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Convergence to self-similar solutions for the homogeneous Boltzmann equation

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Abstract. The Boltzmann H-theorem implies that the solution to the Boltzmann equation tends to an equilibrium, that is, a Maxwellian when time tends to infinity. This has been proved in various settings when the initial energy is finite. However, when the initial energy is infinite, the time asymptotic state is no longer described by a Maxwellian, but a self-similar solution obtained by Bobylev–Cercignani. The purpose of this paper is to rigorously justify this for the spatially homogeneous problem with a Maxwellian molecule type cross section without angular cutoff.

Keywords. Measure valued solution, infinite energy, self-similar solutions, time asymptotic states

1. Introduction

Consider the homogeneous Boltzmann equation

$$\partial_t f(t,v) = Q(f,f)(t,v), \qquad v \in \mathbb{R}^3,\ t \in \mathbb{R}^+, \tag{1.1}$$

with initial data

$$f(0,v) = f_0(v) \geq 0, \qquad v \in \mathbb{R}^3, \tag{1.2}$$

where the non-negative unknown function $f(t,v)$ is the distribution density function of particles with velocity $v \in \mathbb{R}^3$ at time $t \in \mathbb{R}^+$. The right hand side of (1.1) is the Boltzmann bilinear collision operator corresponding to the Maxwellian molecule type cross section

$$Q(g,f)(v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \mathcal{B}\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right)\big(f(v')g(v'_*) - f(v)g(v_*)\big)\,d\sigma\,dv_*. \tag{1.3}$$

Here for $\sigma \in \mathbb{S}^2$,

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \qquad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma,$$

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and from the conservation of momentum and energy,

$$v' + v'_* = v + v_*, \qquad |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2.$$

The *Maxwellian molecule type cross section* $\mathcal{B}(\tau)$ in (1.3) is a non-negative function depending only on the deviation angle $\theta = \cos^{-1}\big(\frac{v-v_*}{|v-v_*|} \cdot \sigma\big)$. As usual, θ is restricted to $0 \le \theta \le \pi/2$ by replacing $\mathcal{B}(\cos\theta)$ by its "symmetrized" version $[\mathcal{B}(\cos\theta) + \mathcal{B}(\pi - \cos\theta)]\mathbf{1}_{0\le\theta\le\pi/2}$. Moreover, motivated by inverse power laws, throughout this paper, we assume

$$\lim_{\theta\to 0_+} \mathcal{B}(\cos\theta)\theta^{2+2s} = b_0 \tag{1.4}$$

for positive constants $s \in (0, 1)$ and $b_0 > 0$.

As in [7–9, 11, 16], the Cauchy problem (1.1), (1.2) is considered in the set of probability measures on $\mathbb{R}^3$. We first introduce some function spaces defined in the previous literature. For $\alpha \in [0, 2]$, $\mathcal{P}^\alpha(\mathbb{R}^3)$ denotes the set of probability density functions f on $\mathbb{R}^3$ such that

$$\int_{\mathbb{R}^3} |v|^\alpha f(v)\, dv < \infty,$$

and moreover, when $\alpha \ge 1$,

$$\int_{\mathbb{R}^3} v_j f(v)\, dv = 0, \qquad j = 1, 2, 3.$$

Following [7], the characteristic function $\varphi(t, \xi)$ is the Fourier transform of $f(t, v) \in \mathcal{P}^0(\mathbb{R}^3)$ with respect to v:

$$\varphi(t, \xi) = \hat{f}(t, \xi) = \mathcal{F}(f)(t, \xi) = \int_{\mathbb{R}^3} e^{-iv\cdot\xi} f(t, v)\, dv. \tag{1.5}$$

For each $\alpha \in [0, 2]$, set $\widetilde{\mathcal{P}}^\alpha(\mathbb{R}^3) = \mathcal{F}^{-1}(\mathcal{K}^\alpha(\mathbb{R}^3))$ with $\mathcal{K}(\mathbb{R}^3) = \mathcal{F}(\mathcal{P}^0(\mathbb{R}^3))$ and

$$\mathcal{K}^\alpha(\mathbb{R}^3) = \{\varphi \in \mathcal{K}(\mathbb{R}^3) : \|\varphi - 1\|_{\mathcal{D}^\alpha} < \infty\}.$$

Here the *Toscani distance* $\mathcal{D}^\alpha$ with $\alpha > 0$ between two suitable functions $\varphi(\xi)$ and $\tilde{\varphi}(\xi)$ is defined by

$$\|\varphi - \tilde{\varphi}\|_{\mathcal{D}^\alpha} \equiv \sup_{0\ne\xi\in\mathbb{R}^3} \frac{|\varphi(\xi) - \tilde{\varphi}(\xi)|}{|\xi|^\alpha}.$$

The set $\mathcal{K}^\alpha(\mathbb{R}^3)$ endowed with the $\mathcal{D}^\alpha$-distance is a complete metric space. It follows from [7, Lemma 3.12] that $\mathcal{K}^\alpha(\mathbb{R}^3) = \{1\}$ for all $\alpha > 2$, and $\{1\} \subset \mathcal{K}^\alpha(\mathbb{R}^3) \subset \mathcal{K}^\beta(\mathbb{R}^3) \subset \mathcal{K}(\mathbb{R}^3)$ for $2 \ge \alpha \ge \beta \ge 0$.

The advantage of considering the Maxwellian molecule cross section is that the Bobylev formula [5, 6] is in a simple form. Namely, taking the Fourier transform (1.5) of equation (1.1) leads to the following equation for the new unknown $\varphi = \varphi(t, \xi)$:

$$\partial_t \varphi(t, \xi) = \int_{\mathbb{S}^2} \mathcal{B}\left(\frac{\xi \cdot \sigma}{|\xi|}\right)\big(\varphi(t, \xi^+)\varphi(t, \xi^-) - \varphi(t, \xi)\big)\, d\sigma, \tag{1.6}$$

where we have used

$$\varphi(t,0) = \int_{\mathbb{R}^3} f(t,v)\,dv = 1.$$

Here,

$$\xi^+ = \frac{\xi + |\xi|\sigma}{2}, \qquad \xi^- = \frac{\xi - |\xi|\sigma}{2}, \tag{1.7}$$

which satisfy

$$\xi^+ + \xi^- = \xi, \qquad |\xi^+|^2 + |\xi^-|^2 = |\xi|^2. \tag{1.8}$$

From now on, we consider the Cauchy problem for (1.6) with initial condition

$$\varphi(0,\xi) = \varphi_0(\xi). \tag{1.9}$$

For $\alpha \in (2s, 2]$, it is shown in [7, 11, 12] that this Cauchy problem admits a unique global solution $\varphi(t,\xi) \in C([0,\infty), \mathcal{K}^\alpha(\mathbb{R}^3))$ for every $\varphi_0(\xi) \in \mathcal{K}^\alpha(\mathbb{R}^3)$. Moreover, $f(t,\cdot) \in L^1(\mathbb{R}^3) \cap H^\infty(\mathbb{R}^3)$ for any $t > 0$ if $\mathcal{F}^{-1}(\varphi_0)(v)$ is not a Dirac mass [12, 13].

The large time behavior of the solution depends on whether the initial energy is finite or not, and in the above setting, it depends on the parameter α (see [2, 7–9, 11, 12, 14–16] and the references cited therein):

- When $\alpha = 2$, the initial datum has finite energy so that the solution tends to the Maxwellian defined by the initial datum. This was indeed proved in the early work by Tanaka [15] using weak convergence in probability. And it was also proved later in [9, 14, 16] by using analytic methods with convergence in Toscani metrics. Moreover, if some moment higher than the second order is assumed to be bounded, the convergence in the $\mathcal{D}^{2+\delta}$-distance with $\delta > 0$ is shown to be exponential in time [9].
- When $2s < \alpha < 2$, the initial energy is infinite so that the solution will no longer tend to an equilibrium, but to a self-similar solution

$$f_{\alpha,K}(t,v) = e^{-3\mu_\alpha t}\Psi_{\alpha,K}(ve^{-\mu_\alpha t})$$

constructed in [5, 6], where

$$\mu_\alpha = \frac{\lambda_\alpha}{\alpha}, \qquad \lambda_\alpha \equiv \int_{\mathbb{S}^2} \mathcal{B}\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left(\frac{|\xi^-|^\alpha + |\xi^+|^\alpha}{|\xi|^\alpha} - 1\right) d\sigma. \tag{1.10}$$

Here, $K > 0$ is any given constant and $\Psi_{\alpha,K}(v)$ is a radially symmetric non-negative function satisfying

$$\int_{\mathbb{R}^3} \Psi_{\alpha,K}(v)\,dv = 1, \qquad \hat{\Psi}_{\alpha,K}(\xi) \in \mathcal{K}^\alpha(\mathbb{R}^3), \qquad \lim_{|\eta|\to 0} \frac{1-\hat{\Psi}_{\alpha,K}(\eta)}{|\eta|^\alpha} = K. \tag{1.11}$$

The $H^\infty(\mathbb{R}^3)$-regularity of the self-similar solution was proved in [12, 13]. However, convergence to the self-similar solution $f_{\alpha,K}(t,v)$ is not well understood even though there are some works [6–8] about pointwise convergence in the radially symmetric setting or in weak topology with scaling. In fact, even how to show convergence in the sense of distributions has been a problem.

The main difficulties in studying convergence to self-similar solutions come from the fact that a self-similar solution has infinite energy and it decays to zero exponentially in time except in L_1-norm. The purpose of this paper is to show strong convergence when $\alpha \in (\max\{2s, 1\}, 2]$ under some conditions on the initial perturbation.

For this, we first consider the $\mathcal{D}^{2+\delta}$-distance between two solutions. For $f_0(v) \in \widetilde{\mathcal{P}}^\alpha(\mathbb{R}^3)$ and $g_0(v) \in \widetilde{\mathcal{P}}^\alpha(\mathbb{R}^3)$, as in [9, 10], set

$$\begin{aligned} \widetilde{P}(t,\xi) &= e^{-At}\widetilde{P}(0,\xi), \\ \widetilde{P}(0,\xi) &= \frac{1}{2}\sum_{j,l=1}^{3} \xi_j \xi_l P_{jl}(0) X(\xi), \\ P_{jl}(0) &= \int_{\mathbb{R}^3} \left(v_j v_l - \frac{\delta_{jl}}{3}|v|^2 \right) (f_0(v) - g_0(v))\, dv, \end{aligned} \tag{1.12}$$

where

$$A = \frac{3}{4}\int_{\mathbb{S}^2} \mathcal{B}\left(\frac{\sigma\cdot\xi}{|\xi|}\right)\left(1 - \left(\frac{\sigma\cdot\xi}{|\xi|}\right)^2\right) d\sigma, \tag{1.13}$$

δ_{jl} is the Kronecker delta and $X(\xi) \equiv X(|\xi|)$ is a smooth radially symmetric function satisfying $0 \le X(\xi) \le 1$ and $X(\xi) = 1$ for $|\xi| \le 1$ and $X(\xi) = 0$ for $|\xi| \ge 2$.

The first result in this paper, on the $\mathcal{D}^{2+\delta}$ time asymptotic stability of the solutions, is given by

Theorem 1.1. *Suppose $f_0(v), g_0(v) \in \widetilde{\mathcal{P}}^\alpha(\mathbb{R}^3)$ for $\alpha \in (\max\{2s, 1\}, 2]$. Let $\hat{f}(t,\xi)$ and $\hat{g}(t,\xi)$ be the corresponding two global solutions of the Cauchy problem (1.6) with initial data $\hat{f}_0(\xi)$ and $\hat{g}_0(\xi)$ respectively. Assume for some $\delta \in (0,\alpha] \cap (0, A/\mu_\alpha)$, the initial data satisfy*

$$\int_{\mathbb{R}^3} |v|^2 (f_0(v) - g_0(v))\, dv = 0, \tag{1.14}$$

$$\begin{cases} \displaystyle\int_{\mathbb{R}^3} |v|^2 |f_0(v) - g_0(v)|\, dv < \infty, \\ \|\hat{f}_0(\cdot) - \hat{g}_0(\cdot) - \widetilde{P}(0,\cdot)\|_{\mathcal{D}^{2+\delta}} < \infty. \end{cases} \tag{1.15}$$

Then there exists some positive constant $C_1 > 0$ independent of t and ξ such that

$$\|\hat{f}(t,\cdot) - \hat{g}(t,\cdot) - \widetilde{P}(t,\cdot)\|_{\mathcal{D}^{2+\delta}} \le C_1 e^{-\eta_0 t}, \quad t \ge 0. \tag{1.16}$$

Here, $\eta_0 = \min\{A - \delta\mu_\alpha, B\}$ and

$$B = \int_{\mathbb{S}^2} \mathcal{B}\left(\frac{\sigma\cdot\xi}{|\xi|}\right)\left(1 - \left|\cos\frac{\theta}{2}\right|^{2+\delta} - \left|\sin\frac{\theta}{2}\right|^{2+\delta}\right) d\sigma, \quad \cos\theta = \frac{\sigma\cdot\xi}{|\xi|}. \tag{1.17}$$

Note that for $\mathcal{D}^{2+\delta}$-convergence to a self-similar solution, one can simply take $g_0 = \Psi_{\alpha,K}(v)$. Based on this, in order to obtain convergence in the strong topology, such as in the Sobolev norms, we will give a uniform in time estimate of the solution in H^N-norm:

Theorem 1.2. *For* $\max\{1, 2s\} < \alpha < 2$*, assume that* $f_0(v) \in \widetilde{\mathcal{P}}^\alpha(\mathbb{R}^3)$ *satisfies* (1.14)–(1.15) *and is not a Dirac mass, and* $g_0(v) = \Psi_{\alpha,K}(v)$*. Then for any given positive constant* $t_1 > 0$ *and any* $N \in \mathbb{N}$*, there exists a positive constant* $C_2(t_1, N)$ *independent of* t *such that*

$$\sup_{t\in[t_1,\infty)} \|f(t,\cdot)\|_{H^N} \le C_2(t_1, N). \tag{1.18}$$

Consequently, there exists a positive constant $C_3(t_1, N)$ *independent of* t *such that*

$$\|f(t,\cdot) - f_{\alpha,K}(t,\cdot)\|_{H^N} = \|f(t,v) - e^{-3\mu_\alpha t}\Psi_{\alpha,K}(ve^{-\mu_\alpha t})\|_{H^N} \le C_3(t_1, N)e^{-(\eta_0-\epsilon)t} \tag{1.19}$$

for any $t \ge t_1$ *and any* $\epsilon \in (0, \eta_0)$.

Since

$$e^{-3\mu_\alpha t/2}\|\Psi_{\alpha,K}(\cdot)\|_{L^2} \le \|f_{\alpha,K}(t,\cdot)\|_{H^N} \le e^{-3\mu_\alpha t/2}\|\Psi_{\alpha,K}(\cdot)\|_{H^N}, \tag{1.20}$$

(1.19) and (1.20) imply that when

$$3\mu_\alpha < 2\eta_0, \tag{1.21}$$

the convergence rate given in (1.19) is faster than the decay rate of the self-similar solution itself. Hence in this case, the infinite energy solution $f(t, v)$ converges to the self-similar solution $f_{\alpha,K}(t, v)$ exponentially in time.

Based on the estimate (1.19), we can further deduce the following $L^1(\mathbb{R}^3)$-convergence result:

$$\int_{\mathbb{R}^3} |f(t,v) - e^{-3\mu_\alpha t}\Psi_{\alpha,K}(e^{-\mu_\alpha t}v)|\,dv \le C_4(t_1, N)e^{-(\eta_0/q-\alpha\mu_\alpha/p-\epsilon)t}, \qquad t \ge t_1. \tag{1.22}$$

Here $C_4(t_1, N)$ is some positive constant depending only on t_1, N; $\epsilon > 0$ is any sufficiently small positive constant; and $p, q > 1$ satisfy

$$1/p + 1/q = 1, \qquad q/p > 3/(2\alpha). \tag{1.23}$$

Remark 1.1. Since $\mu_\alpha \to 0+$ as $\alpha \to 2$, the condition (1.21) holds when α is close to 2.

For the case with finite energy, the above stability estimates give a better convergence description of the solution than in the previous literature. This extends the exponential convergence result in the Toscani metrics $\mathcal{D}^{2+\delta}$ with $\delta > 0$ (cf. [9]) to the Sobolev space $H^N(\mathbb{R}^3)$ for any $N \in \mathbb{N}$. In fact, we have

Corollary 1.1. *Suppose that* $f_0(v) \in \mathcal{P}^2(\mathbb{R}^3)$ *is not a Dirac mass and satisfies*

$$\int_{\mathbb{R}^3} |v|^2 f_0(v)\,dv = 3, \qquad \|\hat{f}_0(\cdot) - \mu(\cdot) - \widetilde{P}(0,\cdot)\|_{\mathcal{D}^{2+\delta}} < \infty, \tag{1.24}$$

for some positive constant $\delta \in (0, 2]$ *with* $\mu = (2\pi)^{-3/2}e^{-|v|^2/2}$*. Then for any* $N \in \mathbb{N}$*, there exist positive constants* $C_5, C_6(t_1, N) > 0$ *independent of* t *such that*

$$\|\hat{f}(t,\cdot) - \mu(\cdot) - \widetilde{P}(t,\cdot)\|_{\mathcal{D}^{2+\delta}} \le C_5 e^{-\eta_1 t}, \qquad t > 0, \tag{1.25}$$

and

$$\sup_{t\in[t_1,\infty)} \|f(t,\cdot)\|_{H^N} \le C_6(t_1,N), \qquad t \ge t_1. \tag{1.26}$$

Here $t_1 > 0$ *is any given positive constant and* $\eta_1 = \min\{A, B\}$.

A direct consequence of (1.25) and (1.26) is

$$\|f(t,\cdot) - \mu(\cdot)\|_{H^N} \le C_7(t_1,N)e^{-(\eta_1-\epsilon)t} \tag{1.27}$$

and

$$\int_{\mathbb{R}^3} |f(t,v) - \mu(v)|\, dv \le C_8(t_1,N)e^{-(\eta_1/q-\epsilon)t}, \qquad t \ge t_1, \tag{1.28}$$

for any sufficiently small positive constant ϵ, some positive constants $C_7(t_1,N)$ and $C_8(t_1,N)$ depending only on t_1 and N, and $p,q > 1$ with $p^{-1} + q^{-1} = 1$ and $4q > 3p$.

Remark 1.2. Two comments on the above two theorems are in order:

- By Lemma 2.6, sufficient conditions for the requirements (1.15) and (1.24) are

  $$\int_{\mathbb{R}^3} |v|^{2+\delta} |f_0(v) - g_0(v)|\, dv < \infty$$

  and

  $$\int_{\mathbb{R}^3} |v|^{2+\delta} |f_0(v) - \mu(v)|\, dv < \infty,$$

  respectively.
- The convergence rate in Corollary 1.1 is faster than in Theorems 1.1 and 1.2.

We now list some notation used throughout the paper. Firstly, C, C_i with $i \in \mathbb{N}$, and $O(1)$ are used for generic large positive constants, and ε, κ stand for generic small positive constants. When the dependence needs to be specified, the notation like $C(\cdot,\cdot)$ is used. For a multi-index $\beta = (\beta_1,\beta_2,\beta_3)$, $\partial_v^\beta = \partial_{v_1}^{\beta_1}\partial_{v_2}^{\beta_2}\partial_{v_3}^{\beta_3}$. Finally, $A \lesssim B$ means that there is a constant $C > 0$ such that $A \le CB$, and $A \sim B$ means $A \lesssim B$ and $B \lesssim A$.

The rest of this paper is organized as follows: Some known results concerning the global solvability, stability, and regularity of solutions to the Cauchy problem (1.6), (1.9) in $\mathcal{K}^\alpha(\mathbb{R}^3)$ are recalled in Section 2. Moreover, some properties of approximation of the initial data in $\mathcal{K}^\alpha(\mathbb{R}^3)$ are also given there. The proofs of Theorem 1.1, Theorem 1.2, and Corollary 1.1 will be given in the next three sections respectively.

2. Preliminaries

In this section, we first recall the global solvability, stability and regularity results on the Cauchy problem (1.6), (1.9) obtained in [5–8, 11–13]. Then we study the properties of the approximation $f_{0R}(v)$ to the initial data $f_0(v)$ defined in (2.3) for later stability estimates.

For the Cauchy problem (1.6), (1.9), the following estimates are proved in [7, 8, 11–13].

Lemma 2.1. *For $\alpha \in (2s, 2]$, if $\varphi_0(\xi) \in \mathcal{K}^\alpha(\mathbb{R}^3)$, then the Cauchy problem (1.6), (1.9) admits a unique global classical solution $\varphi(t,\xi) \equiv \hat{f}(t,\xi) \in C([0,\infty), \mathcal{K}^\alpha(\mathbb{R}^3))$ satisfying*

$$\|\varphi(t,\cdot) - 1\|_{\mathcal{D}^\alpha} \le e^{\lambda_\alpha t}\|\varphi_0(\cdot) - 1\|_{\mathcal{D}^\alpha}. \tag{2.1}$$

If $\psi(t,\xi) \in C([0,\infty), \mathcal{K}^\alpha(\mathbb{R}^3))$ is another solution with initial data $\psi_0(\xi) \in \mathcal{K}^\alpha(\mathbb{R}^3)$, then

$$\|\varphi(t,\cdot) - \psi(t,\cdot)\|_{\mathcal{D}^\alpha} \le e^{\lambda_\alpha t}\|\varphi_0(\cdot) - \psi_0(\cdot)\|_{\mathcal{D}^\alpha}. \tag{2.2}$$

Furthermore, if $f_0(v) = \mathcal{F}^{-1}(\varphi_0)(v)$ is not a Dirac mass, then $f(t,\cdot) \in L^1(\mathbb{R}^3) \cap \mathcal{P}^\beta(\mathbb{R}^3) \cap H^\infty(\mathbb{R}^3)$ for $t > 0$ and $0 < \beta < \alpha$.

And for the self-similar solution $f_{\alpha,K}(t,v)$ constructed in [5, 6], by [12, 13], we have

Lemma 2.2. *For $\alpha \in (2s, 2)$ and a constant $K > 0$, there exists a radially symmetric function $\hat{\Psi}_{\alpha,K}(\xi) \in \mathcal{K}^\alpha(\mathbb{R}^3)$ satisfying (1.11) such that*

$$f_{\alpha,K}(t,v) = e^{-3\mu_\alpha t}\Psi_{\alpha,K}(ve^{-\mu_\alpha t})$$

is a solution of the Cauchy problem for (1.1) with initial datum $\Psi_{\alpha,K}(v)$. Moreover, $\Psi_{\alpha,K}(t,\cdot) \in L^1(\mathbb{R}^3) \cap \mathcal{P}^\beta(\mathbb{R}^3) \cap H^\infty(\mathbb{R}^3)$ for $0 < \beta < \alpha$.

The relation between $\mathcal{P}^\alpha(\mathbb{R}^3)$ and $\mathcal{K}^\alpha(\mathbb{R}^3)$ was given in [7] and [12] and it can be stated as follows.

Lemma 2.3. (i) *For $\alpha \in (0,2]$, if $h(v) \in \mathcal{P}^\alpha(\mathbb{R}^3)$, then $\hat{h}(\xi) \in \mathcal{K}^\alpha(\mathbb{R}^3)$.*
(ii) *For $\alpha \in (0,2]$, if $\hat{h}(\xi) \in \mathcal{K}^\alpha(\mathbb{R}^3)$, then $h(v) \in \mathcal{P}^\beta(\mathbb{R}^3)$ for any $0 < \beta < \alpha$.*

Since the energy of the initial data is infinite, we will first approximate the solution by a cutoff at large velocity so that the moment of any order is bounded. Then it remains to show that the solution with this kind of approximation has a uniform bound independent of the cutoff parameter. On the other hand, the approximate solution cannot be arbitrary because it has to be in the function space $\mathcal{K}^\alpha$.

For $\alpha \in (2s, 2]$ and $f_0(v) \in \widetilde{\mathcal{P}}^\alpha(\mathbb{R}^3)$, let $X(v)$ be the smooth function defined in the construction of $\widetilde{P}(t,\xi)$, set $X_R(v) = X(v/R)$, and define

$$f_{0R}(v) = \widetilde{f}_{0R}(v + a_R^f), \qquad \widetilde{f}_{0R}(v) = \frac{f_0(v)X_R(v)}{\int_{\mathbb{R}^3} f_0(v)X_R(v)\,dv} \tag{2.3}$$

with

$$a_R^f = \int_{\mathbb{R}^3} v\widetilde{f}_{0R}(v)\,dv = \frac{\int_{\mathbb{R}^3} vf_0(v)X_R(v)\,dv}{\int_{\mathbb{R}^3} f_0(v)X_R(v)\,dv}. \tag{2.4}$$

The properties of the approximation function are given in

Lemma 2.4. *For $1 < \beta < \alpha \le 2$, if we choose $R > 0$ sufficiently large, then:*
(i) *$\hat{f}_{0R}(\xi), \hat{g}_{0R}(\xi) \in \mathcal{K}^2(\mathbb{R}^3)$, and for sufficiently large $R > 0$,*

$$\|\hat{f}_{0R}(\cdot) - \hat{g}_{0R}(\cdot)\|_{\mathcal{D}^2} \le C_9\left(1 + \int_{\mathbb{R}^3}|v|(f_0(v) + g_0(v))\,dv + \int_{\mathbb{R}^3}|v|^2|f_0(v) - g_0(v)|\,dv\right). \tag{2.5}$$

Here the positive constant C_9 depends only on $\int_{\mathbb{R}^3}(1 + |v|^\beta)(f_0(v) + g_0(v))\,dv$.

(ii) *For* $1 < \beta < \alpha \le 2$ *and sufficiently large* $R > 0$, $f_{0R}(v) \in \mathcal{P}^\beta(\mathbb{R}^3)$ *with* $\mathcal{P}^\beta(\mathbb{R}^3)$*-norm being uniformly bounded, precisely,*

$$\int_{\mathbb{R}^3} |v|^\beta f_{0R}(v)\,dv \lesssim \int_{\mathbb{R}^3} (1+|v|^\beta) f_0(v)\,dv. \tag{2.6}$$

Thus

$$\|\hat{f}_{0R}(\cdot) - 1\|_{\mathcal{D}^\beta} \lesssim 1, \qquad \|\hat{f}_{0R}(\cdot) - \hat{f}_0(\cdot)\|_{\mathcal{D}^\beta} \lesssim 1, \tag{2.7}$$

$$\lim_{R\to\infty} \|\hat{f}_{0R}(\cdot) - \hat{f}_0(\cdot)\|_{\mathcal{D}^\beta} = 0. \tag{2.8}$$

Proof. We first prove (2.6)–(2.8). Since it is straightforward to verify (2.6), and (2.7) is a direct consequence of (2.6) and Lemma 2.3, we only prove (2.8). For this, note that

$$\lim_{R\to\infty} \int_{\mathbb{R}^3} f_0(v) X_R(v)\,dv = 1.$$

Choosing R sufficiently large, we have

$$\int_{\mathbb{R}^3} f_0(v) X_R(v)\,dv \ge \frac{1}{2}, \qquad \int_{\mathbb{R}^3} g_0(v) X_R(v)\,dv \ge \frac{1}{2}. \tag{2.9}$$

Thus

$$\begin{aligned} |a_R^f| &\le 2\left|\int_{\mathbb{R}^3} v f_0(v) X_R(v)\,dv\right| \\ &= 2\left|\int_{\mathbb{R}^3} v f_0(v)(1 - X_R(v))\,dv\right| \le 2R^{1-\beta} \int_{\mathbb{R}^3} |v|^\beta f_0(v)\,dv. \end{aligned} \tag{2.10}$$

Similarly,

$$|a_R^g| \le 2R^{1-\beta} \int_{\mathbb{R}^3} |v|^\beta g_0(v)\,dv. \tag{2.11}$$

From (2.9), (2.10), and the fact that

$$\begin{aligned} |\hat{f}_{0R}(\xi) - \hat{f}_0(\xi)| &\le \int_{\mathbb{R}^3} |f_{0R}(v) - f_0(v)|\,dv \\ &\le \left(\int_{\mathbb{R}^3} f_0(v) X_R(v)\,dv\right)^{-1} \int_{\mathbb{R}^3} |f_0(v + a_R^f) - f_0(v)|\,dv \\ &\quad + \left(\int_{\mathbb{R}^3} f_0(v) X_R(v)\,dv\right)^{-1} \int_{\mathbb{R}^3} f_0(v)\left|X_R(v + a_R^f) - \int_{\mathbb{R}^3} f_0(v) X_R(v)\,dv\right|\,dv, \end{aligned}$$

we obtain

$$\lim_{R\to\infty} \sup_{\xi\in\mathbb{R}^3} |\hat{f}_{0R}(\xi) - \hat{f}_0(\xi)| = 0. \tag{2.12}$$

On the other hand, $\hat{f}_0(\xi) \in \mathcal{K}^\alpha(\mathbb{R}^3)$ implies that $\|1 - \hat{f}_0\|_{\mathcal{D}^\alpha} \lesssim 1$. Consequently, for $1 < \beta < \alpha \le 2$,

$$\frac{|1 - \hat{f}_0(\eta)|}{|\eta|^\beta} \le \|1 - \hat{f}_0\|_{\mathcal{D}^\alpha} |\eta|^{\alpha-\beta},$$

so that for each $\varepsilon > 0$, there exists a $\delta_1(\varepsilon) > 0$ such that

$$\frac{|1 - \hat{f}_0(\eta)|}{|\eta|^\beta} < \frac{\varepsilon}{2} \tag{2.13}$$

for any $|\eta| < \delta_1$.

Choose $\widetilde{\beta} \in (\beta, \alpha)$ so that $\|1 - \hat{f}_{0R}\|_{\mathcal{D}^{\widetilde{\beta}}}$ is bounded by a constant independent of R, using (2.7). Then

$$\frac{|1 - \hat{f}_{0R}(\eta)|}{|\eta|^\beta} \le \|1 - \hat{f}_{0R}\|_{\mathcal{D}^{\widetilde{\beta}}} |\eta|^{\widetilde{\beta}-\beta} \lesssim |\eta|^{\widetilde{\beta}-\beta} < \frac{\varepsilon}{2} \tag{2.14}$$

provided that $|\eta| < \delta_2(\varepsilon)$ for some sufficiently small $\delta_2 > 0$.

(2.13) together with (2.14) implies that for any $\varepsilon > 0$ and $|\eta| < \delta = \min\{\delta_1, \delta_2\}$,

$$\frac{|\hat{f}_{0R}(\eta) - \hat{f}_0(\eta)|}{|\eta|^\beta} \le \frac{|1 - \hat{f}_{0R}(\eta)|}{|\eta|^\beta} + \frac{|1 - \hat{f}_0(\eta)|}{|\eta|^\beta} < \varepsilon. \tag{2.15}$$

Now (2.8) follows directly from (2.12) and (2.15).

It remains to prove (2.5). Set

$$k(v, \xi) = \frac{e^{-iv\cdot\xi} + iv\cdot\xi - 1}{|\xi|^2}. \tag{2.16}$$

then

$$\begin{aligned}
&\frac{|\hat{f}_{0R}(\xi) - \hat{g}_{0R}(\xi)|}{|\xi|^2} \\
&\quad = \left| \int_{\mathbb{R}^3} \left(k(v - a_R^f, \xi) \frac{f_0(v) X_R(v)}{\int_{\mathbb{R}^3} f_0(v) X_R(v)\,dv} - k(v - a_R^g, \xi) \frac{g_0(v) X_R(v)}{\int_{\mathbb{R}^3} g_0(v) X_R(v)\,dv} \right) dv \right| \\
&\quad \lesssim \left| \int_{\mathbb{R}^3} (k(v - a_R^f, \xi) - k(v - a_R^g, \xi)) \frac{f_0(v) X_R(v)}{\int_{\mathbb{R}^3} f_0(v) X_R(v)\,dv}\,dv \right| \\
&\qquad + \left| \int_{\mathbb{R}^3} k(v - a_R^g, \xi) \left(\frac{f_0(v) X_R(v)}{\int_{\mathbb{R}^3} f_0(v) X_R(v)\,dv} - \frac{g_0(v) X_R(v)}{\int_{\mathbb{R}^3} g_0(v) X_R(v)\,dv} \right) dv \right| \\
&\quad =: I_1 + I_2.
\end{aligned} \tag{2.17}$$

First, from (2.9), (2.11) and the fact that $|k(v-a^g_R,\xi)| \lesssim |v-a^g_R|^2$, we have

$$\begin{aligned}
I_2 &\le 2\left|\int_{\mathbb{R}^3} k(v-a^g_R,\xi)(f_0(v)-g_0(v))X_R(v)\,dv\right| \\
&\quad + 4\left|\int_{\mathbb{R}^3} k(v-a^g_R,\xi)g_0(v)X_R(v)\left(\int_{\mathbb{R}^3}(f_0(v)-g_0(v))X_R(v)\,dv\right)dv\right| \\
&\lesssim \int_{\mathbb{R}^3}(1+|v|^2)|f_0(v)-g_0(v)|\,dv \\
&\quad + \int_{\mathbb{R}^3}(1+|v|^2)g_0(v)X_R(v)\left|\int_{\mathbb{R}^3}(f_0(v)-g_0(v))(1-X_R(v))\,dv\right|dv \\
&\lesssim \int_{\mathbb{R}^3}(1+|v|^2)|f_0(v)-g_0(v)|\,dv \\
&\quad + \int_{\mathbb{R}^3}(1+|v|^\beta)g_0(v)\,dv\cdot R^{2-\beta}\cdot\left|\int_{\mathbb{R}^3}(f_0(v)-g_0(v))(1-X_R(v))\,dv\right| \\
&\lesssim \int_{\mathbb{R}^3}(1+|v|^2)|f_0(v)-g_0(v)|\,dv \\
&\quad + \int_{\mathbb{R}^3}(1+|v|^\beta)g_0(v)\,dv\cdot\int_{\mathbb{R}^3}|v|^{2-\beta}|f_0(v)-g_0(v)|\,dv \\
&\lesssim \int_{\mathbb{R}^3}(1+|v|^2)|f_0(v)-g_0(v)|\,dv\cdot\left(1+\int_{\mathbb{R}^3}(1+|v|^\beta)g_0(v)\,dv\right).
\end{aligned} \tag{2.18}$$

For I_1, noticing that

$$|k(v-a^f_R,\xi)-k(v-a^g_R,\xi)| = |\xi|^{-2}|e^{-i(v-a^f_R)\cdot\xi}(e^{-i(a^g_R-a^f_R)\cdot\xi}-1)+i(a^g_R-a^f_R)\cdot\xi|, \tag{2.19}$$

for $|\xi|\ge 1$ we have

$$|k(v-a^f_R,\xi)-k(v-a^g_R,\xi)| \lesssim |a^g_R-a^f_R| \lesssim R^{1-\beta}. \tag{2.20}$$

For $|\xi|\le 1$,

$$\begin{aligned}
&|k(v-a^f_R,\xi)-k(v-a^g_R,\xi)| \\
&\quad\lesssim |\xi|^{-2}\big|\big[(1+O(1)|v-a^g_R|\,|\xi|)\big(-i(a^g_R-a^f_R)\cdot\xi+O(1)|a^g_R-a^f_R|^2|\xi|^2\big) \\
&\qquad\qquad + i(a^g_R-a^f_R)\cdot\xi\big]\big| \\
&\quad\lesssim |a^g_R-a^f_R|^2+|v-a^g_R|\,|a^g_R-a^f_R|+|v-a^g_R|\,|a^g_R-a^f_R|^2|\xi| \\
&\quad\lesssim (1+|v|)R^{1-\beta}.
\end{aligned} \tag{2.21}$$

Thus, (2.19)–(2.21) imply that

$$|k(v-a^f_R,\xi)-k(v-a^g_R,\xi)| \lesssim 1+|v|, \tag{2.22}$$

and consequently

$$I_1 \lesssim \int_{\mathbb{R}^3} (1+|v|) f_0(v)\, dv. \tag{2.23}$$

Inserting (2.18) and (2.23) into (2.17) yields (2.5). □

Now let

$$P_{jl}^R(0) \equiv \int_{\mathbb{R}^3} \left(v_j v_l - \frac{\delta_{jl}}{3}|v|^2 \right) (f_{0R}(v) - g_{0R}(v))\, dv \tag{2.24}$$

be the approximation of $P_{jl}(0)$ defined by $(1.12)_3$. The following lemma gives the convergence of $P_{jl}^R(0)$ to $P_{jl}(0)$ as $R \to \infty$.

Lemma 2.5. *Assume*

$$\int_{\mathbb{R}^3} |v|^2 |f_0(v) - g_0(v)|\, dv < \infty. \tag{2.25}$$

Then

$$\lim_{R\to\infty} P_{jl}^R(0) = P_{jl}(0). \tag{2.26}$$

Proof. Notice that

$$\begin{aligned}
P_{jl}^R(0) &= \int_{\mathbb{R}^3} \left(v_j v_l - \frac{\delta_{jl}}{3}|v|^2 \right) \frac{f_0(v+a_R^f) X_R(v+a_R^f) - f_0(v+a_R^g) X_R(v+a_R^g)}{\int_{\mathbb{R}^3} f_0(v) X_R(v)\, dv}\, dv \\
&+ \left(\int_{\mathbb{R}^3} f_0(v) X_R(v)\, dv \right)^{-1} \int_{\mathbb{R}^3} \left(v_j v_l - \frac{\delta_{jl}}{3}|v|^2 \right) (f_0(v+a_R^g) - g_0(v+a_R^g)) X_R(v+a_R^g)\, dv \\
&+ \int_{\mathbb{R}^3} \left(v_j v_l - \frac{\delta_{jl}}{3}|v|^2 \right) g_0(v+a_R^g) X_R(v+a_R^g) \\
&\qquad \times \left(\left(\int_{\mathbb{R}^3} f_0(v) X_R(v)\, dv \right)^{-1} - \left(\int_{\mathbb{R}^3} g_0(v) X_R(v)\, dv \right)^{-1} \right) dv \\
&= \int_{\mathbb{R}^3} \left[\left((v_j - a_{Rj}^f)(v_l - a_{Rl}^f) - \frac{\delta_{jl}}{3}|v - a_R^f|^2 \right) - \left((v_j - a_{Rj}^g)(v_l - a_{Rl}^g) - \frac{\delta_{jl}}{3}|v - a_R^g|^2 \right) \right] \\
&\qquad \times \frac{f_0(v) X_R(v)}{\int_{\mathbb{R}^3} f_0(v) X_R(v)\, dv}\, dv \\
&+ \int_{\mathbb{R}^3} \left((v_j - a_{Rj}^g)(v_l - a_{Rl}^g) - \frac{\delta_{jl}}{3}|v - a_R^g|^2 \right) \frac{(f_0(v) - g_0(v)) X_R(v)}{\int_{\mathbb{R}^3} f_0(v) X_R(v)\, dv}\, dv \\
&+ \int_{\mathbb{R}^3} \left(v_j v_l - \frac{\delta_{jl}}{3}|v|^2 \right) g_0(v+a_R^g) X_R(v+a_R^g) \frac{\int_{\mathbb{R}^3} (f_0(v) - g_0(v)) X_R(v)\, dv}{\int_{\mathbb{R}^3} f_0(v) X_R(v)\, dv \cdot \int_{\mathbb{R}^3} g_0(v) X_R(v)\, dv}\, dv \\
&=: I_3 + I_4 + I_5.
\end{aligned} \tag{2.27}$$

From

$$\begin{aligned}
&\left| \left((v_j - a_{Rj}^f)(v_l - a_{Rl}^f) - \frac{\delta_{jl}}{3}|v - a_R^f|^2 \right) - \left((v_j - a_{Rj}^g)(v_l - a_{Rl}^g) - \frac{\delta_{jl}}{3}|v - a_R^g|^2 \right) \right| \\
&\qquad \lesssim |a_R^f - a_R^g|\,|v| + |a_R^g|^2 + |a_R^f|^2
\end{aligned}$$

and (2.9) we have, for $R \geq R_1$,

$$|I_3| \lesssim (|a_R^g| + |a_R^f|) \int_{\mathbb{R}^3} (1 + |v|) f_0(v)\, dv.$$

This together with (2.10)–(2.11) implies that

$$\lim_{R\to\infty} I_3 = 0. \tag{2.28}$$

For I_5, if R is sufficiently large, from (2.9), (2.11) and the assumption (2.25) we have

$$\begin{aligned}
|I_5| &\lesssim \left| \int_{\mathbb{R}^3} \left(v_j v_l - \frac{\delta_{jl}}{3}|v|^2 \right) g_0(v + a_R^g) X_R(v + a_R^g) \left(\int_{\mathbb{R}^3} (f_0(v) - g_0(v)) X_R(v)\, dv \right) dv \right| \\
&\lesssim \int_{\mathbb{R}^3} (1 + |v - a_R^g|^2) g_0(v + a_R^g) X_R(v + a_R^g) \left| \int_{\mathbb{R}^3} (f_0(v) - g_0(v))(1 - X_R(v))\, dv \right| dv \\
&\lesssim \int_{\mathbb{R}^3} (1 + |v|^\beta) R^{2-\beta} g_0(v) X_R(v) \left| \int_{\mathbb{R}^3} (f_0(v) - g_0(v))(1 - X_R(v))\, dv \right| dv \\
&\lesssim R^{2(1-\beta)} \left(\int_{\mathbb{R}^3} (1 + |v|^\beta) g_0(v)\, dv \right) \left(\int_{\mathbb{R}^3} |v|^\beta |f_0(v) - g_0(v)|\, dv \right) \lesssim R^{2(1-\beta)}.
\end{aligned}$$

Thus

$$\lim_{R\to\infty} I_5 = 0. \tag{2.29}$$

Finally for I_4, from (2.10), (2.11), (2.25) and $\lim_{R\to\infty} \int_{\mathbb{R}^3} f_0(v) X_R(v)\, dv = 1$, the dominated convergence theorem yields

$$\lim_{R\to\infty} I_4 = \int_{\mathbb{R}^3} \left(v_j v_l - \frac{\delta_{jl}}{3}|v|^2 \right) (f_0(v) - g_0(v))\, dv = P_{jl}(0). \tag{2.30}$$

Inserting (2.28)–(2.30) into (2.27) gives (2.26). □

In the last lemma of this section, a sufficient condition on $f_0(v) - g_0(v)$ is given for (1.15) in Theorem 1.1 to hold.

Lemma 2.6. *Let* $0 < \delta \leq 1$. *Then*

$$\|\hat{f}_0(\cdot) - \hat{g}_0(\cdot) - \widetilde{P}(0, \cdot)\|_{\mathcal{D}^{2+\delta}} \lesssim \int_{\mathbb{R}^3} (1 + |v|^{2+\delta}) |f_0(v) - g_0(v)|\, dv. \tag{2.31}$$

Proof. In fact, by the assumption (1.14), we have

$$\begin{aligned}
&\hat{f}_0(\xi) - \hat{g}_0(\xi) - \widetilde{P}(0, \xi) \\
&\quad = \int_{\mathbb{R}^3} \left[e^{-iv\cdot\xi} - 1 + iv\cdot\xi - \frac{1}{2}\sum_{j,l=1}^{3} \xi_j \xi_l X(\xi) \left(v_j v_l - \frac{\delta_{jl}}{3}|v|^2 \right) \right] (f_0(v) - g_0(v))\, dv \\
&\quad = \int_{\mathbb{R}^3} \left[e^{-iv\cdot\xi} - 1 + iv\cdot\xi - \frac{1}{2}\sum_{j,l=1}^{3} \xi_j \xi_l X(\xi) v_j v_l \right] (f_0(v) - g_0(v))\, dv.
\end{aligned}$$

The Taylor expansion of $e^{-iv\cdot\xi} - 1 + iv\cdot\xi$ to the second order implies that

$$\left| e^{-iv\cdot\xi} - 1 + iv\cdot\xi - \frac{1}{2}\sum_{j,l=1}^{3} \xi_j\xi_l X(\xi) v_j v_l \right| \lesssim |v|^2|\xi|^2,$$

and the Taylor expansion of $e^{-iv\cdot\xi} - 1 + iv\cdot\xi - \frac{1}{2}\sum_{j,l=1}^{3} \xi_j\xi_l X(\xi) v_j v_l$ to the third order gives

$$\left| e^{-iv\cdot\xi} - 1 + iv\cdot\xi - \frac{1}{2}\sum_{j,l=1}^{3} \xi_j\xi_l X(\xi) v_j v_l \right| \lesssim (1+|v|^3)|\xi|^3.$$

Thus interpolation yields

$$\left| e^{-iv\cdot\xi} - 1 + iv\cdot\xi - \frac{1}{2}\sum_{j,l=1}^{3} \xi_j\xi_l X(\xi) v_j v_l \right| \lesssim (1+|v|^{2+\delta})|\xi|^{2+\delta}$$

for all $0 < \delta \le 1$. Hence (2.31) follows. □

3. Proof of Theorem 1.1

To prove Theorem 1.1, as in [7], we first approximate the cross section by a sequence of bounded cross sections defined by

$$\mathcal{B}_n(s) = \min\{\mathcal{B}(s), n\}, \qquad n \in \mathbb{N}. \tag{3.1}$$

Then consider

$$\partial_t H_n + H_n = \frac{1}{\overline{\sigma}_n}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \mathcal{B}_n\left(\frac{(v-v_*)\cdot\sigma}{|v-v_*|}\right) H_n(v')H_n(v'_*)\,d\sigma\,dv_*, \tag{3.2}$$

$$H_n(0,v) = H_0(v). \tag{3.3}$$

Here

$$\overline{\sigma}_n = \int_{\mathbb{S}^2} \mathcal{B}_n\left(\frac{\xi\cdot\sigma}{|\xi|}\right) d\sigma. \tag{3.4}$$

For $\alpha \in (\max\{2s,1\},2]$ and $f_0(v), g_0(v) \in \widetilde{\mathcal{P}}^\alpha(\mathbb{R}^3)$, let $f_{0R}(v)$ and $g_{0R}(v)$ be the approximations of $f_0(v)$ and $g_0(v)$ constructed in the previous section. Since $f_{0R}(v), g_{0R}(v) \in \mathcal{P}^2(\mathbb{R}^3) \subset \mathcal{K}^2(\mathbb{R}^3)$, it follows from Lemma 2.1 that the Cauchy problem (3.2)–(3.3) with $H_0(v) = f_{0R}(v)$ $[H_0(v) = g_{0R}(v)]$ admits a unique non-negative global solution $F_R^n(t,v)$ $[G_R^n(t,v)]$ satisfying $\hat{F}_R^n(t,\xi) \in C([0,\infty),\mathcal{K}^2(\mathbb{R}^3))$ $[\hat{G}_R^n(t,\xi) \in C([0,\infty),\mathcal{K}^2(\mathbb{R}^3))]$. Moreover, for $\max\{2s,1\} < \beta < \alpha \le 2$, (2.2) and Lemmas 2.3 and 2.4 imply that

$$\begin{cases} \|\hat{F}_R^n(t,\cdot) - \hat{F}_n(t,\cdot)\|_{\mathcal{D}^\beta} \le e^{\lambda_\beta^n t}\|\hat{f}_{0R}(\cdot) - \hat{f}_0(\cdot)\|_{\mathcal{D}^\beta} \lesssim e^{\lambda_\beta^n t}, \\ \|\hat{G}_R^n(t,\cdot) - \hat{G}_n(t,\cdot)\|_{\mathcal{D}^\beta} \le e^{\lambda_\beta^n t}\|\hat{g}_{0R}(\cdot) - \hat{g}_0(\cdot)\|_{\mathcal{D}^\beta} \lesssim e^{\lambda_\beta^n t}. \end{cases} \tag{3.5}$$

Here $F_n(t,v)$ and $G_n(t,v)$ denote the unique non-negative solutions of the Cauchy problem (3.2)–(3.3) with initial data $f_0(v) \in \widetilde{\mathcal{P}}^\alpha(\mathbb{R}^3)$ and $g_0(v) \in \widetilde{\mathcal{P}}^\alpha(\mathbb{R}^3)$ respectively, and

$$\lambda_\alpha^n = \frac{1}{\overline{\sigma}_n} \int_{\mathbb{S}^2} \mathcal{B}_n\left(\frac{\xi \cdot \sigma}{|\xi|}\right)\left(\frac{|\xi^+|^\alpha + |\xi^-|^\alpha}{|\xi|^\alpha} - 1\right) d\sigma. \tag{3.6}$$

Furthermore,

$$\int_{\mathbb{R}^3} |v|^2 F_R^n(t,v)\,dv = \int_{\mathbb{R}^3} |v|^2 f_{0R}(v)\,dv < \infty,$$
$$\int_{\mathbb{R}^3} |v|^2 G_R^n(t,v)\,dv = \int_{\mathbb{R}^3} |v|^2 g_{0R}(v)\,dv < \infty.$$

Consequently, Lemma 2.1 yields

$$\|\hat{F}_R^n(t,\cdot) - \hat{G}_R^n(t,\cdot)\|_{\mathcal{D}^2} \le \|\hat{f}_{0R}(\cdot) - \hat{g}_{0R}(\cdot)\|_{\mathcal{D}^2} \lesssim 1, \qquad t > 0. \tag{3.7}$$

Noticing that

$$|\hat{F}_R^n(t,\xi) - \hat{F}_n(t,\xi)| \le |\xi|^\beta \|\hat{F}_R^n(t,\cdot) - \hat{F}_n(t,\cdot)\|_{\mathcal{D}^\beta} \le |\xi|^\beta e^{\lambda_\beta^n t} \|\hat{f}_{0R}(\cdot) - \hat{f}_0(\cdot)\|_{\mathcal{D}^\beta},$$

where (3.5) has been used, from (2.8) we have

Lemma 3.1. *The limit*

$$\lim_{R\to\infty} (\hat{F}_R^n(t,\xi), \hat{G}_R^n(t,\xi)) = (\hat{F}_n(t,\xi), \hat{G}_n(t,\xi))$$

holds locally uniformly with respect to $t \in \mathbb{R}^+$ *and* $\xi \in \mathbb{R}^3$.

Setting

$$\Phi_1^{nR}(t,\xi) = \hat{F}_R^n(t,\xi) - \hat{G}_R^n(t,\xi) - \widetilde{P}_R^n(t,\xi) \tag{3.8}$$

with

$$\begin{aligned} \widetilde{P}_R^n(t,\xi) &= \frac{1}{2} e^{-A_n t} \sum_{j,l=1}^{3} P_{jl}^R(0)\xi_j\xi_l X(\xi), \\ A_n &= \frac{3}{4\overline{\sigma}_n} \int_{\mathbb{S}^2} \mathcal{B}_n\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left[1 - \left(\frac{\xi\cdot\sigma}{|\xi|}\right)^2\right] d\sigma, \end{aligned} \tag{3.9}$$

we now deduce the equation for $\Phi_1^{nR}(t,\xi)$. Set

$$\hat{Q}_n^+(\hat{F},\hat{G}) = \frac{1}{\overline{\sigma}_n} \int_{\mathbb{S}^2} \mathcal{B}_n\left(\frac{\xi\cdot\sigma}{|\xi|}\right) \hat{F}(t,\xi^+)\hat{G}(t,\xi^-)\,d\sigma. \tag{3.10}$$

Since $\hat{F}_R^n(t,\xi)$ and $\hat{G}_R^n(t,\xi)$ satisfy

$$\partial_t \hat{F}_R^n + \hat{F}_R^n = \hat{Q}^+(\hat{F}_R^n, \hat{F}_R^n), \qquad \partial_t \hat{G}_R^n + \hat{G}_R^n = \hat{Q}^+(\hat{G}_R^n, \hat{G}_R^n),$$

we have

$$\begin{aligned}\partial_t \Phi_1^{nR} + \Phi_1^{nR} &= -(\partial_t \widetilde{P}_R^n + \widetilde{P}_R^n) + \hat{Q}^+(\Phi_1^{nR}, \hat{F}_R^n) + \hat{Q}^+(\hat{G}_R^n, \Phi_1^{nR}) \\ &\quad + \hat{Q}^+(\widetilde{P}_R^n, \hat{F}_R^n) + \hat{Q}^+(\hat{G}_R^n, \widetilde{P}_R^n), \\ \Phi_1^{nR}(0,\xi) &= \hat{f}_{0R}(\xi) - \hat{g}_{0R}(\xi) - \widetilde{P}_R^n(0,\xi).\end{aligned} \tag{3.11}$$

Let

$$\Phi_1^n(t,\xi) = \hat{F}^n(t,\xi) - \hat{G}^n(t,\xi) - \widetilde{P}^n(t,\xi). \tag{3.12}$$

By taking $R \to \infty$, from Lemmas 3.1 and 2.5 we deduce that $\Phi_1^{nR}(t,\xi) \to \Phi_1^n(t,\xi)$ as $R \to \infty$, locally uniformly with respect to $t \in \mathbb{R}^+$ and $\xi \in \mathbb{R}^3$. To derive the equation for $\Phi_1^n(t,\xi)$, we first study

$$E_R^n(t,\xi) = \hat{Q}^+(\widetilde{P}_R^n, \hat{F}_R^n) + \hat{Q}^+(\hat{G}_R^n, \widetilde{P}_R^n) - (\partial_t \widetilde{P}_R^n(t,\xi) + \widetilde{P}_R^n(t,\xi)). \tag{3.13}$$

In fact, for $E_R^n(t,\xi)$, we have

Lemma 3.2. *We have*

$$\lim_{R\to\infty} E_R^n(t,\xi) = E^n(t,\xi), \tag{3.14}$$

locally uniformly with respect to $t \in \mathbb{R}^+$ and $\xi \in \mathbb{R}^3$. Moreover,

$$|E^n(t,\xi)| \le \begin{cases} O(1)|\xi|^{2+\delta} e^{-(A_n - \delta\lambda_\alpha^n/\alpha)t}, & |\xi| \le 1, \\ O(1) e^{-A_n t}, & |\xi| \ge 1, \end{cases} \tag{3.15}$$

for any $\delta \in (0,\alpha] \cap (0, \alpha A_n/\lambda_\alpha^n)$ and some positive constant $O(1)$ independent of t, ξ, R and n.

Proof. Since

$$\begin{aligned}&E_R^n(t,\xi) = -(\partial_t \widetilde{P}_R^n(t,\xi) + \widetilde{P}_R^n(t,\xi)) \\ &+ \frac{1}{2\overline{\sigma}_n} \sum_{j,l=1}^3 e^{-A_n t} P_{jl}^R(0) \int_{\mathbb{S}^2} \mathcal{B}_n\left(\frac{\xi\cdot\sigma}{|\xi|}\right)[\xi_j^+\xi_l^+ X(\xi^+) + \xi_j^-\xi_l^- X(\xi^-)]\, d\sigma \\ &+ \frac{1}{2\overline{\sigma}_n} \sum_{j,l=1}^3 e^{-A_n t} P_{jl}^R(0) \\ &\quad \times \int_{\mathbb{S}^2} \mathcal{B}_n\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\big[\xi_j^+\xi_l^+ X(\xi^+)(\hat{F}_R^n(t,\xi^-) - 1) + \xi_j^-\xi_l^- X(\xi^-)(\hat{G}_R^n(t,\xi^+) - 1)\big]\, d\sigma,\end{aligned}$$

it follows from Lemmas 3.1 and 2.5 that

$$\lim_{R\to\infty} E_R^n(t,\xi) = E^n(t,\xi), \tag{3.16}$$

locally uniformly with respect to $t \in \mathbb{R}^+$ and $\xi \in \mathbb{R}^3$. Here

$$\begin{aligned} E^n(t,\xi) = &-(\partial_t \widetilde{P}^n(t,\xi) + \widetilde{P}^n(t,\xi)) \\ &+ \frac{1}{2\overline{\sigma}_n}\sum_{j,l=1}^{3} e^{-A_n t} P_{jl}(0)\int_{\mathbb{S}^2} \mathcal{B}_n\left(\frac{\xi\cdot\sigma}{|\xi|}\right)[\xi_j^+\xi_l^+ X(\xi^+) + \xi_j^-\xi_l^- X(\xi^-)]\,d\sigma \\ &+ \frac{1}{2\overline{\sigma}_n}\sum_{j,l=1}^{3} e^{-A_n t} P_{jl}(0)\int_{\mathbb{S}^2} \mathcal{B}_n\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\Big[\xi_j^+\xi_l^+ X(\xi^+)(\hat{F}^n(t,\xi^-)-1) \\ &\qquad + \xi_j^-\xi_l^- X(\xi^-)(\hat{G}^n(t,\xi^+)-1)\Big]\,d\sigma \\ =: &-(\partial_t \widetilde{P}^n(t,\xi) + \widetilde{P}^n(t,\xi)) + I_6 + I_7 \end{aligned} \tag{3.17}$$

and

$$\widetilde{P}^n(t,\xi) = \frac{1}{2}e^{-A_n t}\sum_{j,l=1}^{3} P_{jl}(0)\xi_j\xi_l X(\xi). \tag{3.18}$$

To bound I_6 and I_7, first note that for $|\xi| \ge 1$, the estimates

$$|\hat{F}^n(t,\xi)| \le 1, \qquad |\hat{G}^n(t,\xi)| \le 1$$

imply that

$$|E^n(t,\xi)| \le O(1)(1+A_n)e^{-A_n t} \le O(1)e^{-A_n t}. \tag{3.19}$$

Here we have used the fact that A_n has a uniform upper bound for any $n \in \mathbb{N}$.

If $|\xi| \le 1$, then $|\xi^\pm| \le 1$ so that $X(\xi^\pm) \equiv 1$. Hence, as in [9], we have

$$\xi_j^-\xi_l^- + \xi_j^+\xi_l^+ = \frac{1}{2}(\xi_j\xi_l + |\xi|^2\sigma_j\sigma_l),$$

and

$$\frac{1}{\overline{\sigma}_n}\int_{\mathbb{S}^2} \mathcal{B}_n\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\sigma_j\sigma_l\,d\sigma = \frac{2A_n}{3}\delta_{jl} + (1-2A_n)\frac{\xi_j\xi_l}{|\xi|^2}.$$

Then

$$\begin{aligned} I_6 &= \frac{1}{2\overline{\sigma}_n}\sum_{j,l=1}^{3} e^{-A_n t} P_{jl}(0)\int_{\mathbb{S}^2} \mathcal{B}_n\left(\frac{\xi\cdot\sigma}{|\xi|}\right)[\xi_j^+\xi_l^+ + \xi_j^-\xi_l^-]\,d\sigma \\ &= \frac{1}{4\overline{\sigma}_n}\sum_{j,l=1}^{3} e^{-A_n t} P_{jl}(0)\int_{\mathbb{S}^2} \mathcal{B}_n\left(\frac{\xi\cdot\sigma}{|\xi|}\right)[\xi_j\xi_l + |\xi|^2\sigma_j\sigma_l]\,d\sigma \\ &= \frac{1}{4}\sum_{j,l=1}^{3} e^{-A_n t} P_{jl}(0)\xi_j\xi_l + \frac{1}{4}\sum_{j,l=1}^{3} e^{-A_n t} P_{jl}(0)\left[(1-2A_n)\xi_j\xi_l + \frac{2A_n}{3}\delta_{jl}|\xi|^2\right] \\ &= (1-A_n)\widetilde{P}^n(t,\xi) = \partial_t \widetilde{P}^n(t,\xi) + \widetilde{P}^n(t,\xi). \end{aligned}$$

Thus for $|\xi| \le 1$,

$$I_6 - (\partial_t \widetilde{P}^n(t,\xi) + \widetilde{P}^n(t,\xi)) = 0. \tag{3.20}$$

For I_7, when $|\xi| \le 1$ we have, from the assumption $f_0(v), g_0(v) \in \widetilde{\mathcal{P}}^\alpha(\mathbb{R}^3)$ and Lemma 2.1,

$$|\hat{F}_n(t,\xi) - 1| \le |\xi|^\alpha \|\hat{F}_n(t,\cdot) - 1\|_{\mathcal{D}^\alpha} \le e^{\lambda_\alpha^n t} |\xi|^\alpha \|\hat{f}_0(\cdot) - 1\|_{\mathcal{D}^\alpha},$$
$$|\hat{G}_n(t,\xi) - 1| \le |\xi|^\alpha \|\hat{G}_n(t,\cdot) - 1\|_{\mathcal{D}^\alpha} \le e^{\lambda_\alpha^n t} |\xi|^\alpha \|\hat{g}_0(\cdot) - 1\|_{\mathcal{D}^\alpha}.$$

The above estimates together with $|\hat{F}^n(t,\xi)| \le 1$ and $|\hat{G}^n(t,\xi)| \le 1$ imply that

$$|\hat{F}_n(t,\xi) - 1| + |\hat{G}_n(t,\xi) - 1| \le O(1)|\xi|^{\varepsilon\alpha} e^{\varepsilon\lambda_\alpha^n t} \tag{3.21}$$

for any $\varepsilon \in (0, 1]$. Consequently, for $|\xi| \le 1$,

$$|I_7| \le O(1)|\xi|^{2+\varepsilon\alpha} e^{-(A_n - \varepsilon\lambda_\alpha^n)t}. \tag{3.22}$$

(3.20) together with (3.22) implies that

$$|E^n(t,\xi)| \le O(1)|\xi|^{2+\varepsilon\alpha} e^{-(A_n - \varepsilon\lambda_\alpha^n)t}, \qquad |\xi| \le 1. \tag{3.23}$$

From (3.19) and (3.23), the estimate (3.15) follows immediately by writing $\delta = \varepsilon\alpha$. □

Now by letting $R \to \infty$ in (3.11), we infer from Lemmas 2.5, 3.1 and 3.2 that $\Phi_1^n(t,\xi) = \hat{F}^n(t,\xi) - \hat{G}^n(t,\xi) - \widetilde{P}^n(t,\xi)$ solves

$$\partial_t \Phi_1^n + \Phi_1^n = \hat{Q}^+(\Phi_1^n, \hat{F}^n) + \hat{Q}^+(\hat{G}^n, \Phi_1^n) + E^n(t,\xi), \tag{3.24}$$
$$\Phi_1^n(0,\xi) = \hat{f}_0(\xi) - \hat{g}_0(\xi) - \widetilde{P}^n(0,\xi). \tag{3.25}$$

Here $E^n(t,\xi)$ satisfies (3.15). By Lemmas 2.5, 3.1 and 3.2, $\Phi_1^n(t,\xi)$, $\hat{F}^n(t,\xi)$, $\hat{G}_n(t,\xi)$ and $E^n(t,\xi)$ are continuous functions of $(t,\xi) \in \mathbb{R}^+ \times \mathbb{R}^3$ and satisfy (3.24) in the sense of distributions. Since $\Phi_1^n(t,\xi)$, $\hat{F}^n(t,\xi)$, $\hat{G}_n(t,\xi)$, and $E^n(t,\xi)$ are uniformly bounded, so is $\partial_t \Phi_1^n(t,\xi)$, so that $\Phi_1^n(t,\xi)$ is globally Lipschitz continuous with respect to t. Hence (3.24) holds almost everywhere. Furthermore, by the continuity of $\Phi_1^n(t,\xi)$, $\hat{F}^n(t,\xi)$, $\hat{G}_n(t,\xi)$ and $E^n(t,\xi)$, we see that $\partial_t \Phi_1^n(t,\xi)$ is a continuous function of $(t,\xi) \in \mathbb{R}^+ \times \mathbb{R}^3$ and consequently $\Phi_1^n(t,\xi)$ satisfies (3.24) everywhere.

The next lemma gives an upper bound on $\|\Phi_1^n(t,\cdot)\|_{\mathcal{D}^{2+\delta}}$ for $\delta \in (0,\alpha] \cap (0, \alpha A_n/\lambda_n)$.

Lemma 3.3. *If $\|\Phi_1^n(0,\cdot)\|_{\mathcal{D}^{2+\delta}} < \infty$ with $\delta \in (0,\alpha] \cap (0, \alpha A_n/\lambda_n)$, then*

$$\|\Phi_1^n(t,\cdot)\|_{\mathcal{D}^{2+\delta}} \lesssim e^{-\eta_0^n t}. \tag{3.26}$$

Here $\eta_0^n = \min\{B_n, A_n - \delta\lambda_\alpha^n/\alpha\}$ with

$$B_n = \frac{1}{\overline{\sigma}_n} \int_{\mathbb{S}^2} \mathcal{B}_n\left(\frac{\sigma\cdot\xi}{|\xi|}\right)\left(1 - \left|\cos\frac{\theta}{2}\right|^{2+\delta} - \left|\sin\frac{\theta}{2}\right|^{2+\delta}\right) d\sigma, \qquad \cos\theta = \frac{\sigma\cdot\xi}{|\xi|}. \tag{3.27}$$

Proof. The proof is divided into two steps. The first step is to show that $\Phi_1^n(t,\xi)/|\xi|^{2+\delta} \in L^\infty(\mathbb{R}^3)$. Indeed, for $\kappa > 0$, from (3.24) we have

$$\begin{aligned}
&\left(\frac{\Phi_1^n(t,\xi)}{|\xi|^2(|\xi|^\delta+\kappa)}\right)_t + \frac{\Phi_1^n(t,\xi)}{|\xi|^2(|\xi|^\delta+\kappa)} \\
&\quad = \frac{1}{\overline{\sigma}_n}\int_{\mathbb{S}^2} \mathcal{B}_n\left(\frac{\sigma\cdot\xi}{|\xi|}\right)\Bigg[\frac{\Phi_1^n(t,\xi^+)}{|\xi^+|^2(|\xi^+|^\delta+\kappa)}\frac{|\xi^+|^2(|\xi^+|^\delta+\kappa)}{|\xi|^2(|\xi|^\delta+\kappa)}\hat{F}^n(t,\xi^-) \\
&\qquad + \hat{G}^n(t,\xi^+)\frac{\Phi_1^n(t,\xi^-)}{|\xi^-|^2(|\xi^-|^\delta+\kappa)}\frac{|\xi^-|^2(|\xi^-|^\delta+\kappa)}{|\xi|^2(|\xi|^\delta+\kappa)}\Bigg]d\sigma + \frac{E^n(t,\xi)}{|\xi|^2(|\xi|^\delta+\kappa)}. \qquad (3.28)
\end{aligned}$$

On the other hand, by letting $R\to\infty$ in (3.7), we see from Lemma 3.1 that for $t>0$,

$$\|\hat{F}^n(t,\cdot)-\hat{G}^n(t,\cdot)\|_{\mathcal{D}^2} \le \|\hat{f}_0(\cdot)-\hat{g}_0(\cdot)\|_{\mathcal{D}^2} \lesssim 1. \qquad (3.29)$$

This together with the definition of $\widetilde{P}^n(t,\xi)$ implies that

$$\frac{\Phi_1^n(t,\xi)}{|\xi|^2(|\xi|^\delta+\kappa)} \in L^\infty(\mathbb{R}^3), \qquad t>0,$$

for any $\kappa>0$. Hence, by (3.28), Lemma 3.2 and the fact that $|\xi^\pm|\le|\xi|$, $|\hat{F}^n(t,\xi)|\le 1$, $|\hat{G}^n(t,\xi)|\le 1$, we can deduce by using the Gronwall inequality that there exists a positive constant $C(T)>0$ independent of κ, n and ξ such that

$$\sup_{0\ne\xi\in\mathbb{R}^3}\frac{|\Phi_1^n(t,\xi)|}{|\xi|^2(|\xi|^\delta+\kappa)} \le C(T) \qquad (3.30)$$

for $0\le t\le T$. Here $T>0$ is any given positive constant.

Since the positive constant $C(T)>0$ in (3.30) is independent of κ, we see from (3.30) by letting $\kappa\to 0_+$ that

$$\sup_{0\ne\xi\in\mathbb{R}^3}\frac{|\Phi_1^n(t,\xi)|}{|\xi|^{2+\delta}} \le C(T), \qquad 0\le t\le T. \qquad (3.31)$$

Set

$$\Phi_2^n(t,\xi) = \frac{\Phi_1^n(t,\xi)}{|\xi|^{2+\delta}}. \qquad (3.32)$$

From (3.24) and the fact that $|\xi^\pm|^2 = \frac{|\xi|^2\pm\xi\cdot\sigma|\xi|}{2} = \frac{|\xi|^2(1\pm\cos\theta)}{2}$ we can deduce that

$$\begin{aligned}
\partial_t\Phi_2^n + \Phi_2^n = \frac{1}{\overline{\sigma}_n}\int_{\mathbb{S}^2}\mathcal{B}_n\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\Bigg[\Phi_2^n(t,\xi^+)\hat{F}^n(t,\xi^-)\left|\cos\frac{\theta}{2}\right|^{2+\delta} \\
+ \hat{G}^n(t,\xi^+)\Phi_2^n(t,\xi^-)\left|\sin\frac{\theta}{2}\right|^{2+\delta}\Bigg]d\sigma + \frac{E^n(t,\xi)}{|\xi|^{2+\delta}}. \qquad (3.33)
\end{aligned}$$

A direct consequence of (3.33) is

$$|\partial_t \Phi_2^n + \Phi_2^n| \le (1 - B_n)\|\Phi_2^n(t,\cdot)\|_{L^\infty} + O(1)e^{-(A_n - \delta\lambda_\alpha^n/\alpha)t}. \tag{3.34}$$

Since $0 < B_n < 1$, we can apply the argument used in [9] to obtain

$$|\Phi_2^n(t,\xi)| \le O(1)e^{-\eta_0 t}, \tag{3.35}$$

so that (3.26) follows. □

We now turn to the proof of Theorem 1.1. Let $F_n(t,v)$ and $G_n(t,v)$ be the unique solutions of the Cauchy problem (3.2)–(3.3) with initial data $f_0(v)$ and $g_0(v)$ respectively. Then

$$f_n(t,v) \equiv F_n(\overline{\sigma}_n t, v), \qquad g_n(t,v) \equiv G_n(\overline{\sigma}_n t, v) \tag{3.36}$$

solve

$$\begin{cases} \partial_t f_n + \overline{\sigma}_n f_n = \displaystyle\int_{\mathbb{S}^2} \mathcal{B}_n\left(\frac{\sigma\cdot(v-v_*)}{|v-v_*|}\right) f_n(v') f_n(v'_*)\, dv_*\, d\sigma, \\ f_n(0,v) = f_0(v), \end{cases} \tag{3.37}$$

and

$$\begin{cases} \partial_t g_n + \overline{\sigma}_n g_n = \displaystyle\int_{\mathbb{S}^2} \mathcal{B}_n\left(\frac{\sigma\cdot(v-v_*)}{|v-v_*|}\right) g_n(v') g_n(v'_*)\, dv_*\, d\sigma, \\ g_n(0,v) = g_0(v), \end{cases} \tag{3.38}$$

respectively.

The estimate (3.26) in Lemma 3.3 gives

$$\|\hat{F}_n(t,\cdot) - \hat{G}_n(t,\cdot) - \widetilde{P}^n(t,\cdot)\|_{\mathcal{D}^{2+\delta}} \le O(1)e^{-\eta_0^n t}. \tag{3.39}$$

Putting (3.36) and (3.39) together yields

$$\|\hat{f}_n(t,\cdot) - \hat{g}_n(t,\cdot) - \widetilde{P}^n(\overline{\sigma}_n t,\cdot)\|_{\mathcal{D}^{2+\delta}} \le O(1)e^{-\overline{\sigma}_n\eta_0^n t}. \tag{3.40}$$

Noticing that

$$\lim_{n\to\infty} \overline{\sigma}_n A_n = A, \qquad \lim_{n\to\infty} \overline{\sigma}_n B_n = B, \qquad \lim_{n\to\infty} \overline{\sigma}_n \lambda_\alpha^n = \lambda_\alpha,$$

we have

$$\lim_{n\to\infty} \overline{\sigma}_n \eta_0^n = \eta_0, \qquad \lim_{n\to\infty} \widetilde{P}^n(\overline{\sigma}_n t,\xi) = \widetilde{P}(t,\xi). \tag{3.41}$$

On the other hand, it is shown in [7, 11] that $(\hat{f}_n(t,\xi), \hat{g}_n(t,\xi)) \to (\hat{f}(t,\xi), \hat{g}(t,\xi))$ as $n \to \infty$, locally uniformly with respect to $(t,\xi) \in \mathbb{R}^+ \times \mathbb{R}^3$. By (3.40), we deduce from (3.41) that

$$\|\hat{f}(t,\cdot) - \hat{g}(t,\cdot) - \widetilde{P}(t,\cdot)\|_{\mathcal{D}^{2+\delta}} \lesssim O(1)e^{-\eta_0 t}. \tag{3.42}$$

This is exactly (1.16), and thus the proof of Theorem 1.1 is complete.

4. Proof of Theorem 1.2

To prove Theorem 1.2, compared with Theorem 1.1, we only need to obtain a uniform $H^N(\mathbb{R}^3)$-estimate (1.18) on $f(t,v)$, and the key point is to deduce the following coercivity estimate.

Lemma 4.1. *There exists a sufficiently large positive constant $t_1 > 0$ such that if $|\xi| \geq 2$ then*

$$\int_{\mathbb{S}^2} \mathcal{B}\left(\frac{\xi \cdot \sigma}{|\xi|}\right)(1 - |\hat{f}(t, \xi^-)|)\, d\sigma \geq \kappa e^{2s\mu_\alpha t}|\xi|^{2s} \tag{4.1}$$

for any $t \geq t_1$ and some positive constant $\kappa > 0$ which depends only on t_1.

Proof. Noticing that

$$\hat{\Psi}_{\alpha,K}(\xi) \in \mathcal{K}^\alpha(\mathbb{R}^3), \qquad \Psi_{\alpha,K}(v) \in L^1(\mathbb{R}^3) \cap H^\infty(\mathbb{R}^3), \qquad \int_{\mathbb{R}^3} \Psi_{\alpha,K}(v)\, dv = 1,$$

we know from [12, Theorem 1.1] that $\Psi_{\alpha,K}(v) \in \mathcal{P}^\beta(\mathbb{R}^3)$ for $1 < \beta < \alpha < 2$, and consequently [1, Lemma 3] shows that there exists a positive constant $\kappa_1 > 0$ independent of t and ξ such that

$$1 - |\hat{\Psi}_{\alpha,K}(\xi)| \geq \kappa_1 \min\{1, |\xi|^2\}.$$

Hence,

$$1 - |\hat{\Psi}_{\alpha,K}(e^{\mu_\alpha t}\xi)| \geq \kappa_1 \min\{1, |e^{\mu_\alpha t}\xi|^2\}, \qquad \forall (t,\xi) \in \mathbb{R}^+ \times \mathbb{R}^3. \tag{4.2}$$

On the other hand, from the $\mathcal{D}^{2+\delta}$-stability estimate in Theorem 1.1 we have

$$\begin{aligned} |\hat{f}(t,\xi) - \hat{\Psi}_{\alpha,K}(e^{\mu_\alpha t}\xi)| &\leq O(1)|\xi|^{2+\delta}e^{-\eta_0 t} + |\widetilde{P}(t,\xi)| \\ &\leq \kappa_2(|\xi|^{2+\delta} + |\xi|^2)e^{-\eta_0 t} \end{aligned} \tag{4.3}$$

for any $(t,\xi) \in \mathbb{R}^+ \times \mathbb{R}^3$ with a constant $\kappa_2 > 0$ independent of t and ξ.

A direct consequence of (4.2) and (4.3) is

$$\begin{aligned} 1 - |\hat{f}(t,\xi)| &\geq (1 - |\hat{\Psi}_{\alpha,K}(e^{\mu_\alpha t}\xi)|) - |\hat{f}(t,\xi) - \hat{\Psi}_{\alpha,K}(e^{\mu_\alpha t}\xi)| \\ &\geq \kappa_1 \min\{1, |e^{\mu_\alpha t}\xi|^2\} - \kappa_2(|\xi|^{2+\delta} + |\xi|^2)e^{-\eta_0 t}, \qquad \forall (t,\xi) \in \mathbb{R}^+ \times \mathbb{R}^3. \end{aligned} \tag{4.4}$$

Thus

$$1 - |\hat{f}(t,\xi)| \geq \max\left\{0, \kappa_1 \min\{1, |e^{\mu_\alpha t}\xi|^2\} - \kappa_2(|\xi|^{2+\delta} + |\xi|^2)e^{-\eta_0 t}\right\}. \tag{4.5}$$

Having (4.5), we now turn to the proof of (4.1). First, note that $|\xi^-|^2 = |\xi|^2 \sin^2(\theta/2)$. If we choose $t_1 > 0$ so large that

$$\kappa_1 e^{2\mu_\alpha t} - 2\kappa_2 e^{-\eta_0 t} \geq \frac{\kappa_1}{2}e^{2\mu_\alpha t} + \frac{\kappa_1}{2}e^{2\mu_\alpha t_1} - 2\kappa_2 e^{-\eta_0 t_1} \geq \frac{\kappa_1}{2}e^{2\mu_\alpha t}, \qquad \forall t \geq t_1, \tag{4.6}$$

then for $t \geq t_1$, $|\xi| \geq 2$ and θ so small that

$$\theta \in \left[0, \frac{2}{e^{\mu_\alpha t}|\xi|}\right] \subset \left[0, \frac{\pi}{2}\right), \tag{4.7}$$

we have

$$\begin{aligned}\kappa_1 \min\{1, |e^{\mu_\alpha t}\xi^-|^2\} - \kappa_2(|\xi^-|^{2+\delta} + |\xi^-|^2)e^{-\eta_0 t} \\ &\geq \kappa_1|e^{\mu_\alpha t}\xi^-|^2 - \kappa_2(|\xi^-|^{2+\delta} + |\xi^-|^2)e^{-\eta_0 t} \\ &\geq (\kappa_1 e^{2\mu_\alpha t} - 2\kappa_2 e^{-\eta_0 t})|\xi^-|^2 \geq \frac{\kappa_1}{2} e^{2\mu_\alpha t}|\xi^-|^2. \end{aligned} \tag{4.8}$$

Thus when $t \geq t_1$, $|\xi| \geq 2$ and θ satisfies (4.7), one can deduce from the assumption (1.4) and the estimates (4.5), (4.6) and (4.8) that

$$\begin{aligned}\int_{\mathbb{S}^2} \mathcal{B}\left(\frac{\xi\cdot\sigma}{|\xi|}\right)(1 - |\hat{f}(t,\xi^-)|)\, d\sigma \\ &\geq 2\pi \int_0^{2/(e^{\mu_\alpha t}|\xi|)} \mathcal{B}(\cos\theta)(\kappa_1 e^{2\mu_\alpha t} - 2\kappa_2 e^{-\eta_0 t})|\xi^-|^2 \sin\theta\, d\theta \\ &\geq \pi(\kappa_1 e^{2\mu_\alpha t})|\xi|^2 \int_0^{2/(e^{\mu_\alpha t}|\xi|)} \mathcal{B}(\cos\theta)\sin^2\frac{\theta}{2}\sin\theta\, d\theta \\ &\geq \frac{2\kappa_1 e^{2\mu_\alpha t}}{\pi^2}|\xi|^2 \int_0^{2/(e^{\mu_\alpha t}|\xi|)} \mathcal{B}(\cos\theta)\theta^3\, d\theta \\ &\geq \frac{b_0\kappa_1 e^{2\mu_\alpha t}}{\pi^2}|\xi|^2 \int_0^{2/(e^{\mu_\alpha t}|\xi|)} \theta^{1-2s}\, d\theta = \frac{2^{1-2s} b_0\kappa_1 e^{2s\mu_\alpha t}}{(1-s)\pi^2}|\xi|^{2s}.\end{aligned}$$

Here we have used the fact that $\sin\theta \geq 2\theta/\pi$ for $0 \leq \theta \leq \pi/2$. This completes the proof of the lemma. □

With Lemma 4.1, we now deduce a uniform estimate on $f(t,v)$. Let $\varphi(t,\xi)$ be the Fourier transform of $f(t,v)$ with respect to v. For any $N \in \mathbb{N}$, let $M(\xi) = \widetilde{M}(|\xi|^2)(1 - X(|\xi|^2/4))$ with $\widetilde{M}(t) = t^N$ and $X(t)$ defined as in Theorem 1.1. Multiplying (1.6) by $2M^2(\xi)\overline{\varphi}(t,\xi)$ with $\overline{\varphi}(t,\xi)$ being the complex conjugate of $\varphi(t,\xi)$ gives

$$\begin{aligned}&\frac{d}{dt}\left(\int_{\mathbb{R}^3} |M(\xi)\varphi(t,\xi)|^2\, d\xi\right) \\ &= 2\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \mathcal{B}\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\Re\big\{(\varphi(t,\xi^+)\varphi(t,\xi^-) - \varphi(t,\xi))M^2(\xi)\overline{\varphi}(t,\xi)\big\}\, d\sigma\, d\xi \\ &= -\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \mathcal{B}\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\big(|M(\xi)\varphi(t,\xi)|^2 + |M(\xi^+)\varphi(t,\xi^+)|^2 \\ &\qquad - 2\Re\big\{\varphi(t,\xi^-)(M(\xi^+)\varphi(t,\xi^+))\overline{M(\xi)\varphi(t,\xi)}\big\}\big)\, d\sigma\, d\xi \\ &\quad - \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \mathcal{B}\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\big(|M(\xi)\varphi(t,\xi)|^2 - |M(\xi^+)\varphi(t,\xi^+)|^2\big)\, d\sigma\, d\xi \\ &\quad - 2\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \mathcal{B}\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\Re\big\{\varphi(t,\xi^-)(M(\xi) - M(\xi^+))\varphi(t,\xi^+)\overline{M(\xi)\varphi(t,\xi)}\big\}\, d\sigma\, d\xi \\ &=: J_1 - J_2 - 2J_3.\end{aligned} \tag{4.9}$$

We estimate J_1, J_2 and J_3 consecutively as follows. Since $\operatorname{supp} M(\xi) \subset \{\xi \in \mathbb{R}^3 : |\xi| \geq 2\}$, it follows first from Lemma 4.1 that

$$J_1 \lesssim -e^{2s\mu_\alpha t} \int_{\mathbb{R}^3} |\xi|^{2s} |M(\xi)\varphi(t,\xi)|^2 \, d\xi \,, \tag{4.10}$$

because

$$\begin{aligned}
|M(\xi)\varphi(t,\xi)|^2 + |M(\xi^+)\varphi(t,\xi^+)|^2 - 2\Re\big\{\varphi(t,\xi^-)(M(\xi^+)\varphi(t,\xi^+))\overline{M(\xi)\varphi(t,\xi)}\big\} \\
\geq (1 - |\varphi(t,\xi^-)|)\big(|M(\xi)\varphi(t,\xi)|^2 + |M(\xi^+)\varphi(t,\xi^+)|^2\big) \\
\geq (1 - |\varphi(t,\xi^-)|)|M(\xi)\varphi(t,\xi)|^2.
\end{aligned}$$

For J_2, if we use the change of variable $\xi \to \xi^+$ for the term $M(\xi^+)\varphi(t,\xi^+)$, the cancelation lemma [1, Lemma 1] implies that

$$\begin{aligned}
|J_2| &= 2\pi \left| \int_{\mathbb{R}^3} |M(\xi)\varphi(t,\xi)|^2 \left(\int_0^{\pi/2} \mathcal{B}(\cos\theta) \sin\theta \, (1 - \cos^{-3}(\theta/2)) \, d\theta \right) d\xi \right| \\
&\lesssim \int_{\mathbb{R}^3} |M(\xi)\varphi(t,\xi)|^2 \, d\xi.
\end{aligned} \tag{4.11}$$

For J_3, note that

$$\begin{aligned}
J_3 &= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \mathcal{B}\left(\frac{\xi\cdot\sigma}{|\xi|}\right) \Re\big\{\varphi(t,\xi^-)(\widetilde{M}(\xi) - \widetilde{M}(\xi^+))\{1 - X(|\xi^+|^2/4)\} \\
&\qquad\qquad \times \varphi(t,\xi^+)\overline{M(\xi)\varphi(t,\xi)}\big\} \, d\sigma \, d\xi \\
&\quad + \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \mathcal{B}\left(\frac{\xi\cdot\sigma}{|\xi|}\right) \Re\big\{\varphi(t,\xi^-)\widetilde{M}(\xi)(X(|\xi^+|^2/4) - X(|\xi|^2/4)) \\
&\qquad\qquad \times \varphi(t,\xi^+)\overline{M(\xi)\varphi(t,\xi)}\big\} \, d\sigma \, d\xi \\
&=: J_3^1 + J_3^2.
\end{aligned}$$

We estimate J_3^1 and J_3^2 separately.

For J_3^1, since $|\xi^+|^2 = |\xi|^2\cos^2(\theta/2) \sim |\xi|^2$ for $\theta \in [0,\pi/2]$ and $|\xi|^2 - |\xi^+|^2 = |\xi|^2\sin^2(\theta/2)$, we have

$$|\widetilde{M}(\xi) - \widetilde{M}(\xi^+)| \lesssim \sin^2(\theta/2)\widetilde{M}(\xi^+),$$

and consequently

$$\begin{aligned}
|J_3^1| &\lesssim \int_{\mathbb{R}^3} \left(\int_0^{\pi/2} \mathcal{B}(\cos\theta) \sin\theta \sin^2(\theta/2) \, d\theta \right) |M(\xi)\varphi(t,\xi)| \cdot |M(\xi^+)\varphi(t,\xi^+)| \, d\xi \\
&\lesssim \int_{\mathbb{R}^3} |M(\xi)\varphi(t,\xi)|^2 \, d\xi.
\end{aligned} \tag{4.12}$$

Here we have used the fact that $|\varphi(t,\xi^-)| \leq 1$.

For J_3^2, since

$$X(|\xi|^2/4) - X(|\xi^+|^2/4) = X'\left(\frac{\eta|\xi|^2 + (1-\eta)|\xi^+|^2}{4}\right)\frac{|\xi|^2 - |\xi^+|^2}{4}$$
$$= \frac{|\xi|^2 \sin^2(\theta/2)}{4} X'\left(\frac{\eta|\xi|^2 + (1-\eta)|\xi^+|^2}{4}\right), \qquad \eta \in [0,1],$$

and

$$|\xi^+|^2 \le |\xi|^2 \le 2|\xi^+|^2, \qquad \mathrm{supp}\{X'(|\xi|^2/4)\} \subset \{\xi \in \mathbb{R}^3 : 4 \le |\xi|^2 \le 8\},$$

we obtain

$$\mathrm{supp}\{\widetilde{M}(\xi)(X(|\xi|^2/4) - X(|\xi^+|^2/4))\} \subset \{\xi \in \mathbb{R}^3 : 4 \le |\xi|^2 \le 16\}.$$

Hence, there exists a constant $C_N > 0$ depending on N such that

$$|J_3^2| \le 4^{2N} \int_{|\xi|\le 4} \left(\int_0^{\pi/2} \mathcal{B}(\cos\theta)\sin\theta \sin^2(\theta/2)\, d\theta\right) |\varphi(t,\xi)| \cdot |\varphi(t,\xi^+)|\, d\xi$$
$$\le C_N \tag{4.13}$$

because $|\varphi(t,\xi)| \le 1$. Now (4.12) together with (4.13) shows that there exists a $C_1 > 0$ such that

$$|J_3| \le C_1 \int_{\mathbb{R}^3} |\xi|^{2s} |M(\xi)\varphi(t,\xi)|^2\, d\xi + C_N\,. \tag{4.14}$$

Inserting (4.10), (4.11) and (4.14) into (4.9), we have, for another $C_N' > 0$,

$$\frac{d}{dt}\left(\int_{\mathbb{R}^3} |M(\xi)\varphi(t,\xi)|^2\, d\xi\right) + \int_{\mathbb{R}^3} |M(\xi)\varphi(t,\xi)|^2\, d\xi \le C_N',$$

which gives

$$\int_{\mathbb{R}^3} |M(\xi)\varphi(t,\xi)|^2\, d\xi \le e^{-(t-t_1)} \int_{\mathbb{R}^3} |M(\xi)\varphi(t_1,\xi)|^2\, d\xi + C_N', \qquad t \ge t_1. \tag{4.15}$$

Noting $|\varphi(\xi)| \le 1$ again, by means of (4.15) we see that for any $N \in \mathbb{N}$ there exists a $C(t_1, N) > 0$ such that

$$\sup_{t\in[t_1,\infty)} \|f(t)\|_{H^N} \le C(t_1,N) < \infty, \qquad \forall N \in \mathbb{N}.$$

This and (1.20) give

$$\sup_{t\in[t_1,\infty)} \|f(t,\cdot) - f_{\alpha,K}(t,\cdot)\|_{H^N} \le C(t_1,N) < \infty, \qquad \forall N \in \mathbb{N},\ t \ge t_1. \tag{4.16}$$

Moreover, for any $\epsilon \in (0,2)$ we have

$$
\begin{aligned}
&|\hat f(t,\xi) - \hat\Psi_{\alpha,K}(e^{\mu_\alpha t}\xi)|^2 \\
&\quad\lesssim |\hat f(t,\xi) - \hat\Psi_{\alpha,K}(e^{\mu_\alpha t}\xi) - \widetilde P(t,\xi)|^2 + |\widetilde P(t,\xi)|^2 \\
&\quad\lesssim [e^{-\eta_0 t}|\xi|^{2+\delta}]^{2-\epsilon}|\hat f(t,\xi) - \hat\Psi_{\alpha,K}(e^{\mu_\alpha t}\xi) - \widetilde P(t,\xi)|^\epsilon + |\widetilde P(t,\xi)|^2 \\
&\quad\lesssim [e^{-\eta_0 t}|\xi|^{2+\delta}]^{2-\epsilon}|\hat f(t,\xi) - \hat\Psi_{\alpha,K}(e^{\mu_\alpha t}\xi)|^\epsilon \\
&\qquad + |\xi|^4 X(\xi)^\epsilon\big(|\xi|^{\delta(2-\epsilon)}X(\xi)^\epsilon e^{-(2-\epsilon)\eta_0 t-\epsilon A t} + e^{-2At}X(\xi)^{(2-\epsilon)}\big) \\
&\quad\lesssim [e^{-\eta_0 t}|\xi|^{2+\delta}]^{2-\epsilon}|\hat f(t,\xi) - \hat\Psi_{\alpha,K}(e^{\mu_\alpha t}\xi)|^\epsilon \\
&\qquad + e^{-2\eta_0 t}\big(|\xi|^{(2+\delta)(2-\epsilon)}X(\xi)^\epsilon + |\xi|^4 X(\xi)^2\big).
\end{aligned}
\tag{4.17}
$$

(4.16) and (4.17) yield

$$
\begin{aligned}
&\|f(t,\cdot) - f_{\alpha,K}(t,\cdot)\|^2_{H^N} \\
&\quad= \int_{\mathbb{R}^3}(1+|\xi|^2)^N|\hat f(t,\xi) - \hat\Psi_{\alpha,K}(e^{\mu_\alpha t}\xi)|^2\,d\xi \\
&\quad\lesssim e^{-(2-\epsilon)\eta_0 t}\int_{\mathbb{R}^3}(1+|\xi|^2)^N|\xi|^{(2+\delta)(2-\epsilon)}|\hat f(t,\xi) - \hat\Psi_{\alpha,K}(e^{\mu_\alpha t}\xi)|^\epsilon\,d\xi \\
&\qquad + e^{-2\eta_0 t}\int_{\mathbb{R}^3}\big(X(\xi)^2|\xi|^4 + X(\xi)^\epsilon|\xi|^{(2+\delta)(2-\epsilon)}\big)\,d\xi \lesssim e^{-(2-\epsilon)\eta_0 t}.
\end{aligned}
\tag{4.18}
$$

Here we have used the fact that

$$
\begin{aligned}
&\int_{\mathbb{R}^3}(1+|\xi|^2)^N|\xi|^{(2+\delta)(2-\epsilon)}|\hat f(t,\xi) - \hat\Psi_{\alpha,K}(e^{\mu_\alpha t}\xi)|^\epsilon\,d\xi \\
&\qquad\le \left(\int_{\mathbb{R}^3}(1+|\xi|^2)^{\frac{2(N+2-\epsilon)}{\epsilon}}|\xi|^{\frac{2(2+\delta)(2-\epsilon)}{\epsilon}}|\hat f(t,\xi) - \hat\Psi_{\alpha,K}(e^{\mu_\alpha t}\xi)|^2\,d\xi\right)^{\epsilon/2} \\
&\qquad\quad\times\left(\int_{\mathbb{R}^3}(1+|\xi|^2)^{-2}\,d\xi\right)^{(2-\epsilon)/2} \\
&\qquad\le C(t_1,N), \qquad t\ge t_1.
\end{aligned}
$$

Note that (4.18) is exactly (1.19).

To complete the proof of Theorem 1.2 and of the remarks following it, we have only to deduce the $L^1(\mathbb{R}^3)$-convergence result (1.22). For this purpose, we deduce from [12] that for any $1<\beta<\alpha\le 2$,

$$
\int_{\mathbb{R}^3}|v|^\beta f(t,v)\,dv \lesssim e^{\lambda_\beta t}, \tag{4.19}
$$

thus for $1/p + 1/q = 1$ $(p,q>1)$ we get

$$\begin{aligned}
&\int_{\mathbb{R}^3} |f(t,v) - e^{-3\mu_\alpha t}\Psi_{\alpha,K}(e^{-\mu_\alpha t}v)|\,dv \\
&\quad\lesssim \int_{\mathbb{R}^3} [(1+|v|)^\ell |f(t,v) - e^{-3\mu_\alpha t}\Psi_{\alpha,K}(e^{-\mu_\alpha t}v)|]^{1/p} \\
&\qquad \times (1+|v|)^{-\ell/p}[|f(t,v) - e^{-3\mu_\alpha t}\Psi_{\alpha,K}(e^{-\mu_\alpha t}v)|]^{1/q}\,dv \\
&\quad\lesssim \left(\int_{\mathbb{R}^3} (1+|v|)^\ell |f(t,v) - e^{-3\mu_\alpha t}\Psi_{\alpha,K}(e^{-\mu_\alpha t}v)|\,dv\right)^{1/p} \\
&\qquad \times \left(\int_{\mathbb{R}^3} (1+|v|)^{-2q\ell/p}\,dv\right)^{-1/(2q)} \|f(t,v) - e^{-3\mu_\alpha t}\Psi_{\alpha,K}(e^{-\mu_\alpha t}v)\|^{1/q} \\
&\quad\lesssim e^{-\frac{2\eta_0-\epsilon}{2q}t}\left(\int_{\mathbb{R}^3} (1+|v|)^\ell |f(t,v) - e^{-3\mu_\alpha t}\Psi_{\alpha,K}(e^{-\mu_\alpha t}v)|\,dv\right)^{1/p}
\end{aligned} \tag{4.20}$$

provided that

$$2q\ell/p > 3. \tag{4.21}$$

Since

$$\left(\int_{\mathbb{R}^3} (1+|v|)^\ell e^{-3\mu_\alpha t}\Psi_{\alpha,K}(e^{-\mu_\alpha t}v)\,dv\right)^{1/p} \lesssim e^{\ell\mu_\alpha t/p}, \tag{4.22}$$

and by choosing $\beta = \ell p_1 \in (1,\alpha)$, $1/p_1 + 1/q_1 = 1$ $(p_1, q_1 > 1)$, (4.19) and Hölder's inequality imply that

$$\begin{aligned}
\left(\int_{\mathbb{R}^3} (1+|v|)^\ell f(t,v)\,dv\right)^{1/p} &= \left(\int_{\mathbb{R}^3} (1+|v|)^\ell |f(t,v)|^{1/p_1}|f(t,v)|^{1/q_1}\,dv\right)^{1/p} \\
&\lesssim \left(\int_{\mathbb{R}^3} (1+|v|)^{p_1\ell} f(t,v)\,dv\right)^{1/(pp_1)} \\
&\lesssim e^{\lambda_\beta t/(pp_1)} = e^{\ell\mu_\beta t/p},
\end{aligned} \tag{4.23}$$

we conclude from (4.22) and (4.23) that

$$\left(\int_{\mathbb{R}^3} (1+|v|)^\ell |f(t,v) - e^{-3\mu_\alpha t}\Psi_{\alpha,K}(e^{-\mu_\alpha t}v)|\,dv\right)^{1/p} \lesssim e^{\max\{\ell\mu_\beta t/p,\,\ell\mu_\alpha t/p\}}. \tag{4.24}$$

Putting (4.20) and (4.24) together implies

$$\int_{\mathbb{R}^3} |f(t,v) - e^{-3\mu_\alpha t}\Psi_{\alpha,K}(e^{-\mu_\alpha t}v)|\,dv \lesssim e^{\max\{\ell\mu_\beta t/p,\,\ell\mu_\alpha t/p\}-(2\eta_0-\epsilon)t/(2q)}. \tag{4.25}$$

Thus if the assumption (1.21) is true, so we can choose β sufficiently close to α such that

$$3\mu_\beta < 2\eta_0, \tag{4.26}$$

we can also choose ℓ and $p, q > 1$ such that

$$\begin{aligned}
&\ell\mu_\alpha/p < \eta_0/q, \quad \ell\mu_\beta/p < \eta_0/q, \quad 2q\ell/p > 3, \\
&\ell < \ell p_1 = \beta \in (1,\alpha), \quad 1/p + 1/q = 1, \quad p_1 > 1.
\end{aligned} \tag{4.27}$$

Consequently,

$$\max\{\ell\mu_\beta/p, \ell\mu_\alpha/p\} < \eta_0/q, \tag{4.28}$$

and (4.25) implies the exponential decay of $\int_{\mathbb{R}^3} |f(t,v) - e^{-3\mu_\alpha t}\Psi_{\alpha,K}(e^{-\mu_\alpha t}v)|\,dv$.

Moreover, since we can choose $\beta < \alpha$ sufficiently close to α, $p_1 > 1$ close to 1, and $\ell < \beta < \alpha$ close to α, we can deduce from the above analysis that there exists a sufficiently small positive constant $\epsilon > 0$ such that

$$\int_{\mathbb{R}^3} |f(t,v) - e^{-3\mu_\alpha t}\Psi_{\alpha,K}(e^{-\mu_\alpha t}v)|\,dv \lesssim e^{-(\eta_0/q-\alpha\mu_\alpha/p-\epsilon)t}. \tag{4.29}$$

Here $p, q > 1$ satisfy

$$1/p + 1/q = 1, \qquad q/p > 3/(2\alpha). \tag{4.30}$$

(4.29) is exactly the estimate (1.22), and thus the proof of Theorem 1.2, and of the remarks that follow it, is complete.

5. Proof of Corollary 1.1

First of all, note that Theorems 1.1 and 1.2 hold for $\alpha = 2$. The purpose of Corollary 1.1 is to have a better convergence rate in the case of finite energy.

In fact, compared with Theorems 1.1 and 1.2, the main difference is that now the initial data $f_0(v)$ is of finite energy and consequently the corresponding global solution $F_n(t,v)$ of the Cauchy problem (3.2)–(3.3) with $H_0(v) = f_{0R}(v)$ also has finite energy, i.e.

$$\int_{\mathbb{R}^3} |v|^2 F_n^R(t,v)\,dv = \int_{\mathbb{R}^3} |v|^2 f_{0R}(v)\,dv \lesssim 1 + \int_{\mathbb{R}^3} |v|^2 f_0(v)\,dv < \infty. \tag{5.1}$$

With the use of (5.1), it is straightforward to show that

$$|\hat{F}_n(t,\xi) - 1| \le O(1)|\xi|^2, \qquad |\mu(\xi) - 1| \le O(1)|\xi|^2. \tag{5.2}$$

Consequently, the term I_7 in (3.17) can be estimated by

$$|I_7| \le \begin{cases} O(1)|\xi|^4 e^{-A_n t}, & |\xi| \le 1,\ t \in \mathbb{R}^+, \\ O(1)e^{-A_n t}, & |\xi| \ge 1,\ t \in \mathbb{R}^+. \end{cases} \tag{5.3}$$

Here, $G_n(t,v) = \mu(v)$.

Having (5.3), the proof of Corollary 1.1 is the same as the ones of Theorems 1.1 and 1.2. Therefore, we omit the details.

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