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Solution of the parametric center problem for the Abel differential equation

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Abstract. The Abel differential equation $y' = p(x)y^2 + q(x)y^3$ with $p, q \in \mathbb{R}[x]$ is said to have a center on an interval $[a, b]$ if all its solutions with the initial value $y(a)$ small enough satisfy the condition $y(b) = y(a)$. The problem of description of conditions implying that the Abel equation has a center may be interpreted as a simplified version of the classical center-focus problem of Poincaré. The Abel equation is said to have a "parametric center" if for each $\varepsilon \in \mathbb{R}$ the equation $y' = p(x)y^2 + \varepsilon q(x)y^3$ has a center. In this paper we show that the Abel equation has a parametric center if and only if the antiderivatives $P = \int p(x)\,dx$, $Q = \int q(x)\,dx$ satisfy the equalities $P = \widetilde{P} \circ W$, $Q = \widetilde{Q} \circ W$ for some polynomials $\widetilde{P}$, $\widetilde{Q}$, and W such that $W(a) = W(b)$. We also show that the last condition is necessary and sufficient for the "generalized moments" $\int_a^b P^i\,dQ$ and $\int_a^b Q^i\,dP$ to vanish for all $i \geq 0$.

Keywords. Periodic orbits, centers, Abel equation, moment problem, composition conjecture

1. Introduction

Let

$$y' = p(x)y^2 + q(x)y^3 \tag{1}$$

be the Abel differential equation, where x is real and $p(x)$ and $q(x)$ are continuous. Equation (1) is said to *have a center on an interval* $[a, b]$ if all its solutions with the initial value $y(a)$ small enough satisfy the condition $y(b) = y(a)$.

The problem of description of conditions implying a center for (1) is closely related to the classical Poincaré center-focus problem about conditions implying that all trajectories of the system

$$\begin{cases} \dot{x} = -y + F(x, y), \\ \dot{y} = x + G(x, y), \end{cases} \tag{2}$$

where $F(x, y)$, $G(x, y)$ are polynomials without constant or linear terms, are closed in a neighborhood of the origin. Namely, it was shown in [11] that if $F(x, y)$, $G(x, y)$ are homogeneous polynomials of the same degree, then one can construct trigonometric polynomials $f(\cos\varphi, \sin\varphi)$, $g(\cos\varphi, \sin\varphi)$ such that (2) has a center if and only if all solutions

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of the equation

$$\frac{dr}{d\varphi} = f(\cos\varphi, \sin\varphi)r^2 + g(\cos\varphi, \sin\varphi)r^3$$

with $r(0)$ small enough satisfy the condition $r(2\pi) = r(0)$.

Set

$$P(x) = \int_0^x p(s)\,ds, \qquad Q(x) = \int_0^x q(s)\,ds. \tag{3}$$

The following *composition condition* introduced in [3] is sufficient for equation (1) to have a center: there exist C^1-functions $\widetilde{P}$, $\widetilde{Q}$, W such that

$$P(x) = \widetilde{P}(W(x)), \qquad Q(x) = \widetilde{Q}(W(x)), \qquad W(a) = W(b). \tag{4}$$

Indeed, if (4) holds, then any solution of (1) has the form $y(x) = \widetilde{y}(W(x))$, where $\widetilde{y}$ is a solution of the equation

$$y' = \widetilde{P}'(x)y^3 + \widetilde{Q}'(x)y^2,$$

implying that $y(a) = y(b)$, since $W(a) = W(b)$.

It is known that in general the composition condition is not necessary for (1) to have a center [2]. However, it is believed that in the case where $p(x)$ and $q(x)$ are polynomials, equation (1) has a center if and only if the composition condition (4) holds for some polynomials $\widetilde{P}$, $\widetilde{Q}$, $W \in \mathbb{R}[x]$ (see [7], [9] for some partial results in this direction).

In this paper we study the following "parametric center problem" for equation (1) with polynomial coefficients: *under what conditions the equation*

$$y' = p(x)y^2 + \varepsilon q(x)y^3, \qquad p, q \in \mathbb{R}[x], \tag{5}$$

has a center for any $\varepsilon \in \mathbb{R}$? Posed for the first time in [4–6], this problem turned out to be very stimulating and resulted in a whole area of new ideas and methods related to the so called "polynomial moment problem" (see the discussion below). Along with the parametric center problem some other weakened versions of the center problem for the Abel differential equation have been introduced and studied (see e.g. [10], [13], [14]). However, the parametric center problem has remained unsolved (see the recent paper [8] for the state of the art), and the goal of this paper is to fill this gap.

Our main result is the following theorem.

Theorem 1.1. *The Abel differential equation* (5) *has a center on an interval* $[a, b]$ *for any* $\varepsilon \in \mathbb{R}$ *if and only if the antiderivatives* $P = \int p(x)\,dx$ *and* $Q = \int q(x)\,dx$ *satisfy the composition condition* (4) *for some polynomials* $\widetilde{P}$, $\widetilde{Q}$, W.

The proof of Theorem 1.1 is based on a link between the parametric center problem and the vanishing of certain "polynomial moments". Namely, it was shown in [6] that the parametric center implies the equalities

$$\int_a^b P^i\,dQ = 0, \quad i \geq 0, \qquad \int_a^b Q^i\,dP = 0, \quad i \geq 0, \tag{6}$$

and in fact we prove the following "moment" counterpart of Theorem 1.1.

Theorem 1.2. *Polynomials $P, Q \in \mathbb{R}[x]$ satisfy* (6) *if and only if they satisfy the composition condition* (4) *for some $\widetilde{P}, \widetilde{Q}, W \in \mathbb{R}[x]$.*

The problem of description of polynomial solutions of the system

$$\int_a^b P^i \, dQ = 0, \qquad i \geq 0, \tag{7}$$

called the "polynomial moment problem", has been studied in many recent papers (see e.g. [4–6, 12, 15–21, 23]). Again, the composition condition (4) is sufficient for equalities (7) to be satisfied, although in general it is not necessary [15]. A complete solution of the polynomial moment problem was obtained in [20], [19]. Namely, it was shown in [20] that if polynomials P, Q satisfy (7), then there exist polynomials Q_j such that $Q = \sum_j Q_j$ and

$$P(x) = P_j(W_j(x)), \qquad Q_j(x) = V_j(W_j(x)), \qquad W_j(a) = W_j(b) \tag{8}$$

for some polynomials $P_j(z), V_j(z), W_j(z)$. Moreover, in [19] polynomial solutions of (7) were described in explicit form (see Section 2 below).

In this paper we apply the results of [19] to each of the two systems in (6) separately and show that the restrictions obtained imply that any solution P, Q of "mixed polynomial moment problem" (6) satisfies the composition condition (4). The main difficulties of the proof stem from the fact that a separate solution of systems in (6) leads to functional equations of the type

$$\sum_{j=1}^{r} V_j(W_j(x)) = A(B(x)), \tag{9}$$

where V_j, W_j, A, B are polynomials, and r equals 2 or 3. Such equations can be considered as generalizations of the functional equation

$$A(B(x)) = C(D(x)), \tag{10}$$

studied by Ritt [22]. However, the well established methods for studying (10) related to monodromy cannot be applied to (9) for $r > 1$, and other methods are required.

Although the center problem for the Abel equation with polynomial coefficients can be considered in the complex setting, in this paper we work in the classical real framework. Thus, an adaptation to the real case of the results of [19] obtained over $\mathbb{C}$ is needed. This is done in Section 2. We show that possible "types" of solutions of the polynomial moment problem over $\mathbb{R}$ remain the same, although one of these types becomes "smaller" (Theorem 2.10). Moreover, in Section 2 we establish some important restrictions of the arithmetical nature on points a, b for which there exist solutions of (7) which do not satisfy the composition condition (Corollary 2.5).

In Section 3 we apply the results of Section 2 to each of the two systems in (6) and prove Theorem 1.2. Our approach consists in a painstaking analysis of systems of equations for the coefficients of the polynomials appearing in corresponding equalities (9). Eventually, Theorem 1.2 is deduced from the restrictions on P and Q obtained from these systems combined with the restrictions on possible values of a and b.

2. Polynomial moment problem over $\mathbb{C}$ and over $\mathbb{R}$

2.1. Solution of the polynomial moment problem over $\mathbb{C}$

In this subsection we briefly recall the description of $P, Q \in \mathbb{C}[z]$ satisfying (7) for $a, b \in \mathbb{C}$, obtained in [19]. For more details we refer the reader to [19].

Recall that the Chebyshev polynomials of the first kind T_n can be defined by the formula $T_n(\cos\varphi) = \cos(n\varphi)$. It follows directly from this definition that

$$T_n(1) = 1, \quad T_n(-1) = (-1)^n, \quad n \geq 0, \qquad (11)$$
$$T_n \circ T_m = T_m \circ T_n = T_{mn}, \quad n, m \geq 1,$$

where $(A \circ B)(z) = A(B(z))$.

An explicit expression for T_n is given by the formula

$$T_n(x) = \frac{n}{2}\sum_{k=0}^{[n/2]}(-1)^k\frac{(n-k-1)!}{k!(n-2k)!}(2x)^{n-2k}, \qquad (12)$$

implying in particular that

$$T_n(-x) = (-1)^n T_n(x) \qquad (13)$$

(see e.g. [1, Chapter 22]).

Following [19], we will call a solution P, Q of (7) *reducible* if the composition condition (4) holds for some $\widetilde{P}, \widetilde{Q}, W \in \mathbb{C}[z]$.

Theorem 2.1 ([19]). *Let P, Q be non-constant complex polynomials and a, b distinct complex numbers such that equalities (7) hold. Then either P, Q is a reducible solution of (7), or there exist complex polynomials P_j, Q_j, V_j, W_j, $1 \leq j \leq r$, such that*

$$Q = \sum_{j=1}^{r} Q_j, \quad P = P_j \circ W_j, \quad Q_j = V_j \circ W_j, \quad W_j(a) = W_j(b).$$

Moreover, one of the following conditions holds:

(i) $r = 2$ *and*

$$P = U \circ z^{sn}R^n(z^n) \circ V, \quad W_1 = z^n \circ V, \quad W_2 = z^s R(z^n) \circ V,$$

where R, U, V are complex polynomials, $n > 1$, $s > 0$, and $\mathrm{GCD}(s, n) = 1$;

(ii) $r = 2$ *and*

$$P = U \circ T_{m_1 m_2} \circ V, \quad W_1 = T_{m_1} \circ V, \quad W_2 = T_{m_2} \circ V,$$

where U, V are complex polynomials, $m_1 > 1$, $m_2 > 1$, and $\mathrm{GCD}(m_1, m_2) = 1$;

(iii) $r = 3$ *and*

$$P = U \circ z^2 R^2(z^2) \circ T_{m_1 m_2} \circ V,$$
$$W_1 = T_{2m_1} \circ V, \quad W_2 = T_{2m_2} \circ V, \quad W_3 = (zR(z^2) \circ T_{m_1 m_2}) \circ V,$$

where R, U, V are complex polynomials, $m_1 > 1$ and $m_2 > 1$ are odd, and $\mathrm{GCD}(m_1, m_2) = 1$.

It is assumed that in the above formulas $V(a) \neq V(b)$, since otherwise P, Q is reducible. We will call solutions appearing in (i)–(iii) of Theorem 2.1 solutions of the *first*, *second*, and *third type*, respectively.

Notice that these sets of solutions are not disjoint. For example, if one of the parameters n, m of a solution of the second type equals 2, then this solution is also of the first type. Indeed, if say $n = 2$, then $W_1 = T_2 \circ V = \mu \circ z^2 \circ V$, where $\mu = 2z - 1$. On the other hand, since m is odd in view of $\mathrm{GCD}(n, m) = 1$, the polynomial $W_2 = T_m \circ V$ has the form $W_2 = zR(z^2) \circ V$ by (12). Therefore,

$$P = (U \circ \mu) \circ z^2 R^2(z^2) \circ V, \qquad Q = ((V_1 \circ \mu) \circ z^2 + V_2 \circ zR(z^2)) \circ V,$$

and for the polynomials $\widetilde{W}_1 = z^2 \circ V$ and $\widetilde{W}_2 = W_2 = zR(z^2) \circ V$ we have

$$\widetilde{W}_1(a) = \widetilde{W}_1(b), \qquad \widetilde{W}_2(a) = \widetilde{W}_2(b). \tag{14}$$

Similarly, if the parameters a, b of a solution of the third type satisfy $V(a) = -V(b)$, then this solution is also of the first type. Indeed,

$$V_1 \circ T_{2m_1} + V_2 \circ T_{2m_2} = \widetilde{V}_1 \circ z^2$$

for some $\widetilde{V}_1 \in \mathbb{C}[z]$, while

$$zR(z^2) \circ T_{m_1 m_2} = z\widetilde{R}(z^2)$$

for some $\widetilde{R} \in \mathbb{C}[z]$, since m_1, m_2 are odd. Therefore,

$$P = U \circ z^2 \widetilde{R}^2(z^2) \circ V, \qquad Q = (\widetilde{V}_1 \circ z^2 + V_3 \circ z\widetilde{R}(z^2)) \circ V,$$

and $\widetilde{W}_1 = z^2 \circ V$ satisfies $\widetilde{W}_1(a) = \widetilde{W}_1(b)$, since $V(a) = -V(b)$.

Finally, it is easy to check that a solution of the third type is also of the second type if $(T_{m_i} \circ V)(a) = (T_{m_i} \circ V)(b)$ for $i = 1$ or 2 (see [19, pp. 725–726] for details and a further discussion of interrelations between different types of solutions).

2.2. *Lemmas related to a, b*

It is clear that if $T_l(a) = T_l(b)$ for some distinct $a, b \in \mathbb{C}$, then $T_{m_1}(a) = T_{m_1}(b)$ and $T_{m_2}(a) = T_{m_2}(b)$ for any m_1 and m_2 divisible by l. The following lemma shows that for generic points a and b this is the only reason for two Chebyshev polynomials to take equal values at a and b.

Lemma 2.2. *Let T_{m_1}, T_{m_2}, T_{m_3} be the Chebyshev polynomials and a, b be distinct complex numbers.*

(a) *Assume that*

$$T_{m_1}(a) = T_{m_1}(b), \qquad T_{m_2}(a) = T_{m_2}(b). \tag{15}$$

Then either $T_l(a) = T_l(b)$ for $l = \mathrm{GCD}(m_1, m_2)$, or

$$T'_{m_1 m_2}(a) = T'_{m_1 m_2}(b) = 0. \tag{16}$$

(b) *Assume that*

$$T_{m_1}(a)=T_{m_1}(b),\qquad T_{m_2}(a)=T_{m_2}(b),\qquad T_{m_3}(a)=T_{m_3}(b). \tag{17}$$

Then there exist distinct indices i_1, i_2, $1\leq i_1, i_2\leq 3$, *such that* $T_l(a)=T_l(b)$ *for* $l=\mathrm{GCD}(m_{i_1}, m_{i_2})$.

Proof. Choose $\alpha,\beta\in\mathbb{C}$ such that $\cos\alpha=a$, $\cos\beta=b$. Then equalities (15) imply

$$m_1\alpha=\varepsilon_1 m_1\beta+2\pi k_1,\qquad m_2\alpha=\varepsilon_2 m_2\beta+2\pi k_2, \tag{18}$$

where $\varepsilon_1=\pm 1$, $\varepsilon_2=\pm 1$, and $k_1, k_2\in\mathbb{Z}$. Assume first that $\varepsilon_1=\varepsilon_2$. Let u, v be integers satisfying

$$um_1+vm_2=l. \tag{19}$$

Multiplying the first equality in (18) by u and adding the second equality multiplied by v, we see that

$$l\alpha=\varepsilon_1 l\beta+2\pi k_1 u+2\pi k_2 v,$$

implying that $T_l(a)=T_l(b)$.

Assume now that $\varepsilon_2=-\varepsilon_1$. Then similarly we conclude that

$$l\alpha=\varepsilon_1\beta(um_1-vm_2)+2\pi k_1 u+2\pi k_2 v. \tag{20}$$

Furthermore, eliminating α from (18) we obtain

$$\varepsilon_1 m_1 m_2\beta=\pi k_2 m_1-\pi k_1 m_2. \tag{21}$$

Since

$$T_n'(\cos\varphi)=n(\sin n\varphi/\sin\varphi), \tag{22}$$

equality (21) implies that $T'_{m_1 m_2}(b)=0$ unless

$$\beta=\pi k_3,\qquad k_3\in\mathbb{Z}. \tag{23}$$

If (23) holds, then $b=1$ if k_3 is even, and $b=-1$ if k_3 is odd, implying that

$$T_l(b)=(-1)^{k_3 l},$$

in view of (11). On the other hand, (23) implies by (20) that

$$T_l(a)=(-1)^{k_3(um_1-vm_2)}.$$

Since the sum and difference of any two numbers have the same parity, this implies

$$T_l(a)=(-1)^{k_3(um_1+vm_2)}=(-1)^{k_3 l}=T_l(b).$$

Similarly, one can see that $T_l(a)=T_l(b)$ unless $T'_{m_1 m_2}(a)=0$.

In order to prove (b) observe that equalities (17) imply

$$m_1\alpha=\varepsilon_1 m_1\beta+2\pi k_1,\qquad m_2\alpha=\varepsilon_2 m_2\beta+2\pi k_2,\qquad m_3\alpha=\varepsilon_3 m_3\beta+2\pi k_3, \tag{24}$$

where $\varepsilon_1=\pm 1$, $\varepsilon_2=\pm 1$, $\varepsilon_3=\pm 1$, and $k_1, k_2, k_3\in\mathbb{Z}$. Clearly, among the numbers ε_1, ε_2, ε_3 at least two are equal and we conclude as above that $T_l(a)=T_l(b)$ for $l=\mathrm{GCD}(m_{i_1}, m_{i_2})$, where $\varepsilon_{i_1}=\varepsilon_{i_2}$. □

Corollary 2.3. *Let T_{m_1}, T_{m_2} be the Chebyshev polynomials and a, b be distinct complex numbers. Assume that*

$$T_{m_1}(a) = T_{m_1}(b), \qquad T_{m_2}(a) = T_{m_2}(b),$$

and $\mathrm{GCD}(m_1, m_2) = 1$. *Then*

$$T'_{m_1 m_2}(a) = T'_{m_1 m_2}(b) = 0.$$

Proof. Follows from Lemma 2.2(a) since the equality $S(a) = S(b)$ for some polynomial S and $a \neq b$ obviously implies that $\deg S > 1$. □

Recall that a number $\gamma \in \mathbb{C}$ is called *algebraic* if it is a root of a polynomial of positive degree with rational coefficients. The set of all algebraic numbers is a subfield of $\mathbb{C}$. The monic polynomial $p(x) \in \mathbb{Q}[x]$ of minimal degree such that $p(\gamma) = 0$ is called the *minimal polynomial* of γ. A minimal polynomial is irreducible over $\mathbb{Q}$. An algebraic number γ is called an *algebraic integer* if its minimal polynomial has integer coefficients. In fact, this condition may be replaced by the condition that γ is a root of *some* monic polynomial with integer coefficients. The set of all algebraic integers is closed under addition and multiplication.

Lemma 2.4. *Assume that $a \in \mathbb{C}$ is a root of T'_n. Then $a \in \mathbb{R}$, and $2a$ is an algebraic integer.*

Proof. Since equality (22) shows that T'_n has $n - 1$ distinct real roots, all roots of T'_n are real. The other statements follow from the formulas

$$T'_n = nU_{n-1} \quad \text{and} \quad U_n = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} (2x)^{n-2k},$$

where U_n denotes the Chebyshev polynomial of the second kind (see [1]). □

Corollary 2.5. *In the notation of Theorem 2.1 assume that P, Q is a solution of (7) of the second type, or a solution of the third type which cannot be represented as a solution of the first type. Then $2V(a)$ and $2V(b)$ are algebraic integers.*

Proof. Without loss of generality we may assume that $V = x$. If Q is of the second type, then the statement follows from Corollary 2.3 and Lemma 2.4.

If Q is of the third type, then applying Lemmas 2.2(a) and 2.4 to the equalities

$$T_{2m_1}(a) = T_{2m_1}(b), \qquad T_{2m_2}(a) = T_{2m_2}(b),$$

we conclude that $2a$ and $2b$ are algebraic integers unless $T_2(a) = T_2(b)$. However, the last equality yields $a = -b$, implying, as observed above, that Q can be represented as a solution of the first type. □

2.3. Decompositions of polynomials with real coefficients

In this subsection we collect necessary results concerning decomposition of polynomials with real coefficients into compositions of polynomials of lesser degree.

The following lemma is well known (see e.g. [19, Corollary 2.2]).

Lemma 2.6. *Assume that*

$$P = A \circ B = \widetilde{A} \circ \widetilde{B},$$

where $P, A, B, \widetilde{A}, \widetilde{B} \in \mathbb{C}[z]$ *and* $\deg A = \deg \widetilde{A}$. *Then there exists a polynomial* $\mu \in \mathbb{C}[z]$ *of degree one such that*

$$\widetilde{A} = A \circ \mu^{-1}, \qquad \widetilde{B} = \mu \circ B. \qquad \square$$

Corollary 2.7. *Let* $P = U \circ V$, *where* $P \in \mathbb{R}[z]$, *while* $U, V \in \mathbb{C}[z]$. *Assume that the leading coefficient of* V *and its constant term are real numbers. Then* $U, V \in \mathbb{R}[z]$.

Proof. Since $P \in \mathbb{R}[z]$, we have

$$P = U \circ V = \overline{U} \circ \overline{V}, \tag{25}$$

where $\overline{U}, \overline{V}$ are polynomials obtained from U, V by complex conjugation of all coefficients. By Lemma 2.6, (25) implies that

$$\overline{U} = U \circ \mu^{-1}, \qquad \overline{V} = \mu \circ V, \tag{26}$$

where $\mu = \alpha z + \beta$ for some $\alpha, \beta \in \mathbb{C}$. Since the leading coefficient of V is real, the second equality in (26) implies $\alpha = 1$. Now the equality $\overline{V} = V + \beta$ yields $\beta = 0$, since the constant term of V is real. Thus, $\overline{U} = U$, $\overline{V} = V$ and hence $U, V \in \mathbb{R}[z]$. $\square$

Corollary 2.8. *Assume that* $P = U \circ V$, *where* $P \in \mathbb{R}[z]$, *while* $U, V \in \mathbb{C}[z]$. *Then there exists a polynomial* $\mu \in \mathbb{C}[z]$ *of degree one such that the polynomials*

$$U_1 = U \circ \mu^{-1}, \qquad V_1 = \mu \circ V$$

are in $\mathbb{R}[z]$. *In particular, if* P *is decomposable over* $\mathbb{C}$, *it is decomposable over* $\mathbb{R}$.

Proof. Let μ be any polynomial of degree one such that the leading coefficient and the constant term of the polynomial $V_1 = \mu \circ V$ are real. Then U_1 and V_1 are in $\mathbb{R}[z]$ by Corollary 2.7. $\square$

Lemma 2.9. *Let* μ_1, μ_2 *be complex polynomials of degree one.*

(a) *Assume that the polynomial* $\mu_1 \circ z^n \circ \mu_2$, $n \geq 2$, *has real coefficients. Then there exist* $\widetilde{\mu} \in \mathbb{R}[z]$ *and* $c \in \mathbb{C}$ *such that* $\mu_2 = c\widetilde{\mu}$.
(b) *Assume that the polynomial* $\mu_1 \circ T_n \circ \mu_2$, $n \geq 2$, *has real coefficients. Then either* $\mu_2 \in \mathbb{R}[z]$, *or there exists* $\widetilde{\mu} \in \mathbb{R}[z]$ *such that* $\mu_2 = i\widetilde{\mu}$.

Proof. Let $\mu_1 = \alpha_1 z + \beta_1$ and $\mu_2 = \alpha_2 z + \beta_2$, where $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{C}$. Then the coefficients of z^n and z^{n-1} of the polynomial $\mu_1 \circ z^n \circ \mu_2$ are $c_n = \alpha_1 \alpha_2^n$ and $c_{n-1} = \alpha_1 \alpha_2^{n-1} \beta_2 n$ respectively. Since by assumption these numbers are real, we conclude that the number

$$c_{n-1}/c_n = n\beta_2/\alpha_2$$

is also real. Therefore, $\beta_2/\alpha_2 \in \mathbb{R}$, and hence $\mu_2 = \alpha_2 \widetilde{\mu}$, where $\widetilde{\mu} = z + \beta_2/\alpha_2 \in \mathbb{R}[z]$.

Similarly, since

$$T_n(x) = 2^{n-1}x^n - n2^{n-3}x^{n-2} + \cdots$$

by (12), the coefficients of z^n, z^{n-1}, z^{n-2} of the polynomial $\mu_1 \circ T_n \circ \mu_2$ are

$$c_n = \alpha_1 2^{n-1} \alpha_2^n, \qquad c_{n-1} = \alpha_1 2^{n-1} n \alpha_2^{n-1} \beta_2,$$
$$c_{n-2} = \alpha_1 2^{n-2} n(n-1) \alpha_2^{n-2} \beta_2^2 - \alpha_1 2^{n-3} n \alpha_2^{n-2},$$

respectively. As above, $c_n, c_{n-1} \in \mathbb{R}$ implies that $\beta_2/\alpha_2 \in \mathbb{R}$. Since

$$c_{n-2} = \frac{n(n-1)c_n}{2}\left(\frac{\beta_2}{\alpha_2}\right)^2 - \frac{nc_n}{4}\left(\frac{1}{\alpha_2}\right)^2,$$

it now follows from $c_{n-2} \in \mathbb{R}$ that $\alpha_2^2 \in \mathbb{R}$. Since $\mu_2 = \alpha_2 \widetilde{\mu}$, where $\widetilde{\mu} = z + \beta_2/\alpha_2$ is in $\mathbb{R}[z]$, this proves the statement. □

2.4. Solution of the polynomial moment problem over $\mathbb{R}$

In this subsection we deduce from Theorem 2.1 a description of polynomials P, Q with *real* coefficients satisfying (7) for $a, b \in \mathbb{R}$.

The theorem below is a "real" analogue of Theorem 2.1. Keeping the above notation, we will call a solution $P, Q \in \mathbb{R}[x]$ of (7) *reducible* if (4) holds for some $\widetilde{P}, \widetilde{Q}, W \in \mathbb{R}[x]$. We will also call solutions in (i)–(iii) described below solutions of the *first*, *second*, and *third type*. Notice that the set of "real" solutions of the first type is "smaller" than the set of "complex" solutions.

Theorem 2.10. *Let P, Q be non-constant real polynomials and a, b distinct real numbers such that equalities (7) hold. Then either P, Q is a reducible solution of (7), or there exist real polynomials P_j, Q_j, V_j, W_j, $1 \le j \le r$, such that*

$$Q = \sum_{j=1}^{r} Q_j, \qquad P = P_j \circ W_j, \qquad Q_j = V_j \circ W_j, \qquad W_j(a) = W_j(b).$$

Moreover, one of the following conditions holds:

(i) $r = 2$ *and*

$$P = U \circ x^2 R^2(x^2) \circ V, \qquad W_1 = x^2 \circ V, \qquad W_2 = xR(x^2) \circ V,$$

where R, U, V are real polynomials;

(ii) $r = 2$ *and*

$$P = U \circ T_{m_1 m_2} \circ V, \qquad W_1 = T_{m_1} \circ V, \qquad W_2 = T_{m_2} \circ V,$$

where U, V *are real polynomials,* $m_1, m_2 > 1$, *and* $\mathrm{GCD}(m_1, m_2) = 1$;

(iii) $r = 3$ *and*

$$P = U \circ x^2 R^2(x^2) \circ T_{m_1 m_2} \circ V,$$
$$W_1 = T_{2m_1} \circ V, \qquad W_2 = T_{2m_2} \circ V, \qquad W_3 = (x R(x^2) \circ T_{m_1 m_2}) \circ V,$$

where R, U, V *are real polynomials,* $m_1, m_2 > 1$ *are odd, and* $\mathrm{GCD}(m_1, m_2) = 1$.

Proof. Our strategy is to apply Theorem 2.1 and to use the condition that $P, Q \in \mathbb{R}[x]$ and $a, b \in \mathbb{R}$. Assume first that (4) holds for some $\widetilde{P}, \widetilde{Q}, W \in \mathbb{C}[x]$. Applying Corollary 2.8 to the equality $P = \widetilde{P} \circ W$ we conclude that without loss of generality we may assume that $\widetilde{P}$ and W are in $\mathbb{R}[x]$. Now the equality $Q = \widetilde{Q} \circ W$ implies by Corollary 2.7 that $\widetilde{Q}$ is also in $\mathbb{R}[x]$.

Assume that P, Q is a solution of the first type. By Corollary 2.8 it follows from $P = P_1 \circ W_1$ that there exists a complex polynomial μ_1 of degree one such that the polynomial $\mu_1 \circ W_1$ has real coefficients. Further, applying Corollary 2.8 to the equality $\mu_1 \circ W_1 = \mu_1 \circ x^n \circ V$, we conclude that there exists a complex polynomial μ_2 of degree one such that the polynomials $\mu_1 \circ x^n \circ \mu_2$ and $\mu_2^{-1} \circ V$ have real coefficients. By Lemma 2.9(a) this implies that there exist $\widetilde{\mu} \in \mathbb{R}[x]$ and $c \in \mathbb{C}$ such that $\mu_2 = c\widetilde{\mu}$. Since $\mu_2^{-1} = \widetilde{\mu}^{-1} \circ x/c$, it now follows from $\mu_2^{-1} \circ V \in \mathbb{R}[x]$ that $V/c \in \mathbb{R}[x]$. Therefore, changing V to V/c, and modifying P_1, V_1, U, and R in an obvious way, without loss of generality we may assume that $V \in \mathbb{R}[x]$.

Clearly, $V \in \mathbb{R}[x]$ implies that $W_1 = x^n \circ V \in \mathbb{R}[x]$. It now follows from $P = P_1 \circ W_1$ by Corollary 2.7 that $P_1 \in \mathbb{R}[x]$. Furthermore, it follows from $W_1(a) = W_1(b)$ and $a, b \in \mathbb{R}$ that $n = 2k$ and $V(a) = -V(b)$. Since $\mathrm{GCD}(s, n) = 1$, the evenness of n implies that $x^s R(x^n) = x\widetilde{R}(x^2)$ for some $\widetilde{R} \in \mathbb{C}[z]$. Moreover, for such $\widetilde{R}$ obviously $x^{sn} R^n(x^n) = x^{n/2} \circ x^2 \widetilde{R}^2(x^2)$. Thus, changing P_1 to $P_1 \circ x^{n/2}$, V_1 to $V_1 \circ x^{n/2}$, U to $U \circ x^{n/2}$, and $x^s R(x^n)$ to $x\widetilde{R}(x^2)$, we may assume that $W_1 = x^2 \circ V$ and $W_2 = x R(x^2) \circ V$. Applying Corollary 2.7 to $P = (P_2 \circ x R(x^2)) \circ V$ we see that $P_2 \circ x R(x^2) \in \mathbb{R}[x]$. Therefore, taking into account that the constant term of $x R(x^2)$ is zero, Corollary 2.7 implies that for $c \in \mathbb{C}$ such that the leading coefficient of $cx R(x^2)$ is real the polynomials $P_2 \circ x/c$ and $cx R(x^2)$ are in $\mathbb{R}[x]$. Thus, modifying P_2 and R we can assume that they are in $\mathbb{R}[x]$. Now Corollary 2.7 applied to $P = U \circ (x^2 R^2(x^2) \circ V)$ implies $U \in \mathbb{R}[x]$.

Finally, the equality

$$Q = V_1 \circ W_1 + V_2 \circ W_2 = \overline{V}_1 \circ W_1 + \overline{V}_2 \circ W_2$$

implies that

$$Q = \frac{V_1 + \overline{V}_1}{2} \circ W_1 + \frac{V_2 + \overline{V}_2}{2} \circ W_2.$$

Therefore, changing if necessary V_1 to $(V_1 + \overline{V}_1)/2$ and V_2 to $(V_2 + \overline{V}_2)/2$, we may assume that $V_1, V_2 \in \mathbb{R}[x]$.

Assume now that P, Q is a solution of the third type. We may assume $V(a) \neq -V(b)$, for otherwise, as observed after Theorem 2.1, this solution also belongs to the first type considered earlier. As above, there exist complex polynomials μ_1 and μ_2 of degree one such that the polynomials $\mu_1 \circ T_{2m_1} \circ \mu_2$ and $\mu_2^{-1} \circ V$ have real coefficients. By Lemma 2.9(b), this implies that either $\mu_2 \in \mathbb{R}[x]$, or there exists $\widetilde{\mu} \in \mathbb{R}[x]$ such that $\mu_2 = i\widetilde{\mu}$. Since $\mu_2^{-1} \circ V \in \mathbb{R}[x]$, in the first case we have $V \in \mathbb{R}[x]$, while in the second one,

$$V = i\widetilde{V}, \qquad \widetilde{V} \in \mathbb{R}[x]. \tag{27}$$

Let us show that (27) is impossible. Indeed, applying Lemma 2.2(a) to the equalities $W_1(a) = W_1(b)$, $W_2(a) = W_2(b)$ and arguing as in Corollary 2.5, we conclude that $V(a)$ and $V(b)$ are roots of the polynomial $T'_{4m_1m_2}$, for otherwise $V(a) = -V(b)$. Since T'_n has only real zeroes, we conclude that $V(a), V(b) \in \mathbb{R}$, and hence (27) is impossible as $a, b \in \mathbb{R}$. Thus, $V \in \mathbb{R}[x]$.

Applying now Corollary 2.7 to the equality $P = (P_3 \circ xR(x^2)) \circ (T_{m_1m_2} \circ V)$ we deduce that $P_3 \circ xR(x^2) \in \mathbb{R}[x]$. Furthermore, arguing as above, we conclude that we may assume that $P_3, R \in \mathbb{R}[x]$ as well as $P_1, P_2, U \in \mathbb{R}[x]$ and $V_1, V_2, V_3 \in \mathbb{R}[x]$.

The proof in the case where P, Q is a solution of the second type is similar with obvious simplifications. □

3. Proof of Theorem 1.2

3.1. *Plan of the proof*

In the rest of the paper we will always assume that all the polynomials considered have real coefficients. Let us describe the general plan of the proof of Theorem 1.2. Let $\mathbb{R}(P, Q)$ be the subfield of $\mathbb{C}(x)$ generated by P and Q. By the Lüroth theorem, $\mathbb{R}(P, Q) = \mathbb{R}(W)$ for some $W \in \mathbb{R}(x)$ with $\deg W \geq 2$, implying that

$$P = \widetilde{P} \circ W, \qquad Q = \widetilde{Q} \circ W$$

for some $\widetilde{P}, \widetilde{Q} \in \mathbb{R}(x)$ such that $\mathbb{R}(\widetilde{P}, \widetilde{Q}) = \mathbb{R}(x)$. Moreover, since $P, Q \in \mathbb{R}[x]$, it is easy to see that we may assume that $\widetilde{P}, \widetilde{Q}, W \in \mathbb{R}[x]$. Therefore, since equalities (6) imply

$$\int_{W(a)}^{W(b)} \widetilde{P}^i d\widetilde{Q} = 0, \qquad \int_{W(a)}^{W(b)} \widetilde{Q}^i d\widetilde{P} = 0, \qquad i \geq 0, \tag{28}$$

in order to prove Theorem 1.2 it is enough to show that if polynomials P and Q satisfy

$$\mathbb{R}(P, Q) = \mathbb{R}(x) \tag{29}$$

and $a \neq b$, then P and Q cannot satisfy (6).

Applying Theorem 2.10 to the first and to the second system of equations in (6) separately, we arrive at nine different "cases" depending on types of solutions. For example, "the case (2,1)" means that Q is a solution of the second type of the polynomial moment problem (7), while P is a solution of the first type of the polynomial moment problem

$$\int_a^b Q^i \, dP = 0, \qquad i \geq 0.$$

In more detail, this means that, on the one hand,

$$Q = V_1 \circ W_1 + V_2 \circ W_2, \qquad P = U \circ T_{nm} \circ V,$$

where

$$\begin{aligned} &W_1 = T_n \circ V, &&W_2 = T_m \circ V, \\ &W_1(a) = W_1(b), &&W_2(a) = W_2(b), \end{aligned} \tag{30}$$

while, on the other hand,

$$P = \widetilde{V}_1 \circ \widetilde{W}_1 + \widetilde{V}_2 \circ \widetilde{W}_2, \qquad Q = \widetilde{U} \circ x^2 R^2(x^2) \circ \widetilde{V},$$

where

$$\begin{aligned} &\widetilde{W}_1 = x^2 \circ \widetilde{V}, &&\widetilde{W}_2 = x R(x^2) \circ \widetilde{V}, \\ &\widetilde{W}_1(a) = \widetilde{W}_1(b), &&\widetilde{W}_2(a) = \widetilde{W}_2(b). \end{aligned}$$

In view of assumption (29), the polynomial V (as well as $\widetilde{V}$) is of degree one, for otherwise

$$\mathbb{R}(P, Q) \subseteq \mathbb{R}(V) \subsetneq \mathbb{R}(x).$$

Furthermore, it is clear that we may assume that one of the polynomials V and $\widetilde{V}$ equals x. Our strategy will be to show that such systems of equations always imply that equalities (4) hold, in contradiction with (29) (recall that the condition $W(a) = W(b)$ implies that $\deg W > 1$).

Since we may exchange P and Q, it is only necessary to consider the cases (1,1), (2,1), (3,1), (2,2), (3,2), and (3,3). Finally, we may impose some additional restrictions related to the fact that a solution of the polynomial moment problem may belong to different types. For example, assuming that the theorem is already proved in the case (1,1), considering the case (2,1) we may assume that $n, m > 2$ in (30), since otherwise the solution P, Q also belongs to the case (1,1).

For a polynomial

$$P = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \qquad a_i \in \mathbb{R},\ 0 \leq i \leq n,$$

of degree n, set

$$C_i(P) = a_{n-i}, \qquad 0 \leq i \leq n.$$

The following simple lemma permits us to control initial terms in a composition of two polynomials and is widely used in the following.

Lemma 3.1. *Let T be a polynomial of degree d. Then for any polynomial S of degree r with leading coefficient c we have*

$$C_i(S \circ T) = C_i(cx^r \circ T), \qquad 0 \le i \le d-1. \tag{31}$$

In particular, for any two polynomials S_1, S_2 of equal degree with equal leading coefficients,

$$C_i(S_1 \circ T) = C_i(S_2 \circ T), \qquad 0 \le i \le d-1.$$

Proof. Indeed, deg $(S - cx^r) \circ T = dr - d$. Therefore, (31) holds. □

Corollary 3.2. *Let T be a polynomial of degree $d \ge 2$ with $C_1(T) = 0$, U an arbitrary polynomial, and $\alpha, \beta \in \mathbb{R}$, $\alpha \ne 0$. Then $C_1(U \circ T \circ (\alpha x + \beta)) = 0$ if and only if $\beta = 0$.*

Proof. Indeed, if $\deg U = r$ and $C_0(U) = c$, then $C_1(U \circ T) = C_1(cz^r \circ T)$ by Lemma 3.1. On the other hand, $C_1(cz^r \circ T) = 0$, since $C_1(T) = 0$. Therefore, $C_1(U \circ T) = 0$, and it is clear that for any polynomial F such that $C_1(F) = 0$ the equality $C_1(F \circ (\alpha x + \beta)) = 0$ holds if and only if $\beta = 0$. □

3.2. *Proof of Theorem 1.2 in the case (1,1)*

Lemma 3.3. *Let W_1, W_2 be polynomials of degree two such that $W_1(a) = W_1(b)$ and $W_2(a) = W_2(b)$ for distinct $a, b \in \mathbb{R}$. Then $W_2 = \lambda_1 W_1 + \lambda_2$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$.*

Proof. Let

$$W_1 = \alpha_1 x^2 + \beta_1 x + \gamma_1, \qquad W_2 = \alpha_2 x^2 + \beta_2 x + \gamma_2,$$

where $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2 \in \mathbb{R}$. Then the assunptions of the lemma yield

$$\alpha_1(a+b) + \beta_1 = 0, \qquad \alpha_2(a+b) + \beta_2 = 0.$$

Therefore, $\beta_1/\alpha_1 = \beta_2/\alpha_2$, implying the statement. □

In order to prove the theorem in the case (1,1) it is enough to observe that in this case there exist $U, R, \widetilde{U}, \widetilde{R} \in \mathbb{R}[x]$ such that

$$P = (U \circ xR(x^2)) \circ W_1, \qquad Q = (\widetilde{U} \circ x\widetilde{R}(x^2)) \circ \widetilde{W}_1$$

where

$$W_1 = x^2 \circ V, \qquad \widetilde{W}_1 = x^2 \circ \widetilde{V}$$

are polynomials of degree two such that $W_1(a) = W_1(b)$ and $\widetilde{W}_1(a) = \widetilde{W}_1(b)$. By Lemma 3.3, we have $\widetilde{W}_1 = \lambda_1 W_1 + \lambda_2$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$, and hence (4) holds for $W = W_1$.

3.3. Proof of Theorem 1.2 in the case (2,1)

If P, Q is a solution of (6) corresponding to the case (2,1), then without loss of generality we may assume that there exist polynomials V_1, V_2, U, $\widetilde{V}_1$, $\widetilde{V}_2$, $\widetilde{U}$, R and $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$, such that

$$\begin{aligned} Q &= V_1 \circ T_{m_1} + V_2 \circ T_{m_2} = \widetilde{U} \circ x^2 R^2(x^2) \circ (\alpha x + \beta), \\ P &= \widetilde{V}_1 \circ x^2 \circ (\alpha x + \beta) + \widetilde{V}_2 \circ x R(x^2) \circ (\alpha x + \beta) = U \circ T_{m_1 m_2}, \end{aligned} \tag{32}$$

where $\mathrm{GCD}(m_1, m_2) = 1$. In addition, for the polynomials

$$W_1 = T_{m_1}, \quad W_2 = T_{m_2}, \quad \widetilde{W}_1 = x^2 \circ (\alpha x + \beta), \quad \widetilde{W}_2 = x R(x^2) \circ (\alpha x + \beta)$$

we have

$$W_1(a) = W_1(b), \quad W_2(a) = W_2(b), \quad \widetilde{W}_1(a) = \widetilde{W}_1(b), \quad \widetilde{W}_2(a) = \widetilde{W}_2(b). \tag{33}$$

If at least one of the numbers m_1, m_2 equals 2, then P, Q belongs to the type (1,1) considered above. So, we may assume that $m_1, m_2 \geq 3$. Notice that the second representation for Q in (32) implies that $n = \deg Q$ is even. Moreover, $n \geq 6$, for otherwise $\deg R = 0$ in contradiction with $\widetilde{W}_2(a) = \widetilde{W}_2(b)$.

Although the above conditions seem to be very strong, it is difficult to use them in their full generality since they contain many unknown parameters. Thus, actually we will mostly use only the fact that in (32) the right side is a polynomial in $x^2 \circ (\alpha x + \beta)$, and the first three equalities in (33).

First of all observe that any polynomial of the form $Q = V_1 \circ T_{m_1} + V_2 \circ T_{m_2}$ can be represented in the form

$$Q = d_n T_n + d_{n-1} T_{n-1} + \cdots + d_1 T_1 + d_0, \qquad d_i \in \mathbb{R}, \tag{34}$$

where $d_i = 0$ unless i is divisible by m_1 or m_2. Indeed, it is clear that $T_0, T_1, \ldots, T_r$ is a basis of the subspace of $\mathbb{R}[x]$ consisting of all polynomials of degree $\leq r$. Therefore, a polynomial S can be represented in the form

$$S = V \circ T_m = (a_r x^r + a_{r-1} x^{r-1} + \cdots + a_1 x + a_0) \circ T_m$$

if and only if

$$S = (b_r T_r + b_{r-1} T_{r-1} + \cdots + b_1 T_1 + b_0) \circ T_m = b_r T_{rm} + b_{r-1} T_{(r-1)m} + \cdots + b_1 T_m + b_0.$$

Consequently, $Q = V_1 \circ T_{m_1} + V_2 \circ T_{m_2}$ can be represented in the required form. Moreover, it is clear that such a representation is unique.

Define $C(n, m_1, m_2)$ as the set of all polynomials (34) such that $d_i = 0$ unless i is divisible by m_1 or m_2, and $d_n \neq 0$. To be definite, we will always assume that n is divisible by m_1. Similarly to the notation $C_i(Q)$ introduced above, for a polynomial $Q \in C(n, m_1, m_2)$ given by (34) set

$$Ch_i(Q) = d_{n-i}, \qquad 0 \leq i \leq n.$$

Lemma 3.4. *Let $Q \in C(n, m_1, m_2)$, where $m_1, m_2 \geq 3$ and $n \geq 5$.*

(i) *If $Ch_1(Q) \neq 0$, then $Ch_2(Q) = 0$.*
(ii) *If $Ch_1(Q), Ch_3(Q) \neq 0$, then $m_1 = 3$ and $Ch_4(Q) = 0$,*
(iii) *If $Ch_1(Q), Ch_3(Q), Ch_5(Q) \neq 0$, then $m_1 = 3$ and $m_2 = 4$.*

Proof. If $Ch_1(Q) \neq 0$, then $m_2 \,|\, n-1$ since otherwise both n and $n-1$ are divisible by m_1. Further,

$$|m_i k_1 - m_i k_2| \geq m_i \geq 3, \qquad i = 1, 2,\ k_1, k_2 \in \mathbb{N}, \tag{35}$$

unless $k_1 = k_2$, implying that $n-2$ can be divisible neither by m_1, since $m_1 \,|\, n$, nor by m_2, since $m_2 \,|\, n-1$. Thus, $Ch_2(Q) = 0$.

Assume that additionally $Ch_3(Q) \neq 0$. Since $m_2 \,|\, n-1$, the number $n-3$ cannot be divisible by m_2 in view of (35). Therefore, $n-3$ is divisible by m_1, implying that $m_1 = 3$. Furthermore, $Ch_4(Q) = 0$ for otherwise $m_2 = 3$ by (35), implying that both n and $n-1$ are divisible by 3.

Finally, if also $Ch_5(Q) \neq 0$, it follows from $m_1 = 3$ that $m_2 \,|\, n-5$, implying that $m_2 = 4$. □

Notice that the restriction $n \geq 5$ in Lemma 3.4 is imposed since $Ch_s(Q)$, $s \geq 1$, is defined only for polynomials Q of degree at least s.

Corollary 3.5. *Let $Q \in C(n, m_1, m_2)$, where $m_1, m_2 \geq 3$, and $n \geq 5$ is even. If $Ch_1(Q) \neq 0$, then either $Ch_2(Q) = Ch_3(Q) = 0$, or $Ch_2(Q) = Ch_4(Q) = Ch_5(Q) = 0$.*

Proof. Since $m_2 \,|\, n-1$ and n is even, m_2 is odd. Therefore, $m_2 \neq 4$, and the statement follows from Lemma 3.4. □

Lemma 3.6. *Let Q and F be polynomials such that $Q = F \circ x^2 \circ (x - \delta)$ for some $\delta \in \mathbb{R}$ and $n = \deg Q \geq 6$.*

(i) *If $Ch_2(Q) = Ch_3(Q) = 0$, then either $\delta = 0$, or*

$$(2\delta)^2 = \frac{3}{(n-1)(n-2)}. \tag{36}$$

(ii) *If $Ch_2(Q) = Ch_4(Q) = Ch_5(Q) = 0$, then either $\delta = 0$, or 2δ satisfies the equation*

$$\frac{2}{15}(n-1)(n-2)(n-3)(n-4)t^4 - (n-2)(n-3)t^2 + 1 = 0. \tag{37}$$

Proof. If $Ch_2(Q) = Ch_3(Q) = 0$, then $Q = c_0 T_n + c_1 T_{n-1} + R_1$, where $\deg R_1 \leq n-4$. Set

$$T_s^*(x) = 2T_s(x/2), \qquad s \geq 1.$$

Clearly, the equality

$$c_0 T_n + c_1 T_{n-1} + R_1 = F \circ x^2 \circ (x - \delta)$$

implies

$$c_0 T_n^* + c_1 T_{n-1}^* + \widetilde{R}_1 = \widetilde{F} \circ x^2 \circ (x - \gamma), \tag{38}$$

where $\widetilde{R}_1 = 2R_1(x/2)$, $\widetilde{F} = 2F(x/4)$, and $\gamma = 2\delta$. Furthermore, without loss of generality we may assume that $c_0 = 1$.

Since the right side of (38) is a polynomial in $(x-\gamma)^2$, it follows by taking into account $\deg R_1 \le n-4$ that the derivatives of $T_n^* + c_1 T_{n-1}^*$ of orders $n-1$ and $n-3$ at γ vanish, that is,

$$T_n^{*(n-1)}(\gamma) + c_1 T_{n-1}^{*(n-1)}(\gamma) = 0, \qquad T_n^{*(n-3)}(\gamma) + c_1 T_{n-1}^{*(n-3)}(\gamma) = 0.$$

Since

$$T_s^* = x^s - sx^{s-2} + \frac{s(s-3)}{2}x^{s-4} - \frac{s(s-4)(s-5)}{6}x^{s-6} + \cdots \tag{39}$$

by (12), this implies that

$$\begin{aligned}
&n!\gamma + c_1(n-1)! = 0,\\
&\frac{n!}{3!}\gamma^3 - n(n-2)!\gamma + c_1\left(\frac{(n-1)!}{2!}\gamma^2 - (n-1)(n-3)!\right) = 0.
\end{aligned}$$

The first of these equalities implies that $c_1 = -n\gamma$. Substituting this in the second equality we obtain

$$\frac{n!}{3!}\gamma^3 - n(n-2)!\gamma - n\gamma\left(\frac{(n-1)!}{2!}\gamma^2 - (n-1)(n-3)!\right) = -\frac{n!}{3}\gamma^3 + n(n-3)!\gamma = 0,$$

implying that $2\delta = \gamma$ satisfies (36) unless $\delta = 0$.

Similarly, if $Ch_2(Q) = Ch_4(Q) = Ch_5(Q) = 0$, we arrive at

$$T_n^* + c_1 T_{n-1}^* + c_3 T_{n-3}^* + \widetilde{R}_1 = \widetilde{F} \circ x^2 \circ (x-\gamma), \tag{40}$$

where $\deg \widetilde{R}_1 \le n-6$ and $\gamma = 2\delta$, implying that the derivatives of $T_n^* + c_1 T_{n-1}^* + c_3 T_{n-3}^*$ of orders $n-1$, $n-3$, and $n-5$ at γ vanish. Thus,

$$\begin{aligned}
&T_n^{*(n-1)}(\gamma) + c_1 T_{n-1}^{*(n-1)}(\gamma) = 0,\\
&T_n^{*(n-3)}(\gamma) + c_1 T_{n-1}^{*(n-3)}(\gamma) + c_3 T_{n-3}^{*(n-3)}(\gamma) = 0,\\
&T_n^{*(n-5)}(\gamma) + c_1 T_{n-1}^{*(n-5)}(\gamma) + c_3 T_{n-3}^{*(n-5)}(\gamma) = 0,
\end{aligned}$$

By (39), these equalities are equivalent to

$$\begin{aligned}
&n!\gamma + c_1(n-1)! = 0,\\
&\frac{n!}{3!}\gamma^3 - n(n-2)!\gamma + c_1\left(\frac{(n-1)!}{2!}\gamma^2 - (n-1)(n-3)!\right) + c_3(n-3)! = 0,\\
&\frac{n!}{5!}\gamma^5 - \frac{n(n-2)!}{3!}\gamma^3 + \frac{n(n-3)(n-4)!}{2}\gamma\\
&\qquad + c_1\left(\frac{(n-1)!}{4!}\gamma^4 - \frac{(n-1)(n-3)!}{2!}\gamma^2 + \frac{(n-1)(n-4)(n-5)!}{2}\right)\\
&\qquad + c_3\left(\frac{(n-3)!}{2!}\gamma^2 - (n-3)(n-5)!\right) = 0.
\end{aligned}$$

As above, it follows from the first of these equalities that $c_1 = -n\gamma$, and substituting this in the second equality we obtain

$$\frac{n!}{3!}\gamma^3 - n(n-2)!\gamma - n\gamma\left(\frac{(n-1)!}{2!}\gamma^2 - (n-1)(n-3)!\right) + c_3(n-3)! = 0,$$

implying that

$$c_3 = \frac{n(n-1)(n-2)}{3}\gamma^3 - n\gamma.$$

Now the third equality gives

$$\begin{aligned}\frac{n!}{5!}\gamma^5 - \frac{n(n-2)!}{3!}\gamma^3 + \frac{n(n-3)(n-4)!}{2}\gamma \\ - n\gamma\left(\frac{(n-1)!}{4!}\gamma^4 - \frac{(n-1)(n-3)!}{2!}\gamma^2 + \frac{(n-1)(n-4)(n-5)!}{2}\right) \\ + \left(\frac{n(n-1)(n-2)}{3}\gamma^3 - n\gamma\right)\left(\frac{(n-3)!}{2!}\gamma^2 - (n-3)(n-5)!\right) = 0.\end{aligned}$$

The coefficient of γ^5 in this expression is

$$\frac{n!}{3!}\left(\frac{1}{20} - \frac{1}{4} + 1\right) = \frac{2n!}{15}.$$

The coefficient of γ^3 is

$$\begin{aligned}&-\frac{n(n-2)!}{3!} + \frac{n(n-1)(n-3)!}{2!} - \frac{n(n-1)(n-2)(n-3)(n-5)!}{3} - \frac{n(n-3)!}{2!} \\ &= -\frac{n(n-3)!}{2!}\left(\frac{n-2}{3} - (n-1)\right) - n(n-3)(n-5)!\left(\frac{(n-1)(n-2)}{3} + \frac{(n-4)}{2}\right) \\ &= \frac{(2n-1)n(n-3)!}{6} - n(n-3)(n-5)!\frac{2n^2-3n-8}{6} \\ &= \frac{n(n-3)(n-5)!}{6}\big((2n-1)(n-4) - (2n^2-3n-8)\big) = -n(n-2)(n-3)(n-5)!.\end{aligned}$$

Finally, the coefficient of γ is

$$\begin{aligned}&\frac{n(n-3)(n-4)!}{2} - \frac{n(n-1)(n-4)(n-5)!}{2} + n(n-3)(n-5)! \\ &= n(n-5)!\left(\frac{(n-3)(n-4)}{2} - \frac{(n-1)(n-4)}{2} + (n-3)\right) = n(n-5)!.\end{aligned}$$

Collecting terms and canceling $n(n-5)!$ we see that $2\delta = \gamma$ satisfies (37) unless $\delta = 0$. □

Corollary 3.7. *Let Q and F be polynomials such that $Q = F \circ x^2 \circ (x-\delta)$ for some $\delta \in \mathbb{R}$.*

(i) *If $Ch_2(Q) = Ch_3(Q) = 0$ and $n = \deg Q \geq 6$, then 4δ is not an algebraic integer unless $\delta = 0$.*
(ii) *If $Ch_2(Q) = Ch_4(Q) = Ch_5(Q) = 0$ and $n = \deg Q \geq 10$, then 4δ is not an algebraic integer unless $\delta = 0$.*

Proof. Set $\gamma = 4\delta$. If $Ch_2(Q) = Ch_3(Q) = 0$ and $\delta \neq 0$, then γ is a root of the equation

$$t^2 - \frac{12}{(n-1)(n-2)} = 0.$$

Since for $n \geq 6$ the number $\frac{12}{(n-1)(n-2)}$ is not an integer, this implies that γ cannot be an algebraic integer of degree two. Moreover, γ cannot be an algebraic integer of degree one, for otherwise γ is an integer, implying that so is

$$\gamma^2 = \frac{12}{(n-1)(n-2)}.$$

If $Ch_2(Q) = Ch_4(Q) = Ch_5(Q) = 0$ and $\delta \neq 0$, then γ is a root of the equation

$$(n-1)(n-2)(n-3)(n-4)t^4 - 30(n-2)(n-3)t^2 + 120 = 0. \tag{41}$$

Observe that if γ is a rational root of (41), then $-\gamma$ is also a root of (41), implying that $t^2 - \gamma^2$ divides the polynomial in (41). Hence the degree of an irreducible (over $\mathbb{Q}$) factor of that polynomial cannot be three, so it can only be one, two, or four.

If the polynomial in (41) is irreducible over $\mathbb{Q}$, then γ cannot be an algebraic integer since $(n-1)(n-2)(n-3)(n-4)$ does not divide 120 for $n \geq 10$.

Assume now that γ is an algebraic integer satisfying an irreducible equation of the form $t^2 + c_1 t + c_2 = 0$ with $c_1, c_2 \in \mathbb{Z}$. Then by the Gauss lemma,

$$(n-1)(n-2)(n-3)(n-4)t^4 - 30(n-2)(n-3)t^2 + 120 = (t^2 + c_1 t + c_2)(d_0 t^2 + d_1 t + d_2)$$

for some $d_0, d_1, d_2 \in \mathbb{Z}$. Since the coefficients of t^3 and t on the left side vanish, we have

$$d_1 + c_1 d_0 = 0, \qquad c_1 d_2 + c_2 d_1 = 0,$$

implying that, unless

$$c_1 = 0, \qquad d_1 = 0, \tag{42}$$

we have

$$d_1 = -c_1 d_0, \qquad d_2 = c_2 d_0. \tag{43}$$

If (43) holds, then $120 = c_2 d_2 = c_2^2 d_0$ contrary to $d_0 = (n-1)(n-2)(n-3)(n-4)$ and $n \geq 10$. Similarly, if (42) holds, then $\gamma^2 = -c_2$ is an integer, and it follows from (41) that $(n-2)(n-3)\gamma^2$ divides 120, easily implying a contradiction with the condition $n \geq 10$.

Finally, if γ is an integer, then so is γ^2, implying as above that $(n-2)(n-3)\gamma^2$ divides 120 in contradiction with $n \geq 10$. □

Proof of Theorem 1.2 in the case (2,1). Observe first that if $\beta = 0$ in (32), then the conclusion is true. Indeed, in this case the condition $\widetilde{W}_1(a) = \widetilde{W}_1(b)$ is equivalent to $T_2(a) = T_2(b)$. Therefore, applying Lemma 2.2(b) to the equalities

$$T_{m_1}(a) = T_{m_1}(b), \qquad T_{m_2}(a) = T_{m_2}(b), \qquad T_2(a) = T_2(b),$$

we conclude that at least one of the numbers m_1, m_2 is even, and hence

$$Q = \widetilde{U} \circ xR^2(x) \circ (\alpha x)^2 \quad \text{and} \quad P = U \circ T_{m_1 m_2} = U \circ T_{m_1 m_2/2} \circ T_2$$

satisfy (4) for $W = x^2$.

Further, observe that (32) implies that

$$Q = \widetilde{U} \circ xR^2(x) \circ x^2 \circ (\alpha x + \beta) = F \circ x^2 \circ (x + \beta/\alpha), \tag{44}$$

where $F = \widetilde{U} \circ xR^2(x) \circ \alpha^2 x$, while the condition $\widetilde{W}_1(a) = \widetilde{W}_1(b)$ yields

$$a + b = -2\beta/\alpha. \tag{45}$$

If $Ch_1(Q) = 0$, then (34) implies that also $C_1(Q) = 0$, since $C_1(T_n) = 0$ by (12). In its turn, $C_1(Q) = 0$ implies that $\beta = 0$ by Corollary 3.2 applied to (44). So, assume that $Ch_1(Q) \neq 0$. By Corollary 3.5, this implies that either $Ch_2(Q)$ and $Ch_3(Q)$ vanish, or $Ch_2(Q)$, $Ch_4(Q)$ and $Ch_5(Q)$ vanish (recall that $n = \deg Q$ is even and $n \geq 6$ so that we can use Corollary 3.5).

If $Ch_2(Q) = Ch_3(Q) = 0$, then Corollary 3.7(i) applied to (44) implies that $-4\beta/\alpha$ is not an algebraic integer unless $\beta = 0$. On the other hand, (45) implies that $-4\beta/\alpha$ is an algebraic integer, since $2a$ and $2b$ are algebraic integers by Corollary 2.5. Thus, we conclude again that $\beta = 0$.

Similarly, assuming that $Ch_3(Q) \neq 0$, while $Ch_2(Q) = Ch_4(Q) = Ch_5(Q) = 0$, we may apply Corollary 3.7(ii) whenever $n \geq 10$. Since n is even and $Ch_3(Q) \neq 0$ implies that $3 \mid n$ in view of $m_1 = 3$ (see Lemma 3.4), the only possible value for n which does not satisfy $n \geq 10$ is 6. Substituting $n = 6$ in (41) we obtain the equation $t^4 - 3t^2 + 1 = 0$ whose roots are algebraic integers. Thus, for $n = 6$ the previous reasoning fails.

In order to prove the theorem in the remaining case, observe first that for $n = 6$ the condition $Ch_1(Q) \neq 0$ implies that $m_2 = 5$. Thus, $Q \in C(6, 3, 5)$ and hence

$$Q = T_6 + c_1 T_5 + c_3 T_3 + c_6,$$

where $c_1, c_3, c_6 \in \mathbb{R}$. Recall that instead of (32) we have used in the proof the weaker condition that the right side of (32) has the form $F \circ x^2 \circ (x - \delta)$. Therefore, to finish the proof it is enough to show that the equality

$$T_6 + c_1 T_5 + c_3 T_3 + c_6 = c(x(x^2 - d))^2 \circ (x - \beta), \tag{46}$$

where $c_1, c_3, c_6, c, d, \beta \in \mathbb{R}$, implies that $\beta = 0$. This may be verified by a direct calculation. Namely, the comparison of the leading coefficients of both sides of (46) implies that $c=32$, while the comparison of the other coefficients gives

$$\begin{aligned} 16c_1+192\beta&=0, & -480\beta^2+64d-48&=0,\\ 640\beta^3-256\beta d-20c_1+4c_3&=0, & -480\beta^4+384\beta^2 d-32d^2+18&=0,\\ 192\beta^5-256\beta^3 d+64\beta d^2+5c_1-3c_3&=0, & c_6&=-32\beta^6+64\beta^4 d-32\beta^2 d^2-1. \end{aligned}$$

We leave it to the reader to verify (for example, with the help of Maple) that the only solution of the above system is

$$c_1=0, \quad c_3=0, \quad c_6=1, \quad \beta=0, \quad d=3/4$$

(for these values of the parameters, equality (46) simply reduces to $T_6=T_2\circ T_3$). □

3.4. *Proof of Theorem 1.2 in the case (2,2)*

First, observe that Theorem 1.2 in the case (2,2) follows from the following statement.

Proposition 3.8. *Let $V_1, V_2, U, \widetilde{V}_1, \widetilde{V}_2, \widetilde{U}\in\mathbb{R}[x]$ and $\alpha,\beta\in\mathbb{R}$, $\alpha\neq 0$, satisfy*

$$V_1\circ T_{m_1}+V_2\circ T_{m_2}=\widetilde{U}\circ T_{\widetilde{m}_1\widetilde{m}_2}\circ(\alpha x+\beta), \tag{47}$$

$$\widetilde{V}_1\circ T_{\widetilde{m}_1}\circ(\alpha x+\beta)+\widetilde{V}_2\circ T_{\widetilde{m}_2}\circ(\alpha x+\beta)=U\circ T_{m_1m_2}, \tag{48}$$

where

$$\mathrm{GCD}(m_1,m_2)=\mathrm{GCD}(\widetilde{m}_1,\widetilde{m}_2)=1, \tag{49}$$

and $m_1, m_2, \widetilde{m}_1, \widetilde{m}_2\geq 3$. Then $\alpha=\pm 1$ and $\beta=0$.

Indeed, in the case (2,2) the equalities

$$Q=V_1\circ T_{m_1}+V_2\circ T_{m_2}=\widetilde{U}\circ T_{\widetilde{m}_1\widetilde{m}_2}\circ(\alpha x+\beta), \tag{50}$$

$$P=\widetilde{V}_1\circ T_{\widetilde{m}_1}\circ(\alpha x+\beta)+\widetilde{V}_2\circ T_{\widetilde{m}_2}\circ(\alpha x+\beta)=U\circ T_{m_1m_2}, \tag{51}$$

and (49) hold for some $m_1, m_2, \widetilde{m}_1, \widetilde{m}_2\geq 2$. Additionally,

$$T_{m_1}(a)=T_{m_1}(b), \qquad T_{m_2}(a)=T_{m_2}(b), \tag{52}$$

$$T_{\widetilde{m}_1}(\alpha a+\beta)=T_{\widetilde{m}_1}(\alpha b+\beta), \qquad T_{\widetilde{m}_2}(\alpha a+\beta)=T_{\widetilde{m}_2}(\alpha b+\beta). \tag{53}$$

If at least one of m_1, m_2 equals 2, then P, Q belongs to the type (1,2) considered above. So, we may assume that $m_1, m_2\geq 3$. Similarly, we may assume that $\widetilde{m}_1, \widetilde{m}_2\geq 3$, since otherwise P, Q belongs to the type (2,1). Since under these conditions Proposition 3.8 implies that $\alpha=\pm 1$ and $\beta=0$, it follows from the equalities

$$T_{m_1}(a)=T_{m_1}(b), \qquad T_{m_2}(a)=T_{m_2}(b), \qquad T_{\widetilde{m}_1}(\pm a)=T_{\widetilde{m}_1}(\pm b)$$

by Lemma 2.2(b), taking into account (13), that we have $T_s(a)=T_s(b)$, where either $s=\mathrm{GCD}(m_1,\widetilde{m}_1)$, or $s=\mathrm{GCD}(m_2,\widetilde{m}_1)$. Since

$$Q=\widetilde{U}\circ T_{\widetilde{m}_1\widetilde{m}_2}\circ(\pm x)=\widetilde{U}\circ(\pm T_{\widetilde{m}_1\widetilde{m}_2})=\widetilde{U}\circ(\pm T_{\widetilde{m}_1\widetilde{m}_2/s})\circ T_s,$$

$$P=U\circ T_{m_1m_2}=U\circ T_{m_1m_2/s}\circ T_s,$$

we conclude that (4) holds for $W = T_s$. This completes the proof of Theorem 1.2 in the case (2,2) assuming Proposition 3.8.

The next lemmas, used in the proof of Proposition 3.8, are similar to Lemmas 3.4 and 3.6 and are used for imposing restrictions on possible values of α and β in (47), (48), and eventually to show that $\alpha = \pm 1$ and $\beta = 0$.

Lemma 3.9. *Let Q and F be polynomials such that $Q = F \circ T_s \circ (\alpha x + \beta)$, where $s \geq 5$, $\alpha, \beta \in \mathbb{R}$, and $\alpha \neq 0$. Set $n = \deg Q$.*

(i) *If $Ch_2(Q) = Ch_3(Q) = 0$, then either $\alpha = \pm 1$, $\beta = 0$, or*

$$4\beta^2 = \frac{6}{(n-1)(2n-1)}, \qquad \alpha^2 = \frac{2n-4}{2n-1}. \tag{54}$$

(ii) *If $Ch_2(Q) = Ch_4(Q) = 0$, then either $\alpha = \pm 1$ and $\beta = 0$, or*

$$4\beta^2 = \frac{12}{(n-1)(2n-1)}, \qquad \alpha^2 = \frac{2n-7}{2n-1}. \tag{55}$$

In particular, in both cases $\alpha^2 < 1$ and $\beta \neq 0$ unless $\alpha = \pm 1$ and $\beta = 0$.

Proof. If $Ch_2(Q) = 0$, then

$$Q = c_0 T_n + c_1 T_{n-1} + c_3 T_{n-3} + c_4 T_{n-4} + R_1,$$

where R_1 is a polynomial such that $\deg R_1 \leq n - 5$ and $c_0, c_1, c_3, c_4 \in \mathbb{R}$. Further, by Lemma 3.1, for some $b_0 \in \mathbb{R}$ we have

$$\begin{aligned} C_i(Q) &= C_i(F \circ T_s \circ (\alpha x + \beta)) = C_i(b_0 T_{n/s} \circ T_s \circ (\alpha x + \beta)) \\ &= C_i(b_0 T_n \circ (\alpha x + \beta)), \qquad 0 \leq i \leq s - 1, \end{aligned}$$

implying that

$$Q = (b_0 T_n) \circ (\alpha x + \beta) + R_2,$$

where R_2 is a polynomial such that $\deg R_2 \leq n - s$. Thus,

$$Q = c_0 T_n + c_1 T_{n-1} + c_3 T_{n-3} + c_4 T_{n-4} + R_1 = b_0 T_n \circ (\alpha x + \beta) + R_2, \tag{56}$$

where $\deg R_1, \deg R_2 \leq n - 5$ and $c_0, c_1, c_3, c_4, b_0 \in \mathbb{R}$. Changing x to $x/2$, and $R_i(x)$ to $2R_i(x/2)$, we obtain, in the notation of Lemma 3.6, a similar equality

$$Q = c_0 T_n^* + c_1 T_{n-1}^* + c_3 T_{n-3}^* + c_4 T_{n-4}^* + R_1 = b_0 T_n^* \circ (\widetilde{\alpha} x + \widetilde{\beta}) + R_2, \tag{57}$$

where $\widetilde{\beta} = 2\beta$, $\widetilde{\alpha} = \alpha$. Furthermore, we may assume that $c_0 = 1$, implying $b_0 = 1/\widetilde{\alpha}^n$ and $c_1 = \widetilde{\beta} n / \widetilde{\alpha}$. Thus, we can rewrite (57) in the form

$$Q = T_n^* + \frac{\widetilde{\beta} n}{\widetilde{\alpha}} T_{n-1}^* + c_3 T_{n-3}^* + c_4 T_{n-4}^* + R_1 = \frac{1}{\widetilde{\alpha}^n} T_n^* \circ (\widetilde{\alpha} x + \widetilde{\beta}) + R_2. \tag{58}$$

Calculating $C_2(Q)$, $C_3(Q)$, $C_4(Q)$ for both representations of Q in (58) using (39) (and also the Taylor formula, for the second representation), we obtain

$$-n = \frac{1}{(n-2)!\widetilde{\alpha}^2}\left[\frac{n!\widetilde{\beta}^2}{2!} - n(n-2)!\right],$$
$$\frac{-\widetilde{\beta}n(n-1)}{\widetilde{\alpha}} + c_3 = \frac{1}{(n-3)!\widetilde{\alpha}^3}\left[\frac{n!\widetilde{\beta}^3}{3!} - n(n-2)!\widetilde{\beta}\right],$$
$$\frac{n(n-3)}{2} + c_4 = \frac{1}{(n-4)!\widetilde{\alpha}^4}\left[\frac{n!\widetilde{\beta}^4}{4!} - \frac{n(n-2)!\widetilde{\beta}^2}{2!} + \frac{n(n-3)(n-4)!}{2}\right].$$

It follows from the first of these equalities that

$$\widetilde{\alpha}^2 = 1 - (n-1)\widetilde{\beta}^2/2. \tag{59}$$

Further, if $Ch_3(Q) = c_3 = 0$, then substituting this value of $\widetilde{\alpha}^2$ into the second equality we conclude that either (54) holds, or $\beta = 0$ and hence $\alpha = \pm 1$ by (59). Similarly, if $Ch_4(Q) = c_4 = 0$, then substituting (59) into the third equality we obtain

$$\frac{n!\widetilde{\beta}^4}{4!} - \frac{n(n-2)!\widetilde{\beta}^2}{2!} + \frac{n(n-3)(n-4)!}{2} = \frac{n(n-3)(n-4)!}{2}\left[\frac{(n-1)^2\widetilde{\beta}^4}{4} - (n-1)\widetilde{\beta}^2 + 1\right],$$

implying that either $\widetilde{\alpha} = \pm 1$ and $\widetilde{\beta} = 0$, or (55) holds.

Finally, it is clear that $\alpha^2 < 1$ and $\beta \neq 0$ unless $\alpha = \pm 1$ and $\beta = 0$. □

Lemma 3.10. *Let $Q \in C(n, m_1, m_2)$, where $m_1, m_2 \geq 3$ and $n \geq 6$. Then at least one of the coefficients $Ch_2(Q)$, $Ch_4(Q)$, $Ch_6(Q)$ vanishes.*

Proof. Assuming $Ch_2(Q), Ch_4(Q) \neq 0$, we will show $Ch_6(Q) = 0$. First, $Ch_2(Q) \neq 0$ implies by (35) that $m_2 \,|\, n-2$. It now follows from $Ch_4(Q) \neq 0$ by (35) that $m_1 = 4$. Therefore, $Ch_6(Q) = 0$, for otherwise $m_2 = 4$ by (35), implying that both n and $n-2$ are divisible by 4. □

Lemma 3.11. *Let Q and F be polynomials such that $Q = F \circ T_s \circ \alpha x$, where $F \in \mathbb{R}[x]$, $\alpha \in \mathbb{R} \setminus \{0\}$, and $s \geq 7$. Set $n = \deg Q$.*

(i) *If $Ch_2(Q) = 0$, then $\alpha^2 = 1$.*
(ii) *If $Ch_4(Q) = 0$, then either $\alpha^2 = 1$, or $\alpha^2 = \frac{n-3}{n-1}$.*
(iii) *If $Ch_6(Q) = 0$, then either $\alpha^2 = 1$, or α^2 is a root of the equation*

$$(n^2 - 3n + 2)t^2 + (-2n^2 + 12n - 16)t + (n^2 - 9n + 20) = 0. \tag{60}$$

In particular, in all these cases the inequality $\alpha^2 < 1$ holds unless $\alpha^2 = 1$.

Proof. As in Lemma 3.9 we can write the equality $Q = F \circ T_s \circ \alpha x$ in the form

$$T_n^* + c_1 T_{n-1}^* + c_2 T_{n-2}^* + c_3 T_{n-3}^* + c_4 T_{n-4}^* + c_5 T_{n-5}^* + c_6 T_{n-6}^* + R_1 = \frac{1}{\widetilde{\alpha}^n} T_n^* \circ (\alpha x) + R_2,$$

where R_1, R_2 are polynomials such that $\deg R_1, \deg R_2 \le n-7$, and $c_i \in \mathbb{R}$, $1 \le i \le 6$. Furthermore, it follows from (12) that $c_1 = 0$, implying inductively that also $c_3 = 0$ and $c_5 = 0$. Calculating now $C_2(Q)$, $C_4(Q)$, $C_6(Q)$ for both representations of Q in

$$T_n^* + c_2 T_{n-2}^* + c_4 T_{n-4}^* + c_6 T_{n-6}^* + R_1 = \frac{1}{\widetilde{\alpha}^n} T_n^* \circ (\alpha x) + R_2, \tag{61}$$

we obtain

$$\begin{aligned}
&-n + c_2 = -\frac{n}{\alpha^2},\\
&\frac{n(n-3)}{2} - (n-2)c_2 + c_4 = \frac{n(n-3)}{2\alpha^4},\\
&-\frac{n(n-4)(n-5)}{6} + \frac{(n-2)(n-5)c_2}{2} - (n-4)c_4 + c_6 = -\frac{n(n-4)(n-5)}{6\alpha^6}.
\end{aligned}$$

It follows from the first equality that

$$c_2 = n(\alpha^2 - 1)/\alpha^2,$$

implying that if $c_2 = 0$, then $\alpha^2 = 1$. Substituting this value of c_2 into the second equality we obtain

$$c_4 = \frac{n(n-3)}{2\alpha^4} - \frac{n(n-3)}{2} + \frac{n(n-2)(\alpha^2-1)}{\alpha^2} = n\frac{(n-1)\alpha^4 - 2(n-2)\alpha^2 + (n-3)}{2\alpha^4}.$$

Since

$$(n-1)\alpha^4 - 2(n-2)\alpha^2 + (n-3) = (n-1)(\alpha^2-1)\left(\alpha^2 - \frac{n-3}{n-1}\right),$$

this implies that if $c_4 = 0$, then either $\alpha^2 = 1$, or $\alpha^2 = \frac{n-3}{n-1}$.

Finally, since

$$\begin{aligned}
c_6 &= -\frac{n(n-4)(n-5)}{6\alpha^6} + \frac{n(n-4)(n-5)}{6} - \frac{(n-2)(n-5)c_2}{2} + (n-4)c_4\\
&= -\frac{n(n-4)(n-5)}{6\alpha^6} + \frac{n(n-4)(n-5)}{6} - \frac{(n-2)(n-5)}{2}\frac{n(\alpha^2-1)}{\alpha^2}\\
&\quad + (n-4)n\frac{(n-1)\alpha^4 - 2(n-2)\alpha^2 + (n-3)}{2\alpha^4}\\
&= n\frac{(n^2-3n+2)\alpha^6 + (-3n^2+15n-18)\alpha^4 + (3n^2-21n+36)\alpha^2 - n^2 + 9n - 20}{6\alpha^6},
\end{aligned}$$

it follows from the factorization

$$(n^2-3n+2)\alpha^6+(-3n^2+15n-18)\alpha^4+(3n^2-21n+36)\alpha^2-n^2+9n-20$$
$$=(\alpha^2-1)\big((n^2-3n+2)\alpha^4+(-2n^2+12n-16)\alpha^2+(n^2-9n+20)\big)$$

that either $\alpha^2=1$, or α^2 is a root of (60). Solving (60), we find two roots

$$t_1=\frac{n^2-6n+8-\sqrt{3n^2-18n+24}}{n^2-3n+2}=\frac{\sqrt{(n-2)(n-4)}(\sqrt{(n-2)(n-4)}-\sqrt{3})}{(n-1)(n-2)},$$
$$t_2=\frac{n^2-6n+8+\sqrt{3n^2-18n+24}}{n^2-3n+2}=\frac{\sqrt{(n-2)(n-4)}(\sqrt{(n-2)(n-4)}+\sqrt{3})}{(n-1)(n-2)}.$$

Since $n\geq 7$, they are real and positive. Moreover, $t_1<t_2$ and

$$t_2-1=\frac{\sqrt{3(n-2)(n-4)}-3n+6}{(n-1)(n-2)}=\frac{\sqrt{3(n-2)}(\sqrt{n-4}-\sqrt{3(n-2)})}{(n-1)(n-2)}<0. \quad \square$$

Proof of Proposition 3.8. Denote the polynomials (47), (48) by Q and P as in (50), (51). Observe that (50), (51) and the restrictions imposed on $m_1, m_2, \widetilde{m}_1, \widetilde{m}_2$ imply that

$$\deg P\geq m_1m_2\geq 12, \qquad \deg Q\geq \widetilde{m}_1\widetilde{m}_2\geq 12.$$

Assume that $Ch_1(Q)\neq 0$. Then Lemma 3.4 implies that either $Ch_2(Q)=Ch_3(Q)=0$, or $Ch_2(Q)=Ch_4(Q)=0$. Applying Lemma 3.9 to equality (47) we see that, unless $\alpha=\pm 1$ and $\beta=0$, the conditions $\alpha<1$ and $\beta\neq 0$ hold. So, assume that $\alpha<1$ and $\beta\neq 0$. Rewrite (51) in the form

$$P=\widetilde{V}_1\circ T_{\widetilde{m}_1}+\widetilde{V}_2\circ T_{\widetilde{m}_2}=U\circ T_{m_1m_2}\circ\left(\frac{x-\beta}{\alpha}\right). \tag{62}$$

Since $\beta\neq 0$, Corollary 3.2 applied to (62) implies that $C_1(P)\neq 0$. Applying now Lemmas 3.4 and 3.9 to (62) in the same way as before to (50), we conclude that $1/\alpha<1$. The contradiction obtained proves that $\alpha=\pm 1$ and $\beta=0$.

Assume now that $Ch_1(Q)=0$. Then $\beta=0$, by Corollary 3.2. Furthermore, by Lemma 3.10 at least one of the coefficients $Ch_2(Q), Ch_4(Q), Ch_6(Q)$ vanishes, implying by Lemma 3.11 that, unless $\alpha=\pm 1$, $\beta=0$, the condition $\alpha<1$ holds. Since $\beta=0$ implies by Corollary 3.2 that $C_1(P)=0$ in view of (62), we conclude as above that the assumption $\alpha<1$ leads to a contradiction. $\square$

3.5. *Proof of Theorem 1.2 in the cases (3,1), (3,2), (3,3)*

The case (3,1) reduces to the case (2,1) as follows. We start from the equality

$$Q=V_1\circ T_{2m_1}+V_2\circ T_{2m_2}+V_3\circ xR(x^2)\circ T_{m_1m_2}=\widetilde{U}\circ x^2\widetilde{R}^2(x^2)\circ(\alpha x+\beta), \tag{63}$$

where $V_1, V_2, V_3, R, \widetilde{R}, \widetilde{U}\in\mathbb{R}[x]$, $\alpha,\beta\in\mathbb{R}$, $\alpha\neq 0$, and $m_1, m_2\geq 3$ are coprime and odd. It follows from the first representation for Q in (63) that Q can be written in the form

$$Q=d_nT_n+d_{n-1}T_{n-1}+\cdots+d_1T_1+d_0, \qquad d_i\in\mathbb{R}, \tag{64}$$

where $d_i = 0$ unless i is divisible by $2m_1$, $2m_2$, or m_1m_2. Clearly, the conditions imposed on m_1, m_2 imply that

$$\begin{aligned} |2m_1k_1 - 2m_2k_2| &\geq 2 \quad \text{unless} \quad |2m_1k_1 - 2m_2k_2| = 0, \\ |2m_ik_1 - 2m_ik_2| &\geq 2m_i \geq 6 \quad \text{unless} \quad |2m_ik_1 - 2m_ik_2| = 0, \quad i = 1, 2, \\ |2m_ik_1 - m_1m_2k_2| &\geq m_i \geq 3 \quad \text{unless} \quad |2m_ik_1 - m_1m_2k_2| = 0, \quad i = 1, 2, \\ |m_1m_2k_1 - m_1m_2k_2| &\geq m_1m_2 \geq 15 \quad \text{unless} \quad |m_1m_2k_1 - m_1m_2k_2| = 0. \end{aligned}$$

Therefore, $Ch_1(Q) = 0$, implying that $C_1(Q) = 0$, since $C_1(T_n) = 0$. It now follows from the second representation for Q in (63) by Corollary 3.2 that $\beta = 0$. Since the polynomial $\widetilde{W}_1 = z^2 \circ (\alpha x + \beta)$ satisfies $\widetilde{W}_1(a) = \widetilde{W}_1(b)$, this implies that $a = -b$. Therefore, the solution P, Q also belongs to the case (2,1) considered earlier (see the remarks after Theorem 2.1).

In the case (3,2) there exist $V_1, V_2, V_3, U, R, \widetilde{V}_1, \widetilde{V}_2, \widetilde{U} \in \mathbb{R}[x]$ and $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$, such that

$$Q = V_1 \circ T_{2m_1} + V_2 \circ T_{2m_2} + V_3 \circ xR(x^2) \circ T_{m_1m_2} = \widetilde{U} \circ T_{\widetilde{m}_1\widetilde{m}_2} \circ (\alpha x + \beta), \tag{65}$$

$$P = \widetilde{V}_1 \circ T_{\widetilde{m}_1} \circ (\alpha x + \beta) + \widetilde{V}_2 \circ T_{\widetilde{m}_2} \circ (\alpha x + \beta) = U \circ x^2R^2(x^2) \circ T_{m_1m_2}, \tag{66}$$

where $m_1, m_2 \geq 3$ are odd and coprime, and $\widetilde{m}_1, \widetilde{m}_2 \geq 2$ are coprime. Further, without loss of generality we may assume that $a \neq -b$, for otherwise P, Q belongs to the case $(2, 2)$. Moreover, we may assume that $\widetilde{m}_1, \widetilde{m}_2 \geq 3$, for otherwise P, Q belongs to the case $(3, 1)$.

Since equalities (65), (66) may be written in the form

$$Q = (V_1 \circ T_2 + V_3 \circ xR(x^2) \circ T_{m_2}) \circ T_{m_1} + (V_2 \circ T_2) \circ T_{m_2} = \widetilde{U} \circ T_{\widetilde{m}_1\widetilde{m}_2} \circ (\alpha x + \beta),$$
$$P = \widetilde{V}_1 \circ T_{\widetilde{m}_1} \circ (\alpha x + \beta) + \widetilde{V}_2 \circ T_{\widetilde{m}_2} \circ (\alpha x + \beta) = (U \circ x^2R^2(x^2)) \circ T_{m_1m_2},$$

it follows from Proposition 3.8 that $\alpha = \pm 1$ and $\beta = 0$. Since we have assumed that $a \neq -b$, it follows now from the equalities

$$T_{2m_1}(a) = T_{2m_1}(b), \quad T_{2m_2}(a) = T_{2m_2}(b), \quad T_{\widetilde{m}_1}(\pm a) = T_{\widetilde{m}_1}(\pm b)$$

by Lemma 2.2(b), taking into account (13), that we have $T_s(a) = T_s(b)$ either for $s = \mathrm{GCD}(2m_1, \widetilde{m}_1)$, or for $s = \mathrm{GCD}(2m_2, \widetilde{m}_1)$. Finally, since in any case $s \mid \widetilde{m}_1\widetilde{m}_2$ and $s \mid 2m_1m_2$, we obtain

$$\begin{aligned} Q &= \widetilde{U} \circ T_{\widetilde{m}_1\widetilde{m}_2} \circ (\pm x) = \widetilde{U} \circ (\pm T_{\widetilde{m}_1\widetilde{m}_2}) = \widetilde{U} \circ (\pm T_{\widetilde{m}_1\widetilde{m}_2/s}) \circ T_s, \\ P &= U \circ x^2R^2(x^2) \circ T_{m_1m_2} = U \circ xR^2(x) \circ x^2 \circ T_{m_1m_2} = F \circ T_{2m_1m_2} \\ &= F \circ T_{2m_1m_2/s} \circ T_s, \end{aligned}$$

where

$$F = U \circ xR^2(x) \circ \frac{x+1}{2}.$$

Thus, (4) holds for $W = T_s$.

In the case (3,3) the proof is similar: there exist $V_1, V_2, V_3, U, R, \widetilde{V}_1, \widetilde{V}_2, \widetilde{V}_3, \widetilde{U}, \widetilde{R}$ in $\mathbb{R}[x]$ and $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$, such that

$$\begin{aligned} Q &= V_1 \circ T_{2m_1} + V_2 \circ T_{2m_2} + V_3 \circ x R(x^2) \circ T_{m_1 m_2} \\ &= \widetilde{U} \circ x^2 \widetilde{R}^2(x^2) \circ T_{\widetilde{m}_1 \widetilde{m}_2} \circ (\alpha x + \beta), \\ P &= \widetilde{V}_1 \circ T_{2\widetilde{m}_1} \circ (\alpha x + \beta) + \widetilde{V}_2 \circ T_{2\widetilde{m}_2} \circ (\alpha x + \beta) + \widetilde{V}_3 \circ x \widetilde{R}(x^2) \circ T_{\widetilde{m}_1 \widetilde{m}_2} \circ (\alpha x + \beta) \\ &= U \circ x^2 R^2(x^2) \circ T_{m_1 m_2}, \end{aligned}$$

where $m_1, m_2, \widetilde{m}_1, \widetilde{m}_2 \geq 3$ are odd and $\mathrm{GCD}(m_1, m_2) = \mathrm{GCD}(\widetilde{m}_1, \widetilde{m}_2) = 1$. Moreover, without loss of generality we may assume that $a \neq -b$. Using Proposition 3.8 we conclude as above that $\alpha = \pm 1$ and $\beta = 0$. Finally, it follows from the equalities

$$T_{2m_1}(a) = T_{2m_1}(b), \qquad T_{2m_2}(a) = T_{2m_2}(b), \qquad T_{2\widetilde{m}_1}(\pm a) = T_{2\widetilde{m}_1}(\pm b)$$

that (4) holds for $W = T_s$, where either $s = \mathrm{GCD}(2m_1, 2\widetilde{m}_1)$, or $s = \mathrm{GCD}(2m_2, 2\widetilde{m}_1)$.

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