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V. Georgescu · C. Gérard · D. Häfner

Asymptotic completeness for superradiant Klein–Gordon equations and applications to the De Sitter–Kerr metric

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Abstract. We show asymptotic completeness for a class of superradiant Klein–Gordon equations. Our results are applied to the Klein–Gordon equation on the De Sitter–Kerr metric with small angular momentum of the black hole. For this equation we obtain asymptotic completeness for fixed angular momentum of the field.

Keywords. Asymptotic completeness, Klein–Gordon equation, De Sitter–Kerr metric, superradiance

1. Introduction

1.1. Introduction

Asymptotic completeness is one of the fundamental properties one might want to show for a Hamiltonian describing the dynamics of a physical system. Roughly speaking it states that the Hamiltonian of the system is equivalent to a free Hamiltonian for which the dynamics is well understood. The dynamics that we want to understand behaves then at large times like this free dynamics modulo possible eigenvalues. In the case when the Hamiltonian is selfadjoint with respect to some suitable Hilbert space inner product, an enormous amount of literature has been dedicated to this question. The question is much less studied in the case when the Hamiltonian is not selfadjoint. This situation occurs for example for the Klein–Gordon equation when the field is coupled to a (strong) electric field. This system has been studied by Kako [25] in the short range case and by C. Gérard [16] in the long range case. In this situation the Hamiltonian, although not selfadjoint on a Hilbert space, is selfadjoint on a so called *Krein space*. In a previous paper [17] we addressed the question of boundary values of the resolvent for selfadjoint operators on Krein spaces. Applications to propagation estimates for the Klein–Gordon equation are given in [18].

V. Georgescu: Département de Mathématiques, Université de Cergy-Pontoise,
95302 Cergy-Pontoise Cedex, France; e-mail: Vladimir.Georgescu@math.cnrs.fr

C. Gérard: Département de Mathématiques, Université de Paris XI,
91405 Orsay Cedex, France; e-mail: christian.gerard@math.u-psud.fr

D. Häfner: Univ. Grenoble Alpes, CNRS, Institut Fourier,
F-38000 Grenoble, France; e-mail: Dietrich.Hafner@univ-grenoble-alpes.fr

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The Klein–Gordon equation can be written in a quite general setting in the form

$$(\partial_t^2 - 2ik\partial_t + h)\phi = 0, \quad \phi : \mathbb{R} \to \mathcal{H}, \tag{1.1}$$

with selfadjoint operators k and h. However, if h is not positive the natural conserved energy for (1.1),

$$\|\partial_t\phi\|^2 + (\phi|h\phi),$$

is not positive and in general no positive conserved energy is available. This happens in particular when the equation is associated to a Lorentzian manifold with no global time-like Killing vector field. In this situation natural positive energies can grow in time, and we will loosely speak about *superradiance*, the most famous example being the (De Sitter) Kerr metric which describes rotating black holes. This example does not enter into the framework of our previous papers because the Hamiltonian can no longer be realized as a selfadjoint operator on a Krein space whose topology is given by some natural positive (but not conserved) energy. The problem comes from the fact that the operator k has different "limit operators" at the different ends of the manifold. In the one-dimensional case scattering results for this situation have been obtained by Bachelot [4].

Asymptotic completeness for wave equations on Lorentzian manifolds has been studied for a long time since the works of Dimock and Kay in the 1980's (see e.g. [10]). The main motivation came from the *Hawking effect*. Such results are a necessary step to give mathematically rigorous descriptions of the Hawking effect (see Bachelot [3] and Häfner [22]). The most complete scattering results exist in the Schwarzschild metric (see e.g. Bachelot [2]). Asymptotic completeness has also been shown on the Kerr metric for nonsuperradiant modes of the Klein–Gordon equation (see Häfner [21]) and for the Dirac equation (for which no superradiance occurs; see Häfner–Nicolas [23]). In this setting asymptotic completeness can be understood as an existence and uniqueness result for the characteristic Cauchy problem in energy space at null infinity (see [23] for details). As far as we are aware, asymptotic completeness has not been addressed in the setting of superradiant equations on the (De Sitter) Kerr background. Note however that scattering results have been obtained by Dafermos, Holzegel and Rodnianski [6] in the difficult nonlinear setting of the Einstein equations supposing exponential decay for the scattering data on the future event horizon and at future null infinity. Also there has been enormous progress in the last years on a somewhat related question of decay of the local energy for the wave equation on the (De Sitter) Kerr metric. In this context we mention the papers of Andersson–Blue [1], Dyatlov [12], Dafermos–Rodnianski [7], Dafermos–Rodnianski–Shlapentokh-Rothman [8], Finster–Kamran-Smoller–Yau [14], [15], Tataru–Tohaneanu [29] and Vasy [30] as well as references therein for an overview.

Let us make some comments on the similarities and differences between asymptotic completeness results and decay of the local energy:

- For a hyperbolic equation like the wave equation, the essential ingredients for asymptotic completeness are *minimal velocity estimates* stating that the energy in cones inside the light cone goes to zero. No precise rate is required, but no loss of derivatives is permitted in the estimates.

– *Energy estimates* are necessary for asymptotic completeness. One needs to estimate the energy at null infinity by the energy on a $t = 0$ slice and the energy at the time $t = 0$ slice by the energy at null infinity. Leaving aside the question of loss of derivatives, the first estimate can probably be deduced from the local energy decay estimates. For the other direction however a new argument is needed. Indeed, most of the local energy decay estimates use the redshift, which becomes a blueshift in the inverse sense of time (see [6]).
– The choice of coordinates has probably to be different. Whereas coordinates that extend smoothly across the event horizon are well adapted to the question of decay of local energy, they do not seem to be well adapted for showing asymptotic completeness results.

We refer to [27] for a more detailed discussion on the link between local energy decay and asymptotic completeness results.

In this paper we show asymptotic completeness results for the superradiant Klein–Gordon equation in a quite general setting. Our abstract Klein–Gordon operators have to be understood as operators acting on $\mathbb{R}_t \times \Sigma$, where Σ is a manifold with two ends, both asymptotically hyperbolic. Another important feature is that the operators are independent of t, which allows us to reduce the Klein–Gordon equation to the form (1.1) and to rewrite it as a first order evolution equation

$$\partial_t u = iHu, \quad u = (\phi, i^{-1}\partial_t\phi)$$

(see Subsect. 2.2). The generator H will be called the *Hamiltonian* in what follows. A more detailed description is given in Sect. 2.

We also impose the existence of "limit Hamiltonians" at the ends which can be realized as selfadjoint operators on a Hilbert space. In this setting the nonreal spectrum of the Hamiltonian consists of a finite number of complex eigenvalues with finite multiplicity, we can define a smooth functional calculus for the Hamiltonian, and the truncated resolvent can be extended meromorphically across the real axis. We show propagation estimates for initial data which in energy are supported outside so called singular points. These singular points are closely related to real resonances, but unlike the selfadjoint setting there may be singular points which are not real resonances.

From the propagation estimates it follows in particular that the evolution is uniformly bounded for data supported in energy outside the singular points. The same holds true for high energy data for which no superradiance appears.

We then apply our results to the De Sitter–Kerr metric with small angular momentum. We show asymptotic completeness for a fixed but arbitrary angular momentum n of the field. For $n \neq 0$ the absence of real resonances follows from the results of Dyatlov [11]. However, some additional work is required to show the absence of singular points. As usual for asymptotic completeness results and to simplify the exposition, we consider only the limit as $t \to \infty$. All the results in this paper also hold for the $t \to -\infty$ limit and the proofs are the same.

Our main example is the Klein–Gordon equation for the De Sitter–Kerr metric. Nevertheless we have found it appropriate to present several parts of this paper in more abstract settings. We have tried to present each type of results (definition of energy spaces, generator of the dynamics, meromorphic extension of the resolvent, propagation estimates etc.) in its natural generality. On the one hand, this permits one to better understand the mechanisms behind the proofs, on the other hand the method seems general enough to include other examples in the future. For example, the charged Klein–Gordon equation for the De Sitter–Reissner–Norström metric can certainly be handled by this method. For this equation there is also no positive conserved energy if the product of the charge of the black hole and the charge of the field is too big. However, the method does not seem to be general enough to treat the charged Klein–Gordon equation on the De Sitter–Kerr Newman metric. In this case the operator k has limits which depend on the angle θ and the limit operator does not commute with the equation, which is an essential feature of our construction. A summary of our results for a class of Klein–Gordon equations including the case of the De Sitter–Kerr spacetime is given in the next section.

Note added in proof. After this paper was completed, Dafermos, Rodnianski and Shlapentokh-Rothman [9] have shown existence and completeness of wave operators in the Kerr spacetime by quite different methods making use of their decay results in [8]. Our restriction to fixed angular momentum is dropped in [9].

1.2. Plan of the paper

- Sect. 2 contains a summary of the main results of this paper.
- In Sect. 3 we collect some results on general abstract Klein–Gordon equations and give some basic resolvent estimates. It turns out that already in this abstract setting superradiance can only occur at low frequencies, as expressed in the estimates of Lemma 3.6. This fact is already known for the Kerr metric (see e.g. Dafermos–Rodnianski [7]), but not in this spectral formulation. We also study *gauge transformations* in Sect. 3.5.3. In a more geometric language, they correspond to choices of different Killing fields.
- In Sect. 4 we recall some elements of *meromorphic Fredholm theory*. We show that a meromorphic extension of the truncated resolvent of h gives a meromorphic extension of the weighted resolvent of H.
- In Sect. 5 we describe the abstract setting for a Klein–Gordon operator on a manifold with two ends. Our assumptions ensure that the asymptotic Hamiltonians at the ends are selfadjoint. Gluing the resolvents of the asymptotic operators together gives the resolvent for H by using the Fredholm theory of Sect. 4. In this way we obtain resolvent estimates for the Hamiltonian H which are sufficient to construct a smooth functional calculus for H.
- In Sect. 6 we prove propagation estimates which are needed for the proof of the asymptotic completeness result. We also introduce the notion of *singular points*. Singular points are obstacles to uniform boundedness of the evolution and therefore also to asymptotic completeness. A useful criterion for the absence of singular points is given.
- In Sect. 7 we show uniform boundedness of the evolution for data which are spectrally supported outside the singular points.
- Asymptotic completeness is shown in the abstract setting in Sect. 8. The scattering space corresponds to data which are supported in energy outside the singular points.

– The geometric setting introduced in Sect. 2 is developed in Sect. 9. The main task is to check that the operators there fulfill the hypotheses of the abstract setting. In particular, existence of meromorphic extensions of weighted resolvents is deduced from a result of Mazzeo–Melrose [26].
– In Sect. 10 we apply our general result of Sect. 8 to the geometric setting and obtain an asymptotic completeness result in that setting.
– In Sect. 11 we describe the Klein–Gordon equation for the De Sitter–Kerr metric.
– In Sect. 12 the main results are formulated in the De Sitter–Kerr setting. Two types of results are established: comparison to spherically symmetric asymptotic dynamics on the same energy space and comparison to asymptotic profiles. These asymptotic profiles give rise to energy spaces which are bigger than the original ones. The wave operators can therefore only be defined as limits on dense subspaces. They then extend by continuity to the whole energy space for the profiles. Inverse wave operators exist on the whole energy space as limits.
– The proofs of the theorems in the De Sitter–Kerr setting are given in Sect. 13. We apply our earlier abstract theorems. To obtain the meromorphic extensions of the different truncated resolvents it is crucial that the cosmological constant is strictly positive. The absence of real resonances and complex eigenvalues follows from the work of Dyatlov [11] for a compactly supported cut-off resolvent. A hypoellipticity argument enables us to use an exponential weight. Our general criterion of Section 6 then yields the absence of singular points.

The paper contains a certain number of hypotheses. Hypotheses (A), (ME), (TE), (PE) and (B) are formulated in an abstract Hilbert space setting. The more concrete geometric hypotheses (G) imply hypotheses (A), (ME), (TE), (PE), and (B). Below we list the places where the different hypotheses are introduced.

- The geometric hypotheses (G) are introduced in Sect. 2.1.
- The abstract hypotheses (A) are introduced in Sect. 3.4.
- The hypotheses on meromorphic extensions (ME) are introduced in Sect. 4.2.
- The hypotheses on Klein–Gordon operators “with two ends” (TE) are introduced in Sects. 5.1 and 5.2.
- The additional hypothesis to obtain propagation estimates (PE) is introduced in Sect. 6.4.
- The additional hypothesis for boundedness (B) is introduced at the beginning of Sect. 7.

2. Summary of the results

In this section we will present a summary of the results of our paper about *concrete* Klein–Gordon equations. We introduce a class of Klein–Gordon equations which, while being more general than the Klein–Gordon equations on De Sitter–Kerr spacetimes, retain their essential features: invariance under time translations and axial rotations, and existence of an ergosphere. These Klein–Gordon equations fit into the general framework

$$(\partial_t^2 - 2ik\partial_t + h)\phi = 0, \tag{2.1}$$

where $\phi : \mathbb{R} \to \mathcal{H}$ and h, k are two selfadjoint operators on some Hilbert space $\mathcal{H}$, to which a large part of the paper is devoted.

2.1. *A class of Klein–Gordon equations*

We set $\mathcal{M} =]r_-, r_+[_r \times \mathbb{S}_\omega^{d-1}$. We consider Klein–Gordon equations on $\mathbb{R}_t \times \mathcal{M}$ of the form

$$(\partial_t^2 - 2ik\partial_t + h)\phi = 0, \tag{2.2}$$

where k, h are differential operators on $\mathcal{M}$ of order 1 and 2 respectively *independent of* t, and are perturbations of simpler *separable* operators, which we now introduce.

To measure the size of a perturbation, we set $q(r) := \sqrt{(r_+ - r)(r - r_-)}$ and define

$$T^\sigma = \{f \in C^\infty(\mathcal{M}) : \partial_r^\alpha \partial_\omega^\beta f \in \mathcal{O}(q(r)^{\sigma - 2\alpha})\}, \tag{2.3}$$

so that $f \in T^\sigma$ for $\sigma > 0$ vanishes at $r = r_\pm$. We also fix cut-off functions $i_\pm \in C^\infty([r_-, r_+])$ with $i_- = 0$ in a neighborhood of r_+, $i_+ = 0$ in a neighborhood of r_- and $i_-^2 + i_+^2 = 1$.

2.1.1. *Separable Klein–Gordon equations.*

We fix

$$P = \sum_{i,j=1}^{d-1} D_i^* \alpha_{ij}(\omega) D_j \geq 0,$$

a symmetric elliptic operator on $L^2(\mathbb{S}^{d-1}, d\omega)$, so that $(P, H^2(\mathbb{S}^{d-1}, d\omega))$ is selfadjoint. We assume that for a suitable choice of coordinate θ_1 (the azimuthal angle φ if $d = 3$), $L^2(\mathbb{S}^{d-1}, d\omega)$ has a basis of eigenfunctions of D_{θ_1}. Let Y^n be the eigenspace corresponding to the eigenvalue $n \in \mathbb{Z}$. Then we have

$$L^2(\mathbb{S}^{d-1}, d\omega) = \bigoplus_{n \in \mathbb{Z}} Y^n.$$

Our first assumption is

$$[P, D_{\theta_1}] = 0, \quad \text{i.e. the } \alpha_{ij} \text{ are independent of } \theta_1. \tag{G1}$$

We fix a second order differential operator on $\mathcal{M}$ of the form

$$h_{0,s} = \alpha_1 D_r \alpha_2^2 D_r \alpha_1 + \alpha_3^2 P + \alpha_4^2. \tag{2.4}$$

Here α_i, $1 \leq i \leq 4$, are smooth functions depending only on r. We suppose that there exist $\alpha_j^\pm \in \mathbb{R}$, $1 \leq j \leq 4$, such that for some $\delta > 0$,

$$\alpha_j - q(r)(i_-\alpha_j^- + i_+\alpha_j^+) \in T^{1+\delta}, \quad \alpha_j \gtrsim q(r). \tag{G2}$$

Note that (G2) implies

$$\alpha_j \in T^1, \quad \alpha_j \lesssim q(r). \tag{2.5}$$

We will also need a first order operator on $\mathcal{M}$ of the form

$$k_s = k_{s,r} D_{\theta_1} + k_{s,v}. \tag{2.6}$$

Here $k_{s,r}$ and $k_{s,v}$ are smooth functions depending only on r. We suppose that there exist $k_{s,v}^-$, $k_{s,r}^- \in \mathbb{R}$ such that for some $\delta > 0$,

$$\begin{cases} i_+ k_{s,r},\, i_+ k_{s,v} \in T^2, \\ i_-(k_{s,r} - k_{s,r}^-) \in T^2, \\ i_-(k_{s,v} - k_{s,v}^-) \in T^2. \end{cases} \tag{G3}$$

We set

$$h_s = h_{0,s} - k_s^2.$$

The associated separable Klein–Gordon equation is

$$(\partial_t^2 - 2ik_s\partial_t + h_s)\phi = 0. \tag{2.7}$$

2.1.2. Perturbed Klein–Gordon equations. We fix a perturbation h_0 of $h_{0,s}$ of the form

$$\begin{aligned} h_0|_{C_0^\infty(\mathcal{M})} &= h_{0,s} + \sum_{i,j=1}^{d-1} D_i^* g^{ij} D_j + \sum_{i=1}^{d-1} (g^i D_i + D_i^* \overline{g}^i) \\ &\quad + D_r g^{rr} D_r + g^r D_r + D_r \overline{g}^r + f \\ &=: h_{0,s} + h_p. \end{aligned} \tag{2.8}$$

We also fix a perturbation k of k_s of the form

$$k := k_s + (k_{p,r} D_{\theta_1} + k_{p,v}) =: k_s + k_p, \tag{2.9}$$

We assume

$$\text{the functions } g^{ij},\ g^i,\ g^{rr},\ g^r,\ f, k_{p,r},\ k_{p,v} \text{ are independent of } \theta_1, \tag{G4}$$

$$\begin{aligned} h_0 &\gtrsim \alpha_1(r)(D_r q^2(r) D_r + P + 1)\alpha_1(r), \\ h_{0,s} &\gtrsim \alpha_1(r)(D_r q^2(r) D_r + P + 1)\alpha_1(r), \end{aligned} \tag{G5}$$

where the notation $\gtrsim$ is explained in Subsect. 3.1.

The asymptotic behavior of the various coefficients of the perturbations h_p, k_p is assumed to be as follows:

$$\begin{aligned} &g^{ij} \in T^{2+\delta}, \quad g^{rr} \in T^{4+\delta}, \quad g^r \in T^{2+\delta}, \quad \text{for some } \delta > 0, \\ &g^i, k_{p,r}, k_{p,v}, f \in T^2. \end{aligned} \tag{G6}$$

We set

$$h := h_0 - k^2,$$

and consider the *perturbed* Klein–Gordon equation

$$(\partial_t^2 - 2ik\partial_t + h)\phi = 0. \tag{2.10}$$

We introduce the Hilbert spaces $\mathcal{H} = L^2(]r_-, r_+[_r \times \mathbb{S}^{d-1}_\omega; dr d\omega)$ and $\mathcal{H}^n = \mathcal{H} \cap \mathrm{Ker}(D_{\theta_1} - n)$. All operators have natural restrictions to the space $\mathcal{H}^n$, which we denote by a superscript n, for example h_0^n is the restriction of h_0 to $\mathcal{H}^n$. The operator k^n is bounded and $(h^n, D(h_0^n))$ is selfadjoint. Let $j_\pm \in C^\infty(]r_-, r_+[)$ be such that $j_- = 1$ close to r_-, $j_- = 0$ close to r_+, $j_+ = 1$ close to r_+, $j_+ = 0$ close to r_- and

$$j_\pm i_\pm = j_\pm, \quad i_+ j_- = i_- j_+ = 0.$$

Let $\ell := n k^-_{s,r} + k^-_{s,v}$ and

$$k_\pm := k \mp \ell j_\mp^2, \quad h_\pm := h_0 - k_\pm^2. \tag{2.11}$$

We also set

$$\tilde{h}_- := h_- + 2\ell k_- - \ell^2 = h_0 - (\ell - k_-)^2. \tag{2.12}$$

We require

$$\exists \ell \in \mathbb{R} \text{ such that } (h_+, k_+) \text{ and } (\tilde{h}_-, k_- - \ell) \text{ satisfy (G5) with } h_0 \text{ replaced by } h_+^n,\ \tilde{h}_-^n. \tag{G7}$$

Remark 2.1. Let us now make some comments on the various hypotheses (G): (G1) and (G4) express the axisymmetry of the problem, (G2), (G3), (G5) and (G6) the behavior at the two infinities $r = r_\pm$ of the various operators. In other words, h_0 tends to $h_{0,s}$ at $r = r_\pm$, while k tends to 0 at r_+ and to $k^-_{s,r}(r_-)D_{\theta_1} + k^-_{s,v}(r_-)$ at r_-. It is an essential feature of the problem that $k_{s,r}(r_\pm)$ and $k_{s,v}(r_\pm)$ are constants independent of θ. The positivity condition (G5) expresses the fact that the Cauchy surface $\{t = 0\}$ is spacelike. Finally, condition (G7) can be interpreted as follows. Whereas the operator h is not positive, it is positive near r_+. Further, h_+ is an operator which is equal to h near r_+ and differs from it near r_- so that it becomes positive. This change can be seen as only effected on k, we keep h_0 which is the good positive operator. The change of k gives a change of h via the formula $h = h_0 - k^2$. Condition (G7) ensures that the asymptotic operators so constructed give rise to a selfadjoint problem on an appropriate energy space. The situation near r_- seems at first glance a little different, but can be reduced to the situation near r_+ by considering $v = e^{-it\ell}u$ instead of u. The function v fulfills a Klein–Gordon equation, where h is replaced by $h_0 - (k - \ell)^2$ and k is replaced by $k - \ell$. We then apply this same procedure to the new operators $h_0 - (k - \ell)^2$, $k - \ell$. Rather than a change in the unknown function, this can also be seen as a change of the energy we consider. In the De Sitter–Kerr case this means that near r_- we consider an energy associated to $\partial_t + c\partial_\varphi$ for some appropriate c rather than ∂_t. We refer to Sect. 3.5.3 for details.

It will be shown in Sect. 11 that Klein–Gordon equations in De Sitter–Kerr spacetimes can be reduced to (2.10), with assumptions (G) satisfied, after some changes of unknown function.

2.2. *Energy spaces*

Set $\mathcal{H} = L^2(]r_-, r_+[\times \mathbb{S}^{d-1}, dr d\omega)$ and consider the *Cauchy problem* for (2.10):

$$\begin{cases} (\partial_t^2 - 2ik\partial_t + h)\phi = 0, \\ \phi|_{t=0} = u_0, \\ i^{-1}\partial_t\phi|_{t=0} = u_1, \end{cases} \tag{2.13}$$

for $\phi : \mathbb{R} \to \mathcal{H}$. The hyperbolic nature of the equation is expressed by the condition

$$h_0 := h - k^2 \geq 0.$$

Setting $u(t) = (\phi(t), i^{-1}\partial_t\phi(t))$, we can formally rewrite (2.13) as

$$u(t) = e^{itH}u, \quad u = (u_0, u_1),$$

for

$$H = \begin{pmatrix} 0 & \mathbb{1} \\ h & 2k \end{pmatrix}.$$

It is well known that there are two natural hermitian forms formally conserved by the evolution e^{itH}. The first is the *charge*:

$$q(u, u) = (u_1|u_0)_{\mathcal{H}} + (u_0|u_1)_{\mathcal{H}} - 2(u_0|ku_0)_{\mathcal{H}} \tag{2.14}$$

where $(\cdot|\cdot)_{\mathcal{H}}$ is the scalar product for the Hilbert space $\mathcal{H} = L^2(]r_-, r_+[_r \times \mathbb{S}^{d-1}_\omega, dr d\omega)$. The charge is of course related to the symplectic nature of Klein–Gordon equations.

The second is the *energy*

$$E(u, u) := (u_1|u_1)_{\mathcal{H}} + (u_0|hu_0)_{\mathcal{H}} \tag{2.15}$$

related in concrete models to the Killing vector field ∂_t. On De Sitter–Kerr spacetimes, there exists no global time-like Killing vector fields, which implies that $E(u, u)$ is not positive. Therefore E cannot be directly used to equip the space of Cauchy data with a topology.

A large part of our paper will be devoted to the proof of *resolvent estimates* for $(H - z)^{-1}$ when z approaches the real axis. The natural functional framework is as follows:

One defines the *homogeneous energy space* $\dot{\mathcal{E}}$ to be the completion of $C_0^\infty(\mathcal{M}) \oplus C_0^\infty(\mathcal{M})$ for the norm

$$\|u\|^2_{\dot{\mathcal{E}}} := \|u_1 - ku_0\|^2_{\mathcal{H}} + (u_0|h_0u_0)_{\mathcal{H}} \tag{2.16}$$

(see Subsect. 3.5). One can then show that the formal expression H has a natural meaning as a closed, densely defined operator on $\dot{\mathcal{E}}$, denoted by $\dot{H}$ (see Subsect. 3.5), which is the generator of a strongly continuous group on $\dot{\mathcal{E}}$, henceforth denoted by $e^{it\dot{H}}$ (see Subsect. 3.6).

One then denotes by $\dot{\mathcal{E}}^n$, $\dot{H}^n$ the energy spaces and generators associated to $\mathcal{H}^n$.

Remark 2.2. Let $v = e^{-ikt}u$. Then u is a solution of (2.1) if and only if v is a solution of

$$(\partial_t^2 + h(t))v = 0, \qquad h(t) = e^{-ikt}h_0 e^{ikt}, \qquad h_0 = h + k^2 \geq 0.$$

The natural energy for v is

$$\|\partial_t v\|^2 + (h(t)v|v).$$

Rewriting this energy for u gives (2.16).

2.3. *Propagation estimates*

Applying the abstract results of Sect. 5 to our concrete situation, we obtain the following intermediate results.

2.3.1. Smooth functional calculus. The operators $\dot{H}^n$ admit a *smooth functional calculus*, i.e. there exists a map

$$C_0^\infty(\mathbb{R}) \ni \chi \mapsto \chi(\dot{H}^n) \in \mathcal{B}(\dot{\mathcal{E}}^n)$$

which is a $*$-algebra morphism.

The above functional calculus extends trivially to the space $\mathbb{C}\mathbb{1} + C_0^\infty(\mathbb{R})$ of smooth functions constant near ∞ by setting $(\lambda + \chi)(\dot{H}^n) = \lambda\mathbb{1} + \chi(\dot{H}^n)$ for $\lambda \in \mathbb{C}$ and $\chi \in C_0^\infty(\mathbb{R})$.

Moreover if $\sigma_{\rm pp}^{\mathbb{C}}(\dot{H}^n) = \emptyset$ (i.e. $\dot{H}^n$ has no complex eigenvalues), then for $\chi \in C_0^\infty(\mathbb{R})$ with $\chi \equiv 1$ near 0 one has

$$\text{s-}\lim_{R\to\infty} \chi(R^{-1}\dot{H}^n) = \mathbb{1}$$

(see Prop. 5.11).

2.3.2. Boundedness of the evolution away from singular points. From Sect. 5 one also obtains the existence of a *bounded, closed, discrete set* $\mathcal{S}^n \subset \mathbb{R}$ such that if $\chi \in \mathbb{C}\mathbb{1} + C_0^\infty(\mathbb{R})$ vanishes near $\mathcal{S}^n$ then

$$\sup_{t\in\mathbb{R}} \|e^{it\dot{H}^n}\chi(\dot{H}^n)\|_{\mathcal{B}(\dot{\mathcal{E}}^n)} < \infty. \tag{2.17}$$

We call the elements of $\mathcal{S} = \bigcup_{n\in\mathbb{Z}} \mathcal{S}^n$ *singular points* for $\dot{H}$.

2.3.3. Propagation estimates away from singular points. Another consequence of the abstract results of Sect. 6 is *propagation estimates* for $e^{it\dot{H}^n}$ away from $\mathcal{S}$: if $\chi \in C_0^\infty(\mathbb{R}\backslash\mathcal{S})$ and $\epsilon > 0$ then

$$\int_{\mathbb{R}} \|q(r)^\epsilon e^{-it\dot{H}^n}\chi(\dot{H}^n)u\|^2_{\dot{\mathcal{E}}^n}\, dt \leq C_n\|u\|^2_{\dot{\mathcal{E}}^n}. \tag{2.18}$$

2.3.4. Absence of singular points for De Sitter–Kerr. One of the key intermediate results of our paper is Prop. 13.1, which asserts that for *De Sitter–Kerr* Klein–Gordon equations, $\mathcal{S}$ is actually *empty*, provided the blackhole angular momentum a is small enough. We deduce this from the work of Dyatlov [11].

2.4. Scattering theory

If we know that $\mathcal{S} = \emptyset$, the estimates (2.17) and (2.18) are sufficient to apply the general methods of *time-dependent scattering theory* to the (nonselfadjoint) operators $\dot{H}^n$, provided one first defines appropriate *comparison dynamics*.

Since $\mathcal{M}$ has two ends $r = r_\pm$, we need two comparison dynamics, generated by Hamiltonians $\dot{H}_{\pm\infty}$. For $l = l(n) = nk^-_{s,r} + k^-_{s,v} \in \mathbb{R}$ we set (see (G3))

$$h_{+\infty} := h_{0,s}, \quad h_{-\infty} := h_{+\infty} - \ell^2, \quad k_{+\infty} := 0, \quad k_{-\infty} := \ell.$$

The asymptotic Hamiltonians are then

$$\dot{H}_{+\infty} := \begin{pmatrix} 0 & \mathbb{1} \\ h_{+\infty} & 0 \end{pmatrix}, \quad \dot{H}_{-\infty} := \begin{pmatrix} 0 & \mathbb{1} \\ h_{-\infty} & 2\ell \end{pmatrix},$$

which are selfadjoint as operators on their natural energy spaces $\dot{\mathcal{E}}_{\pm\infty}$ (see Sect. 10). Moreover the cut-off maps $i_\pm$ introduced in Subsect. 2.1 map $\dot{\mathcal{E}}_{\pm\infty}$ into $\dot{\mathcal{E}}$ and $\dot{\mathcal{E}}$ into $\dot{\mathcal{E}}_{\pm\infty}$.

The main result of our paper is then the following theorem (see Thm. 10.5), which by 2.3.4 applies to Klein–Gordon equations on De Sitter–Kerr spacetimes for a small enough:

Theorem 2.3. *Assume hypotheses* (G) *and that $\mathcal{S} = \emptyset$. Then:*

(i) *For all $\varphi^\pm \in \dot{\mathcal{E}}_{\pm\infty}$ there exist $\psi^\pm \in \dot{\mathcal{E}}$ such that*

$$e^{-it\dot{H}^n}\psi^\pm - i_\pm e^{-it\dot{H}^n_{\pm\infty}}\varphi^\pm \to 0, \quad t \to \infty, \quad \text{in } \dot{\mathcal{E}}.$$

(ii) *For all $\psi^\pm \in \dot{\mathcal{E}}$ there exist $\varphi^\pm \in \dot{\mathcal{E}}_{\pm\infty}$ such that*

$$e^{-it\dot{H}^n_{\pm\infty}}\varphi^\pm - i_\pm e^{-it\dot{H}^n}\psi^\pm \to 0, \quad t \to \infty, \quad \text{in } \dot{\mathcal{E}}_{\pm\infty}.$$

Remark 2.4. (i) Theorem 2.3 is an asymptotic completeness result. We fix the angular momentum n. Then for every data in the asymptotic energy space we find data in the full energy space such that the difference between the asymptotic solution multiplied by a suitable cut-off and the full solution goes to zero (part (i)). Similarly for every data in the full energy space we find data in the asymptotic energy space such that an analogous difference goes to zero (part (ii)). Part (i) asserts the existence of direct wave operators, and (ii) the existence of inverse wave operators.

(ii) An important point in the theorem is that $\dot{H}^n_\pm$ are selfadjoint operators on their energy space. The associated dynamics can now be compared to even simpler asymptotic dynamics by the usual Hilbert space methods. We will illustrate this point in the concrete case of the De Sitter–Kerr metric.

2.5. The Klein–Gordon equation for the De Sitter–Kerr metric

We refer to Section 11 for an introduction to the De Sitter–Kerr metric and the Klein–Gordon equation associated to it. Let $\dot{H}^n$ and $\dot{\mathcal{E}}$ be the first order Klein–Gordon operator and homogeneous energy space associated to the Klein–Gordon equation with fixed angular momentum n for the De Sitter–Kerr metric with angular momentum a.

Remark 2.5 (Energy spaces in the De Sitter–Kerr case). In the De Sitter–Kerr case the energy $\|u\|_{\dot{\mathcal{E}}}$ follows the local rotation of the spacetime. To see this, first observe that in this case

$$\partial_t - ik = \frac{\nabla^a t}{(\nabla_b t \nabla^b t)^{1/2}} =: T,$$

and T is the four-velocity of a locally nonrotating observer. The energy $\|u\|_{\dot{\mathcal{E}}}$ is associated to T rather than ∂_t. We also observe that $k = \Omega D_\varphi$ and Ω has finite limits $\Omega_{-/+}$ when $r \to r_{\mp}$. Here r_- corresponds to the black hole horizon and r_+ to the cosmological horizon. These limits are called the angular velocities of the horizons. The Killing fields $\partial_t - \Omega_{-/+}\partial_\varphi$ on the De Sitter–Kerr metric are timelike close to the black hole $(-)$ resp. cosmological (+) horizon. Working with these Killing fields rather than with ∂_t leads to the conserved energies

$$\|u\|^2_{\tilde{\dot{\mathcal{E}}}_{-/+}} = \|(\partial_t - \Omega_{-/+}\partial_\varphi)u\|^2 + (h_0 - (k - \Omega_{-/+}D_\varphi)^2 u \mid u).$$

Note that in the limit $k \to \Omega_{-/+}D_\varphi$ the expressions of $\|u\|_{\dot{\mathcal{E}}}$ and $\|u\|_{\tilde{\dot{\mathcal{E}}}_{-/+}}$ coincide.

Our first result is

Theorem 2.6. *There exists $a_0 > 0$ such that for all $|a| < a_0$ and $n \in \mathbb{Z}$, there exists $C_n > 0$ such that*

$$\|e^{-it\dot{H}^n}u\|_{\dot{\mathcal{E}}^n} \le C_n\|u\|_{\dot{\mathcal{E}}^n}, \quad u \in \dot{\mathcal{E}}^n,\ t \in \mathbb{R}. \tag{2.19}$$

To describe our asymptotic completeness result we introduce a Regge–Wheeler type coordinate x. This change of coordinate gives rise to a change of the Hamiltonian, the Hilbert space $\mathcal{H}^n$ and the energy space. We denote the resulting Hamiltonian and spaces again by $\dot{H}^n$, $\mathcal{H}^n$ and $\dot{\mathcal{E}}^n$. We now introduce the Hamiltonians $\dot{H}_+$, $\dot{H}_-$ which describe the simplest possible asymptotic comparison dynamics. Let $\ell_\pm := \Omega_\pm n$ and

$$h^n_{+/-} := -\partial_x^2 - \ell^2_{+/-}, \quad k_{+/-} := \ell_{+/-},$$

acting on $\mathcal{H}^n$. We associate to these operators the natural homogeneous energy spaces $\dot{\mathcal{E}}^n_{-/+}$ and Hamiltonians $\dot{H}^n_{-/+}$.

Theorem 2.7. *There exists $a_0 > 0$ such that for all $|a| < a_0$ and $n \in \mathbb{Z}\setminus\{0\}$ the following holds:*

(i) *For all $u \in \mathcal{E}^{\text{fin},n}_{+/-}$ the limits*

$$W_{+/-}u := \lim_{t\to\infty} e^{it\dot{H}^n} i^2_{+/-} e^{-it\dot{H}^n_{+/-}} u$$

exist in $\dot{\mathcal{E}}^n$. The operators $W_{+/-}$ extend to bounded operators $W_{+/-} \in \mathcal{B}(\dot{\mathcal{E}}^n_{+/-}; \dot{\mathcal{E}}^n)$.

(ii) *The inverse wave operators*

$$\Omega_{+/-} := \text{s-}\lim_{t\to\infty} e^{it\dot{H}^n_{+/-}} i^2_{+/-} e^{-it\dot{H}^n}$$

exist in $\mathcal{B}(\dot{\mathcal{E}}^n; \dot{\mathcal{E}}^n_{+/-})$.

Statements (i) *and* (ii) *also hold for* $n = 0$ *if* $m > 0$.

Remark 2.8. (i) The limits in (i) cannot exist on the whole asymptotic homogeneous energy space. The spaces $\mathcal{E}^{\text{fin},n}_{+/-}$ are subspaces of the asymptotic homogeneous energy spaces. The elements of these energy spaces have a finite number of eigenmodes with respect to a certain elliptic reference operator (see Sect. 12.3 for details). This problem does not exist if one compares for example to a separable comparison dynamics (see Sect. 12.2).

(ii) The dynamics in the above theorem can be computed explicitly (see Sect. 12.3 for details).

(iii) Other comparison dynamics are natural, in particular in the massless case. In this case an interesting comparison dynamics is the one that pushes the first component along the flow of incoming principal null geodesics and the second component along the flow of outgoing principal null geodesics. In this case an asymptotic completeness result can be interpreted as an existence and uniqueness result for the characteristic Cauchy problem at infinity. We refer to [27] for the Schwarzschild case and to [23] for the Dirac equation for the Kerr–Newman metric.

(iv) Theorem 2.7 is a result for fixed angular momentum. It seems nevertheless to be a good starting point if one wants to establish a mathematically precise description of the Hawking effect for bosons in the De Sitter–Kerr setting. Indeed the chemical potential of the Hawking state will depend on the angular momentum (see [22]).

3. Background on abstract Klein–Gordon operators

3.1. Notation

- If X, Y are sets and $f : X \to Y$, we write $f : X \tilde{\to} Y$ if f is bijective. We use the same notation if X, Y are topological spaces and f is a homeomorphism.
- If $\mathcal{H}$ is a Banach space we denote by $\mathcal{H}^*$ its adjoint space, the set of continuous anti-linear functionals on $\mathcal{H}$ equipped with the natural Banach space structure. Thus the canonical anti-duality $\langle u, w\rangle$, where $u \in \mathcal{H}$ and $w \in \mathcal{H}^*$, is anti-linear in u and linear in w. In general we denote by $\langle\cdot|\cdot\rangle$ hermitian forms on $\mathcal{H}$, again anti-linear in the first argument and linear in the second one, but if $\mathcal{H}$ is a Hilbert space its scalar product is denoted by $(\cdot|\cdot)$.
- $\mathcal{B}(\mathcal{H})$ is the space of bounded operators on $\mathcal{H}$, and $\mathcal{B}_\infty(\mathcal{H})$ the subspace of compact operators.
- If S is a closed densely defined operator on a Banach space, then $D(S)$, $\rho(S)$, $\sigma(S)$ are its domain, resolvent set and spectrum. We use the notation $\langle S\rangle = (1 + S^2)^{1/2}$ if S is an operator for which this expression has a meaning, in particular if S is a real number.

– If S is a selfadjoint operator on a Hilbert space then $S > 0$ means $S \geq 0$ and $\operatorname{Ker} S = \{0\}$.
– If A, B are two selfadjoint operators or real numbers (possibly depending on some parameters), we write $A \lesssim B$ if $A \leq CB$ for some constant $C > 0$ (uniformly with respect to the parameters).

3.2. Scales of Hilbert spaces

Let $\mathcal{H}$ be a Hilbert space identified with its adjoint space $\mathcal{H}^* = \mathcal{H}$ via the Riesz isomorphism. If h is a selfadjoint operator on $\mathcal{H}$ we associate to it the *nonhomogeneous Sobolev spaces*

$$\langle h\rangle^{-s}\mathcal{H} := \operatorname{Dom}|h|^s, \quad \langle h\rangle^{s}\mathcal{H} := (\langle h\rangle^{-s}\mathcal{H})^*, \quad s \geq 0.$$

The spaces $\langle h\rangle^{-s}\mathcal{H}$ are equipped with the graph norm $\|\langle h\rangle^s u\|$. We keep the notation

$$(u|v), \quad u \in \langle h\rangle^{-s}\mathcal{H},\ v \in \langle h\rangle^{s}\mathcal{H},$$

for the duality bracket between $\langle h\rangle^{-s}\mathcal{H}$ and $\langle h\rangle^{s}\mathcal{H}$.

If $\operatorname{Ker} h = \{0\}$ then we also define the *homogeneous Sobolev space* $|h|^s\mathcal{H}$ equal to the completion of $\operatorname{Dom}|h|^{-s}$ for the norm $\|\,|h|^{-s}u\|$. The notation $\langle h\rangle^s\mathcal{H}$ or $|h|^s\mathcal{H}$ is convenient but somewhat ambiguous because usually $a\mathcal{H}$ denotes the image of $\mathcal{H}$ under the linear operator a. We refer to [18, Subsect. 2.1] for a complete discussion of this question.

Let us mention some properties of the scales of spaces defined above:

- $\langle h\rangle^{-s}\mathcal{H} \subset \langle h\rangle^{-t}\mathcal{H}$ if $t \leq s$, $\langle h\rangle^{-s}\mathcal{H} \subset |h|^{-s}\mathcal{H}$ and $|h|^{s}\mathcal{H} \subset \langle h\rangle^{s}\mathcal{H}$ if $s \geq 0$,
- $\langle h\rangle^{0}\mathcal{H} = |h|^{0}\mathcal{H} = \mathcal{H}$, $\langle h\rangle^{s}\mathcal{H} = (\langle h\rangle^{-s}\mathcal{H})^*$, $|h|^{s}\mathcal{H} = (|h|^{-s}\mathcal{H})^*$,
- $0 \in \rho(h) \Leftrightarrow \langle h\rangle^{s}\mathcal{H} = |h|^{s}\mathcal{H}$ for some $s \neq 0 \Leftrightarrow \langle h\rangle^{s}\mathcal{H} = |h|^{s}\mathcal{H}$ for all s,
- the operator $|h|^s$ is unitary from $|h|^{-t}\mathcal{H}$ to $|h|^{s-t}\mathcal{H}$ for all $s, t \in \mathbb{R}$.

3.3. Quadratic pencils

Let $\mathcal{H}$ be a Hilbert space, h a selfadjoint operator on $\mathcal{H}$, and $k \in \mathcal{B}(\mathcal{H})$ a *bounded* symmetric operator. Then $h_0 = h + k^2$ is a selfadjoint operator on $\mathcal{H}$ with the same domain as h, hence $\langle h\rangle^s\mathcal{H} = \langle h_0\rangle^s\mathcal{H}$ for $s \in [-1, 1]$. Thus the operators h and h_0 define the same scale of Sobolev spaces for $s \in [-1, 1]$, which we shall denote

$$\mathcal{H}^s := \langle h\rangle^{-s}\mathcal{H} = \langle h_0\rangle^{-s}\mathcal{H} \quad \text{if } -1 \leq s \leq 1.$$

We define the *quadratic pencil*

$$p(z) = h + z(2k - z) = h_0 - (k - z)^2, \quad z \in \mathbb{C}.$$

A priori these are operators on $\mathcal{H}$ with domain $\mathcal{H}^1$ and we clearly have $p(z)^* = p(\overline{z})$ as operators on $\mathcal{H}$. Moreover, for each $s \in [0, 1]$ they extend to operators in $\mathcal{B}(\mathcal{H}^s; \mathcal{H}^{s-1})$ and, for example, the relation $p(z)^* = p(\overline{z})$ holds as operators $\mathcal{H}^{1/2} \to \mathcal{H}^{-1/2}$. From this it is easy to deduce the following lemma (see [17, Lemma 8.1] for a more general result).

Lemma 3.1. *The following conditions are equivalent:*

(1) $p(z) : \mathcal{H}^1 \tilde{\to} \mathcal{H}$, (2) $p(\overline{z}) : \mathcal{H}^1 \tilde{\to} \mathcal{H}$,
(3) $p(z) : \mathcal{H}^{1/2} \tilde{\to} \mathcal{H}^{-1/2}$, (4) $p(\overline{z}) : \mathcal{H}^{1/2} \tilde{\to} \mathcal{H}^{-1/2}$,
(5) $p(z) : \mathcal{H} \tilde{\to} \mathcal{H}^{-1}$, (6) $p(\overline{z}) : \mathcal{H} \tilde{\to} \mathcal{H}^{-1}$.

In particular, the set

$$\rho(h, k) := \{z \in \mathbb{C} : p(z) : \mathcal{H}^{1/2} \tilde{\to} \mathcal{H}^{-1/2}\} = \{z \in \mathbb{C} : p(z) : \mathcal{H}^1 \tilde{\to} \mathcal{H}\} \tag{3.1}$$

is invariant under complex conjugation.

The next result is easy to prove in the present context; one can find a proof under more general conditions in [17, Lemma 8.2].

Proposition 3.2. *If h is bounded below then there exists $c_0 > 0$ such that*

$$\{z : |\mathrm{Im}\, z| > |\mathrm{Re}\, z| + c_0\} \subset \rho(h, k).$$

We shall now prove some estimates on $p^{-1}(z)$ for $z \in \rho(h, k)$. Note that they are valid under much more general assumptions on h and k than those imposed in this paper.

Lemma 3.3. *Assume that $h + c \geq 0$ for some $c \geq 0$ and let $b > 1$. If $z \in \rho(h, k)$ then*

$$\|p(z)^{-1}\| \leq \frac{b}{|z\ \mathrm{Im}\, z|} \quad \textit{if } |z|^2 \geq \frac{bc}{b-1}. \tag{3.2}$$

Proof. We abbreviate $p = p(z)$ and $\mu = \mathrm{Im}\, z$. The main point is the identity

$$\mathrm{Im}\, \frac{z}{\mu p} = \frac{1}{p^*}(h + |z|^2)\frac{1}{p}, \tag{3.3}$$

which is rather obvious:

$$\frac{z}{p} - \frac{\overline{z}}{p^*} = \frac{1}{p^*}(zp^* - \overline{z}p)\frac{1}{p} = (z - \overline{z})\frac{1}{p^*}(h + |z|^2)\frac{1}{p}.$$

Then (3.3) gives $(|z|^2 - c)\frac{1}{p^*}\frac{1}{p} \leq \mathrm{Im}\, \frac{z}{\mu p}$, hence

$$|\mu|(|z|^2 - c)\|p^{-1}u\|^2 \leq |\,\mathrm{Im}(u|zp^{-1}u)| \leq |z|\,\|u\|\,\|p^{-1}u\|,$$

hence $|\mu|(|z|^2 - c)\|p^{-1}u\|^2 \leq |z|\,\|u\|$, which is more than required. □

Lemma 3.4. *Assume $h_0 \geq 0$ and $k^2 \leq \alpha h_0 + \beta$ with $\alpha < 1$. Then h is bounded from below and if $h + c \geq 0$ and $\varepsilon > 0$ then there is a number C such that for $z \in \rho(h, k)$ and $|z| \geq \sqrt{c} + \varepsilon$,*

$$\|h_0^{1/2} p(z)^{-1}\| \leq C|\mathrm{Im}\, z|^{-1}. \tag{3.4}$$

Proof. If we set $q = p^{-1}$ then (3.3) implies $q^*((1-\alpha)h_0 + |z|^2)q \leq \beta q^* q + \mu^{-1}\,\mathrm{Im}\, zq$, hence

$$(1 - \alpha)\|h_0^{1/2} p^{-1}u\|^2 \leq \beta\|p^{-1}u\|^2 + |z/\mu|\,\|u\|\,\|p^{-1}u\| \leq \frac{\beta b^2}{|z\mu|^2}\|u\|^2 + \frac{b}{\mu^2}\|u\|^2$$

if $|z|^2 \geq \frac{bc}{b-1}$. This estimate is more precise than (3.4). Note that we may take $c = \beta$. □

3.4. Spaces

The operators h, k, h_0 and the spaces $\mathcal{H}^s$ are as in the preceding subsection, in particular k is a bounded operator in $\mathcal{H}$, but now we shall impose much stronger conditions.

From now on *we always assume* that

$$h_0 := h + k^2 > 0. \tag{A1}$$

Then the homogeneous scale $h_0^s\mathcal{H}$ associated to h_0 is well defined. Note that, if h is injective, the spaces $|h|^{-1}\mathcal{H}$ and $h_0^{-1}\mathcal{H}$ are quite different in general, although $\langle h\rangle^{-1}\mathcal{H} = \langle h_0\rangle^{-1}\mathcal{H}$.

We shall require k to behave well with respect to the homogeneous h_0-scale:

$$\begin{cases} k \in \mathcal{B}(h_0^{-1/2}\mathcal{H}); \\ \text{if } z \notin \mathbb{R} \text{ then } (k-z)^{-1} \in \mathcal{B}(h_0^{-1/2}\mathcal{H}) \text{ and} \\ \|(k-z)^{-1}\|_{\mathcal{B}(h_0^{-1/2}\mathcal{H})} \lesssim |\mathrm{Im}\, z|^{-n} \text{ for some } n > 0; \\ \text{there exists } m > 0 \text{ such that if } |z| \geq m\|k\|_{\mathcal{B}(\mathcal{H})} \text{ then} \\ \|(k-z)^{-1}\|_{\mathcal{B}(h_0^{-1/2}\mathcal{H})} \lesssim \big||z| - \|k\|_{\mathcal{B}(\mathcal{H})}\big|^{-1}. \end{cases} \tag{A2}$$

The next comments will clarify the meaning of these conditions. Recall that $h_0^{-s}\mathcal{H}$ and $h_0^s\mathcal{H}$ are adjoints to each other but they are not comparable, and neither are they comparable with $\mathcal{H}$. The first assumption says that the operator k leaves $D(h_0^{1/2})$ invariant and that its restriction to $D(h_0^{1/2})$ extends to a bounded operator, say $\overline{k}$, in $h_0^{-1/2}\mathcal{H}$. The rest of the assumption concerns the resolvent of $\overline{k}$ in this space. In order not to overcharge the notation we keep the notation k for $\overline{k}$.

The preceding assumptions allow us to get a new estimate on the quadratic pencil p.

Lemma 3.5. *Under conditions* (A1) *and* (A2)*, there are numbers* $C, M > 0$ *such that*

$$\|h_0^{1/2} p(z)^{-1}(k-z)u\| \leq C|\mathrm{Im}\, z|^{-1}\|h_0^{1/2}u\| \quad \text{if } |z| \geq M\|k\|_{\mathcal{B}(\mathcal{H})}. \tag{3.5}$$

Proof. We abbreviate $p = p(z)$ and $m = z - k$, so that $m^* = \overline{z} - k$ and $p = h_0 - m^2$. We have

$$\frac{z}{m}h_0\frac{1}{p}m - m^*\frac{1}{p^*}h_0\frac{\overline{z}}{m^*} = m^*\frac{1}{p^*}\left(p^*\frac{z}{|m|^2}h_0 - h_0\frac{\overline{z}}{|m|^2}p\right)\frac{1}{p}m.$$

If we replace here p by $h_0 - m^2$ and then develop and rearrange the terms, we get

$$m^*\frac{1}{p^*}\left((z-\overline{z})h_0\frac{1}{|m|^2}h_0 + h_0\frac{\overline{z}m}{m^*} - \frac{zm^*}{m}h_0\right)\frac{1}{p}m.$$

Since $\frac{m^*}{m} = 1 - \frac{z-\overline{z}}{m}$ and $\frac{z}{m} = 1 + \frac{k}{m}$, a simple computation gives

$$h_0\frac{\overline{z}m}{m^*} - \frac{zm^*}{m}h_0 = (z-\overline{z})\left(h_0 + h_0\frac{k}{m^*} + \frac{k}{m}h_0\right).$$

To conclude, we have proved, with $\mu = \operatorname{Im} z$,

$$\frac{1}{\mu}\operatorname{Im}\left(\frac{z}{m}h_0\frac{1}{p}m\right) = m^*\frac{1}{p^*}\left(h_0 + 2\operatorname{Re}\left(\frac{k}{m}h_0\right) + h_0\frac{1}{|m|^2}h_0\right)\frac{1}{p}m.$$

We may also write this as follows:

$$\begin{aligned}\frac{1}{\mu}\operatorname{Im}(u|zm^{-1}h_0p^{-1}mu) = \|h_0^{1/2}p^{-1}mu\|^2 + 2\operatorname{Re}(p^{-1}mu|km^{-1}h_0p^{-1}mu)\\ + \|m^{-1}h_0p^{-1}mu\|^2.\end{aligned}$$

Since the last term is positive, we get

$$\begin{aligned}\|h_0^{1/2}p^{-1}mu\|^2 &\le \frac{1}{\mu}\operatorname{Im}(u|zm^{-1}h_0p^{-1}mu) - 2\operatorname{Re}(p^{-1}mu|km^{-1}h_0p^{-1}mu)\\ &= \frac{1}{\mu}\operatorname{Im}(h_0^{1/2}u|h_0^{-1/2}zm^{-1}h_0^{1/2}\cdot h_0^{1/2}p^{-1}mu)\\ &\quad - 2\operatorname{Re}(h_0^{1/2}p^{-1}mu|h_0^{-1/2}km^{-1}h_0^{1/2}\cdot h_0^{1/2}p^{-1}mu).\end{aligned}$$

Set $a(z) = \|h_0^{-1/2}zm^{-1}h_0^{1/2}\|$ and $b(z) = 2\|h_0^{-\frac{1}{2}}km^{-1}h_0^{1/2}\|$. Since $zm^{-1} = 1 + km^{-1}$, assumption (A2) implies the boundedness of $a(z)$ for large z and $b(z) \to 0$ if $z \to \infty$. Finally, we have

$$(1 - b(z))\|h_0^{1/2}p^{-1}mu\| \le a(z)|\mu|^{-1}\|h_0^{1/2}u\|,$$

which proves the lemma. □

For easier reference later on, in the next proposition we summarize a particular case of the estimates we got in Lemmas 3.3–3.5.

Proposition 3.6. *Assume that conditions* (A1) *and* (A2) *are satisfied and let* $\varepsilon > 0$. *Then there are numbers* $C, M > 0$ *such that*

$$\|p^{-1}(z)\| \le C|z|^{-1}|\operatorname{Im} z|^{-1} \quad \text{if } |z| \ge (1+\epsilon)\|k\|_{\mathcal{B}(\mathcal{H})}, \tag{3.6}$$

$$\|h_0^{1/2}p^{-1}(z)\| \le C|\operatorname{Im} z|^{-1} \quad \text{if } |z| \ge (1+\epsilon)\|k\|_{\mathcal{B}(\mathcal{H})}, \tag{3.7}$$

$$\|h_0^{1/2}p^{-1}(z)(k-z)u\| \le C|\operatorname{Im} z|^{-1}\|h_0^{1/2}u\| \quad \text{if } |z| \ge M\|k\|_{\mathcal{B}(\mathcal{H})}. \tag{3.8}$$

Sometimes it is useful to consider also the homogeneous h-scale. The following assumption will be convenient in such situations:

$$h \ge ck^2 \quad \text{for some real } c > 0. \tag{A3}$$

This means that h is positive and $\|ku\| \le c^{-1/2}\|h^{1/2}u\|$ for all $u \in D(h^{1/2})$.

Lemma 3.7. *If* $c > 0$ *is real then* $h \ge ck^2 \Leftrightarrow h \ge \frac{c}{1+c}h_0$. *Thus* (A3) *is satisfied if and only if there is* $b > 0$ *real such that* $bh_0 \le h \le h_0$. *If* (A1) *and* (A3) *hold then* $h > 0$.

Proof. Note that $h = h_0 - k^2 \le h_0$. On the other hand, if $c > 0$ is real then we clearly have

$$h \ge ck^2 \;\Leftrightarrow\; h_0 \ge (1+c)k^2 \;\Leftrightarrow\; h \ge \frac{c}{1+c} h_0, \tag{3.9}$$

and this implies the assertions of the lemma. □

Corollary 3.8. *If* (A1) *and* (A3) *are satisfied then* $h_0^s \mathcal{H} = h^s \mathcal{H}$ *for all* $-1/2 \le s \le 1/2$.

Proof. Indeed, from $bh_0 \le h \le h_0$ we get $b^\theta h_0^\theta \le h^\theta \le h_0^\theta$ if $0 \le \theta \le 1$. □

3.4.1. Inhomogeneous energy spaces. The *inhomogeneous* energy space is the vector space

$$\mathcal{E} := \mathcal{H}^{1/2} \oplus \mathcal{H},$$

equipped with the natural direct sum topology which makes it a Hilbertizable space. For consistency with the norm that we introduce later in the homogeneous case, we take

$$\left\| \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right\|_{\mathcal{E}}^2 := \|u_1 - ku_0\|^2 + ((h_0 + 1)u_0 \mid u_0), \tag{3.10}$$

but of course we could replace here k by zero. It is convenient, as explained in [17], to identify its adjoint space $\mathcal{E}^*$ with $\mathcal{H} \oplus \mathcal{H}^{-1/2}$ the anti-duality being given by

$$\langle u, v \rangle := (u_0 \mid v_1 - kv_0) + (u_1 - ku_0 \mid v_0) \quad \text{if } u = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in \mathcal{E},\; v = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \in \mathcal{E}^*, \tag{3.11}$$

usually called the *charge*. Observe that $\mathcal{E} \subset \mathcal{E}^*$ continuously and densely. We identify $\mathcal{E}^{**} = \mathcal{E}$ as in the Hilbert space case by setting $\langle v, u \rangle = \overline{\langle u, v \rangle}$.

In what follows it will often be convenient to use the operator

$$\Phi(k) := \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}. \tag{3.12}$$

Note that $\Phi(k) : \mathcal{E} \xrightarrow{\sim} \mathcal{E}$ and $\Phi(k) : \mathcal{E}^* \xrightarrow{\sim} \mathcal{E}^*$ with $\Phi^{-1}(k) = \Phi(-k)$ and we may write

$$\mathcal{E} = \Phi(k)(\langle h_0 \rangle^{-1/2}\mathcal{H} \oplus \mathcal{H}) \quad \text{and} \quad \mathcal{E}^* = \Phi(k)(\mathcal{H} \oplus \langle h_0 \rangle^{1/2}\mathcal{H}), \tag{3.13}$$

which explains the choice of the norm in (3.10) and makes the connection with (3.15).

3.4.2. Homogeneous energy spaces. We define the *homogeneous energy space* $\dot{\mathcal{E}}$ as the completion of $\mathcal{E}$ under the norm defined by

$$\left\| \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right\|_{\dot{\mathcal{E}}}^2 := \|u_1 - ku_0\|^2 + (h_0 u_0 | u_0). \tag{3.14}$$

The completion is the set of couples $u = \binom{u_0}{u_1}$ with $u_0 \in h_0^{-1/2}\mathcal{H}$, $u_1 \in (1 + h_0^{-1/2})\mathcal{H}$, and such that $u_1 - ku_0 \in \mathcal{H}$. We shall realize its adjoint space $\dot{\mathcal{E}}^*$ with the help of the charge anti-duality defined as in (3.11). Observe that since $k \in \mathcal{B}(h_0^{-1/2}\mathcal{H})$ by (A2), we also have

$$\dot{\mathcal{E}} = \Phi(k)(h_0^{-1/2}\mathcal{H} \oplus \mathcal{H}), \quad \dot{\mathcal{E}}^* = \Phi(k)(\mathcal{H} \oplus h_0^{1/2}\mathcal{H}). \tag{3.15}$$

If assumption (A3) is satisfied then we also define the h-homogeneous energy spaces

$$\dot{\dot{\mathcal{E}}} := h^{-1/2}\mathcal{H} \oplus \mathcal{H}, \quad \dot{\dot{\mathcal{E}}}^* := \mathcal{H} \oplus h^{1/2}\mathcal{H}.$$

Here the direct sums are in the Hilbert space sense, and the identification of $\dot{\dot{\mathcal{E}}}^*$ with the space adjoint to $\dot{\dot{\mathcal{E}}}$ is done with the help of the sesquilinear form defined as in (3.11) but with $k = 0$.

Lemma 3.9. *Assume (A1)–(A3). Then $\dot{\mathcal{E}} = \dot{\dot{\mathcal{E}}}$ and the norms $\|\cdot\|_{\dot{\mathcal{E}}}$ and $\|\cdot\|_{\dot{\dot{\mathcal{E}}}}$ are equivalent.*

Proof. We have to prove that $\|u_1 - ku_0\|^2 + (h_0u_0|u_0) \simeq \|u_1\|^2 + (hu_0|u_0)$. But this is obvious by (A3) and Lemma 3.7. □

3.4.3. Conserved quantities. On $\mathcal{E}$ we introduce for $\ell \in \mathbb{R}$ the hermitian forms

$$\langle u|u\rangle_\ell := (u_1 - \ell u_0 \mid u_1 - \ell u_0) + (p(\ell)u_0 \mid u_0), \tag{3.16}$$

where $p(\ell) := h_0 - (k - \ell)^2$. If $u = (\phi, i^{-1}\partial_t\phi)$ and ϕ is a solution of the Klein–Gordon equation (2.1) then these forms are formally conserved. Indeed, we compute

$$\begin{aligned}
\frac{d}{dt}\langle u|u\rangle_\ell &= \frac{d}{dt}\big((\partial_t\phi - i\ell\phi \mid \partial_t\phi - i\ell\phi) + ((h + 2k\ell - \ell^2)\phi \mid \phi)\big) \\
&= 2\,\mathrm{Re}(\partial_t^2\phi - i\ell\partial_t\phi \mid \partial_t\phi - i\ell\phi) + 2\,\mathrm{Re}((h + 2k\ell - \ell^2)\phi \mid \partial_t\phi) \\
&= 2\,\mathrm{Re}(i(2k - \ell)\partial_t\phi - h\phi \mid \partial_t\phi - i\ell\phi) + 2\,\mathrm{Re}((h + 2k\ell - \ell^2)\phi \mid \partial_t\phi) \\
&= -2\,\mathrm{Re}((2k - \ell)\partial_t\phi \mid \ell\phi) + 2\,\mathrm{Re}((2k\ell - \ell^2)\phi \mid \partial_t\phi) = 0.
\end{aligned}$$

These forms are however in general not positive.

Lemma 3.10. *For all $\ell \in \mathbb{R}$, $\langle\cdot|\cdot\rangle_\ell$ is continuous with respect to the norm $\|\cdot\|_{\mathcal{E}}$.*

Proof. Due to the polarization identity it suffices to show $|\langle u|u\rangle_\ell| \lesssim \|u\|_{\mathcal{E}}^2$ for all $u \in \mathcal{E}$. Since

$$\langle u|u\rangle_\ell = \|u_1 - \ell u_0\|^2 + \|h_0^{1/2}u_0\|^2 - \|(k - \ell)u_0\|^2 \tag{3.17}$$

and k is bounded, this is obvious. □

Lemma 3.11. *For all $\ell \in \mathbb{R}$, $\langle\cdot|\cdot\rangle_\ell$ is continuous with respect to the norm $\|\cdot\|_{\dot{\mathcal{E}}}$ if and only if*

$$h_0 \gtrsim (k - \ell)^2. \tag{3.18}$$

Proof. We have $\|u\|^2_{\dot{\mathcal{E}}} = \|u_1 - ku_0\|^2 + \|h_0^{1/2}u_0\|^2$ and we have to decide when $|\langle u|u\rangle_\ell| \lesssim \|u\|^2_{\dot{\mathcal{E}}}$. By (3.17) this holds if and only if there is a number c such that

$$\big|\|u_1 - \ell u_0\|^2 - \|(k-\ell)u_0\|^2\big| \le c\|u_1 - ku_0\|^2 + c\|h_0^{1/2}u_0\|^2.$$

If this holds, let $u_1 = ku_0 + \varepsilon(k-\ell)u_0$ with $\varepsilon > 0$. Then $a\|(k-\ell)u_0\|^2 \le c\|h_0^{1/2}u_0\|^2$ with $a = 2\varepsilon + (1-c)\varepsilon^2$ and $a > 0$ for small ε, hence (3.18) is satisfied. The converse is obvious. □

3.5. *Energy Klein–Gordon operators*

Let

$$\hat{H} := \begin{pmatrix} 0 & 1 \\ h & 2k \end{pmatrix} = \Phi(k)\hat{K}\Phi^{-1}(k) \quad \text{where} \quad \hat{K} := \begin{pmatrix} k & 1 \\ h_0 & k \end{pmatrix}. \tag{3.19}$$

The *energy Klein–Gordon operators* will be various realizations of $\hat{H}$. The operator $\hat{K}$ is a *charge Klein–Gordon operator* and will only play a technical role.

3.5.1. *Klein–Gordon operator on the inhomogeneous energy space.*

The *inhomogeneous Klein–Gordon operator* is the operator H induced by $\hat{H}$ on $\mathcal{E}$. This means that its domain is

$$D(H) := \{u \in \mathcal{E} : \hat{H}u \in \mathcal{E}\} = \mathcal{H}^1 \oplus \mathcal{H}^{1/2},$$

and for $u \in D(H)$ we have $Hu = \hat{H}u$. For the second equality above, see [18, Sect. 5.2]. We also recall [18, Prop. 5.3]:

Proposition 3.12. *Assume* (A1) *and* (A2).

- *One has* $\rho(H) = \rho(h,k)$.
- *In particular, if* $\rho(h,k) \ne \emptyset$ *then* H *is a closed densely defined operator in* $\mathcal{E}$ *and its spectrum is invariant under complex conjugation.*
- *If* $z \in \rho(h,k)$, *then*

$$R(z) := (H-z)^{-1} = p^{-1}(z)\begin{pmatrix} z-2k & 1 \\ h & z \end{pmatrix}. \tag{3.20}$$

We may similarly define the operator K induced by $\hat{K}$ in $\mathcal{E}$ and one may easily check, under the same conditions (A1) and (A2), that $\Phi(k) : \mathcal{H}^1 \oplus \mathcal{H}^{1/2} \xrightarrow{\sim} \mathcal{H}^1 \oplus \mathcal{H}^{1/2}$ with inverse $\Phi(-k)$, hence H and K have the same domain and $H = \Phi(k)K\Phi(-k)$. This implies

$$(K-z)^{-1} = \begin{pmatrix} p^{-1}(z)(z-k) & p^{-1}(z) \\ 1 + (z-k)p^{-1}(z)(z-k) & (z-k)p^{-1}(z) \end{pmatrix}. \tag{3.21}$$

3.5.2. Klein–Gordon operator on the homogeneous energy space. The *homogeneous Klein–Gordon operator* is the operator $\dot{H}$ induced by $\hat{H}$ on $\dot{\mathcal{E}}$. This means that its domain is

$$D(\dot{H}) = \{u \in \dot{\mathcal{E}} : \hat{H}u \in \dot{\mathcal{E}}\}$$

and for $u \in D(\dot{H})$ we have $\dot{H}u = \hat{H}u$. The proofs will involve the homogeneous operator $\dot{K}$ associated to the auxiliary operator $\hat{K}$ and acting in the space $h_0^{-1/2}\mathcal{H} \oplus \mathcal{H}$ with domain

$$D(\dot{K}) = \{v \in h_0^{-1/2}\mathcal{H} \oplus \mathcal{H} : \hat{K}v \in h_0^{-1/2}\mathcal{H} \oplus \mathcal{H}\}.$$

From (3.15) and (3.19) we see that $\Phi(k)$ induces an isomorphism of $h_0^{-1/2}\mathcal{H} \oplus \mathcal{H}$ with $\dot{\mathcal{E}}$ whose inverse is $\Phi(-k)$. Clearly then $\dot{H} = \Phi(k)\dot{K}\Phi(-k)$.

Lemma 3.13. *Under conditions* (A1) *and* (A2) *we have*

$$D(\dot{H}) = \Phi(k)\big((h_0^{-1/2}\mathcal{H} \cap h_0^{-1}\mathcal{H}) \oplus \mathcal{H}^{1/2}\big).$$

Proof. From the preceding comments we see that the assertion of the lemma is equivalent to

$$D(\dot{K}) = (h_0^{-1/2}\mathcal{H} \cap h_0^{-1}\mathcal{H}) \oplus \mathcal{H}^{1/2} \tag{3.22}$$

Since $\hat{K}v = \binom{kv_0+v_1}{h_0v_0+kv_1}$, if v belongs to the right hand side above then $kv_0 + v_1 \in h_0^{-1/2}\mathcal{H}$ and $h_0v_0 + kv_1 \in \mathcal{H}$, thus $\hat{K}v \in h_0^{-1/2}\mathcal{H} \oplus \mathcal{H}$, hence $v \in D(\dot{K})$. Conversely, if $v \in D(\dot{K})$ then

$$v_0 \in h_0^{-1/2}\mathcal{H}, \quad v_1 \in \mathcal{H}, \quad kv_0 + v_1 \in h_0^{-1/2}\mathcal{H}, \quad h_0v_0 + kv_1 \in \mathcal{H}.$$

We have to show $v_0 \in h_0^{-1/2}\mathcal{H} \cap h_0^{-1}\mathcal{H}$ and $v_1 \in \mathcal{H}^{1/2}$, which follow from $v_0 \in h_0^{-1}\mathcal{H}$ and $v_1 \in h_0^{-1/2}\mathcal{H}$. The last relation is a consequence of $kv_0 + v_1 \in h_0^{-1/2}\mathcal{H}$ because $k \in \mathcal{B}(h_0^{-1/2}\mathcal{H})$. Since k is bounded on $\mathcal{H}$, we finally get $h_0v_0 \in \mathcal{H} - kv_1 \subset \mathcal{H}$, hence $v_0 \in h_0^{-1}\mathcal{H}$. □

Lemma 3.14. *Assume that conditions* (A1) *and* (A2) *are satisfied and let* $z \in \rho(h,k)\backslash\mathbb{R}$. *Then the maps* $p(z)^{-1}$ *and* $p(z)^{-1}h_0$ *induce continuous operators* $h_0^{-1/2}\mathcal{H} \to h_0^{-1/2}\mathcal{H} \cap h_0^{-1}\mathcal{H}$ *and* $h_0^{-1/2}\mathcal{H} \to \mathcal{H}^{1/2}$ *respectively.*

Proof. We set $m = z - k$ and, to simplify the writing, we do not specify z unless this is really necessary, e.g. we write p for $p(z)$ and $p = h_0 - m^2$. From (A2) it follows that m induces bounded invertible operators in all the spaces $\mathcal{H}^s$ with $-1/2 \le s \le 1/2$ and in the space $h_0^{-1/2}\mathcal{H}$ (in all $h_0^s\mathcal{H}$ with $-1/2 \le s \le 1/2$, in fact). Since h_0 extends to a unitary operator $h_0^{-1/2}\mathcal{H} \to h_0^{1/2}\mathcal{H}$ and $h_0^{1/2}\mathcal{H}$ is a dense subspace of $\mathcal{H}^{-1/2}$, the operator $p^{-1}h_0$ extends to a bounded map $p^{-1}h_0 : h_0^{-1/2}\mathcal{H} \to \mathcal{H}^{1/2}$. Then we write

$$p^{-1} = p^{-1}(m^2 - h_0 + h_0)m^{-2} = p^{-1}h_0m^{-2} - m^{-2},$$

from which it follows that p^{-1} extends to an operator in $\mathcal{B}(h_0^{-1/2}\mathcal{H})$. We still have to prove that p^{-1} sends $h_0^{-1/2}\mathcal{H}$ into $h_0^{-1}\mathcal{H}$. For this we note that

$$h_0 p^{-1} = (h_0 - m^2 + m^2)p^{-1} = 1 + m^2 p^{-1},$$

and thus, by what we have just proved, we see that $h_0 p^{-1} h_0^{-1/2}\mathcal{H} \subset h_0^{-1/2}\mathcal{H}$, hence p^{-1} sends $h_0^{-1/2}\mathcal{H}$ into $h_0^{-3/2}\mathcal{H} \cap h_0^{-1/2}\mathcal{H} \subset h_0^{-1}\mathcal{H}$, which clearly proves the assertion. □

Proposition 3.15. *If* (A1) *and* (A2) *are true then* $\rho(\dot{H}) \setminus \mathbb{R} = \rho(h,k) \setminus \mathbb{R}$, *and for* z *in this set,*

$$\dot{R}(z) := (\dot{H} - z)^{-1} = \Phi(k) \begin{pmatrix} p^{-1}(z)(z-k) & p^{-1}(z) \\ 1 + (z-k)p^{-1}(z)(z-k) & (z-k)p^{-1}(z) \end{pmatrix} \Phi(-k). \tag{3.23}$$

Proof. As in the proof of Lemma 3.13, we prove the corresponding statement for the operator $\dot{K}$. Fix $z \in \rho(h,k) \setminus \mathbb{R}$ and adopt the notation of the proof of Lemma 3.14. We show that $z \in \rho(\dot{K})$ and that $(\dot{K} - z)^{-1}$ is just the matrix in (3.23) or in (3.21):

$$(\dot{K} - z)^{-1} = \begin{pmatrix} p^{-1}m & p^{-1} \\ 1 + mp^{-1}m & mp^{-1} \end{pmatrix}. \tag{3.24}$$

We denote by S the matrix on the right hand side of (3.24) and first show that S sends $h_0^{-1/2}\mathcal{H} \oplus \mathcal{H}$ into $D(\dot{K})$ as defined in (3.22). Thus, if $v_0 \in h_0^{-1/2}\mathcal{H}$ and $v_1 \in \mathcal{H}$ we must prove that

$$p^{-1}mv_0 + p^{-1}v_1 \in h_0^{-1/2}\mathcal{H} \cap h_0^{-1}\mathcal{H} \quad \text{and} \quad (1 + mp^{-1}m)v_0 + mp^{-1}v_1 \in \mathcal{H}^{1/2}. \tag{3.25}$$

From Lemma 3.1 we get $p^{-1}v_1 \in \mathcal{H}^1 \subset h_0^{-1/2}\mathcal{H} \cap h_0^{-1}\mathcal{H}$, hence also $mp^{-1}v_1 \in \mathcal{H}^{1/2}$. Thus it remains to treat the terms involving v_0. From Lemma 3.14 we get $p^{-1}mv_0 \in h_0^{-1/2}\mathcal{H} \cap h_0^{-1}\mathcal{H}$. On the other hand, since $p = h_0 - m^2$, Lemma 3.1 and a simple computation give

$$1 + mp^{-1}m = mp^{-1}h_0 m^{-1} \tag{3.26}$$

in the sense of bounded operators $\mathcal{H}^{-1/2} \to \mathcal{H}^{1/2}$. From this relation and Lemma 3.14 we get $(1 + mp^{-1}m)v_0 \in \mathcal{H}^{1/2}$.

Thus $S : h_0^{-1/2}\mathcal{H} \oplus \mathcal{H} \to D(\dot{K})$ and a straightforward computation gives $(\dot{K} - z)Sv = v$ for all $v \in h_0^{-1/2}\mathcal{H} \oplus \mathcal{H}$. On the other hand, if $u \in D(\dot{K})$ and $v = (\dot{K} - z)u$ then it is easy to show that $u = Sv$. This finishes the proof of the relation $S = (\dot{K} - z)^{-1}$, i.e. of (3.24).

It remains to show that $\rho(\dot{K}) \setminus \mathbb{R} \subset \rho(h,k)$. Assume that $z \notin \mathbb{R}$ and

$$\dot{K} - z : (h_0^{-1/2}\mathcal{H} \cap h_0^{-1}\mathcal{H}) \oplus \mathcal{H}^{1/2} \to h_0^{-1/2}\mathcal{H} \oplus \mathcal{H}$$

is bijective. Then for any $v = \binom{0}{v_1} \in h_0^{-1/2}\mathcal{H} \oplus \mathcal{H}$ there is a unique $u = \binom{u_0}{u_1}$ with $u_0 \in h_0^{-1/2}\mathcal{H} \cap h_0^{-1}\mathcal{H}$ and $u_1 \in \mathcal{H}^{1/2}$ such that $-mu_0 + u_1 = 0$ and $h_0u_0 - mu_1 = v_1$. Then

$u_0 = m^{-1}u_1 \in \mathcal{H}^{1/2}$ and also $u_0 \in h_0^{-1}\mathcal{H}$, hence $u \in \mathcal{H}^1$ and $pu_0 = (h_0 - m^2)u_0 = v_1$. Thus $p : \mathcal{H}^1 \to \mathcal{H}$ is surjective. It is also injective, because if $pu_0 = 0$ for some $u_0 \in \mathcal{H}^1$ then $u = \binom{u_0}{mu_0} \in D(\dot{K})$ and $(\dot{K} - z)u = 0$, hence $u = 0$. □

We point out a simple relation between H and $\dot{H}$ (a similar statement holds for K and $\dot{K}$). Recall that $\mathcal{E} \subset \dot{\mathcal{E}}$ continuously and densely.

Lemma 3.16. *$\dot{H}$ coincides with the closure of H in $\dot{\mathcal{E}}$.*

Proof. If $z \in \rho(h, k) \setminus \mathbb{R}$ then z belongs to $\rho(H) \cap \rho(\dot{H})$ and the resolvents $R := (H - z)^{-1}$ and $\dot{R} = (\dot{H} - z)^{-1}$ are bounded operators in $\mathcal{E}$ and $\dot{\mathcal{E}}$ respectively. Moreover, $\dot{R}$ is clearly a continuous extension of R to $\dot{\mathcal{E}}$, so it is the closure of R in $\dot{\mathcal{E}}$. By thinking in terms of graphs one easily sees that $\dot{H} - z = \dot{R}^{-1}$ is the closure of $H - z = R^{-1}$ in $\dot{\mathcal{E}}$. □

We will often consider the case where (A3) is fulfilled. In this case

$$D(\dot{H}) = (h^{-1/2}\mathcal{H} \cap h^{-1}\mathcal{H}) \oplus \mathcal{H}^{1/2}$$

and $\dot{H}$ is selfadjoint (see e.g. [21, Lemme 2.1.1]). Note also that the resolvent of $\dot{H}$ is then given by (see [18, Prop. 5.7])

$$\dot{R}(z) = \begin{pmatrix} z^{-1}p^{-1}(z)h - z^{-1} & p^{-1}(z) \\ p^{-1}(z)h & zp^{-1}(z) \end{pmatrix}. \tag{3.27}$$

Moreover, if we assume (A3), then $\|\dot{R}(z)\|_{\mathcal{B}(\dot{\mathcal{E}})} \le |\mathrm{Im}\, z|^{-1}$. Using [18, Prop. 5.10] we obtain the following resolvent estimate for H:

Proposition 3.17. *Assume* (A1)–(A2). *Then*

$$\|R(z)\|_{\mathcal{B}(\mathcal{E})} \lesssim (1 + |z|^{-1})\|\dot{R}(z)\|_{\mathcal{B}(\dot{\mathcal{E}})} + |z|^{-1}. \tag{3.28}$$

Assume in addition (A3). *Then*

$$\|R(z)\|_{\mathcal{B}(\mathcal{E})} \lesssim (1 + |z|^{-1})|\mathrm{Im}\, z|^{-1}. \tag{3.29}$$

3.5.3. Gauge transformations. Let us recall that our starting point was the Klein–Gordon equation

$$(\partial_t - ik)^2 u + h_0 u = 0. \tag{3.30}$$

If u is a solution of (3.30) and $\ell \in \mathbb{R}$, then $v = e^{-it\ell}u$ solves

$$(\partial_t - i(k - \ell))^2 v + h_0 v = 0. \tag{3.31}$$

Let us formulate this in terms of generators: if

$$\Phi(\ell)H\Phi^{-1}(\ell) =: H_\ell + \ell,$$

then

$$H_\ell = \begin{pmatrix} 0 & 1 \\ p(\ell) & 2(k - \ell) \end{pmatrix}, \qquad p(\ell) = h_0 - (k - \ell)^2.$$

It follows that if there exists $\ell \in \mathbb{R}$ such that (A3) is fulfilled with h replaced by $p(\ell)$ and k by $k - \ell$, then $\dot{H}$ is selfadjoint on the homogeneous energy space

$$\dot{\mathcal{E}} = \Phi(\ell)(p(\ell)^{-1/2}\mathcal{H} \oplus \mathcal{H}).$$

3.6. Existence of the dynamics

From [17, Cor. 8.6] we obtain:

Lemma 3.18. *H is the generator of a C_0-group e^{-itH} on $\mathcal{E}$.*

Now we show that e^{-itH} extends to a C_0-group on $\dot{\mathcal{E}}$.

Lemma 3.19. *$\dot{H}$ is the generator of a C_0-group on $\dot{\mathcal{E}}$ and for each real t the operator $e^{-it\dot{H}}$ coincides with the continuous extension of e^{-itH} to $\dot{\mathcal{E}}$.*

Proof. We start by proving that for some constants $C, \omega > 0$ we have

$$\|e^{-itH}\varphi\|_{\dot{\mathcal{E}}} \le Ce^{\omega|t|}\|\varphi\|_{\dot{\mathcal{E}}} \quad \forall\varphi \in \mathcal{E}. \tag{3.32}$$

Let first $\varphi \in D(H)$. We compute, by using (3.14) for $u = (u_0, u_1) = e^{-itH}\varphi$,

$$\begin{aligned}\frac{d}{dt}\|u\|^2_{\dot{\mathcal{E}}} &= 2\,\mathrm{Re}(ihu_0 + iku_1 \mid u_1 - ku_0) + 2\,\mathrm{Re}(h_0u_0|iu_1)\\ &= ([ik, h]u_0 \mid u_0) \lesssim (h_0u_0|u_0) \lesssim \|u\|^2_{\dot{\mathcal{E}}}.\end{aligned}$$

The inequality (3.32) then follows for $\varphi \in D(H)$ by Gronwall's lemma and for $\varphi \in \mathcal{E}$ by density. From (3.32) we see that e^{-itH} extends to a continuous operator V_t on $\dot{\mathcal{E}}$ such that $\|V_t\|_{\dot{\mathcal{E}}} \le Ce^{\omega|t|}$. This clearly implies that V_t is a C_0-group on $\dot{\mathcal{E}}$, and from Nelson's invariant domain theorem it follows that its generator is the closure of H in $\dot{\mathcal{E}}$, which by Lemma 3.16 is just $\dot{H}$. □

4. Meromorphic extensions

In this section we discuss various facts related to meromorphic extensions of quadratic pencils.

4.1. Background and definitions

Definition 4.1. Let $\mathcal{H}$ be a Hilbert space. For $z_0 \in \mathbb{C}$, let $\mathcal{U}$ be a neighborhood of z_0, and let $F : \mathcal{U} \setminus \{z_0\} \to \mathcal{B}(\mathcal{H})$ be a holomorphic function. We say that F is *finite meromorphic* at z_0 if the Laurent expansion of F at z_0 has the form

$$F(z) = \sum_{n=m}^{\infty}(z - z_0)^n A_n, \qquad m > -\infty,$$

the operators $A_m, \dots, A_{-1}$ being of finite rank if $m < 0$. If, in addition, A_0 is a Fredholm operator, then F is called *Fredholm at z_0*.

We will need the following fact (cf. [19, Prop. 4.1.4]):

Proposition 4.2. *Let $\mathcal{D} \subset \mathbb{C}$ be a connected open set, let $Z \subset \mathcal{D}$ be a discrete and closed subset of $\mathcal{D}$, and let $F : \mathcal{D} \setminus Z \to \mathcal{B}(\mathcal{H})$ be a holomorphic function. Assume that*

– *F is finite meromorphic and Fredholm at each point of $\mathcal{D}$;*
– *there exists $z_0 \in \mathcal{D} \setminus Z$ such that $F(z_0)$ is invertible.*

Then there exists a discrete closed subset Z' of $\mathcal{D}$ such that $Z \subset Z'$ and

- *$F(z)$ is invertible for each $z \in \mathcal{D} \setminus Z'$;*
- *$F^{-1} : \mathcal{D} \setminus Z' \to \mathcal{B}(\mathcal{H})$ is finite meromorphic and Fredholm at each point of $\mathcal{D}$.*

4.2. Meromorphic extensions of weighted resolvents

Let w be a positive selfadjoint operator on $\mathcal{H}$ with bounded inverse w^{-1}. One should think of w as a *weight function*. Both w and w^{-1} will act on $\mathcal{E}$, $\dot{\mathcal{E}}$ by $w(u_0, u_1) = (wu_0, wu_1)$ etc. In this subsection we will require (A3).

We need the following hypotheses:

$$\begin{cases} \text{(a) } wkw \in \mathcal{B}(\mathcal{H}), \\ \text{(b) } [k, w] = 0, \\ \text{(c) } h^{-1/2}[h, w^{-\epsilon}]w^{\epsilon/2} \in \mathcal{B}(\mathcal{H}) \text{ for all } 0 < \epsilon \le 1, \\ \text{(d) if } \epsilon > 0 \text{ then } \|w^{-\epsilon}u\| \lesssim \|h^{1/2}u\| \text{ for all } u \in h^{-1/2}\mathcal{H}, \\ \text{(e) } w^{-1}\langle h\rangle^{-1} \in \mathcal{B}_\infty(\mathcal{H}). \end{cases} \tag{ME1}$$

Note that part (d) of (ME1) is a Hardy type inequality and it implies the boundedness of the operators $w^{-\epsilon}h^{-1/2}$ and $h^{-1/2}w^{-\epsilon}$. Later on we shall see that these two operators are compact if (ME1) is satisfied (see the proof of Lemma 4.3).

Observe that from part (c) we also get $w^{\epsilon/2}[h, w^{-\epsilon}]h^{-1/2} \in \mathcal{B}(\mathcal{H})$. Moreover, we shall have $w^{-\epsilon}\langle h\rangle^{-\tilde{\epsilon}} \in \mathcal{B}_\infty(\mathcal{H})$ for all $\epsilon, \tilde{\epsilon} > 0$. Indeed, $w^{-z}\langle h\rangle^{-z} \in \mathcal{B}_\infty(\mathcal{H})$ is an analytic function of z in the region $\operatorname{Re} z > 0$, and by (e) this is a compact operator for $\operatorname{Re} z \ge 1$, hence for any z.

We also need the assumption

$$\begin{cases} \forall \epsilon > 0\ \exists \delta_\epsilon > 0 \text{ such that } w^{-\epsilon}(h - z^2)^{-1}w^{-\epsilon} \text{ extends from } \operatorname{Im} z > 0 \\ \text{to } \operatorname{Im} z > -\delta_\epsilon \text{ as a finite meromorphic function with values in } \mathcal{B}_\infty(\mathcal{H}). \end{cases} \tag{ME2}$$

Lemma 4.3. *Assume* (A1)–(A3) *and* (ME1)–(ME2) *and let* $0 < \epsilon \le 1$. *Then the operators*

(i) $w^{-\epsilon}p^{-1}(z)w^{-\epsilon}$,
(ii) $(h + 1)^{1/2}w^{-\epsilon}p^{-1}(z)w^{-\epsilon}$,
(iii) $w^{-\epsilon}p^{-1}(z)hw^{-\epsilon}h^{-1/2}$,
(iv) $h^{1/2}w^{-\epsilon}p^{-1}(z)(z - 2k)w^{-\epsilon}h^{-1/2}$

extend from $\{\operatorname{Im} z > 0\}$ *to* $\{\operatorname{Im} z > -\delta_{\epsilon/2}\}$ *as finite meromorphic functions with values in* $\mathcal{B}_\infty(\mathcal{H})$.

Proof. The relation $p(z) = (1 + 2zk(h - z^2)^{-1})(h - z^2)$ yields

$$w^{-\epsilon}p^{-1}(z)w^{-\epsilon} = w^{-\epsilon}(h - z^2)^{-1}w^{-\epsilon}\big(1 + 2zw^{\epsilon}kw^{\epsilon} \cdot w^{-\epsilon}(h - z^2)^{-1}w^{-\epsilon}\big)^{-1}.$$

Applying Prop. 4.2 to

$$F(z) = 1 + 2zw^{\epsilon}kw^{\epsilon} \cdot w^{-\epsilon}(h - z^2)^{-1}w^{-\epsilon}$$

proves (i).

We now write

$$\begin{aligned} hw^{-\epsilon}p^{-1}(z)w^{-\epsilon} &= w^{-\epsilon}hp^{-1}(z)w^{-\epsilon} + [h, w^{-\epsilon}]w^{\epsilon/2}w^{-\epsilon/2}p^{-1}(z)w^{-\epsilon} \\ &= w^{-2\epsilon} + z(z-2k)w^{-\epsilon}p^{-1}(z)w^{-\epsilon} + [h, w^{-\epsilon}]w^{\epsilon/2}w^{-\epsilon/2}p^{-1}(z)w^{-\epsilon}. \end{aligned}$$

This allows us to compute the second operator:

$$\begin{aligned} &(h+1)^{1/2}w^{-\epsilon}p^{-1}(z)w^{-\epsilon} \\ &\quad = (h+1)^{-1/2}w^{-2\epsilon} + (h+1)^{-1/2}z(z-2k)\cdot w^{-\epsilon}p^{-1}(z)w^{-\epsilon} \\ &\qquad + (h+1)^{-1/2}[h, w^{-\epsilon}]w^{\epsilon/2}\cdot w^{-\epsilon/2}p^{-1}(z)w^{-\epsilon} + (h+1)^{-1/2}w^{-\epsilon}p^{-1}(z)w^{-\epsilon}. \end{aligned}$$

By using (i) and hypotheses (c), (e) of (ME1) we get (ii).

Let us now prove (iii). Let $\chi \in C_0^\infty(\mathbb{R})$ with $\chi = 1$ in a neighborhood of 0. We write

$$\begin{aligned} w^{-\epsilon}p^{-1}(z)hw^{-\epsilon}h^{-1/2} &= w^{-\epsilon}p^{-1}(z)hw^{-\epsilon}h^{-1/2}(1-\chi(h)) \\ &\quad + w^{-\epsilon}p^{-1}(z)hw^{-\epsilon}h^{-1/2}\chi(h) =: T_1 + T_2. \end{aligned}$$

We have

$$T_1 = w^{-2\epsilon}h^{-1/2}(1-\chi(h)) + w^{-\epsilon}p^{-1}(z)w^{-\epsilon}\cdot z(z-2k)h^{-1/2}(1-\chi(h)).$$

The first term is compact by the second comment after hypothesis (ME1), and the second is compact outside the poles of $w^{-\epsilon}p^{-1}(z)w^{-\epsilon}$ by part (i). We have

$$T_2 = w^{-\epsilon}p^{-1}(z)w^{-\epsilon/2}\cdot w^{\epsilon/2}[h, w^{-\epsilon}]h^{-1/2}\chi(h) + w^{-\epsilon}p^{-1}(z)w^{-\epsilon}\cdot h^{1/2}\chi(h).$$

By the same comment we see that both terms here extend to finite meromorphic functions in $\{\operatorname{Im} z > -\delta_{\epsilon/2}\}$ with values in $\mathcal{B}_\infty(\mathcal{H})$. Thus (iii) is proved.

Note that since $h = p + z(z-2k)$ we have

$$w^{-\epsilon}p^{-1}(z)hw^{-\epsilon}h^{-1/2} = w^{-2\epsilon}h^{-1/2} + w^{-\epsilon}p^{-1}(z)w^{-\epsilon/2}\cdot z(z-2k)w^{-\epsilon/2}h^{-1/2}.$$

The left hand side here is a compact operator by what we have just proved, and the last term is also compact by (i) and because $w^{-\epsilon/2}h^{-1/2}$ is bounded by (ME1)(d). Since $0 < \epsilon \le 1$ is arbitrary, we see that $w^{-\epsilon}h^{-1/2}$ and $h^{-1/2}w^{-\epsilon}$ are compact operators if $0 < \epsilon \le 1$.

Finally, we prove (iv). We have

$$\begin{aligned} &h^{1/2}w^{-\epsilon}p^{-1}(z)(z-2k)w^{-\epsilon}h^{-1/2} \\ &\qquad = h^{-1/2}[h, w^{-\epsilon}]w^{\epsilon/2}\cdot w^{-\epsilon/2}p^{-1}(z)w^{-\epsilon/2}\cdot(z-2k)w^{-\epsilon/2}h^{-1/2} \\ &\qquad\quad + h^{-1/2}w^{-\epsilon/2}\cdot w^{-\epsilon/2}hp^{-1}(z)w^{-\epsilon/2}\cdot(z-2k)w^{-\epsilon/2}h^{-1/2}. \end{aligned}$$

For the first term of the right hand side we use (ME1)(c) as well as (i) and the boundedness of $w^{-\epsilon/2}h^{-1/2}$. For the last term we note that it is equal to

$$\begin{aligned} &h^{-1/2}w^{-\epsilon}(z-2k)w^{-\epsilon}h^{-1/2} \\ &\qquad + h^{-1/2}w^{-\epsilon/2}z(z-2k)\cdot w^{-\epsilon/2}p^{-1}(z)w^{-\epsilon/2}\cdot(z-2k)w^{-\epsilon/2}h^{-1/2}. \end{aligned}$$

The first term is a holomorphic function with values in $\mathcal{B}_\infty(\mathcal{H})$ because $w^{-\epsilon}h^{-1/2}$ and $h^{-1/2}w^{-\epsilon}$ are in $\mathcal{B}_\infty(\mathcal{H})$. The last line is treated as before. This proves (iv). □

Using this lemma we obtain a meromorphic extension of the truncated resolvent of H.

Proposition 4.4. *Assume the hypotheses of Lemma 4.3 and let $\epsilon > 0$. Then $w^{-\epsilon}R(z)w^{-\epsilon}$ and $w^{-\epsilon}\dot{R}(z)w^{-\epsilon}$ extend finite-meromorphically to $\{\mathrm{Im}\, z > -\delta_{\epsilon/2}\}$ as operator valued functions with values in $\mathcal{B}_\infty(\mathcal{E})$ and $\mathcal{B}_\infty(\dot{\mathcal{E}})$ respectively.*

Proof. We first prove the assertion concerning $R(z)$. Using (3.20) we see that

$$\begin{aligned} w^{-\epsilon}R(z)w^{-\epsilon} &= w^{-\epsilon}p(z)^{-1}\begin{pmatrix} z-2k & 1 \\ h & z \end{pmatrix} w^{-\epsilon} \\ &= \begin{pmatrix} 0 & 0 \\ w^{-2\epsilon} & 0 \end{pmatrix} + w^{-\epsilon}p^{-1}w^{-\epsilon}\begin{pmatrix} z-2k & 1 \\ z(z-2k) & z \end{pmatrix}. \end{aligned}$$

We then use Lemma 4.3(i), (ii) as well as assumption (ME1)(e).

Let us now treat $\dot{R}(z)$. Recall that under hypothesis (A3) we have

$$\dot{R}(z) = \begin{pmatrix} z^{-1}p^{-1}(z)h - z^{-1} & p^{-1}(z) \\ p^{-1}(z)h & zp^{-1}(z) \end{pmatrix}.$$

Using the fact that $w^{-\epsilon}h^{-1/2}$ is bounded by hypothesis (ME1)(d), we can write

$$w^{-\epsilon}\dot{R}(z)w^{-\epsilon} = w^{-\epsilon}p^{-1}(z)\begin{pmatrix} z-2k & 1 \\ h & z \end{pmatrix} w^{-\epsilon}.$$

We therefore have to show that

$$\begin{aligned} &h^{1/2}w^{-\epsilon}p^{-1}(z)(z-2k)w^{-\epsilon}h^{-1/2}, \quad h^{1/2}w^{-\epsilon}p^{-1}(z)w^{-\epsilon}, \\ &w^{-\epsilon}p^{-1}(z)hw^{-\epsilon}h^{-1/2}, \quad w^{-\epsilon}p^{-1}(z)zw^{-\epsilon} \end{aligned}$$

all extend finite-meromorphically with values in $\mathcal{B}_\infty(\mathcal{H})$. This follows from Lemma 4.3. □

5. Klein–Gordon operators with "two ends"

In this section we discuss an abstract framework corresponding to Klein–Gordon operators on manifolds with "two ends". The essential condition is that the asymptotic Hamiltonians in both ends are selfadjoint for a positive energy norm, modulo some gauge transformation.

5.1. Assumptions

We assume that there exists a selfadjoint operator x on $\mathcal{H}$ with $\sigma(x) = \sigma_{\mathrm{ac}}(x) = \mathbb{R}$ such that w is a smooth function of x, k commutes with x, and h_0 is *local* in x in the following sense: if $\chi_1, \chi_2 \in C^\infty(\mathbb{R})$ are bounded together with all their derivatives and if $\mathrm{supp}\,\chi_1 \cap \mathrm{supp}\,\chi_2 = \emptyset$, then $\chi_1(x)h_0\chi_2(x) = 0$. To summarize, we assume

$$\begin{cases} [x,k] = 0, \\ w = w(x) \text{ with } w \in C^\infty(\mathbb{R}), \\ h_0 \text{ is local in } x. \end{cases} \tag{TE1}$$

Let $i_\pm \in C^\infty(\mathbb{R})$ be such that $i_+^2 + i_-^2 = 1$, $0 \le i_\pm \le 1$, and

$$\operatorname{supp} i_- \subset]-\infty, 1[, \quad i_- = 1 \quad \text{on }]-\infty, -1],$$
$$\operatorname{supp} i_+ \subset]-1, \infty[, \quad i_+ = 1 \quad \text{on } [1, \infty[.$$

Let

$$j_\pm := i_\pm(x \mp 3).$$

We then have

$$j_\pm i_\pm = j_\pm, \quad i_+ j_- = i_- j_+ = 0.$$

Let

$$k_\pm := k \mp \ell j_\mp^2, \quad h_\pm := h_0 - k_\pm^2. \tag{5.1}$$

We also set

$$\tilde{h}_- := h_- + 2\ell k_- - \ell^2 = h_0 - (\ell - k_-)^2. \tag{5.2}$$

We require

$$\text{there exists } \ell \in \mathbb{R} \text{ such that } (h_+, k_+) \text{ and } (\tilde{h}_-, k_- - \ell) \text{ satisfy (A3).} \tag{TE2}$$

We also set $p_\pm(z) := h_\pm + z(2k_\pm - z)$. Note that $\tilde{h}_- = p_-(\ell)$.

Remark 5.1 (on assumptions (TE1)–(TE2)). x has to be thought of as a position variable and w as a weight function depending on x. The mixed term k has to commute with the position variable. (TE2) ensures that the asymptotic Hamiltonians define a selfadjoint problem.

5.2. *Asymptotic Hamiltonians*

We introduce the homogeneous energy spaces

$$\dot{\mathcal{E}}_+ := h_+^{-1/2}\mathcal{H} \oplus \mathcal{H}, \quad \dot{\mathcal{E}}_- := \Phi(\ell)(\tilde{h}_-^{-1/2}\mathcal{H} \oplus \mathcal{H}).$$

Then the operators

$$\dot{H}_\pm = \begin{pmatrix} 0 & 1 \\ h_\pm & 2k_\pm \end{pmatrix} \tag{5.3}$$

are selfadjoint with domains

$$D(\dot{H}_+) = h_+^{-1/2}\mathcal{H} \cap h_+^{-1}\mathcal{H} \oplus \langle h_+\rangle^{-1/2}\mathcal{H},$$
$$D(\dot{H}_-) = \Phi(\ell)\big((\tilde{h}_-^{-1/2}\mathcal{H} \cap \tilde{h}_-^{-1}\mathcal{H}) \oplus \langle \tilde{h}_-\rangle^{-1/2}\mathcal{H}\big).$$

We denote $\dot{R}_\pm(z) := (\dot{H}_\pm - z)^{-1}$.

We will also need the following assumption for ℓ as in (TE2). Let $\tilde{i} \in C_0^\infty(]-2,2[)$ with $\tilde{i} = 1$ on $[-1,1]$.

$$\begin{cases} \text{(a) } wi_+ki_+w,\ wi_-(k-\ell)i_-w \in \mathcal{B}(\mathcal{H}), \\ \text{(b) } [h,i_\pm] = \tilde{i}[h,i_\pm]\tilde{i}, \\ \text{(c) } (h_+,k_+,w) \text{ and } (\tilde{h}_-,k_- - \ell, w) \text{ fulfill (ME1), (ME2)}, \\ \text{(d) } h_\pm^{1/2} i_\pm h_\pm^{-1/2},\ h_0^{1/2} i_\pm h_0^{-1/2} \in \mathcal{B}(\mathcal{H}), \\ \text{(e) the operators } w[h,i_\pm]wh_\pm^{-1/2},\ w[h,i_\pm]wh_0^{-1/2},\ [h,i_\pm]h_\pm^{-1/2}, \\ \quad [h,i_\pm]h_0^{-1/2},\ h_0^{-1/2}[w^{-1},h_0]w \text{ are bounded on } \mathcal{H}, \\ \text{(f) if } \epsilon > 0 \text{ then } \|w^{-\epsilon}u\| \lesssim \|h_0^{1/2}u\| \text{ for all } u \in h_0^{-1/2}\mathcal{H}. \end{cases} \tag{TE3}$$

Remark 5.2 (on assumption (TE3)). Large parts of assumption (TE3) just ensure that the operations linked to commutators are local also in this abstract setting. (TE3)(a) states that k has finite limits at $\pm\infty$ and the convergence rate can be measured by the weight function w. (TE3)(c) will ensure that the weighted resolvents of the asymptotic Hamiltonians have meromorphic extensions. (TE3)(f) is some abstract Hardy type inequality.

As a direct consequence of Proposition 4.4 we obtain

Proposition 5.3. *For any $\epsilon > 0$ the functions $w^{-\epsilon}R_\pm(z)w^{-\epsilon}$ and $w^{-\epsilon}\dot{R}(z)w^{-\epsilon}$ extend finite-meromorphically to $\{\operatorname{Im} z > -\delta_{\epsilon/2}\}$ with values in $\mathcal{B}_\infty(\mathcal{E}_\pm)$ and $\mathcal{B}(\dot{\mathcal{E}}_\pm)$ respectively.*

5.3. *Construction of the resolvent*

We will need the following lemma:

Lemma 5.4. *The linear maps*

$$i_\pm : \dot{\mathcal{E}} \to \dot{\mathcal{E}}, \quad i_\pm : \dot{\mathcal{E}}_\pm \to \dot{\mathcal{E}}_\pm, \quad i_\pm : \dot{\mathcal{E}}_\pm \to \dot{\mathcal{E}}, \quad i_\pm : \dot{\mathcal{E}} \to \dot{\mathcal{E}}_\pm$$

are bounded.

Proof. First note that condition (TE3)(d) and the relation $[k,i_\pm]=0$ give the continuity of the maps $i_\pm : \dot{\mathcal{E}} \to \dot{\mathcal{E}}$ and $i_\pm : \dot{\mathcal{E}}_\pm \to \dot{\mathcal{E}}_\pm$. Then note that

$$i_+(h_+ + k^2)i_+ = i_+h_0i_+, \qquad i_-(\tilde{h}_- + (k-\ell)^2)i_- = i_-h_0i_-.$$

This together with (TE3)(d) gives the continuity of $i_\pm : \dot{\mathcal{E}} \to \dot{\mathcal{E}}_\pm$. We then claim that

$$i_+(h_0 + k^2)i_+ \lesssim i_+h_+i_+, \tag{5.4}$$
$$i_-(h_0 + (k-\ell)^2)i_- \lesssim i_-\tilde{h}_-i_-. \tag{5.5}$$

Indeed, (5.4) follows from

$$i_+(h_0+k^2)i_+ \lesssim i_+h_+i_+ + w^{-2}i_+^2 \lesssim i_+h_+i_+.$$

Here we have used (TE3)(c). Then (5.5) follows from

$$i_-(h_0 + (k-\ell)^2)i_- \lesssim i_-\tilde{h}_-i_- + i_-(k-\ell)^2 i_- \lesssim i_-\tilde{h}_-i_- + w^{-2}i_-^2 \lesssim i_-\tilde{h}_-i_-.$$

Finally, (5.4), (5.5) and (TE3)(d) give the continuity of $i_+ : \dot{\mathcal{E}}_+ \to \dot{\mathcal{E}}$ and $i_- : \dot{\mathcal{E}}_- \to \dot{\mathcal{E}}$. □

We now introduce a new operator:

$$Q(z) := i_-(\dot{H}_- - z)^{-1}i_- + i_+(\dot{H}_+ - z)^{-1}i_+. \tag{5.6}$$

Thanks to Lemma 5.4, $Q(z)$ is well defined as a bounded operator on $\dot{\mathcal{E}}$. We now compute

$$(\dot{H} - z)Q(z) = 1 + [\dot{H}, i_-](\dot{H}_- - z)^{-1}i_- + [\dot{H}, i_+](\dot{H}_+ - z)^{-1}i_+.$$

Note that

$$[\dot{H}, i_\pm] = \begin{pmatrix} 0 & 0 \\ [h, i_\pm] & 0 \end{pmatrix}.$$

Let

$$K_\pm(z) := \begin{pmatrix} 0 & 0 \\ [h, i_\pm] & 0 \end{pmatrix} \dot{R}_\pm(z) i_\pm, \qquad \tilde{K}_\pm(z) := i_\pm \dot{R}_\pm(z) \begin{pmatrix} 0 & 0 \\ [h, i_\pm] & 0 \end{pmatrix}. \tag{5.7}$$

Note that by assumptions (TE1) and (TE3) the operators

$$[\dot{H}, i_\pm]w^\epsilon \quad \text{and} \quad i_\pm(1 - j_\pm)w^\epsilon \quad \text{are bounded on } \dot{\mathcal{E}} \text{ for all } \epsilon > 0. \tag{5.8}$$

We set

$$A(z) := K_-(z)(1 - j_-) + K_+(z)(1 - j_+) : \mathbb{C}^+ \to \mathcal{B}(\dot{\mathcal{E}}).$$

Using Proposition 5.3 and Lemma 5.4 we see that $A(z)$ extends meromorphically to $\operatorname{Im} z > -\delta$ with values in $\mathcal{B}_\infty(\dot{\mathcal{E}})$ for some $\delta > 0$. As $\dot{H}_\pm$ are selfadjoint, it follows that

$$\|A(z)\|_{\mathcal{B}(\dot{\mathcal{E}})} \le 1/2$$

for $\operatorname{Im} z$ sufficiently large. Thus $(1 + A(z))^{-1}$ exists for $\operatorname{Im} z$ large enough. By Proposition 4.2 there exists a closed discrete subset Z^+ of the half-plane $\{\operatorname{Im} z > -\delta\}$ such that $(1 + A(z))^{-1}$ exists if $\operatorname{Im} z > -\delta$ and $z \notin Z^+$, and $(1 + A(z))^{-1}$ is finite meromorphic in $\{\operatorname{Im} z > -\delta\}$ and analytic in $\{\operatorname{Im} > -\delta\} \setminus Z^+$. Let

$$K(z) = K_-(z) + K_+(z).$$

Now observe that $j_a K_b = 0$ for $a = \pm,\ b = \pm$ by assumption (TE3)(b). Therefore

$$\begin{aligned} &1 + K(z) = \big(1 + K_-(z)j_- + K_+(z)j_+\big)\big(1 + K_-(z)(1 - j_-) + K_+(z)(1 - j_+)\big), \\ &\big(1 + K_-(z)j_- + K_+(z)j_+\big)^{-1} = 1 - K_-(z)j_- - K_+(z)j_+. \end{aligned}$$

We can now construct the resolvent of $\dot{H}$ by

$$
\begin{aligned}
R_{\dot{H}}(z) &:= Q(z)(1+K(z))^{-1} \\
&= Q(z)\big(1+K_-(z)(1-j_-)+K_+(z)(1-j_+)\big)^{-1}\big(1-K_-(z)j_- - K_+j_+\big). \qquad (5.9)
\end{aligned}
$$

The same considerations are valid in the lower half-plane, and we obtain a set of poles Z^-. The set $(Z^- \cap \mathbb{C}^-) \cup (Z^+ \cap \mathbb{C}^+)$ is clearly finite.

Proposition 5.5. *If conditions (A1)–(A2) and (TE1)–(TE3) are satisfied then there is a finite set $Z \subset \mathbb{C} \setminus \mathbb{R}$ with $\overline{Z} = Z$ such that the spectra of H and $\dot{H}$ are included in $\mathbb{R} \cup Z$ and the resolvents R and $\dot{R}$ are finite meromorphic functions on $\mathbb{C} \setminus \mathbb{R}$. Moreover, the point spectrum of H coincides with the point spectrum of $\dot{H}$, and the set Z consists of eigenvalues of finite multiplicity of H and of $\dot{H}$.*

Proof. From the previous arguments it follows that if we define $R_H(z) := R_{\dot{H}}(z)|_{\mathcal{E}}$ then $R_H(z) = R(z)$ and $R_{\dot{H}}(z) = \dot{R}(z)$ for z with sufficiently large (positive or negative) imaginary part. We know by Proposition 3.15 that $\rho(\dot{H}) \cap (\mathbb{C} \setminus \mathbb{R}) = \rho(h,k) \cap (\mathbb{C} \setminus \mathbb{R})$. Then we use Proposition 3.6 to see that all the poles of $\dot{H}$ in $\mathbb{C} \setminus \mathbb{R}$ are in a finite ball. But in this ball $(1 + A(z))^{-1}$ has only a finite number of poles. By using (5.9) and an analyticity argument we see that $\dot{R}(z)$ has only a finite number of poles in $\mathbb{C} \setminus \mathbb{R}$. From the analyticity properties of a resolvent family it follows then that the nonreal spectrum Z of $\dot{H}$ coincides with the set of nonreal poles of its resolvent, in particular it is finite.

From $\rho(\dot{H}) \setminus \mathbb{R} = \rho(h,k) \setminus \mathbb{R}$ and Lemma 3.1 we see that the complex spectrum is invariant under conjugation. Note also that every eigenvector of $\dot{H}$ for a nonzero eigenvalue is in $D(H)$, and thus is an eigenvector of H. It remains to show that the complex point spectrum consists exactly of the complex eigenvalues of $\dot{H}$ and that the corresponding eigenspaces are finite-dimensional. If z_0 is a pole of $\dot{R}(z)$, then on a neighborhood of z_0 we may write $\dot{R}(z) = \sum_{n=1}^{N}(z_0 - z)^{-n} S_n + S(z)$ with S holomorphic near z_0 and the $S_n \neq 0$ of finite rank because the function $A(z)$ is finite meromorphic. From this it follows that z_0 is an eigenvalue of finite multiplicity of $\dot{H}$ (see [32, Ch. VIII, Sect. 8] for details). □

From now on we denote by $\sigma^{\mathbb{C}}_{\mathrm{pp}}(\dot{H})$ the set of nonreal eigenvalues of $\dot{H}$. For $z \in \sigma^{\mathbb{C}}_{\mathrm{pp}}(\dot{H})$ the Riesz projector is defined by

$$
E(z, \dot{H}) := \frac{i}{2\pi} \oint_\gamma (\dot{H} - z)^{-1}\, dz,
$$

where γ is a small curve in $\rho(\dot{H})$ surrounding z. Let

$$
\mathbb{1}^{\mathbb{C}}_{\mathrm{pp}}(\dot{H}) := \bigoplus_{z \in \sigma^{\mathbb{C}}_{\mathrm{pp}}(H)} E(z, \dot{H}) \quad \text{and} \quad \mathcal{E}^{\mathbb{C}}_{\mathrm{pp}}(\dot{H}) := \mathbb{1}^{\mathbb{C}}_{\mathrm{pp}}(\dot{H})\dot{\mathcal{E}}.
$$

Let furthermore

$$
\mathbb{1}_{\mathbb{R}}(\dot{H}) := \mathbb{1} - \mathbb{1}^{\mathbb{C}}_{\mathrm{pp}}(\dot{H}), \qquad \mathcal{E}_{\mathbb{R}}(\dot{H}) := \mathbb{1}_{\mathbb{R}}(\dot{H})\dot{\mathcal{E}}.
$$

We clearly have $\dot{\mathcal{E}} = \mathcal{E}_{\mathbb{R}}(\dot{H}) \oplus \mathcal{E}^{\mathbb{C}}_{\mathrm{pp}}(\dot{H})$ and both spaces are invariant under $e^{-it\dot{H}}$.

5.4. Resolvent estimates

For $R,\ \delta > 0$ we set

$$\mathcal{U}_0(R,\delta) = \{z \in \mathbb{C} : 0 < |\mathrm{Im}\, z| \le \delta,\ |\mathrm{Re}\, z| \le R\}.$$

Lemma 5.6. *Assume (A1)–(A2) and (TE1)–(TE3). Then for each $R > 0$ there are $M, \delta > 0$ such that $\sigma(H) \setminus \mathbb{R}$ does not intersect $\mathcal{U}_0(R,\delta)$ and for all $z \in \mathcal{U}_0(R,\delta)$,*

$$\|\dot{R}(z)\|_{\mathcal{B}(\dot{\mathcal{E}})} \lesssim |\mathrm{Im}\, z|^{-M}, \tag{5.10}$$

$$\|R(z)\|_{\mathcal{B}(\mathcal{E})} \lesssim (1 + |z|^{-1})|\mathrm{Im}\, z|^{-M} + |z|^{-1}. \tag{5.11}$$

Proof. Recall that

$$\dot{R}(z) = Q(z)(1 + A(z))^{-1}\big(1 - K_-(z) j_- - K_+(z) j_+\big).$$

We choose $\delta > 0$ sufficiently small such that $(1 + A(z))^{-1}$ has no poles in $\mathcal{U}_0(R,\delta)$. Then

$$\|Q(z)\|_{\mathcal{B}(\dot{\mathcal{E}})} \lesssim |\mathrm{Im}\, z|^{-1}.$$

The meromorphic extension of $(1 + A(z))^{-1}$ has only a finite number of real poles in $\overline{\mathcal{U}_0(R,\delta)}$, hence

$$\|(1 + A(z))^{-1}\|_{\mathcal{B}(\dot{\mathcal{E}})} \lesssim |\mathrm{Im}\, z|^{-M_1}, \qquad M_1 > 0.$$

Noting that $[H, i_\pm] : \dot{\mathcal{E}} \to \dot{\mathcal{E}}$ is bounded by assumption (TE3) we obtain

$$\|1 - K_-(z) j_- - K_+(z) j_+\|_{\mathcal{B}(\dot{\mathcal{E}})} \lesssim |\mathrm{Im}\, z|^{-1}.$$

This gives (5.10) with $M = M_1 + 2$; and (5.11) now follows from Proposition 3.17. □

Remark 5.7. If $\sigma^{\mathbb{C}}_{\mathrm{pp}}(\dot{H}) = \emptyset$, then we can choose δ independently of R.

Lemma 5.8. *Let $R \ge M\|k\|_{\mathcal{B}(\mathcal{H})}$ with M as in Proposition 3.6. Then*

$$\|\dot{R}(z)\|_{\mathcal{B}(\dot{\mathcal{E}})} \lesssim |\mathrm{Im}\, z|^{-1} \quad \textit{if } |z| \ge R.$$

Proof. Recall from (3.23) that

$$\dot{R}(z) := (\dot{H} - z)^{-1} = \Phi(k)\begin{pmatrix} -p^{-1}(z)(k - z) & p^{-1}(z) \\ 1 + (k - z)p^{-1}(z)(k - z) & -(k - z)p^{-1}(z) \end{pmatrix}\Phi(-k).$$

Therefore it is sufficient to show

$$\|h_0^{1/2} p^{-1}(z)(k - z)u\| \lesssim \frac{1}{|\mathrm{Im}\, z|}\|h_0^{1/2} u\|, \tag{5.12}$$

$$\|h_0^{1/2} p^{-1}(z)u\| \lesssim \frac{1}{|\mathrm{Im}\, z|}\|u\|, \tag{5.13}$$

$$\|(k - z)p^{-1}(z)u\| \lesssim \frac{1}{|\mathrm{Im}\, z|}\|u\|, \tag{5.14}$$

$$\big\|\big(1 + (k - z)p^{-1}(z)(k - z)\big)u\big\| \lesssim \frac{1}{|\mathrm{Im}\, z|}\|h_0^{1/2} u\|, \tag{5.15}$$

for $|z| \geq R$. Estimates (5.12)–(5.14) follow from Proposition 3.6. To show (5.15) we use (3.26) and write

$$1 + (k - z)p^{-1}(z)(k - z) = (k - z)p^{-1}(z)h_0^{1/2}h_0^{1/2}(k - z)^{-1}h_0^{-1/2}h_0^{1/2}.$$

Then using (3.7) and (A2) we obtain

$$\left\| \big(1 + (k - z)p^{-1}(z)(k - z)\big)u \right\| \lesssim (\|k\|_{\mathcal{B}(\mathcal{H})} + |z|)\frac{1}{|\mathrm{Im}\, z|}\frac{1}{|z| - \|k\|_{\mathcal{B}(\mathcal{H})}}\|h_0^{1/2}u\|.$$

We can suppose $M \geq 2$ and obtain

$$(\|k\|_{\mathcal{B}(\mathcal{H})} + |z|)\frac{1}{|z| - \|k\|_{\mathcal{B}(\mathcal{H})}} \lesssim 1,$$

which finishes the proof of the lemma. □

5.5. *Smooth functional calculus*

The resolvent estimates in Lemma 5.6 easily allow us to construct a smooth functional calculus for $\dot{H}$. For $f \in C_0^\infty(\mathbb{R})$ we denote by $\tilde{f} \in C_0^\infty(\mathbb{C})$ an *almost analytic extension* of f, satisfying

$$\tilde{f}|_{\mathbb{R}} = f,$$
$$\left|\frac{\partial \tilde{f}(z)}{\partial \bar{z}}\right| \leq C_N |\mathrm{Im}\, z|^N, \qquad N \in \mathbb{N}.$$

Proposition 5.9. *Assume* (A1)–(A2) *and* (TE1)–(TE3).

(i) *Let $f \in C_0^\infty(\mathbb{R})$. Let $\tilde{f}$ be an almost analytic extension of f with* $\mathrm{supp}\, \tilde{f} \cap \sigma_{pp}^{\mathbb{C}}(\dot{H}) = \emptyset$. *Then the integral*

$$f(\dot{H}) := \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}}(z)\dot{R}(z)\, dz \wedge d\bar{z}$$

is norm convergent in $\mathcal{B}(\dot{\mathcal{E}})$ and is independent of the choice of $\tilde{f}$.

(ii) *The map $C_0^\infty(\mathbb{R}) \ni f \mapsto f(\dot{H}) \in \mathcal{B}(\dot{\mathcal{E}})$ is a continuous algebra morphism if we equip $C_0^\infty(\mathbb{R})$ with its canonical topology.*

Remark 5.10. (i) The condition $\mathrm{supp}\, \tilde{f} \cap \sigma_{pp}^{\mathbb{C}}(\dot{H}) = \emptyset$ can always be satisfied by choosing $\mathrm{supp}\, \tilde{f}$ close enough to the real axis.

(ii) If $\chi \in C^\infty(\mathbb{R})$ with $\chi = 1$ on $\mathbb{R} \setminus]-R, R[$ then we define $\chi(\dot{H}) := \mathbb{1}_{\mathbb{R}}(\dot{H}) - (1 - \chi)(\dot{H})$.

(iii) In the same way we define a smooth functional calculus for H, $H_\pm$, $\dot{H}_\pm$. For $\dot{H}_\pm$ this coincides with the smooth functional calculus for selfadjoint operators.

Proposition 5.11. *If $\sigma_{pp}^{\mathbb{C}}(\dot{H}) = \emptyset$ and $\chi \in C_0^\infty(\mathbb{R})$ with $\chi = 1$ in a neighborhood of zero, then*

$$\text{s-}\lim_{L \to \infty} \chi(L^{-1}\dot{H}) = 1.$$

Proof. First note that for some $M > 0$,

$$\|\dot{R}(z)\|_{\mathcal{B}(\dot{\mathcal{E}})} \lesssim \frac{1}{|\mathrm{Im}\, z|} + \frac{1}{|\mathrm{Im}\, z|^M}, \qquad |\mathrm{Im}\, z| > 0. \tag{5.16}$$

Indeed, we first choose $R > 0$ as in Lemma 5.8. Then

$$\|\dot{R}(z)\|_{\mathcal{B}(\dot{\mathcal{E}})} \lesssim \frac{1}{|\mathrm{Im}\, z|}, \qquad \forall z \in \mathbb{C} \setminus B(0, R).$$

By Remark 5.7 we can choose $\delta = R$ in Lemma 5.6 to obtain

$$\|\dot{R}(z)\|_{\mathcal{B}(\dot{\mathcal{E}})} \lesssim \frac{1}{|\mathrm{Im}\, z|^M}, \qquad \forall z \in B(0, R) \subset \mathcal{U}_0(R, R).$$

In particular we can choose the same almost analytic extension of χ to define $\chi(L^{-1}\dot{H})$ for all $L > 0$. We now show that

$$\lim_{L\to\infty} \chi(L^{-1}\dot{H}) - i_-\chi(L^{-1}\dot{H}_-)i_- - i_+\chi(L^{-1}\dot{H}_+)i_+ = 0. \tag{5.17}$$

We have

$$\begin{aligned}\chi(L^{-1}\dot{H}) &- i_-\chi(L^{-1}\dot{H}_-)i_- - i_+\chi(L^{-1}\dot{H}_+)i_+ \\ &= \frac{1}{2\pi i}\int \overline{\partial}\tilde{\chi}(z) L(\dot{R}(Lz) - Q(Lz))\, dz \wedge d\overline{z}.\end{aligned}$$

Now recall that $\dot{R}(z) = Q(z)(1 + K(z))^{-1}$, thus

$$\dot{R}(Lz) - Q(Lz) = -\dot{R}(Lz)K(Lz).$$

Thanks to (5.16), for $L \geq 1$ we have the estimate

$$\|\overline{\partial}\tilde{\chi}(z) L\dot{R}(Lz)K(Lz)\| \lesssim 1/L \to 0.$$

This implies (5.17). As $\dot{H}_\pm$ is selfadjoint in $\dot{\mathcal{E}}_\pm$, using Lemma 5.4 we find

$$\text{s-}\lim_{L\to\infty} i_\pm\chi(L^{-1}\dot{H}_\pm)i_\pm = i_\pm^2.$$

Thus

$$\text{s-}\lim_{L\to\infty} \chi(L^{-1}\dot{H}) = i_-^2 + i_+^2 = 1. \qquad \square$$

6. Propagation estimates

In this section we derive resolvent and propagation estimates for $\dot{H}$, similar to those obtained for selfadjoint operators. The key ingredients are the meromorphic extension of $\dot{R}(z)$ in Sect. 4 and the fact that the asymptotic Hamiltonians $\dot{H}_\pm$ are selfadjoint for their energy norms. There is however a new difficulty not present in the selfadjoint case: in addition to resolvent poles and thresholds, additional spectral singularities may appear. In the theory of selfadjoint operators on Krein spaces used in our previous works [17, 18] these spectral singularities are known as *critical points*.

6.1. Resonances and boundary values of the resolvent

By the usual arguments the operator

$$A_w(z) := w^{\epsilon} K_-(z)(1 - j_-)w^{-\epsilon} + w^{\epsilon} K_+(z)(1 - j_+)w^{-\epsilon}$$

can also be extended meromorphically from the upper half-plane to $\{\operatorname{Im} z > -\delta_{\epsilon/2}\}$ with values in $\mathcal{B}_\infty(\dot{\mathcal{E}})$. By the same argument as in the construction of the resolvent, for $\operatorname{Im} z$ large enough we have

$$\|A_w(z)\|_{\mathcal{B}(\dot{\mathcal{E}})} \le 1/2.$$

Using Proposition 4.2 we see that $(1 + A_w(z))^{-1}$ is meromorphic in $\{\operatorname{Im} z > -\delta_{\epsilon/2}\}$. Let S_w be the set of its poles. Now we have

$$\begin{aligned} w^{-\epsilon}\dot{R}(z)w^{-\epsilon} &= w^{-\epsilon} Q(z) w^{-\epsilon}(1 + A_w(z))^{-1} \\ &\quad \times \left(1 - w^{\epsilon} K_-(z) j_- w^{-\epsilon} - w^{\epsilon} K_+(z) j_+ w^{-\epsilon}\right). \end{aligned} \tag{6.1}$$

Using (6.1) we see that $w^{-\epsilon}\dot{R}(z)w^{-\epsilon}$ can be extended meromorphically from the upper half-plane to $\{\operatorname{Im} z > -\delta_{\epsilon/2}\}$ with values in $\mathcal{B}_\infty(\dot{\mathcal{E}})$. The same result also holds for the resolvents of $\dot{H}_\pm$, by assumption (TE3)(c).

Definition 6.1. The poles in $\{\operatorname{Im} z \le 0\}$ of the meromorphic extension of $w^{-\epsilon}\dot{R}(z)w^{-\epsilon}$ are called *resonances* of $\dot{H}$. The set of *real resonances* of $\dot{H}$, resp. $\dot{H}_\pm$, is denoted by $\mathcal{T}$, resp. $\mathcal{T}_\pm$.

Note that $\mathcal{T}$, $\mathcal{T}_\pm$ are obviously closed discrete sets. As a consequence of the meromorphic extensions of $w^{-\epsilon}\dot{R}(z)w^{-\epsilon}$ and $w^{-\epsilon}\dot{R}_\pm(z)w^{-\epsilon}$ we obtain:

Proposition 6.2. *Assume* (A1)–(A2) *and* (TE1)–(TE3). *Let* $\epsilon > 0$.

– *There exists* $\nu > 0$ *such that for all* $\chi \in C_0^\infty(\mathbb{R} \setminus \mathcal{T})$ *and all* $k \in \mathbb{N}$ *we have*

$$\sup_{\nu \ge \delta > 0,\, \lambda \in \mathbb{R}} \|w^{-\epsilon}\chi(\lambda)\dot{R}^k(\lambda \pm i\delta)w^{-\epsilon}\|_{\mathcal{B}(\dot{\mathcal{E}})} < \infty. \tag{6.2}$$

– *For all* $\chi \in C_0^\infty(\mathbb{R} \setminus \mathcal{T}_\pm)$ *and all* $k \in \mathbb{N}$ *we have*

$$\sup_{\delta > 0,\, \lambda \in \mathbb{R}} \|w^{-\epsilon}\chi(\lambda)\dot{R}_\pm^k(\lambda \pm i\delta)w^{-\epsilon}\|_{\mathcal{B}(\dot{\mathcal{E}}_\pm)} < \infty. \tag{6.3}$$

We apply [31, Thm. 4.3.1] to obtain:

Corollary 6.3. *Assume* (A1)–(A2) *and* (TE1)–(TE3). *Let* $\epsilon > 0$ *and* $\mathcal{T}_\pm$ *be as in Proposition* 6.2(ii). *Then for all* $\chi \in C_0^\infty(\mathbb{R} \setminus \mathcal{T}_\pm)$,

$$\sup_{\|u\|_{\dot{\mathcal{E}}_\pm}=1,\, \delta \ne 0} \int_{\mathbb{R}} \|w^{-\epsilon}\dot{R}_\pm(\lambda + i\delta)\chi(\lambda)u\|^2_{\dot{\mathcal{E}}_\pm}\, d\lambda < \infty. \tag{6.4}$$

Note that we cannot directly apply [31, Thm. 4.3.1] to $\dot{H}$, because the selfadjointness of the operator is crucial in this theorem. To discuss this further let us introduce a definition.

Definition 6.4. We call $\lambda \in \mathbb{R}$ a *regular point* of $\dot{H}$ if there exists $\chi \in C_0^\infty(\mathbb{R})$ with $\chi(\lambda) = 1$ and $\nu > 0$ such that

$$\sup_{\|u\|_{\dot{\mathcal{E}}_\pm}=1,\, \nu>|\delta|>0} \int_{\mathbb{R}} \|w^{-\epsilon} \dot{R}(\lambda + i\delta)\chi(\lambda)u\|^2_{\dot{\mathcal{E}}_\pm} \, d\lambda < \infty. \tag{6.5}$$

Otherwise we call it a *singular point*. We denote by $\mathcal{S}$ the set of singular points of $\dot{H}$.

Remark 6.5. Denoting by $\mathcal{S}_\pm$ the analog of $\mathcal{S}$ for $\dot{H}_\pm$ we see that $\mathcal{S}_\pm = \mathcal{T}_\pm$ by Kato's theory of H-smoothness (see [31, Ch. 4, Sect. 3]).

In our situation it is still possible to control the set of singular points. Recall that

$$Q(z) = (1 - \tilde{K}_-(z) - \tilde{K}_+(z))\dot{R}(z).$$

We then have

$$w^{-\epsilon} Q(z) = \big(1 - w^{-\epsilon}\tilde{K}_-(z)w^\epsilon - w^{-\epsilon}\tilde{K}_+(z)w^\epsilon\big)w^{-\epsilon}\dot{R}(z).$$

Let

$$\tilde{A}_w(z) := -w^{-\epsilon}\tilde{K}_-(z)w^\epsilon - w^{-\epsilon}\tilde{K}_+(z)w^\epsilon.$$

By the usual arguments $\tilde{A}_w$ is meromorphic in $\{\mathrm{Im}\, z > -\delta_{\epsilon/2}\}$ with values in $\mathcal{B}_\infty(\dot{\mathcal{E}})$. Also $\|\tilde{A}_w(z)\|_{\mathcal{B}(\dot{\mathcal{E}})} \le 1/2$ for $\mathrm{Im}\, z$ sufficiently large. We can therefore again apply Proposition 4.2 to see that $(1 + \tilde{A}_w(z))^{-1}$ is meromorphic for $\{\mathrm{Im}\, z > -\delta_{\epsilon/2}\}$. We then have

$$w^{-\epsilon}\dot{R}(z) = (1 + \tilde{A}_w(z))^{-1} w^{-\epsilon} Q(z). \tag{6.6}$$

Proposition 6.6. *Assume* (A1)–(A2) *and* (TE1)–(TE3).

(i) *Let $\mathbb{N}_w$ be the set of real poles of $\tilde{A}_w(z)$. Then*

$$\mathcal{S} \subset \mathbb{N}_w \cup \mathcal{T}_+ \cup \mathcal{T}_-.$$

It follows that $\mathcal{S}$ is a closed and discrete set.

(ii) *Let λ be a regular point of $\dot{H}$. Then there exists $\chi \in C_0^\infty(\mathbb{R})$ with $\chi(\lambda) = 1$ and $\nu > 0$ such that*

$$\sup_{\|u\|_{\dot{\mathcal{E}}_\pm}=1,\, \nu>|\delta|>0} \int_{\mathbb{R}} \|w^{-\epsilon} \dot{R}(\lambda + i\delta)\chi(\dot{H})u\|^2_{\dot{\mathcal{E}}_\pm} \, d\lambda < \infty.$$

Proof. The first part follows from (6.6) and Corollary 6.3. For the second part we have to show that we can replace $\chi(\lambda)$ by $\chi(\dot{H})$ at a regular point. We choose $\tilde{\chi} \in C_0^\infty(I)$ with $\tilde{\chi}\chi = \chi$ and write

$$\begin{aligned} \|w^{-\epsilon}\dot{R}(\lambda \pm i\delta)\chi(\dot{H})f\|^2_{\dot{\mathcal{E}}} &\lesssim \|w^{-\epsilon}\dot{R}(\lambda \pm i\delta)\tilde{\chi}(\lambda)\chi(\dot{H})f\|^2_{\dot{\mathcal{E}}} \\ &\quad + \|w^{-\epsilon}\dot{R}(\lambda \pm i\delta)(1 - \tilde{\chi}(\lambda))\chi(\dot{H})f\|^2_{\dot{\mathcal{E}}}. \end{aligned} \tag{6.7}$$

The estimate for the first term follows from the definition of regular points and (6.5). Let us treat the second term. We claim

$$\|w^{-\epsilon}\dot{R}(\lambda \pm i\delta)(1-\tilde{\chi}(\lambda))\chi(\dot{H})\|_{\mathcal{B}(\dot{\mathcal{E}})} \lesssim \langle\lambda\rangle^{-1},$$

uniformly in δ. In fact let

$$f_\lambda^\epsilon(x) := \langle\lambda\rangle\frac{1}{x-(\lambda+i\delta)}(1-\tilde{\chi}(\lambda))\chi(x).$$

It is sufficient to show that all the seminorms $\|f_\lambda^\epsilon\|_m$ are uniformly bounded with respect to $\lambda,\ \delta$. Note that $g_\lambda(x) = (1-\tilde{\chi}(\lambda))\chi(x)$ vanishes to all orders at $x=\lambda$. If supp $\chi \subset [-C, C]$ this is enough to ensure that $\|f_\lambda^\epsilon\|_m$ is uniformly bounded in $\lambda \in [-2C, 2C]$ and $\delta > 0$. For $|\lambda| \geq 2C$ we observe that

$$\left|\langle\lambda\rangle\frac{1}{x-(\lambda+i\delta)}\right| \lesssim 1,$$

with analogous estimates for the derivatives. This gives the integrability of the second term in (6.7). □

6.2. *Propagation estimates*

As an immediate consequence of Proposition 6.2 we obtain:

Proposition 6.7. *Assume* (A1)–(A2) *and* (TE1)–(TE3). *Let* $\epsilon > 0$.

– *For all* $\chi \in C_0^\infty(\mathbb{R}\setminus\mathcal{T})$ *and* $k \in \mathbb{N}$ *we have*

$$\|w^{-\epsilon}e^{-it\dot{H}}\chi(\dot{H})w^{-\epsilon}\|_{\mathcal{B}(\dot{\mathcal{E}})} \lesssim \langle t\rangle^{-k}. \tag{6.8}$$

– *For all* $\chi \in C_0^\infty(\mathbb{R}\setminus\mathcal{T}_\pm)$ *and* $k \in \mathbb{N}$ *we have*

$$\|w^{-\epsilon}e^{-it\dot{H}_\pm}\chi(\dot{H}_\pm)w^{-\epsilon}\|_{\mathcal{B}(\dot{\mathcal{E}}_\pm)} \lesssim \langle t\rangle^{-k}. \tag{6.9}$$

Proof. We only prove (i), the proof of (ii) being analogous. We have

$$w^{-\epsilon}e^{-it\dot{H}}\chi(\dot{H})w^{-\epsilon} = \frac{1}{2\pi i}\int \chi(\lambda)e^{-it\lambda}w^{-\epsilon}\big(\dot{R}(\lambda+i0)-\dot{R}(\lambda-i0)\big)w^{-\epsilon}\,d\lambda.$$

Integration by parts gives

$$w^{-\epsilon}e^{-it\dot{H}}\chi(\dot{H})w^{-\epsilon} = \frac{1}{2\pi i}\frac{1}{(it)^k}\sum_{\pm}\sum_{j=1}^{k+1}\pm C_k^{j-1}\int \chi_j(\lambda)e^{-it\lambda}w^{-\epsilon}\dot{R}^j(\lambda\pm i0)w^{-\epsilon}\,d\lambda$$

with $\chi_j := \chi^{(k+1-j)}$. The estimate then follows from Proposition 6.2. □

Proposition 6.8. *Assume* (A1)–(A2) *and* (TE1)–(TE3). *Let* $\epsilon > 0$. *Then for all* $\chi \in C_0^\infty(\mathbb{R}\setminus\mathcal{S})$,

$$\int_{\mathbb{R}}\|w^{-\epsilon}e^{-it\dot{H}}\chi(\dot{H})\varphi\|_{\dot{\mathcal{E}}}^2\,dt \lesssim \|\varphi\|_{\dot{\mathcal{E}}}^2. \tag{6.10}$$

Proof. We write

$$w^{-\epsilon}\big(\dot R(\lambda+i\delta)-\dot R(\lambda-i\delta)\big)\chi(\dot H)f=i\int_{\mathbb{R}} w^{-\epsilon}e^{-\delta|t|}e^{i\lambda t}e^{-i\dot H t}\chi(\dot H)f\,dt.$$

By Plancherel's formula this yields

$$\int_{\mathbb{R}}\big\|w^{-\epsilon}\big(\dot R(\lambda+i\delta)-\dot R(\lambda-i\delta)\big)\chi(\dot H)f\big\|_{\dot{\mathcal{E}}}^2\,d\lambda=\int_{\mathbb{R}} e^{-2\delta|t|}\|w^{-\epsilon}e^{-it\dot H}\chi(\dot H)f\|_{\dot{\mathcal{E}}}^2\,dt.$$

By Proposition 6.6(ii) the left hand side of this equation is uniformly bounded in δ for δ small enough. □

Corollary 6.9. *If* (A1)–(A2) *and* (TE1)–(TE3) *hold and* λ *is a real eigenvalue of* $\dot H$ *then* $\lambda\in\mathcal{S}$.

6.3. Estimates on singular points

It will be important in applications to prove that $\dot H$ has no singular points. To do this we will use the following proposition.

Proposition 6.10. *Assume* (A1)–(A2) *and* (TE1)–(TE3). *Then*

$$\mathcal{S}\subset\mathcal{T}\cup\mathcal{T}_-\cup\mathcal{T}_+.$$

Proof. From (5.9) we obtain, for $\operatorname{Im} z\gg 1$,

$$\dot R(z)=Q(z)(\mathbb{1}+K(z))^{-1}=Q(z)-Q(z)(\mathbb{1}+K(z))^{-1}K(z),$$

hence

$$\begin{aligned} w^{-\epsilon}\dot R(z)&=w^{-\epsilon}Q(z)-w^{-\epsilon}Q(z)(\mathbb{1}+K(z))^{-1}w^{-\epsilon}w^{\epsilon}K(z)\\ &=w^{-\epsilon}Q(z)-w^{-\epsilon}\dot R(z)w^{-\epsilon}w^{\epsilon}K(z).\end{aligned}$$

Next we write $w^{\epsilon}K(z)=w^{\epsilon}K_-(z)+w^{\epsilon}K_+(z)$ and deduce from the expression (5.7) for $K_\pm(z)$ that $w^{\epsilon}K_\pm(z)=m_\epsilon\dot R_\pm(z)i_\pm$ for $m_\epsilon\in\mathcal{B}(\dot{\mathcal{E}})$. It then suffices to recall the expression (5.6) for $Q(z)$, and apply Remark 6.5. □

6.4. Additional resolvent estimates

In this subsection we make the link between the poles of $\eta p^{-1}(z)\eta$ and those of $\eta\dot R(z)\eta$ for $\eta\in C_0^\infty(\mathbb{R})$.

We will need the following hypothesis:

$$\begin{cases}\text{(a) } \psi\in C_0^\infty(\mathbb{R})\Rightarrow h_0^{1/2}\psi(x)h_0^{-1/2}\in\mathcal{B}(\mathcal{H}),\\ \text{(b) } \psi\in C_0^\infty(\mathbb{R}),\ \psi\geq 0,\ \psi=1 \text{ near } 0\ \Rightarrow \text{s-lim}_{n\to\infty}\psi(x/n)=1 \text{ in } h_0^{-1/2}\mathcal{H}.\end{cases}\qquad\text{(PE)}$$

Lemma 6.11. *Let $\eta, \tilde\eta \in C_0^\infty(\mathbb{R})$ with $\tilde\eta\eta = \eta$. If z is not a pole of $\tilde\eta p^{-1}(z)\tilde\eta$ then z is not a pole of $\eta R(z)\eta$ or of $\eta\dot R(z)\eta$, and if $\mathcal{P}(z) := \|\tilde\eta p^{-1}(z)\tilde\eta\|_{\mathcal{B}(\mathcal{H})}$ then*

$$\|\eta R(z)\eta\|_{\mathcal{B}(\mathcal{E})} \lesssim \langle z\rangle^2(1+\langle z\rangle\mathcal{P}^2(z)), \tag{6.11}$$

$$\|\eta \dot R(z)\eta\|_{\mathcal{B}(\dot{\mathcal{E}})} \lesssim \langle z\rangle^2(1+\langle z\rangle\mathcal{P}^2(z)). \tag{6.12}$$

Proof. We choose $\eta_1, \eta_0 \in C_0^\infty(\mathbb{R})$ with $\eta_0\eta = \eta$, $\eta_1\eta_0 = \eta_0$ and $\tilde\eta\eta_1 = \eta_1$. We first notice that (6.12) follows from (6.11) because

$$\|\eta\dot R(z)\eta u\|_{\dot{\mathcal{E}}} \lesssim \|\eta R(z)\eta u\|_{\mathcal{E}} \lesssim \langle z\rangle^2(1+\langle z\rangle\mathcal{P}^2(z))\|\eta_0 u\|_{\mathcal{E}} \lesssim \langle z\rangle^2(1+\langle z\rangle\mathcal{P}^2(z))\|u\|_{\dot{\mathcal{E}}},$$

where we have used Hardy's inequality, (TE3)(f). Now recall from (3.23) that

$$\dot R(z) := (\dot H - z)^{-1} = \Phi(k)\begin{pmatrix} -p^{-1}(z)(k-z) & p^{-1}(z) \\ 1+(k-z)p^{-1}(z)(k-z) & -(k-z)p^{-1}(z)\end{pmatrix}\Phi(-k).$$

It is therefore sufficient to show that

$$\|\eta p^{-1}(z)(k-z)\eta u\|_{\mathcal{H}^1} \lesssim \langle z\rangle(1+\langle z\rangle^2\mathcal{P}(z))\|u\|_{\mathcal{H}^1}, \tag{6.13}$$

$$\|\eta p^{-1}(z)\eta u\|_{\mathcal{H}^1} \lesssim (1+\langle z\rangle^2\mathcal{P}(z))\|u\|_{\mathcal{H}}, \tag{6.14}$$

$$\|\eta(1+(k-z)p^{-1}(z)(k-z))\eta u\|_{\mathcal{H}} \lesssim \langle z\rangle(1+\langle z\rangle^2\mathcal{P}^2(z))\|u\|_{\mathcal{H}^1}, \tag{6.15}$$

$$\|\eta(k-z)p^{-1}(z)\eta u\|_{\mathcal{H}} \lesssim \langle z\rangle\mathcal{P}(z)\|u\|_{\mathcal{H}}. \tag{6.16}$$

First, (6.16) is clear; let us consider (6.14). By complex interpolation (6.14) will follow from

$$\|\eta p^{-1}(z)\eta u\|_{\mathcal{H}^2} \lesssim (\langle z\rangle^2\mathcal{P}(z)+1)\|u\|_{\mathcal{H}}. \tag{6.17}$$

We compute

$$\begin{aligned} h_0\eta p^{-1}(z)\eta &= [h_0,\eta]p^{-1}(z)\eta + \eta h_0 p^{-1}(z)\eta \\ &= (h_0+1)^{-1}[h_0,\eta]\eta_0(h_0+1)p^{-1}(z)\eta \\ &\quad + (h_0+1)^{-1}[h_0,[h_0,\eta]]\eta_0 p^{-1}(z)\eta + \eta h_0 p^{-1}(z)\eta, \\ \eta_0 h_0 p^{-1}(z)\eta &= \eta + (k-z)^2\eta_0 p^{-1}(z)\eta. \end{aligned}$$

Thus

$$\|\eta_0 h_0 p^{-1}(z)\eta u\| \lesssim (1+\langle z\rangle^2\mathcal{P}(z))\|u\|_{\mathcal{H}}.$$

As $(h_0+1)^{-1}[h_0,[h_0,\eta]]$ is bounded, this gives (6.17) and thus (6.14).

Let us now consider (6.13). First note that $\|(k-z)u\|_{\mathcal{H}^1} \lesssim \langle z\rangle\|u\|_{\mathcal{H}^1}$. We then estimate, using (6.14),

$$\|\eta p^{-1}(z)\eta u\|_{\mathcal{H}^1} \lesssim (\langle z\rangle^2\mathcal{P}(z)+1)\|u\|_{\mathcal{H}^1}.$$

This gives (6.13).

Let us now show (6.15). We write

$$\eta\big(1+(k-z)p^{-1}(z)(k-z)\big)\eta \\ = \eta p^{-1}(z)\eta_1[h_0,k\eta_0]h_0^{-1/2}h_0^{1/2}\eta_1 p^{-1}(z)(k-z)\eta + \eta p^{-1}(z)(k-z)^2\eta.$$

Using (6.13) we have

$$\|\eta p^{-1}(z)\eta_1[h_0,k\eta_0]h_0^{-1/2}h_0^{1/2}\eta_1 p^{-1}(z)(k-z)\eta u\|_{\mathcal{H}} \\ \lesssim \mathcal{P}(z)\|h_0^{1/2}\eta_1 p^{-1}(z)(k-z)\eta\|_{\mathcal{B}(\mathcal{H})}\|u\|_{\mathcal{H}} \lesssim \langle z\rangle\mathcal{P}(z)(1+\langle z\rangle^2\mathcal{P}(z))\|u\|_{\mathcal{H}^1}.$$

This proves (6.15). □

Corollary 6.12. *If $w^{-\epsilon}p^{-1}(z)w^{-\epsilon}$ has no real poles then $w^{-\epsilon}\dot{R}(z)w^{-\epsilon}$ has no real poles.*

Proof. By the preceding lemmas $\eta\dot{R}(z)\eta$ has no real poles for all $\eta \in C_0^\infty(\mathbb{R})$. Suppose that $w^{-\epsilon}\dot{R}(z)w^{-\epsilon}$ has a pole at $z = z_0 \in \mathbb{R}$. In a neighborhood of $z = z_0$ we have

$$w^{-\epsilon}\dot{R}(z)w^{-\epsilon} = \sum_{j=1}^{m}\frac{A_j}{(z-z_0)^j} + H(z),$$

where $H(z)$ is holomorphic and the A_j are of finite rank. Let $\eta_1, \eta_2 \in C_0^\infty(\mathbb{R})$. We have

$$w^{-\epsilon}\eta_1\dot{R}(z)\eta_2 w^{-\epsilon} = \sum_{j=1}^{m}\frac{\eta_1 A_j\eta_2}{(z-z_0)^j} + \eta_1 H(z)\eta_2.$$

As $\eta_1\dot{R}(z)\eta_2$ does not have a pole at $z = z_0$, we have

$$\eta_1 A_j\eta_2 = 0, \quad \forall \eta_1,\ \eta_2 \in C_0^\infty(\mathbb{R}),\ j = 1,\dots,m.$$

It follows that

$$A_j\eta = 0, \quad \forall \eta \in C_0^\infty(\mathbb{R}),\ j = 1,\dots,m.$$

In view of (PE) this implies that $A_j = 0$. □

7. Boundedness of the evolution 1: abstract setting

The aim of this section is to show that the evolution is bounded outside the complex eigenvalues and the singular points of $\dot{H}$. We assume

$$w^{-1}: D(h_0)\to D(h_0),\ \text{and } [-ik,h] \lesssim w^{-1}h_0w^{-1} \text{ as quadratic forms on } D(h_0). \quad \text{(B)}$$

For $\chi \in C^\infty(\mathbb{R})$ and $\mu > 0$ we set $\chi_\mu(\lambda) := \chi(\lambda/\mu)$.

Theorem 7.1. *Assume* (A1)–(A2)*,* (TE1)–(TE3)*,* (PE)*, and* (B)*. Assume furthermore* $\sigma_{pp}^{\mathbb{C}}(\dot{H}) = \emptyset$.

(i) *Let* $\chi \in C^{\infty}(\mathbb{R})$ *with* $\chi = 0$ *on* $[-1, 1]$ *and* $\chi = 1$ *outside* $[-2, 2]$*. Then there are* $\mu_0, C_1 > 0$ *such that for all* $\mu \geq \mu_0$ *and* $t \in \mathbb{R}$*,*

$$\|e^{-it\dot{H}}\chi_\mu(\dot{H})u\|_{\dot{\mathcal{E}}} \leq C_1\|\chi_\mu(\dot{H})u\|_{\dot{\mathcal{E}}}, \quad u \in \dot{\mathcal{E}}. \tag{7.1}$$

(ii) *If* $\chi \in C_0^{\infty}(\mathbb{R} \setminus \mathcal{S})$ *then there is* $C_2 > 0$ *such that for all* $u \in \dot{\mathcal{E}}$ *and* $t \in \mathbb{R}$ *we have*

$$\|e^{-it\dot{H}}\chi(\dot{H})u\|_{\dot{\mathcal{E}}} \leq C_2\|u\|_{\dot{\mathcal{E}}}. \tag{7.2}$$

Remark 7.2. If $\sigma_{pp}^{\mathbb{C}}(\dot{H}) \neq \emptyset$, then the theorem still holds for $e^{-it\dot{H}}|_{\mathcal{E}_{\mathbb{R}}(\dot{H})}$. Here $\mathcal{E}_{\mathbb{R}}(\dot{H}) = \mathbb{1}_{\mathbb{R}}(\dot{H})\dot{\mathcal{E}}$ (see Sect. 5.3).

The proof of Thm. 7.1 is divided into a high frequency analysis (part (i)) and a low frequency analysis (part (ii)).

7.1. High frequency analysis

Lemma 7.3. *Assume* (A1)–(A2)*,* (TE1)–(TE3)*,* (PE)*, and* (B)*. If* χ *is as in the statement of Thm.* 7.1 *then for* $\mu > 0$ *sufficiently large,*

$$\|(\chi_\mu(\dot{H})u)_0\|_{\mathcal{H}} \lesssim \frac{1}{\mu}\|\chi_\mu(\dot{H})u\|_{\dot{\mathcal{E}}}.$$

Proof. Let $\hat{\chi}$ be as χ with $\hat{\chi}\chi = \chi$. Set $\varphi = \hat{\chi} - 1$ and observe that $\varphi = -1$ on $]-1, 1[$. Let $\tilde{\varphi}$ be some (finite order) almost analytic extension of φ given by (for some $N \geq 1$)

$$\tilde{\varphi}(x + iy) = \sum_{r=0}^{N} \varphi^{(r)}(x)\frac{(iy)^r}{r!}\tau\left(\frac{y}{\delta\langle x\rangle}\right)$$

with $\tau \in C_0^{\infty}(\mathbb{R})$, $\tau(s) = 1$ in $|s| \leq 1/2$, and $\tau(s) = 0$ in $|s| \geq 1$. Here δ is chosen such that $\dot{R}(z)$ has no poles in $|\mathrm{Im}\, z| \leq \delta\langle x\rangle$ if $x \in \operatorname{supp}\varphi$. We compute

$$\begin{aligned}
\overline{\partial}\tilde{\varphi}(z) &= \hat{\chi}^{(N+1)}(x)\frac{(iy)^{(N+1)}}{(N+1)!}\tau\left(\frac{y}{\delta\langle x\rangle}\right) \\
&\quad + \left(\sum_{r=0}^{N} \varphi^{(r)}(x)\frac{(iy)^r}{r!}\right)\tau'\left(\frac{y}{\delta\langle x\rangle}\right)\left(\frac{i}{\delta\langle x\rangle} + \frac{yx}{\delta\langle x\rangle^2}\right) \\
&=: \tilde{\varphi}_1(x + iy) + \tilde{\varphi}_2(x + iy).
\end{aligned}$$

Let $\mu \geq \mu_0 = \max\{(1+\varepsilon)\|k\|_{\mathcal{B}(\mathcal{H})}, 2(1+\varepsilon)\|k\|_{\mathcal{B}(\mathcal{H})}/\delta\}$. Then $\operatorname{supp}\overline{\partial}\tilde{\varphi} \subset K := \{z \in \mathbb{C} : |\mu z| \geq (1+\varepsilon)\|k\|_{\mathcal{B}(\mathcal{H}))}\} \cap \{z \in \mathbb{C} : |z| \geq \min\{1, \delta/2\}\}$. Indeed, on $\operatorname{supp}\tilde{\varphi}_1$ we have $|z| \geq 1$ and thus

$$|\mu z| \geq \mu_0 = (1+\varepsilon)\|k\|_{\mathcal{B}(\mathcal{H})}.$$

On supp $\tilde{\varphi}_2$ we have $|z| \geq \delta|z|/2$ and thus

$$|\mu z| \geq \mu\delta/2 \geq (1+\varepsilon)\|k\|_{\mathcal{B}(\mathcal{H})}.$$

Note that

$$1 = -(\chi - 1)(0) = \frac{1}{2\pi i}\int \overline{\partial}\tilde{\varphi}(z)\frac{1}{z}\,dz \wedge d\overline{z}.$$

We have

$$\begin{aligned}\hat{\chi}_\mu(\dot{H}) &= \varphi_\mu(\dot{H}) + 1\\ &= \frac{1}{2\pi i}\int \overline{\partial}\tilde{\varphi}(z)\left(\left(\frac{\dot{H}}{\mu} - z\right)^{-1} + \frac{1}{z}\right)dz \wedge d\overline{z}\\ &= -\frac{1}{2\pi i}\int \overline{\partial}\tilde{\varphi}(z)(\dot{H} - \mu z)^{-1}\frac{\dot{H}}{z}\,dz \wedge d\overline{z}. \qquad (7.3)\end{aligned}$$

Let $v^\mu = \chi_\mu(\dot{H})u$. We compute

$$\left((\dot{H} - \mu z)^{-1}\frac{\dot{H}}{z}\begin{pmatrix} v_0^\mu \\ v_1^\mu \end{pmatrix}\right)_0 = \frac{1}{z}p^{-1}(\mu z)(\mu z v_1^\mu + h v_0^\mu).$$

For $z \in \operatorname{supp} \overline{\partial}\tilde{\varphi}(z)$ we estimate, using Proposition 3.6,

$$\begin{aligned}\|p^{-1}(\mu z)z\mu v_1^\mu\|_{\mathcal{H}} &\lesssim \|p^{-1}(\mu z)z\mu(v_1^\mu - k v_0^\mu)\|_{\mathcal{H}} + \|p^{-1}(\mu z)z\mu k v_0^\mu\|_{\mathcal{H}}\\ &\lesssim \frac{1}{|\mathrm{Im}\, z|\mu}\|(v_1^\mu - k v_0^\mu)\|_{\mathcal{H}} + \frac{1}{\mu|\mathrm{Im}\, z|}\|v_0^\mu\|_{\mathcal{H}},\\ \|p^{-1}(\mu z)h v_0^\mu\|_{\mathcal{H}} &\lesssim \|p^{-1}(z)h_0 v_0^\mu\|_{\mathcal{H}} + \|p^{-1}(\mu z)k^2 v_0^\mu\|_{\mathcal{H}}\\ &\lesssim \frac{1}{|\mathrm{Im}\, z|\mu}\|h_0^{1/2}v_0^\mu\|_{\mathcal{H}} + \frac{1}{\mu^2|\mathrm{Im}\, z|}\|v_0^\mu\|_{\mathcal{H}}\\ &\lesssim \frac{1}{|\mathrm{Im}\, z|\mu}\|h_0^{1/2}v_0^\mu\|_{\mathcal{H}} + \frac{1}{\mu^2|\mathrm{Im}\, z|}\|v_0^\mu\|_{\mathcal{H}}\end{aligned}$$

Using (7.3) we obtain

$$\|(\chi_\mu(\dot{H}^n)u)_0\|_{\mathcal{H}} \lesssim \frac{1}{\mu}\|\chi_\mu(\dot{H}^n)u\|_{\dot{\mathcal{E}}} + \frac{1}{\mu^2}\|(\chi_\mu(\dot{H}^n)u)_0\|_{\mathcal{H}}.$$

This gives the conclusion for μ sufficiently large. □

Corollary 7.4. *Assume (A1)–(A2), (TE1)–(TE3), (PE), and (B). Let χ be as in Thm. 7.1. Then for $\mu > 0$ sufficiently large there exists $\varepsilon > 0$ such that for all $u \in \mathcal{E}$,*

$$\langle \chi_\mu(\dot{H})u, \chi_\mu(\dot{H})u\rangle_0 \geq \varepsilon\|\chi_\mu(\dot{H})u\|^2_{\dot{\mathcal{E}}}.$$

Proof. By Lemma 7.3 we have

$$\begin{aligned}\langle \chi_\mu(\dot{H})u, \chi_\mu(\dot{H})u\rangle_0 &\geq \|\chi_\mu(\dot{H})u\|^2_{\dot{\mathcal{E}}} - 2\|k(\chi_\mu(\dot{H})u)_0\|^2_{\mathcal{H}} \\ &\geq (1 - C/\mu^2)\|\chi_\mu(\dot{H})u\|^2_{\dot{\mathcal{E}}},\end{aligned}$$

which gives the conclusion for μ sufficiently large. □

Corollary 7.5. *Assume (A1)–(A2), (TE1)–(TE3), (PE), and (B). Let χ be as in Thm. 7.1. Then there exists $C_1 > 0$ such that for all $u \in \dot{\mathcal{E}}$ and $t \in \mathbb{R}$,*

$$\|e^{-it\dot{H}}\chi_\mu(\dot{H})u\|_{\dot{\mathcal{E}}} \leq C_1\|\chi_\mu(\dot{H})u\|_{\dot{\mathcal{E}}}.$$

Proof. We use the fact that $\langle e^{-it\dot{H}}\chi_\mu(\dot{H})u, e^{-it\dot{H}}\chi_\mu(\dot{H})u\rangle_0$ is conserved. By Corollary 7.4 we have

$$\begin{aligned}\|e^{-itH}\chi_\mu(\dot{H})u\|^2_{\dot{\mathcal{E}}} &\lesssim \langle e^{-it\dot{H}}\chi_\mu(\dot{H})u, e^{-it\dot{H}}\chi_\mu(\dot{H})u\rangle_0 \\ &= \langle \chi_\mu(\dot{H})u, \chi_\mu(\dot{H})u\rangle_0 \lesssim \|\chi_\mu(\dot{H})u\|^2_{\dot{\mathcal{E}}}.\end{aligned}$$ □

7.2. *Low frequency analysis*

Part (ii) of Thm. 7.1 follows from the following

Lemma 7.6. *Assume (A1)–(A2), (TE1)–(TE3), (PE), and (B). Let $\chi \in C_0^\infty(\mathbb{R}\setminus\mathcal{S})$. Then there exists $C > 0$ such that*

$$\|e^{-it\dot{H}}\chi(\dot{H})u\|_{\dot{\mathcal{E}}} \leq C\|u\|_{\dot{\mathcal{E}}}, \quad u \in \dot{\mathcal{E}},\ t \in \mathbb{R}. \tag{7.4}$$

Proof. Let $u \in \dot{\mathcal{E}}$. Let

$$\psi(t) := (\psi_0(t), \psi_1(t)) := e^{-itk}\begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} e^{-it\dot{H}}\chi(\dot{H})u.$$

Note that

$$\|e^{-it\dot{H}}\chi(\dot{H})u\|^2_{\dot{\mathcal{E}}} = \|\psi_1\|^2 + (h(t)\psi_0(t)|\psi_0(t)) =: \|\psi(t)\|^2_{\dot{\mathcal{E}}(t)}$$

with $h(t) = e^{-itk}(h + k^2)e^{itk}$, and that solves the wave equation

$$(\partial_t^2 + h(t))\psi_0(t) = 0, \qquad \psi_1(t) = -i\partial_t\psi_0(t).$$

Thus

$$\frac{d}{dt}\|e^{-it\dot{H}}\chi(\dot{H})u\|^2_{\dot{\mathcal{E}}} = (h'(t)\psi_0(t)|\psi_0(t)),$$

where

$$h'(t) = e^{-itk}[-ik, h]e^{ikt} \lesssim w^{-1}e^{-ikt}h_0 e^{ikt}w^{-1},$$

by (B). Therefore

$$\frac{d}{dt}\|e^{-it\dot{H}}\chi(\dot{H})u\|^2_{\dot{\mathcal{E}}} \lesssim \|w^{-1}\psi(t)\|^2_{\dot{\mathcal{E}}(t)} \lesssim \|w^{-1}e^{-it\dot{H}}\chi(\dot{H})u\|^2_{\dot{\mathcal{E}}}, \tag{7.5}$$

where we have used the fact that w^{-1} commutes with $e^{-itk\left(\begin{smallmatrix}1&0\\k&1\end{smallmatrix}\right)}$ (see (TE1)). Integrating (7.5) we obtain

$$\|e^{-it\dot{H}}\chi(\dot{H})u\|^2_{\dot{\mathcal{E}}} \lesssim \|u\|^2_{\dot{\mathcal{E}}} + \int_0^t \|w^{-1}e^{-it'\dot{H}}\chi(\dot{H})u\|^2_{\dot{\mathcal{E}}}\, dt' \lesssim \|u\|^2_{\dot{\mathcal{E}}},$$

by Prop. 6.8. □

We end this section by proving a weak convergence result, which will be important in Sect. 8.

Lemma 7.7. *Assume* (A1)–(A2), (TE1)–(TE3), (PE)*, and* (B)*. Then*

$$e^{-it\dot{H}}\chi(\dot{H}) \rightharpoonup 0, \qquad \forall \chi \in C_0^\infty(\mathbb{R}\setminus\mathcal{S}).$$

Proof. Since $e^{-it\dot{H}}\chi(\dot{H})$ is uniformly bounded in t by Thm. 7.3, it suffices to prove that $\langle v|e^{-it\dot{H}}\chi(\dot{H})u\rangle_{\dot{\mathcal{E}}} \to 0$ for u, v in a dense subspace of $\dot{\mathcal{E}}$, where $\langle\cdot|\cdot\rangle_{\dot{\mathcal{E}}}$ is the scalar product associated to the norm of $\dot{\mathcal{E}}$. By (PE) the space $\{u \in \dot{\mathcal{E}} : u = \chi(x)u,\ \chi \in C_0^\infty(\mathbb{R})\}$ is dense in $\dot{\mathcal{E}}$. For such u, v the convergence to 0 follows from Prop. 6.7. □

8. Asymptotic completeness 1: abstract setting

In this section we prove existence and completeness of wave operators, comparing the full dynamics $e^{-it\dot{H}}$ with the two asymptotic dynamics $e^{-it\dot{H}_\pm}$, for energies away from the set $\mathcal{S}$ of singular points. We first define the spaces of *scattering states*.

Definition 8.1. We call $\chi \in C^\infty(\mathbb{R})$ an *admissible* cut-off function for $\dot{H}$ if

- $\chi = 0$ in a neighborhood of $\mathcal{S}$, and
- $\chi = 0$ or $\chi = 1$ on $\mathbb{R}\setminus\,]-R, R[$ for some $R > 0$.

We denote by $\mathcal{C}^H$ the set of all admissible cut-offs for $\dot{H}$.

Definition 8.2. The spaces of *scattering states* are defined as

$$\dot{\mathcal{E}}_{\text{scatt}} := \{\chi(\dot{H})u : u \in \dot{\mathcal{E}},\ \chi \in \mathcal{C}^H\}, \qquad \dot{\mathcal{E}}_{\text{scatt}\pm} := \{\chi(\dot{H}_\pm)u : u \in \dot{\mathcal{E}}_\pm,\ \chi \in \mathcal{C}^H\}.$$

We will need the following three lemmas.

Lemma 8.3. *Assume* (A1)–(A2) *and* (TE1)–(TE3)*. Then* $w[\dot{H}, i_\pm]w \in \mathcal{B}(\dot{\mathcal{E}}; \dot{\mathcal{E}}_\pm)$.

Proof. We have

$$w[\dot{H}, i_\pm]w = \begin{pmatrix} 0 & 0 \\ w[h, i_\pm]w & 0 \end{pmatrix} \in \mathcal{B}(\dot{\mathcal{E}}; \dot{\mathcal{E}}_\pm),$$

by hypothesis (TE3)(e). □

Lemma 8.4. *Assume* (A1)–(A2) *and* (TE1)–(TE3)*.*

(i) *Let* $\chi \in C_0^\infty(\mathbb{R})$*. Then*

$$i_\pm\chi(\dot{H}_\pm) - \chi(\dot{H})i_\pm \in \mathcal{B}_\infty(\dot{\mathcal{E}}_\pm; \dot{\mathcal{E}}).$$

(ii) *Let* $\chi \in C^\infty(\mathbb{R})$ *be such that* $\chi = 1$ *outside* $]-R, R[$ *for some* $R > 0$*. Then*

$$i_\pm\chi(\dot{H}_\pm) - \chi(\dot{H})i_\pm \in \mathcal{B}_\infty(\dot{\mathcal{E}}_\pm; \dot{\mathcal{E}}).$$

Proof. Note first that (ii) follows from (i), by replacing χ by $1-\chi$. We therefore only have to prove (i). Let $\tilde{\chi}$ be an almost analytic extension of χ such that $\operatorname{supp}\tilde{\chi}$ does not contain complex poles of $\dot{R}(z)$. We have

$$i_\pm\chi(\dot{H}_\pm)-\chi(\dot{H})i_\pm=\frac{1}{2\pi i}\int\overline{\partial}\tilde{\chi}(z)\dot{R}(z)[\dot{H},i_\pm]\dot{R}_\pm(z)\,dz\wedge d\overline{z}.$$

By hypotheses (TE3)(b) and (TE3)(e) we have

$$[\dot{H},i_\pm]\dot{R}_\pm(z)\in\mathcal{B}_\infty(\dot{\mathcal{E}}_\pm;\dot{\mathcal{E}}).$$

Then we apply the estimates in Lemma 5.6. □

Theorem 8.5. *Assume* (A1)–(A2), (TE1)–(TE3), (PE), *and* (B).

(i) *For all* $\varphi^\pm\in\dot{\mathcal{E}}_{\mathrm{scatt}\pm}$ *there exists* $\psi^\pm\in\dot{\mathcal{E}}_{\mathrm{scatt}}$ *such that*

$$e^{-it\dot{H}}\psi^\pm-i_\pm e^{-it\dot{H}_\pm}\varphi^\pm\to0,\quad t\to\infty,\quad \text{in } \dot{\mathcal{E}}.$$

(ii) *For all* $\psi\in\dot{\mathcal{E}}_{\mathrm{scatt}}$ *there exist* $\varphi^\pm\in\dot{\mathcal{E}}_{\mathrm{scatt}\pm}$ *such that*

$$e^{-it\dot{H}_\pm}\varphi^\pm-i_\pm e^{-it\dot{H}}\psi\to0,\quad t\to\infty,\quad \text{in } \dot{\mathcal{E}}_\pm.$$

Proof. Let $\chi\in\mathcal{C}^H$. We only prove (i), the proof of (ii) being analogous. We first show that the limit

$$W^\pm\varphi:=\lim_{t\to\infty}e^{it\dot{H}}\chi(\dot{H})i_\pm e^{-it\dot{H}_\pm}\chi(\dot{H}_\pm)\varphi \tag{8.1}$$

exists for all $\varphi\in\dot{\mathcal{E}}_\pm$. We first treat the case $\chi=0$ on $\mathbb{R}\setminus\,]-R,R[$. Using Thm. 7.1(i), Lemma 5.4 and the fact that $\dot{H}_\pm$ are selfadjoint, we obtain

$$\|e^{it\dot{H}}\chi(\dot{H})i_\pm e^{-it\dot{H}_\pm}\chi(\dot{H}_\pm)\varphi\|_{\dot{\mathcal{E}}}\lesssim\|\varphi\|_{\dot{\mathcal{E}}_\pm}. \tag{8.2}$$

By (8.2) and assumption (PE) we may assume that $\varphi\in D(w)$. We compute

$$\frac{d}{dt}e^{it\dot{H}}\chi(\dot{H})i_\pm e^{-it\dot{H}_\pm}\chi(\dot{H}_\pm)=e^{it\dot{H}}\chi(\dot{H})[\dot{H},i_\pm]e^{-it\dot{H}_\pm}\chi(\dot{H}_\pm). \tag{8.3}$$

Integrating (8.3) and using Lemma 8.3 and Proposition 6.7 we obtain

$$\begin{aligned}&\|e^{it\dot{H}}\chi(\dot{H})i_\pm e^{-it\dot{H}_\pm}\chi(\dot{H}_\pm)-e^{is\dot{H}}\chi(\dot{H})i_\pm e^{-is\dot{H}_\pm}\chi(\dot{H}_\pm)\varphi\|\\&\quad\lesssim\int_s^t\|w^{-1}e^{-it'\dot{H}_\pm}\chi(\dot{H}_\pm)\varphi\|_{\dot{\mathcal{E}}_\pm}\,dt'\lesssim\int_s^t\langle t'\rangle^{-2}\,dt'\,\|w\varphi\|_{\dot{\mathcal{E}}_\pm}\to0,\quad s,t\to\infty.\end{aligned}$$

This gives the existence of the limit (8.1). Let now $\chi=1$ on $\mathbb{R}\setminus]-R,R[$. Let $\hat{\chi}\in C_0^\infty(\mathbb{R})$ with $\hat{\chi}=1$ in a neighborhood of 0. Using the fact that $e^{it\dot{H}}\chi(\dot{H})$ is uniformly bounded by Thm. 7.1(ii), and Lemmas 8.4 and 6.11, we see that

$$\begin{aligned}&\operatorname*{s-lim}_{t\to\infty}e^{it\dot{H}}\chi(\dot{H})i_\pm e^{-it\dot{H}_\pm}\hat{\chi}^2(L^{-1}\dot{H}_\pm)\chi(\dot{H}_\pm)\\&\qquad=\operatorname*{s-lim}_{t\to\infty}e^{it\dot{H}}\chi(\dot{H})\hat{\chi}(L^{-1}\dot{H})i_\pm e^{-it\dot{H}_\pm}\hat{\chi}(L^{-1}\dot{H}_\pm)\chi(\dot{H}_\pm)\end{aligned}$$

exists, since $\hat{\chi}(L^{-1}\cdot)$ is compactly supported. Let $\epsilon > 0$. We estimate

$$\begin{aligned}&\|e^{it\dot{H}}\chi(\dot{H})i_\pm e^{-it\dot{H}_\pm}\chi(\dot{H}_\pm)\varphi^\pm - e^{is\dot{H}}\chi(\dot{H})i_\pm e^{-is\dot{H}_\pm}\chi(\dot{H}_\pm)\varphi^\pm\|_{\dot{\mathcal{E}}}\\ &\le \|e^{it\dot{H}}\chi(\dot{H})i_\pm e^{-it\dot{H}_\pm}\hat{\chi}^2(L^{-1}\dot{H}_\pm)\chi(\dot{H}_\pm)\varphi^\pm\\ &\quad - e^{is\dot{H}}\chi(\dot{H})i_\pm e^{-is\dot{H}_\pm}\hat{\chi}^2(L^{-1}\dot{H}_\pm)\chi(\dot{H}_\pm)\varphi^\pm\|_{\dot{\mathcal{E}}} + 2\|(1-\hat{\chi}^2(L^{-1}\dot{H}_\pm))\varphi^\pm\|_{\dot{\mathcal{E}}} < \epsilon,\end{aligned}$$

if we choose first L and then t, s large enough. This shows the existence of the limit (8.1) if $\chi = 1$ on $\mathbb{R}\setminus\,]-R, R[$.

For $\phi^\pm \in \dot{\mathcal{E}}_\pm$ let

$$\psi_t^\pm := e^{it\dot{H}}\chi(\dot{H})i_\pm e^{-it\dot{H}_\pm}\chi(\dot{H}_\pm)\phi^\pm, \qquad \psi^\pm := \lim_{t\to\infty}\psi_t^\pm.$$

Let us write

$$\psi_t^\pm = \psi^\pm + r(t), \qquad r(t)\to 0,\ t\to\infty.$$

Let $\tilde{\chi} \in \mathcal{C}^H$ with $\tilde{\chi}\chi = \chi$. We clearly have $\tilde{\chi}(\dot{H})\psi_t^\pm = \psi_t^\pm$ and thus

$$\tilde{\chi}(\dot{H})\psi^\pm + \tilde{\chi}(\dot{H})r(t) = \psi^\pm + r(t).$$

Taking the limit $t\to\infty$ we find

$$\tilde{\chi}(\dot{H})\psi^\pm = \psi^\pm, \quad \text{in particular} \quad \psi^\pm \in \dot{\mathcal{E}}_{\text{scatt}},$$

hence $e^{-it\dot{H}}\psi^\pm$ is uniformly bounded by Thm. 7.1. It follows that

$$e^{-it\dot{H}}\psi^\pm - \chi(\dot{H})i_\pm e^{-it\dot{H}_\pm}\chi(\dot{H}_\pm)\phi^\pm \to 0.$$

Applying Lemma 8.4 we find

$$e^{-it\dot{H}}\psi^\pm - i_\pm e^{-it\dot{H}_\pm}\chi^2(\dot{H}_\pm)\phi^\pm \to 0, \qquad t\to\infty. \tag{8.4}$$

Applying once more a density argument we obtain (i). □

9. Geometric setting

In this section we consider the geometric framework introduced in Subsect. 2.1. The main task will be to check that the hypotheses (G) imply the abstract hypotheses of Sects. 4, 5. We will check the hypotheses for the restrictions of the operators to $\mathcal{H}^n$ and the corresponding energy spaces. In the following, we will drop the index n.

9.1. Asymptotic Hamiltonians

To apply the framework of Sect. 5 we need a coordinate function x on $\mathcal{M}$ with range equal to $\mathbb{R}$. This is easily done with the change of variables given by

$$\frac{dx}{dr} = \alpha_1^{-2}(r).$$

Note that there is a freedom in the choice of the integration constant. This choice, however, is not important for what follows. For $r \to r_-$ we find

$$x(r) - x(r_-) = \int_{r_-}^{r} \frac{1}{\alpha_-^2 q^2}\, dr' + \int_{r_-}^{r} h(r')\, dr'$$

with $h(r) \in \mathcal{O}((r - r_-)^{-1+\delta})$. Here we have used (G2). Recalling that $q(r) = \sqrt{(r_+ - r)(r - r_-)}$ we find, for r close to r_-,

$$x(r) - x(r_-) \geq \frac{1}{\kappa_-} \ln(r - r_-) - C$$

with $\kappa_- = (\alpha_-)^2(r_+ - r_-)$. It follows that

$$r - r_- \lesssim e^{\kappa_- x}, \qquad r \to r_-.$$

In a similar way we obtain

$$r_+ - r \lesssim e^{-\kappa_+ x}, \qquad r \to r_+,$$

where $\kappa_+ = (\alpha_+)^2(r_+ - r_-)$. Since $\partial_x = \alpha_1^{-2}(r)\partial_r$, we find that

$$f(r) \in T^\sigma \quad \text{iff} \quad f(r(x)) \in T_x^\sigma \quad \text{for}$$

$$T_x^\sigma := \left\{ f \in C^\infty(\mathbb{R} \times \mathbb{S}^{d-1}) : \partial_x^\alpha \partial_\omega^\beta f \in \left\{ \begin{array}{c} \mathcal{O}(e^{\sigma\kappa_- x/2}),\ x \to -\infty, \\ \mathcal{O}(e^{-\sigma\kappa_+ x/2}),\ x \to \infty \end{array} \right\} \right\}.$$

This change of variables gives rise to the unitary transformation

$$\mathcal{U}_1 :\ L^2(]r_-, r_+[\times \mathbb{S}^{d-1}, dr d\omega) \to L^2(\mathbb{R} \times \mathbb{S}^{d-1}, dx d\omega) =: \mathcal{H}_1,$$
$$v(r, \omega) \mapsto \alpha_1(r(x)) v(x, \omega).$$

We set

$$\mathcal{E}_{\pm,1} := (\mathcal{U}_1 \oplus \mathcal{U}_1)\mathcal{E}_\pm, \qquad h_\pm^1 := \mathcal{U}_1 h_\pm \mathcal{U}_1^{-1}, \qquad k_\pm^1 := \mathcal{U}_1 k_\pm \mathcal{U}_1^{-1}.$$

We compute

$$\begin{aligned} h_0^1 = \mathcal{U}_1 h_0 \mathcal{U}_1^{-1} &= \mathcal{U}_1 h_{0,s} \mathcal{U}_1^{-1} + \sum_{i,j=1}^{d-1} D_i^* g^{ij} D_j + \sum_{i=1}^{d-1} (g^i D_i + D_i^* \overline{g}^i) \\ &\quad + \alpha_1^{-1}(r) D_x g^{rr} \alpha_1^{-2}(r) D_x \alpha_1^{-1}(r) + \alpha_1^{-1}(r) g^r D_x \alpha_1^{-1}(r) \\ &\quad + \alpha_1^{-1}(r) D_x \overline{g}^r \alpha_1^{-1}(r) + f, \end{aligned}$$

$$\mathcal{U}_1 h_{0,s} \mathcal{U}_1 = D_x \alpha_2^2 \alpha_1^{-2} D_x + \alpha_3^2 P + \alpha_4^2.$$

We will often drop the exponent 1 when it is clear which coordinate system is used.

9.2. *Meromorphic extensions*

In this subsection we will check that $h_+, \tilde{h}_-$ satisfy (ME2). To do so we use a result of Mazzeo and Melrose [26] about the meromorphic extension of the truncated resolvent for the Laplace operator on asymptotically hyperbolic manifolds. We start by briefly recalling this result.

9.2.1. A result of Mazzeo–Melrose. Let Y be a compact n-dimensional manifold with boundary given by the defining function y:

$$\partial Y = \{y = 0\}, \quad dy|_{\partial Y} \neq 0, \quad y|_{Y^0} > 0.$$

Let g be a complete metric on Y of the form

$$g = h/y^2, \tag{9.1}$$

where h is a C^∞ metric on Y. One is usually interested in the Laplace–Beltrami operator Δ_g. We have to consider slightly more general operators. Let

$$\mathcal{V}_0(Y) = \{V \in C^\infty(Y; TY) : V|_{\partial Y} = 0\},$$

the space of vector fields vanishing on the boundary. In local coordinates (y, x) near ∂Y the vector fields $y\partial_y$, $y\partial_{x_j}$ span $\mathcal{V}_0(Y)$.

We now need the definition of the *normal operator*. For $p \in \partial Y$ the tangent space T_pY is divided into two half-spaces by the hypersurface $T_p\partial Y$. We will denote by Y_p the half-space on the "Y" side (that is, spanned by $T_p\partial Y$ and the inward normal vector at p). Then any smooth coefficient polynomial Q in $\mathcal{V}_0(Y)$ defines a natural constant coefficient operator on Y_p:

$$N_p(Q)u := \lim_{r\to 0} R_r^* f^* Q (f^{-1})^* R_{1/r}^* u, \tag{9.2}$$

where $u \in C^\infty(Y_p)$, R_r is the natural $\mathbb{R}_+$-action on $Y_p \simeq N_+T_p\partial Y$ given by multiplication by r on the fibers, and f is a local diffeomorphism from $\Omega \subset Y$, $p \in \Omega$:

$$f : \Omega \to \Omega', \quad \Omega' \subset T_pY, \quad f(p) = 0, \quad df_p = I, \quad f(\partial Y) \subset T_p\partial Y.$$

The definition is independent of f. The normal operator freezes the coefficients at a point p, one obtains a polynomial in the elements of $\mathcal{V}_0(Y_p)$. The following result is implicit in [26]. We use here the formulation in [28].

Proposition 9.1 (Mazzeo–Melrose, 1987). *Let Q be a second order differential operator on Y which is a polynomial in $\mathcal{V}_0(Y)$ with coefficients in $C^\infty(Y)$. Assume that*

(i) *the principal part of Q is an elliptic polynomial in the elements of $\mathcal{V}_0(Y)$ uniformly on Y,*
(ii) *for every $p \in \partial Y$ the normal operator of Q defined by (9.2) is given by*

$$N_p(Q) = -K\left[z_1^2 D_{z_1}^2 + i(n-2)z_1 D_{z_1} + z_1^2 \sum_{i,j=2}^{n} h_{ij}(p) D_{z_i} D_{z_j} - \left(\frac{n-1}{2}\right)^2\right],$$
$$Y_p = \{z \in \mathbb{R}^n : z_1 \geq 0\}, \quad [h_{ij}] \geq C\mathbb{1}, \ C > 0,$$

where $K < 0$ is constant on the components of ∂Y.

Then for any metric g of the form (9.1),

$$R_Q(z) = (Q - z^2)^{-1} : L^2(Y, \mathrm{dvol}_g) \to L^2(Y, \mathrm{dvol}_g)$$

is holomorphic in $\{\operatorname{Im} z \gg 1\}$*. For $N > 0$ the operator $y^N R_Q(z) y^N$ extends to a meromorphic operator in* $\{\operatorname{Im} z > -\delta\}$ *for some* $\delta > 0$.

Remark 9.2. The width of the strip onto which one can extend the truncated resolvent is of order N if one removes some special points along the imaginary axis. At these points, which are given by $(-K)^{1/2}(-i)\left(\frac{2k-1}{2}\right)^{1/2}$, $k \in \mathbb{N}$, essential singularities might occur. If the operator is the Laplacian associated to an asymptotic hyperbolic metric and the metric is even, then no essential singularities appear—see [20] for a detailed discussion of these questions. In our case it is sufficient to know that there exists a meromorphic extension in some strip, and we will not study the type of singularities.

9.2.2. Meromorphic extensions of the resolvents of $h_+, \tilde{h}_-$

Lemma 9.3. *Assume hypotheses* (G)*. Then* (h_+, k_+, w) *and* $(\tilde{h}_-, k_- - \ell, w)$ *satisfy* (ME1)–(ME2) *for* $w = q(r)^{-1}$.

Proof. We will show that $w^{-\epsilon}(h^n_\pm - z^2)^{-1} w^{-\epsilon}$ has a meromorphic extension to a strip $\{\operatorname{Im} z > -\delta_\epsilon\}$, $\delta_\epsilon > 0$. Let us start with h_+.

We want to apply Prop. 9.1 for $Y = \overline{\mathcal{M}} = [r_-, r_+] \times \mathbb{S}^{d-1}$. The principal part of h^n_+ is an elliptic polynomial in the elements of $\mathcal{V}_0(Y)$ and hypothesis (i) of Prop. 9.1 is fulfilled. Near each boundary component we set $z = r - r_-$ and $z = r_+ - r$ resp. We change the C^∞ structure on Y (as a manifold with boundary) and allow a new smooth coordinate $y = \sqrt{z}$. We will denote the new manifold by $Y_{1/2}$ and think of $Y_{1/2}$ as a conformally compact manifold in the sense of having a metric of the form (9.1).

Near $r = r_-$ the operator h^n_+ becomes

$$\begin{aligned} h_+ = {}& \tfrac{1}{4}(\alpha_1^+)^2(\alpha_2^+)^2(r_+ - r_-)^4 D_y y D_y y + (\alpha_3^+)^2 (r_+ - r_-)^2 y^2 \sum_{i,j=1}^{d-1} D_i^* \alpha_{ij} D_j \\ & - (k^+_{s,r} n + k^+_{s,v})^2 + \mathcal{O}(y^\delta)(D_y y)^2 + \mathcal{O}(y^\delta) y^2 \sum_{i,j=1}^{d-1} D_i^* \alpha_{ij} D_j + \mathcal{O}(y^\delta). \end{aligned}$$

We now conjugate h_+ by a weight function (see [28]) and set

$$Q = ((1-\chi) + \chi y^2) h_+ ((1-\chi) + \chi y^2)^{-1}, \tag{9.3}$$

where $\chi \in C^\infty(Y)$ with $\chi = 1$ for $y < \epsilon < 1/2$ and $\chi = 0$ for $y > 2\epsilon$. It follows that the normal operator becomes

$$\begin{aligned} N_p(Q) = {}& \tfrac{1}{4}(\alpha_1^+)^2(\alpha_2^+)^2(r_+ - r_-)^4 \Bigg(y^2 \Bigg(D_y^2 + \frac{4(\alpha_3^+)^2}{(\alpha_1^+)^2(\alpha_2^+)^2(r_+ - r_-)^2} \sum_{i,j=1}^{d-1} D_i^* \alpha_{ij}(p) D_j \Bigg) \\ & + i y D_y - 1 - \frac{4(k^+_{s,r} n + k^+_{s,v})^2}{(\alpha_1^+)^2(\alpha_2^+)^2(r_+ - r_-)^4} \Bigg). \end{aligned}$$

This operator is shifted with respect to the model operator of Prop. 9.1, and the points where essential singularities may occur are now given by $z^2 = (-K)\big(\beta - \big(\frac{1-2k}{2}\big)^2\big)$, $k \in \mathbb{N}$, where $-K = \frac{1}{4}(\alpha_1^+)^2(\alpha_2^+)^2(r_+ - r_-)^4$ and $\beta = -\frac{4(k_{s,r}^- n + k_{s,v})^2}{(\alpha_1^+)^2(\alpha^+)^2(r_+-r_-)^4}$. As β is negative, all these points have strictly negative imaginary part. Hence we obtain a meromorphic continuation of $w^{-\epsilon}(Q - z^2)^{-1}w^{-\epsilon}$, where Q is given by (9.3). Since for $\mathrm{Im}\, z \gg 0$ we have

$$(h_+ - z^2)^{-1} = ((1-\chi) + \chi y^2)^{-1}(Q - z^2)^{-1}((1-\chi) + \chi y),$$

we obtain a meromorphic continuation of $w^{-\epsilon}(h_+ - z^2)^{-1}w^{-\epsilon}$. The proofs near $r = r_+$ and for $\tilde{h}_-$ are similar. □

9.3. Verification of the abstract hypotheses

Proposition 9.4. *Assume hypotheses* (G). *Then conditions* (A1)–(A2)*,* (TE1)–(TE3)*,* (PE)*, and* (B) *are satisfied.*

The rest of the subsection is devoted to the proof of Prop. 9.4. We start by some preparations.

9.3.1. Some useful facts. By (G5) we have estimates

$$\|q(r)D_r\alpha_1(r)u\| \lesssim \|h_0^{1/2}u\|, \tag{9.4}$$

$$\|q(r)D_j u\| \lesssim \|h_0^{1/2}u\|, \quad j = 1, \dots, d-1, \tag{9.5}$$

$$\|q(r)u\| \lesssim \|h_0^{1/2}u\|. \tag{9.6}$$

The estimates (9.4)–(9.6) also hold with h_0 replaced by h_+ or $\tilde{h}_-$. We will also need the following Hardy type estimate.

Lemma 9.5. *We have*

$$\text{(i) } \|\langle x(r)\rangle^{-1}u\|_{\mathcal{H}} \lesssim \|h_0^{1/2}u\|_{\mathcal{H}},$$

$$\text{(ii) } \|fu\|_{\mathcal{H}} \lesssim \|h_0^{1/2}u\|_{\mathcal{H}}, \quad f \in T^{\delta},\ \delta > 0.$$

Proof. Since $\langle x(r)\rangle \sim |\ln(r - r_\pm)|$ as $r \to r_\pm$, (ii) follows from (i). We recall a version of Hardy's inequality:

$$\int_0^\infty |v(x)|^2 x^{-2}\, dx \le 4\int_0^\infty |v'(x)|^2\, dx, \quad v \in C_0^\infty(\mathbb{R} \setminus \{0\}). \tag{9.7}$$

Let $\chi_1 \in C_0^\infty(\mathbb{R})$ with $\chi_1(0) = 1$, and $\chi_2 \in C^\infty(\mathbb{R})$ with $\chi_1 + \chi_2 = 1$. We have

$$\|\langle x\rangle^{-1}\chi_1 u\|^2_{\mathcal{H}_1} \lesssim \|\chi_1 u\|^2_{\mathcal{H}_1} \lesssim ((-\partial_x^2 + \alpha_1^2)u \mid u)_{\mathcal{H}_1}$$

because $\alpha_1^2 \gtrsim \chi_1^2$. Now applying (9.7) to $\chi_2 u$ gives

$$\begin{aligned}\|\langle x\rangle^{-1}\chi_2 u\|_{\mathcal{H}_1}^2 &\lesssim \int_{\mathbb{R}\times\mathbb{S}^2} |\partial_x(\chi_2 u)|^2\,dx\,d\omega \\ &\lesssim \int_{\mathbb{R}\times\mathbb{S}^2} (\chi_2')^2|u|^2\,dx\,d\omega + \int_{\mathbb{R}\times\mathbb{S}^2} \chi_2^2|u'|^2\,dx\,d\omega \lesssim ((-\partial_x^2+\alpha_1^2)u \mid u)_{\mathcal{H}_1}\end{aligned}$$

because $(\chi_2')^2 \lesssim \alpha_1^2$. It follows that

$$\int \langle x\rangle^{-2}|u|^2\,dx\,d\omega \lesssim \int (|D_x u|^2 + \alpha_1^2|u|^2)\,dx\,d\omega.$$

Changing to (r,ω) coordinates yields, since $dx = \frac{1}{\alpha_1^2}dr$ and $D_x = \alpha_1^2 D_r$,

$$\int \langle x(r)\rangle^{-4}|u|^2\frac{1}{\alpha_1^2}\,dr\,d\omega \lesssim \int (|\alpha_1 D_r u|^2 + |u|^2)\,dr\,d\omega.$$

Setting $v = \frac{1}{\alpha_1}u$ gives

$$\begin{aligned}\int \langle x(r)\rangle^{-4}|v|^2\,dr\,d\omega &\lesssim \int (|\alpha_1 D_r \alpha_1 v|^2 + \alpha_1^2|v|^2)\,dr\,d\omega, \\ &\lesssim \int (|q D_r \alpha_1 v|^2 + \alpha_1^2|v|^2)\,dr\,d\omega,\end{aligned}$$

and using $h_0 \gtrsim \alpha_1(D_r q^2 D_r + 1)\alpha_1$ and (G5) we complete the proof. □

Lemma 9.6. *Let $f, g \in C^\infty(\mathbb{R})$ with $\lim_{|x|\to\infty} f(x) = \lim_{|x|\to\infty} g(x) = 0$. Then the operators $f(x)g(h_+)$ and $f(x)g(\tilde{h}_-)$ are compact.*

Proof. We only prove the lemma for h_+, the proof for $\tilde{h}_-$ being analogous. We may assume that $f, g \in C_0^\infty(\mathbb{R})$. Let Ω be a bounded domain which contains supp f. Then $f(x)g(h_+)$ sends $L^2(\mathcal{M})$ to $H^2(\Omega)$. But $H^2(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L^2(\mathcal{M})$ and the first embedding is compact. □

9.3.2. Verification of hypotheses (A1)–(A2). We have already noticed that (G5) implies (A1) (in particular (G5) implies $0 \notin \sigma_{\rm pp}(h_0)$). Let us check (A2). We first check that $h_0^{1/2}kh_0^{-1/2} \in \mathcal{B}(\mathcal{H})$. This will follow from

$$kh_0k \lesssim h_0 \quad \text{on } C_0^\infty(\mathcal{M}). \tag{9.8}$$

Several terms have to be estimated:

$$\begin{aligned}-k\alpha_1\partial_r\alpha_2^2\partial_r\alpha_1 k &= -\alpha_1\partial_r k^2\alpha_2^2\partial_r\alpha_1 - \alpha_1\partial_r k\alpha_2^2\alpha_1 k' + \alpha_1^2 k'^2\alpha_2^2 + \alpha_1 k'\alpha_2^2 k\partial_r\alpha_1 \\ &\lesssim -\alpha_1\partial_r q^2\partial_r\alpha_1 - \alpha_1\partial_r k\alpha_2^2\alpha_1 k' + \alpha_1^2 k'^2\alpha_2^2 + \alpha_1 k'\alpha_2 2k\partial_r\alpha_1, \\ -k\partial_r g^{rr}\partial_r k &= -\alpha_1\partial_r\left(\frac{k}{\alpha_1}\right)^2 g^{rr}\partial_r\alpha_1 - \alpha_1\partial_r k g^{rr}\left(\frac{k}{\alpha_1}\right)' + \left(\frac{k}{\alpha_1}\right)' g^{rr}k\partial_r\alpha_1 \\ &\lesssim -\alpha_1\partial_r q^2\partial_r\alpha_1 - \alpha_1\partial_r k\left(\frac{k}{\alpha_1}\right)' g^{rr} + \left(\frac{k}{\alpha_1}\right)' g^{rr}k\partial_r\alpha_1,\end{aligned}$$

and

$$kg^r D_r k = \frac{k^2}{\alpha_1} g^r D_r \alpha_1 + \frac{1}{i} k g^r \left(\frac{k}{\alpha_1}\right)' \alpha_1.$$

Summing and adding the angular terms we find

$$\begin{aligned}
kh_0 k \lesssim{}& \alpha_1 D_r q^2 D_r \alpha_1 - \alpha_1 \partial_r k \alpha_2^2 \alpha_1 k' + \alpha_1^2 (k')^2 \alpha_2^2 + \alpha_1 k' \alpha_2^2 k \partial_r \alpha_1 - \alpha_1 \partial_r k \left(\frac{k}{\alpha_1}\right)' g^{rr} \\
&+ \left(\frac{k}{\alpha_1}\right)' k g^{rr} \partial_r \alpha_1 + \frac{1}{i} k g^r \left(\frac{k}{\alpha_1}\right)' \alpha_1 + \frac{k^2}{\alpha_1} g^r D_r \alpha_1 + \alpha_1 D_r \frac{k^2}{\alpha_1} \overline{g}^r - \frac{1}{i} \left(\frac{k}{\alpha_1}\right)' k \alpha_1 \overline{g}^r \\
&+ \sum_{i,j} \alpha_3 D_i^* \alpha_{ij} k^2 D_j \alpha_3 + k^2 \alpha_4^2 + \sum_{i,j} D_i^* k^2 g^{ij} D_j + \sum_i (g^i k^2 D_i + D_i^* k^2 \overline{g}^i) + k^2 f \\
&+ \sum_{i,j} D_i^* k \alpha_3^2 \alpha_{ij} (D_j k) - \sum_{i,j} \alpha_3^2 (D_i^* k) \alpha_{ij} k D_j - \sum_{i,j} \alpha_3^2 (D_i^* k) \alpha_{ij} (D_j k) \\
&+ \sum_{i,j} D_i^* k g^{ij} (D_j k) - \sum_{i,j} (D_i^* k) g^{ij} k D_j - \sum_{i,j} (D_i^* k) g^{ij} (D_j k) \\
&+ \sum_i k g^i (D_i k) - \sum_i (D_i^* k) k \overline{g}^i + 2 \operatorname{Im} g^r k k'.
\end{aligned}$$

We have

$$k\alpha_2^2 k' \in T^2, \quad \alpha_1^2 (k')^2 \alpha_2^2 \in T^4, \quad k' \alpha_2^2 k \in T^2, \quad \frac{k}{q} \left(\frac{k}{\alpha_1}\right)' g^{rr} \in T^\delta, \quad k \alpha_1 g^r \left(\frac{k}{\alpha_1}\right)' \in T^\delta,$$

$$\frac{k^2}{\alpha_1} g^r \frac{1}{q} \in T^\delta, \; g^i k^2 \in T^2, \quad k\alpha_3^2 \alpha_{ij} (D_i k) \in T^4, \quad \alpha_3^2 (D_i^* k) \alpha_{ij} (D_j k) \in T^6,$$

$$k g^{ij} (D_j k) \in T^{4+\delta}, \quad (D_i^* k) g^{ij} (D_j k) \in T^{6+\delta}, \quad k g^i (D_i k) \in T^4, \quad g^r k k' \in T^{2+\delta}.$$

This gives (9.8) by using (G5), (9.4)–(9.6) and Lemma 9.5. The estimate for

$$\|(k-z)^{-1}\|_{\mathcal{B}(h_0^{-1/2}\mathcal{H})}$$

is exactly the same with the derivatives of k replaced by $\frac{\partial k}{(k-z)^2}$. Here we also use the inequalities

$$\|(k-z)^{-1}\|_{\mathcal{B}(\mathcal{H})} \le \frac{1}{|\operatorname{Im} z|}, \qquad \|(k-z)^{-1}\|_{\mathcal{B}(\mathcal{H})} \le \frac{1}{|z| - \|k\|_{\mathcal{B}(\mathcal{H})}},$$

the second one being valid for $|z| \ge (1+\epsilon)\|k\|_{\mathcal{B}(\mathcal{H})}$, $\epsilon > 0$.

9.3.3. Verification of hypotheses (TE1)–(TE3). (TE1) is obvious; let us check (TE2). We check it for h_+, the proof for $\tilde{h}_-$ being analogous. First note that (G5) for h_+ implies $0 \notin \sigma_{\mathrm{pp}}(h_+)$. We have $k_+^2 \in T^2$. This implies the estimate

$$\|k_+ u\| \lesssim \|h_+^{1/2} u\|.$$

We now check that (TE3) is fulfilled. Recall that $w = q^{-1}$.

– (TE3)(a) follows from (G3).
– (TE3)(b) is clear.
– (TE3)(c): We have already shown in Sect. 9.2.2 that $h_+, \tilde{h}_-$ fulfill (ME2). Let us check (ME1):

  – (ME1)(a) follows from (G3).
  – (ME1)(b) is clear.
  – (ME1)(c): Let us show that $h_+^{-1/2}[h_+, w^{-\epsilon}]w^{\epsilon/2}$ is bounded. We have

$$\begin{aligned}[ih_+, w^{-\epsilon}] &= \alpha_1 D_r \alpha_2^2 \alpha_1 (w^{-\epsilon})' + D_r g^{rr}(w^{-\epsilon})' + g^r(w^{-\epsilon})' + hc\\ &= \alpha_1 D_r \alpha_2^2 \alpha_1 (w^{-\epsilon})' + \alpha_1 D_r \frac{g^{rr}}{\alpha_1}(w^{-\epsilon})' - \frac{1}{i}\alpha_1 \left(\frac{1}{\alpha_1}\right)' g^{rr}(w^{-\epsilon})'\\ &\quad + g^r(w^{-\epsilon})' + hc\\ &= \alpha_1 D_r q\alpha + \beta\end{aligned}$$

  with $\alpha \in T^\epsilon$ and $\beta \in T^{\epsilon+\delta}$. We have $h_+^{-1/2}\beta w^{\epsilon/2} \in \mathcal{B}(\mathcal{H})$ by Lemma 9.5. We have $h_+^{-1/2}\alpha_1 D_r q\alpha w^{\epsilon/2} \in \mathcal{B}(\mathcal{H})$ by (9.4). The proof for $\tilde{h}_-$ is analogous.
  – (ME1)(d) follows from Lemma 9.5.
  – (ME1)(e) follows from Lemma 9.6.

– (TE3)(d): The proof is exactly the same as for (A2), we omit the details.
– (TE3)(e): We start with $w[h, i_+]wh_+^{-1/2}$. We have

$$w[ih, i_+]w = w(\alpha_1 D_r \alpha_2^2 \alpha_1 i_+' + D_r g^{rr} i_+' + g^r i_+' + hc)w = \alpha q D_r \alpha_1 + \beta$$

  with $\alpha \in T^\infty$ and $\beta \in T^\infty$. This gives $w[h, i_+]wh_+^{-1/2} \in \mathcal{B}(\mathcal{H})$. The proof for the other operators is the same, except for $h_0^{-1/2}[w^{-1}, h_0]w^{1/2}$ for which it is analogous to the proof for $h_+^{-1/2}[h, w^{-\epsilon}]w^{\epsilon/2}$. We omit the details.
– (TE3)(f) follows from Hardy's inequality of Lemma 9.5.

9.3.4. Verification of hypotheses (PE) *and* (B). For (PE), thanks to (9.4)–(9.6) we see that $\|h_0^{1/2}u\|^2$ is equivalent to

$$\|D_x u\|^2 + \sum_{j=1}^{d-1} \|q(r(x))D_j u\|^2 + \|q(r(x))u\|^2.$$

As $\psi(x/n)u \to u$ in $L^2(\mathbb{R} \times \mathbb{S}^{d-1})$, we only have to show that

$$[iD_x, \psi(x/n)]u \to 0$$

for $u \in h_0^{-1/2}\mathcal{H}$. We have

$$\left[iD_x, \psi\left(\frac{x}{n}\right)\right]u = \frac{x}{n}\psi'\left(\frac{x}{n}\right)\frac{1}{x}h_0^{-1/2}h_0^{1/2}u.$$

By Lemma 9.5 it is sufficient to show that

$$\frac{x}{n}\psi'\left(\frac{x}{n}\right)v \to 0 \quad \text{for } v \in L^2(\mathbb{R}\times\mathbb{S}^{d-1}),$$

which is obvious.

For (B), first note that $w^{-\epsilon}$ clearly sends $D(h_0)$ into itself. We compute

$$[ih_{0,s}, k] = \alpha_1 D_r \alpha_2^2 k' \alpha_1 + \sum_{i,j} D_i^* \alpha_{ij}(\partial_j k)\alpha_3^2 + hc =: C_r + C_\omega.$$

We have

$$w^\epsilon C_r w^\epsilon = \alpha_1 w^\epsilon D_r \alpha_2^2 k' \alpha_1 w^\epsilon + hc = \alpha_1 D_r \alpha_2^2 k' \alpha_1 w^{2\epsilon} k' \alpha_1 - \frac{1}{i}(w^\epsilon)' \alpha_2^2 \alpha_1^2 w^\epsilon + hc.$$

Using

$$\alpha_2^2 w^{2\epsilon} k' \in T^{2-2\epsilon}, \quad (w^\epsilon)'\alpha_2^2\alpha_1^2 w^\epsilon \in T^{2-2\epsilon}, \quad \alpha_3^2 \alpha_{ij}(\partial_j k) w^{2\epsilon} \in T^{4-2\epsilon},$$

we find that for $\epsilon < 1$,

$$w^\epsilon C_r w^\epsilon \lesssim h_0, \quad w^\epsilon C_\omega w^\epsilon \lesssim h_0,$$

by Lemma 9.5. We now compute

$$\begin{aligned} w^\epsilon[i(h_0 - h_{0,s}), k]w^\epsilon = &\sum_{i,j} D_i^* q \frac{g^{ij}(iD_j k)w^{2\epsilon}}{q} + \sum_i g^i (D_i k) w^{2\epsilon} \\ &+ \alpha_1 D_r w^{2\epsilon} \alpha_1^{-1} g^{rr} k' - \frac{1}{i}\alpha_1 \left(\frac{w^\epsilon}{\alpha_1}\right)' g^{rr} k' w^\epsilon + hc. \end{aligned}$$

Noting that

$$\begin{aligned} &\frac{g^{ij}}{q}(D_j k) w^{2\epsilon} \in T^{3+\delta-2\epsilon}, \quad g^i (iD_j k) w^{2\epsilon} \in T^{3+\delta-2\epsilon}, \\ &g^r k' w^{2\epsilon} \in T^{2+\delta-2\epsilon}, \quad \frac{w^{2\epsilon} g^{rr} k'}{q\alpha_1} \in T^{2+\delta-2\epsilon}, \\ &\left(\frac{w^\epsilon}{\alpha_1}\right)' \alpha_1 g^{rr} k' w^\epsilon \in T^{2+\delta-2\epsilon}, \end{aligned}$$

and using (9.4)–(9.6) and Lemma 9.5, we find that, for $\epsilon > 0$ sufficiently small,

$$w^\epsilon[i(h_0 - h_{0,s}), k]w^\epsilon \lesssim h_0.$$

Thus (B) is fulfilled.

10. Asymptotic completeness 2: geometric setting

In this section we will compare the full dynamics to the asymptotic spherically symmetric dynamics. We set

$$h_{+\infty} := h_{0,s}, \quad h_{-\infty} := h_{+\infty} - \ell^2, \quad k_{+\infty} := 0, \quad k_{-\infty} := \ell.$$

The operators

$$\dot{H}_{+\infty} := \begin{pmatrix} 0 & \mathbb{1} \\ h_{+\infty} & 0 \end{pmatrix}, \quad \dot{H}_{-\infty} := \begin{pmatrix} 0 & \mathbb{1} \\ h_{-\infty} & 2\ell \end{pmatrix}$$

are selfadjoint on

$$\dot{\mathcal{E}}_{+\infty} := (h_{+\infty})^{-1/2}\mathcal{H} \oplus \mathcal{H} \quad \text{resp.} \quad \dot{\mathcal{E}}_{-\infty} := \Phi(\ell)((h_{+\infty})^{-1/2}\mathcal{H} \oplus \mathcal{H})$$

with domains

$$D(\dot{H}_{+\infty}) = (h_{+\infty})^{-1/2}\mathcal{H} \cap (h_{+\infty})^{-1}\mathcal{H} \oplus \langle h_{+\infty}\rangle^{-1/2}\mathcal{H}, \qquad D(\dot{H}_{-\infty}) = \Phi(\ell)D(\dot{H}_{+\infty}).$$

Remark 10.1. We have $\sigma_{\rm pp}(\dot{H}_{\pm\infty}) = \emptyset$. This follows from [21, Lemme 4.2.1].

Lemma 10.2. *Assume* (G1)–(G7). *Then* $\dot{\mathcal{E}}_{+\infty} = \dot{\mathcal{E}}_{+}$ *and* $\dot{\mathcal{E}}_{-\infty} = \dot{\mathcal{E}}_{-}$ *with equivalent norms.*

Proof. We have to show

$$h_{+\infty} \lesssim h_{+} \lesssim h_{+\infty}, \qquad h_{+\infty} \lesssim \tilde{h}_{-} \lesssim h_{+\infty}. \tag{10.1}$$

Recalling that $h_{0,s} = h_{+\infty}$, it is sufficient to show (10.1) for $h_{+\infty}$ replaced by $h_{0,s}$. First note that

$$h_{0,s} \lesssim \alpha_1(D_r q^2 D_r + P + 1)\alpha_1. \tag{10.2}$$

(G7) then gives

$$h_{0,s} \lesssim h_{+}, \qquad h_{0,s} \lesssim \tilde{h}_{-}.$$

By (G5), (G6), the Hardy inequality of Lemma 9.5, and the estimates (9.4)–(9.6) we have

$$h_0 \lesssim h_{0,s}.$$

Now,

$$h_{+} = h_0 - k_{+}^2 \lesssim h_0 \lesssim h_{0,s}, \qquad \tilde{h}_{-} = h_0 - (k_{-} - \ell)^2 \lesssim h_0 \lesssim h_{0,s},$$

which finishes the proof. □

Lemma 10.3. *Assume* (G1)–(G7). *For* $\chi \in C_0^\infty(\mathbb{R})$ *we have*

$$\chi(\dot{H}_{\pm\infty}) - \chi(\dot{H}_{\pm}) \in \mathcal{B}_\infty(\dot{\mathcal{E}}_{\pm}).$$

Proof. Let $\chi \in C_0^\infty(\mathbb{R})$. We prove the conclusion for $\chi(\dot{H}_{+\infty}) - \chi(\dot{H}_{+})$, the proof for $\chi(\dot{H}_{-\infty}) - \chi(\dot{H}_{-})$ being analogous. Let us introduce, for a positive selfadjoint operator h, the transformation

$$\mathcal{U}(h) := \frac{1}{\sqrt{2}}\begin{pmatrix} h^{1/2} & i \\ h^{1/2} & -i \end{pmatrix}, \qquad \mathcal{U}^{-1}(h) = \frac{1}{\sqrt{2}}\begin{pmatrix} h^{-1/2} & h^{-1/2} \\ -i & i \end{pmatrix}.$$

Note that

$$\mathcal{U}(h_{+}) : \dot{\mathcal{E}}_{\pm} \to \mathcal{H} \oplus \mathcal{H}, \qquad \mathcal{U}(h_{+\infty}) : \dot{\mathcal{E}}_{+\infty} \to \mathcal{H} \oplus \mathcal{H}$$

are unitary. We set

$$L_{+\infty} := \mathcal{U}(h_{+\infty})\dot{H}_{+\infty}\mathcal{U}^*(h_{+\infty}), \qquad L_+ := \mathcal{U}(h_+)\dot{H}_+\mathcal{U}^*(h_+).$$

By [21, Lemmes 6.1.3, A.4.4] we have

$$(\mathbb{1} - \mathcal{U}(h_{+\infty})\mathcal{U}^*(h_+))\chi(L_+),\ \chi(L_{+\infty}) - \chi(L_+) \in \mathcal{B}_\infty(\mathcal{H} \oplus \mathcal{H}). \tag{10.3}$$

We now write

$$\begin{aligned}\chi(\dot{H}_{+\infty}) - \chi(\dot{H}_+) = {} & \mathcal{U}^*(h_{+\infty})(\mathbb{1} - \mathcal{U}(h_{+\infty})\mathcal{U}^*(h_+))\chi(L_+)\mathcal{U}(h_+) \\ & + \mathcal{U}^*(h_{+\infty})(\chi(L_{+\infty}) - \chi(L_+))\mathcal{U}(h_{+\infty}) \\ & + \mathcal{U}^*(h_{+\infty})\chi(L_+)(\mathbb{1} - \mathcal{U}(h_+)\mathcal{U}^*(h_{+\infty}))\mathcal{U}(h_{+\infty}),\end{aligned}$$

which is compact by (10.3). □

Let

$$\dot{R}_{\pm\infty}(z) := (\dot{H}_{\pm\infty} - z)^{-1}.$$

In the same way as for $\dot{H}_\pm$ we can show:

Proposition 10.4. *Assume* (G1)–(G7). *Let* $\epsilon > 0$. *There exists a discrete and closed set* $\mathcal{T}_{\pm\infty} \subset \mathbb{R}$ *such that for all* $\chi \in C_0^\infty(\mathbb{R} \setminus \mathcal{T}_{\pm\infty})$ *and all* $k \in \mathbb{N}$ *we have*

$$\sup_{\epsilon>0,\, \lambda\in\mathbb{R}} \|w^{-\epsilon}\chi(\lambda)\dot{R}^k_{\pm\infty}(\lambda \pm i\epsilon)w^{-\epsilon}\|_{\mathcal{B}(\dot{\mathcal{E}}_{\pm\infty})} < \infty. \tag{10.4}$$

Let $\hat{\mathcal{T}} := \mathcal{S} \cup \mathcal{T}_{\pm\infty}$. The admissible energy cut-offs for $\dot{H}_{\pm\infty}$ are now defined in exactly the same manner as for $\dot{H}$, with $\mathcal{S}$ replaced by $\hat{\mathcal{T}}$ in the definition. Let $\mathcal{C}^{H_{\pm\infty}}$ be the set of all admissible cut-off functions for $\dot{H}_{\pm\infty}$. We define

$$\begin{aligned}\dot{\mathcal{E}}_{\mathrm{scatt},\pm\infty} &:= \{\chi(\dot{H}_{\pm\infty})u : \chi \in \mathcal{C}^{H_{\pm\infty}},\ u \in \dot{\mathcal{E}}_{\pm\infty}\}, \\ \dot{\mathcal{E}}_{\mathrm{scatt}} &:= \{\chi(\dot{H})u : \chi \in \mathcal{C}^{H_{\pm\infty}},\ u \in \dot{\mathcal{E}}\}.\end{aligned}$$

Theorem 10.5. *Assume* (G1)–(G7).

(i) *For all* $\varphi^\pm \in \dot{\mathcal{E}}_{\mathrm{scatt},\pm\infty}$ *there exists* $\psi^\pm \in \dot{\mathcal{E}}_{\mathrm{scatt}}$ *such that*

$$e^{-it\dot{H}}\psi^\pm - i_\pm e^{-it\dot{H}_{\pm\infty}}\varphi^\pm \to 0, \quad t \to \infty, \quad \text{in } \dot{\mathcal{E}}.$$

(ii) *For all* $\psi \in \dot{\mathcal{E}}_{\mathrm{scatt}}$ *there exists* $\varphi^\pm \in \dot{\mathcal{E}}_{\mathrm{scatt},\pm\infty}$ *such that*

$$e^{-it\dot{H}_{\pm\infty}}\varphi^\pm - i_\pm e^{-it\dot{H}}\psi \to 0, \quad t \to \infty, \quad \text{in } \dot{\mathcal{E}}_{\pm\infty}.$$

Proof. Let $\chi \in \mathcal{C}^{H_{\pm\infty}}$. By Thm. 8.5 it is sufficient to show the existence of the wave operators

$$W_\chi^\pm := \text{s-}\lim_{t\to\infty} e^{it\dot{H}_\pm} e^{-it\dot{H}_{\pm\infty}} \chi(\dot{H}_{\pm\infty}) \quad \text{in } \dot{\mathcal{E}}_\pm,$$
$$\Omega_\chi^\pm := \text{s-}\lim_{t\to\infty} e^{it\dot{H}_{\pm\infty}} e^{-it\dot{H}_\pm} \chi(\dot{H}_\pm) \quad \text{in } \dot{\mathcal{E}}_\pm,$$

and that

$$\tilde{\chi}(\dot{H}_\pm) W_\chi^\pm = W_\chi^\pm, \quad \tilde{\chi}(\dot{H}_{\pm\infty}) \Omega_\chi^\pm = \Omega_\chi^\pm \tag{10.5}$$

for $\tilde{\chi} \in \mathcal{C}^{H_{\pm\infty}}$ with $\tilde{\chi}\chi = \chi$. The existence of W_χ^+ and Ω_χ^+ follows directly from [21, Thm. 6.2.2]. For the existence of W_χ^-, Ω_χ^- note that

$$\Phi(\ell)\dot{H}_{-\infty}\Phi^{-1}(\ell) = \dot{H}_{-\infty}^\ell + \ell\mathbb{1}, \quad \Phi(\ell)\dot{H}_-\Phi^{-1}(\ell) = \dot{H}_-^\ell + \ell\mathbb{1},$$

where

$$\dot{H}_{-\infty}^\ell := \begin{pmatrix} 0 & \mathbb{1} \\ h_{0,s} & 0 \end{pmatrix}, \quad \dot{H}_-^\ell := \begin{pmatrix} 0 & \mathbb{1} \\ h_0 - (k_- - \ell)^2 & 2(k_- - \ell) \end{pmatrix}.$$

Again the existence of

$$\tilde{W}_\chi^- := \text{s-}\lim_{t\to\infty} e^{it\dot{H}_-^\ell} e^{-it\dot{H}_{-\infty}^\ell} \chi(\dot{H}_{-\infty}^\ell), \quad \tilde{\Omega}_\chi^- := \text{s-}\lim_{t\to\infty} e^{it\dot{H}_{-\infty}^\ell} e^{-it\dot{H}_-^\ell} \chi(\dot{H}_-^\ell)$$

follows from [21, Thm. 6.2.2]. The existence of W_χ^-, Ω_χ^- then follows by applying the transformation $\Phi(\ell)$. The identity (10.5) follows from Lemma 10.3. □

Remark 10.6. (i) Note that the results of [21] apply here although the situation considered in [21] is slightly different. In [21] the cylindrical manifold $\mathbb{R} \times \mathbb{S}^{d-1}$ has one asymptotically Euclidean end and one asymptotically hyperbolic end, whereas here we consider two asymptotically hyperbolic ends. The latter situation is simpler, in particular no gluing of the two conjugate operators for the ends in the setting of Mourre theory is necessary.

(ii) As $\sigma_{\text{pp}}(\dot{H}_{\pm\infty}) = \emptyset$ and $\dot{H}_\pm$, $\dot{H}_{\pm\infty}$ are selfadjoint, the wave operators

$$W^\pm := \text{s-}\lim_{t\to\infty} e^{it\dot{H}_\pm} e^{-it\dot{H}_{\pm\infty}}$$

exist. In a similar way we obtain the existence of the wave operators

$$\Omega^\pm := \text{s-}\lim_{t\to\infty} e^{it\dot{H}_{\pm\infty}} e^{-itH_\pm} \mathbb{1}^{\text{ac}}(\dot{H}_\pm),$$

where $\mathbb{1}^{\text{ac}}(\dot{H}_\pm)$ is the projection on the absolutely continuous subspace of $\dot{H}_\pm$.

11. The Klein–Gordon equation on the De Sitter–Kerr spacetime

In this section we recall the Klein–Gordon equation on the De Sitter–Kerr spacetime, which will be our main example of the geometric framework from Sect. 9.

11.1. The De Sitter–Kerr metric in Boyer–Lindquist coordinates

In Boyer–Lindquist coordinates the De Sitter–Kerr spacetime is described by a smooth 4-dimensional Lorentzian manifold $\mathcal{M}_{BH} = \mathbb{R}_t \times \mathbb{R}_r \times \mathbb{S}^2_\omega$, whose spacetime metric is given by

$$\begin{aligned} g &:= \frac{\Delta_r - a^2 \sin^2\theta\, \Delta_\theta}{\lambda^2\rho^2} dt^2 + \frac{2a \sin^2\theta\, ((r^2+a^2)\Delta_\theta - \Delta_r)}{\lambda^2\rho^2} dt d\varphi \\ &\quad - \frac{\rho^2}{\Delta_r} dr^2 - \frac{\rho^2}{\Delta_\theta} d\theta^2 - \frac{\sin^2\theta\sigma^2}{\lambda^2\rho^2} d\varphi^2, \qquad (11.1) \\ \rho^2 &:= r^2 + a^2\cos^2\theta, \quad \Delta_r := \big(1 - \tfrac{1}{3}\Lambda r^2\big)(r^2+a^2) - 2Mr, \\ \Delta_\theta &:= 1 + \tfrac{1}{3}\Lambda a^2 \cos^2\theta, \quad \sigma^2 := (r^2+a^2)^2\Delta_\theta - a^2\Delta_r \sin^2\theta, \quad \lambda := 1 + \tfrac{1}{3}\Lambda a^2. \end{aligned}$$

Here $\Lambda > 0$ is the cosmological constant, $M > 0$ is the mass of the black hole and a its angular momentum per unit mass. The metric is defined for $\Delta_r > 0$; we assume that this holds on an open interval $]r_-, r_+[$. (For $a = 0$, this is true when $9\Lambda M^2 < 1$; it remains true if we take a small enough.)

Note that the vector fields ∂_t and ∂_φ are Killing. The De Sitter–Schwarzschild metric ($a = 0$) is a special case of the above. The set $\{\rho^2 = 0\}$ is a true curvature singularity. In contrast to ρ^2, the roots of Δ_r are mere coordinate singularities. r_- and r_+ represent *event horizons* and we will only be interested in the region $r_- < r < r_+$. This region is not stationary in the sense that there exists no global time-like Killing vector field. In particular there are regions in $\mathbb{R}_t \times]r_-, r_+[_r \times \mathbb{S}^2_\omega$ in which ∂_t becomes space-like. Indeed, the function Δ_r' has a single zero $r_{\max}$ on $]r_-, r_+[$. The function Δ_r is strictly increasing on $]r_-, r_{\max}[$, and strictly decreasing on $]r_{\max}, r_+[$. Therefore there exist $r_1(\theta)$, $r_2(\theta)$ defined on $]0, \pi[$ such that

$$\Delta_r - a^2 \sin^2\theta\, \Delta_\theta \begin{cases} < 0 & \text{on }]r_-, r_1(\theta)[, \\ > 0 & \text{on }]r_1(\theta), r_2(\theta)[, \\ < 0 & \text{on }]r_2(\theta), r_+[. \end{cases}$$

As a consequence the vector field ∂_t is

- time-like on $\{(t, r, \theta, \varphi) : r_1(\theta) < r < r_2(\theta)\}$,
- space-like on $\{(t, r, \theta, \varphi) : r_- < r < r_1(\theta)\} \cup \{(t, r, \theta, \varphi : r_2(\theta) < r < r_+\} =: \mathcal{A}_- \cup \mathcal{A}_+$.

The regions $\mathcal{A}_\pm$ are called *ergospheres*. Of particular interest are the *locally nonrotating observers*. These observers have four-velocity

$$u^a = \frac{\nabla^a t}{(\nabla_b t \nabla^b t)^{1/2}}.$$

They rotate with coordinate angular velocity

$$\Omega = -\frac{g_{t\varphi}}{g_{\varphi\varphi}} = \frac{a((r^2+a^2)\Delta_\theta - \Delta_r)}{\sigma^2}. \qquad (11.2)$$

Their four-velocity is then (see also Remark 2.5)

$$u^a = \partial_t - \Omega \partial_\varphi.$$

Note that the angular velocity has finite limits at both horizons:

$$\Omega_\pm := \Omega(r_\pm, \theta) = \frac{a}{r_\pm^2 + a^2}. \tag{11.3}$$

11.2. The Klein–Gordon equation on the De Sitter–Kerr spacetime

We now reduce the Klein–Gordon equation on the De Sitter–Kerr spacetime to the abstract form (2.1).

A standard computation using $\Box_g = |g|^{-1/2}\partial_a |g|^{1/2} g^{ab}\partial_b$ yields, for the De Sitter–Kerr metric,

$$\left(\frac{\sigma^2\lambda^2}{\rho^2\Delta_\theta\Delta_r}\partial_t^2 - 2\frac{a(\Delta_r - (r^2+a^2)\Delta_\theta)\lambda^2}{\rho^2\Delta_\theta\Delta_r}\partial_\varphi\partial_t - \frac{(\Delta_r - a^2\sin^2\theta\,\Delta_\theta)\lambda^2}{\rho^2\Delta_\theta\Delta_r\sin^2\theta}\partial_\varphi^2 \right.$$
$$\left. - \frac{1}{\rho^2}\partial_r\Delta_r\partial_r - \frac{1}{\sin\theta\,\rho^2}\partial_\theta\sin\theta\,\Delta_\theta\partial_\theta + m^2\right)\phi = 0. \tag{11.4}$$

We multiply the equation on the left by $c^2 = \frac{\rho^2\Delta_r\Delta_\theta}{\lambda^2\sigma^2}$ to obtain

$$\left(\partial_t^2 - 2\frac{a(\Delta_r - (r^2+a^2)\Delta_\theta)}{\sigma^2}\partial_\varphi\partial_t - \frac{(\Delta_r - a^2\sin^2\theta\,\Delta_\theta)}{\sin^2\theta\,\sigma^2}\partial_\varphi^2 \right.$$
$$\left. - \frac{\Delta_r\Delta_\theta}{\lambda^2\sigma^2}\partial_r\Delta_r\partial_r - \frac{\Delta_r\Delta_\theta}{\lambda^2\sin\theta\,\sigma^2}\partial_\theta\sin\theta\,\Delta_\theta\partial_\theta + \frac{\rho^2\Delta_r\Delta_\theta}{\lambda^2\sigma^2}m^2\right)\phi = 0. \tag{11.5}$$

We now consider the unitary transform

$$U : L^2\left(\mathcal{M}, \frac{\sigma^2}{\Delta_r\Delta_\theta}dr\,d\omega\right) \to L^2(\mathcal{M}, dr\,d\omega), \qquad \phi \mapsto \frac{\sigma}{\sqrt{\Delta_r\Delta_\theta}}\phi.$$

If ϕ solves (11.5), then $u = U\phi$ solves

$$\left(\partial_t^2 - 2\frac{a(\Delta_r - (r^2+a^2)\Delta_\theta)}{\sigma^2}\partial_\varphi\partial_t - \frac{\Delta_r - a^2\sin^2\theta\,\Delta_\theta}{\sin^2\theta\sigma^2}\partial_\varphi^2 - \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma}\partial_r\Delta_r\partial_r\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} \right.$$
$$\left. - \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sin\theta\,\sigma}\partial_\theta\sin\theta\,\Delta_\theta\partial_\theta\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} + \frac{\rho^2\Delta_r\Delta_\theta}{\lambda^2\sigma^2}m^2\right)u = 0. \tag{11.6}$$

We introduce a Regge–Wheeler type coordinate x by the requirement

$$\frac{dx}{dr} = \lambda\frac{r^2+a^2}{\Delta_r}.$$

We then introduce the unitary transform

$$\mathcal{V} : L^2(]r_-, r_+[\times\mathbb{S}^2) \to L^2(\mathbb{R}\times\mathbb{S}^2,\, dx\,d\omega), \qquad v(r,\omega) \mapsto \sqrt{\frac{\Delta_r}{\lambda(r^2+a^2)}}\,v(r(x),\omega).$$

Let u be a solution of the Klein–Gordon equation (11.6) and $\psi = \sqrt{\frac{\Delta_r}{\lambda(r^2+a^2)}}u$. Then

$$\begin{aligned}\Big(\partial_t^2 - 2\frac{a(\Delta_r - (r^2+a^2)\Delta_\theta)}{\sigma^2}\partial_\varphi\partial_t - \frac{\Delta_r - a^2\sin^2\theta\,\Delta_\theta}{\sin^2\theta\,\sigma^2}\partial_\varphi^2 \\ - \frac{\sqrt{(r^2+a^2)\Delta_\theta}}{\sigma}\partial_x(r^2+a^2)\partial_x\frac{\sqrt{(r^2+a^2)\Delta_\theta}}{\sigma} \\ - \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sin\theta\,\sigma}\partial_\theta\sin\theta\,\Delta_\theta\partial_\theta\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} + \frac{\rho^2\Delta_r\Delta_\theta}{\lambda^2\sigma^2}m^2\Big)\psi = 0. \quad (11.7)\end{aligned}$$

12. Asymptotic completeness 3: The De Sitter–Kerr case

In this section we state the main theorems for the De Sitter–Kerr spacetime. The proofs are given in Sect. 13.

We consider the Klein–Gordon equation (11.7) and write it in the usual form

$$(\partial_t^2 - 2ik\partial_t + h)\psi = 0.$$

Let

$$\mathcal{H}^n = \{u \in L^2(\mathbb{R}\times\mathbb{S}^2) : (D_\varphi - n)u = 0\}, \quad n \in \mathbb{Z}. \quad (12.1)$$

We construct the energy spaces $\dot{\mathcal{E}}^n$, $\mathcal{E}^n$ as well as the Klein–Gordon operators H^n, $\dot{H}^n$ as in Sect. 3. Also let $i_\pm \in C^\infty(\mathbb{R})$, $i_- = 0$ in a neighborhood of ∞, $i_+ = 0$ in a neighborhood of $-\infty$ and $i_-^2 + i_+^2 = 1$. We will use two types of comparison dynamics:

– a separable comparison dynamics,
– asymptotic profiles.

12.1. Uniform boundedness of the evolution

Theorem 12.1. *There exists $a_0 > 0$ such that for all $|a| < a_0$ and all $n \in \mathbb{Z}$, there exists $C_n > 0$ such that*

$$\|e^{-it\dot{H}^n}u\|_{\dot{\mathcal{E}}^n} \le C_n\|u\|_{\dot{\mathcal{E}}^n}, \quad u \in \dot{\mathcal{E}}^n,\ t \in \mathbb{R}. \quad (12.2)$$

Note that for $n = 0$ the Hamiltonian $\dot{H}^n = \dot{H}^0$ is selfadjoint, therefore the only issue is $n \neq 0$.

Because of the existence of a zero resonance the evolution is not expected to be uniformly bounded on the inhomogeneous energy space. This is already the case for the De Sitter–Schwarzschild metric, i.e. if $a = 0$. In fact from [5, Thm. 1.3], denoting by r the zero resonance state, for $\chi \in C_0^\infty(\mathbb{R})$ we have

$$\chi e^{-itH}\chi u = \gamma\begin{pmatrix} r\chi(\chi r|u_1) \\ 0\end{pmatrix} + E(t) \quad \text{for some } \gamma > 0, \quad (12.3)$$

$$\|E(t)\|_{\mathcal{E}} \lesssim e^{-\epsilon t}\|\langle -\Delta_{\mathbb{S}^2}\rangle u\|_{\mathcal{E}}, \quad (12.4)$$

with some $\epsilon > 0$. Note that in [5] the norm

$$\|u_1\|^2 + (h_0 u|u) + \int_0^1 \int_{\mathbb{S}^2} |u_0|^2(x,\omega)\, dx\, d\omega$$

is used, but the same proof also gives (12.4). Now suppose that e^{-itH} is uniformly bounded on $\mathcal{E}$. Then from (12.3) we obtain, as $t \to \infty$,

$$\|r\chi(r\chi|u_1)\|_{\mathcal{E}} \lesssim \|u\|_{\mathcal{E}}, \qquad u \in C_0^\infty(\mathbb{R}\times\mathbb{S}^2) \oplus C_0^\infty(\mathbb{R}\times\mathbb{S}^2),$$

and thus

$$\|r\chi(r\chi|u_1)\|_{\mathcal{H}} \lesssim \|u\|_{\mathcal{E}}, \qquad u \in \mathcal{E},$$

by density. Here $\mathcal{H} = L^2(\mathbb{R}\times\mathbb{S}^2, dx d\omega)$. It follows that

$$\|r\chi(r\chi|v)\|_{\mathcal{H}} \lesssim \|v\|_{\mathcal{H}}, \qquad v \in \mathcal{H}.$$

Thus $\|r\chi\|_{\mathcal{H}} \lesssim 1$ uniformly in χ, which implies $r \in \mathcal{H}$ which is false. Therefore the evolution is not uniformly bounded on $\mathcal{E}$, nor on $\mathcal{E}^0$. It is however bounded on $\mathcal{E}^n$ for all $n \neq 0$.

12.2. Separable comparison dynamics

Let $\ell_\pm := \Omega_\pm n$. We set

$$h_{\pm\infty} := -\ell_\pm^2 - \partial_x^2 + \frac{\Delta_r}{\lambda^2(r^2+a^2)} P + \Delta_r m^2, \qquad k_{\pm\infty} := \ell_\pm,$$

where

$$P := -\frac{\lambda^2}{\sin^2\theta}\partial_\varphi^2 - \frac{1}{\sin\theta}\partial_\theta \sin\theta\, \Delta_\theta \partial_\theta.$$

For $n = 0$, P might have a zero eigenvalue and the natural energy spaces associated to h_0 and $h_{\pm\infty}$ may be different in the massless case. We will therefore consider the case $n = 0$ only in the massive case. Let $\dot{\mathcal{E}}^n_{\pm\infty}$, $\dot{H}^n_{\pm\infty}$ be the homogeneous energy spaces and operators associated to $(h_{\pm\infty}, k_{\pm\infty})$ according to Sect. 3.

Theorem 12.2. *There exists $a_0 > 0$ such that for all $|a| < a_0$ and $n \in \mathbb{Z}\setminus\{0\}$ the following holds:*

– *The wave operators*

$$W^\pm = \text{s-}\lim_{t\to\infty} e^{it\dot{H}^n} i_\pm e^{-it\dot{H}^n_{\pm\infty}} \tag{12.5}$$

  exist as bounded operators $W^\pm \in \mathcal{B}(\dot{\mathcal{E}}^n_{\pm\infty}; \dot{\mathcal{E}}^n)$.

– *The inverse wave operators*

$$\Omega^\pm = \text{s-}\lim_{t\to\infty} e^{it\dot{H}^n_{\pm\infty}} i_\pm e^{-it\dot{H}^n} \tag{12.6}$$

  exist as bounded operators $\Omega^\pm \in \mathcal{B}(\dot{\mathcal{E}}^n; \dot{\mathcal{E}}^n_{\pm\infty})$.

(12.5) *and* (12.6) *also hold for $n = 0$ if $m > 0$.*

12.3. Asymptotic profiles

We now introduce the Hamiltonians $\dot{H}_r$, $\dot{H}_l$ which describe the simplest possible asymptotic comparison dynamics. Let

$$h^n_{r/l} = -\partial_x^2 - \ell^2_{+/-}, \qquad k_{r/l} = \ell_{+/-},$$

acting on $\mathcal{H}^n$ defined in (12.1).

We associate to these operators the natural homogeneous energy spaces $\dot{\mathcal{E}}^n_{r/l}$ and Hamiltonians $\dot{H}^n_{r/l}$. Note that the solution of

$$\begin{cases} (\partial_t^2 - 2i\ell_\pm\partial_t - \partial_x^2 - \ell_\pm^2)u = 0, \\ u|_{t=0} = u_0, \\ \partial_t u|_{t=0} = u_1 \end{cases} \tag{12.7}$$

can be computed explicitly. Indeed, if u is the solution of (12.7), then $v = e^{-i\ell_\pm t}u$ fulfills

$$\begin{cases} (\partial_t^2 - \partial_x^2)v = 0, \\ v|_{t=0} = u_0, \\ \partial_t v|_{t=0} = u_1 - i\ell_\pm u_0. \end{cases} \tag{12.8}$$

Thus for smooth data the explicit solution of (12.7) is given by

$$u_0(t,x,\omega) = \frac{e^{i\ell_\pm t}}{2}\left(u_0(x+t,\omega) + u_0(x-t,\omega) + \int_{x-t}^{x+t} (u_1(\tau,\omega) - i\ell_\pm u_0(\tau,\omega))\,d\tau\right).$$

Let us denote the cut-offs $i_{+/-}$ by $i_{r/l}$.

The spaces $i_l\dot{\mathcal{E}}^n_l$ and $i_r\dot{\mathcal{E}}^n_r$ are not included in $\dot{\mathcal{E}}^n$ and the group $e^{-it\dot{H}^n_{r/l}}$ does not improve regularity. There is therefore no chance that the limits

$$W^+u = \lim_{t\to\infty} e^{it\dot{H}^n} i_{r/l} e^{-it\dot{H}^n_{r/l}} u$$

exist for all $u \in \dot{\mathcal{E}}^n_{r/l}$. We will first show the existence of the limits on smaller spaces and then extend the wave operators by continuity. Let $\{\lambda_q : q \in \mathbb{N}\} = \sigma(P)$ and $Z_q = \mathbb{1}_{\{\lambda_q\}}(P)\mathcal{H}$. Then

$$D(h_0) = D(h_{0,s}) = \left\{u \in \mathcal{H} : \sum_{q\in\mathbb{N}} \|h_0^{s,q}\mathbb{1}_{\{\lambda_q\}}(P)u\|^2 < \infty\right\},$$

where $h_0^{s,q}$ is the restriction of $h_{0,s}$ to $L^2(\mathbb{R}) \otimes Z_q$. Let

$$W_q := (L^2(\mathbb{R}) \otimes Z_q) \oplus (L^2(\mathbb{R}) \otimes Z_q), \qquad \mathcal{E}^{q,n}_{r/l} := \mathcal{E}^n_{r/l} \cap W_q,$$

$$\mathcal{E}^{\mathrm{fin},n}_{r/l} := \left\{u \in \mathcal{E}^n_{r/l} : \exists Q > 0,\ u \in \bigoplus_{q\le Q} \mathcal{E}^{q,n}_{r/l}\right\}.$$

Theorem 12.3. *There exists $a_0 > 0$ such that for all $|a| < a_0$ and $n \in \mathbb{Z} \setminus \{0\}$ the following holds:*

(i) *For all $u \in \mathcal{E}^{\mathrm{fin},n}_{r/l}$ the limits*

$$W_{r/l} u = \lim_{t\to\infty} e^{it\dot{H}^n} i^2_{r/l} e^{-it\dot{H}^n_{r/l}} u$$

exist in $\dot{\mathcal{E}}^n$. The operators $W_{r/l}$ extend to bounded operators $W_{r/l} \in \mathcal{B}(\dot{\mathcal{E}}^n_{r/l}; \dot{\mathcal{E}}^n)$.

(ii) *The inverse wave operators*

$$\Omega_{r/l} = \text{s-}\lim_{t\to\infty} e^{it\dot{H}^n_{r/l}} i^2_{r/l} e^{-it\dot{H}^n}$$

exist in $\mathcal{B}(\dot{\mathcal{E}}^n; \dot{\mathcal{E}}^n_{r/l})$.

Statements (i) *and* (ii) *also hold for $n = 0$ if $m > 0$.*

13. Proof of the main theorems for the De Sitter–Kerr spacetime

We want to apply the geometric setting developed in Sect. 9. To do so, we have to reduce the setting to $\ell_+ = 0$ by a change of coordinates. We introduce the new coordinate

$$\tilde{\varphi} = \varphi - \frac{a}{r_+^2 + a^2} t,$$

the other coordinates remain unchanged. We will denote $\tilde{\varphi}$ again by φ in the following. In the new coordinates, (11.7) reads

$$\begin{aligned}\Bigg(\Big(\partial_t - \frac{a}{r_+^2 + a^2}\partial_\varphi\Big)^2 - 2\frac{a(\Delta_r - (r^2 + a^2)\Delta_\theta)}{\sigma^2}\partial_\varphi\Big(\partial_t - \frac{a}{r_+^2 + a^2}\partial_\varphi\Big) \\ - \frac{\Delta_r - a^2 \sin^2\theta\, \Delta_\theta}{\sin^2\theta\, \sigma^2}\partial_\varphi^2 - \frac{\sqrt{\Delta_r \Delta_\theta}}{\lambda\sigma}\partial_r \Delta_r \partial_r \frac{\sqrt{\Delta_r \Delta_\theta}}{\lambda\sigma} \\ - \frac{\sqrt{\Delta_r \Delta_\theta}}{\lambda \sin\theta\, \sigma}\partial_\theta \sin\theta\, \Delta_\theta \partial_\theta \frac{\sqrt{\Delta_r \Delta_\theta}}{\lambda\sigma} + \frac{\rho^2 \Delta_r \Delta_\theta}{\lambda^2\sigma^2} m^2\Bigg)\psi = 0, \end{aligned} \tag{13.1}$$

i.e.

$$\begin{aligned}\Bigg(\partial_t^2 - 2\Big(\frac{a}{r_+^2 + a^2} + \frac{a(\Delta_r - (r^2 + a^2)\Delta_\theta)}{\sigma^2}\Big)\partial_\varphi \partial_t \\ + \Big(\frac{a^2}{(r_+^2 + a^2)^2} + 2\frac{a^2(\Delta_r - (r^2 + a^2)\Delta_\theta)}{\sigma^2(r_+^2 + a^2)} - \frac{\Delta_r - a^2\sin^2\theta\,\Delta_\theta}{\sin^2\theta\,\sigma^2}\Big)\partial_\varphi^2 \\ - \frac{\sqrt{\Delta_r \Delta_\theta}}{\lambda\sigma}\partial_r \Delta_r \partial_r \frac{\sqrt{\Delta_r \Delta_\theta}}{\lambda\sigma} - \frac{\sqrt{\Delta_r \Delta_\theta}}{\lambda \sin\theta\,\sigma}\partial_\theta \sin\theta\,\Delta_\theta \partial_\theta \frac{\sqrt{\Delta_r \Delta_\theta}}{\lambda\sigma} + \frac{\rho^2 \Delta_r \Delta_\theta}{\lambda^2\sigma^2} m^2\Bigg)\psi = 0.\end{aligned} \tag{13.2}$$

Set

$$k := \left(\frac{a}{r_+^2 + a^2} + \frac{a(\Delta_r - (r^2 + a^2)\Delta_\theta)}{\sigma^2}\right) D_\varphi,$$

$$h := \left(\frac{a^2}{(r_+^2 + a^2)^2} + 2\frac{a^2(\Delta_r - (r^2 + a^2)\Delta_\theta)}{\sigma^2(r_+^2 + a^2)} - \frac{\Delta_r - a^2\sin^2\theta\,\Delta_\theta}{\sin^2\theta\,\sigma^2}\right)\partial_\varphi^2$$
$$- \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma}\partial_r\Delta_r\partial_r\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} - \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sin\theta\,\sigma}\partial_\theta\sin\theta\,\Delta_\theta\partial_\theta\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} + \frac{\rho^2\Delta_r\Delta_\theta}{\lambda^2\sigma^2}m^2.$$

Noting that the coordinate change $\varphi \mapsto \tilde{\varphi}$ corresponds to the unitary transform $e^{-it\Omega_+ D_\varphi}$, and using Subsect. 3.5.3, we see that it is sufficient to show the corresponding theorems of Sect. 12 for the operators h, k. We set $h_0 := h + k^2$. A tedious calculation gives

$$h_0 = -\frac{\rho^4\Delta_r\Delta_\theta}{\sigma^4\sin^2\theta}\partial_\varphi^2 - \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma}\partial_r\Delta_r\partial_r\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma}$$
$$- \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sin\theta\,\sigma}\partial_\theta\sin\theta\,\Delta_\theta\partial_\theta\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} + \frac{\rho^2\Delta_r\Delta_\theta}{\lambda^2\sigma^2}m^2. \tag{13.3}$$

We set

$$h_0^n := \frac{(\rho^4 - \sigma^2)\Delta_r\Delta_\theta}{\sigma^4\sin^2\theta}n^2$$
$$- \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma}\partial_r\Delta_r\partial_r\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} + \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma}P\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} + \frac{\rho^2\Delta_r\Delta_\theta}{\lambda^2\sigma^2}m^2, \tag{13.4}$$

$$k^n := \left(\frac{a}{r_+^2 + a^2} + \frac{a(\Delta_r - (r^2 + a^2)\Delta_\theta)}{\sigma^2}\right)n. \tag{13.5}$$

In the following we will drop the index n which is implicit in the operators.

13.1. *Verification of the geometric hypotheses*

Let us recall that

$$P = -\frac{\lambda^2}{\sin^2\theta}\partial_\varphi^2 - \frac{1}{\sin\theta}\partial_\theta\sin\theta\,\Delta_\theta\partial_\theta.$$

With this choice of P, (G1) is clearly fulfilled. We now set

$$h_{0,s} := -\frac{\sqrt{\Delta_r}}{\lambda(r^2 + a^2)}\partial_r\Delta_r\partial_r\frac{\sqrt{\Delta_r}}{\lambda(r^2 + a^2)} + \frac{\sqrt{\Delta_r}}{\lambda(r^2 + a^2)}P\frac{\sqrt{\Delta_r}}{\lambda(r^2 + a^2)} + \Delta_r m^2.$$

Recall that $q(r) = \sqrt{(r_+ - r)(r - r_-)}$. We write $\Delta_r = q^2(r)P_2(r)$, where P_2 is a polynomial of degree 2. It is easy to see that (G2) is fulfilled with

$$\alpha_1^\pm = \alpha_3^\pm := \frac{\sqrt{P_2(r_\pm)}}{\lambda(r_\pm^2 + a^2)}, \quad \alpha_2^\pm := \sqrt{P_2(r_\pm)}, \quad \alpha_4^\pm := m^2\frac{\sqrt{P_2(r_\pm)}}{(r_\pm^2 + a^2)\lambda^2}.$$

We also set

$$\begin{aligned} k_{s,v} &:= k_s^n := \left(\frac{a}{r_+^2+a^2} - \frac{a}{r^2+a^2}\right)n, \\ k_{s,v}^- &:= \frac{an}{(r_+^2+a^2)(r_-^2+a^2)}(r_- - r_+)(r_- + r_+), \\ k_{s,r} &= k_{s,r}^- = 0. \end{aligned}$$

With these choices (G3) is clearly fulfilled. (G4) follows from (13.4). Let

$$\begin{aligned} g_1 &:= \frac{\sqrt{\Delta_r \Delta_\theta}}{\lambda\sigma} \in T^1, \quad g_0 := \frac{\sqrt{\Delta_r}}{\lambda(r^2+a^2)} \in T^1, \\ p_1 &:= \frac{\sqrt{\Delta_r \Delta_\theta}}{\sigma} \in T^1, \quad p_0 := \frac{\sqrt{\Delta_r}}{r^2+a^2} \in T^1. \end{aligned}$$

We have

$$g_1 - g_0 \in T^3, \qquad p_1 - p_0 \in T^3.$$

An elementary calculation gives

$$\begin{aligned} g^{rr} &= (g_0 - g_1)(g_0 + g_1)\Delta_r \in T^5, \\ g^r &= i((\partial_r g_1)g_1 - (\partial_r g_0)g_0)\Delta_r \in T^3, \\ g^{\theta\theta} &= (p_0 - p_1)(p_0 + p_1) \in T^3, \\ g^\theta &= i((\partial_\theta p_1)p_1 - (\partial_\theta p_0)p_0) \in T^2, \\ f &= ((\partial_r g_0)^2 - (\partial_r g_1)^2)\Delta_r + ((\partial_\theta p_0)^2 - (\partial_\theta p_1)^2) + \frac{m^2\Delta_r\rho^2\Delta_\theta}{\lambda^2\sigma^2} - \Delta_r m^2 \\ &\quad + \left(\frac{\rho^4\Delta_r\Delta_\theta}{\sigma^4\sin^2\theta} - \frac{\Delta_r}{(r^2+a^2)^2\sin^2\theta}\right)n^2 \in T^2, \\ g^{\varphi\varphi} &= g^\varphi = 0. \end{aligned}$$

Note that because of the diagonalization with respect to D_φ, we can put $g^{\varphi\varphi}$ and g^φ into f. We also have

$$k_{p,v} = \left(\frac{a\Delta_r}{\sigma^2} + \frac{a^3\sin^2\theta\,\Delta_r}{(r^2+a^2)\sigma^2}\right)n \in T^2, \qquad k_{p,r} = 0.$$

It follows that hypothesis (G6) is fulfilled.

Let us now check (G5). We consider the case $n = 0$ only if $m > 0$. Also (G5) will only be satisfied if $|a| < a_1$ for some a_1 independent of n. We first show that (G5) is fulfilled for $h_{0,s}^n$. We have

$$h_{0,s}^n = \alpha_1(D_r\Delta_r D_r + P + (r^2+a^2)^2 m^2)\alpha_1 \gtrsim \alpha_1(D_r q^2 D_r + P + 1)\alpha_1$$

with $\alpha_1 = \frac{\sqrt{\Delta_r}}{\lambda(r^2+a^2)}$ because $P \gtrsim 1$ for $n \neq 0$ and we suppose $m > 0$ for $n = 0$. Let now

$$\begin{aligned}\tilde{h}_0^n &= \frac{\rho^4 \Delta_r \Delta_\theta}{\sigma^2 \Delta_\theta \sin^2\theta} n^2 - \frac{\sqrt{\Delta_r}}{\lambda(r^2+a^2)} \partial_r \Delta_r \partial_r \frac{\sqrt{\Delta_r}}{\lambda(r^2+a^2)} \\ &\quad - \frac{\sqrt{\Delta_r}}{\lambda(r^2+a^2)\sin\theta} \partial_\theta \sin\theta\, \Delta_\theta \partial_\theta \frac{\sqrt{\Delta_r}}{\lambda(r^2+a^2)} + \frac{\rho^2 \Delta_r m^2 \Delta_\theta}{\lambda^2\sigma^2} \\ &\gtrsim \alpha_1 (D_r q^2 D_r + P + 1)\alpha_1.\end{aligned}$$

We then compute

$$\begin{aligned}h_0^n - \tilde{h}_0^n = &-\left(\frac{1}{\sigma} - \frac{1}{(r^2+a^2)\sqrt{\Delta_\theta}}\right) \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda} \partial_r \Delta_r \partial_r \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} \\ &- \frac{\sqrt{\Delta_r}}{\lambda(r^2+a^2)} \partial_r \Delta_r \partial_r \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda} \left(\frac{1}{\sigma} - \frac{1}{(r^2+a^2)\sqrt{\Delta_\theta}}\right) \\ &- \left(\frac{1}{\sigma} - \frac{1}{(r^2+a^2)\sqrt{\Delta_\theta}}\right) \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sin\theta} \partial_\theta \sin\theta\, \Delta_\theta \partial_\theta \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} \\ &- \frac{\sqrt{\Delta_r}}{(r^2+a^2)\lambda\sin\theta} \partial_\theta \sin\theta\, \Delta_\theta \partial_\theta \frac{\sqrt{\Delta_r}}{\lambda} \left(\frac{1}{\sigma} - \frac{1}{(r^2+a^2)\sqrt{\Delta_\theta}}\right).\end{aligned}$$

We compute

$$\left(\frac{1}{\sigma} - \frac{1}{(r^2+a^2)\sqrt{\Delta_\theta}}\right) = \frac{a^2 \Delta_r \sin^2\theta}{\sigma^2 (r^2+a^2)^2 \Delta_\theta} \left(\frac{1}{\sigma} + \frac{1}{(r^2+a^2)\Delta_\theta}\right)^{-1} =: a^2 g_a.$$

We have $g_a \in T^2$ uniformly in a, meaning that

$$\forall \alpha, \beta \in \mathbb{N}, \quad |\partial_r^\alpha \partial_\omega^\beta g_a| \leq C_{\alpha\beta} q(r)^{2-2\alpha}$$

with $C_{\alpha,\beta}$ independent of a. We then compute

$$\begin{aligned}-a^2 g_a \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda} \partial_r \Delta_r \partial_r \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} &= -a^2 \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} \partial_r \tilde{g}_a \Delta_r \partial_r \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} \\ &\quad + a^2 \tilde{g}_a' \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} \Delta_r \partial_r \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} \\ &\gtrsim -a^2 h_0^n - a^2 \tilde{h}_0^n,\end{aligned}$$

where $\tilde{g}_a = g_a \sigma \in T^2$ uniformly in a. Using similar arguments for the other terms we find

$$h_0^n - \tilde{h}_0^n \gtrsim -a^2 h_0^n - a^2 \tilde{h}_0^n,$$

and thus for a small enough (independently of n),

$$h_0^n \gtrsim \tilde{h}_0^n \gtrsim \alpha_1 (D_r q^2 D_r + P + 1)\alpha_1. \tag{13.6}$$

This is (G5) for h_0^n.

Let us now check (G7). Recall that $\ell = k_{s,v}^-$. We construct h_+, $\tilde{h}_-$ and $k_\pm$ as in Subsect. 2.1. We have

$$h_+ = h_0 - k_+^2, \quad k_+^2 \leq C_+ a^2 n^2 (r_+ - r)(r - r_-).$$

Observing that $P \geq n^2/\sin^2\theta$ we obtain, using (13.6),

$$h_0^n \gtrsim \alpha_1(D_r q^2 D_r + P + 1)\alpha_1 + n^2(r_+ - r)(r - r_-).$$

Thus there exists $a_2 > 0$ (independent of n) such that for all $|a| < a_2$ and $n \in \mathbb{Z}$ we have

$$h_+, \tilde{h}_- \gtrsim \alpha_1(r)(D_r q^2(r) D_r + P + 1)\alpha_1(r).$$

In particular (G5) is fulfilled for h_+, $\tilde{h}_-$ if a is small enough. Thus (G7) is fulfilled.

In the following we assume $|a| < a_0$, where a_0 is such that all the geometric hypotheses are fulfilled for all $n \in \mathbb{Z} \setminus \{0\}$ $(m = 0)$ resp. all $n \in \mathbb{Z}$ $(m > 0)$ if $|a| < a_0$.

13.2. Proof of Thm. 12.1

Thm. 12.1 will follow from Thm. 7.1, provided we show that the set $\mathcal{S}$ of singular points is empty. We recall that the sets $\mathcal{S}, \mathcal{T}, \mathcal{T}_\pm$ were defined in Subsect. 6.1 and that we showed in Prop. 6.10 that $\mathcal{S} \subset \mathcal{T} \cup \mathcal{T}_- \cup \mathcal{T}_+$. Therefore Thm. 12.1 will follow from

Proposition 13.1. (i) *There exists $a_1 > 0$ such that for $|a| < a_1$ and $n \neq 0$*

$$\sigma_{\mathrm{pp}}^{\mathbb{C}}(\dot{H}) = \mathcal{T} = \emptyset.$$

(ii) *$\mathcal{T}_\pm = \emptyset$ for $n \neq 0$.*

Proof. (i) essentially follows from the work of Dyatlov [11]. Let us first prove that $\sigma^{\mathbb{C}}(\dot{H}) = \emptyset$. By [11, Thm. 4] we have $\rho(h,k) \cap \{\mathrm{Im}\, z > 0\} = \emptyset$ for $a > 0$ sufficiently small. Then we apply Prop. 3.15.

Let us now prove that $\mathcal{T} = \emptyset$, i.e. $r(z) := w^{-\epsilon} p^{-1}(z) w^{-\epsilon}$ has no real poles. We replace the weight $w^{-\epsilon}$ by $\cosh(\epsilon x)^{-1}$ which is equivalent and holomorphic in a neighborhood of the real axis. We will denote this new weight again by $w^{-\epsilon}$. We know that $r(z)$ has a meromorphic extension to $\{\mathrm{Im}\, z > -\delta_\epsilon\}$ for some $\delta_\epsilon > 0$. We still call this meromorphic extension $r(z)$. Let

$$\tilde{p}(z) = \tilde{p}(z, x, \partial_x) := w^\epsilon p(z) w^\epsilon.$$

This is an elliptic second order operator with analytic coefficients. We clearly have

$$r(z) \circ \tilde{p}(z) = \tilde{p}(z) \circ r(z) = \mathbb{1}, \tag{13.7}$$

first for $\mathrm{Im}\, z$ sufficiently large and then in $\{\mathrm{Im}\, z > -\delta_\epsilon\}$ by meromorphic extension. Let $K_z(x, x')$ be the distribution kernel of $r(z)$. We have

$$\begin{aligned} \tilde{p}(z)(x, \partial_x) K_z(x, x') &= \delta(x, x'), \quad z \in \Omega, \\ \tilde{p}(z)^t (x', \partial_{x'}) K_z(x, x') &= \delta(x, x'), \quad z \in \Omega, \end{aligned}$$

where $\tilde{p}(z)^t$ is the transpose of $\tilde{p}(z)$, and is also elliptic with analytic coefficients. By the Morrey–Nirenberg theorem [24, Thm. 7.5.1], $\tilde{p}(z)$ and $\tilde{p}(z)^t$ are *analytic hypoelliptic*, which implies that $K_z(x, x')$ is analytic in x, x' outside the diagonal for $\{\mathrm{Im}\, z > -\delta_\epsilon\}$.

Recall from [11] that there exists $\delta_r > 0$ such that for all $\eta \in C_0^\infty(]r_- + \delta_r, r_+ - \delta_r[)$ and some $\delta_0 > 0$, $\eta p^{-1}(z)\eta$ has no poles in $\{\mathrm{Im}\, z > -\delta_0\}$. Let now $z_0 \in \{\mathrm{Im}\, z > -\delta_0\}$ be a possible pole of $r(z)$. We write

$$r(z) = \sum_{j=1}^{N} P_j (z - z_0)^{-j} + H(z),$$

where the P_j are finite rank operators and $H(z)$ is holomorphic close to z_0 and $P_N \neq 0$. We want to show that all the P_j are zero. Clearly it is sufficient to show that $P_N = 0$.

We have

$$P_N = \frac{1}{2i\pi} \oint_\gamma (z - z_0)^{N-1} r(z)\, dz,$$

which shows that the kernel $P_N(x, x')$ of P_N is analytic outside the diagonal. But as $\eta p(z)^{-1}\eta$ has no poles, we necessarily have $P_N(x, x') = 0$ for distinct $x, x' \in \mathrm{supp}\, \eta$. By analytic continuation we therefore have $P_N(x, x') = 0$ for $x \neq x'$. We then have

$$\begin{aligned}
\tilde{p}(z_0) P_N &= \frac{1}{2i\pi} \oint_\gamma \tilde{p}(z_0)(z - z_0)^{N-1} r(z)\, dz \\
&= \frac{1}{2i\pi} \oint_\gamma (z - z_0)^{N-1} (\tilde{p}(z_0) - \tilde{p}(z)) r(z)\, dz \\
&\quad + \frac{1}{2i\pi} \oint_\gamma (z - z_0)^{N-1} \tilde{p}(z) r(z)\, dz.
\end{aligned}$$

As $\tilde{p}(z) - \tilde{p}(z_0) = (z - z_0)T(z)$ with $T(z)$ holomorphic close to z_0, the first term is zero; the second is zero because $\tilde{p}(z)r(z) = \mathbb{1}$. It follows that $\tilde{p}(z_0)P_N = 0$.

Let us show that this implies $P_N = 0$. Let $u \in L^2(\mathbb{R} \times \mathbb{S}^2)$ with compact support. As the distribution kernel of P_N is supported on the diagonal, $v = P_N u$ also has compact support and $\tilde{p}(z_0)v = 0$. Again by analytic hypoellipticity of $\tilde{p}(z_0)$, v is analytic with compact support, thus $v = 0$. By a density argument $P_N = 0$. This completes the proof of (i).

Let us now prove (ii). By [18, Prop. 9.3] we know that $\mathcal{T}_\pm \cap \mathbb{R} \setminus \{0\} = \emptyset$. By Corollary 6.12 it is sufficient to show that 0 is not a resonance of $w^{-\epsilon} p_\pm^{-1}(z) w^{-\epsilon}$. We treat the $+$ case, the $-$ case being analogous. Suppose that 0 is a resonance. First note that $p_+(0)$ is an elliptic operator with $p_+(0) \gtrsim n^2 w^{-2}$. In particular $w^\epsilon p_+(0) w^\epsilon v = 0$ implies $v = 0$. Let $r(z) = w^{-\epsilon} p_+^{-1}(z) w^{-\epsilon}$. Suppose that $r(z)$ has a pole at $z = 0$:

$$r(z) = \sum_{j=1}^{N} \frac{P_j}{z^j} + H(z), \qquad P_N \neq 0.$$

Here the P_j are of finite rank and $H(z)$ is holomorphic. Let $u \in \mathcal{H}$ with $P_N u \neq 0$. Then

$$z^N u = \sum_{j=1}^{N} z^{N-j} w^\epsilon p_+(z) w^\epsilon P_j u + z^N w^\epsilon p_+(z) w^\epsilon H(z) u.$$

In the limit $z \to 0$ we obtain $w^\epsilon p_+(0) w^\epsilon P_N u = 0$ and so $P_N u = 0$, a contradiction. □

13.3. Proof of Thm. 12.2

We will apply the results of Sect. 10. First note that in our new setting (i.e. after rotation) we have to consider

$$\begin{aligned} h_{-\infty} &:= -\ell^2 - \partial_x^2 + \frac{\Delta_r}{r^2+a^2} P + \frac{\Delta_r m^2}{\lambda^2 (r^2+a^2)}, \\ k_{-\infty} &:= \ell, \\ h_{+\infty} &:= -\partial_x^2 + \frac{\Delta_r}{r^2+a^2} P + \frac{\Delta_r m^2}{\lambda^2 (r^2+a^2)}, \\ k_{+\infty} &:= 0, \\ \ell &:= \left(\frac{a}{r_+^2+a^2} - \frac{a}{r_-^2+a^2} \right) n. \end{aligned}$$

We associate to these operators the operators $H_{\pm\infty}$, $\dot{H}_{\pm\infty}$ and spaces $\mathcal{E}_{\pm\infty}$, $\dot{\mathcal{E}}_{\pm\infty}$ as in Sect. 3. Let $\mathcal{T}_{\pm\infty}$ be the set of singular points of $\dot{H}_{\pm\infty}$.

Lemma 13.2. *For $n \neq 0$ we have $\mathcal{T}_{\pm\infty} = \emptyset$.*

Proof. As $\dot{H}_{\pm\infty}$ is selfadjoint, we can use the Kato theory of H-smoothness. The proof for the absence of real resonances is analogous to the proof of Prop. 13.1(ii); we omit the details. □

13.4. Proof of Thm. 12.2

We first consider the case $n \neq 0$. By Prop. 13.1 we know that $\sigma_{pp}^{\mathbb{C}}(\dot{H}) = \mathcal{S} = \mathcal{T}_{\pm\infty} = \emptyset$. Thus $\mathbb{1} = \mathbb{1}_{\mathbb{R}}(\dot{H})$ is an admissible energy cut-off. Using in addition the fact that $e^{-it\dot{H}}$, $e^{-it\dot{H}_{\pm\infty}}$ are uniformly bounded, we deduce the theorem from Thm. 10.5. For $n = 0$ all operators are selfadjoint. This case follows from [21]; we omit the details. □

13.5. Proof of Thm. 12.3

We first write the comparison dynamics which we obtain after rotation:

$$h_r = -\partial_x^2, \quad h_l = -\partial_x^2 - \ell^2, \quad k_r = 0, \quad k_l = \ell.$$

We associate to these operators the natural homogeneous energy spaces $\dot{\mathcal{E}}_{r/l}$. Let

$$\dot{H}_r = \begin{pmatrix} 0 & \mathbb{1} \\ h_r & 2k_r \end{pmatrix}, \quad \dot{H}_l = \begin{pmatrix} 0 & \mathbb{1} \\ h_l & 2k_l \end{pmatrix}.$$

We now further analyze the energy spaces. Note that

$$\dot{\mathcal{E}}_l = \Phi(\ell)(H^1(\mathbb{R}; L^2(\mathbb{S}^2)) \oplus L^2(\mathbb{R} \times \mathbb{S}^2)), \quad \dot{\mathcal{E}}_r = H^1(\mathbb{R}; L^2(\mathbb{S}^2)) \oplus L^2(\mathbb{R} \times \mathbb{S}^2).$$

We will need the following subspaces:

$$\dot{\mathcal{E}}_l^L = \Big\{(u_0, u_1) \in \dot{\mathcal{E}}_l : u_1 - i\ell u_0 \in L^1(\mathbb{R}; L^2(\mathbb{S}^2)), \int (u_1 - i\ell u_0)(x,\omega)\,dx = 0 \text{ a.e. in } \omega\Big\},$$

$$\dot{\mathcal{E}}_r^L = \Big\{(u_0, u_1) \in \dot{\mathcal{E}}_r : u_1 \in L^1(\mathbb{R}; L^2(\mathbb{S}^2)), \int u_1(x,\omega)\,dx = 0 \text{ a.e. in } \omega\Big\}.$$

We define the spaces of *incoming/outgoing* initial data:

$$\dot{\mathcal{E}}_l^{\text{in}} = \{u \in \dot{\mathcal{E}}_l^L : u_1 = \partial_x u_0 + i\ell u_0\}, \qquad \dot{\mathcal{E}}_r^{\text{in}} = \{u \in \dot{\mathcal{E}}_r^L : u_1 = \partial_x u_0\},$$
$$\dot{\mathcal{E}}_l^{\text{out}} = \{u \in \dot{\mathcal{E}}_l^L : u_1 = -\partial_x u_0 + i\ell u_0\}, \qquad \dot{\mathcal{E}}_r^{\text{out}} = \{u \in \dot{\mathcal{E}}_r^L : u_1 = -\partial_x u_0\}.$$

If $(u_0, u_1) \in \dot{\mathcal{E}}_l^{\text{in}}$, then the solution of (12.7) is given by

$$u_0(t, x, \omega) = e^{i\ell t} u_0(x + t, \omega),$$

which is clearly incoming.

Lemma 13.3. *We have*

$$\dot{\mathcal{E}}_l^L = \dot{\mathcal{E}}_l^{\text{in}} \oplus \dot{\mathcal{E}}_l^{\text{out}}, \qquad \dot{\mathcal{E}}_r^L = \dot{\mathcal{E}}_r^{\text{in}} \oplus \dot{\mathcal{E}}_r^{\text{out}}.$$

Proof. We only show the lemma for $\dot{\mathcal{E}}_l^L$, $\dot{\mathcal{E}}_r^L$ being the special case $\ell = 0$. For $u = (u_0, u_1) \in \dot{\mathcal{E}}_l^L$ we define

$$\begin{aligned}
u_0^{\text{in}} &= \tfrac{1}{2}\int_x^\infty (-\partial_x u_0 - (u_1 - i\ell u_0))(\tau, \omega)\,d\tau,\\
u_1^{\text{in}} &= \tfrac{1}{2}(u_1 - i\ell u_0 + \partial_x u_0) + \frac{i\ell}{2}\int_x^\infty (-\partial_x u_0 - (u_1 - i\ell u_0))(\tau, \omega)\,d\tau,\\
u_0^{\text{out}} &= \tfrac{1}{2}\int_{-\infty}^x (\partial_x u_0 - (u_1 - i\ell u_0))(\tau, \omega)\,d\tau,\\
u_1^{\text{out}} &= \tfrac{1}{2}(u_1 - i\ell u_0 - \partial_x u_0) + \frac{i\ell}{2}\int_{-\infty}^x (\partial_x u_0 - (u_1 - i\ell u_0))(\tau, \omega)\,d\tau,\\
u^{\text{in/out}} &= (u_0^{\text{in/out}}, u_1^{\text{in/out}}).
\end{aligned} \tag{13.8}$$

It is easy to check that

$$u = u^{\text{in}} + u^{\text{out}},\ u^{\text{int/out}} \in \dot{\mathcal{E}}_\ell^{\text{in/out}},$$

which shows that $\dot{\mathcal{E}}_\ell^{\text{in}} + \dot{\mathcal{E}}_\ell^{\text{out}} = \dot{\mathcal{E}}_\ell$. Next if $v \in \dot{\mathcal{E}}_\ell^{\text{in}} \cap \dot{\mathcal{E}}_\ell^{\text{out}}$ we have $\partial_x v_0 = 0$, $v_0 \in L^2$, hence $v_0 = 0$, hence $v_1 = 0$. □

Remark 13.4. If $(u_0, u_1) \in \dot{\mathcal{E}}_l^L$ or $(u_0, u_1) \in \dot{\mathcal{E}}_r^L$ and $\operatorname{supp} u_0, \operatorname{supp} u_1 \subset]R_1, R_2[\times \mathbb{S}^2$, then

$$\operatorname{supp} u_{0,1}^{\text{in}} \subset]-\infty, R_2[\times \mathbb{S}^2, \qquad \operatorname{supp} u_{0,1}^{\text{out}} \subset]R_1, \infty[\times \mathbb{S}^2. \tag{13.9}$$

The spaces $\mathcal{E}^q_{r/l}$, $\mathcal{E}^{\text{fin}}_{r/l}$ are defined as before but starting with slightly modified operators (due to rotation). Let

$$\mathcal{D}^{\text{fin}}_{r/l} = C_0^\infty(\mathbb{R} \times \mathbb{S}^2) \times C_0^\infty(\mathbb{R} \times \mathbb{S}^2) \cap \mathcal{E}^{\text{fin}}_{r/l} \cap \dot{\mathcal{E}}^L_{r/l}.$$

Lemma 13.5. *$\mathcal{D}^{\text{fin}}_{r/l}$ is dense in $\mathcal{E}^{\text{fin}}_{r/l}$.*

Proof. We prove the lemma in two steps. First, $C_0^\infty(\mathbb{R} \times \mathbb{S}^2) \times C_0^\infty(\mathbb{R} \times \mathbb{S}^2) \cap \mathcal{E}^{\text{fin}}_{r/l}$ is dense in $\mathcal{E}^{\text{fin}}_{r/l}$. This follows easily from the usual regularization procedures.

Secondly, $C_0^\infty(\mathbb{R}\times\mathbb{S}^2)\times C_0^\infty(\mathbb{R}\times\mathbb{S}^2)\cap\mathcal{E}^{\text{fin}}_{r/l}\cap\dot{\mathcal{E}}^L_{r/l}$ is dense in $C_0^\infty(\mathbb{R}\times\mathbb{S}^2)\times C_0^\infty(\mathbb{R}\times\mathbb{S}^2) \cap \mathcal{E}^{\text{fin}}_{r/l}$. We can clearly replace $\mathcal{E}^{\text{fin}}_{r/l}$ by $\mathcal{E}^q_{r/l}$ in this statement. We only treat the l-case. Let

$$u = (u_0, u_1) \in C_0^\infty(\mathbb{R} \times \mathbb{S}^2) \times C_0^\infty(\mathbb{R} \times \mathbb{S}^2) \cap \mathcal{E}^q_l.$$

We will consider u as a function of x alone. We set

$$v = \Phi(-\ell)u.$$

Let $\psi \in C_0^\infty(\mathbb{R}),\ \psi \geq 0,\ \psi = 1$ in a neighborhood of zero and $\int \psi(x)dx = 1$. We set

$$v_0^n = v_0, \quad v_1^n = v_1 - n^{-1}\psi(n^{-1}x)\int v_1(x)\,dx,$$

so that $\int v_1^n(x)\,dx = 0$. We then estimate

$$\|v_1 - v_1^n\|_{L^2} \leq n^{-1/2}\|v_1\|_{L^1}\|n^{-1/2}\psi(n^{-1}\cdot)\|_{L^2} \leq Cn^{-1/2}\|v_1\|_{L^1} \to 0,$$

which completes the proof. □

We need an additional fact:

Lemma 13.6. *There exists $C > 0$ such that*

$$\|i_{r/l}u\|_{\dot{\mathcal{E}}_{r/l}} \leq C\|u\|_{\dot{\mathcal{E}}}, \quad u \in \dot{\mathcal{E}}.$$

Proof. We have

$$\begin{aligned}\|i_r u\|^2_{\dot{\mathcal{E}}_r} &= \|i_r u_1\|^2_{\mathcal{H}} + (i_r h_{+\infty} i_r u_0|u_0)\\ &\lesssim \|u_1 - ku_0\|^2_{\mathcal{H}} + (i_r(h_{+\infty} + k^2)i_r u_0 \mid u_0)\\ &\lesssim \|u_1 - ku_0\|^2_{\mathcal{H}} + (h_0 u_0|u_0) = \|u\|^2_{\dot{\mathcal{E}}}.\end{aligned}$$

Now recall that

$$\tilde{h}_{-\infty} = -\partial_x^2 + \frac{\Delta_r}{r^2 + a^2}P + m^2\Delta_r.$$

We then estimate

$$\begin{aligned}\|i_l u\|^2_{\dot{\mathcal{E}}_l} &= \|i_l(u_1 - \ell u_0)\|^2 + (i_l \tilde{h}_{-\infty} i_l u_0 | u_0) \\ &\lesssim \|i_l(u_1 - k u_0)\|^2_{\mathcal{H}} + (i_l(\tilde{h}_{-\infty} + (k-\ell)^2) i_l u_0 \mid u_0) \\ &\lesssim \|u_1 - k u_0\|^2_{\mathcal{H}} + (h_0 u_0 | u_0) = \|u\|^2_{\dot{\mathcal{E}}}.\end{aligned}$$

□

Proof of Thm. 12.3. We first show for $u \in \mathcal{E}^{\text{fin}}_{r/l}$ the existence of the limit

$$\tilde{W}_{r/l} u = \lim_{t\to\infty} e^{it\dot{H}_{\pm\infty}} i_{r/l} e^{-it\dot{H}_{r/l}} u$$

in $\dot{\mathcal{E}}_{\pm\infty}$. Let $u \in \bigoplus_{|q|\le Q} \mathcal{E}^q$. Using the estimate

$$\|e^{it\dot{H}_{\pm\infty}} i_{r/l} e^{-it\dot{H}_{r/l}} u\|_{\dot{\mathcal{E}}_{\pm\infty}} \le C(Q)\|u\|_{\dot{\mathcal{E}}_{r/l}}$$

as well as the spherical symmetry of the problem, it is sufficient to show for all $|q| \le Q$ the existence of the limit

$$\lim_{t\to\infty} e^{it\dot{H}^q_{\pm\infty}} i_{r/l} e^{-it\dot{H}^q_{r/l}} u^q,$$

where $u^q \in \mathcal{E}^q$ and $\dot{H}^q_{\pm\infty}$ resp. $\dot{H}^q_{r/l}$ are the restrictions of $\dot{H}_{\pm\infty}$ resp. $\dot{H}^l_{r/l}$ to $\dot{\mathcal{E}}^q_{r/l}$. The existence of this limit follows from standard arguments using the exponential decay of Δ_r at $\pm\infty$. Using Thm. 12.2 we obtain the existence of the limit

$$\lim_{t\to\infty} e^{it\dot{H}} i^2_{r/l} e^{-it\dot{H}_{r/l}} u = W_{r/l} u.$$

We now want to show that there exists $C > 0$ such that for all $u \in \mathcal{E}^{\text{fin}}_{r/l}$,

$$\|W_{r/l} u\|_{\dot{\mathcal{E}}} \le C\|u\|_{\dot{\mathcal{E}}_{r/l}}. \tag{13.10}$$

We first consider W_l. By Lemma 13.5 we can suppose $(u_0, u_1) \in \mathcal{D}^{\text{fin}}_l$. Let $\text{supp}\, u_0$, $\text{supp}\, u_1 \subset]R_1, R_2[$. We decompose (u_0, u_1) into incoming and outgoing solutions according to the discussion at the beginning of this subsection:

$$u_0 = u^{\text{in}}_{0,l} + u^{\text{out}}_{0,l}, \qquad u_1 = u^{\text{in}}_{1,l} + u^{\text{out}}_{1,l}.$$

By Remark 13.4 we have

$$\text{supp}\, u^{\text{in}}_{0,l}, \text{supp}\, u^{\text{in}}_{1,l} \subset]-\infty, R_2[\times \mathbb{S}^2, \qquad \text{supp}\, u^{\text{out}}_{0,l}, \text{supp}\, u^{\text{out}}_{1,l} \subset]R_1, \infty[\times \mathbb{S}^2.$$

Let $u^{\text{in}}_l = (u^{\text{in}}_{0,l}, u^{\text{in}}_{1,l})$ and $u^{\text{out}}_l = (u^{\text{out}}_{0,l}, u^{\text{out}}_{1,l})$. We have $W_l u^{\text{out}}_l = 0$, because $i^2_l e^{-it\dot{H}_l} u^{\text{out}}_l = 0$ for t sufficiently large. We have

$$\text{supp}\, e^{-it\dot{H}_l} u^{\text{in}}_l \subset (]-\infty, R_2 - t[\times \mathbb{S}^2) \times (]-\infty, R_2 - t[\times \mathbb{S}^2).$$

We then estimate for t large

$$\begin{aligned}\|e^{it\dot{H}}i_l^2 e^{-it\dot{H}_l}u^{\mathrm{in}}\|_{\dot{\mathcal{E}}} &\lesssim \|i_l^2 e^{-it\dot{H}_l}u^{\mathrm{in}}\|_{\dot{\mathcal{E}}}\\ &\lesssim \|u^{\mathrm{in}}\|^2_{\dot{\mathcal{E}}_l} + \left(\left(\frac{\Delta_r}{r^2+a^2}P + \Delta_r m^2\right)(e^{-it\dot{H}_l}u^{\mathrm{in}})_0 \,\middle|\, (e^{-it\dot{H}_l}u^{\mathrm{in}})_0\right)\\ &\lesssim \|u^{\mathrm{in}}\|^2_{\dot{\mathcal{E}}_l} + e^{-\kappa_- t}(Q+1)\|u^{\mathrm{in}}\|^2_{\mathcal{H}}\\ &\to \|u^{\mathrm{in}}\|^2_{\dot{\mathcal{E}}_l}, \qquad t\to\infty.\end{aligned}$$

It follows that $\|W_l u\|_{\dot{\mathcal{E}}} \le C\|u\|_{\dot{\mathcal{E}}_l}$, which is the required estimate. The proof for W_r is analogous. Part (ii) is shown in the same way. The required estimate follows from Lemma 13.6. □

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