8 + 48c_1^6 - 48c_1^4 - 8c_1^4c_2^4 + 16c_1^2 - 20c_1^2c_2^4 - c_2^8 + c_2^4 = 0.$$



Thus the boundary of  $\mathcal{D}_p^\circ(1)$  is contained in the zero set of  $q$ . Since  $q$  is irreducible,  $\partial\mathcal{D}_p^\circ(1)$  in fact equals the zero set of  $q$ . In particular,  $\mathcal{D}_p^\circ(1) = \{(x, y) \in \mathbb{R}^2 : q(x, y) \geq 0\}$  is not a spectrahedron, since it fails the line test in [HV07].



Polar dual  $\mathcal{D}_p^\circ(1)$  of the bent TV screen.

**Example 8.5.** Recall the free bent TV screen is the nonnegativity set  $\mathcal{D}_p$  for the polynomial  $p = 1 - x^2 - y^4$  (see Example 8.3). Let  $\mathcal{K}$  denote the closed matrix convex hull of  $\mathcal{D}_p$ . Then  $\mathcal{K}(1) = \mathcal{D}_p(1)$ , and hence, by Proposition 4.3 and Example 8.4,  $\mathcal{D}_p^\circ(1) = \mathcal{D}_p(1)^\circ$  is not a spectrahedron. Consequently,  $\mathcal{D}_p^\circ$  is not a free spectrahedron. In particular,  $\mathcal{K}$  cannot be represented by a single  $\Omega$  as in Theorem 4.6.

**Example 8.6.** Tracial and contractively tracial hulls need not be convex (levelwise), as this example shows. Consider

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

To show that  $D = \frac{1}{2}(A + B)$  is not in  $\text{thull}(\{A, B\})$ , suppose there exist  $2 \times 2$  matrices  $V_1, \dots, V_m$  such that  $\sum V_j^* V_j = I$  and

$$\sum V_j A V_j^* = D.$$

On the one hand, the trace of  $D$  is zero, on the other hand,  $\sum V_j A V_j^*$  has trace 1. Hence  $D$  is not in the tracial hull of  $A$ . A similar argument shows that  $D$  is not in the tracial hull of  $B$ . Hence by Lemma 6.3,  $D \notin \text{thull}(\{A, B\})$ .

Now consider the tuples  $A = (A_1, A_2)$  and  $B = (B_1, B_2)$  defined by

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = -B_2, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = -B_1.$$

In this case  $D = \frac{1}{2}(A + B)$  is

$$D = (D_1, D_2) = \frac{1}{2} \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right).$$

Suppose  $\sum C_j^* C_j \preceq I$ . Let

$$F_k = \sum C_j A_k C_j^*$$

and note  $\operatorname{tr}(F_k) \geq 0$ . On the other hand,  $\operatorname{tr}(D_k) = 0$ . Hence, if  $F_k = D_k$ , then

$$0 = \operatorname{tr}(F_k) = \sum_j \operatorname{tr}(C_j A_k C_j^*) \geq 0.$$

But then, for each  $j$ ,

$$0 = \operatorname{tr}((C_j(A_1 + A_2)C_j^*) = \operatorname{tr}(C_j C_j^*).$$

It follows that  $C_j = 0$  for each  $j$ , and thus  $F_k = 0$ , a contradiction. Thus,  $D$  is not in the contractive tracial hull of  $A$ , and by symmetry it is not in the contractive tracial hull of  $B$ . By Lemma 6.4,  $D$  is not in the contractive tracial hull generated by  $\{A, B\}$ .

The following example shows that a contractively stable set need not be convex.

**Example 8.7.** Consider the  $2 \times 2$  matrices  $A, B$  from Example 8.6. The smallest contractively stable set containing  $A, B$  is the levelwise closed set

$$\mathcal{Y} = \left\{ \sum C_j^* A C_j : \sum C_j^* C_j \preceq I \right\} \cup \left\{ \sum D_j^* B D_j : \sum D_j^* D_j \preceq I \right\}.$$

Each matrix in  $\mathcal{Y}$  is either positive semidefinite or negative semidefinite, so  $\frac{1}{2}(A+B) \notin \mathcal{Y}$ .

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