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The classification of Nichols algebras over groups with finite root system of rank two

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Abstract. We classify all groups G and all pairs (V, W) of absolutely simple Yetter–Drinfeld modules over G such that the support of $V \oplus W$ generates G, $c_{W,V}c_{V,W} \neq \mathrm{id}$, and the Nichols algebra of the direct sum of V and W admits a finite root system. As a byproduct, we determine the dimensions of such Nichols algebras, and several new families of finite-dimensional Nichols algebras are obtained. Our main tool is the Weyl groupoid of pairs of absolutely simple Yetter–Drinfeld modules over groups.

Keywords. Hopf algebra, Nichols algebra, Weyl groupoid

Contents

Introduction

In the last years, Nichols algebras turned out to be important in many branches of mathematics such as Hopf algebras and quantum groups [30], [7], [27], [32], [34, 35], Schubert calculus [14], [11], and mathematical physics [28], [33]. Nichols algebras appeared first

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in a work of Nichols [31], where he studies and classifies certain pointed Hopf algebras. Pointed Hopf algebras have applications in conformal field theory [15].

Let $\mathbb{K}$ be a field and let G be a group. The Lifting Method of Andruskiewitsch and Schneider [6] (see also [7]) provides the best known approach to the classification of finite-dimensional pointed Hopf algebras. First, the method asks to determine all finite-dimensional Nichols algebras over G and to provide a presentation by generators and relations. Whereas for abelian groups the situation is understood to a great extent [19], [20], [8], [9, 10], less is known for non-abelian groups.

One idea to approach the problem is to adapt the method applied for abelian groups. The problem here is that the structure of the Nichols algebra of a simple Yetter–Drinfeld module over G is very complicated. Only few finite-dimensional examples are known [18], [21], and for the important examples of Fomin–Kirillov algebras [14] it is not even known whether they are Nichols algebras or whether they are finite-dimensional. Nevertheless, any direct sum of simple Yetter–Drinfeld modules having a finite-dimensional Nichols algebra gives rise to the structure of a Weyl groupoid [5], and surprisingly, the finiteness of the Weyl groupoid implies strong restrictions on the direct summands. Therefore it is reasonable to attack the classification of semisimple Yetter–Drinfeld modules with finite-dimensional Nichols algebras before looking at the simple objects. The situation is even more astonishing: The functoriality of the Nichols algebra [7, Cor. 2.3] allows one to look at Yetter–Drinfeld submodules of simple objects, which are semisimple with respect to a smaller group. Then information in the semisimple setting can be used for simple objects [2], [3, 4].

First ideas to analyze in detail the Nichols algebra of a semisimple Yetter–Drinfeld module were developed in [22]. That work is based on the notion of the Weyl groupoid of tuples of simple Yetter–Drinfeld modules over arbitrary Hopf algebras with bijective antipode [5], [23], [24]. Using the classification of finite Weyl groupoids of rank two [12], a breakthrough in the approach was achieved in [26].

Recall that for any group G, a $\mathbb{K}G$-module V is *absolutely simple* if $\mathbb{L} \otimes_{\mathbb{K}} V$ is a simple $\mathbb{L}G$-module for any field extension $\mathbb{L}$ of $\mathbb{K}$. We say that a Yetter–Drinfeld module V over a group algebra $\mathbb{K}G$ is absolutely simple if $\mathbb{L} \otimes_{\mathbb{K}} V$ is a simple Yetter–Drinfeld module over $\mathbb{L}G$ for any field extension $\mathbb{L}$ of $\mathbb{K}$. Recall from [22] that the groups Γ_n are central extensions of dihedral groups, whereas the group T, defined in [26], is a central extension of $\mathbf{SL}(2, 3)$.

Theorem ([26, Thm. 4.5]). *Let G be a non-abelian group, and V and W be two absolutely simple Yetter–Drinfeld modules over G such that G is generated by the support of $V \oplus W$. Assume that the Nichols algebra of $V \oplus W$ is finite-dimensional. If $c_{W,V} c_{V,W} \neq \mathrm{id}_{V\otimes W}$, then G is an epimorphic image of T or of Γ_n for $n \in \{2, 3, 4\}$.*

Note that if the square of the braiding between V and W is the identity, then $\mathfrak{B}(V \oplus W)$ and $\mathfrak{B}(V) \otimes \mathfrak{B}(W)$ are isomorphic as $\mathbb{N}_0$-graded objects in ${}^G_G\mathcal{YD}$ by [16, Thm. 2.2]. On the other hand, the assumption that G is generated by $\mathrm{supp}(V \oplus W)$ is natural since the braiding of $V \oplus W$, and hence the structure of $\mathfrak{B}(V \oplus W)$ as a braided Hopf algebra, depends only on the action and coaction of the subgroup of G generated by $\mathrm{supp}(V \oplus W)$.

Already in [22], Nichols algebras of pairs of simple Yetter–Drinfeld modules over non-abelian epimorphic images of Γ_2 were studied and new Nichols algebras of dimension 1296 over fields of characteristic 3 were found. In [25], the Nichols algebras over non-abelian epimorphic images of T and Γ_4 were studied and new Nichols algebras of dimensions 80621568, 262144 (if $\operatorname{char} \mathbb{K} \neq 2$) and 1259712, 65536 (if $\operatorname{char} \mathbb{K} = 2$) were found. The situation is more complicated when G is a non-abelian epimorphic image of Γ_3, and it is studied in this work. It is the first case where one meets a finite-dimensional Nichols algebra not of diagonal type which has a non-standard Weyl groupoid. We obtain several new families of Nichols algebras, the ranks and dimensions of which can be read off from Table 1.

Table 1. Nichols algebras with finite root system of rank two

Rank	Group	Dimension	char $\mathbb{K}$	Support	Reference
4	Γ_2	64		$Z_2^{2,2}$	Example 1.2
4	Γ_2	1296	3	$Z_2^{2,2}$	Example 1.3
4	Γ_3	10368	$\neq 2, 3$	$Z_3^{3,1}$	Thm. 8.2
4	Γ_3	5184	2	$Z_3^{3,1}$	Thm. 8.2
4	Γ_3	1152	3	$Z_3^{3,1}$	Thm. 8.2
4	Γ_3	2239488	2	$Z_3^{3,1}$	Thms. 8.6, 8.8
5	Γ_3	10368	$\neq 2, 3$	$Z_3^{3,2}$	Thm. 8.1
5	Γ_3	5184	2	$Z_3^{3,2}$	Thm. 8.1
5	Γ_3	1152	3	$Z_3^{3,2}$	Thm. 8.1
5	Γ_3	2304		$Z_3^{3,2}$	Thm. 8.3
5	Γ_3	2304		$Z_3^{3,1}$	Thm. 8.4
5	Γ_3	2239488	2	$Z_3^{3,2}$	Thm. 8.7
5	T	80621568	$\neq 2$	$Z_T^{4,1}$	Example 1.7
5	T	1259712	2	$Z_T^{4,1}$	Example 1.7
6	Γ_4	262144	$\neq 2$	$Z_4^{4,2}$	Example 1.5
6	Γ_4	65536	2	$Z_4^{4,2}$	Example 1.5

Having studied Nichols algebras over non-abelian epimorphic images of Γ_2, Γ_3, Γ_4, and T, we are able to classify all pairs (V, W) of absolutely simple Yetter–Drinfeld modules over a non-abelian group G such that the Nichols algebra of $V \oplus W$ is finite-dimensional. Moreover, we determine the Hilbert series and the decomposition of the Nichols algebra of $V \oplus W$ into the tensor product of Nichols algebras of simple Yetter–Drinfeld modules. The finite-dimensional Nichols algebras appearing in our classification are listed in Table 1. The pairs (V, W) of absolutely simple Yetter–Drinfeld modules over G appear in Section 1. Our main theorem is the following.

Theorem. *Let G be a non-abelian group and let V and W be absolutely simple Yetter–Drinfeld modules over G such that G is generated by the support of $V \oplus W$. Assume that* $(\mathrm{id} - c_{W,V}c_{V,W})(V \otimes W)$ *is non-zero and the Nichols algebra* $\mathfrak{B}(V \oplus W)$ *is finite-dimensional. Then* $\mathfrak{B}(V \oplus W)$ *is one of the Nichols algebras of Table 1.*

See Theorem 2.1 for a more precise statement. Let us explain briefly how the proof of Theorem 2.1 goes. We have to study in detail Nichols algebras over non-abelian epimorphic images of the groups Γ_2, Γ_3, Γ_4 and T. The analysis concerning the group Γ_2 was done in [22] and the groups Γ_4 and T were studied in [25]. The classification of finite-dimensional Nichols algebras associated with Γ_3 is one of the main results of this paper and requires several steps. We need to deal with three different pairs (V, W) of absolutely simple Yetter–Drinfeld modules over non-abelian epimorphic images of Γ_3. We first determine when $(\mathrm{ad}\, V)^m(W)$ and $(\mathrm{ad}\, W)^m(V)$ are absolutely simple or zero, and then we compute the Cartan matrix of (V, W). Then we prove that these pairs are essentially the only pairs which we need to consider, and the reflections of these pairs are computed. With this information we compute the finite root systems of rank two associated with Nichols algebras over non-abelian epimorphic images of Γ_3. This information allows us to determine the structure of such Nichols algebras.

The main result of our paper is expected to lead to powerful applications. We intend to attack the classification of finite-dimensional Nichols algebras of finite direct sums of absolutely simple Yetter–Drinfeld modules over groups. For this project it is very useful that the reflections of the absolutely simple pairs are already calculated. On the other hand, we are confident that our classification will be useful to study Nichols algebras over simple Yetter–Drinfeld modules, as was done for example in [3, 4].

We do not know the defining relations of the Nichols algebras appearing in our classification. In the spirit of [1, Question 5.9], one then has the following problem: Give a nice presentation by generators and relations of the Nichols algebras appearing in Table 1. To attack this, the ideas of [9, 10] could be useful.

The paper is organized as follows. In Section 1 we list all the finite-dimensional Nichols algebras appearing in our classification. In Section 2 we state the main result of the paper, Theorem 2.1. This theorem classifies Nichols algebras of group type over the sum of two absolutely simple Yetter–Drinfeld modules. Sections 3–8 are devoted to the structure and the root systems of finite-dimensional Nichols algebras over non-abelian epimorphic images of Γ_3. Finally, in Section 9, we prove our main result, Theorem 2.1.

1. The examples

Before stating our main result, we collect all the examples of Nichols algebras with finite root systems obtained over non-abelian epimorphic images of Γ_2, Γ_3, Γ_4 and T. These are the examples which appear in our classification in Theorem 2.1.

Recall from [22] that Γ_n for $n \geq 2$ is the group given by generators a, b, ν and relations

$$ba = \nu ab, \qquad \nu a = a\nu^{-1}, \qquad \nu b = b\nu, \qquad \nu^n = 1,$$

and T is the group given by generators ζ, χ_1, χ_2 and relations

$$\zeta\chi_1 = \chi_1\zeta, \quad \zeta\chi_2 = \chi_2\zeta, \quad \chi_1\chi_2\chi_1 = \chi_2\chi_1\chi_2, \quad \chi_1^3 = \chi_2^3.$$

Remark 1.1. The groups Γ_2, Γ_3, Γ_4 and T are isomorphic to the enveloping groups of the quandles $Z_2^{2,2}$, $Z_3^{3,1}$, $Z_3^{3,2}$, $Z_4^{4,2}$ and $Z_T^{4,1}$ (see [26, §2] and [25] for an alternative description of these quandles). An epimorphic image G of any of these enveloping groups G_X is non-abelian if and only if the restriction of the epimorphism $G_X \to G$ to the quandle X is injective.

By [22], Γ_3 is isomorphic to the group given by generators ν, ζ, γ and relations

$$\gamma\nu = \nu^2\gamma, \quad \zeta\gamma = \gamma\zeta, \quad \zeta\nu = \nu\zeta, \quad \nu^3 = 1.$$

1.1. Epimorphic images of Γ_2

In [22, §4], Nichols algebras over non-abelian epimorphic images of Γ_2 were studied. Let G be a non-abelian group. Let $g, h, \epsilon \in G$, and assume that there is a group epimorphism

$$\Gamma_2 \to G, \quad a \mapsto g,\ b \mapsto h,\ \nu \mapsto \epsilon.$$

Example 1.2. Let $V, W \in {}^G_G\mathcal{YD}$. Assume that $V \simeq M(g, \rho)$, where ρ is a character of $G^g = \langle \epsilon, g, h^2 \rangle$, and $W \simeq M(h, \sigma)$, where σ is a character of $G^h = \langle \epsilon, h, g^2 \rangle$. Let $v \in V_g$ with $v \neq 0$. Then $\{v, hv\}$ is a basis of V and the degrees of these basis vectors are g and ϵg, respectively. Similarly, let $w \in W_h$ with $w \neq 0$. Then $\{w, gw\}$ is a basis of W and the degrees of these vectors are h and ϵh, respectively. The action of G on V and W is given by the following tables:

V	v	hv
ϵ	$\rho(\epsilon)v$	$\rho(\epsilon)hv$
h	hv	$\rho(h^2)v$
g	$\rho(g)v$	$\rho(\epsilon)\rho(g)hv$

W	w	gw
ϵ	$\sigma(\epsilon)w$	$\sigma(\epsilon)gw$
h	$\sigma(h)w$	$\sigma(\epsilon)\sigma(h)gw$
g	gw	$\sigma(g^2)w$

Assume that

$$\rho(\epsilon h^2)\sigma(\epsilon g^2) = 1, \quad \rho(g) = \sigma(h) = -1.$$

Then, by [22, Thm. 4.6], $\dim \mathfrak{B}(V \oplus W) = 64$ and the Hilbert series of the Nichols algebra $\mathfrak{B}(V \oplus W)$ is

$$\mathcal{H}(t_1, t_2) = (1 + t_1)^2(1 + t_1t_2)^2(1 + t_2)^2.$$

A special case of this example appeared first in [29, Example 6.5].

Example 1.3. Let $V, W \in {}^G_G\mathcal{YD}$. Assume that $V \simeq M(g, \rho)$, where ρ is a character of $G^g = \langle \epsilon, g, h^2 \rangle$, and $W \simeq M(h, \sigma)$, where σ is a character of $G^h = \langle \epsilon, h, g^2 \rangle$. Assume also that $\operatorname{char} \mathbb{K} = 3$ and that

$$\rho(\epsilon h^2)\sigma(\epsilon g^2) = 1, \quad \rho(g) = 1, \quad \sigma(h) = -1. \tag{1.1}$$

Then, by [22, Thm. 4.7], $\dim \mathfrak{B}(V \oplus W) = 1296$ and the Hilbert series of the Nichols algebra $\mathfrak{B}(V \oplus W)$ is

$$\mathcal{H}(t_1, t_2) = (1 + t_1 + t_1^2)^2(1 + t_2)^2(1 + t_1t_2 + t_1^2t_2^2)^2(1 + t_1^2t_2)^2.$$

Remark 1.4. The braiding of the Yetter–Drinfeld modules of Examples 1.2 and 1.3 can be obtained from the following table:

	v	hv	w	gw
g	$\rho(g)v$	$\rho(\epsilon g)hv$	gw	$\sigma(g^2)w$
ϵg	$\rho(\epsilon g)v$	$\rho(g)hv$	$\sigma(\epsilon)gw$	$\sigma(\epsilon g^2)w$
h	hv	$\rho(h^2)v$	$\sigma(h)w$	$\sigma(\epsilon h)gw$
ϵh	$\rho(\epsilon)hv$	$\rho(\epsilon h^2)v$	$\sigma(\epsilon h)w$	$\sigma(h)gw$

1.2. Epimorphic images of Γ_4

Finite-dimensional Nichols algebras over non-abelian epimorphic images of Γ_4 were computed in [25, §5]. Let G be a non-abelian group and let $g, h, \epsilon \in G$. Assume that there is a group epimorphism

$$\Gamma_4 \to G, \quad a \mapsto g,\ b \mapsto h,\ \nu \mapsto \epsilon,$$

such that $\epsilon^2 \neq 1$.

Example 1.5. Let $V, W \in {}^G_G\mathcal{YD}$. Assume that $V \simeq M(h, \rho)$, where ρ is a character of the centralizer $G^h = \langle \epsilon, h, g^2 \rangle$ with $\rho(h) = -1$. Let $v \in V_h$ with $v \neq 0$. Then the elements v, gv form a basis of V, and the degrees of these basis vectors are h and $ghg^{-1} = \epsilon^{-1}h$, respectively. The action of G on V is given by the following table:

V	v	gv
ϵ	$\rho(\epsilon)v$	$\rho(\epsilon)^{-1}gv$
h	$-v$	$-\rho(\epsilon)^{-1}gv$
g	gv	$\rho(g^2)v$

Assume that $W \simeq M(g, \sigma)$, where σ is a character of $G^g = \langle \epsilon^2, \epsilon^{-1}h^2, g \rangle$ with $\sigma(g) = -1$. Let $w \in W_g$ with $w \neq 0$. The elements $w, hw, \epsilon w, \epsilon hw$ form a basis of W. The degrees of these basis vectors are $g, \epsilon g, \epsilon^2 g$ and $\epsilon^3 g$, respectively. The action of G on W is given by the following table:

W	w	hw	ϵw	ϵhw
ϵ	ϵw	ϵhw	$\sigma(\epsilon^2)w$	$\sigma(\epsilon^2)hw$
h	hw	$\sigma(\epsilon^{-1}h^2)\epsilon w$	ϵhw	$\sigma(\epsilon h^2)w$
g	$-w$	$-\sigma(\epsilon^2)\epsilon hw$	$-\sigma(\epsilon^2)\epsilon w$	$-\sigma(\epsilon^2)hw$

Assume further that

$$\rho(\epsilon) = \rho(g^2)\sigma(\epsilon^{-1}h^2), \quad \rho(\epsilon)^2 = -1.$$

Then, by [25, Thm. 5.4],

$$\mathcal{H}(t_1, t_2) = (1 + t_2)^4(1 + t_2^2)^2(1 + t_1t_2)^4(1 + t_1^2t_2^2)^2 q(t_1t_2^2)q(t_1),$$

where

$$q(t)=\begin{cases}(1+t)^2(1+t^2) & \text{if } \operatorname{char}\mathbb{K}\neq 2,\\ (1+t)^2 & \text{if } \operatorname{char}\mathbb{K}=2.\end{cases}$$

In particular,

$$\dim\mathfrak{B}(V\oplus W)=\begin{cases}8^2 64^2=262144 & \text{if } \operatorname{char}\mathbb{K}\neq 2,\\ 4^2 64^2=65536 & \text{if } \operatorname{char}\mathbb{K}=2.\end{cases}$$

Remark 1.6. The braiding of the Yetter–Drinfeld module of Example 1.5 can be obtained from the following table:

	v	gv	w	hw	ϵw	ϵhw
h	$-v$	$-\rho(\epsilon)^{-1}gv$	hw	$\sigma(\epsilon^{-1}h^2)\epsilon w$	ϵhw	$\sigma(\epsilon h^2)w$
$\epsilon^3 h$	$-\rho(\epsilon)^3 v$	$-gv$	$\sigma(\epsilon^2)\epsilon hw$	$\sigma(\epsilon^{-1}h^2)w$	hw	$\sigma(\epsilon^{-1}h^2)\epsilon w$
g	gv	$\rho(g^2)v$	$-w$	$-\sigma(\epsilon^2)\epsilon hw$	$-\sigma(\epsilon^2)\epsilon w$	$-\sigma(\epsilon^2)hw$
ϵg	$\rho(\epsilon^3)gv$	$\rho(\epsilon g^2)v$	$-\epsilon w$	$-hw$	$-w$	$-\sigma(\epsilon^2)\epsilon hw$
$\epsilon^2 g$	$\rho(\epsilon^2)gv$	$\rho(\epsilon^2 g^2)v$	$-\sigma(\epsilon^2)w$	$-\epsilon hw$	$-\epsilon w$	$-hw$
$\epsilon^3 g$	$\rho(\epsilon)gv$	$\rho(\epsilon^3 g^2)v$	$-\sigma(\epsilon^2)\epsilon w$	$-\sigma(\epsilon^2)hw$	$-\sigma(\epsilon^2)w$	$-\epsilon hw$

1.3. Epimorphic images of T

Nichols algebras over non-abelian epimorphic images of the group T were studied in [25, §2]. Let G be a non-abelian group, and let $x_1,x_2,z\in G$. Assume that there is a group epimorphism

$$T\to G,\qquad \zeta\mapsto z,\ \chi_1\mapsto x_1,\ \chi_2\mapsto x_2.$$

Clearly, z is a central element of G. Moreover, the elements

$$x_1,\ x_2,\ x_3:=x_2x_1x_2^{-1},\ x_4:=x_1x_2x_1^{-1}$$

form a conjugacy class of G.

Example 1.7. Let $V,W\in {}^G_G\mathcal{YD}$. Assume that $V\simeq M(z,\rho)$, where ρ is a character of the centralizer $G^z=G$, and $W=M(x_1,\sigma)$, where σ is a character of $G^{x_1}=\langle x_1,x_2x_3,z\rangle$ with $\sigma(x_1)=-1$ and $\sigma(x_2x_3)=1$. Let $v\in V_z\setminus\{0\}$. Then $\{v\}$ is basis of V. The action of G on V is given by

$$zv=\rho(z)v,\qquad x_iv=\rho(x_1)v\qquad \text{for all } i\in\{1,2,3,4\}.$$

Let $w_1\in W_{x_1}$ be such that $w_1\neq 0$. Then the vectors

$$w_1,\quad w_2:=-x_4w_1,\quad w_3:=-x_2w_1,\quad w_4:=-x_3w_1$$

form a basis of W. The degrees of these vectors are x_1, x_2, x_3 and x_4, respectively. The action of G on W is given by the following table:

W	w_1	w_2	w_3	w_4
x_1	$-w_1$	$-w_4$	$-w_2$	$-w_3$
x_2	$-w_3$	$-w_2$	$-w_4$	$-w_1$
x_3	$-w_4$	$-w_1$	$-w_3$	$-w_2$
x_4	$-w_2$	$-w_3$	$-w_1$	$-w_4$
z	$\sigma(z)w_1$	$\sigma(z)w_2$	$\sigma(z)w_3$	$\sigma(z)w_4$

Assume further that

$$(\rho(x_1)\sigma(z))^2 - \rho(x_1)\sigma(z) + 1 = 0, \quad \rho(x_1 z)\sigma(z) = 1.$$

Then, by [25, Thm. 2.8], $\mathfrak{B}(V \oplus W)$ is finite-dimensional. If $\operatorname{char} \mathbb{K} \neq 2$, then

$$\mathcal{H}(t_1, t_2) = (6)_{t_1}(6)_{t_1 t_2^3}(6)_{t_1^2 t_2^3}(2)_{t_2}^2(3)_{t_2}(6)_{t_2}(2)_{t_1 t_2}^2(3)_{t_1 t_2}(6)_{t_1 t_2}(2)_{t_1 t_2^2}^2(3)_{t_1 t_2^2}(6)_{t_1 t_2^2}$$

and $\dim \mathfrak{B}(V \oplus W) = 6^3\, 72^3 = 80621568$, and if $\operatorname{char} \mathbb{K} = 2$, then

$$\mathcal{H}(t_1, t_2) = (3)_{t_1}(3)_{t_1 t_2^3}(3)_{t_1^2 t_2^3}(2)_{t_2}^2(3)_{t_2}^2(2)_{t_1 t_2}^2(3)_{t_1 t_2}^2(2)_{t_1 t_2^2}^2(3)_{t_1 t_2^2}^2$$

and $\dim \mathfrak{B}(V \oplus W) = 3^3\, 36^3 = 1259712$.

Remark 1.8. The structure of the Yetter–Drinfeld module of Example 1.7 can be read off from the following table:

	v	w_1	w_2	w_3	w_4
z	$\rho(z)v$	$\sigma(z)w_1$	$\sigma(z)w_2$	$\sigma(z)w_3$	$\sigma(z)w_4$
x_1	$\rho(x_1)v$	$-w_1$	$-w_4$	$-w_2$	$-w_3$
x_2	$\rho(x_1)v$	$-w_3$	$-w_2$	$-w_4$	$-w_1$
x_3	$\rho(x_1)v$	$-w_4$	$-w_1$	$-w_3$	$-w_2$
x_4	$\rho(x_1)v$	$-w_2$	$-w_3$	$-w_1$	$-w_4$

1.4. Epimorphic images of Γ_3

The results of this section will be proved in Section 8. Let G be a non-abelian group. Let $g, \epsilon, z \in G$, and assume that there is a group epimorphism

$$\Gamma_3 \to G, \quad \gamma \mapsto g,\ \nu \mapsto \epsilon,\ \zeta \mapsto z.$$

Example 1.9. Let $V \in {}^G_G\mathcal{YD}$. Assume that $V \simeq M(g, \rho)$, where ρ is a character of $G^g = \langle g, z \rangle$. Let $v \in V_g$ with $v \neq 0$. The elements v, ϵv and $\epsilon^2 v$ form a basis of V. The degrees of these vectors are g, $g\epsilon$ and $g\epsilon^2$, respectively.

Similarly, let $W \in {}^G_G\mathcal{YD}$ be such that $W \simeq M(\epsilon z, \sigma)$, where σ is a character of the centralizer $G^\epsilon = \langle \epsilon, z, g^2 \rangle$. Let $w \in W_{\epsilon z}$ with $w \neq 0$. Then w, gw is a basis of W. The degrees of these basis vectors are ϵz and $\epsilon^2 z$, respectively. The actions of G on V and W are given in the following tables:

V	v	ϵv	$\epsilon^2 v$
ϵ	ϵv	$\epsilon^2 v$	v
z	$\rho(z)v$	$\rho(z)\epsilon v$	$\rho(z)\epsilon^2 v$
g	$\rho(g)v$	$\rho(g)\epsilon^2 v$	$\rho(g)\epsilon v$

W	w	gw
ϵ	$\sigma(\epsilon)w$	$\sigma(\epsilon)^2 gw$
z	$\sigma(z)w$	$\sigma(z)gw$
g	gw	$\sigma(g^2)w$

If $\rho(g) = \sigma(\epsilon z) = -1$, $\rho(z)^2\sigma(\epsilon g^2) = 1$, and $1 + \sigma(\epsilon) + \sigma(\epsilon)^2 = 0$, then

$$\dim \mathfrak{B}(V \oplus W) = \begin{cases} 10368 & \text{if } \operatorname{char} \mathbb{K} \notin \{2, 3\}, \\ 5184 & \text{if } \operatorname{char} \mathbb{K} = 2, \\ 1152 & \text{if } \operatorname{char} \mathbb{K} = 3, \end{cases}$$

and

$$\mathcal{H}(t_1, t_2) = (2)_{t_2}(h'_p)_{t_2}(2)^2_{t_1t_2}(3)_{t_1t_2}(h_p)_{t_1^2t_2}(2)^2_{t_1}(3)_{t_1},$$

where $p = \operatorname{char} \mathbb{K}$, $h_2 = 3$, $h_3 = 2$, and $h_p = 6$ for all $p \notin \{2, 3\}$, and $h'_3 = 2$, $h'_p = 6$ for all $p \neq 3$, by Theorem 8.1.

If $\rho(g) = \sigma(\epsilon z) = -1$, $\rho(z)^2\sigma(\epsilon g^2) = \sigma(\epsilon) = 1$, and $\operatorname{char} \mathbb{K} \neq 3$, then

$$\mathcal{H}(t_1, t_2) = (2)^2_{t_2}(2)^2_{t_1t_2}(3)_{t_1t_2}(2)^2_{t_1^2t_2}(2)^2_{t_1}(3)_{t_1}$$

and $\dim \mathfrak{B}(V \oplus W) = 2304$, by Theorem 8.3.

If $\operatorname{char} \mathbb{K} = 2$, $\rho(g) = 1$, $(3)_{\sigma(\epsilon)} = 0$, $\sigma(z) = \sigma(\epsilon)$, and $\rho(z)^2\sigma(\epsilon g^2) = 1$, then $\dim \mathfrak{B}(V \oplus W) = 2239488$ and

$$\mathcal{H}(t_1, t_2) = (3)^2_{t_2}(2)^2_{t_1t_2^2}(3)_{t_1t_2^2}(2)_{t_1^2t_2^3}(3)_{t_1t_2}(4)_{t_1t_2}(6)_{t_1t_2}(6)_{t_1^2t_2^2}(2)_{t_1^2t_2}(2)^2_{t_1}(3)_{t_1},$$

by Theorem 8.7.

Example 1.10. Let $V \in {}^G_G\mathcal{YD}$. Assume that $V \simeq M(g, \rho)$, where ρ is a character of $G^g = \langle g, z\rangle$. Let $v \in V_g$ with $v \neq 0$. The elements v, ϵv and $\epsilon^2 v$ form a basis of V. The degrees of these vectors are g, $g\epsilon$ and $g\epsilon^2$, respectively.

Let $W \in {}^G_G\mathcal{YD}$ with $W \simeq M(z, \sigma)$, where σ is a character of G. Let $w \in W_z$ with $w \neq 0$. Then w is a basis of W. The action of G on V can be obtained from Example 1.9.

Let $p = \operatorname{char} \mathbb{K}$. If $\rho(g) = -1$, $(3)_{-\rho(z)\sigma(g)} = 0$, and $\rho(z)\sigma(gz) = 1$, then

$$\dim \mathfrak{B}(V \oplus W) = \begin{cases} 10368 & \text{if } \operatorname{char} \mathbb{K} \notin \{2, 3\}, \\ 5184 & \text{if } \operatorname{char} \mathbb{K} = 2, \\ 1152 & \text{if } \operatorname{char} \mathbb{K} = 3, \end{cases}$$

and

$$\mathcal{H}(t_1, t_2) = (h_p)_{t_2}(2)^2_{t_1t_2}(3)_{t_1t_2}(2)_{t_1^2t_2}(h'_p)_{t_1^2t_2}(2)^2_{t_1}(3)_{t_1}$$

by Theorem 8.2, where h_p and h'_p are as in Example 1.9.

If $\operatorname{char} \mathbb{K} = 2$, $\sigma(z) = 1$, $(3)_{\rho(z)\sigma(g)} = 0$, and $(\rho(g) - 1)(\rho(gz)\sigma(g) - 1) = 0$, then $\dim \mathfrak{B}(V \oplus W) = 2239488$, and

$$\mathcal{H}(t_1, t_2) = (2)_{t_2}(3)_{t_1t_2}(4)_{t_1t_2}(6)_{t_1t_2}(6)_{t_1^2t_2^2}(2)_{t_1^4t_2^3}(2)^2_{t_1^3t_2^2}(3)_{t_1^3t_2^2}(3)^2_{t_1^2t_2}(2)^2_{t_1}(3)_{t_1},$$

or

$$\mathcal{H}(t_1, t_2) = (2)_{t_2}(2)^2_{t_1t_2}(3)_{t_1t_2}(3)^2_{t_1^2t_2}(2)^2_{t_1^3t_2}(3)_{t_1^3t_2}(2)_{t_1^4t_2}(3)_{t_1}(4)_{t_1}(6)_{t_1}(6)_{t_1^2},$$

by Theorems 8.6 and 8.8.

Example 1.11. Let $V \in {}^G_G\mathcal{YD}$. Assume that $V \simeq M(g,\rho)$, where ρ is a character of $G^g = \langle g, z\rangle$. Let $v \in V_g$ with $v \neq 0$. The elements v, ϵv and $\epsilon^2 v$ form a basis of V. The degrees of these vectors are g, $g\epsilon$ and $g\epsilon^2$, respectively. The action of G on V can be obtained from Example 1.9.

Let $W \in {}^G_G\mathcal{YD}$ be such that $W \simeq M(z,\sigma)$, where σ is an absolutely irreducible representation of G of degree ≥ 2. Then $\operatorname{char}\mathbb{K} \neq 3$, $\deg\sigma = 2$, $\sigma(1+\epsilon+\epsilon^2) = 0$, and the isomorphism class of W is uniquely determined by the constants $\sigma(g^2)$ and $\sigma(z)$ (see Lemma 3.2).

Assume that $\rho(g) = \sigma(z) = -1$ and $\rho(z^2)\sigma(g^2) = 1$. Then

$$\mathcal{H}(t_1,t_2) = (2)^2_{t_2}(2)^2_{t_1t_2}(3)_{t_1t_2}(2)^2_{t_1^2t_2}(2)^2_{t_1}(3)_{t_1}$$

and $\dim \mathfrak{B}(V \oplus W) = 2304$, by Theorem 8.4.

Remark 1.12. The braidings of the Yetter–Drinfeld modules of Examples 1.9, 1.10, and 1.11, respectively, can be obtained from the following tables:

	v	ϵv	$\epsilon^2 v$	w	gw
g	$\rho(g)v$	$\rho(g)\epsilon^2 v$	$\rho(g)\epsilon v$	gw	$\sigma(g^2)w$
$g\epsilon$	$\rho(g)\epsilon^2 v$	$\rho(g)\epsilon v$	$\rho(g)v$	$\sigma(\epsilon)gw$	$\sigma(\epsilon^2 g^2)w$
$g\epsilon^2$	$\rho(g)\epsilon v$	$\rho(g)v$	$\rho(g)\epsilon^2 v$	$\sigma(\epsilon)^2 gw$	$\sigma(\epsilon g^2)w$
ϵz	$\rho(z)\epsilon v$	$\rho(z)\epsilon^2 v$	$\rho(z)v$	$\sigma(\epsilon z)w$	$\sigma(\epsilon^2 z)gw$
$\epsilon^2 z$	$\rho(z)\epsilon^2 v$	$\rho(z)v$	$\rho(z)\epsilon v$	$\sigma(\epsilon^2 z)w$	$\sigma(\epsilon z)gw$

	v	ϵv	$\epsilon^2 v$	w
g	$\rho(g)v$	$\rho(g)\epsilon^2 v$	$\rho(g)\epsilon v$	$\sigma(g)w$
$g\epsilon$	$\rho(g)\epsilon^2 v$	$\rho(g)\epsilon v$	$\rho(g)v$	$\sigma(g\epsilon)w$
$g\epsilon^2$	$\rho(g)\epsilon v$	$\rho(g)v$	$\rho(g)\epsilon^2 v$	$\sigma(g\epsilon^2)w$
z	$\rho(z)v$	$\rho(z)\epsilon v$	$\rho(z)\epsilon^2 v$	$\sigma(z)w$

	v	ϵv	$\epsilon^2 v$	w	gw
g	$\rho(g)v$	$\rho(g)\epsilon^2 v$	$\rho(g)\epsilon v$	gw	$\sigma(g^2)w$
$g\epsilon$	$\rho(g)\epsilon^2 v$	$\rho(g)\epsilon v$	$\rho(g)v$	λgw	$\lambda^2\sigma(g^2)w$
$g\epsilon^2$	$\rho(g)\epsilon v$	$\rho(g)v$	$\rho(g)\epsilon^2 v$	$\lambda^2 gw$	$\lambda\sigma(g^2)w$
z	$\rho(z)v$	$\rho(z)\epsilon v$	$\rho(z)\epsilon^2 v$	$\sigma(z)w$	$\sigma(z)gw$

2. The Classification Theorem

Now we state the main theorem of the paper. It provides the classification of a class of finite-dimensional Nichols algebras of group type. We have listed important data of these Nichols algebras in Table 1.

Theorem 2.1. *Let G be a non-abelian group and let V and W be finite-dimensional absolutely simple Yetter–Drinfeld modules over G. Assume that $c_{W,V}c_{V,W} \neq \mathrm{id}_{V\otimes W}$ and that the support of $V \oplus W$ generates the group G. Then the following are equivalent:*

(1) *The Nichols algebra $\mathfrak{B}(V \oplus W)$ is finite-dimensional.*
(2) *The pair (V, W) admits all reflections and the Weyl groupoid of (V, W) is finite.*
(3) *Up to permutation of its entries, the pair (V, W) is one of the pairs of Examples 1.2, 1.3, 1.5, 1.7, 1.9, 1.10, and 1.11 of Section 1.*

In this case, the rank and the dimension of $\mathfrak{B}(V \oplus W)$ appear in Table 1.

Theorem 2.1 will be proved in Section 9.

3. Preliminaries

Let us first state some useful results from [23, 22]. Recall that $S_n \in \mathrm{End}(V^{\otimes n})$, where $n \in \mathbb{N}$, denotes the quantum symmetrizer.

Lemma 3.1 ([22, Thm. 1.1]). *Let V and W be Yetter–Drinfeld modules over a Hopf algebra H with bijective antipode. Let $\varphi_0 = 0$, $X_0^{V,W} = W$, and*

$$\varphi_m = \mathrm{id} - c_{V^{\otimes(m-1)}\otimes W,V}\, c_{V,V^{\otimes(m-1)}\otimes W} + (\mathrm{id} \otimes \varphi_{m-1})c_{1,2} \in \mathrm{End}(V^{\otimes m} \otimes W),$$
$$X_m^{V,W} = \varphi_m(V \otimes X_{m-1}) \subseteq V^{\otimes m} \otimes W$$

for all $m \geq 1$. Then $(\mathrm{ad}\, V)^n(W) \simeq X_n^{V,W}$ for all $n \in \mathbb{N}_0$.

Now we collect information on Γ_3 we will need. By [22, §3], the center of Γ_3 is $Z(\Gamma_3) = \langle \zeta, \gamma^2 \rangle$, and the conjugacy classes of Γ_3 are

$$\delta^G = \{\delta\}, \quad (\nu\delta)^G = \{\nu\delta, \nu^2\delta\}, \quad (\gamma\delta)^G = \{\nu^j\gamma\delta \mid 0 \leq j \leq 2\}, \tag{3.1}$$

where δ runs over all elements of $Z(\Gamma_3)$. In this section, let G be a group. Assume that there exist $g, \epsilon, z \in G$, $\epsilon \neq 1$, such that there is a group epimorphism $\Gamma_3 \to G$ with $\gamma \mapsto g$, $\nu \mapsto \epsilon$ and $\zeta \mapsto z$. Note that the condition $\epsilon \neq 1$ just means that G is non-abelian.

Lemma 3.2. *Assume that $\mathbb{K}$ is algebraically closed. Let V be a simple $\mathbb{K}G$-module and let $\rho : \mathbb{K}G \to \mathrm{End}(V)$ be the corresponding representation of $\mathbb{K}G$. Then $\dim V \leq 2$. Moreover, if $\dim V = 2$ then $\mathrm{char}\, \mathbb{K} \neq 3$, $\rho(1 + \epsilon + \epsilon^2) = 0$, and the isomorphism class of V is uniquely determined by the scalars $\rho(g^2)$ and $\rho(z)$.*

Proof. Let $v \in V$ be an eigenvector of $\rho(\epsilon)$. Hence $V = \mathbb{K}v + \mathbb{K}gv$, and so $\dim V \leq 2$.

Assume that $\dim V = 2$. Then $gv \notin \mathbb{K}v$. If $\epsilon v = 1$, then $\epsilon gv = g\epsilon^2 v = gv$. Then $\rho(\epsilon) = \mathrm{id}_V$, and $\mathbb{K}w$ for an eigenvector w of $\rho(g)$ is a $\mathbb{K}G$-invariant subspace of V. This contradicts the simplicity of V. Since $\epsilon^3 = 1$ and $1 + \epsilon + \epsilon^2 \in Z(\mathbb{K}G)$, we conclude that $\rho(1 + \epsilon + \epsilon^2) = 0$ and $\mathrm{char}\, \mathbb{K} \neq 3$. Thus v, gv is a basis of V consisting of eigenvectors of $\rho(\epsilon)$.

Let now W be a simple $\mathbb{K}G$-module with $\dim W = 2$, and let $w \in W \setminus \{0\}$ and $\lambda \in \mathbb{K}$ be such that $\epsilon v = \lambda v$, $\epsilon w = \lambda w$. Assume that $g^2 w = \rho(g^2)w$ and $zw = \rho(z)w$. Then the map $f : V \to W$, $v \mapsto w$, $gv \mapsto gw$, is an isomorphism of $\mathbb{K}G$-modules. This proves the lemma. ◻

4. Reflections of the first pair

Let G be a non-abelian group, and let $g, \epsilon, z \in G$. Assume that there is an epimorphism $\Gamma_3 \to G$ with $\gamma \mapsto g$, $\nu \mapsto \epsilon$, $\zeta \mapsto z$. Let $V \in {}^G_G\mathcal{YD}$ be such that $V \simeq M(g, \rho)$, where ρ is an absolutely irreducible representation of the centralizer $G^g = \langle z, g\rangle$. Since this centralizer is abelian, $\deg \rho = 1$. Let $v \in V_g$ with $v \neq 0$. The elements $v, \epsilon v, \epsilon^2 v$ form a basis of V. The degrees of these basis vectors are g, $g\epsilon$ and $g\epsilon^2$, respectively.

Remark 4.1. The action of G on V is given by the following table:

$M(g,\rho)$	v	ϵv	$\epsilon^2 v$
ϵ	ϵv	$\epsilon^2 v$	v
z	$\rho(z)v$	$\rho(z)\epsilon v$	$\rho(z)\epsilon^2 v$
g	$\rho(g)v$	$\rho(g)\epsilon^2 v$	$\rho(g)\epsilon v$

Let $W \in {}^G_G\mathcal{YD}$ be such that $W \simeq M(\epsilon z, \sigma)$, where σ is an absolutely irreducible representation of $G^{\epsilon z} = G^{\epsilon} = \langle \epsilon, z, g^2\rangle$. Since G^ϵ is abelian, $\deg\sigma = 1$. Let $w \in W_{\epsilon z}$ with $w \neq 0$. Then w, gw is a basis of W. The degrees of these basis vectors are ϵz and $\epsilon^2 z$, respectively.

Remark 4.2. The action of G on W is given by the following table:

W	w	gw
ϵ	$\sigma(\epsilon)w$	$\sigma(\epsilon)^2 gw$
z	$\sigma(z)w$	$\sigma(z)gw$
g	gw	$\sigma(g^2)w$

In order to calculate $R_1(V, W)$, we first compute the modules $(\operatorname{ad} V)^n(W)$ for $n \in \mathbb{N}$. For $n \in \mathbb{N}$ we write $X_n = X_n^{V,W}$ and $\varphi_n = \varphi_n^{V,W}$.

Lemma 4.3. *The Yetter–Drinfeld module $X_1^{V,W}$ is absolutely simple if and only if $\rho(z)^2\sigma(\epsilon g^2) = 1$. In this case, $X_1^{V,W} \simeq M(gz, \sigma_1)$, where σ_1 is the character of $G^{gz} = \langle g, z\rangle$ with*

$$\sigma_1(g) = -\rho(gz^{-1})\sigma(\epsilon), \quad \sigma_1(z) = \rho(z)\sigma(z).$$

Let $w' = \varphi_1(\epsilon^2 v \otimes w)$. Then $w' \in (V \otimes W)_{gz}$ is non-zero. Moreover, w', $\epsilon w'$, and $\epsilon^2 w'$ form a basis of $X_1^{V,W}$. The degrees of these basis vectors are gz, $g\epsilon z$, and $g\epsilon^2 z$, respectively.

Proof. By [22, Lemma 1.7], $X_1^{V,W} \simeq \mathbb{K}G\varphi_1(\epsilon^2 v \otimes w)$. Using the actions of G on V and W we obtain

$$w' = (\mathrm{id} - c_{W,V}c_{V,W})(\epsilon^2 v \otimes w) = \epsilon^2 v \otimes w - \rho(z)\sigma(\epsilon)^2\epsilon v \otimes gw, \tag{4.1}$$

and hence $w' \in (V \otimes W)_{gz}$ is non-zero. We compute

$$\begin{aligned} gw' &= g\epsilon^2 v \otimes gw - \sigma(\epsilon)^2\rho(z)g\epsilon v \otimes g^2 w \\ &= \rho(g)\epsilon v \otimes gw - \sigma(\epsilon)^2\rho(gz)\sigma(g^2)\epsilon^2 v \otimes w. \end{aligned}$$

Since V and W are absolutely simple and $G^{gz} = \langle g\rangle Z(G)$, the Yetter–Drinfeld module $X_1^{V,W}$ is absolutely simple if and only if $gw' \in \mathbb{K}w'$. By the above calculations, this is equivalent to $\rho(z)^2\sigma(\epsilon g^2) = 1$, and in this case $gw' = -\rho(gz^{-1})\sigma(\epsilon)w'$. The remaining claims are clear. □

Now we compute $X_2^{V,W}$. Since (g, gz) and $(g\epsilon^2, gz)$ represent the two orbits of $g^G \times (gz)^G$ under the diagonal action of G, we conclude that

$$X_2^{V,W} = \varphi_2(V \otimes X_1^{V,W}) = \mathbb{K}G\{\varphi_2(v \otimes w'), \varphi_2(\epsilon^2 v \otimes w')\}. \tag{4.2}$$

For the computation of $\varphi_2(v \otimes w')$ and $\varphi_2(\epsilon^2 v \otimes w')$ we need the following.

Lemma 4.4. *Assume that $\rho(z)^2\sigma(\epsilon g^2) = 1$. Let $w' = \varphi_1(\epsilon^2 v \otimes w)$. Then*

$$\varphi_1(v \otimes w) = \sigma(\epsilon)^{-1}\epsilon w', \tag{4.3}$$
$$\varphi_1(v \otimes gw) = -\rho(z)^{-1}\epsilon^2 w', \tag{4.4}$$
$$\varphi_1(\epsilon^2 v \otimes gw) = -\rho(z)^{-1}\sigma(\epsilon)^{-1}\epsilon w'. \tag{4.5}$$

Proof. We first prove (4.3). Since

$$w' = \varphi_1(\epsilon^2 v \otimes w) = \epsilon^2\varphi_1(v \otimes \epsilon w) = \epsilon^2\sigma(\epsilon)\varphi_1(v \otimes w),$$

equality (4.3) holds. Now apply g to (4.3) and use Lemma 4.3 to obtain (4.4). Finally, to obtain (4.5) apply ϵ^2 to (4.4). □

Lemma 4.5. *Assume that $\rho(z)^2\sigma(\epsilon g^2) = 1$. Then*

$$\varphi_2(\epsilon^2 v \otimes w') = (1 + \rho(g))(\epsilon^2 v \otimes w' + \rho(g)\sigma(\epsilon)v \otimes \epsilon w'), \tag{4.6}$$

$$\begin{aligned}\varphi_2(v \otimes w') = {}&(1 + \rho(g)^2\sigma(\epsilon))v \otimes w' + \rho(g)\sigma(\epsilon)^2\epsilon v \otimes \epsilon w'\\ &+ \rho(g)\sigma(\epsilon)^2\epsilon^2 v \otimes \epsilon^2 w'.\end{aligned} \tag{4.7}$$

In particular, $\varphi_2(\epsilon^2 v \otimes w') = 0$ if and only if $\rho(g) = -1$. Let $w'' = \varphi_2(v \otimes w')$. Then $w'' \in (V \otimes V \otimes W)_{g^2 z}$ is non-zero.

Proof. Recall that $\varphi_2 = \mathrm{id} - c_{X_1^{V,W},V}c_{V,X_1^{V,W}} + (\mathrm{id} \otimes \varphi_1)c_{1,2}$. Thus Lemma 4.3 and (4.1) imply that

$$\begin{aligned}\varphi_2(\epsilon^2 v \otimes w') = {}&\epsilon^2 v \otimes w' + \rho(g)^2\sigma(\epsilon)v \otimes \epsilon w'\\ &+ \rho(g)\epsilon^2 v \otimes \varphi_1(\epsilon^2 v \otimes w) - \rho(gz)\sigma(\epsilon)^2 v \otimes \varphi_1(\epsilon^2 v \otimes gw).\end{aligned}$$

Hence (4.6) follows from Lemma 4.4. Similarly, we compute

$$\begin{aligned}\varphi_2(v \otimes w') = {}&(1 + \rho(g)^2\sigma(\epsilon))v \otimes w'\\ &+ \rho(g)\epsilon v \otimes \varphi_1(v \otimes w) - \rho(gz)\sigma(\epsilon)^2\epsilon^2 v \otimes \varphi_1(v \otimes gw),\end{aligned}$$

and then use Lemma 4.4 to obtain (4.7). From this the lemma follows. □

Lemma 4.6. *Assume that $\rho(z)^2\sigma(\epsilon g^2)=1$. Let $w_1''=\varphi_2(v\otimes w_1')$. Then $X_2^{V,W}$ is absolutely simple if and only if $\rho(g)=-1$. In this case one of the following holds:*

(1) *If $\sigma(\epsilon)^2+\sigma(\epsilon)+1=0$, then $X_2^{V,W}=\mathbb{K}w_1''\simeq M(g^2z,\sigma_2)$, where σ_2 is the character of G given by*

$$\sigma_2(g)=-\rho(z)^{-1}\sigma(\epsilon),\quad \sigma_2(\epsilon)=1,\quad \sigma_2(z)=\rho(z)^2\sigma(z).$$

(2) *If $\sigma(\epsilon)=1$ and $\operatorname{char}\mathbb{K}\neq 3$, then $X_2^{V,W}\simeq M(g^2z,\sigma_2)$, where σ_2 is the two-dimensional absolutely irreducible representation of G with basis $\{w'',\epsilon w''\}$ and*

$$gw''=-\rho(z)^{-1}w'',\quad \epsilon^2w''=-w''-\epsilon w'',\quad zw''=\rho(z)^2\sigma(z)w''.$$

Proof. Since $z\in Z(G)$, we conclude that $zw''=\rho(z)^2\sigma(z)w''$. Further,

$$gw''=\varphi_2(gv\otimes gw')=-\rho(g^2z^{-1})\sigma(\epsilon)w''$$

by Lemma 4.3. Since g^2z and ϵg^2z are not conjugate in G, (4.2) and Lemma 4.5 imply that $X_2^{V,W}$ is absolutely simple if and only if $\rho(g)=-1$ and $\mathbb{K}Gw''$ is absolutely simple.

Assume that $\rho(g)=-1$. Since $\epsilon^3=1$ in G, we know that $\sigma(\epsilon)^3=1$. Hence $\sigma(\epsilon)^2+\sigma(\epsilon)+1=0$ or $\sigma(\epsilon)=1$. Using (4.7) and Lemma 4.3 one directly computes

$$(1-\epsilon)w''=(1+\sigma(\epsilon)+\sigma(\epsilon)^2)(v\otimes w'-\epsilon v\otimes\epsilon w'),\tag{4.8}$$

and similarly

$$(1+\epsilon+\epsilon^2)w''=(1-\sigma(\epsilon)^2)^2(1+\epsilon+\epsilon^2)(v\otimes w').\tag{4.9}$$

Suppose first that $\sigma(\epsilon)^2+\sigma(\epsilon)+1=0$. Then (4.8) becomes $(1-\epsilon)w''=0$, and hence the claim follows.

Suppose now that $\sigma(\epsilon)=1$ and $\operatorname{char}\mathbb{K}\neq 3$. Then $\{w'',\epsilon w''\}$ is linearly independent and $(1+\epsilon+\epsilon^2)w''=0$ by (4.9). Hence the lemma follows. □

Lemma 4.7. *Assume that $\rho(z)^2\sigma(\epsilon g^2)=1$ and $\rho(g)=-1$. Then $X_3^{V,W}=0$.*

Proof. Lemma 4.6 implies that $X_3^{V,W}=\mathbb{K}G\varphi_3(\epsilon v\otimes X_2^{V,W})$. Now observe that $\epsilon^2g(\epsilon v\otimes w'')\in\mathbb{K}\epsilon v\otimes\epsilon^2w''$, and hence it is enough to prove that $\varphi_3(\epsilon v\otimes w'')=0$.

Since $\rho(g)=-1$, (4.6) implies that $\varphi_2(\epsilon^2v\otimes w')=0$. Apply to this equality g and ϵ^2 and use Lemma 4.3 to conclude that $\varphi_2(\epsilon v\otimes w')=0$ and $\varphi_2(\epsilon v\otimes\epsilon^2w')=0$. A direct calculation using Lemmas 4.5 and 4.6 then shows that

$$\begin{aligned}\varphi_3(\epsilon v\otimes w'')&=\epsilon v\otimes w''+\sigma(\epsilon)\epsilon v\otimes\epsilon^2w''+(1+\sigma(\epsilon))g\epsilon v\otimes\varphi_2(\epsilon v\otimes w')\\&\quad-\sigma(\epsilon)^2g\epsilon^2v\otimes\varphi_2(\epsilon v\otimes\epsilon w')-\sigma(\epsilon)^2gv\otimes\varphi_2(\epsilon v\otimes\epsilon^2w')\\&=\epsilon v\otimes(w''+\sigma(\epsilon)\epsilon^2w''+\sigma(\epsilon)^2\epsilon w'').\end{aligned}$$

By Lemma 4.6, if $\sigma(\epsilon)^2+\sigma(\epsilon)+1=0$ then $\epsilon w''=w''$, and otherwise $\sigma(\epsilon)=1$ and $(1+\epsilon+\epsilon^2)w''=0$. Hence $\varphi_3(\epsilon v\otimes w'')=0$. □

Now we compute the modules $(\operatorname{ad} W)^n(V)$ for $n \in \mathbb{N}$. For $n \in \mathbb{N}$ let $\varphi_n = \varphi_n^{W,V}$. Let $X_1^{W,V} = \varphi_1(W \otimes V)$. By [22, Lemma 1.7], $X_1^{W,V} = \mathbb{K}G\varphi_1(w \otimes \epsilon v)$. Let $v_1' = \varphi_1(w \otimes \epsilon v)$. A direct calculation yields

$$v_1' = w \otimes \epsilon v - \rho(z)\sigma(\epsilon^2) g w \otimes \epsilon^2 v. \tag{4.10}$$

Hence $v_1' \in (W \otimes V)_{gz}$ is non-zero.

Lemma 4.8. *Assume that $\rho(z)^2\sigma(\epsilon g^2) = 1$. Then $X_1^{W,V}$ is an absolutely simple Yetter–Drinfeld module, and $X_1^{W,V} \simeq M(gz, \rho_1)$, where $\rho_1 = \sigma_1$. Moreover, v_1', $\epsilon v_1'$, $\epsilon^2 v_1'$ is a basis of $X_1^{W,V}$, and the degrees of these basis vectors are gz, $g\epsilon z$, $g\epsilon^2 z$, respectively.*

Proof. Since $X_1^{V,W} \simeq X_1^{W,V}$ via $c_{V,W}$, the claim follows from Lemma 4.3. □

Lemma 4.9. *Assume that $\rho(z)^2\sigma(\epsilon g^2) = 1$ and $\rho(g) = -1$. Then $X_2^{W,V} = 0$ if and only if $\sigma(\epsilon z) = -1$. Moreover, $X_2^{W,V}$ is absolutely simple if and only if $\sigma(\epsilon z) \neq -1$ and $\sigma(\epsilon z^2) = 1$. In this case $X_2^{W,V} \simeq M(gz^2, \rho_2)$, where ρ_2 is the character of G^g given by*

$$\rho_2(g) = -\rho(z)^{-2}\sigma(z)^{-1}, \quad \rho_2(z) = \rho(z)\sigma(z)^2.$$

Proof. First we apply ϵ and ϵ^2 to $v_1' = \varphi_1(w \otimes \epsilon v)$ to obtain

$$\varphi_1(w \otimes \epsilon^2 v) = \sigma(\epsilon)^2 \epsilon v_1', \quad \varphi_1(w \otimes v) = \sigma(\epsilon)\epsilon^2 v_1'. \tag{4.11}$$

Since G acts transitively on $(\epsilon z)^G \times (gz)^G$ via the diagonal action, we deduce from [22, Lemma 1.7] that $X_2^{W,V} = \mathbb{K}G\varphi_2(w \otimes \epsilon v_1')$. We compute

$$\begin{aligned}\varphi_2(w \otimes \epsilon v_1') = {} & w \otimes \epsilon v_1' - \rho(z)\sigma(\epsilon^2 z^2) g w \otimes \epsilon^2 v_1' \\ & + \sigma(\epsilon^2 z) w \otimes \varphi_1(w \otimes \epsilon^2 v) - \rho(z)\sigma(z) g w \otimes \varphi_1(w \otimes v).\end{aligned}$$

Equalities (4.11) imply that

$$\varphi_2(w \otimes \epsilon v_1') = (1 + \sigma(\epsilon z))\big(w \otimes \epsilon v_1' - \rho(z)\sigma(\epsilon z) g w \otimes \epsilon^2 v_1'\big). \tag{4.12}$$

Hence $\varphi_2(w \otimes \epsilon v_1') = 0$ if and only if $\sigma(\epsilon z) = -1$. Let

$$v_1'' = w \otimes \epsilon v_1' - \rho(z)\sigma(\epsilon z) g w \otimes \epsilon^2 v_1'.$$

Then $v_1'' \in (W \otimes X_1^{W,V})_{gz^2}$ is non-zero. Since $\rho(z)^2\sigma(\epsilon g^2) = 1$,

$$\begin{aligned} g v_1'' &= g w \otimes \epsilon^2 g v_1' - \rho(z)\sigma(\epsilon z)\sigma(g)^2 w \otimes \epsilon g v_1' \\ &= g w \otimes \rho_1(g)\epsilon^2 v_1' - \rho(z)^{-1}\sigma(z)\rho_1(g) w \otimes \epsilon v_1' \\ &= -\rho_1(g)\rho(z)^{-1}\sigma(z)\big(w \otimes \epsilon v_1' - \rho(z)\sigma(z)^{-1} g w \otimes \epsilon^2 v_1'\big).\end{aligned}$$

Thus $X_2^{W,V}$ is absolutely simple if and only if $\sigma(\epsilon z) \neq 1$ and $g v_1'' \in \mathbb{K} v_1''$. This is equivalent to $\sigma(\epsilon z) \neq 1$ and $\sigma(\epsilon z^2) = 1$. Finally, the equality $z v_1' = \rho(z)\sigma(z)^2 v_1'$ follows from $v_1' \in W \otimes W \otimes V$ and $z \in Z(G)$. □

Lemma 4.10. *Assume that* $\rho(z)^2\sigma(\epsilon g^2)=1$, $\rho(g)=-1$, $\sigma(\epsilon z)\neq -1$, *and* $\sigma(\epsilon z^2)=1$. *Define inductively* $y_0=v$ *and*

$$y_n = w\otimes \epsilon y_{n-1} - \rho(z)\sigma(z^{3n-1})gw\otimes \epsilon^2 y_{n-1}$$

for all $n\geq 1$. *Then* $y_n\in (W^{\otimes n}\otimes V)_{gz^n}$ *and*

$$\varphi_n(w\otimes \epsilon y_{n-1}) = (1+\sigma(z)^{-1}\cdots+\sigma(z)^{-n+1})y_n \tag{4.13}$$

for all $n\geq 1$.

Proof. We proceed by induction on n. For $n=1$ the claim holds by (4.10) and since $\sigma(\epsilon)^2=\sigma(\epsilon)^{-1}=\sigma(z^2)$. It is also clear that $y_n\in (W^{\otimes n}\otimes V)_{gz^n}$ for all $n\geq 0$.

Assume that (4.13) holds for some $n\geq 1$. Apply ϵ and ϵ^2 to (4.13) to obtain

$$\varphi_n(w\otimes \epsilon^2 y_{n-1}) = (1+\sigma(z)^{-1}+\cdots+\sigma(z)^{-n+1})\sigma(\epsilon)^2\epsilon y_n, \tag{4.14}$$

$$\varphi_n(w\otimes y_{n-1}) = (1+\sigma(z)^{-1}+\cdots+\sigma(z)^{-n+1})\sigma(\epsilon)\epsilon^2 y_n. \tag{4.15}$$

Since $\sigma(\epsilon z^2)=1$,

$$\begin{aligned}\varphi_{n+1}(w\otimes \epsilon y_n) &= w\otimes \epsilon y_n - \rho(z)\sigma(z)^{2n}\sigma(\epsilon)^2 gw\otimes \epsilon^2 y_n\\ &\quad + \sigma(\epsilon^2 z)w\otimes \varphi_n(w\otimes \epsilon^2 y_{n-1}) - \rho(z)\sigma(z^{3n-1})\sigma(\epsilon z)gw\otimes \varphi_n(w\otimes y_{n-1}).\end{aligned}$$

Using (4.14) and (4.15) and $\sigma(\epsilon z^2)=1$ one obtains

$$\varphi_{n+1}(w\otimes \epsilon y_n) = (1+\sigma(z)^{-1}+\cdots+\sigma(z)^{-n})y_{n+1},$$

as desired. □

Lemma 4.11. *Assume that* $\rho(z)^2\sigma(\epsilon g^2)=1$, $\rho(g)=-1$, $\sigma(\epsilon z)\neq -1$, *and* $\sigma(\epsilon z^2)=1$. *Then* $X_n^{W,V}\simeq M(gz^n,\rho_n)$ *for all* $n\geq 1$, *where* ρ_n *is the character of* G^g *given by*

$$\rho_n(g) = (-1)^{n+1}\rho(z)^{-n}\sigma(z)^{(3n+5)n/2}, \qquad \rho_n(z)=\rho(z)\sigma(z)^n.$$

Moreover, $X_n^{W,V}=0$ *if and only if* $(n)^!_{\sigma(z)}=0$.

Proof. For all $n\geq 0$ let y_n be as in Lemma 4.10. Then $zy_n=\rho(z)\sigma(z)^n y_n$, since $y_n\in W^{\otimes n}\otimes V$ and $z\in Z(G)$. To prove that $gy_n=\rho_n(g)y_n$ for all $n\geq 1$ we proceed by induction. For $n=1$ this holds by Lemmas 4.8 and 4.3. Suppose now that $gy_n=\rho_n(g)y_n$ for some $n\geq 1$. Then

$$\begin{aligned}gy_{n+1} &= gw\otimes \epsilon^2 gy_n - \rho(z)\sigma(z^{3n+2})\sigma(g^2)w\otimes \epsilon gy_n\\ &= \rho_n(g)\big(gw\otimes \epsilon^2 y_n - \rho(z)\sigma(z^{3n+2})\sigma(g^2)w\otimes \epsilon y_n\big).\end{aligned}$$

Since $\sigma(\epsilon z^2)=1$ and $\rho(z)^2\sigma(\epsilon g^2)=1$, we conlude that the expression for $\rho_{n+1}(g)$ given in the claim can be written as

$$\rho_{n+1}(g) = -\rho_n(g)\rho(z)^{-1}\sigma(z^{3n-2}).$$

This implies the claim.

To complete the proof of the lemma we observe that

$$X_n^{W,V} = \mathbb{K}G\varphi_n(w \otimes \epsilon y_n),$$

and hence equality (4.13) of Lemma 4.10 implies that $X_n^{W,V} = 0$ if and only if $(k)_{\sigma(z)} = 0$ for some $k \le n$, which is equivalent to $(n)^!_{\sigma(z)} = 0$. □

We collect all the results of this section in the following proposition. We write $a_{i,j}^{(V,W)}$ for the entries of the Cartan matrix of the pair (V, W).

Proposition 4.12. *Let $V, W \in {}^G_G\mathcal{YD}$ be such that $V \simeq M(g, \rho)$, where ρ is a character of G^g, and $W \simeq M(\epsilon z, \sigma)$, where σ is a character of $G^{\epsilon z}$. Then:*

(1) $(\mathrm{ad}\, V)^m(W)$ *and* $(\mathrm{ad}\, W)^m(V)$ *are absolutely simple or zero for all $m \in \mathbb{N}_0$ if and only if*

$$\rho(z)^2\sigma(\epsilon g^2) = 1, \quad \rho(g) = -1, \quad (\sigma(\epsilon z) + 1)(\sigma(\epsilon z^2) - 1) = 0.$$

In particular, these equalities imply that $\sigma(z)^6 = 1$.

(2) *Assume that the equalities for ρ and σ in (1) hold. Then the Cartan matrix of (V, W) satisfies $a_{1,2}^{(V,W)} = -2$ and $X_2^{V,W} \simeq M(g^2 z, \sigma_2)$, where*

(a) *if $1 + \sigma(\epsilon) + \sigma(\epsilon)^2 = 0$, then σ_2 is the character of G given by $\sigma_2(\epsilon) = 1$, $\sigma_2(g) = -\rho(z)^{-1}\sigma(\epsilon)$, $\sigma_2(z) = \rho(z)^2\sigma(z)$, and*

(b) *if $\sigma(\epsilon) = 1$ and $\mathrm{char}\,\mathbb{K} \neq 3$, then σ_2 is a two-dimensional absolutely irreducible representation of G with $\sigma_2(g^2) = \rho(z)^{-2}$ and $\sigma_2(z) = \rho(z)^2\sigma(z)$.*

Moreover,

$$a_{2,1}^{(V,W)} = \begin{cases} -1 & \text{if } \sigma(\epsilon z) = -1, \\ -2 & \text{if } \sigma(z) = \sigma(\epsilon) \text{ and } (3)_{\sigma(\epsilon)} = 0, \\ -5 & \text{if } \sigma(z) + \sigma(\epsilon) = (3)_{\sigma(\epsilon)} = 0 \text{ and } \mathrm{char}\,\mathbb{K} \neq 2, 3, \\ 1 - p & \text{if } \mathrm{char}\,\mathbb{K} = p \geq 5 \text{ and } \sigma(\epsilon) = \sigma(z) = 1, \end{cases}$$

and otherwise $(\mathrm{ad}\, W)^m(V) \neq 0$ *for all $m \in \mathbb{N}_0$. In these cases $X_m^{W,V} \simeq M(gz^m, \rho_m)$ for $m = -a_{2,1}^{(V,W)}$, where ρ_m is the character of G^g with $\rho_1(z) = \rho(z)\sigma(z)$, $\rho_1(g) = \rho(z)^{-1}\sigma(\epsilon)$, and*

$$\rho_m(z) = \rho(z)\sigma(z)^m, \qquad \rho_m(g) = (-1)^{m+1}\rho(z)^{-m}\sigma(z)^{(3m+5)m/2}$$

for all $m \geq 2$.

Proof. (1) follows from Lemmas 4.3–4.9 and 4.11. Further, Lemmas 4.6 and 4.7 yield the claim concerning $a_{1,2}^{(V,W)}$. Now we prove the claim concerning $a_{2,1}^{(V,W)}$. Since $(\sigma(\epsilon z) + 1)(\sigma(\epsilon z^2) - 1) = 0$ and $\sigma(\epsilon)^3 = 1$, we need to consider the following cases:

- $\sigma(\epsilon z) = -1$ and $\sigma(\epsilon)^3 = 1$,
- $\sigma(\epsilon) = \sigma(z)$ and $\sigma(\epsilon)^2 + \sigma(\epsilon) + 1 = 0$,

- $\sigma(\epsilon) = \sigma(z) = 1$ and $\operatorname{char} \mathbb{K} = p \geq 5$,
- $\sigma(\epsilon) = \sigma(z) = 1$ and $\operatorname{char} \mathbb{K} = 0$,
- $\sigma(\epsilon) = -\sigma(z)$, $\sigma(\epsilon)^2 + \sigma(\epsilon) + 1 = 0$ and $\operatorname{char} \mathbb{K} \neq 2, 3$.

Using the equivalence between $(\operatorname{ad} W)^m(V) = 0$ and $(m)^!_{\sigma(z)} = 0$ of Lemma 4.11, one easily completes the proof of (2). □

Corollary 4.13. *Let $V, W \in {}^G_G\mathcal{YD}$ with $V \simeq M(g, \rho)$ and $W \simeq M(\epsilon z, \sigma)$, where ρ is a character of G^g and σ is a character of G^ϵ. Assume that*

$$\rho(g) = \sigma(\epsilon z) = -1, \quad \rho(z)^2\sigma(\epsilon g^2) = 1, \quad (3)_{\sigma(\epsilon)} = 0.$$

Let $g' = g^{-1}$, $\epsilon' = \epsilon$, $z' = g^2 z$, let ρ' be the representation of $G^{g'}$ dual to ρ, and let σ' be the character of G given by $\sigma'(g) = -\rho(z)^{-1}\sigma(\epsilon)$, $\sigma'(\epsilon) = 1$, $\sigma'(z) = \rho(z)^2\sigma(z)$. Then $a_{1,2}^{(V,W)} = -2$ and

$$R_1(V, W) = (V^*, X_2^{V,W})$$

with $V^* \simeq M(g', \rho')$, $X_2^{V,W} \simeq M(z', \sigma')$, and

$$\rho'(g') = -1, \quad \rho'(z')\sigma'(z'g') = 1, \quad 1 - \rho'(z')\sigma'(g') + \rho'(z')^2\sigma'(g')^2 = 0.$$

Proof. Using Proposition 4.12 one finds that $a_{1,2}^{(V,W)} = -2$, and hence the description of $R_1(V, W)$ follows. It is clear that $\rho'(g') = \rho(g) = -1$. A direct calculation yields

$$\rho'(z')\sigma'(z'g') = -\sigma(\epsilon z) = 1,$$
$$1 - \rho'(z')\sigma'(g') + \rho'(z')^2\sigma'(g')^2 = 1 + \sigma(\epsilon)^{-1} + \sigma(\epsilon)^{-2} = 0.$$ □

Corollary 4.14. *Let $V, W \in {}^G_G\mathcal{YD}$ with $V \simeq M(g, \rho)$ and $W \simeq M(\epsilon z, \sigma)$, where ρ is a character of G^g and σ is a character of G^ϵ. Assume that*

$$\rho(g) = \sigma(\epsilon z) = -1, \quad \rho(z)^2\sigma(\epsilon g^2) = 1, \quad (3)_{\sigma(\epsilon)} = 0.$$

Let $g'' = gz$, $\epsilon'' = \epsilon^{-1}$, $z'' = z^{-1}$, let ρ'' be the character of G^g given by $\rho''(g) = \rho(z)^{-1}\sigma(\epsilon)$ and $\rho''(z) = \rho(z)\sigma(z)$, and let σ'' be the representation of $G^{\epsilon z}$ dual to σ. Then $a_{2,1}^{(V,W)} = -1$ and

$$R_2(V, W) = (X_1^{W,V}, W^*)$$

with $X_1^{W,V} \simeq M(g'', \rho'')$, $W^* \simeq M(\epsilon'' z'', \sigma'')$, and

$$\rho''(g'') = \sigma''(\epsilon'' z'') = -1, \quad \rho''(z'')^2\sigma''(\epsilon'' g''^2) = 1, \quad (3)_{\sigma''(\epsilon'')} = 0.$$

Proof. The assumptions on ρ and σ and Proposition 4.12 yield $a_{2,1}^{(V,W)} = -1$, and hence the description of $R_2(V, W)$ follows. Then we compute

$$\sigma''(\epsilon'' z'') = \sigma''(\epsilon^{-1} z^{-1}) = \sigma(\epsilon z) = -1,$$
$$\rho''(g'') = \rho''(gz) = \rho''(g)\rho''(z) = -1,$$
$$1 + \sigma''(\epsilon'') + \sigma''(\epsilon'')^2 = 1 + \sigma(\epsilon) + \sigma(\epsilon)^2 = 0.$$

As g and z commute, we obtain

$$\rho''(z'')^2\sigma''(\epsilon''g''^2) = \rho''(z)^{-2}\sigma''(\epsilon^{-1}g^2z^2) = \rho(z)^{-2}\sigma(\epsilon)\sigma(g)^{-2}\sigma(z)^{-4}.$$

Since $\sigma(\epsilon z) = -1$, we obtain $\sigma(\epsilon z^{-4}) = \sigma(\epsilon)^{-1}$. Hence we conclude that

$$\rho''(z'')^2\sigma''(\epsilon''g''^2) = (\rho(z)^2\sigma(\epsilon g^2))^{-1} = 1.$$

This completes the proof. □

Corollary 4.15. *Let $V, W \in {}^G_G\mathcal{YD}$ with $V \simeq M(g,\rho)$ and $W \simeq M(\epsilon z,\sigma)$, where ρ is a character of G^g and σ is a character of G^ϵ. Assume that* $\operatorname{char}\mathbb{K} \neq 3$ *and*

$$\rho(g) = -1, \quad \rho(z^2)\sigma(\epsilon g^2) = 1, \quad \sigma(\epsilon) = 1, \quad \sigma(z) = -1.$$

Further, let $g' = g^{-1}$, $\epsilon' = \epsilon$, $z' = g^2z$, let ρ' be the irreducible representation of G^g dual to ρ, and let σ' be an absolutely irreducible representation of G with $\deg\sigma' = 2$, *$\sigma'(g^2) = \rho(z)^{-2}$, and $\sigma'(z) = -\rho(z)^2$. Then $a^{(V,W)}_{1,2} = -2$ and*

$$R_1(V,W) = (V^*, X^{V,W}_2)$$

with $V^* \simeq M(g',\rho')$, $X^{V,W}_2 \simeq M(z',\sigma')$, and

$$\rho'(g') = -1, \quad \rho'(z')^2\sigma'(g'^2) = 1, \quad \sigma'(z') = -1.$$

Proof. It is similar to the proof of Corollary 4.13. □

Corollary 4.16. *Let $V, W \in {}^G_G\mathcal{YD}$ with $V \simeq M(g,\rho)$ and $W \simeq M(\epsilon z,\sigma)$, where ρ is a character of G^g and σ is a character of G^ϵ. Assume that* $\operatorname{char}\mathbb{K} \neq 3$ *and*

$$\rho(g) = -1, \quad \rho(z)^2\sigma(\epsilon g^2) = 1, \quad \sigma(\epsilon) = 1, \quad \sigma(z) = -1.$$

Further, let $g'' = gz$, $\epsilon'' = \epsilon^{-1}$, $z'' = z^{-1}$, let ρ'' be the character of G^g given by $\rho''(z) = -\rho(z)$ and $\rho''(g) = \rho(z)^{-1}$, and let σ'' be the character of G^ϵ dual to σ. Then $a^{(V,W)}_{2,1} = -1$ and

$$R_2(V,W) = (X^{W,V}_1, W^*)$$

with $X^{W,V}_1 \simeq M(g'',\rho'')$, $W^* \simeq M(\epsilon''z'',\sigma'')$, and

$$\rho''(g'') = -1, \quad \rho''(z'')^2\sigma''(\epsilon''g''^2) = 1, \quad \sigma''(\epsilon'') = 1, \quad \sigma''(z'') = -1.$$

Proof. It is similar to the proof of Corollary 4.14. □

5. Reflections of the second pair

In this section we have to deal with an irreducible representation of G of degree two. Therefore we assume that $\mathbb{K}$ is algebraically closed. This will not be relevant for our classification of Nichols algebras.

Let G be a non-abelian group, and let $g, \epsilon, z \in G$. Assume that there is an epimorphism $\Gamma_3 \to G$ with $\gamma \mapsto g$, $\nu \mapsto \epsilon$, $\zeta \mapsto z$. Let $V, W \in {}^{G}_{G}\mathcal{YD}$ be such that $V \simeq M(g, \rho)$ and $W \simeq M(z, \sigma)$, where ρ is a character of G^g and σ is a two-dimensional irreducible representation of G. Let λ be an eigenvalue of $\sigma(\epsilon)$ and let w be a corresponding eigenvector. Then, by Lemma 3.2, $\operatorname{char}\mathbb{K} \neq 3$ and $1 + \lambda + \lambda^2 = 0$. Hence $\epsilon g w = \lambda^{-1} g w$, $\lambda^{-1} \neq \lambda$, and $\{w, gw\}$ is a basis of $W_z = W$.

Remark 5.1. By the above discussion, we obtain the following table for the action of G on W:

W	w	gw
ϵ	λw	$\lambda^2 gw$
z	$\sigma(z)w$	$\sigma(z)gw$
g	gw	$\sigma(g^2)w$

Now we compute the modules $(\operatorname{ad} V)^m(W)$ for $m \in \mathbb{N}$. First we deduce that $X_1^{V,W} = \varphi_1(V \otimes W) = \mathbb{K}G\varphi_1(v \otimes w)$ and

$$\varphi_1(v \otimes w) = v \otimes (w - \rho(z)gw). \tag{5.1}$$

Hence $w' := \varphi_1(v \otimes w) \in (V \otimes W)_{gz}$ is non-zero.

Lemma 5.2. *The Yetter–Drinfeld module $X_1^{V,W}$ is simple if and only if $\rho(z)^2\sigma(g^2) = 1$. In this case, $X_1^{V,W} \simeq M(gz, \sigma_1)$, where σ_1 is the character of G^g with*

$$\sigma_1(z) = \rho(z)\sigma(z), \qquad \sigma_1(g) = -\rho(gz^{-1}).$$

A basis for $X_1^{V,W}$ is given by $\{w', \epsilon w', \epsilon^2 w'\}$. The degrees of these basis vectors are gz, $g\epsilon z$, and $g\epsilon^2 z$, respectively.

Proof. Since $X_1^{V,W} = \mathbb{K}Gw'$ and $w' \in (V \otimes W)_{gz}$, the module $X_1^{V,W}$ is simple if and only if $gw' = \mathbb{K}w'$. Since

$$gw' = gv \otimes (gw - \rho(z)g^2 w) = \rho(g)v \otimes \big(-\rho(z)\sigma(g^2)w + gw\big),$$

the latter is equivalent to $gw' = -\rho(gz)\sigma(g^2)w'$, $\rho(z)^2\sigma(g^2) = 1$. From this the claim follows. □

Remark 5.3. The action of G on $X_1^{V,W}$ can be displayed in a table similar to the one in Remark 4.1, where v has to be replaced by w', and ρ by σ_1.

Lemma 5.4. *Assume that* $\rho(z)^2\sigma(g^2)=1$. *Then* $X_2^{V,W}\neq 0$, *and* $X_2^{V,W}$ *is simple if and only* $\rho(g)=-1$. *In this case,* $X_2^{V,W}\simeq M(\epsilon g^2 z,\sigma_2)$, *where* σ_2 *is the character of* G^ϵ *given by*

$$\sigma_2(\epsilon)=1,\qquad \sigma_2(z)=\rho(z)^2\sigma(z),\qquad \sigma_2(g^2)=\rho(z)^{-2}.$$

The set $\{w'', gw''\}$ *is a basis of* $X_2^{V,W}$. *The degrees of these vectors are* $\epsilon g^2 z$ *and* $\epsilon^2 g^2 z$, *respectively.*

Proof. Observe that $X_2^{V,W}$ is the direct sum $\mathbb{K}G\varphi_2(v\otimes w')\oplus\mathbb{K}G\varphi_2(\epsilon^2 v\otimes w')$ of two Yetter–Drinfeld submodules. Applying g to (5.1) and using Lemma 5.2 we obtain

$$\varphi_1(v\otimes gw)=-\rho(z)^{-1}w'. \tag{5.2}$$

A direct computation shows that

$$\begin{aligned}\varphi_2(v\otimes w')&=v\otimes w'+\rho(g^2)v\otimes w'\\&\quad+\rho(g)v\otimes\varphi_1(v\otimes w)-\rho(gz)v\otimes\varphi_1(v\otimes gw).\end{aligned}$$

Then $\varphi_2(v\otimes w')=(1+\rho(g))^2 v\otimes w'$. To compute $\varphi_2(\epsilon^2 v\otimes w')$ we apply ϵ^2 to (5.1) and (5.2) to obtain

$$\varphi_1(\epsilon^2 v\otimes w)=\lambda\epsilon^2 w',\qquad \varphi_1(\epsilon^2 v\otimes gw)=-\lambda^2\rho(z)^{-1}\epsilon^2 w'.$$

Then one finds that

$$\begin{aligned}\varphi_2(\epsilon^2 v\otimes w')&=\epsilon^2 v\otimes w'+\rho(g)^2 v\otimes\epsilon w'+\rho(g)\epsilon v\otimes\varphi_1(\epsilon^2 v\otimes w)\\&\quad-\rho(z)\rho(g)\epsilon v\otimes\varphi_1(\epsilon^2 v\otimes gw),\end{aligned}$$

and hence

$$\varphi_2(\epsilon^2 v\otimes w')=\epsilon^2 v\otimes w'+\rho(g^2)v\otimes\epsilon w'-\rho(g)\epsilon v\otimes\epsilon^2 w'. \tag{5.3}$$

Therefore $w'':=\varphi_2(\epsilon^2 v\otimes w')\in(V\otimes V\otimes W)_{\epsilon g^2 z}$ is non-zero. We conclude that $\mathbb{K}G\varphi_2(v\otimes w')=0$, that is, $\rho(g)=-1$. In this case one easily deduces from (5.3) that $\epsilon w''=w''$. The equalities $g^2 w''=\rho(z)^{-2}w''$ and $zw''=\rho(z)^2\sigma(z)w''$ are clear since $g^2, z\in Z(G)$ and $\sigma(g^2)=\rho(z)^{-2}$. □

Lemma 5.5. *Assume that* $\rho(z)^2\sigma(g^2)=1$ *and* $\rho(g)=-1$. *Then* $X_3^{V,W}=0$.

Proof. Apply ϵ and $g\epsilon$ to $w''=\varphi_2(\epsilon^2 v\otimes w')$ and use Lemmas 5.4 and 5.2 to obtain $w''=\varphi_2(v\otimes\epsilon w')$ and $gw''=-\rho(z)^{-1}\varphi_2(v\otimes\epsilon^2 w')$, respectively. Now we calculate

$$\begin{aligned}\varphi_3(v\otimes w'')&=v\otimes w''-\rho(z)\epsilon^2 v\otimes gw''-\epsilon v\otimes\varphi_2(v\otimes w')\\&\quad-v\otimes\varphi_2(v\otimes\epsilon w')-\epsilon^2 v\otimes\varphi_2(v\otimes\epsilon^2 w'),\end{aligned}$$

and hence $\varphi_3(v\otimes w'')=0$. Since $X_3^{V,W}=\mathbb{K}G\varphi_3(v\otimes w'')$, the proof is complete. □

Now we compute the modules $(\operatorname{ad} W)^n(V)$ for $n\in\mathbb{N}$. For $n\in\mathbb{N}$ we write $\varphi_n=\varphi_n^{W,V}$.

First note that
$$X_1^{W,V} = \varphi_1(W \otimes V) = \mathbb{K}G\varphi_1(w \otimes v).$$
Further,
$$v_1' := \varphi_1(w \otimes v) = (w - \rho(z)gw) \otimes v \in (W \otimes V)_{gz} \tag{5.4}$$
is non-zero.

Lemma 5.6. *Assume that $\rho(z)^2\sigma(g^2)=1$. Then $X_1^{W,V}$ is simple and $X_1^{W,V} \simeq M(gz, \rho_1)$, where ρ_1 is the character of G^{gz} defined by*
$$\rho_1(g) = -\rho(gz^{-1}), \qquad \rho_1(z) = \rho(z)\sigma(z).$$
The set $\{v_1', \epsilon v_1', \epsilon^2 v_1'\}$ is a basis of $X_1^{W,V}$. The degrees of these basis vectors are gz, $g\epsilon z$, and $g\epsilon^2 z$, respectively.

Proof. This follows from Lemma 5.2. □

Lemma 5.7. *Assume that $\rho(z)^2\sigma(g^2) = 1$ and $\rho(g) = -1$. Then:*

(1) $X_2^{W,V} = 0$ *if and only if* $\sigma(z) = -1$.
(2) $X_2^{W,V}$ *is simple if and only if* $\sigma(z) = 1$ *and* $\sigma(z) \neq -1$. *In this case,* $X_2^{W,V} \simeq M(gz^2, \rho_2)$, *where ρ_2 is the character of G^g with*
$$\rho_2(g) = -\rho(z)^{-2}, \qquad \rho_2(z) = \rho(z).$$

Proof. Since $X_2^{W,V} = \varphi_2(W \otimes X_1^{W,V}) = \mathbb{K}G\varphi_2(w \otimes v_1')$, we need to compute $\varphi_2(w \otimes v_1')$. Using (5.4) we obtain
$$\begin{aligned}\varphi_2(w \otimes v_1') &= w \otimes v_1' - c_{X_1^{W,V},W}c_{W,X_1^{W,V}}(w \otimes v_1') + (\mathrm{id} \otimes \varphi_1)c_{1,2}(w \otimes v_1')\\ &= (1 + \sigma(z))(w \otimes v_1' - \rho(z)\sigma(z)gw \otimes v_1'),\end{aligned}$$
and hence (1) follows. Now assume that $\sigma(z) \neq -1$ and let $v_1'' := (w - \rho(z)\sigma(z)gw) \otimes v_1'$. Then $v_1'' \in (X_2^{W,V})_{gz^2}$ is non-zero. Further, using Lemma 5.6 we obtain
$$gv_1'' = (gw - \rho(z)\sigma(z)g^2w) \otimes gv_1' = \rho(z)^{-1}(gw - \rho(z)\sigma(g^2z)w) \otimes v_1'.$$
Observe that $gv_1'' = \mathbb{K}v_1''$ if and only if $\sigma(z^2) = 1$, and then $gv_1'' = -\sigma(g^2z)v_1''$. Thus $X_2^{W,V}$ is simple if and only if $\sigma(z) \neq -1$ and $\sigma(z) = 1$. In this case, since $\sigma(z) = 1$, we conclude that $X_2^{W,V} \simeq M(gz^2, \rho_2)$, where $\rho_2(z) = \rho(z)$ and $\rho_2(g) = -\rho(z)^{-2}$. □

Now we define $y_n = (w - \rho(z)gw)^{\otimes n} \otimes v$ for all $n \geq 0$.

Lemma 5.8. *Assume that $\rho(z)^2\sigma(g^2) = 1$, $\rho(g) = -1$, $\sigma(z) = 1$, and $\sigma(z) \neq -1$. Let $n \geq 1$. Then $\varphi_n(w \otimes y_{n-1}) = ny_n$ and $X_n^{W,V} = n!\mathbb{K}Gy_n$. Moreover, if $n! \neq 0$ then $X_n^{W,V} \simeq M(gz^n, \rho_n)$, where ρ_n is the character on G^g given by*
$$\rho_n(g) = (-1)^{n+1}\rho(z)^{-n}, \qquad \rho_n(z) = \rho(z).$$

Proof. It is clear that $y_n \in (W^{\otimes n} \otimes V)_{gz^n}$ for all $n \in \mathbb{N}$. Moreover,

$$g(w - \rho(z)gw) = gw - \rho(z)^{-1}w = -\rho(z)^{-1}(w - \rho(z)gw),$$

and hence $gy_n = (-1)^{n+1}\rho(z)^{-n}y_n$ for all $n \in \mathbb{N}$. To prove the other claims we proceed by induction on n. For $n = 1$ the claim holds by Lemma 5.6. So assume that the claim is valid for some $n \geq 1$. Then

$$X^{W,V}_{n+1} = \varphi_{n+1}(w \otimes X^{W,V}_n) = \varphi_{n+1}(w \otimes n!y_n).$$

Since $\sigma(z) = 1$, we obtain

$$\varphi_{n+1}(w \otimes y_n) = w \otimes y_n - \rho(z)gw \otimes y_n + (w - \rho(z)gw) \otimes \varphi_n(w \otimes y_{n-1}).$$

Since $\varphi_n(w \otimes y_{n-1}) = ny_n$ by induction hypothesis, the latter equality implies that $\varphi_{n+1}(w \otimes y_n) = (n+1)y_{n+1}$. Therefore $X^{W,V}_{n+1} = (n+1)!\mathbb{K}Gy_{n+1}$. The remaining claims are clear. □

In the remaining part of this section we do not use any assumption on the field $\mathbb{K}$.

Proposition 5.9. *Let $V, W \in {}^G_G\mathcal{YD}$ be such that $V \simeq M(g, \rho)$, where ρ is a character of G^g, and $W \simeq M(z, \sigma)$, where σ is an absolutely irreducible representation of G of degree two. Then* $\operatorname{char}\mathbb{K} \neq 3$ *and:*

(1) $(\operatorname{ad} V)^m(W)$ *and* $(\operatorname{ad} W)^m(V)$ *are absolutely simple or zero for all $m \in \mathbb{N}_0$ if and only if*

$$\rho(z)^2\sigma(g^2) = 1, \quad \rho(g) = -1, \quad \sigma(z)^2 = 1.$$

(2) *Assume that the conditions on ρ, σ in (1) hold. Then the Cartan matrix of (V, W) satisfies $a^{(V,W)}_{1,2} = -2$ and $X^{V,W}_2 \simeq M(\epsilon g^2 z, \sigma_2)$, where σ_2 is the character of G^ϵ given by*

$$\sigma_2(\epsilon) = 1, \quad \sigma_2(z) = \rho(z)^2\sigma(z), \quad \sigma_2(g^2) = \sigma(g^2).$$

Moreover,

$$a^{(V,W)}_{2,1} = \begin{cases} -1 & \text{if } \sigma(z) = -1, \\ 1 - p & \text{if } \sigma(z) = 1 \text{ and } \operatorname{char}\mathbb{K} = p \geq 5, \end{cases}$$

and $X^{W,V}_m \simeq M(gz^n, \rho_m)$, where $m = -a^{(V,W)}_{2,1}$ and ρ_m is the character of G^g given by

$$\rho_m(g) = (-1)^{m+1}\rho(z)^{-m}, \quad \rho_m(z) = \rho(z)\sigma(z).$$

Proof. Since W is absolutely simple and $z \in Z(G)$, the representation σ is absolutely irreducible. Hence $\operatorname{char}\mathbb{K} \neq 3$ and $\sigma(1 + \epsilon + \epsilon^2) = 0$ by Lemma 3.2.

Since $(\operatorname{ad}(\mathbb{L} \otimes_{\mathbb{K}} V))^m(\mathbb{L} \otimes_{\mathbb{K}} W) \simeq \mathbb{L} \otimes_{\mathbb{K}} (\operatorname{ad} V)^m(W)$ for all $m \in \mathbb{N}$ and all field extensions $\mathbb{L}$ of $\mathbb{K}$, for (1) we may assume that $\mathbb{K}$ is algebraically closed. Then (1) follows from Lemmas 5.2 and 5.4–5.8. Further, under the conditions on ρ and σ in (1) we find that $a^{(V,W)}_{12} = -2$ by Lemmas 5.4 and 5.5, and the structure of $X^{V,W}_2$ is given by Lemma 5.4. Similarly, the value of $a^{(V,W)}_{21}$ and the structure of $X^{W,V}_m$ are obtained from Lemmas 5.7 and 5.8. □

Corollary 5.10. *Assume that* char $\mathbb{K} \neq 3$. *Let* $V, W \in {}^{G}_{G}\mathcal{YD}$ *be such that* $V \simeq M(g,\rho)$ *and* $W \simeq M(z,\sigma)$, *where* ρ *is a character of* G^g, *and* σ *is an absolutely irreducible representation of* G *of degree two. Assume that*

$$\rho(z)^2\sigma(g^2) = 1, \qquad \rho(g) = -1, \qquad \sigma(z) = -1.$$

Further, let $g' = g^{-1}$, $\epsilon' = \epsilon$, $z' = g^2 z$, *let* ρ' *be the character of* $G^{g'} = G^g$ *dual to* ρ, *and let* σ' *be the character of* G^ϵ *given by* $\sigma'(\epsilon) = 1$, $\sigma'(z) = -\rho(z)^2$, $\sigma'(g^2) = \sigma(g^2)$. *Then* $a_{1,2}^{(V,W)} = -2$ *and*

$$R_1(V,W) = (V^*, X_2^{V,W})$$

with $V^* \simeq M(g',\rho')$, $X_2^{V,W} \simeq M(\epsilon' z', \sigma')$, *and*

$$\rho'(g') = -1, \qquad \sigma'(z') = -1, \qquad \rho'(z')^2\sigma'(\epsilon' g'^2) = 1, \qquad \sigma'(\epsilon') = 1.$$

Proof. By Proposition 5.9 we obtain $a_{1,2}^{(V,W)} = -2$, and hence the description of $R_1(V,W)$ follows. Then

$$\sigma'(z') = \sigma'(g^2)\sigma'(z) = -\sigma(g^2)\rho(z^2) = -1.$$

Similarly one proves the other formulas. □

Corollary 5.11. *Assume that* char $\mathbb{K} \neq 3$. *Let* $V, W \in {}^{G}_{G}\mathcal{YD}$ *be such that* $V \simeq M(g,\rho)$ *and* $W \simeq M(z,\sigma)$, *where* ρ *is a character of* G^g, *and* σ *is an absolutely irreducible representation of* G *of degree two. Assume that*

$$\rho(z)^2\sigma(g^2) = 1, \qquad \rho(g) = -1, \qquad \sigma(z) = -1.$$

Let $g'' = gz$, $\epsilon'' = \epsilon^{-1}$, $z'' = z^{-1}$, *let* ρ'' *be the character of* G^g *given by* $\rho''(g) = \rho(z)^{-1}$ *and* $\rho''(z) = -\rho(z)$, *and let* σ'' *be the degree two representation of* G *dual to* σ. *Then* $a_{2,1}^{(V,W)} = -1$ *and*

$$R_2(V,W) = (X_1^{W,V}, W^*)$$

with $X_1^{W,V} \simeq M(g'',\rho'')$, $W^* \simeq M(z'',\sigma'')$, *and*

$$\rho''(z'')^2\sigma''(g''^2) = 1, \qquad \rho''(g'') = -1, \qquad \sigma''(z'') = -1.$$

Proof. It is similar to the proof of Corollary 5.10. □

6. Reflections of the third pair

Let $V, W \in {}^{G}_{G}\mathcal{YD}$ be such that $V \simeq M(g,\rho)$, where ρ is a character of G^g, and $W \simeq M(z,\sigma)$, where σ is a character of G. Let $w \in W = W_z$ with $w \neq 0$. Then $\{w\}$ is a basis of W. Since $g\epsilon = \epsilon^{-1}g$ and $\epsilon^3 = 1$, we obtain

$$gw = \sigma(g)w, \qquad \epsilon w = w, \qquad zw = \sigma(z)w.$$

We first compute the modules $(\operatorname{ad} V)^m(W)$ for $m \in \mathbb{N}$.

Lemma 6.1. *The Yetter–Drinfeld module $X_1^{V,W}$ is non-zero if and only if $\rho(z)\sigma(g)\neq 1$. In this case, $X_1^{V,W}$ is absolutely simple and $X_1^{V,W}\simeq M(gz,\sigma_1)$, where σ_1 is the character of G^g given by*

$$\sigma_1(g)=\rho(g)\sigma(g),\qquad \sigma_1(z)=\rho(z)\sigma(z).$$

Let $w'=v\otimes w$. Then $\{w',\epsilon w',\epsilon^2 w'\}$ is a basis of $X_1^{V,W}$. The degrees of these basis vectors are gz, $g\epsilon z$, and $g\epsilon^2 z$, respectively.

Again, the action of G on $X_1^{V,W}$ can be displayed in a table similar to the one in Remark 4.1, where v has to be replaced by w' and ρ has to be replaced by σ_1.

Proof of Lemma 6.1. Write $X_1^{V,W}=\varphi_1(V\otimes W)=\mathbb{K}G\varphi_1(v\otimes w)$ and compute

$$\varphi_1(v\otimes w)=v\otimes w-c_{W,V}c_{V,W}(v\otimes w)=(1-\rho(z)\sigma(g))v\otimes w.$$

Then $w'=v\otimes w\in(V\otimes W)_{gz}$ is non-zero and $X_1^{V,W}=0$ if and only if $\rho(z)\sigma(g)=1$. Assume that $\rho(z)\sigma(g)\neq 1$. Then

$$gw'=\rho(g)\sigma(g)w',\qquad zw'=\rho(z)\sigma(z)w'.$$

Hence $X_1^{V,W}=\mathbb{K}Gw'\simeq M(gz,\sigma_1)$, and the lemma follows. □

Lemma 6.2. *Assume that $\rho(z)\sigma(g)\neq 1$. Then $X_2^{V,W}\neq 0$. Moreover, $X_2^{V,W}$ is absolutely simple if and only if*

$$\rho(g)=-1,\quad (3)_{-\rho(z)\sigma(g)}=0\quad \text{or}\quad \rho(gz)\sigma(g)=1,\quad (3)_{-\rho(g)}=0.$$

In both cases, $X_2^{V,W}\simeq M(\epsilon g^2 z,\sigma_2)$, where σ_2 is the character of G^ϵ with

$$\sigma_2(\epsilon)=\rho(g)(1-\rho(z)\sigma(g)),\qquad \sigma_2(g^2)=\rho(g^4)\sigma(g^2),\qquad \sigma_2(z)=\rho(z^2)\sigma(z).$$

Let $w''=\varphi_2(\epsilon^2 v\otimes w')$. Then a basis of $X_2^{V,W}$ is given by $\{w'',gw''\}$. The degrees of these basis vectors are $\epsilon g^2 z$ and $\epsilon^2 g^2 z$, respectively.

Proof. Write $X_2^{V,W}=\mathbb{K}G\varphi_2(v\otimes w')\oplus\mathbb{K}G\varphi_2(\epsilon^2 v\otimes w')$. Then we compute

$$\begin{aligned}\varphi_2(v\otimes w')&=(\mathrm{id}-c_{X_1^{V,W},V}c_{V,X_1^{V,W}})(v\otimes w')+(\mathrm{id}\otimes\varphi_1)c_{1,2}(v\otimes v\otimes w)\\&=(1+\rho(g))(1-\rho(gz)\sigma(g))v\otimes w',\end{aligned}$$

and using $\varphi_1(\epsilon^2 v\otimes w)=(1-\rho(z)\sigma(g))\epsilon^2 w'$, we find that

$$\begin{aligned}w''&=\varphi_2(\epsilon^2 v\otimes w')\\&=(\mathrm{id}-c_{X_1^{V,W},V}c_{V,X_1^{V,W}})(\epsilon^2 v\otimes w')+(\mathrm{id}\otimes\varphi_1)c_{1,2}(\epsilon^2 v\otimes v\otimes w)\\&=\epsilon^2 v\otimes w'-\rho(g^2 z)\sigma(g)v\otimes\epsilon w'+\rho(g)(1-\rho(z)\sigma(g))\epsilon v\otimes\epsilon^2 w'.\end{aligned}$$

Hence $w'' \in (V \otimes X_1^{V,W})_{\epsilon g^2 z}$ is non-zero. Therefore w'' is absolutely simple if and only if $(1+\rho(g))(1-\rho(gz)\sigma(g)) = 0$ and $\epsilon w'' = \mathbb{K}w''$. Since

$$\epsilon w'' = v \otimes \epsilon w' - \rho(g^2 z)\sigma(g)\epsilon v \otimes \epsilon^2 w' + \rho(g)(1-\rho(z)\sigma(g))\epsilon^2 v \otimes w',$$

in the case $\rho(g) = -1$ we have $\epsilon w'' = \mathbb{K}w''$ if and only if $(3)_{-\rho(z)\sigma(g)} = 0$, and then $\epsilon w'' = (\rho(z)\sigma(g) - 1)w''$. Similarly, if $\rho(gz)\sigma(g) = 1$, then $\epsilon w'' = \mathbb{K}w''$ if and only if $(3)_{-\rho(g)} = 0$, and then $\epsilon w'' = \rho(g)(1-\rho(z)\sigma(g))w''$. The rest is clear. □

Lemma 6.3. *Assume that* $\rho(z)\sigma(g) \neq 1$ *and* $X_2^{V,W}$ *is absolutely simple. Then* $X_3^{V,W} \neq 0$ *if and only if* $\rho(g) \neq -1$. *In this case,* $\operatorname{char}\mathbb{K} \neq 3$, $\rho(gz)\sigma(g) = 1$, $(3)_{-\rho(g)} = 0$, *the Yetter–Drinfeld module* $X_3^{V,W}$ *is absolutely simple, and* $X_3^{V,W} \simeq M(g^3 z, \sigma_3)$, *where*

$$\sigma_3(g) = \sigma(g), \qquad \sigma_3(z) = \rho(z)^3\sigma(z).$$

Proof. Applying ϵg to $w'' = \varphi_2(\epsilon^2 v \otimes w')$ we obtain

$$\varphi_2(\epsilon^2 v \otimes \epsilon w') = \sigma(g)^{-1}(1-\rho(z)\sigma(g))^2 g w''.$$

Further, $\varphi_2(\epsilon^2 v \otimes \epsilon^2 w') = \epsilon^2\varphi_2(v \otimes w') = 0$. A direct calculation using these formulas and the expression for w'' yields

$$\varphi_3(\epsilon^2 v \otimes w'') = (1+\rho(g))\big(\epsilon^2 v \otimes w'' - \rho(g^3 z)(1-\rho(z)\sigma(g))^2 \epsilon v \otimes g w''\big).$$

Thus $X_3^{V,W} = 0$ if and only if $\rho(g) = -1$. Assume now that $\rho(g) \neq -1$. Since $X_2^{V,W}$ is absolutely simple, Lemma 6.2 implies that $\rho(gz)\sigma(g) = 1$ and $(3)_{-\rho(g)} = 0$. Then $\operatorname{char}\mathbb{K} \neq 3$, since otherwise $(1+\rho(g))^2 = 0$, contradicting $\rho(g) \neq -1$. Let $w''' = (1+\rho(g))^{-1}\varphi_3(\epsilon^2 v \otimes w'')$. Then

$$w''' = \epsilon^2 v \otimes w'' + \rho(g^2 z)\epsilon v \otimes g w'',$$

and hence $w''' \in (V \otimes X_2^{V,W})_{g^3 z}$ is non-zero, $X_3^{V,W} = \mathbb{K}Gw'''$, $gw''' = \sigma(g)w'''$, and $zw''' = \rho(z)^3\sigma(z)w'''$. Therefore $X_3^{V,W} \simeq M(g^3 z, \sigma_3)$, where σ_3 is the character of G^g with $\sigma_3(g) = \sigma(g)$ and $\sigma_3(z) = \rho(z)^3\sigma(z)$. □

Lemma 6.4. *Assume that* $\operatorname{char}\mathbb{K} \neq 3$, $\rho(gz)\sigma(g) = 1$, *and* $(3)_{-\rho(g)} = 0$. *Then* $X_4^{V,W} \neq 0$, *and* $X_4^{V,W}$ *is absolutely simple if and only if* $\operatorname{char}\mathbb{K} = 2$. *In this case,* $X_4^{V,W} \simeq M(g^4 z, \sigma_4)$, *where* σ_4 *is the character of* G *given by*

$$\sigma_4(g) = \rho(g)\sigma(g), \qquad \sigma_4(\epsilon) = 1, \qquad \sigma_4(z) = \rho(z)^4\sigma(z),$$

and $X_5^{V,W} = 0$.

Proof. The assumptions imply that $\rho(z)\sigma(g) = \rho(g)^{-1} \neq 1$, and $\rho(g) \neq -1$ since $\operatorname{char}\mathbb{K} \neq 3$. Therefore $X_n^{V,W}$ is absolutely simple for all $n \in \{1,2,3\}$ by Lemmas 6.1–6.3. By looking at the support of $V \otimes X_3^{V,W}$ we also know that $X_4^{V,W} = \mathbb{K}G\varphi_4(\epsilon^2 v \otimes w''') \oplus \mathbb{K}G\varphi_4(v \otimes w''')$.

We first obtain

$$\begin{aligned}\varphi_4(v \otimes w''') = v \otimes w''' - c_{X_3,V}\, c_{V,X_3}(v \otimes w''') + \rho(g)\epsilon v \otimes \varphi_3(v \otimes w'')\\ + \rho(g^3 z)\epsilon^2 v \otimes \varphi_3(v \otimes g w'').\end{aligned}$$

Now apply ϵ and $g\epsilon$ to

$$\varphi_3(\epsilon^2 v \otimes w'') = (1 + \rho(g)) w''' \tag{6.1}$$

to obtain

$$\begin{aligned}\varphi_3(v \otimes w'') &= \rho(g)^{-2}(1+\rho(g))\epsilon w''',\\ \varphi_3(v \otimes g w'') &= \rho(g)^{-3}(1+\rho(g))\sigma(g)\epsilon^2 w''',\end{aligned}$$

respectively. Then

$$\varphi_4(v \otimes w''') = (1 + \rho(g)^{-1})(v \otimes w''' + \epsilon v \otimes \epsilon w''' + \epsilon^2 v \otimes \epsilon^2 w'''), \tag{6.2}$$

and hence $\varphi_4(v \otimes w''') \in (V \otimes X_3^{V,W})_{g^4 z}$ is non-zero.

Apply ϵg to (6.1) and use the fact that $\sigma_2(\epsilon) = \rho(g^2)$. Thus

$$\varphi_3(\epsilon^2 v \otimes g w'') = (1+\rho(g))\rho(g)\sigma(g)\epsilon w'''.$$

Now compute

$$\begin{aligned}\varphi_4(\epsilon^2 v \otimes w''') &= \epsilon^2 v \otimes w''' - \rho(g^2) v \otimes \epsilon w'''\\ &\quad + \rho(g)\epsilon^2 v \otimes \varphi_3(\epsilon^2 v \otimes w'') - \rho(z) v \otimes \varphi_3(\epsilon^2 v \otimes g w'')\\ &= (1 + \rho(g) + \rho(g)^2)(\epsilon^2 v \otimes w''' - v \otimes \epsilon w''').\end{aligned}$$

Thus $X_4^{V,W}$ is absolutely simple if and only if $(3)_{\rho(g)} = 0$ (in which case $\operatorname{char}\mathbb{K} = 2$) and $\mathbb{K}G\varphi_4(v \otimes w''')$ is absolutely simple. Let

$$w'''' = v \otimes w''' + \epsilon v \otimes \epsilon w''' + \epsilon^2 v \otimes \epsilon^2 w'''.$$

Then $\varphi_4(v \otimes w''') = (1 + \rho(g)^{-1}) w''''$ by (6.2), $g w'''' = \rho(g)\sigma(g) w''''$, and hence $X_4^{V,W} \simeq M(g^4 z, \sigma_4)$, where σ_4 is the character of G given by $\sigma_4(\epsilon) = 1$, $\sigma_4(g) = \rho(g)\sigma(g)$, and $\sigma_4(z) = \rho(z)^4\sigma(z)$.

Now we prove that $X_5^{V,W} = 0$. Observe that

$$X_5^{V,W} = \varphi_5(V \otimes X_4^{V,W}) = \mathbb{K}G\varphi_5(v \otimes w'''').$$

By applying ϵ and $g\epsilon$ to $\varphi_4(\epsilon^2 v \otimes w''') = 0$ we deduce that $\varphi_4(v \otimes \epsilon w''') = 0$ and $\varphi_4(v \otimes \epsilon^2 w''') = 0$, respectively. A direct calculation yields

$$\varphi_5(v \otimes w'''') = (1 + \rho(g))v \otimes w'''' + \rho(g)v \otimes \varphi_4(v \otimes w''') = 0,$$

which proves the claim. □

Now we compute the modules $(\operatorname{ad} W)^m(V)$ for $m \geq 1$. As before, we write $\varphi_n = \varphi_n^{W,V}$.

Lemma 6.5. *Let $y_0 = v$ and let $y_n = w \otimes y_{n-1}$ for all $n \geq 1$. Then $y_n \in (W^{\otimes n} \otimes V)_{gz^n}$ and $\mathbb{K}Gy_n \simeq M(gz^n, \rho_n)$, where ρ_n is the character of G^g given by*

$$\rho_n(g) = \rho(g)\sigma(g)^n, \qquad \rho_n(z) = \rho(z)\sigma(z)^n,$$

and $\varphi_n(w \otimes y_{n-1}) = \gamma_n y_n$ for all $n \in \mathbb{N}$, where

$$\gamma_n = (n)_{\sigma(z)}(1 - \rho(z)\sigma(gz^{n-1})).$$

Moreover, $X_n^{W,V} \simeq \mathbb{K}G((\gamma_1 \cdots \gamma_n)y_n)$ for all $n \in \mathbb{N}_0$.

Proof. We prove by induction on n that $\varphi_n(w \otimes y_{n-1}) = \gamma_n y_n$ for all $n \geq 1$. The remaining claims are then easily shown.

It is clear that $\varphi_1(w \otimes v) = w \otimes v - gw \otimes zv = (1 - \rho(z)\sigma(g))y_1$. Let now $n \geq 1$. Then

$$\varphi_{n+1}(w \otimes y_n) = w \otimes y_n - gz^n w \otimes zy_n + \sigma(z)w \otimes \varphi_n(w \otimes y_{n-1}),$$

and hence the induction hypothesis implies that

$$\varphi_{n+1}(w \otimes y_n) = (n+1)_{\sigma(z)}(1 - \rho(z)\sigma(gz^n))y_{n+1}.$$

This proves the claimed formula. □

Proposition 6.6. *Let $V, W \in {}^G_G\mathcal{YD}$ be such that $V \simeq M(g, \rho)$, where ρ is a character of G^g, and $W \simeq M(z, \sigma)$, where σ is a character of G. Then:*

(1) $(\operatorname{ad} V)^m(W)$ *and* $(\operatorname{ad} W)^m(V)$ *are absolutely simple or zero for all $m \in \mathbb{N}_0$ if and only if*
   (a) $\rho(z)\sigma(g) = 1$, *or*
   (b) $\rho(g) = -1$ *and* $(3)_{-\rho(z)\sigma(g)} = 0$, *or*
   (c) $\rho(gz)\sigma(g) = 1$, $(3)_{\rho(g)} = 0$, *and* $\operatorname{char} \mathbb{K} = 2$.

(2) *Assume that in one of* (1a), (1b), (1c) *holds. Then the Cartan matrix of the pair (V, W) satisfies*

$$a_{1,2}^{(V,W)} = \begin{cases} 0 & \textit{in case } (1a), \\ -2 & \textit{in case } (1b), \\ -4 & \textit{in case } (1c). \end{cases}$$

Moreover,

   (a) *if $a_{1,2}^{(V,W)} = -2$ then $X_2^{V,W} \simeq M(\epsilon g^2 z, \sigma_2)$, where σ_2 is the character of G^ϵ given by $\sigma_2(\epsilon) = -\rho(z)^{-1}\sigma(g)^{-1}$, $\sigma_2(z) = \rho(z)^2\sigma(z)$ and $\sigma_2(g^2) = \sigma(g^2)$,*

(b) *if* $a_{1,2}^{(V,W)} = -4$ *then* $X_4^{V,W} \simeq M(g^4z, \sigma_4)$*, where* σ_4 *is the character of* G *given by* $\sigma_4(g) = \rho(g)\sigma(g)$*,* $\sigma_4(\epsilon) = 1$*, and* $\sigma_4(z) = \rho(z)^4\sigma(z)$*.*

(3) *Assume that in one of* (1a), (1b), (1c) *holds and let* $m \in \mathbb{N}_0$*. Then* $a_{2,1}^{(V,W)} = -m$ *if and only if* $\gamma_{m+1} = 0$ *and* $\gamma_1 \cdots \gamma_m \neq 0$*, where*

$$\gamma_k = (k)_{\sigma(z)}(1 - \rho(z)\sigma(gz^{k-1}))$$

for all $k \in \mathbb{N}_0$*. In this case,* $X_m^{W,V} \simeq M(gz^m, \rho_m)$*, where* ρ_m *is the character of* G^g *given by*

$$\rho_m(g) = \rho(g)\sigma(g)^m, \qquad \rho_m(z) = \rho(z)\sigma(z)^m.$$

Proof. By Lemmas 6.1–6.4, $(\operatorname{ad} V)^m(W)$ is absolutely simple or zero for all $m \in \mathbb{N}$ if and only if $\rho(z)\sigma(g) = 1$ or $\rho(g) = -1$, $(3)_{-\rho(z)\sigma(g)} = 0$, or $\rho(gz)\sigma(g) = 1$, $(3)_{-\rho(g)} = 0$, $\operatorname{char}\mathbb{K} \neq 3$, $\operatorname{char}\mathbb{K} = 2$. By Lemma 6.5 the Yetter–Drinfeld modules $(\operatorname{ad} W)^m(V)$ for $m \geq 0$ are absolutely simple or zero. This proves (1). Then (2) is easy to get from the same lemmas. □

Corollary 6.7. *Let* $V, W \in {}^G_G\mathcal{YD}$ *be such that* $V \simeq M(g, \rho)$*, where* ρ *is a character of* G^g*, and* $W \simeq M(z, \sigma)$*, where* σ *is a character of* G*, and*

$$\rho(g) = -1, \qquad \rho(z)\sigma(gz) = 1, \qquad 1 - \rho(z)\sigma(g) + \rho(z^2)\sigma(g^2) = 0.$$

Further, let $g' = g^{-1}$*,* $\epsilon' = \epsilon$*,* $z' = g^2z$*, let* ρ' *be the character of* G^g *dual to* ρ*, and let* σ' *the character of* G^ϵ *given by* $\sigma'(\epsilon) = -\rho(z)^{-1}\sigma(g)^{-1}$*,* $\sigma'(z) = \rho(z)^2\sigma(z)$*,* $\sigma'(g^2) = \sigma(g^2)$*. Then* $a_{1,2}^{(V,W)} = -2$ *and*

$$R_1(V, W) = (V^*, X_2^{V,W})$$

with $V^* \simeq M(g', \rho')$*,* $X_2^{V,W} \simeq M(\epsilon'z', \sigma')$*, and*

$$\rho'(g') = \sigma'(\epsilon'z') = -1, \qquad \rho'(z')^2\sigma'(\epsilon'g'^2) = 1, \qquad 1 + \sigma'(\epsilon') + \sigma'(\epsilon')^2 = 0.$$

Proof. Using Proposition 6.6 one obtains $a_{1,2}^{(V,W)} = -2$. The rest of the proof is similar to the proof of Corollary 4.13. □

Corollary 6.8. *Let* $V, W \in {}^G_G\mathcal{YD}$ *be such that* $V \simeq M(g, \rho)$*, where* ρ *is a character of* G^g*, and* $W \simeq M(z, \sigma)$*, where* σ *is a character of* G*, and*

$$\rho(g) = -1, \qquad \rho(z)\sigma(gz) = 1, \qquad 1 - \rho(z)\sigma(g) + \rho(z^2)\sigma(g^2) = 0.$$

Let $g'' = gz$*.* $\epsilon'' = \epsilon^{-1}$*,* $z'' = z^{-1}$*, let* ρ'' *be the character of* G^g *given by* $\rho''(g) = -\sigma(g)$*,* $\rho''(z) = \rho(z)\sigma(z)$*, and let* σ'' *be the character of* G *dual to* σ*. Then* $a_{2,1}^{(V,W)} = -1$ *and*

$$R_2(V, W) = (X_1^{W,V}, W^*)$$

with $X_1^{W,V} \simeq M(g'', \rho'')$*,* $W^* \simeq M(z'', \sigma'')$*, and*

$$\rho''(g'') = -1, \qquad \rho''(z'')\sigma''(g''z'') = 1, \qquad (3)_{-\rho''(z'')\sigma''(g'')} = 0.$$

Proof. Since $\rho(z)\sigma(g) \neq 1$, Proposition 6.6 implies that $a_{2,1}^{(V,W)} = -1$. The rest of the proof is similar to the proof of Corollary 4.14. □

7. Computing the reflections

7.1. Pairs of Yetter–Drinfeld modules

Let G be a non-abelian epimorphic image of Γ_3. We first identify pairs (V, W) of Yetter–Drinfeld modules over G with a Cartan matrix of finite type. More precisely, we define classes $\wp_i$, where $1 \le i \le 6$, of pairs of absolutely simple Yetter–Drinfeld modules over G such that all pairs in these classes can be treated simultaneously with respect to reflections.

Definition 7.1 (The classes $\wp_i$ for $1 \le i \le 6$ of pairs of Yetter–Drinfeld modules). Let V and W be Yetter–Drinfeld modules over G and let $i \in \mathbb{N}$ with $1 \le i \le 6$. We say that $(V, W) \in \wp_i$ if there exist $g, \epsilon, z \in G$ such that:

(1) There is a group epimorphism

$$\Gamma_3 \to G, \quad \gamma \mapsto g,\ \nu \mapsto \epsilon,\ \zeta \mapsto z.$$

(2) $V \simeq M(g, \rho)$ for some character ρ of G, and either $W \simeq M(\epsilon z, \sigma)$ for some character σ of G^ϵ or $W \simeq M(z, \sigma)$ for some absolutely irreducible representation σ of G.

(3) W, ρ, σ, and $\mathbb{K}$ satisfy the conditions in the i-th row of Table 2.

Table 2. Conditions on the classes $\wp_i$, $1 \le i \le 6$

	$[W]$	Conditions on ρ and σ	char $\mathbb{K}$
$\wp_1$	$M(\epsilon z, \sigma)$	$\rho(g) = \sigma(\epsilon z) = -1$, $\rho(z^2)\sigma(\epsilon g^2) = 1$, $(3)_{\sigma(\epsilon)} = 0$	
$\wp_2$	$M(\epsilon z, \sigma)$	$\rho(g) = \sigma(z) = -1$, $\rho(z^2)\sigma(\epsilon g^2) = \sigma(\epsilon)=1$	$\neq 3$
$\wp_3$	$M(z, \sigma)$	$\rho(g) = \sigma(z) = -1$, $\rho(z^2)\sigma(g^2) = 1$, $\deg \sigma = 2$	$\neq 3$
$\wp_4$	$M(z, \sigma)$	$\rho(g) = -1$, $\rho(z)\sigma(gz) = 1$, $(3)_{-\rho(z)\sigma(g)} = 0$, $\deg \sigma = 1$	
$\wp_5$	$M(z, \sigma)$	$\rho(g) = \sigma(z) = 1$, $(3)_{\rho(z)\sigma(g)} = 0$, $\deg \sigma = 1$	2
$\wp_6$	$M(z, \sigma)$	$\rho(g) = \sigma(z) = -1$, $(3)_{-\rho(z)\sigma(g)} = 0$, $\deg \sigma = 1$	$\neq 2, 3$

Proposition 7.2. *Let V, W be absolutely simple Yetter–Drinfeld modules over G. Suppose that $c_{W,V}c_{V,W} \neq \mathrm{id}_{V\otimes W}$, the pair (V, W) admits all reflections, the Weyl groupoid of (V, W) is finite, and $|\mathrm{supp}\, V| \ge |\mathrm{supp}\, W|$. Then the Cartan matrix of (V, W) is of finite type if and only if (V, W) belongs to one of the classes $\wp_i$ for $1 \le i \le 5$ in Table 2. In this case $a_{1,2}^{(V,W)} = -2$ and $a_{2,1}^{(V,W)} = -1$.*

Proof. From [25, Thm. 7.3] we know that the quandle $\mathrm{supp}(V \oplus W)$ is isomorphic to $Z_3^{3,1}$ or $Z_3^{3,2}$. Since $|\mathrm{supp}\, V| \ge |\mathrm{supp}\, W|$, we conclude that $|\mathrm{supp}\, V| = 3$ and $|\mathrm{supp}\, W| \le 2$.

Let $g, g' \in \operatorname{supp} V$ with $g' \neq g$ and let $\epsilon = g'^{-1}g$. Then $\epsilon \neq 1$ and $\epsilon^3 = 1$ by (3.1). Since V is absolutely simple, there exists a character ρ of G such that $V \simeq M(g, \rho)$. Since W is absolutely simple and $\operatorname{supp}(V \oplus W)$ generates G, there exists $z \in Z(G)$ such that either $W \simeq M(\epsilon z, \sigma)$ for a character σ of G^ϵ, or $W \simeq M(z, \sigma)$ for some absolutely irreducible representation σ of G. Then $\deg \sigma \leq 2$ by Lemma 3.2.

Since the pair (V, W) admits all reflections and the Weyl groupoid of (V, W) is finite, we conclude from [22, Theorem 2.5] that $(\operatorname{ad} V)^m(W)$ and $(\operatorname{ad} W)^m(V)$ are absolutely simple or zero for all $m \geq 1$. The condition $c_{W,V}c_{V,W} \neq \mathrm{id}_{V\otimes W}$ just means that the Cartan matrix of (V, W) is not of type $A_1 \times A_1$. Propositions 4.12, 5.9, and 6.6 imply that the Cartan matrix of (V, W) is of finite type if and only if (V, W) belongs to one of the classes $\wp_i$ for $1 \leq i \leq 6$ in Table 2. Moreover, in this case $a_{1,2}^{(V,W)} = -2$ and $a_{2,1}^{(V,W)} = -1$.

We have to exclude the class $\wp_6$. To do so, it suffices to prove that if $(V, W) \in \wp_6$ then not all assumptions of the proposition are fulfilled.

Assume that (V, W) belongs to $\wp_6$. By Proposition 6.6,

$$R_2(V, W) = ((\operatorname{ad} W)(V), W^*) \simeq (M(gz, \rho_1), M(z^{-1}, \sigma^{-1})),$$

where ρ_1 is the character of G^g with $\rho_1(g) = \rho(g)\sigma(g)$ and $\rho_1(z) = \rho(z)\sigma(z)$. Let $(V', W') = R_2(V, W)$. We find that $\rho_1(gz) = \rho(z)\sigma(g) \notin \{1, -1\}$, since $(3)_{-\rho(z)\sigma(g)} = 0$ and $\operatorname{char} \mathbb{K} \neq 3$. Since also $\operatorname{char} \mathbb{K} \neq 2$ by assumption on $\wp_6$, Proposition 6.6 implies that not all of $(\operatorname{ad} V')^m(W')$ and $(\operatorname{ad} W')^m(V')$ for $m \geq 1$ are absolutely simple or zero. This contradicts the assumption that (V, W) admits all reflections and the Weyl groupoid of (V, W) is finite. □

Let (V, W) be a pair of Yetter–Drinfeld modules over G. If $(V, W) \in \wp_1$, then we have $R_1(V, W) \in \wp_4$ and $R_2(V, W) \in \wp_1$ by Corollaries 4.13 and 4.14. On the other hand, if $(V, W) \in \wp_4$, then $R_1(V, W) \in \wp_1$ and $R_2(V, W) \in \wp_4$ by Corollaries 6.7 and 6.8. We display this fact in the following graph:

$$\wp_1 \overset{R_1}{\text{———}} \wp_4 \tag{7.1}$$

We omit R_2 in the graph, since it fixes the classes $\wp_1$ and $\wp_4$. Since the Cartan matrices of (V, W) are the same for all reflections of (V, W) in the classes $\wp_1$ and $\wp_4$, we infer the following.

Lemma 7.3. *Let (V, W) be a pair in $\wp_1$ or $\wp_4$. Then (V, W) admits all reflections and the Weyl groupoid of (V, W) is standard of type B_2.*

Similarly to the previous paragraph, Corollaries 4.15, 4.16, 5.10, and 5.11 imply that the reflections of the pairs (V, W) in $\wp_2$ and $\wp_3$ can be displayed in the following graph:

$$\wp_2 \overset{R_1}{\text{———}} \wp_3 \tag{7.2}$$

Lemma 7.4. *Let (V, W) be a pair in $\wp_2$ or $\wp_3$. Then (V, W) admits all reflections and the Weyl groupoid of (V, W) is standard of type B_2.*

The reflections of the pairs $(V, W) \in \wp_5$ show a more complicated pattern.

Let us assume that char $\mathbb{K} = 2$. Lemmas 7.5–7.10 below imply that the reflections of the pairs (V, W) in $\wp_5$ can be displayed in the following graph:

$$\wp_5' \overset{R_1}{\text{———}} \wp_5 \overset{R_2}{\text{———}} \wp_5'' \tag{7.3}$$

The classes $\wp_5'$ and $\wp_5''$ are defined in the same way as $\wp_5$ in Definition 7.1, except that in the last line one refers to the conditions in Table 3. Since the class $\wp_5$ is non-empty only if char $\mathbb{K} = 2$, in the definitions of $\wp_5'$ and $\wp_5''$ we also assume that char $\mathbb{K} = 2$.

Table 3. Conditions on the classes $\wp_5, \wp_5', \wp_5''$

	$[W]$	Conditions on ρ and σ
$\wp_5$	$M(z,\sigma)$	$\rho(g) = \sigma(z) = 1,\ (3)_{\rho(z)\sigma(g)} = 0,\ \deg\sigma = 1$
$\wp_5'$	$M(\epsilon z,\sigma)$	$(3)_{\sigma(\epsilon)} = 0,\ \sigma(z) = \sigma(\epsilon),\ \rho(z^2)\sigma(\epsilon g^2) = 1,\ \rho(g) = 1$
$\wp_5''$	$M(z,\sigma)$	$(3)_{\rho(g)} = 0,\ \sigma(z) = 1,\ \rho(gz)\sigma(g) = 1,\ \deg\sigma = 1$

Lemma 7.5. *Let $(V, W) \in \wp_5$ and $g, \epsilon, z, \rho, \sigma$ be as in Definition 7.1 such that $V \simeq M(g, \rho)$ and $W \simeq M(z, \sigma)$. Let $g' = g^{-1}$, $\epsilon' = \epsilon$, $z' = g^2 z$, let ρ' be the character of G^g dual to ρ, and let σ' be the character of G^ϵ given by $\sigma'(\epsilon) = -\rho(z)^{-1}\sigma(g)^{-1}$, $\sigma'(z) = \rho(z)^2$, $\sigma'(g^2) = \sigma(g^2)$. Then $a_{1,2}^{(V,W)} = -2$ and*

$$R_1(V, W) = (V^*, X_2^{V,W})$$

with $V^* \simeq M(g', \rho')$, $X_2^{V,W} \simeq M(\epsilon' z', \sigma')$, and

$$\rho'(g') = 1, \quad \rho'(z'^2)\sigma'(\epsilon' g'^2) = 1, \quad \sigma'(z') = \sigma'(\epsilon'), \quad (3)_{\sigma'(\epsilon')} = 0.$$

In particular, $R_1(V, W) \in \wp_5'$.

Proof. Using Proposition 6.6 we obtain $a_{1,2}^{(V,W)} = -2$, and hence the description of $R_1(V, W)$ together with the isomorphisms regarding V^* and $X_2^{V,W}$ follows from the same proposition. The remaining claims are easy to check. □

Lemma 7.6. *Let $(V, W) \in \wp_5'$ and $g, \epsilon, z, \rho, \sigma$ be as in Definition 7.1 such that $V \simeq M(g, \rho)$ and $W \simeq M(\epsilon z, \sigma)$. Let $g' = g^{-1}$, $\epsilon' = \epsilon$, $z' = g^2 z$, let ρ' be the character of G^g dual to ρ, and let σ' be the character of G given by $\sigma'(\epsilon) = 1$, $\sigma'(g) = \rho(z)^{-1}\sigma(\epsilon)$, $\sigma'(z) = \rho(z)^2\sigma(z)$. Then $a_{1,2}^{(V,W)} = -2$ and*

$$R_1(V, W) = (V^*, X_2^{V,W})$$

with $V^* \simeq M(g', \rho')$, $X_2^{V,W} \simeq M(z', \sigma')$, and

$$\rho'(g') = \sigma'(z') = 1, \quad 1 + \rho'(z')\sigma'(g') + (\rho'(z')\sigma'(g'))^2 = 0.$$

In particular, $R_1(V, W) \in \wp_5$.

Proof. It is similar to the proof of Lemma 7.5, but one needs Proposition 4.12. □

Lemma 7.7. *Let* $(V, W) \in \wp_5''$ *and* $g, \epsilon, z, \rho, \sigma$ *be as in Definition 7.1 such that* $V \simeq M(g, \rho)$ *and* $W \simeq M(z, \sigma)$. *Let* $g' = g^{-1}$, $\epsilon' = \epsilon$, $z' = g^4 z$, *let* ρ' *be the character of* G^g *dual to* ρ, *and let* σ' *be the character of* G *given by* $\sigma'(g) = \rho(g)\sigma(g)$, $\sigma'(z) = \rho(z)^4\sigma(z)$, $\sigma'(\epsilon) = 1$. *Then* $a_{1,2}^{(V,W)} = -4$ *and*

$$R_1(V, W) = (V^*, X_4^{V,W})$$

with $V^* \simeq M(g', \rho')$, $X_4^{V,W} \simeq M(z', \sigma')$, *and*

$$\rho'(g'z')\sigma'(g') = 1, \qquad 1 + \rho'(g') + \rho'(g')^2 = 0, \qquad \sigma'(z') = 1.$$

In particular, $R_1(V, W) \in \wp_5''$.

Proof. It is similar to the proof of Lemma 7.5. □

Lemma 7.8. *Let* $(V, W) \in \wp_5$ *and* $g, \epsilon, z, \rho, \sigma$ *be as in Definition 7.1 such that* $V \simeq M(g, \rho)$ *and* $W \simeq M(z, \sigma)$. *Let* $g'' = gz$, $\epsilon'' = \epsilon^{-1}$, $z'' = z^{-1}$, *let* ρ'' *be the character of* G^g *given by* $\rho''(g) = \sigma(g)$, $\rho''(z) = \rho(z)$, *and let* σ'' *be the character of* G *dual to* σ. *Then* $a_{2,1}^{V,W} = -1$ *and*

$$R_2(V, W) = (X_1^{W,V}, W^*)$$

with $X_1^{W,V} \simeq M(g'', \rho'')$, $W^* \simeq M(z'', \sigma'')$, *and*

$$\rho''(g''z'')\sigma''(g'') = 1, \qquad 1 + \rho''(g'') + \rho''(g'')^2 = 0, \qquad \sigma''(z'') = 1.$$

In particular, $R_2(V, W) \in \wp_5''$.

Proof. It is similar to the proof of Lemma 7.5. □

Lemma 7.9. *Let* $(V, W) \in \wp_5'$ *and* $g, \epsilon, z, \rho, \sigma$ *be as in Definition 7.1 such that* $V \simeq M(g, \rho)$ *and* $W \simeq M(\epsilon z, \sigma)$. *Let* $g'' = gz^2$, $\epsilon'' = \epsilon^{-1}$, $z'' = z^{-1}$, *let* ρ'' *be the character of* G^g *given by* $\rho''(g) = \rho(z)^{-2}\sigma(z)^{-1}$, $\rho''(z) = \rho(z)\sigma(z)^2$, *and let* σ'' *be the character of* G *dual to* σ. *Then* $a_{2,1}^{(V,W)} = -2$ *and*

$$R_2(V, W) = (X_2^{W,V}, W^*)$$

with $X_2^{W,V} \simeq M(g'', \rho'')$, $W^* \simeq M(z'', \sigma'')$, *and*

$$\rho''(g'') = 1, \qquad \rho''(z''^2)\sigma''(\epsilon'' g''^2) = 1, \qquad \sigma''(z'') = \sigma''(\epsilon''), \qquad (3)_{\sigma''(\epsilon'')} = 0.$$

In particular, $R_2(V, W) \in \wp_5'$.

Proof. It is similar to the proof of Lemma 7.6. □

Lemma 7.10. *Let $(V,W)\in\wp_5''$ and g,ϵ,z,ρ,σ be as in Definition 7.1 such that $V\simeq M(g,\rho)$ and $W\simeq M(z,\sigma)$. Let $g''=gz$, $\epsilon''=\epsilon^{-1}$, $z''=z^{-1}$, let ρ'' be the character of G^g given by $\rho''(g)=\rho(g)\sigma(g)$, $\rho''(z)=\rho(z)\sigma(z)$, and let σ'' be the character of G dual to σ. Then $a_{2,1}^{(V,W)}=-1$ and*

$$R_2(V,W)=(X_1^{W,V},W^*)$$

with $X_1^{W,V}\simeq M(g'',\rho'')$, $W^*\simeq M(z'',\sigma'')$, and

$$\rho''(g'')=\sigma''(z'')=1,\qquad (3)_{\rho''(z'')\sigma''(g'')}=0.$$

In particular, $R_2(V,W)\in\wp_5$.

Proof. It is similar to the proof of Lemma 7.5. □

8. Nichols algebras over Γ_3

8.1. Simple Yetter–Drinfeld modules

In this section, let G be a non-abelian epimorphic image of Γ_3 and let $g,\epsilon,z\in G$ be such that the group epimorphism $\Gamma_3\to G$ maps $\gamma\mapsto g$, $\nu\mapsto\epsilon$ and $\zeta\mapsto z$.

Let $P=\{0\}\cup\{p\in\mathbb{N}\mid p \text{ is prime}\}$. Let $h_2=3$, $h_3=2$ and $h_p=6$ for all $p\in P\setminus\{2,3\}$, and $h_3'=2$, $h_p'=6$ for all $p\in P\setminus\{3\}$. For all $1\le i\le 8$ let $\mathcal{Y}_i$ be the class of Yetter–Drinfeld modules U over G such that there exist $x,g,\epsilon,z\in G$ and an absolutely irreducible representation τ of G^x such that $U\simeq M(x,\tau)$ and x, τ, and $\mathbb{K}$ satisfy the conditions in Table 4. In Table 4 we also provide the Hilbert series of the Nichols algebras of the Yetter–Drinfeld modules in the classes $\mathcal{Y}_i$ for all $1\le i\le 8$ and the references to these Hilbert series.

Table 4. Classes of Yetter–Drinfeld modules

	x	τ	char $\mathbb{K}$	Hilbert series	Ref.
$\mathcal{Y}_1$	g	$\tau(g)=-1$		$(2)_t^2(3)_t$	[29]
$\mathcal{Y}_2$	g	$(3)_{\tau(g)}=0$	2	$(3)_t(4)_t(6)_t(6)_{t^2}$	[21, Prop. 32]
$\mathcal{Y}_3$	z	$\deg\tau=1$, $\tau(z)=-1$		$(2)_t$	[31, §3.4]
$\mathcal{Y}_4$	z	$\deg\tau=1$, $(3)_{-\tau(z)}=0$	$p\in P$	$(h_p)_t$	[31, §3.4]
$\mathcal{Y}_5$	z	$\deg\tau=2$, $\tau(z)=-1$		$(2)_t^2$	[31, §3.4]
$\mathcal{Y}_6$	ϵz	$\tau(z)=\tau(\epsilon)$, $(3)_{\tau(\epsilon)}=0$		$(3)_t^2$	[31, §3.4]
$\mathcal{Y}_7$	ϵz	$\tau(\epsilon)=1$, $\tau(z)=-1$		$(2)_t^2$	[31, §3.4]
$\mathcal{Y}_8$	ϵz	$\tau(\epsilon z)=-1$, $(3)_{\tau(\epsilon)}=0$	$p\in P$	$(2)_t(h_p')_t$	[17, Prop. 2.11]

8.2. Nichols algebras related to $\wp_1 \xrightarrow{R_1} \wp_4$

Theorem 8.1. *Let $p=\operatorname{char}\mathbb{K}$. Let $V,W\in{}^G_G\mathcal{YD}$ be such that $V\simeq M(g,\rho)$, where ρ is a character of G^g, and $W\simeq M(\epsilon z,\sigma)$, where σ is a character of G^ϵ. Assume that*

$(V,W)\in\wp_1$, that is,

$$\rho(g)=-1,\quad \rho(z^2)\sigma(\epsilon g^2)=1,\quad 1+\sigma(\epsilon)+\sigma(\epsilon)^2=0,\quad \sigma(\epsilon z)=-1.$$

Then $W\in\mathcal{Y}_8$, $(\operatorname{ad}V)(W)\in\mathcal{Y}_1$, $(\operatorname{ad}V)^2(W)\in\mathcal{Y}_4$, $V\in\mathcal{Y}_1$, and

$$\mathfrak{B}(V\oplus W)\simeq\mathfrak{B}(W)\otimes\mathfrak{B}((\operatorname{ad}V)(W))\otimes\mathfrak{B}((\operatorname{ad}V)^2(W))\otimes\mathfrak{B}(V)$$

as $\mathbb{N}_0^2$-graded vector spaces in ${}^G_G\mathcal{YD}$. In particular, the Hilbert series of $\mathfrak{B}(V\oplus W)$ is

$$\mathcal{H}(t_1,t_2)=(2)_{t_2}(h'_p)_{t_2}(2)^2_{t_1t_2}(3)_{t_1t_2}(h_p)_{t_1^2t_2}(2)^2_{t_1}(3)_{t_1}$$

and

$$\dim\mathfrak{B}(V\oplus W)=\begin{cases}10368 & \text{if } \operatorname{char}\mathbb{K}\notin\{2,3\},\\ 5184 & \text{if } \operatorname{char}\mathbb{K}=2,\\ 1152 & \text{if } \operatorname{char}\mathbb{K}=3.\end{cases}$$

Proof. The Cartan scheme of (V,W) is standard of type B_2 by Lemma 7.3. The Yetter–Drinfeld modules $(\operatorname{ad}V)(W)$ and $(\operatorname{ad}V)^2(W)$ are in the claimed classes because of Lemmas 4.3 and 4.6. Then the claims concerning the decomposition and the Hilbert series of $\mathfrak{B}(V\oplus W)$ follow from [22, Cor. 2.7(2) and Thm. 2.6] and Table 4. □

Theorem 8.2. *Let $p=\operatorname{char}\mathbb{K}$. Let $V,W\in{}^G_G\mathcal{YD}$ be such that $V\simeq M(g,\rho)$, where ρ is a character of G^g, and $W\simeq M(z,\sigma)$, where σ is a character of G. Assume that $(V,W)\in\wp_4$, that is,*

$$\rho(g)=-1,\quad \rho(z)\sigma(gz)=1,\quad 1-\rho(z)\sigma(g)+(\rho(z)\sigma(g))^2=0.$$

Then $W\in\mathcal{Y}_4$, $(\operatorname{ad}V)(W)\in\mathcal{Y}_1$, $(\operatorname{ad}V)^2(W)\in\mathcal{Y}_8$, $V\in\mathcal{Y}_1$, and

$$\mathfrak{B}(V\oplus W)\simeq\mathfrak{B}(W)\otimes\mathfrak{B}((\operatorname{ad}V)(W))\otimes\mathfrak{B}((\operatorname{ad}V)^2(W))\otimes\mathfrak{B}(V)$$

as $\mathbb{N}_0^2$-graded vector spaces in ${}^G_G\mathcal{YD}$. In particular, the Hilbert series of $\mathfrak{B}(V\oplus W)$ is

$$\mathcal{H}(t_1,t_2)=(h_p)_{t_2}(2)^2_{t_1t_2}(3)_{t_1t_2}(2)_{t_1^2t_2}(h'_p)_{t_1^2t_2}(2)^2_{t_1}(3)_{t_1}$$

and

$$\dim\mathfrak{B}(V\oplus W)=\begin{cases}10368 & \text{if } \operatorname{char}\mathbb{K}\notin\{2,3\},\\ 5184 & \text{if } \operatorname{char}\mathbb{K}=2,\\ 1152 & \text{if } \operatorname{char}\mathbb{K}=3.\end{cases}$$

Proof. It is similar to the proof of Theorem 8.1. For the structure of $(\operatorname{ad}V)^m(W)$ for $m=1,2$ see Lemmas 6.1 and 6.2. Alternatively, one can deduce the claim from Theorem 8.1 since $R_1(V,W)\in\wp_1$ and $R_1^2(V,W)\simeq(V,W)$. □

8.3. Nichols algebras related to $\wp_2\xrightarrow{R_1}\wp_3$

Theorem 8.3. *Let $V,W\in{}^G_G\mathcal{YD}$ be such that $V\simeq M(g,\rho)$, where ρ is a character of G^g, and $W\simeq M(\epsilon z,\sigma)$, where σ is a character of G^ϵ. Assume that $\operatorname{char}\mathbb{K}\neq 3$ and*

$(V, W) \in \wp_2$, *that is,*

$$\rho(g) = \sigma(z) = -1, \qquad \rho(z^2)\sigma(\epsilon g^2) = \sigma(\epsilon) = 1.$$

Then $W \in \mathcal{Y}_7$, $(\operatorname{ad} V)(W) \in \mathcal{Y}_1$, $(\operatorname{ad} V)^2(W) \in \mathcal{Y}_5$, $V \in \mathcal{Y}_1$, *and*

$$\mathfrak{B}(V \oplus W) \simeq \mathfrak{B}(W) \otimes \mathfrak{B}((\operatorname{ad} V)(W)) \otimes \mathfrak{B}((\operatorname{ad} V)^2(W)) \otimes \mathfrak{B}(V)$$

as $\mathbb{N}_0^2$*-graded vector spaces in* ${}_G^G\mathcal{YD}$*. In particular, the Hilbert series of* $\mathfrak{B}(V \oplus W)$ *is*

$$\mathcal{H}(t_1, t_2) = (2)_{t_2}^2 (2)_{t_1 t_2}^2 (3)_{t_1 t_2} (2)_{t_1^2 t_2}^2 (2)_{t_1}^2 (3)_{t_1}$$

and $\dim \mathfrak{B}(V \oplus W) = 2^8 3^2 = 2304$.

Proof. The Cartan scheme of (V, W) is standard of type B_2 by Lemma 7.4 and the decomposition of $\mathfrak{B}(V \oplus W)$ follows from [22, Cor. 2.7(2) and Thm. 2.6]. It is clear that $V \in \mathcal{Y}_1$ and $W \in \mathcal{Y}_7$. Using Lemmas 4.3 and 4.6 we find that $(\operatorname{ad} V)(W) \in \mathcal{Y}_1$ and $(\operatorname{ad} V)^2(W) \in \mathcal{Y}_5$. Now a direct calculation using Table 4 yields the Hilbert series of $\mathfrak{B}(V \oplus W)$. □

Theorem 8.4. *Let* $V, W \in {}_G^G\mathcal{YD}$ *be such that* $V \simeq M(g, \rho)$*, where* ρ *is a character of* G^g*, and* $W \simeq M(z, \sigma)$*, where* σ *is an absolutely irreducible representation of* G *of degree two. Assume that* $\operatorname{char} \mathbb{K} \neq 3$ *and* $(V, W) \in \wp_3$*, that is,*

$$\rho(g) = \sigma(z) = -1, \qquad \rho(z^2)\sigma(g^2) = 1.$$

Then $W \in \mathcal{Y}_5$, $(\operatorname{ad} V)(W) \in \mathcal{Y}_1$, $(\operatorname{ad} V)^2(W) \in \mathcal{Y}_7$, $V \in \mathcal{Y}_1$, *and*

$$\mathfrak{B}(V \oplus W) \simeq \mathfrak{B}(W) \otimes \mathfrak{B}((\operatorname{ad} V)(W)) \otimes \mathfrak{B}((\operatorname{ad} V)^2(W)) \otimes \mathfrak{B}(V)$$

as $\mathbb{N}_0^2$*-graded vector spaces in* ${}_G^G\mathcal{YD}$*. In particular, the Hilbert series of* $\mathfrak{B}(V \oplus W)$ *is*

$$\mathcal{H}(t_1, t_2) = (2)_{t_2}^2 (2)_{t_1 t_2}^2 (3)_{t_1 t_2} (2)_{t_1^2 t_2}^2 (2)_{t_1}^2 (3)_{t_1}$$

and $\dim \mathfrak{B}(V \oplus W) = 2^8 3^2 = 2304$.

Proof. It is similar to the proof of Theorem 8.3, with the use of Lemmas 5.2 and 5.4. Alternatively, one can deduce the claim from Theorem 8.3 since $R_1(V, W) \in \wp_2$ and $R_1^2(V, W) \simeq (V, W)$. □

8.4. Examples related to $\wp_5' \xrightarrow{R_1} \wp_5 \xrightarrow{R_2} \wp_5''$

Lemma 8.5. *Let* $I = \{1, 2\}$, $\mathcal{X} = \{a, b, c\}$, $r_1 = (a\,b) \in \mathbb{S}_{\mathcal{X}}$, $r_2 = (b\,c) \in \mathbb{S}_{\mathcal{X}}$,

$$A^a = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \quad A^b = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}, \quad A^c = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}.$$

Then:

(1) $\mathcal{C} = \mathcal{C}(I, \mathcal{X}, (r_i)_{i\in I}, (A^X)_{X\in\mathcal{X}})$ *is a Cartan scheme.*
(2) $(\Delta^{\mathrm{re}\,X})_{X\in\mathcal{X}}$ *forms a finite irreducible root system of type* $\mathcal{C}$, *and*

$$\Delta_+^{\mathrm{re}\,a} = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2, \alpha_1+2\alpha_2, 2\alpha_1+3\alpha_2\},$$
$$\Delta_+^{\mathrm{re}\,b} = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2, 4\alpha_1+3\alpha_2\},$$
$$\Delta_+^{\mathrm{re}\,c} = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+\alpha_2, 4\alpha_1+\alpha_2\}.$$

Proof. Both claims follow from the definitions. The Cartan scheme $\mathcal{C}$ appeared already in [13, Thm. 6.1]. □

Theorem 8.6. *Let* $V, W \in {}^G_G\mathcal{YD}$ *be such that* $V \simeq M(g,\rho)$, *where* ρ *is a character of* G^g, *and* $W \simeq M(z,\sigma)$, *where* σ *is a character of* G. *Assume that* $\operatorname{char}\mathbb{K} = 2$ *and* $(V, W) \in \wp_5$, *that is,*

$$\rho(g) = \sigma(z) = 1, \qquad 1 + \rho(z)\sigma(g) + \rho(z)^2\sigma(g)^2 = 0.$$

Then there exist Yetter–Drinfeld submodules $W_1 \in \mathcal{Y}_1$, $W_2 \in \mathcal{Y}_6$, $W_3 \in \mathcal{Y}_1$, $W_4 \in \mathcal{Y}_3$, $W_5 \in \mathcal{Y}_2$, *and* $W_6 \in \mathcal{Y}_3$ *of* $\mathfrak{B}(V \oplus W)$ *of degrees* α_1, $2\alpha_1+\alpha_2$, $3\alpha_1+2\alpha_2$, $4\alpha_1+3\alpha_2$, $\alpha_1+\alpha_2$, *and* α_2, *respectively, such that*

$$\mathfrak{B}(V\oplus W) \simeq \mathfrak{B}(W_6)\otimes\mathfrak{B}(W_5)\otimes\cdots\otimes\mathfrak{B}(W_1)$$

as $\mathbb{N}_0^2$*-graded vector spaces in* ${}^G_G\mathcal{YD}$. *In particular, the Hilbert series of* $\mathfrak{B}(V\oplus W)$ *is*

$$\mathcal{H}(t_1,t_2) = (2)_{t_2}(3)_{t_1t_2}(4)_{t_1t_2}(6)_{t_1t_2}(6)_{t_1^2t_2^2}(2)_{t_1^4t_2^3}(2)^2_{t_1^3t_2^2}(3)_{t_1^3t_2^2}(3)^2_{t_1^2t_2}(2)^2_{t_1}(3)_{t_1}$$

and $\dim\mathfrak{B}(V\oplus W) = 2^{10}3^7 = 2239488$.

Proof. Let V', W' be Yetter–Drinfeld modules over G. If $(V', W') \in \wp_5$, then $V' \in \mathcal{Y}_1$ and $W' \in \mathcal{Y}_3$ by Tables 3 and 4. Similarly, if $(V', W') \in \wp_5'$, then $V' \in \mathcal{Y}_1$ and $W' \in \mathcal{Y}_6$, and if $(V', W') \in \wp_5''$, then $V' \in \mathcal{Y}_2$ and $W' \in \mathcal{Y}_3$.

By assumption, $(V, W) \in \wp_5$. Lemmas 7.5–7.10 and 8.5 imply that the pair (V, W) admits all reflections and that the set of real roots $\Delta^{\mathrm{re}\,(V,W)}$ is finite. More precisely, $|\Delta_+^{\mathrm{re}\,(V,W)}| = 6$. Hence, by [24, Cor. 6.16], there exist absolutely simple Yetter–Drinfeld submodules $W_i \in {}^G_G\mathcal{YD}$ of $\mathfrak{B}(V\oplus W)$ with $1 \le i \le 6$, such that

$$\mathfrak{B}(V\oplus W) \simeq \mathfrak{B}(W_6)\otimes\mathfrak{B}(W_5)\otimes\cdots\otimes\mathfrak{B}(W_1).$$

By the same reference, we may also assume that

$$\deg W_{2i+1} = (s_1s_2)^i(\alpha_1) \quad \text{and} \quad \deg W_{2i+2} = (s_1s_2)^i s_1(\alpha_2)$$

for all $0 \le i \le 2$, and if $1 \le k \le 6$ and $\deg W_k = s_{i_1}\cdots s_{i_m}(\alpha_j)$ for some $m \ge 0$ and $i_1,\dots,i_m, j \in \{1,2\}$, then W_k is isomorphic in ${}^G_G\mathcal{YD}$ to the j-th entry of $R_{i_m}\cdots R_{i_1}(V, W)$. Since $\Delta_+^{\mathrm{re}\,(V,W)}$ consists of the roots

$$s_1(\alpha_2) = 2\alpha_1+\alpha_2, \quad s_1s_2(\alpha_1) = 3\alpha_1+2\alpha_2, \quad s_1s_2s_1(\alpha_2) = 4\alpha_1+3\alpha_2,$$
$$s_1s_2s_1s_2(\alpha_1) = \alpha_1+\alpha_2, \quad s_1s_2s_1s_2s_1(\alpha_2) = \alpha_2, \quad \alpha_1,$$

and since

$$R_1(V, W) \in \wp_5', \quad R_2R_1(V, W) \in \wp_5', \quad R_1R_2R_1(V, W) \in \wp_5,$$
$$R_2R_1R_2R_1(V, W) \in \wp_5'', \quad R_1R_2R_1R_2R_1(V, W) \in \wp_5'',$$

the first paragraph of the proof implies that $W_1 \in \mathcal{Y}_1$, $W_2 \in \mathcal{Y}_6$, $W_3 \in \mathcal{Y}_1$, $W_4 \in \mathcal{Y}_3$, $W_5 \in \mathcal{Y}_2$, $W_6 \in \mathcal{Y}_3$. Finally, a direct calculation using Table 4 yields the Hilbert series of $\mathfrak{B}(V \oplus W)$. □

With similar proofs, or by applying the reflections R_1 and R_2 to pairs in $\wp_5$, one also obtains the following two theorems on Nichols algebras of $V \oplus W$, where $(V, W) \in \wp_5'$ or $(V, W) \in \wp_5''$.

Theorem 8.7. *Let $V, W \in {}^G_G\mathcal{YD}$ be such that $V \simeq M(g, \rho)$, where ρ is a character of G^g, and $W \simeq M(\epsilon z, \sigma)$, where σ is a character of G^ϵ. Assume that* $\operatorname{char} \mathbb{K} = 2$ *and $(V, W) \in \wp_5'$, that is,*

$$\rho(g) = 1, \quad \sigma(z) = \sigma(\epsilon), \quad (3)_{\sigma(\epsilon)} = 0, \quad \rho(z^2)\sigma(\epsilon g^2) = 1.$$

Then there exist Yetter–Drinfeld submodules $W_1 \in \mathcal{Y}_1$, $W_2 \in \mathcal{Y}_3$, $W_3 \in \mathcal{Y}_2$, $W_4 \in \mathcal{Y}_3$, $W_5 \in \mathcal{Y}_1$, and $W_6 \in \mathcal{Y}_6$ of $\mathfrak{B}(V \oplus W)$ of degrees α_1, $2\alpha_1 + \alpha_2$, $\alpha_1 + \alpha_2$, $2\alpha_1 + 3\alpha_2$, $\alpha_1 + 2\alpha_2$, and α_2, respectively, such that

$$\mathfrak{B}(V \oplus W) \simeq \mathfrak{B}(W_6) \otimes \mathfrak{B}(W_5) \otimes \cdots \otimes \mathfrak{B}(W_1)$$

as $\mathbb{N}_0^2$-graded vector spaces in ${}^G_G\mathcal{YD}$. In particular, the Hilbert series of $\mathfrak{B}(V \oplus W)$ is

$$\mathcal{H}(t_1, t_2) = (3)_{t_2}^2 (2)_{t_1t_2^2}^2 (3)_{t_1t_2^2} (2)_{t_1^2t_2^3} (3)_{t_1t_2} (4)_{t_1t_2} (6)_{t_1t_2} (6)_{t_1^2t_2^2} (2)_{t_1^2t_2} (2)_{t_1}^2 (3)_{t_1}$$

and $\dim \mathfrak{B}(V \oplus W) = 2^{10}3^7 = 2239488.$

Theorem 8.8. *Let $V, W \in {}^G_G\mathcal{YD}$ be such that $V \simeq M(g, \rho)$, where ρ is a character of G^g, and $W \simeq M(z, \sigma)$, where σ is a character of G. Assume that* $\operatorname{char} \mathbb{K} = 2$ *and $(V, W) \in \wp_5''$, that is,*

$$(3)_{\rho(g)} = 0, \quad \sigma(z) = 1, \quad \rho(gz)\sigma(g) = 1.$$

Then there exist Yetter–Drinfeld submodules $W_1 \in \mathcal{Y}_2$, $W_2 \in \mathcal{Y}_3$, $W_3 \in \mathcal{Y}_1$, $W_4 \in \mathcal{Y}_6$, $W_5 \in \mathcal{Y}_1$, and $W_6 \in \mathcal{Y}_3$ of $\mathfrak{B}(V \oplus W)$ of degrees α_1, $4\alpha_1 + \alpha_2$, $3\alpha_1 + \alpha_2$, $2\alpha_1 + \alpha_2$, $\alpha_1 + \alpha_2$, and α_2, respectively, such that

$$\mathfrak{B}(V \oplus W) \simeq \mathfrak{B}(W_6) \otimes \mathfrak{B}(W_5) \otimes \cdots \otimes \mathfrak{B}(W_1)$$

as $\mathbb{N}_0^2$-graded vector spaces in ${}^G_G\mathcal{YD}$. In particular, the Hilbert series of $\mathfrak{B}(V \oplus W)$ is

$$\mathcal{H}(t_1, t_2) = (2)_{t_2} (2)_{t_1t_2}^2 (3)_{t_1t_2} (3)_{t_1^2t_2}^2 (2)_{t_1^3t_2}^2 (3)_{t_1^3t_2} (2)_{t_1^4t_2} (3)_{t_1} (4)_{t_1} (6)_{t_1} (6)_{t_1^2}$$

and $\dim \mathfrak{B}(V \oplus W) = 2^{10}3^7 = 2239488.$

9. Proof of Theorem 2.1

Here we prove the main result of the paper, Theorem 2.1.

First we prove (1)⇒(2). Since $\dim \mathfrak{B}(V \oplus W) < \infty$, the pair (V, W) admits all reflections by [5, Cor. 3.18] and the Weyl groupoid is finite by [5, Prop. 3.23].

Now we prove simultaneously (2)⇒(3) and (3)⇒(1). By [25, Thm. 7.3], the quandle $\mathrm{supp}(V \oplus W)$ is isomorphic to

$$Z_2^{2,2},\ Z_3^{3,1},\ Z_3^{3,2},\ Z_4^{4,2} \text{ or } Z_T^{4,1}$$

and G is a non-abelian epimorphic image of its enveloping group:

Quandle	$Z_T^{4,1}$	$Z_2^{2,2}$	$Z_3^{3,1}$	$Z_3^{3,2}$	$Z_4^{4,2}$
Enveloping group	T	Γ_2	Γ_3	Γ_3	Γ_4

We consider each case separately. Suppose first that G is an epimorphic image of Γ_2 and $\mathrm{supp}(V \oplus W) \simeq Z_2^{2,2}$. Hence, by [22, §4], we may assume that $V \simeq M(g, \rho)$ and $W \simeq M(h, \sigma)$. Now the claim follows from [22, Thm. 4.9].

Now assume that G is a non-abelian epimorphic image of Γ_3. We first prove that (2) implies (3). By [26, Prop. 4.3], there is a pair (V', W') of absolutely simple Yetter–Drinfeld modules over G, which represents an object of the Weyl groupoid of (V, W), such that the Cartan matrix of (V', W') is of finite type. The list of examples in Subsection 1.4 contains precisely the pairs in the classes $\wp_i$ for $1 \le i \le 5$ and $\wp_5'$, $\wp_5''$. Hence this list is stable under reflections by Lemmas 7.3, 7.4, and 7.5–7.10. Thus we may assume that the Cartan matrix of (V, W) is of finite type. Now (3) follows from Proposition 7.2.

The implication (3)⇒(1) follows from Theorems 8.1–8.4 and 8.6–8.8.

Assume now that G is a non-abelian epimorphic image of Γ_4, but not of Γ_2, and that $\mathrm{supp}(V \oplus W) \simeq Z_4^{4,2}$. Without loss of generality we may assume that $|\mathrm{supp}\, V| = 2$ and $|\mathrm{supp}\, W| = 4$. Let $h \in \mathrm{supp}\, V$, $g \in \mathrm{supp}\, W$, and $\epsilon = hgh^{-1}g^{-1}$. Then $g^G = \{g, \epsilon g, \epsilon^2 g, \epsilon^3 g\}$ and $h^G = \{h, \epsilon^{-1} h\}$. Since $\mathrm{supp}(V \oplus W)$ generates G, we conclude that G is generated by g and h, and $V \simeq M(h, \rho)$ and $W \simeq M(g, \sigma)$ for some character ρ of G^h and some character σ of G^g. Then the implications (2)⇒(3) and (3)⇒(1) follow from [25, Thm. 5.4].

Finally, suppose that G is an epimorphic image of T and $\mathrm{supp}(V \oplus W) \simeq Z_T^{4,1}$. We may assume that $|\mathrm{supp}\, V| = 1$ and $|\mathrm{supp}\, W| = 4$. Let $z \in \mathrm{supp}\, V$ and $x_1 \in \mathrm{supp}\, W$. Then z and x_1^G generate G. Choose $x_2 \in x_1^G \setminus \{x_1\}$. Then $T \to G$, $\chi_1 \mapsto x_1$, $\chi_2 \mapsto x_2$, $\zeta \mapsto z$ is an epimorphism of groups, and $V \simeq M(z, \rho)$ and $W \simeq M(x_1, \sigma)$ for some absolutely irreducible representations ρ of G and σ of G^{x_1}. Then the implications (2)⇒(3) and (3)⇒(1) follow from [25, Thm. 2.8].

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