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A direct approach to Plateau's problem

Abstract. We provide a compactness principle which is applicable to different formulations of Plateau's problem in codimension one and which is exclusively based on the theory of Radon measures and elementary comparison arguments. Exploiting some additional techniques in geometric measure theory, we can use this principle to give a different proof of a theorem by Harrison and Pugh and to answer a question raised by Guy David.

Keywords. Plateau's problem, minimal surfaces, area minimization, geometric measure theory

1. Introduction

Since the pioneering work of Reifenberg there has been ongoing interest in formulations of Plateau's problem involving the minimization of the Hausdorff measure on closed sets coupled with some notion of "spanning a given boundary". More precisely, consider any closed set $H \subset \mathbb{R}^{n+1}$ and assume we have a class $\mathcal{P}(H)$ of relatively closed subsets K of $\mathbb{R}^{n+1} \setminus H$, which encodes a particular notion of "K bounds H". Correspondingly, there is a formulation of Plateau's problem, where the relevant minimum is

$$m_0 := \inf\{\mathcal{H}^n(K) : K \in \mathcal{P}(H)\}, \qquad (1.1)$$

and a minimizing sequence $\{K_j\} \subset \mathcal{P}(H)$ is characterized by the property $\mathcal{H}^n(K_j) \to m_0$. Two good motivations for considering this kind of approach rather than the one based on integer rectifiable currents are that, first, not every interesting boundary can be realized as an integer rectifiable cycle, and second, area minimizing 2-d currents in $\mathbb{R}^3$ are always smooth away from their boundaries, in contrast to what one observes with real world soap films.

There are substantial difficulties related to the minimization of Hausdorff measures on classes of closed (or even compact) sets. Depending on the convergence adopted, these

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are either related to lack of lower semicontinuity or to compactness issues. In both cases, obtaining existence results in this framework is a quite delicate task, as exemplified in various works by Reifenberg [Rei60, Rei64a, Rei64b], De Pauw [DP09], Feuvrier [Feu09], Harrison and Pugh [Har14, HP16], Fang [Fan16], Liang [Lia13] and David [Dav14].

Our goal here is to show that in some interesting cases these difficulties can be avoided by exploiting Preiss' rectifiability theorem for Radon measures [Pre87, DL08] in combination with the sharp isoperimetric inequality on the sphere and standard variational arguments, notably elementary comparisons with spheres and cones. A precise formulation of our main result is the following:

Definition 1 (Cone and cup competitors). Let $H \subset \mathbb{R}^{n+1}$ be closed. Given $K \subset \mathbb{R}^{n+1} \setminus H$ and $B_{x,r} := \{y \in \mathbb{R}^{n+1} : |x - y| < r\} \subset \mathbb{R}^{n+1} \setminus H$, the *cone competitor* for K in $B_{x,r}$ is the set

$$(K \setminus B_{x,r}) \cup \{\lambda x + (1 - \lambda)z : z \in K \cap \partial B_{x,r}, \ \lambda \in [0, 1]\}; \tag{1.2}$$

a *cup competitor* for K in $B_{x,r}$ is any set of the form

$$(K \setminus B_{x,r}) \cup (\partial B_{x,r} \setminus A), \tag{1.3}$$

where A is a connected component of $\partial B_{x,r} \setminus K$.

Given a family $\mathcal{P}(H)$ of relatively closed subsets $K \subset \mathbb{R}^{n+1} \setminus H$, we say that an element $K \in \mathcal{P}(H)$ has the *good comparison property* in $B_{x,r}$ relative to $\mathcal{P}(H)$ if

$$\inf\{\mathcal{H}^n(J) : J \in \mathcal{P}(H), J \setminus \mathrm{cl}(B_{x,r}) = K \setminus \mathrm{cl}(B_{x,r})\} \leq \mathcal{H}^n(L) \tag{1.4}$$

whenever L is the cone competitor or any cup competitor for K in $B_{x,r}$. The family $\mathcal{P}(H)$ is a *good class* if, for any $K \in \mathcal{P}(H)$ and for every $x \in K$, the set K has the good comparison property in $B_{x,r}$ for a.e. $r \in (0, \mathrm{dist}(x, H))$.

Theorem 2. *Let $H \subset \mathbb{R}^{n+1}$ be closed and $\mathcal{P}(H)$ be a good class. Assume the infimum in Plateau's problem (1.1) is finite and let $\{K_j\} \subset \mathcal{P}(H)$ be a minimizing sequence of countably $\mathcal{H}^n$-rectifiable sets. Then, up to subsequences, the measures $\mu_j := \mathcal{H}^n \llcorner K_j$ converge weakly* in $\mathbb{R}^{n+1} \setminus H$ to a measure $\mu = \theta \mathcal{H}^n \llcorner K$, where $K := \mathrm{spt}\,\mu \setminus H$ is a countably $\mathcal{H}^n$-rectifiable set and $\theta \geq 1$. In particular, $\liminf_j \mathcal{H}^n(K_j) \geq \mathcal{H}^n(K)$.*

Furthermore, for every $x \in K$ the quantity $r^{-n}\mu(B_{x,r})$ is nondecreasing and

$$\theta(x) = \lim_{r \downarrow 0} \frac{\mu(B_r(x))}{\omega_n r^n} \geq 1, \tag{1.5}$$

where ω_n is the Lebesgue measure of the unit ball in $\mathbb{R}^n$.

Our point is that although Theorem 2 does not imply in general the existence of a minimizer in $\mathcal{P}(H)$, this might be achieved with little additional work (but possibly using some heavier machinery from geometric measure theory) in some interesting cases.

We will give two applications. The first one is motivated by a very elegant idea of Harrison, which can be explained as follows. Assume that H is a smooth closed compact $n - 1$-dimensional submanifold of $\mathbb{R}^{n+1}$; then we say that a relatively closed set

$K \subset \mathbb{R}^{n+1} \setminus H$ bounds H if K intersects every smooth curve γ whose linking number with H is 1. A possible formulation of Plateau's problem is then to minimize the Hausdorff measure in this class of sets. Building upon her previous work [Har15] on differential chains, Harrison [Har14] gives a general existence result for a suitable weak version of this problem. In [HP16], Harrison and Pugh prove that the corresponding minimizer yields a closed set K which is a minimizer in the original formulation of the problem, and to which the regularity theory for $(\mathbf{M}, \xi, \delta)$-minimal sets by Almgren and Taylor [Alm76, Tay76] can be applied. In particular, K is analytic off an $\mathcal{H}^n$-negligible singular set, and actually in the physical case $n = 3$ and away from the boundary set H, this singular set obeys the experimental observations known as Plateau's laws. Boundary regularity seems a major issue to be settled.

We can recover the theorem of Harrison and Pugh in a relatively short way from Theorem 2. In fact, our approach allows one to work, with the same effort, in a more general setting.

Definition 3. Let $n \geq 2$ and H be a closed set in $\mathbb{R}^{n+1}$. When H is a closed compact $n-1$-dimensional submanifold, following [HP16] we say that a closed set $K \subset \mathbb{R}^{n+1} \setminus H$ *spans* H if it intersects any smooth embedded closed curve γ in $\mathbb{R}^{n+1} \setminus H$ such that the linking number of H and γ is 1.

More generally, for an arbitrary closed H let us consider the family

$$\mathcal{C}_H := \{\gamma : S^1 \to \mathbb{R}^{n+1} \setminus H : \gamma \text{ is a smooth embedding of } S^1 \text{ into } \mathbb{R}^{n+1}\}.$$

We say that $\mathcal{C} \subset \mathcal{C}_H$ is *closed by homotopy* (with respect to H) if together with any $\gamma \in \mathcal{C}$, the set $\mathcal{C}$ contains all elements belonging to the homotopy class $[\gamma] \in \pi_1(\mathbb{R}^{n+1} \setminus H)$. Given $\mathcal{C} \subset \mathcal{C}_H$ closed by homotopy, we say that a relatively closed subset K of $\mathbb{R}^{n+1} \setminus H$ is a *$\mathcal{C}$-spanning set* of H if

$$K \cap \gamma \neq \emptyset \quad \text{for every } \gamma \in \mathcal{C}. \tag{1.6}$$

We denote by $\mathcal{F}(H, \mathcal{C})$ the family of $\mathcal{C}$-spanning sets of H.

Observe that in some cases, the class $\mathcal{F}(H, \mathcal{C})$ might be trivial: for instance if $\mathcal{C}$ contains a homotopically trivial curve, then any element of $\mathcal{F}(H, \mathcal{C})$ has nonempty interior. Of course, we are interested in classes $\mathcal{F}(H, \mathcal{C})$ admitting at least one element with finite Hausdorff measure.

Theorem 4. *Let $n \geq 2$, H be closed in $\mathbb{R}^{n+1}$ and $\mathcal{C}$ be closed by homotopy with respect to H. Assume the infimum in Plateau's problem corresponding to $\mathcal{P}(H) = \mathcal{F}(H, \mathcal{C})$ is finite. Then:*

(a) *$\mathcal{F}(H, \mathcal{C})$ is a good class in the sense of Definition 1.*
(b) *There is a minimizing sequence $\{K_j\} \subset \mathcal{F}(H, \mathcal{C})$ which consists of $\mathcal{H}^n$-rectifiable sets. If K is any set associated to $\{K_j\}$ by Theorem 2, then $K \in \mathcal{F}(H, \mathcal{C})$, and thus K is a minimizer.*
(c) *The set K in (b) is an $(\mathbf{M}, 0, \infty)$-minimal set in $\mathbb{R}^{n+1} \setminus H$ in the sense of Almgren.*

Remark 5. As already mentioned, the variational problem considered in [Har14, HP16] corresponds to the case where H is a closed compact $(n-1)$-dimensional submanifold of $\mathbb{R}^{n+1}$ and $\mathcal{C} = \{\gamma \in \mathcal{C}_H : \text{the linking number of } H \text{ and } \gamma \text{ is } 1\}$. In fact, there is yet a small technical difference: in [Har14, HP16] the authors minimize the Hausdorff spherical measure, which coincides with the Hausdorff measure $\mathcal{H}^n$ on rectifiable sets, but it is in general larger on unrectifiable sets. After completing this note we learned that Harrison and Pugh have been able to improve their proof in order to minimize the Hausdorff measure as well [HP14]. Finally, we stress that, while points (a) and (c) can be deduced from Theorem 2 using elementary results about Radon measures and isoperimetry, point (b) relies in a substantial way upon the theory of Caccioppoli sets and minimal partitions.

We next exploit Theorem 2 to obtain an existence result in the context of "sliding minimizers" introduced by David [Dav14, Dav13].

Definition 6. Let $H \subset \mathbb{R}^{n+1}$ be closed and $K_0 \subset \mathbb{R}^{n+1} \setminus H$ be relatively closed. We denote by $\Sigma(H)$ the family of Lipschitz maps $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that there exists a continuous map $\Phi : [0,1] \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with $\Phi(1, \cdot) = \varphi$, $\Phi(0, \cdot) = \mathrm{Id}$ and $\Phi(t, H) \subset H$ for every $t \in [0,1]$. We then define

$$\mathcal{A}(H, K_0) := \{K : K = \varphi(K_0) \text{ for some } \varphi \in \Sigma(H)\},$$

and say that K_0 is a *sliding minimizer* if $\mathcal{H}^n(K_0) = \inf\{\mathcal{H}^n(J) : J \in \mathcal{A}(H, K_0)\}$.

We will use the convention that, whenever $E \subset \mathbb{R}^{n+1}$ and $\delta > 0$, $U_\delta(E)$ denotes the δ-neighborhood of E.

Theorem 7. *$\mathcal{A}(H, K_0)$ is a good class in the sense of Definition 1. Moreover, assume that*

(i) *K_0 is bounded and countably $\mathcal{H}^n$-rectifiable with $\mathcal{H}^n(K_0) < \infty$;*
(ii) *$\mathcal{H}^n(H) = 0$ and for every $\eta > 0$ there exist $\delta > 0$ and $\pi \in \Sigma(H)$ such that*

$$\mathrm{Lip}\,\pi \leq 1 + \eta, \quad \pi(U_\delta(H)) \subset H. \tag{1.7}$$

Then, given any minimizing sequence $\{K_j\}$ (in Plateau's problem corresponding to $\mathcal{P}(H) = \mathcal{A}(H, K_0)$) and any set K as in Theorem 2, we have

$$\inf\{\mathcal{H}^n(J) : J \in \mathcal{A}(H, K_0)\} = \mathcal{H}^n(K) = \inf\{\mathcal{H}^n(J) : J \in \mathcal{A}(H, K)\}. \tag{1.8}$$

In particular K is a sliding minimizer.

The proof of the second equality in (1.8) borrows important ideas from the work of De-Pauw and Hardt [DPH03] and it uses in a substantial way the theory of varifolds, in particular Allard's regularity theorem. A different approach to the existence of a K satisfying the left hand equality of (1.8) has been suggested by David [Dav14, Section 7], who also raised the question whether one could deduce the right hand equality. Our theorem gives a positive answer to this question (see below for a stronger one, also raised by David).

Remark 8. It seems hard to conclude something about the existence of a minimizer in the *original* class $\mathcal{A}(H, K_0)$ from our approach, without a deeper analysis of what sliding deformations can do to the starting set K_0. The following example illustrates this difficulty. Let H be the union of two far away parallel circles and K_0 be a cylinder joining them, namely define, for R large,

$$H := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1, |x_3| = R\},$$
$$K_0 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1, |x_3| < R\}.$$

Let $\{K_j\} \subset \mathcal{A}(H, K_0)$ be a minimizing sequence and $\mu_j = \mathcal{H}^2 \llcorner K_j$. We obviously expect that $\mathcal{H}^2 \llcorner K_j \to \mathcal{H}^2 \llcorner K$ where

$$K := \{(x_1, x_2, x_3) : x_1^2 + x_2^2 < 1, |x_3| = R\}.$$

Of course $K \notin \mathcal{A}(H, K_0)$, but we can easily build a map $\varphi \in \Sigma(H)$ which "squeezes" K_0 onto the set $K_1 := K \cup \{(0, 0, t) : |t| \leq R\}$, i.e. the top and bottom disks connected by a vertical segment. K_1 is then a minimizer in $\mathcal{A}(H, K_0)$. On the other hand, $K = \mathrm{spt}(\mathcal{H}^2 \llcorner K_1)$, and thus a purely measure-theoretic approach does not seem to capture this phenomenon. It is however very tempting to conjecture that, upon adding a suitable $\mathcal{H}^n$-negligible set (and possibly some more requirements on the boundary H), any set K as in Theorem 7 is an element of $\mathcal{A}(H, K_0)$ (cf. [Dav14]). We refer the reader to [Whi83] for a result of a similar flavour.

The results presented in this paper are concerned with Plateau's problem in codimension one. This assumption is used in a crucial way when we exploit the cup competitors and the optimal isoperimetric inequality in $\mathbb{S}^n$ to prove Theorem 2. Although we also take advantage of some other typical codimension one tools (such as the degree theory and the theory of minimal partitions), the main obstruction to extending our analysis to higher codimension is really our use of cup competitors. Besides, in higher codimension the available isoperimetric inequalities are more difficult to use (see for instance [Alm86]). This point can be overcome by relying only on Lipschitz deformations to produce competitors. In this way it is possible to give a different notion of "good class", which is more involved than the one considered here, but which still allows a general compactness result in the spirit of Theorem 2. Starting from this compactness principle, one can obtain the higher-codimensional versions of both Theorems 4 and 7. These results are proved in [DDG16] where, together with some of the ideas presented in this paper, new ones need to be introduced, and more refined tools from geometric measure theory are exploited (most notably, a suitable version of the deformation theorem for closed rectifiable sets due to David and Semmes [DS00]).

2. Proof of Theorem 2

We start with the following classical fact. We include a quick proof just for the reader's convenience using sets of finite perimeter; the latter are however not really necessary, in particular it should be possible to prove Theorem 2 without leaving the framework

provided by the theory of Radon measures. In what follows we use the notation

$$\sigma_k := \mathcal{H}^k(\{z \in \mathbb{R}^{k+1} : |z| = 1\}) \quad \text{and} \quad \omega_{k+1} := \mathcal{H}^{k+1}(\{z \in \mathbb{R}^{k+1} : |z| \le 1\}) = \frac{\sigma_k}{k+1}.$$

Lemma 9 (Isoperimetry on the sphere). *If $J \subset \partial B_{x,r}$ is compact and $\{A_h\}_{h=0}^\infty$ is the family of all connected components of $\partial B_{x,r} \setminus J$, ordered so that $\mathcal{H}^n(A_h) \ge \mathcal{H}^n(A_{h+1})$, then*

$$\mathcal{H}^n(\partial B_{x,r} \setminus A_0) \le C(n)\mathcal{H}^{n-1}(J)^{n/(n-1)}. \tag{2.1}$$

Moreover, for every $\eta > 0$ there exists $\delta > 0$ such that

$$\min\{\mathcal{H}^n(A_0), \mathcal{H}^n(A_1)\} = \mathcal{H}^n(A_1) \ge (\sigma_n/2 - \delta)r^n \Rightarrow \mathcal{H}^{n-1}(J) \ge (\sigma_{n-1} - \eta)r^{n-1}. \tag{2.2}$$

The inequality (2.1) *also holds if we replace $\partial B_{x,r}$ with ∂Q for any cube $Q \subset \mathbb{R}^{n+1}$ or with any spherical cap $\partial B_{x,r} \cap \{y : (y - x) \cdot v > \varepsilon r\}$, where $v \in S^n$ and $\varepsilon \in]0, 1[$.*

Proof. We first prove (2.1) with $J \subset \partial B_{x,r}$. The proof can be easily adapted to boundary of cubes and spherical caps. Since $\partial A_h \subset J$ and (without loss of generality) $\mathcal{H}^{n-1}(J) < \infty$, we know [AFP00, Prop. 3.62] that each A_h has finite perimeter and $\partial^* A_h \subset J$ (where $\partial^* A_h$ denotes the reduced boundary). By the properties of the reduced boundary one easily infers that $\sum_h \mathcal{H}^{n-1} \llcorner \partial^* A_h \le 2\mathcal{H}^{n-1} \llcorner J$. By the relative isoperimetric inequality on $\partial B_{x,r}$, if $A \subset \partial B_{x,r}$ is of finite perimeter, then

$$\min\{\mathcal{H}^n(A), \mathcal{H}^n(\partial B_{x,r} \setminus A)\} \le C(n)\mathcal{H}^{n-1}(\partial^* A)^{n/(n-1)}. \tag{2.3}$$

By the ordering property of the $\mathcal{H}^n(A_h)$, we thus find

$$\mathcal{H}^n(A_h) \le C(n)\mathcal{H}^{n-1}(\partial^* A_h)^{n/(n-1)}, \quad \forall h \ge 1.$$

Adding up over $h \ge 1$ and using the superadditivity of the function $t \mapsto t^{n/(n-1)}$ yields

$$\mathcal{H}^n(\partial B_{x,r} \setminus A_0) \le C(n)\Big(\sum_{h \ge 1} \mathcal{H}^{n-1}(\partial^* A_h)\Big)^{n/(n-1)} \le C(n)\mathcal{H}^{n-1}(J)^{n/(n-1)}.$$

(2.2) can be proved via a compactness argument: assuming that it fails for a given $\eta > 0$, we find a sequence J_k of sets, each violating the statement for $\delta = 1/k$. Letting A_0^k and A_1^k be the corresponding connected components, we can use the compactness of Caccioppoli sets to conclude that they converge to two sets A_0^∞, A_1^∞ with

$$\mathcal{H}^n(A_0^\infty) = \mathcal{H}^n(A_1^\infty) = \frac{\sigma_n}{2}r^n, \quad \mathcal{H}^n(A_0^\infty \cap A_1^\infty) = 0, \tag{2.4}$$

$$\max\{\mathcal{H}^{n-1}(\partial^* A_0^\infty), \mathcal{H}^{n-1}(\partial^* A_1^\infty)\} \le (\sigma_{n-1} - \eta)r^{n-1}. \tag{2.5}$$

By (2.4), $\partial^* A_0^\infty = \partial^* A_1^\infty$; but then (2.5) contradicts the sharp isoperimetric inequality on the sphere [BZ88, Theorem 10.2.1]. $\qquad\square$

Proof of Theorem 2. Up to extracting subsequences we can assume the existence of a Radon measure μ on $\mathbb{R}^{n+1} \setminus H$ such that

$$\mu_j \stackrel{*}{\rightharpoonup} \mu \quad \text{as Radon measures on } \mathbb{R}^{n+1} \setminus H, \tag{2.6}$$

where $\mu_j = \mathcal{H}^n \llcorner K_j$. We set $K := \operatorname{spt} \mu \setminus H$ and divide the argument into four steps. The first two steps feature some classical estimates in the theory of minimal surfaces (see for instance [Mag12, Ch. 16–17]) adapted to our context.

Step 1. We show the existence of $\theta_0 = \theta_0(n) > 0$ such that

$$\mu(B_{x,r}) \geq \theta_0 \, \omega_n r^n, \quad \forall x \in \operatorname{spt} \mu, \ \forall r < d_x := \operatorname{dist}(x, H). \tag{2.7}$$

By [Mat95, Theorem 6.9], (2.7) implies

$$\mu \geq \theta_0 \mathcal{H}^n \llcorner K \quad \text{on subsets of } \mathbb{R}^{n+1} \setminus H. \tag{2.8}$$

We now prove (2.7). Let $f(r) := \mu(B_{x,r})$ and $f_j(r) := \mathcal{H}^n(K_j \cap B_{x,r})$, so that

$$f_j(r) - f_j(s) \geq \int_s^r \mathcal{H}^{n-1}(K_j \cap \partial B_{x,t}) \, dt, \quad 0 < s < r < d_x,$$

by the coarea formula [Fed69, 3.2.22]. Since f_j is nondecreasing on $(0, d_x)$, one has

$$Df_j \geq f_j' \mathcal{L}^1 \quad \text{with} \quad f_j'(r) \geq \mathcal{H}^{n-1}(K_j \cap \partial B_{x,r}) \quad \text{for a.e. } r \in (0, d_x)$$

(here Df_j denotes the distributional derivative of f_j, f_j' the pointwise derivative and $\mathcal{L}^1$ the Lebesgue measure). By Fatou's lemma, if we set $g(t) := \liminf_j f_j'(t)$, then

$$f(r) - f(s) = \mu(B_{x,r} \setminus B_{x,s}) \geq \int_s^r g(t) \, dt, \quad \text{provided } \mu(\partial B_{x,r}) = \mu(\partial B_{x,s}) = 0.$$

This shows that $Df \geq g\mathcal{L}^1$. On the other hand, using the differentiability a.e. of f and letting $s \uparrow r$, we also conclude $f' \geq g$ $\mathcal{L}^1$-a.e., whereas $Df \geq f'\mathcal{L}^1$ is a simple consequence of the fact that f is a nondecreasing function.

Let A_j denote a connected component of $\partial B_{x,r} \setminus K_j$ of maximal $\mathcal{H}^n$-measure, and let K_j'' be the corresponding cup competitor of K_j in $B_{x,r}$ (see (1.3)). Since $\mathcal{P}(H)$ is a good class, for a.e. $r < d_x$ by (2.1) we find

$$f_j(r) \leq \mathcal{H}^n(\partial B_{x,r} \setminus A_j) + \varepsilon_j \leq C(n)\left(\mathcal{H}^{n-1}(\partial B_{x,r} \cap K_j)\right)^{n/(n-1)} + \varepsilon_j, \tag{2.9}$$

where $\varepsilon_j \to 0$ takes into account the almost minimality of K_j, namely we assume $\mathcal{H}^n(K_j) \leq \inf\{\mathcal{H}^n(K) : K \in \mathcal{P}(H)\} + \varepsilon_j$. Letting $j \to \infty$ we find that

$$f(r) \leq C(n)g(r)^{n/(n-1)} \leq C(n)f'(r)^{n/(n-1)} \quad \text{for a.e. } r < d_x,$$

from which

$$f(r)^{(n-1)/n} \leq C(n)f'(r) \quad \text{for a.e. } r < d_x,$$

which implies

$$1 \leq C(n)(f(r)^{1/n})', \quad \forall r < d_x.$$

Since the distributional derivative $Df^{1/n}$ is nonnegative, we deduce that $r \leq C(n)(f(r)^{1/n} - f(0)^{1/n})$, hence $\mu(B_{x,r}) \geq \theta_0 \omega_n r^n$ for a suitable value of θ_0.

Step 2. We fix $x \in \operatorname{spt} \mu \setminus H$ and prove that

$$r \mapsto f(r)/r^n = \mu(B_{x,r})/r^n \quad \text{is nondecreasing on } (0, d_x). \tag{2.10}$$

This property can be deduced by using the cone competitor in $B_{x,r}$ in place of the cup competitor: estimate (2.9) now becomes

$$f_j(r) \leq \mathcal{H}^n(K_j' \cap B_{x,r}) + \varepsilon_j = \frac{r}{n} \mathcal{H}^{n-1}(K_j \cap \partial B_{x,r}) + \varepsilon_j \leq \frac{r}{n} f_j'(r) + \varepsilon_j,$$

yielding $f(r) \leq (r/n)g(r) \leq (r/n)f'(r)$ for a.e. $r < d_x$. Again the positivity of the measure $D \log(f)$ implies the claimed monotonicity. By (2.8) and (2.10) the n-dimensional density of the measure μ, namely

$$\theta(x) = \lim_{r \to 0^+} \frac{f(r)}{\omega_n r^n} \geq \theta_0,$$

exists, is finite and positive μ-almost everywhere. By the well known theorem of Preiss [DL08, Theorem 1.1], this property implies that $\mu = \theta \mathcal{H}^n \llcorner \tilde{K}$ for some countably $\mathcal{H}^n$-rectifiable set $\tilde{K}$ and some positive Borel function θ. Since K is the support of μ, we have $\mathcal{H}^n(\tilde{K} \setminus K) = 0$. On the other hand, $\mathcal{H}^n(K \setminus \tilde{K}) = 0$ by (2.8), and thus K must be rectifiable and $\mu = \theta \mathcal{H}^n \llcorner K$.

Step 3. We prove that $\theta(x) \geq 1$ for every $x \in K$ such that the approximate tangent space to K exists (thus, $\mathcal{H}^n$-a.e. on K). Fix any such $x \in K \setminus H$ and suppose, up to rotating the coordinates, that $T = \{x_{n+1} = 0\}$ is the approximate tangent space to K at x; in particular (cf. [DL08, Corollary 4.4]),

$$\mathcal{H}^n \llcorner \frac{K - x}{r} \overset{*}{\rightharpoonup} \mathcal{H}^n \llcorner T \quad \text{as } r \to 0^+.$$

By the density lower bound (2.7), for every $\varepsilon > 0$ there is $\rho > 0$ such that

$$K \cap B_{x,r} \subset x + \{y \in \mathbb{R}^{n+1} : |y_{n+1}| < \varepsilon r\}, \quad \forall r < \rho. \tag{2.11}$$

Indeed, assume r is so small that $\mu(B_{x,2r} \setminus (x + \{|y_{n+1}| < \varepsilon r/2\})) < \theta_0 2^{-n} \varepsilon^n r^n$. Then $K \cap (x + \{|y_{n+1}| \geq \varepsilon r\}) \cap B_{x,r}$ must be empty, since otherwise y belonging to that set would imply

$$\mu(B_{x,2r} \setminus (x + \{|y_{n+1}| < \varepsilon r/2\})) \geq \mu(B_{y,\varepsilon r/2}) \geq \theta_0 \varepsilon^n r^n / 2^n,$$

a contradiction. If we set $c(\varepsilon) = \varepsilon/\sqrt{1 - \varepsilon^2}$, then (2.11) can be equivalently stated as

$$K \cap B_{x,\rho} \subset x + \{(y', y_{n+1}) : |y_{n+1}| < c(\varepsilon)|y'|\}. \tag{2.12}$$

If in addition we choose ρ satisfying $\mathcal{H}^n(K \cap \partial B_{x,\rho}) = 0$, then by the coarea formula [Fed69, 3.2.22],

$$0 = \lim_{j \to \infty} \mu_j\big(\mathrm{cl}(B_{x,\rho}) \cap (x + \{(y', y_{n+1}) : |y_{n+1}| \geq c(\varepsilon)|y'|\})\big)$$

$$\geq \int_0^\rho \liminf_{j \to \infty} \mathcal{H}^{n-1}\big(K_j \cap \partial B_{x,r} \cap (x + \{(y', y_{n+1}) : |y_{n+1}| \geq c(\varepsilon)|y'|\})\big)\,dr.$$

So, if $\partial B^+_{x,r,\varepsilon} := \{y \in \partial B_{x,r} : y_{n+1} > x_{n+1} + \varepsilon r\}$ and $\partial B^-_{x,r,\varepsilon} := \{y \in \partial B_{x,r} : y_{n+1} < x_{n+1} - \varepsilon r\}$, then

$$\liminf_{j \to \infty} \mathcal{H}^{n-1}(K_j \cap \partial B^\pm_{x,r,\varepsilon}) = 0 \quad \text{for a.e. } r < \rho. \tag{2.13}$$

Let us fix $r < \rho$ such that (2.13) holds, $f'(r)$ exists, $f'(r) \geq g(r)$ and each K_j has the good comparison property in $B_{x,r}$ (all these conditions can be ensured for a.e. r). Using Lemma 9, namely the relative isoperimetric inequality in the spherical cap $\partial B^+_{x,r,\varepsilon}$, one finds that if A_j^+ denotes the connected component of $\partial B^+_{x,r,\varepsilon} \setminus K_j$ with largest $\mathcal{H}^n$-measure, then $\mathcal{H}^n(\partial B^+_{x,r,\varepsilon} \setminus A_j^+) \leq C(n)\mathcal{H}^{n-1}(K_j \cap \partial B^+_{x,r,\varepsilon})^{n/(n-1)}$, and thus, by (2.13),

$$\lim_{j \to \infty} \mathcal{H}^n(A_j^+) = \mathcal{H}^n(\partial B^+_{x,\varepsilon,r});$$

similarly, $\mathcal{H}^n(A_j^-) \to \mathcal{H}^n(\partial B^-_{x,\varepsilon,r})$ if A_j^- is the largest connected component of $\partial B^-_{x,r,\varepsilon} \setminus K_j$.

We claim that, for j sufficiently large, A_j^+ and A_j^- cannot belong to the same connected component of $\partial B_{x,r} \setminus K_j$: otherwise, we can compare with the cup competitor of K_j in $B_{x,r}$ defined by the connected component of $\partial B_{x,r} \setminus K_j$ containing $A_j^+ \cup A_j^-$ (which is the largest connected component of $\partial B_{x,r} \setminus K_j$ when j is large enough), obtaining

$$\mu(B_{x,r}) \leq \liminf_{j \to \infty} \mathcal{H}^n(K_j \cap B_{x,r}) \leq \liminf_{j \to \infty} \mathcal{H}^n(\partial B_{x,r} \setminus (A_j^+ \cup A_j^-))$$

$$\leq \mathcal{H}^n(\partial B_{x,r} \cap \{|y_{n+1} - x_{n+1}| < \varepsilon r\}) \leq C\varepsilon r^n,$$

contrary to the density lower bound (2.7).

If we now fix η, then we can choose ε so that Lemma 9 entails, for j large enough,

$$(\sigma_{n-1} - \eta)r^{n-1} \leq \liminf_{j \to \infty} \mathcal{H}^{n-1}(K_j \cap \partial B_{x,r}) \leq f'(r).$$

In conclusion, $f'(r) \geq (\sigma_{n-1} - \eta)r^{n-1}$ for a.e. $r < \rho$. As $f(r) \geq (\sigma_{n-1} - \eta)r^n/n$ for every $r < \rho$, one concludes that $\theta(x) \geq (\sigma_{n-1} - \eta)/(n\omega_n)$. Letting $\eta \to 0$ we obtain $\theta(x) \geq 1$.

To complete the proof of Theorem 2 we recall that a standard consequence of the monotonicity (2.10) is the upper semicontinuity of θ; a simple density argument then shows (1.5) (cf. [Sim83, Corollary 17.8]). $\qquad\square$

3. Proof of Theorem 4

Most of the proof of Theorem 4 relies on the following elementary geometric remark.

Lemma 10. *If* $K \in \mathcal{F}(H, \mathcal{C})$, $B_{x,r} \subset\subset \mathbb{R}^{n+1} \setminus H$, *and* $\gamma \in \mathcal{C}$, *then either* $\gamma \cap (K \setminus B_{x,r}) \neq \emptyset$, *or there exists a connected component* σ *of* $\gamma \cap \mathrm{cl}(B_{x,r})$ *which is homeomorphic to an interval and whose end-points belong to distinct connected components of* $\mathrm{cl}(B_{x,r}) \setminus K$ *(and so to distinct components of* $\partial B_{x,r} \setminus K$). *The same conclusion holds if we replace* $B_{x,r}$ *with an open cube* $Q \subset\subset \mathbb{R}^{n+1} \setminus H$.

Proof. Step 1. It is clearly sufficient to assume $\gamma \cap (K \setminus B_{x,r}) = \emptyset$: since γ must intersect K, it must intersect $\mathrm{cl}(B_{x,r})$. We first prove the lemma under the assumption that γ and $\partial B_{x,r}$ intersect transversally.

Indeed, we can then find finitely many mutually disjoint closed circular arcs $I_i \subset S^1$, $I_i = [a_i, b_i]$, such that $\gamma \cap B_{x,r} = \bigcup_i \gamma((a_i, b_i))$ and $\gamma \cap \partial B_{x,r} = \bigcup_i \{\gamma(a_i), \gamma(b_i)\}$. To reach a contradiction, we assume that for every i there exists a connected component A_i of $\mathrm{cl}(B_{x,r}) \setminus K$ such that $\gamma(a_i), \gamma(b_i) \in A_i$. (Note that, possibly, $A_i = A_j$ for some $i \neq j$.) By connectedness of A_i, for each i we can find a smooth embedding $\tau_i : I_i \to A_i$ such that $\tau_i(a_i) = \gamma(a_i)$ and $\tau_i(b_i) = \gamma(b_i)$; moreover, one can easily achieve this by enforcing $\tau_i(I_i) \cap \tau_j(I_j) = \emptyset$. Finally, we define $\bar{\gamma}$ by setting $\bar{\gamma} = \gamma$ on $S^1 \setminus \bigcup_i I_i$, and $\bar{\gamma} = \tau_i$ on I_i. In this way, $[\bar{\gamma}] = [\gamma]$ in $\pi_1(\mathbb{R}^{n+1} \setminus H)$, with $\bar{\gamma} \cap K \setminus \mathrm{cl}(B_{x,r}) = \gamma \cap K \setminus \mathrm{cl}(B_{x,r}) = \emptyset$ and $\bar{\gamma} \cap K \cap \mathrm{cl}(B_{x,r}) = \emptyset$ by construction; that is, $\bar{\gamma} \cap K = \emptyset$. Since there exists $\widetilde{\gamma} \in \mathcal{C}_H$ with $[\widetilde{\gamma}] = [\bar{\gamma}] = [\gamma]$ in $\pi_1(\mathbb{R}^{n+1} \setminus H)$ which is uniformly close to $\bar{\gamma}$, we get $\widetilde{\gamma} \cap K = \emptyset$, contradicting $K \in \mathcal{F}(H, \mathcal{C})$.

Step 2. We now prove the lemma for any ball $B_{x,r} \subset\subset \mathbb{R}^{n+1} \setminus H$. Since γ is a smooth embedding, by Sard's theorem we know that γ and $\partial B_{x,s}$ intersect transversally for a.e. $s > 0$. In particular, given ε small enough, for any such $s \in (r - \varepsilon, r)$ we can construct a smooth diffeomorphism $f_s : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that $f_s = \mathrm{Id}$ on $\mathbb{R}^{n+1} \setminus B_{x,r+2\varepsilon}$ and $f_s(y) = x + (r/s)(y - x)$ for $y \in B_{x,r+\varepsilon}$, so that

$$f_s \to \mathrm{Id} \quad \text{uniformly on } \mathbb{R}^{n+1} \text{ as } s \to r^-. \tag{3.1}$$

We claim that one can apply Step 1 to $f_s \circ \gamma$. Indeed, the facts that $f_s \circ \gamma \in \mathcal{C}$ and $f_s \circ \gamma$ and $\partial B_{x,r}$ intersect transversally are straightforward; moreover, since $\mathrm{dist}(\gamma, K \cap \partial B_{x,r}) > 0$, (3.1) easily implies that $(f_s \circ \gamma) \cap K \setminus B_{x,r} = \emptyset$. Hence, by Step 1, there exists a proper circular arc $I = [a_s, b_s] \subset S^1$ such that $f_s(\gamma(a_s)) \in A_{i(s)}$ and $f_s(\gamma(b_s)) \in A_{j(s)}$ for some connected components $A_i \neq A_j$ of $\mathrm{cl}(B_{x,r}) \setminus K$, and $(f_s \circ \gamma)(a_s, b_s) \subset B_{x,r}$. Up to subsequences, we can assume that $a_s \to \bar{a}$, $b_s \to \bar{b}$ and the arc $[a_s, b_s]$ converges to $[\bar{a}, \bar{b}]$. It follows that $\gamma(\bar{a})$ and $\gamma(\bar{b})$ must belong to distinct connected components of $\mathrm{cl}(B_{x,r}) \setminus K$, as otherwise by (3.1), $f_s(\gamma(a_s))$ and $f_s(\gamma(b_s))$ would belong to the same connected component for some s close enough to r. By (3.1) we also have $\gamma([\bar{a}, \bar{b}]) \subset \mathrm{cl}(B_{x,r})$.

The argument for cubes Q is a routine modification of the one given above, and is left to the reader. $\qquad\square$

Proof of Theorem 4. Step 1. We start by showing that $\mathcal{F}(H, \mathcal{C})$ is a good class in the sense of Definition 1. To this end, we fix $V \in \mathcal{F}(H, \mathcal{C})$ and $x \in V$, and prove that for a.e. $r \in (0, \text{dist}(x, H))$ one has $V', V'' \in \mathcal{F}(H, \mathcal{C})$, where V' is the cone competitor of V in $B_{x,r}$, and V'' is a cup competitor of V in $B_{x,r}$.

We thus fix $\gamma \in \mathcal{C}$ and, without loss of generality, we assume that $\gamma \cap (V \setminus B_{x,r}) = \emptyset$. By Lemma 10, γ has an arc contained in $\text{cl}(B_{x,r})$ homeomorphic to $[0, 1]$ and whose end-points belong to distinct connected components of $\partial B_{x,r} \setminus V$; we denote by $\sigma : [0, 1] \to \text{cl}(B_{x,r})$ a parametrization of this arc. By construction, either $\sigma(0)$ or $\sigma(1)$ must belong to $\gamma \cap V'' \cap \partial B_{x,r}$. This proves that $V'' \in \mathcal{F}(H, \mathcal{C})$. We now show that $\gamma \cap V' \cap \text{cl}(B_{x,r}) \neq \emptyset$. If $x \in \sigma$, then trivially $V' \cap \sigma \neq \emptyset$; if $x \notin \sigma$, then we can project σ radially on $\partial B_{x,r}$, and the projection $\pi \circ \sigma$ must intersect $V' \cap \partial B_{x,r} = V \cap \partial B_{x,r}$ by connectedness. If z is such an intersection point, then $V' \supset \pi^{-1}(z) \cap \sigma([0, 1]) \neq \emptyset$, as $\pi^{-1}(z) = \lambda z$ for some $\lambda \in (0, 1)$. This proves that $V' \in \mathcal{F}(H, \mathcal{C})$.

Step 2. By Step 1, given a minimizing sequence $\{K_j\} \subset \mathcal{F}(H, \mathcal{C})$ which consists of rectifiable sets, we can find a set K with the properties stated in Theorem 2. In order to prove the second statement in (b) we just need to show that $K \in \mathcal{F}(H, \mathcal{C})$.

Suppose for contradiction that some $\gamma \in \mathcal{C}$ does not intersect K. Since both γ and K are compact, there exists $\varepsilon > 0$ such that the tubular neighborhood $U_{2\varepsilon}(\gamma)$ does not intersect K and is contained in $\mathbb{R}^{n+1} \setminus H$. Hence $\mu(U_{2\varepsilon}(\gamma)) = 0$, and thus

$$\lim_{j \to \infty} \mathcal{H}^n(K_j \cap U_\varepsilon(\gamma)) = 0. \tag{3.2}$$

Observe that if ε is small enough, there is a diffeomorphism $\Phi : S^1 \times D_\varepsilon \to U_\varepsilon(\gamma)$ such that $\Phi|_{S^1 \times \{0\}} = \gamma$, where $D_\rho := \{y \in \mathbb{R}^n : |y| < \rho\}$. Denote by γ_y the parallel curve $\Phi|_{S^1 \times \{y\}}$. Then $\gamma_y \in [\gamma] \in \pi_1(\mathbb{R}^{n+1} \setminus H)$ for every $y \in D_\varepsilon$. Thus we must have $K_j \cap (\gamma \times \{y\}) \neq \emptyset$ for every $y \in D_\varepsilon$ and every $j \in \mathbb{N}$. If we let $\hat{\pi} : S^1 \times D_\varepsilon \to D_\varepsilon$ be the projection on the second factor and define $\pi : U_\varepsilon(\gamma) \to D_\varepsilon$ as $\hat{\pi} \circ \Phi^{-1}$, then π is a Lipschitz map. The coarea formula then implies

$$\mathcal{H}^n(K_j \cap U_\varepsilon(\gamma)) \geq \frac{\omega_n \varepsilon^n}{(\text{Lip}\,\pi)^n} > 0,$$

which contradicts (3.2). This shows that $K \in \mathcal{F}(H, \mathcal{C})$, as claimed.

Step 3. We show that K is $(\mathbf{M}, 0, \infty)$-minimal, i.e.

$$\mathcal{H}^n(K) \leq \mathcal{H}^n(\varphi(K))$$

whenever $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a Lipschitz map such that $\varphi = \text{Id}$ on $\mathbb{R}^{n+1} \setminus B_{x,r}$ and $\varphi(B_{x,r}) \subset B_{x,r}$ for some $x \in \mathbb{R}^{n+1} \setminus H$ and $r < \text{dist}(x, H)$. To this end, it suffices to show that given such a φ, we have $\varphi(K) \in \mathcal{F}(H, \mathcal{C})$.

We fix $\gamma \in \mathcal{C}$ and assume that $\gamma \cap (K \setminus B_{x,\rho}) = \emptyset$ for some $\rho \in (r, \text{dist}(x, H))$. By Lemma 10, there exist two distinct connected components A and A' of $B_{x,\rho} \setminus K$ and a connected component of $\gamma \cap \text{cl}(B_{x,\rho})$ having end-points $p \in \text{cl}(A) \cap \partial B_{x,\rho}$ and

$q \in \mathrm{cl}(A') \cap \partial B_{x,\rho}$. We complete the proof by showing that $p = \varphi(p)$ and $q = \varphi(q)$ are in the closures of distinct connected components of $B_{x,\rho} \setminus \varphi(K)$.

To reach a contradiction, denote by Ω the connected component of $B_{x,\rho} \setminus \varphi(K)$ with $p, q \in \mathrm{cl}(\Omega)$. If h denotes the restriction of φ to $\mathrm{cl}(A)$, then the topological degree of h is defined on $\mathbb{R}^{n+1} \setminus h(\partial A)$, thus in Ω. (Recall that the degree of a continuous function h is an integer valued locally constant function equal to the algebraic sum of the numbers of preimages in $h^{-1}(y)$, the sign being chosen according to whether h preserves or changes orientation; see [Hir94, Ch. 5] for more details.) Since $\varphi = \mathrm{Id}$ in a neighborhood of $\partial B_{x,\rho}$, one has $\deg(h, p') = 1$ for every p' sufficiently close to p; since the degree is locally constant and Ω is connected, $\deg(h, \cdot) = 1$ on Ω. In particular, $\varphi^{-1}(y) \cap A \neq \emptyset$ for every $y \in \Omega$. We apply this with $y = q'$ for some $q' \in \Omega$ sufficiently close to q. Let $w \in \varphi^{-1}(q')$; since $\varphi = \mathrm{Id}$ on $\mathbb{R}^{n+1} \setminus B_{x,r}$, if $|q'| > r$ then $w = q'$, and thus $q' \in A$. In other words, every $q' \in B_{x,\rho}$ sufficiently close to q is contained in A. We may thus connect in A any pair of points $p', q' \in B_{x,\rho}$ which are sufficiently close to p and q respectively, that is, p and q can be connected in A. This contradicts $A \neq A'$, and completes the proof of the fact that K is an $(\mathbf{M}, 0, \infty)$-minimal set. We are thus left to prove (b).

Step 4. We want to show that given $K \in \mathcal{F}(H, \mathcal{C})$ with $\mathcal{H}^n(K) < \infty$, there exists $K' \in \mathcal{F}(H, \mathcal{C})$ rectifiable such that $\mathcal{H}^n(K') \leq \mathcal{H}^n(K)$. The proof is divided into three further steps. By [Fed69, 2.10.25], $0 = (\omega_1 \omega_n / \omega_{n+1}) \mathcal{H}^{n+1}(K) \geq \int_{\mathbb{R}}^* \mathcal{H}^n(K \cap \{x_1 = t\}) \, dt$, thus $\mathcal{L}^1(\{t \in \mathbb{R} : \mathcal{H}^n(K \cap \{x_1 = t\}) > 0\}) = 0$. In particular,

$$\mathcal{L}^1\left(\bigcup_{j \in \mathbb{N}} \left\{ t \in (0, 1) : \mathcal{H}^n\left(K \cap \bigcup_{h \in \mathbb{Z}} \{x_1 = t + h/2^j\}\right) > 0 \right\}\right) = 0,$$

so that, for a suitable $x_1^0 \in (0, 1)$ one has $\mathcal{H}^n(K \cap \{x_1 = x_1^0 + 2^{-j} h\}) = 0$ for every $j \in \mathbb{N}$, $h \in \mathbb{Z}$. This argument can be repeated for each coordinate, giving a point $x^0 \in \mathbb{R}^{n+1}$ such that $\mathcal{H}^n(K \cap \{x_m = x_m^0 + 2^{-j} h\}) = 0$ for every $m \in \{1, \ldots, n+1\}$, $j \in \mathbb{N}$, $h \in \mathbb{Z}$. As a consequence, one finds a grid $\mathcal{Q}$ of open dyadic cubes such that $\mathcal{H}^n(K \cap \partial Q) = 0$ for every $Q \in \mathcal{Q}$. We let $\mathcal{W}$ be the Whitney covering of $\mathbb{R}^{n+1} \setminus H$ obtained from $\mathcal{Q}$ as in [Ste70, Theorem 3, p. 16], so that if Q' is the concentric cube with twice the size of $Q \in \mathcal{W}$, then $Q' \cap H = \emptyset$.

Step 5. First, for every $Q \in \mathcal{W}$ we define a suitable replacement K_Q in the cube Q such that $K_Q \cap \mathrm{cl}(Q)$ is $\mathcal{H}^n$-rectifiable with $\mathcal{H}^n(K_Q \cap \mathrm{cl}(Q)) \leq \mathcal{H}^n(K \cap \mathrm{cl}(Q))$ and $K_Q \setminus \mathrm{cl}(Q) = K \setminus \mathrm{cl}(Q)$. Let us denote by $\{F_i\}_i$ the family of connected components of $Q' \setminus K$ and consider the partitioning problem (into Caccioppoli sets, cf. for instance [AFP00, Section 4.4])

$$\inf\left\{ \mathcal{H}^n\left(Q' \cap \bigcup_i \partial^* E_i\right) : \{E_i\}_i \text{ is a partition modulo } \mathcal{H}^{n+1} \text{ of } Q' \text{ with} \right.$$
$$\left. E_i \setminus Q = F_i \setminus Q \right\}. \quad (3.3)$$

Since F_i is open with $\partial F_i \subset K$ and $\mathcal{H}^n(K) < \infty$, the infimum in (3.3) is finite and there exists a minimizing partition $\{E_i\}_i$ (one can apply, for instance, [AFP00, Theorem 4.19

& Remark 4.20]). Let the closed set K_Q be given by

$$K_Q := (K \setminus Q) \cup \left(\mathrm{cl}(Q) \cap \mathrm{cl}\left(\bigcup_i \partial^* E_i \right) \right).$$

By a slight modification of [Mag12, Lemma 30.2], $\mathcal{H}^n(Q \cap (K_Q \setminus \bigcup_i \partial^* E_i)) = 0$, so that $\mathrm{cl}(Q) \cap K_Q$ is countably $\mathcal{H}^n$-rectifiable. To prove $\mathcal{H}^n(K_Q \cap \mathrm{cl}(Q)) \leq \mathcal{H}^n(K \cap \mathrm{cl}(Q))$ it suffices to show

$$\mathcal{H}^n \left(\mathrm{cl}(Q) \cap \left(K_Q \setminus \bigcup_i \partial^* E_i \right) \right) = 0.$$

Since $\mathcal{H}^n(Q \cap (K_Q \setminus \bigcup_i \partial^* E_i)) = 0$ and $\mathcal{H}^n(K \cap \partial Q) = 0$, we just need to prove

$$\mathcal{H}^n \left(\partial Q \cap \left((K_Q \setminus K) \setminus \bigcup_i \partial^* E_i \right) \right) = 0.$$

In turn, by [Mag12, Corollary 6.5], it is enough to find $c_0 > 0$ such that

$$\mathcal{H}^n \left(B_{x,r} \cap \bigcup_i \partial^* E_i \right) \geq c_0 r^n, \quad \forall x \in \partial Q \cap (K_Q \setminus K), \ \forall r < r_x = \mathrm{dist}(x, K \setminus Q). \quad (3.4)$$

To do so, let i_0 be such that $x \in F_{i_0}$ and, for $r < r_x$, let $G_i = E_i \setminus B_{x,r}$ if $i \neq i_0$, and $G_{i_0} = E_{i_0} \cup B_{x,r}$. Since $\{G_i\}_i$ is admissible in (3.3), we find that

$$f(r) := \mathcal{H}^n \left(\mathrm{cl}(B_{x,r}) \cap \bigcup_i \partial^* E_i \right) \leq \mathcal{H}^n \left(\mathrm{cl}(B_{x,r}) \cap \bigcup_i \partial^* G_i \right) = \mathcal{H}^n \left(\partial B_{x,r} \cap \bigcup_i \partial^* G_i \right).$$

We next denote by $E_i^{(\tau)}$ the set of points x of density τ of the set E_i:

$$\lim_{r \to 0} \frac{\mathcal{H}^{n+1}(B_{x,r} \cap E_i)}{\omega_{n+1} r^{n+1}} = \tau.$$

Now, for a.e. $r < r_x$, one has $\mathcal{H}^n(\partial B_{x,r} \cap (E_{i_0}^{(0)} \triangle \partial^* G_{i_0})) = 0$, as well as

$$\mathcal{H}^n(\partial B_{x,r} \cap (E_i^{(1)} \triangle \partial^* G_i)) = 0, \quad \forall i \neq i_0, \quad \mathcal{H}^n \left(\partial B_{x,r} \cap \left(E_{i_0}^{(0)} \triangle \bigcup_{i \neq i_0} E_i^{(1)} \right) \right) = 0.$$

We thus find that $f(r) \leq \mathcal{H}^n(\partial B_{x,r} \cap E_{i_0}^{(0)})$ for a.e. $r < r_x$; now, again for a.e. $r < r_x$, the set $\partial B_{x,r} \cap E_{i_0}^{(0)}$ has finite perimeter in $\partial B_{x,r}$ with

$$\mathcal{H}^{n-1} \left(\partial_{\partial B_{x,r}}^* (\partial B_{x,r} \cap E_{i_0}^{(0)}) \triangle (\partial B_{x,r} \cap \partial^* E_{i_0}) \right) = 0;$$

since $\mathcal{H}^n(\partial B_{x,r} \setminus E_{i_0}^{(0)}) \geq \mathcal{H}^n(\partial B_{x,r})/2$ by convexity of Q, the isoperimetric inequality on $\partial B_{x,r}$ yields $f(r) \leq C(n) \mathcal{H}^{n-1}(\partial^* E_{i_0} \cap \partial B_{x,r})^{n/(n-1)} \leq C(n) f'(r)$ for a.e. $r < r_x$. By arguing as in Step 1 of the proof of Theorem 2, we complete the proof of (3.4).

Step 6. We finally set $K' := \bigcup_{Q \in \mathcal{W}} K_Q \cap \mathrm{cl}(Q)$. By Step 2, K' is $\mathcal{H}^n$-rectifiable, with

$$\mathcal{H}^n(K') \leq \sum_{Q \in \mathcal{W}} \mathcal{H}^n(K_Q \cap \mathrm{cl}(Q)) \leq \sum_{Q \in \mathcal{W}} \mathcal{H}^n(K \cap \mathrm{cl}(Q)) = \sum_{Q \in \mathcal{W}} \mathcal{H}^n(K \cap Q),$$

where in the last identity we have used Step 4. This shows that $\mathcal{H}^n(K') \leq \mathcal{H}^n(K)$. We now prove that $K' \in \mathcal{F}(H, \mathcal{C})$. Let $\gamma \in \mathcal{C}$, so that $\gamma \cap K \cap \mathrm{cl}(Q) \neq \emptyset$ for some $Q \in \mathcal{W}$. Since $K \cap \partial Q \subset K_Q \cap \partial Q \subset K' \cap \partial Q$, we may directly assume that $\gamma \cap K \cap Q \neq \emptyset$. By Lemma 10, there exists a connected component σ of $\gamma \cap \mathrm{cl}(Q)$ with end-points $p \in F_i \cap \partial Q$ and $q \in F_j \cap \partial Q$ for some distinct connected components F_i and F_j of $\mathrm{cl}(Q) \setminus K$. If either p or q belongs to K_Q there is nothing to prove; otherwise, $p \in E_i$ and $q \in E_j$. In particular, by connectedness of σ, we must have $\sigma \cap K_Q \cap \mathrm{cl}\, Q \neq \emptyset$. This completes the proof of (b). □

4. Proof of Theorem 7

Step 1. In this and in the next step we prove that $\mathcal{A}(H, K_0)$ is a good class in the sense of Definition 1. Let $K \in \mathcal{A}(H, K_0)$; in this step we show (1.4) when L is a cup competitor in $B_{x,r} \subset \mathbb{R}^{n+1} \setminus H$ (at least for a.e. r). We therefore consider $B_{x,r} \subset\subset \mathbb{R}^{n+1} \setminus H$ and assume further $\mathcal{H}^n(K \cap \partial B_{x,r}) = 0$, which holds for a.e. r. Also, for convenience we can translate, rescale and assume $x = 0$ and $r = 1$; we then write B instead of $B_{0,1}$. Consider the cup competitor of K in B defined by a given connected component A of $\partial B \setminus K$. Its Hausdorff measure is $\mathcal{H}^n(K \setminus B) + \mathcal{H}^n(\partial B \setminus A)$. Our goal is thus to show that, for any given $\sigma > 0$, there is $J \in \mathcal{A}(H, K_0)$ with the property that $J \setminus \mathrm{cl}(B) = K \setminus \mathrm{cl}(B)$ and $\mathcal{H}^n(J) \leq \mathcal{H}^n(K \setminus B) + \mathcal{H}^n(\partial B \setminus A) + \sigma$, so

$$\mathcal{H}^n(J \cap \mathrm{cl}(B)) \leq \mathcal{H}^n(\partial B \setminus A) + \sigma. \tag{4.1}$$

By definition we need a map $\phi_3 \in \Sigma(H)$ such that $J = \phi_3(K)$. In fact we will build ϕ_3 so that $\phi_3 = \mathrm{Id}$ on $\mathbb{R}^{n+1} \setminus B_{1+\eta}$ for some sufficiently small η.

ϕ_3 will be constructed building upon two additional maps ϕ_1 and ϕ_2. To construct ϕ_1 we just fix $x_0 \in A$ and a small ρ so that $B_{x_0,\rho} \cap K = \emptyset$. Then ϕ_1 projects $B \setminus B_{x_0,\rho}$ onto ∂B along the rays emanating from x_0, while it "stretches" $B_{x_0,\rho} \cap \mathrm{cl}(B)$ onto $\mathrm{cl}(B)$. In doing so, we achieve that $K_1 = \phi_1(K \cap \mathrm{cl}(B))$ is contained in ∂B and is disjoint from $B_{x_0,\rho}$.

We next claim that there exists a Lipschitz map $\phi_2 : \partial B \to \partial B$ with $\phi_2 = \mathrm{Id}$ on $U_\varepsilon(K \cap \partial B)$ for some positive ε and

$$\mathcal{H}^n(\phi_2(K_1)) \leq \mathcal{H}^n(\partial B \setminus A) + \sigma. \tag{4.2}$$

The existence of ϕ_2 will be shown in a moment.

Corresponding to ε, we can find $\eta > 0$ such that $B_{1+\eta} \subset\subset \mathbb{R}^{n+1} \setminus H$ and

$$\frac{K \cap \partial B_{1+t}}{1+t} \subset U_\varepsilon(K \cap \partial B), \quad \forall t \in (0, \eta).$$

Finally, we define $\phi_3 : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by setting

$$
\phi_3(x) := \begin{cases} \phi_2(\phi_1(x)) & \text{for } |x| < 1, \\ \dfrac{|x| - 1}{\eta} x + \dfrac{1 + \eta - |x|}{\eta} \phi_2(\phi_1(x)) & \text{for } 1 \le |x| < 1 + \eta, \\ x, & \text{for } |x| \ge 1 + \eta. \end{cases}
$$

Notice that ϕ_3 is a Lipschitz map with

$$
\phi_3 = \text{Id} \quad \text{on } (\mathbb{R}^{n+1} \setminus B_{1+\eta}) \cup \{(1 + t) x : t \in (0, \eta), \, x \in U_\varepsilon(K \cap \partial B)\}.
$$

In particular, $J \setminus \text{cl}(B) = \phi_3(K \setminus \text{cl}(B)) = K \setminus \text{cl}(B)$ and $J \cap \text{cl}(B) = \phi_3(K \cap \text{cl}(B)) = \phi_2(K_1)$ and, by (4.2), (4.1) holds.

Thus it remains to construct the map ϕ_2. Up to conjugation with a stereographic projection with pole x_0, the existence of ϕ_2 is reduced to the following problem. Given

(i) a connected open set $\Omega \subset \mathbb{R}^n$ with bounded complement and with $\mathcal{H}^n(\partial \Omega) = 0$,
(ii) a ball $B_R \subset \mathbb{R}^n$ such that $\partial \Omega \subset\subset B_R$ and
(iii) a $\sigma > 0$,

find $\varepsilon > 0$ and a Lipschitz map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ such that

(a) $\phi = \text{Id}$ on $U_\varepsilon(\partial \Omega) \cup (\mathbb{R}^n \setminus \Omega) \cup \mathbb{R}^n \setminus B_{2R}$ and
(b) $\mathcal{H}^n(\phi(B_R \cap \Omega)) < \sigma$.

This can be achieved as follows. Let $\mathcal{W}$ be the Whitney decomposition of $B_{2R} \cap \Omega$, constructed from the standard family of dyadic cubes in $\mathbb{R}^n$. Given $\varepsilon > 0$ we can find a "face connected" finite subfamily $\mathcal{W}_0$ of $\mathcal{W}$ such that

$$
(B_R \cap \Omega) \setminus U_\varepsilon(\partial \Omega) \subset \bigcup_{Q \in \mathcal{W}_0} Q,
$$

and for which there exists $Q_0 \in \mathcal{W}_0$ with $Q_0 \setminus B_R \ne \emptyset$. We now construct a Lipschitz map $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that $f = \text{Id}$ on $\mathbb{R}^{n+1} \setminus \bigcup_{Q \in \mathcal{W}_0} Q$ with

$$
f\left(\bigcup_{Q \in \mathcal{W}_0} Q \cap B_R \right) \subset \bigcup_{Q \in \mathcal{W}_0} \partial Q.
$$

To this end we choose a ball $U_0 \subset\subset Q_0 \setminus B_R$, and then define a Lipschitz map $f_0 : \mathbb{R}^n \to \mathbb{R}^n$ with $f_0 = \text{Id}$ on $\mathbb{R}^n \setminus Q_0$, $f_0(U_0) = Q_0$ and $f_0(Q_0 \setminus U_0) = \partial Q_0$ by projecting $Q_0 \setminus U_0$ radially from the center of U_0 onto ∂Q_0, and then stretching U_0 onto Q_0. Let now $Q_1 \in \mathcal{W}_0$ share a hyperface with Q_0, so that the side-length of Q_1 is at most twice that of Q_0. In case the side of Q_1 is twice that of Q_0, we subdivide Q_1 into 2^n subcubes and denote by $\hat{Q}_1$ the one sharing a hyperface with Q_0; otherwise we set $\hat{Q}_1 = Q_1$. Let $x_1 \in Q_0$ be the reflection of the center of $\hat{Q}_1$ with respect to the common hyperface between Q_0 and $\hat{Q}_1$. Then we can find a ball $U_1 \subset\subset Q_0$ and define a Lipschitz map $\hat{f}_1 : \mathbb{R}^n \to \mathbb{R}^n$ such that $\hat{f}_1 = \text{Id}$ on $\mathbb{R}^n \setminus (Q_0 \cup \hat{Q}_1)$, $\hat{f}_1((\hat{Q}_1 \cup Q_0) \setminus U_1) \subset \partial(Q_0 \cup \hat{Q}_1)$ and $\hat{f}_1(U_1) = \hat{Q}_1 \cup Q_0$. In the case when $\hat{Q}_1 \ne Q_1$ we perform a further radial projection onto ∂Q_1 from a small ball centered at the center of $\hat{Q}_1$. In this way we construct a Lipschitz

map $f_1 : \mathbb{R}^n \to \mathbb{R}^n$ such that $f_1 = \text{Id}$ on $\mathbb{R}^n \setminus (Q_0 \cup Q_1)$, $f_1((Q_1 \cup Q_0) \setminus U_1) \subset \partial Q_0 \cup \partial Q_1$ and $f_1(U_1) = Q_1 \cup Q_0$. Thus $g_1 := f_1 \circ f_0$ is a Lipschitz map such that $g_1 = \text{Id}$ on $\mathbb{R}^n \setminus (Q_0 \cup Q_1)$ and $g_1((Q_0 \cup Q_1) \setminus U_0) \subset \partial Q_0 \cup \partial Q_1$. A simple iteration concludes the proof.

Step 2. In this step we address cone competitors. As before, we consider balls B_r centered at 0 with $B_r \subset\subset \mathbb{R}^{n+1} \setminus H$. We assume in addition that $K \cap \partial B_r$ is $\mathcal{H}^{n-1}$-rectifiable with $\mathcal{H}^{n-1}(K \cap \partial B_r) < \infty$ and that r is a Lebesgue point of $(0, \infty) \ni t \mapsto \mathcal{H}^{n-1}(K \cap \partial B_t)$. All these conditions are fullfilled for a.e. r, and again by scaling we can assume that $r = 1$ and use B instead of B_1. Let K' denote the cone competitor of K in B. For $s \in (0, 1)$ set

$$\varphi_s(r) := \begin{cases} 0, & r \in [0, 1 - s), \\ \dfrac{r - (1 - s)}{s}, & r \in [1 - s, 1], \\ r, & r \geq 1, \end{cases}$$

and $\phi_s(x) := \varphi_s(|x|)x/|x|$ for $x \in \mathbb{R}^{n+1}$. Then $\phi_s : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a Lipschitz map with $\phi_s = \text{Id}$ on $\mathbb{R}^{n+1} \setminus B$. In particular, $\phi_s(K) \setminus B = K \setminus B$, and thus we only need to show that

$$\limsup_{s \to 0^+} \mathcal{H}^n(\phi_s(K \cap B)) \leq \mathcal{H}^n(K' \cap B).$$

Since $\phi_s(K \cap B_{1-s}) = \{0\}$, we just have to show that

$$\limsup_{s \to 0^+} \mathcal{H}^n(\phi_s(K) \cap (B \setminus B_{1-s})) \leq \mathcal{H}^{n-1}(K \cap \partial B)/n.$$

Denoting by $J^K \phi_s$ the tangential Jacobian of ϕ_s with respect to K, we find

$$\mathcal{H}^n(\phi_s(K) \cap (B \setminus B_{1-s})) = \int_{K \cap (B \setminus B_{1-s})} J^K \phi_s \, d\mathcal{H}^n$$

$$= \int_{1-s}^1 dt \int_{K \cap \partial B_t \cap \{v \cdot \hat{x} < 1\}} \frac{J^K \phi_s}{\sqrt{1 - (v \cdot \hat{x})^2}} \, d\mathcal{H}^{n-1} + \int_{K \cap (B \setminus B_{1-s}) \cap \{v \cdot \hat{x} = 1\}} J^K \phi_s \, d\mathcal{H}^n, \quad (4.3)$$

where $v(x) \in S^{n+1} \cap (T_x K)^{\perp}$ for $\mathcal{H}^n$-a.e. $x \in K$ and $\hat{x} := x/|x|$. We first notice that on the set $K \cap (B \setminus B_{1-s}) \cap \{v \cdot \hat{x} = 1\}$ we have $J^K \phi_s \leq 1$. Moreover

$$\lim_{s \to 0} \mathcal{H}^n(K \cap (B \setminus B_{1-s})) = 0,$$

and thus the second term in (4.3) can be ignored. At the same time, for a constant C,

$$J^K \phi_s(x) \leq C + \sqrt{1 - (\hat{x} \cdot v)^2} \, \varphi_s'(|x|)(\varphi_s(|x|)/|x|)^{n-1} \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in K.$$

The constant C gives a negligible contribution to the integral as $s \downarrow 0$; as for the second term, having $\varphi_s' = 1/s$ on $(1 - s, 1)$, we find

$$\int_{1-s}^1 \mathcal{H}^{n-1}(K \cap \partial B_t) \varphi_s'(t) \left(\frac{\varphi_s(t)}{t} \right)^{n-1} dt = \frac{1}{s} \int_{1-s}^1 \mathcal{H}^{n-1}(K \cap \partial B_t) \left(\frac{\varphi_s(t)}{t} \right)^{n-1} dt.$$

Since $t = 1$ is a Lebesgue point of $(0, \infty) \ni t \mapsto \mathcal{H}^{n-1}(K \cap \partial B_t)$, we have

$$\lim_{s \to 0} \frac{1}{s} \int_{1-s}^{1} |\mathcal{H}^{n-1}(K \cap \partial B_t) - \mathcal{H}^{n-1}(K \cap \partial B)| \, dt = 0,$$

so that combining the above remarks we find

$$\limsup_{s \to 0^+} \mathcal{H}^n(\phi_s(K \cap B)) \le \mathcal{H}^{n-1}(K \cap \partial B) \limsup_{s \to 0^+} \frac{1}{s} \int_{1-s}^{1} \left(\frac{\varphi_s(t)}{t} \right)^{n-1} dt$$

$$= \frac{\mathcal{H}^{n-1}(K \cap \partial B)}{n},$$

as required. This completes the proof that $\mathcal{A}(H, K_0)$ is a good class.

Step 3. Having proved the first statement of the theorem, we now show the rest. Under the rectifiability assumption on K_0, any minimizing sequence in $\mathcal{A}(H, K_0)$ consists of rectifiable sets, and we can therefore apply Theorem 2. Thus $\mathcal{H}^n \llcorner K_j \overset{*}{\rightharpoonup} \mu = \theta \mathcal{H}^n \llcorner K$, where K is countably $\mathcal{H}^n$-rectifiable and $\theta \ge 1$. Moreover we assume that $\varepsilon_j \downarrow 0$ quantifies the almost minimality of K_j, namely $\inf\{\mathcal{H}^n(J) : J \in \mathcal{A}(H, K_0)\} \ge \mathcal{H}^n(K_j) - \varepsilon_j$.

In this step we prove that $\theta \le 1$ μ-a.e. For contradiction, assume that $\theta(x) = 1 + \sigma > 1$ for some x where K admits an approximate tangent plane T (cf. Step 3 in the proof of Theorem 2). Without loss of generality we can assume $x = 0$ and $T = \{y : y_{n+1} = 0\}$. By (1.5), we can find $r_0 > 0$ such that

$$K \cap B_r \subset S_{\varepsilon r}, \quad 1 + \sigma \le \frac{\mu(\mathrm{cl}(B_r))}{\omega_n r^n} \le 1 + \sigma + \varepsilon\sigma, \quad \forall r < r_0, \qquad (4.4)$$

where $S_{\varepsilon r} = B_r \cap \{|x_{n+1}| < \varepsilon r\}$. If we fix any $r < r_0$ we then find $j_0 = j_0(r) \in \mathbb{N}$ such that

$$\mathcal{H}^n(K_j \cap B_r) > \left(1 + \frac{\sigma}{2}\right)\omega_n r^n, \quad \mathcal{H}^n((K_j \cap B_r) \setminus S_{\varepsilon r}) < \frac{\sigma}{4}\omega_n r^n, \quad \forall j \ge j_0, \qquad (4.5)$$

and thus

$$\mathcal{H}^n(K_j \cap S_{\varepsilon r}) > \left(1 + \frac{\sigma}{4}\right)\omega_n r^n, \quad \forall j \ge j_0.$$

Set

$$X_{\varepsilon r} := \{x = (x', x_{n+1}) \in S_{\varepsilon r} : |x'| < (1 - \sqrt{\varepsilon})r\},$$

and define $f : X_{\varepsilon r} \cup (\mathbb{R}^{n+1} \setminus B_r) \to \mathbb{R}^{n+1}$ with $f(x) := (x', 0)$ if $x \in X_{\varepsilon r}$ and $f(x) := x$ otherwise. In this way $\mathrm{Lip}\, f \le 1 + C\sqrt{\varepsilon}$, and thus by Kirszbraun's theorem [Fed69, 2.10.43] there exists a Lipschitz extension $\hat{f} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with $\mathrm{Lip}\, \hat{f} \le 1 + C\sqrt{\varepsilon}$. Such an extension belongs to $\Sigma(H)$ and we thus find

$$\mathcal{H}^n(K_j \cap B_r) - \varepsilon_j$$

$$\le \underbrace{\mathcal{H}^n(\hat{f}(K_j \cap X_{\varepsilon r}))}_{I} + \underbrace{\mathcal{H}^n(\hat{f}(K_j \cap (S_{\varepsilon r} \setminus X_{\varepsilon r})))}_{II} + \underbrace{\mathcal{H}^n(\hat{f}(K_j \cap (B_r \setminus S_{\varepsilon r})))}_{III}.$$

By construction, $I \leq \omega_n r^n$, while, by (4.5), $\mathcal{H}^n(K_j \cap B_r) > (1 + \sigma/2)\omega_n r^n$ and

$$III \leq (\operatorname{Lip} \hat{f})^n \mathcal{H}^n(K_j \cap (B_r \setminus S_{\varepsilon r})) < (1 + C\sqrt{\varepsilon})^n \frac{\sigma}{4} \omega_n r^n.$$

Hence, as $j \to \infty$,

$$\left(1 + \frac{\sigma}{2}\right) \omega_n r^n \leq \omega_n r^n + \liminf_{j \to \infty} II + (1 + C\sqrt{\varepsilon})^n \frac{\sigma}{4} \omega_n r^n,$$

that is,

$$\left(\frac{1}{2} - \frac{(1 + C\sqrt{\varepsilon})^n}{4}\right) \sigma \leq \liminf_{j \to \infty} \frac{II}{\omega_n r^n}. \tag{4.6}$$

By (4.4) and again by the monotonicity of $s^{-n}\mu(B_s)$, we finally estimate

$$\limsup_{j \to \infty} II \leq (1 + C\sqrt{\varepsilon})^n \mu(\operatorname{cl}(B_r) \setminus B_{(1-\sqrt{\varepsilon})r})$$

$$\leq (1 + C\sqrt{\varepsilon})^n \big((1 + \sigma + \varepsilon\sigma) - (1 + \sigma)(1 - \sqrt{\varepsilon})^n\big) \omega_n r^n. \tag{4.7}$$

However, since $\sigma > 0$, (4.6) and (4.7) are incompatible when ε is sufficiently small.

Step 4. We show that $\mathcal{H}^n(K_j) \to \mathcal{H}^n(K)$, thus proving the first equality in (1.8). Note that since $\mathcal{H}^n \llcorner K_j$ weakly converges to $\mathcal{H}^n \llcorner K$ in $\mathbb{R}^{n+1} \setminus H$, we only need to exclude any concentration of mass in H in the limit, as well as any loss of mass at infinity.

We first let $R_0 > 0$ be such that $H \subset B_{R_0}$ and consider the Lipschitz map $\varphi(x) := \min\{|x|, R_0\}x/|x|$. Obviously $\varphi \in \Sigma(H)$ and we easily compute

$$\mathcal{H}^n(K_j) - \varepsilon_j \leq \mathcal{H}^n(\varphi(K_j)) \leq \mathcal{H}^n(K_j \cap B_{2R_0}) + \frac{1}{2^n} \mathcal{H}^n(K_j \setminus B_{2R_0}).$$

This implies that $\mathcal{H}^n(K_j \setminus B_{2R_0}) \to 0$. In order to prove $\mathcal{H}^n(K_j) \to \mathcal{H}^n(K)$, it remains to show that there is no loss of mass at H. To this end, fix $\eta > 0$, and consider $\delta > 0$ and the map π as in (1.7). Then, from $\pi \in \Sigma(H)$ and $\mathcal{H}^n(\pi(U_\delta(H))) \leq \mathcal{H}^n(H) = 0$,

$$\mathcal{H}^n(K) \leq \limsup_{j \to \infty} \mathcal{H}^n(K_j) \leq \limsup_{j \to \infty} \mathcal{H}^n(\pi(K_j)) \leq (1 + \eta)^n \limsup_{j \to \infty} \mathcal{H}^n(K_j \setminus U_\delta(H))$$

$$= (1 + \eta)^n \limsup_{j \to \infty} \mathcal{H}^n\big((K_j \cap \operatorname{cl}(B_{2R_0})) \setminus U_\delta(H)\big)$$

$$\leq (1 + \eta)^n \mathcal{H}^n(K \cap \operatorname{cl}(B_{2R_0})) \leq (1 + \eta)^n \mathcal{H}^n(K).$$

The arbitrariness of η implies that $\limsup_j \mathcal{H}^n(K_j) = \mathcal{H}^n(K)$.

Step 5. To complete the proof we need to show the second equality in (1.8). We argue in two steps, borrowing some important ideas from [DPH03]. We show in this step that $\mathcal{H}^n(K) \leq \mathcal{H}^n(\phi(K))$ whenever $\phi \in \Sigma(H)$ is a diffeomorphism. Let $G(n)$ denote the Grassmannian of n-planes in $\mathbb{R}^{n+1}$, let $d(\tau, \sigma)$ denote the geodesic distance on $G(n)$, and let $J^\tau \phi$ be the tangential Jacobian of ϕ with respect to $\tau \in G(n)$. Given $\varepsilon > 0$, by Lusin's

theorem we can find $\delta > 0$ and a compact set $\hat{K} \subset K$ with $\mathcal{H}^n(K \setminus \hat{K}) < \varepsilon$ such that K admits an approximate tangent plane $\tau(x)$ at every $x \in \hat{K}$,

$$\sup_{x \in \hat{K}} \sup_{y \in B_{x,\delta}} |\nabla\phi(x) - \nabla\phi(y)| \leq \varepsilon, \quad \sup_{x \in \hat{K}} \sup_{y \in \hat{K} \cap B_{x,\delta}} d(\tau(x), \tau(y)) < \varepsilon, \quad (4.8)$$

and moreover, if we denote by $S_{x,r}$ the set of points in $B_{x,r}$ at distance at most εr from $x + \tau(x)$, then $K \cap B_{x,r} \subset S_{x,r}$ for every $r < \delta$ and $x \in \hat{K}$. By the Vitali–Besicovitch covering theorem [AFP00, Theorem 2.19] we can find a finite disjoint family of closed balls $\{\text{cl}(B_i)\}$ with $B_i = B_{x_i,r_i} \subset\subset \mathbb{R}^{n+1} \setminus H$, $x_i \in \hat{K}$, and $r_i < \delta$, such that $\mathcal{H}^n(\hat{K} \setminus \bigcup_i \text{cl}(B_i)) = 0$. In particular a finite subfamily of them, still denoted by $\{B_i\}$, satisfies $\mathcal{H}^n(\hat{K} \setminus \bigcup_i \text{cl}(B_i)) < \varepsilon$; furthermore, by slightly enlarging the radii we can still assume that the balls are disjoint, that their radii are less than δ, and that $\mathcal{H}^n(\hat{K} \setminus \bigcup_i B_i) < \varepsilon$. By exploiting the construction of Step 3, we can find $j(\varepsilon) \in \mathbb{N}$ and maps $f_i : \text{cl}(B_i) \to \text{cl}(B_i)$ with $\text{Lip } f_i \leq 1 + C\sqrt{\varepsilon}$ such that, for a certain $X_i \subset S_i = S_{x_i, \varepsilon r_i}$,

$$f_i(X_i) \subset B_i \cap (x_i + \tau(x_i)), \quad (4.9)$$
$$\mathcal{H}^n\big(f_i((K_j \cap B_i) \setminus X_i)\big) \leq C\sqrt{\varepsilon}\,\omega_n r_i^n, \quad \forall j \geq j(\varepsilon). \quad (4.10)$$

By (4.8), (4.9) and the area formula, using $\omega_n r_i^n \leq \mathcal{H}^n(K \cap B_i)$ (thanks to monotonicity) and setting $\alpha_i := \mathcal{H}^n((K \setminus \hat{K}) \cap B_i)$, we obtain

$$\mathcal{H}^n(\phi(f_i(K_j \cap X_i))) = \int_{f_i(K_j \cap X_i)} J^{\tau(x_i)}\phi(x)\, d\mathcal{H}^n(x) \leq (J^{\tau(x_i)}\phi(x_i) + \varepsilon)\omega_n r_i^n$$

$$\leq (J^{\tau(x_i)}\phi(x_i) + \varepsilon)\mathcal{H}^n(K \cap B_i) \leq (J^{\tau(x_i)}\phi(x_i) + \varepsilon)\,(\mathcal{H}^n(\hat{K} \cap B_i) + \alpha_i)$$

$$\leq \int_{\hat{K} \cap B_i} (J^{\tau(x)}\phi(x) + 2\varepsilon)\, d\mathcal{H}^n(x) + ((\text{Lip } \phi)^n + \varepsilon)\alpha_i$$

$$= \mathcal{H}^n(\phi(\hat{K} \cap B_i)) + 2\varepsilon\mathcal{H}^n(K \cap B_i) + ((\text{Lip } \phi)^n + \varepsilon)\alpha_i, \quad (4.11)$$

where in the last identity we have used the injectivity of ϕ. Recalling Step 3, each f_i is the identity on ∂B_i. Since $\{\text{cl}(B_i)\}$ is a finite disjoint family of closed balls, we can define $f : \mathbb{R}^n \to \mathbb{R}^n$ by imposing $f = f_i$ on each B_i and $f = \text{Id}$ on $\mathbb{R}^n \setminus \bigcup_i B_i$. Obviously $f \in \Sigma(H)$. Combining (4.10) with $\omega_n r_i^n \leq \mathcal{H}^n(K \cap B_i)$, adding up over i, and letting $j \to \infty$ we thus find $\mathcal{H}^n(K_j) - \varepsilon_j \leq \mathcal{H}^n(\phi(f(K_j))) \leq \mathcal{H}^n(\phi(\hat{K})) + \varrho(\varepsilon)$ for every $j \geq j(\varepsilon)$, where $\varrho(\varepsilon) \to 0$ as $\varepsilon \to 0^+$ in a way which depends on n, $\text{Lip } \phi$, and $\mathcal{H}^n(K)$ only. We first let $j \to \infty$ and then $\varepsilon \to 0$ to prove our claim.

Step 6. By Step 5, the canonical density one varifold associated to the rectifiable set K turns out to be stationary in $\mathbb{R}^{n+1} \setminus H$. By Allard's regularity theorem [Sim83, Chapter 5] there exists an $\mathcal{H}^n$-negligible closed set $S \subset K$ such that $\Gamma := K \setminus S$ is a real analytic hypersurface.

We may now exploit this fact to improve on Step 5 and show that $\mathcal{H}^n(K) \leq \mathcal{H}^n(\phi(K))$ for every $\phi \in \Sigma(H)$, which proves that K is a sliding minimizer (and hence an $(\mathbf{M}, 0, \infty)$-minimal set). The idea is that, by regularity of Γ, at a fixed distance from

the singular set one can project K_j directly onto K, rather than onto its affine tangent planes localized in balls. More precisely, since $\mathcal{H}^n(H \cup S) = 0$ and $\mathcal{H}^n(K) < \infty$, one has

$$\limsup_{j \to \infty} \mathcal{H}^n(K_j \cap U_\delta(H \cup S)) \leq \mathcal{H}^n(K \cap \text{cl}(U_\delta(H \cup S))) =: \varrho(\delta), \qquad (4.12)$$

where $\varrho(\delta) \to 0$ as $\delta \to 0^+$. If $N_\varepsilon(A)$ denotes the normal ε-neighborhood of $A \subset \Gamma$, then by compactness of $\Gamma_\delta := \Gamma \setminus U_\delta(H \cup S)$ there exists $\varepsilon < \delta$ such that projection onto Γ defines a smooth map $p : N_{2\varepsilon}(\Gamma_\delta) \to \Gamma_\delta$. We now define a Lipschitz map

$$f_{\varepsilon,\delta} : N_\varepsilon(\Gamma_\delta) \cup U_{\delta/2}(H \cup S) \cup (\mathbb{R}^{n+1} \setminus U_\delta(\Gamma)) \to \mathbb{R}^{n+1}$$

by setting $f_{\varepsilon,\delta} = p$ on $N_\varepsilon(\Gamma_\delta)$, and $f_{\varepsilon,\delta} = \text{Id}$ on the remainder. Observe that

$$\lim_{\varepsilon \downarrow 0} \text{Lip } f_{\varepsilon,\delta} = 1 < \infty.$$

For every δ we then choose $\varepsilon < \delta$ so that $f = f_{\varepsilon,\delta}$ has Lipschitz constant at most 2 and extend it to a Lipschitz map $\hat{f}$ on $\mathbb{R}^{n+1}$ with the same Lipschitz constant. Obviously $\hat{f}$ belongs to $\Sigma(H)$. We can then estimate

$$\mathcal{H}^n(\hat{f}(K_j) \setminus \Gamma_\delta) \leq (\text{Lip } \hat{f})^n \mathcal{H}^n(K_j \setminus N_\varepsilon(\Gamma_\delta)). \qquad (4.13)$$

Observe that $\mathbb{R}^{n+1} \setminus N_\varepsilon(\Gamma_\delta) \subset\subset \mathbb{R}^{n+1} \setminus U_{\varepsilon/2}(K) \cup U_{2\delta}(H \cup S)$, and thus

$$\limsup_{j \to \infty} \mathcal{H}^n(K_j \setminus N_\varepsilon(\Gamma_\delta)) \leq \mathcal{H}^n(K \cap U_{2\delta}(H \cap S)) \overset{(4.12)}{\leq} \varrho(2\delta). \qquad (4.14)$$

Combining (4.13) and (4.14) yields

$$\limsup_j \mathcal{H}^n(\hat{f}(K_j) \setminus \Gamma_\delta) \leq 2^n \varrho(2\delta).$$

On the other hand, $\Gamma_\delta \subset K$. Thus, combining (4.13) and (4.14) with a standard diagonal argument we obtain a sequence of maps $f_j \in \Sigma(H)$ such that $\mathcal{H}^n(f_j(K_j) \setminus K) \to 0$. Since each K_j equals $\psi_j(K_0)$ for some $\psi_j \in \Sigma(H)$, we conclude that there exists a sequence $\{\varphi_j\} \subset \Sigma(H)$ such that $\mathcal{H}^n(\varphi_j(K_0) \setminus K) \to 0$.

We are now ready to show the right identity in (1.8). Fix $\phi \in \Sigma(H)$. Then

$$\mathcal{H}^n(\phi(K)) \geq \liminf_{j \to \infty} \mathcal{H}^n(\phi \circ \varphi_j(K_0))$$

$$\geq \inf\{\mathcal{H}^n(J) : J \in \mathcal{A}(H, K_0)\} = \mathcal{H}^n(K).$$

This shows that K is a sliding minimizer. $\square$

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