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Shrawan Kumar

Positivity in T-equivariant K-theory of flag varieties associated to Kac–Moody groups

(with an appendix by M. Kashiwara)

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Abstract. Let $X = G/B$ be the full flag variety associated to a symmetrizable Kac–Moody group G. Let T be the maximal torus of G. The T-equivariant K-theory of X has a certain natural basis defined as the dual of the structure sheaves of the finite-dimensional Schubert varieties. We show that under this basis, the structure constants are polynomials with nonnegative coefficients. This result in the finite case was obtained by Anderson–Griffeth–Miller (following a conjecture by Graham–Kumar).

Keywords. Kac–Moody groups, flag varieties, equivariant K-theory, positivity

1. Introduction

Let G be any symmetrizable Kac–Moody group over $\mathbb{C}$ completed along the negative roots and let $G^{\min} \subset G$ be the 'minimal' Kac–Moody group. Let B be the standard (positive) Borel subgroup, B^- the standard negative Borel subgroup, $H = B \cap B^-$ the standard maximal torus and W the Weyl group. Let $\bar{X} = G/B$ be the 'thick' flag variety (introduced by Kashiwara) which contains the standard KM flag ind-variety $X = G^{\min}/B$. Let T be the quotient torus $H/Z(G^{\min})$, where $Z(G^{\min})$ is the center of $G^{\min}$. Then the action of H on $\bar{X}$ (and X) descends to an action of T. We denote the representation ring of T by $R(T)$. For any $w \in W$, we have the Schubert cell $C_w := BwB/B \subset X$, the Schubert variety $X_w := \overline{C_w} \subset X$, the opposite Schubert cell $C^w := B^-wB/B \subset \bar{X}$, and the opposite Schubert variety $X^w := \overline{C^w} \subset \bar{X}$. When G is a (finite-dimensional) semisimple group, it is referred to as the *finite case*.

Let $K_T^{\mathrm{top}}(X)$ be the T-equivariant topological K-group of the ind-variety X. Let $\{\psi^w\}_{w\in W}$ be the 'basis' of $K_T^{\mathrm{top}}(X)$ given by Kostant–Kumar (Definition 3.2).

Express the product in topological K-theory $K_T^{\mathrm{top}}(X)$:

$$\psi^u \cdot \psi^v = \sum_w p_{u,v}^w \psi^w \quad \text{for } p_{u,v}^w \in R(T). \tag{1}$$

S. Kumar: Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599-3250, USA; e-mail: shrawan@email.unc.edu

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Then the following result is our main theorem (Theorem 4.13). This was conjectured by Graham–Kumar [GK, Conjecture 3.1] in the finite case and proved in this case by Anderson–Griffeth–Miller [AGM, Corollary 5.2].

Theorem 1.1. *For any $u, v, w \in W$,*

$$(-1)^{\ell(u)+\ell(v)+\ell(w)}\, p^w_{u,v} \in \mathbb{Z}_+[(e^{-\alpha_1}-1),\dots,(e^{-\alpha_r}-1)],$$

where $\{\alpha_1,\dots,\alpha_r\}$ are the simple roots, i.e., $(-1)^{\ell(u)+\ell(v)+\ell(w)}\, p^w_{u,v}$ is a polynomial in the variables $x_1 = e^{-\alpha_1}-1,\dots,x_r = e^{-\alpha_r}-1$ with nonnegative integral coefficients.

By a result of Kostant–Kumar [KK, Proposition 3.25],

$$K^{\mathrm{top}}(X) \simeq \mathbb{Z} \otimes_{R(T)} K^{\mathrm{top}}_T(X), \tag{2}$$

where $\mathbb{Z}$ is considered as an $R(T)$-module via the evaluation at 1 and $K^{\mathrm{top}}(X)$ is the topological (nonequivariant) K-group of X. Thus, as an immediate consequence of the above theorem (by evaluating at 1), we obtain the following result (Corollary 4.14). It was conjectured by A. S. Buch in the finite case and proved in this case by Brion [B].

Corollary 1.2. *For any $u, v, w \in W$,*

$$(-1)^{\ell(u)+\ell(v)+\ell(w)}\, a^w_{u,v} \in \mathbb{Z}_+,$$

where $a^w_{u,v}$ are the structure constants of the product in $K^{\mathrm{top}}(X)$ with respect to the basis $\psi^w_o := 1 \otimes \psi^w$.

Further, Theorem 1.1 also gives the positivity for the multiplicative structure constants in the Schubert basis for the T-equivariant cohomology $H^*_T(X,\mathbb{C})$ with complex coefficients as described below.

The representation ring $R(T)$ has a decreasing filtration $\{R(T)_n\}_{n\ge 0}$, where

$$R(T)_n := \{f \in R(T) : \mathrm{mult}_1(f) \ge n\},$$

where $\mathrm{mult}_1(f)$ denotes the multiplicity of the zero of f at 1.

We first recall the following result from [KK, §§2.28–2.30 and Theorem 3.13].

Theorem 1.3. *There exists a decreasing filtration $\{\mathcal{F}_n\}_{n\ge 0}$ of the ring $K^{\mathrm{top}}_T(X)$ compatible with the filtration of $R(T)$ such that there is a ring isomorphism of the associated graded ring,*

$$\beta : \mathbb{C} \otimes_{\mathbb{Z}} \mathrm{gr}(K^{\mathrm{top}}_T(X)) \simeq H^*_T(X,\mathbb{C}).$$

Moreover, for any $w \in W$ we have $\psi^w \in \mathcal{F}_{\ell(w)}$ and under this isomorphism,

$$\beta(\overline{\psi^w}) = \hat{\varepsilon}^w,$$

where $\overline{\psi^w}$ denotes the element $\psi^w \pmod{\mathcal{F}_{\ell(w)+1}}$ in $\mathrm{gr}_{\ell(w)}(K^{\mathrm{top}}_T(X))$ and $\hat{\varepsilon}^w$ is the (equivariant) Schubert basis of $H^*_T(X,\mathbb{C})$ as in [K, *Theorem* 11.3.9].

Express the product in $H_T^*(X)$:

$$\hat{\varepsilon}^u \cdot \hat{\varepsilon}^v = \sum_w h_{u,v}^w \hat{\varepsilon}^w \quad \text{for } h_{u,v}^w \in S(\mathfrak{t}^*),$$

where $\mathfrak{t}$ is the Lie algebra of T and $h_{u,v}^w$ is a homogeneous polynomial of degree $\ell(u) + \ell(v) - \ell(w)$. Combining Theorems 1.1 and 1.3, we obtain the following result proved by Graham [Gr].

Theorem 1.4. *For any $u, v, w \in W$,*

$$h_{u,v}^w \in \mathbb{Z}_+[\alpha_1, \dots, \alpha_r],$$

i.e., $h_{u,v}^w$ is a homogeneous polynomial in $\{\alpha_1, \dots, \alpha_r\}$ of degree $\ell(u) + \ell(v) - \ell(w)$ with nonnegative integral coefficients.

We can further specialize the above theorem to obtain the positivity for the multiplicative structure constants $b_{u,v}^w$ in the standard Schubert basis $\{\varepsilon^w\}_{w \in W}$, obtained from specializing $\hat{\varepsilon}^w$ at 0, for the singular (nonequivariant) cohomology $H^*(X, \mathbb{C})$, because of the following result:

$$H^*(X, \mathbb{C}) \simeq \mathbb{C} \otimes_{S(\mathfrak{t}^*)} H_T^*(X, \mathbb{C}), \tag{3}$$

where $\mathbb{C}$ is considered as an $S(\mathfrak{t}^*)$-module via evaluation at 0 [K, Proposition 11.3.7]. We get the following corollary due to Kumar–Nori [KuN] from Theorem 1.4 by evaluating at 0.

Corollary 1.5. *For any $u, v, w \in W$,*

$$b_{u,v}^w \in \mathbb{Z}_+.$$

The proof of Theorem 1.1 relies heavily on algebro-geometric techniques. We realize the structure constants $p_{u,v}^w$ from (1) as the coproduct structure constants in the structure sheaf basis $\{\mathcal{O}_{X_w}\}_{w \in W}$ of the T-equivariant K-group $K_0^T(X)$ of finitely supported T-equivariant coherent sheaves on X (Proposition 4.1). Let $K_T^0(\bar{X})$ denote the Grothendieck group of T-equivariant coherent $\mathcal{O}_{\bar{X}}$-modules $\mathcal{S}$. Then there is a 'natural' pairing (see §3)

$$\langle\, , \rangle : K_T^0(\bar{X}) \otimes K_0^T(X) \to R(T),$$

coming from the T-equivariant Euler–Poincaré characteristic. For any character e^λ of H, let $\mathcal{L}(\lambda)$ be the G-equivariant line bundle on $\bar{X}$ associated to the character $e^{-\lambda}$ of H (§2). Define the T-equivariant coherent sheaf $\xi^u := e^{-\rho}\mathcal{L}(\rho)\omega_{X^u}$ on $\bar{X}$, where

$$\omega_{X^u} := \mathcal{E}xt_{\mathcal{O}_{\bar{X}}}^{\ell(u)}(\mathcal{O}_{X^u}, \mathcal{O}_{\bar{X}}) \otimes \mathcal{L}(-2\rho)$$

is the dualizing sheaf of X^u. We show that the basis $\{[\xi^w]\}$ is dual to the basis $\{[\mathcal{O}_{X_w}]\}_{w \in W}$ under the above pairing (Proposition 3.6).

Following [AGM], we define the 'mixing group' Γ in Definition 4.6 and prove its connectedness (Lemma 4.8). Then, we prove our main technical result (Theorem 4.10) on vanishing of some Tor sheaves as well as some cohomology vanishing. The proofs of its two parts are given in Sections 5 and 9 respectively.

From the connectedness of Γ and Theorem 4.10, we get Corollary 4.11. This corollary allows us to easily obtain our main theorem (Theorem 1.1).

The rest of the paper is devoted to proving Theorem 4.10.

In Section 5, we prove various local Ext and Tor vanishing results crucially using the 'Acyclicity Lemma' of Peskine–Szpiro (Corollary 5.3). The following is one of the main results of this section (Propositions 5.1 and 5.4).

Proposition 1.6. *For any* $u, w \in W$,

$$\mathscr{E}xt^j_{\mathscr{O}_{\bar{X}}}(\mathscr{O}_{X^u}, \mathscr{O}_{X_w}) = 0 \quad \textit{for all } j \neq \ell(u).$$

Thus,

$$\mathscr{T}or_j^{\mathscr{O}_{\bar{X}}}(\xi^u, \mathscr{O}_{X_w}) = 0 \quad \textit{for all } j > 0.$$

This proposition allows us to prove the (a) part of Theorem 4.10.

We also prove the following local Tor vanishing result (Lemma 5.5 and Corollary 5.7), which is a certain cohomological analogue of the proper intersection property of X^u with X_w.

Lemma 1.7. *For any* $u, w \in W$,

$$\mathscr{T}or_j^{\mathscr{O}_{\bar{X}}}(\mathscr{O}_{X^u}, \mathscr{O}_{X_w}) = \mathscr{T}or_j^{\mathscr{O}_{\bar{X}}}(\mathscr{O}_{\partial X^u}, \mathscr{O}_{X_w}) = 0 \quad \textit{for all } j > 0.$$

In Section 6 we show that the Richardson varieties $X^v_w := X_w \cap X^v \subset \bar{X}$ are irreducible, normal and Cohen–Macaulay, for short CM (Proposition 6.6). Then, we construct a desingularization Z^v_w of X^v_w (Theorem 6.8). In this section, we prove that various maps appearing in the big diagram in Section 7 are smooth or flat morphisms. Though not used in the paper, we determine the dualizing sheaf of the Richardson varieties X^v_w (Lemma 6.14).

In Section 7, we introduce the crucial irreducible scheme $\mathcal{Z}$ and its desingularization $f : \tilde{\mathcal{Z}} \to \mathcal{Z}$. We also introduce a divisor $\partial\mathcal{Z}$ of $\mathcal{Z}$ and show that $\mathcal{Z}$ and $\partial\mathcal{Z}$ are CM (Propositions 7.4 and 7.8 respectively). We further show that $\mathcal{Z}$ is irreducible and normal (Lemma 7.5). We show, in fact, that $\mathcal{Z}$ has rational singularities (Proposition 7.7), which is crucially used in the proof of Theorem 8.5.

In Section 8, we use the relative Kawamata–Viehweg vanishing theorem (Theorem 8.3) to obtain two crucial vanishing results on the higher direct images of the dualizing sheaf of $\tilde{\mathcal{Z}}$ twisted by $\partial\tilde{\mathcal{Z}}$ under $\tilde{\pi}$ and f, where $\partial\tilde{\mathcal{Z}} := f^{-1}\partial\mathcal{Z}$ and $\tilde{\pi} : \tilde{\mathcal{Z}} \to \bar{\Gamma}$ is the map from the big diagram in Section 7 (Proposition 8.4 and Theorem 8.5 respectively). This sets the stage for the proof of our main technical Theorem 4.10(b), which is achieved in Section 9.

Finally, we have included an appendix by M. Kashiwara where he determines the dualizing sheaf of X^u.

An informed reader will notice many ideas taken from very interesting papers [B] and [AGM] by Brion and Anderson–Griffeth–Miller respectively. However, there are several technical difficulties to deal with arising from the infinite-dimensional set-up, which has required various different formulations and more involved proofs. Some of the major differences are:

(1) In the finite case one just works with the opposite Schubert varieties X^u and their very explicit BSDH desingularizations. In our general symmetrizable Kac–Moody set-up, we need to consider the Richardson varieties X_w^u and their desingularizations Z_w^u. Our desingularization Z_w^u is not as explicit as the BSDH desingularization. Then, we need to draw upon the result due to Kumar–Schwede [KuS] that X_w^u has Kawamata log terminal singularities (in particular, rational singularities) and use this result (together with a result due to Elkik) to prove that $\mathcal{Z}$ has rational singularities (Proposition 7.7).

(2) Instead of considering just one flag variety in the finite case, we need to consider the 'thick' flag variety and the standard ind flag variety and the pairing between them. Moreover, the identification of the basis of $K_T^0(\bar{X})$ dual to the basis of $K_0^T(X)$ given by the structure sheaf of the Schubert varieties X_w is more delicate.

(3) In the finite case one uses Kleiman's transversality result for the flag variety X. In our infinite case, to circumvent the absence of Kleiman's transversality result, we needed to prove various local Ext and Tor vanishing results.

We feel that some of the local Ext and Tor vanishing results and the results on the geometry of Richardson varieties (including the construction of their desingularizations) proved in this paper are of independent interest.

2. Notation

We take the base field to be the field $\mathbb{C}$ of complex numbers. By a *variety*, we mean an algebraic variety over $\mathbb{C}$, which is reduced but not necessarily irreducible. For a scheme X and a closed subscheme Y, $\mathcal{O}_X(-Y)$ denotes the ideal sheaf of Y in X.

Let G be any symmetrizable Kac–Moody group over $\mathbb{C}$ completed along the negative roots (as opposed to completed along the positive roots as in [K, Chap. 6]), and let $G^{\min} \subset G$ be the 'minimal' Kac–Moody group as in [K, §7.4]. Let B be the standard (positive) Borel subgroup, B^- the standard negative Borel subgroup, $H = B \cap B^-$ the standard maximal torus and W the Weyl group [K, Chap. 6]. Let

$$\bar{X} = G/B$$

be the 'thick' flag variety which contains the standard KM flag ind-variety

$$X = G^{\min}/B.$$

If G is not of finite type, then $\bar{X}$ is an infinite-dimensional nonquasi-compact scheme [Ka, §4] and X is an ind-projective variety [K, §7.1]. The group $G^{\min}$ (in particular, the maximal torus H) acts on $\bar{X}$ and X. Let T be the quotient $H/Z(G^{\min})$, where $Z(G^{\min})$ is the center of $G^{\min}$. (Recall that, by [K, Lemma 6.2.9(c)], $Z(G^{\min}) = \{h \in H : e^{\alpha_i}(h) = 1$ for all the simple roots $\alpha_i\}$.) Then the action of H on $\bar{X}$ (and X) descends to an action of T.

For any $w \in W$, we have the *Schubert cell*

$$C_w := BwB/B \subset X,$$

the *Schubert variety*

$$X_w := \overline{C_w} \subset X,$$

the *opposite Schubert cell*

$$C^w := B^- wB/B \subset \bar{X},$$

and the *opposite Schubert variety*

$$X^w := \overline{C^w} \subset \bar{X},$$

all endowed with the reduced subscheme structures. Then, X_w is a (finite-dimensional) irreducible projective subvariety of X and X^w is a finite-codimensional irreducible subscheme of $\bar{X}$ ([K, §7.1] and [Ka, §4]). For any integral weight λ (i.e., any character e^λ of H), we have a G-equivariant line bundle $\mathcal{L}(\lambda)$ on $\bar{X}$ associated to the character $e^{-\lambda}$ of H. Explicitly, the character $e^{-\lambda}$ of H extends uniquely to a character (still denoted by $e^{-\lambda}$) of B since $H \simeq B/U$, where U is the unipotent radical of B. Now, let $\mathcal{L}(\lambda)$ be the line bundle over $\bar{X} = G/B$ associated to the principal B-bundle $G \to G/B$ via the one-dimensional representation of B given by the character $e^{-\lambda}$.

We denote the representation ring of T by $R(T)$.

Let $\{\alpha_1, \dots, \alpha_r\} \subset \mathfrak{h}^*$ be the set of simple roots, $\{\alpha_1^\vee, \dots, \alpha_r^\vee\} \subset \mathfrak{h}$ the set of simple coroots and $\{s_1, \dots, s_r\} \subset W$ the corresponding simple reflections, where $\mathfrak{h} := \operatorname{Lie} H$. Let $\rho \in \mathfrak{h}^*$ be any integral weight satisfying

$$\rho(\alpha_i^\vee) = 1 \quad \text{for all } 1 \leq i \leq r.$$

When G is a finite-dimensional semisimple group, ρ is unique, but for a general Kac–Moody group G, it may not be unique.

For any $v \leq w \in W$, consider the *Richardson variety*

$$X_w^v := X^v \cap X_w \subset X$$

and its boundary

$$\partial X_w^v := (\partial X^v) \cap X_w,$$

both endowed with the reduced subvariety structures, where $\partial X^v := X^v \backslash C^v$. We also set $\partial X_w := X_w \backslash C_w$. (By [KuS, Proposition 5.3], X_w^v and ∂X_w^v, endowed with the scheme-theoretic intersection structure, are Frobenius split in char. $p > 0$; in particular, they are reduced. More generally, any scheme-theoretic intersection $X_{w_1} \cap \cdots \cap X_{w_m} \cap X^{v_1} \cap \cdots \cap X^{v_n}$ is reduced by loc. cit.)

3. Identification of the dual of the structure sheaf basis

Definition 3.1. For a quasi-compact scheme Y, an $\mathcal{O}_Y$-module $\mathcal{S}$ is called *coherent* if it is finitely presented as an $\mathcal{O}_Y$-module and any $\mathcal{O}_Y$-submodule of finite type admits a finite presentation.

A subset $S \subset W$ is called an *ideal* if $x \in S$ and $y \leq x$ imply $y \in S$. An $\mathcal{O}_{\bar{X}}$-module $\mathcal{S}$ is called *coherent* if $\mathcal{S}_{|V^S}$ is a coherent $\mathcal{O}_{V^S}$-module for any finite ideal $S \subset W$, where V^S is the quasi-compact open subset of $\bar{X}$ defined by

$$V^S = \bigcup_{w \in S} wU^- B/B,$$

where U^- is the unipotent part of B^-. Let $K^0_T(\bar{X})$ denote the Grothendieck group of T-equivariant coherent $\mathcal{O}_{\bar{X}}$-modules $\mathcal{S}$. Observe that since the coherence condition on $\mathcal{S}$ is imposed only for $\mathcal{S}_{|V^S}$ for finite ideals $S \subset W$, $K^0_T(\bar{X})$ can be thought of as the inverse limit of $K^0_T(V^S)$, as S varies over the finite ideals of W [KS, §2].

Similarly, define $K_0^T(X) := \text{Limit}_{n\to\infty} K_0^T(X_n)$, where $\{X_n\}_{n\geq 1}$ is the filtration of X giving the ind-projective variety structure (i.e., $X_n = \bigcup_{\ell(w)\leq n} BwB/B$) and $K_0^T(X_n)$ is the Grothendieck group of T-equivariant coherent sheaves on the projective variety X_n.

We also define

$$K_T^{\text{top}}(X) := \underset{n\to\infty}{\text{Inv.lt.}}\, K_T^{\text{top}}(X_n),$$

where $K_T^{\text{top}}(X_n)$ is the T-equivariant topological K-group of the projective variety X_n.

Let $* : K_T^{\text{top}}(X_n) \to K_T^{\text{top}}(X_n)$ be the involution induced from the operation which takes a T-equivariant vector bundle to its dual. This of course induces the involution $*$ on $K_T^{\text{top}}(X)$.

We recall the 'basis' $\{\psi^w\}_{w\in W}$ of $K_T^{\text{top}}(X)$ given by Kostant–Kumar. (Actually, our $\psi^w = *\tau^{w^{-1}}$, where τ^w is the original 'basis' given in [KK, §3].)

Definition 3.2. For $w \in W$, fix a reduced decomposition $\mathfrak{w} = (s_{i_1}, \dots, s_{i_n})$ for w (i.e., $w = s_{i_1} \dots s_{i_n}$ is a reduced decomposition) and let $\theta_{\mathfrak{w}} : Z_{\mathfrak{w}} \to X_w$ be the Bott–Samelson–Demazure–Hansen (for short BSDH) desingularization [K, §7.1]. By [KK, Proposition 3.35], $K^0_T(Z_{\mathfrak{w}}) \to K_T^{\text{top}}(Z_{\mathfrak{w}})$ is an isomorphism, where $K^0_T(Z_{\mathfrak{w}})$ is the Grothendieck group associated to the semigroup of T-equivariant algebraic vector bundles on $Z_{\mathfrak{w}}$. (Observe that the action of H on $Z_{\mathfrak{w}}$ descends to an action of T.)

For any $\psi \in K_T^{\text{top}}(X)$ and $w \in W$, define the 'virtual' Euler–Poincaré characteristic by

$$\tilde{\chi}_T(X_w, \psi) := \chi_T(Z_{\mathfrak{w}}, \theta^*_{\mathfrak{w}}(\psi)) \in R(T).$$

By [KK, Proposition 3.36], $\tilde{\chi}_T(X_w, \psi)$ is well defined, i.e., it does not depend upon the particular choice of the reduced decomposition $\mathfrak{w}$ of w.

Now, define $\psi^w \in K_T^{\text{top}}(X)$ as the unique element satisfying

$$\tilde{\chi}_T(X_v, \psi^w) = \delta_{v,w} \quad \text{for all } v \in W. \tag{4}$$

Such a ψ^w exists and is unique [KK, Proposition 3.39]. Moreover, $\{\psi^w\}_{w\in W}$ is a 'basis' in the sense that any element of $K_T^{\text{top}}(X)$ is uniquely written as a linear combination of $\{\psi^w\}_{w\in W}$ with possibly infinitely many nonzero coefficients [KK, Proposition 2.20 and Remark 3.14]. Conversely, an arbitrary linear combination of ψ^w is an element of $K_T^{\text{top}}(X)$.

For any $w \in W$,

$$[\mathscr{O}_{X_w}] \in K_0^T(X).$$

Lemma 3.3. *$\{[\mathscr{O}_{X_w}]\}_{w\in W}$ forms a basis of $K_0^T(X)$ as an $R(T)$-module.*

Proof. Apply [CG, §5.2.14 and Theorem 5.4.17]. □

For $u \in W$, by [KS, §2], $\mathscr{O}_{X^u}$ is a coherent $\mathscr{O}_{\bar{X}}$-module. In particular, $\mathscr{O}_{\bar{X}}$ is a coherent $\mathscr{O}_{\bar{X}}$-module.

Consider the quasi-compact open subset $V^u := uU^-B/B \subset \bar{X}$. The following lemma is due to Kashiwara–Shimozono [KS, Lemma 8.1].

Lemma 3.4. *Any T-equivariant coherent sheaf $\mathcal{S}$ on V^u admits a free resolution in* $\mathrm{Coh}_T(\mathscr{O}_{V^u})$*:*

$$0 \to S_n \otimes \mathscr{O}_{V^u} \to \cdots \to S_1 \otimes \mathscr{O}_{V^u} \to S_0 \otimes \mathscr{O}_{V^u} \to \mathcal{S} \to 0,$$

where S_k are finite-dimensional T-modules and $\mathrm{Coh}_T(\mathscr{O}_{V^u})$ *denotes the abelian category of T-equivariant coherent $\mathscr{O}_{V^u}$-modules.* □

Define a pairing

$$\langle\ ,\ \rangle : K_T^0(\bar{X}) \otimes K_0^T(X) \to R(T), \qquad \langle[\mathcal{S}], [\mathcal{F}]\rangle = \sum_i (-1)^i \chi_T(X_n, \mathscr{T}\!or_i^{\mathscr{O}_{\bar{X}}}(\mathcal{S}, \mathcal{F})),$$

if $\mathcal{S}$ is a T-equivariant coherent sheaf on $\bar{X}$ and $\mathcal{F}$ is a T-equivariant coherent sheaf on X supported in X_n (for some n), where χ_T denotes the T-equivariant Euler–Poincaré characteristic.

Lemma 3.5. *The above pairing is well defined.*

Proof. By Lemma 3.4, for any $u \in W$, there exists $N(u)$ (depending upon $\mathcal{S}$) such that $\mathscr{T}\!or_j^{\mathscr{O}_{\bar{X}}}(\mathcal{S}, \mathcal{F}) = 0$ for all $j > N(u)$ in the open set V^u. Now, let $j > \max_{\ell(u)\leq n} N(u)$, where $\mathcal{F}$ has support in X_n. Then

$$\mathscr{T}\!or_j^{\mathscr{O}_{\bar{X}}}(\mathcal{S}, \mathcal{F}) = 0 \quad \text{on} \quad \bigcup_{\ell(u)\leq n} V^u,$$

and hence $\mathscr{T}\!or_j^{\mathscr{O}_{\bar{X}}}(\mathcal{S}, \mathcal{F}) = 0$ on $\bar{X}$, since $BuB/B \subset uB^-B/B$ and hence $\operatorname{supp} \mathcal{F} \subset X_n \subset \bigcup_{\ell(u)\leq n} V^u$.

Of course, for any $j \geq 0$, $\mathscr{T}\!or_j^{\mathscr{O}_{\bar{X}}}(\mathcal{S}, \mathcal{F})$ is a sheaf supported on X_n and it is $\mathscr{O}_{X_n}$-coherent on the open set $X_n \cap V^u$ of X_n for any $u \in W$. Thus, $\mathscr{T}\!or_j^{\mathscr{O}_{\bar{X}}}(\mathcal{S}, \mathcal{F})$ is an

$\mathscr{O}_{X_n}$-coherent sheaf, and hence

$$\chi_T(\bar{X}, \mathscr{T}or_j^{\mathscr{O}_{\bar{X}}}(\mathcal{S}, \mathcal{F})) = \chi_T(X_n, \mathscr{T}or_j^{\mathscr{O}_{\bar{X}}}(\mathcal{S}, \mathcal{F}))$$

is well defined. This proves the lemma. □

By [KS, proof of Proposition 3.4], for any $u \in W$,

$$\mathscr{E}xt^k_{\mathscr{O}_{\bar{X}}}(\mathscr{O}_{X^u}, \mathscr{O}_{\bar{X}}) = 0, \qquad \forall k \neq \ell(u). \tag{5}$$

Define the sheaf

$$\omega_{X^u} := \mathscr{E}xt^{\ell(u)}_{\mathscr{O}_{\bar{X}}}(\mathscr{O}_{X^u}, \mathscr{O}_{\bar{X}}) \otimes \mathcal{L}(-2\rho), \tag{6}$$

which, by the analogy with the Cohen–Macaulay (for short CM) schemes of finite type, will be called the *dualizing sheaf* of X^u.

Now, set the T-equivariant sheaf on $\bar{X}$,

$$\xi^u := e^{-\rho}\mathcal{L}(\rho)\omega_{X^u} = e^{-\rho}\mathcal{L}(-\rho)\mathscr{E}xt^{\ell(u)}_{\mathscr{O}_{\bar{X}}}(\mathscr{O}_{X^u}, \mathscr{O}_{\bar{X}}).$$

By Theorem 10.4 below, ξ^u is the ideal sheaf of ∂X^u in X^u.

By Lemma 3.4, for any $v \in W$, $\mathscr{O}_{X^u \cap V^v}$ admits a resolution

$$0 \to \mathcal{F}_n \to \cdots \to \mathcal{F}_0 \to \mathscr{O}_{X^u \cap V^v} \to 0$$

by free $\mathscr{O}_{V^v}$-modules of finite rank. Thus, the sheaf $\mathscr{E}xt^{\ell(u)}_{\mathscr{O}_{\bar{X}}}(\mathscr{O}_{X^u}, \mathscr{O}_{\bar{X}})$ restricted to V^v is given by the $\ell(u)$-th cohomology of the sheaf sequence

$$0 \leftarrow \mathscr{H}om_{\mathscr{O}_{\bar{X}}}(\mathcal{F}_n, \mathscr{O}_{\bar{X}}) \leftarrow \mathscr{H}om_{\mathscr{O}_{\bar{X}}}(\mathcal{F}_{n-1}, \mathscr{O}_{\bar{X}}) \leftarrow \cdots \leftarrow \mathscr{H}om_{\mathscr{O}_{\bar{X}}}(\mathcal{F}_0, \mathscr{O}_{\bar{X}}) \leftarrow 0.$$

In particular, $\mathscr{E}xt^{\ell(u)}_{\mathscr{O}_{\bar{X}}}(\mathscr{O}_{X^u}, \mathscr{O}_{\bar{X}})$ restricted to V^v is $\mathscr{O}_{V^v}$-coherent, and hence so is ξ^u as an $\mathscr{O}_{\bar{X}}$-module. Hence,

$$[\mathscr{E}xt^{\ell(u)}_{\mathscr{O}_{\bar{X}}}(\mathscr{O}_{X^u}, \mathscr{O}_{\bar{X}})] \in K^0_T(\bar{X}).$$

Proposition 3.6. *For any $u, w \in W$,*

$$\langle [\xi^u], [\mathscr{O}_{X_w}] \rangle = \delta_{u,w}.$$

Proof.[1] By definition,

$$\langle [\xi^u], [\mathscr{O}_{X_w}] \rangle = \sum_i (-1)^i \chi_T(X_n, \mathscr{T}or_i^{\mathscr{O}_{\bar{X}}}(\xi^u, \mathscr{O}_{X_w})),$$

where n is taken such that $n \geq \ell(w)$. Thus, by (subsequent) Proposition 5.4,

$$\langle [\xi^u], [\mathscr{O}_{X_w}] \rangle = \chi_T(X_n, \xi^u \otimes_{\mathscr{O}_{\bar{X}}} \mathscr{O}_{X_w}). \tag{7}$$

[1] We thank the referee for this shorter proof than our original proof.

By Theorem 10.4 and Corollary 5.7, we have the sheaf exact sequence

$$0 \to \xi^u \otimes_{\mathscr{O}_{\bar{X}}} \mathscr{O}_{X_w} \to \mathscr{O}_{X^u} \otimes_{\mathscr{O}_{\bar{X}}} \mathscr{O}_{X_w} \to \mathscr{O}_{\partial X^u} \otimes_{\mathscr{O}_{\bar{X}}} \mathscr{O}_{X_w} \to 0.$$

Thus,

$$\chi_T(X_n, \xi^u \otimes_{\mathscr{O}_{\bar{X}}} \mathscr{O}_{X_w}) = \chi_T(X_n, \mathscr{O}_{X^u_w}) - \chi_T(X_n, \mathscr{O}_{(\partial X^u)\cap X_w}), \tag{8}$$

since $\mathscr{O}_Y \otimes_{\mathscr{O}_{\bar{X}}} \mathscr{O}_Z = \mathscr{O}_{Y\cap Z}$. By Proposition 6.6, when nonempty, X^u_w is an irreducible variety and hence $(\partial X^u) \cap X_w = \bigcup_{w\geq v>u} X^v_w$ is connected (if nonempty) since $w \in X^v_w$ for all $u < v \leq w$. If $u \not\leq w$, then X^u_w is empty, and hence by (7)–(8),

$$\langle [\xi^u], [\mathscr{O}_{X_w}]\rangle = 0.$$

So, assume that $u \leq w$. In this case, X^u_w is nonempty. Moreover, by [KuS, Corollary 3.2],

$$H^i(X_n, \mathscr{O}_{X^u_w}) = 0, \quad \forall i > 0.$$

Also, by Corollary 5.7,

$$H^i(X_n, \mathscr{O}_{(\partial X^u)\cap X_w}) = 0, \quad \forall i > 0.$$

Thus, for $u \leq w$,

$$\chi_T(X_n, \mathscr{O}_{X^u_w}) = 1, \tag{9}$$

and for $u < w$,

$$\chi_T(X_n, \mathscr{O}_{(\partial X^u)\cap X_w}) = 1. \tag{10}$$

Hence, by (7)–(8),

$$\langle [\xi^u], [\mathscr{O}_{X_w}]\rangle = 0 \quad \text{for } u < w.$$

Finally, $\langle [\xi^w], [\mathscr{O}_{X_w}]\rangle = 1$. This proves the proposition. □

4. Geometric identification of the T-equivariant K-theory structure constants and statements of the main results

Express the product in topological K-theory $K^{\text{top}}_T(X)$:

$$\psi^u \cdot \psi^v = \sum_w p^w_{u,v} \psi^w \quad \text{for } p^w_{u,v} \in R(T).$$

(For fixed $u, v \in W$, infinitely many $p^w_{u,v}$ could be nonzero.)

Also, express the coproduct in $K^T_0(X)$:

$$\Delta_*[\mathscr{O}_{X_w}] = \sum_{u,v} q^w_{u,v} [\mathscr{O}_{X_u}] \otimes [\mathscr{O}_{X_v}],$$

where $\Delta : X \to X \times X$ is the diagonal map.

Proposition 4.1. *For all $u, v, w \in W$,*

$$p^w_{u,v} = q^w_{u,v}.$$

Proof. For $w \in W$, fix a reduced decomposition $\mathfrak{w} = (s_{i_1}, \ldots, s_{i_n})$ for w and let $\theta = \theta_{\mathfrak{w}} : Z_{\mathfrak{w}} \to X_w$ be the BSDH desingularization as in Definition 3.2. By [KK, Proposition 3.39] (where χ_T is the T-equivariant Euler–Poincaré characteristic),

$$\chi_T(\theta^*(\psi^u \cdot \psi^v)) = \chi_T\Big(\sum_{w_1} p^{w_1}_{u,v}\theta^*(\psi^{w_1})\Big) = p^w_{u,v}. \tag{11}$$

On the other hand,

$$\begin{aligned}\theta^*(\psi^u \cdot \psi^v) &= \theta^*\Delta^*(\psi^u \boxtimes \psi^v) = \Delta^*_{\mathfrak{w}}(\theta \times \theta)^*(\psi^u \boxtimes \psi^v)\\ &= \Delta^*_{\mathfrak{w}}(\theta^*\psi^u \boxtimes \theta^*\psi^v),\end{aligned} \tag{12}$$

where $\Delta_{\mathfrak{w}} : Z_{\mathfrak{w}} \to Z_{\mathfrak{w}} \times Z_{\mathfrak{w}}$ is the diagonal map.

In the following proof, for any morphism f of schemes, we abbreviate Rf_* by $f_!$.

Let $\pi : Z_{\mathfrak{w}} \to \mathrm{pt}$ and let $\Delta_{\mathfrak{w}*}[\mathscr{O}_{Z_{\mathfrak{w}}}] = \sum_{\mathfrak{u},\mathfrak{v}\le\mathfrak{w}} \hat{q}^{\mathfrak{w}}_{\mathfrak{u},\mathfrak{v}}[\mathscr{O}_{Z_{\mathfrak{u}}}] \boxtimes [\mathscr{O}_{Z_{\mathfrak{v}}}]$ for some unique $\hat{q}^{\mathfrak{w}}_{\mathfrak{u},\mathfrak{v}} \in R(T)$, where $\mathfrak{u} \le \mathfrak{w}$ means that $\mathfrak{u}$ is a subword of $\mathfrak{w}$. (This decomposition is due to the fact that $[\mathscr{O}_{Z_{\mathfrak{u}}}]_{\mathfrak{u}\le\mathfrak{w}}$ is an $R(T)$-basis of

$$K^T_0(Z_{\mathfrak{w}}) \simeq K^0_T(Z_{\mathfrak{w}}) \simeq K^{\mathrm{top}}_T(Z_{\mathfrak{w}}),$$

where $K^T_0(Z_{\mathfrak{w}})$ is the Grothendieck group associated to the semigroup of T-equivariant coherent sheaves on $Z_{\mathfrak{w}}$. (For the latter isomorphism, see [KK, Proposition 3.35].) Then

$$\begin{aligned}\chi_T(\theta^*(\psi^u \cdot \psi^v)) &= \pi_!\big(\Delta^*_{\mathfrak{w}}(\theta^*\psi^u \boxtimes \theta^*\psi^v)\big) \quad \text{by (12)}\\ &= (\pi\times\pi)_!\big(\Delta_{\mathfrak{w}*}(\Delta^*_{\mathfrak{w}}(\theta^*\psi^u \boxtimes \theta^*\psi^v))\big)\\ &= (\pi\times\pi)_!\big((\theta^*\psi^u \boxtimes \theta^*\psi^v)\cdot(\Delta_{\mathfrak{w}*}[\mathscr{O}_{Z_{\mathfrak{w}}}])\big) \quad \text{by the projection formula}\\ &= (\pi\times\pi)_!\Big((\theta^*\psi^u \boxtimes \theta^*\psi^v)\cdot\Big(\sum_{\mathfrak{u},\mathfrak{v}} \hat{q}^{\mathfrak{w}}_{\mathfrak{u},\mathfrak{v}}[\mathscr{O}_{Z_{\mathfrak{u}}}]\boxtimes[\mathscr{O}_{Z_{\mathfrak{v}}}]\Big)\Big) \quad \text{for some } \hat{q}^{\mathfrak{w}}_{\mathfrak{u},\mathfrak{v}} \in R(T)\\ &= \sum_{\mathfrak{u},\mathfrak{v}} \hat{q}^{\mathfrak{w}}_{\mathfrak{u},\mathfrak{v}}\chi_T(\theta^*\psi^u\cdot[\mathscr{O}_{Z_{\mathfrak{u}}}])\chi_T(\theta^*\psi^v\cdot[\mathscr{O}_{Z_{\mathfrak{v}}}]) = \sum_{\substack{\mu(\mathfrak{u})=u\\ \mu(\mathfrak{v})=v}} \hat{q}^{\mathfrak{w}}_{\mathfrak{u},\mathfrak{v}},\end{aligned} \tag{13}$$

where the last equality follows since

$$\chi_T(\theta^*\psi^u \cdot [\mathscr{O}_{Z_{\mathfrak{u}}}]) = \delta_{u,\mu(\mathfrak{u})}, \tag{14}$$

where $\mu(\mathfrak{u})$ denotes the Weyl group element u if the standard map $Z_{\mathfrak{u}} \to G/B$ has image precisely equal to X_u. To prove (14), use [KK, Propositions 3.36, 3.39] and [K, proof of Corollary 8.1.10]. (Actually, we need the extension of [KK, Proposition 3.36] for nonreduced words $\mathfrak{v}$, but the proof of this extension is identical.)

From the identity

$$\Delta_{\mathfrak{w}*}[\mathscr{O}_{Z_{\mathfrak{w}}}] = \sum_{\mathfrak{u},\mathfrak{v}\le\mathfrak{w}} \hat{q}^{\mathfrak{w}}_{\mathfrak{u},\mathfrak{v}}[\mathscr{O}_{Z_{\mathfrak{u}}}]\boxtimes[\mathscr{O}_{Z_{\mathfrak{v}}}],$$

we get

$$\Delta_*\theta_![\mathscr{O}_{Z_{\mathfrak{w}}}] = (\theta\times\theta)_!\,\Delta_{\mathfrak{w}*}[\mathscr{O}_{Z_{\mathfrak{w}}}] = \sum_{\mathfrak{u},\mathfrak{v}} \hat{q}^{\mathfrak{w}}_{\mathfrak{u},\mathfrak{v}}\theta_![\mathscr{O}_{Z_{\mathfrak{u}}}]\boxtimes\theta_![\mathscr{O}_{Z_{\mathfrak{v}}}]$$

$$= \sum_{u_1,v_1\le w}\ \sum_{\substack{\mu(\mathfrak{u})=u_1\\ \mu(\mathfrak{v})=v_1}} \hat{q}^{\mathfrak{w}}_{\mathfrak{u},\mathfrak{v}}[\mathscr{O}_{X_{u_1}}]\boxtimes[\mathscr{O}_{X_{v_1}}], \tag{15}$$

by [K, Theorem 8.2.2(c)]. Moreover, since

$$\Delta_*\,\theta_![\mathscr{O}_{Z_{\mathfrak{w}}}] = \Delta_*\,[\mathscr{O}_{X_w}] = \sum q^w_{u_1,v_1}[\mathscr{O}_{X_{u_1}}]\boxtimes[\mathscr{O}_{X_{v_1}}], \tag{16}$$

we get (equating (15) and (16)), for any $u_1, v_1 \le w$,

$$q^w_{u_1,v_1} = \sum_{\substack{\mu(\mathfrak{u})=u_1\\ \mu(\mathfrak{v})=v_1}} \hat{q}^{\mathfrak{w}}_{\mathfrak{u},\mathfrak{v}}. \tag{17}$$

Combining (11), (13) and (17), we get $p^w_{u,v} = q^w_{u,v}$. This proves the proposition. □

Lemma 4.2 (due to M. Kashiwara). *The $R(T)$-span of $\{[\xi^u]\}_{u\in W}$ inside $K^0_T(\bar{X})$ (where we allow an arbitrary infinite sum, which makes sense as an element of $K^0_T(\bar{X})$) coincides with $K^0_T(\bar{X})$.*

Proof. To prove this, write $[\xi^u]$ as a linear combination of $[\mathscr{O}_{X^v}]$ by Theorem 10.4. Then it is an upper triangular $R(T)$-matrix with diagonal terms equal to 1. By [KS, §2], $[\mathscr{O}_{X^v}]$ is a 'basis' of $K^0_T(\bar{X})$. This proves the lemma. □

By Proposition 3.6, $\{[\xi^u]\}_{u\in W}$ are independent over $R(T)$ even allowing infinite sums.

Now, express the product in $K^0_T(\bar{X})$:

$$[\xi^u]\cdot[\xi^v] = \sum_w d^w_{u,v}[\xi^w] \quad \text{for } d^w_{u,v}\in R(T).$$

Let $\bar{\Delta} : \bar{X} \to \bar{X}\times\bar{X}$ be the diagonal map. Then

$$[\xi^u]\cdot[\xi^v] = \bar{\Delta}^*([\xi^u\boxtimes\xi^v]).$$

Lemma 4.3. *For all $u, v, w \in W$,*

$$p^w_{u,v} = d^w_{u,v}.$$

Proof. For any $w \in W$,

$$\begin{aligned}\langle\bar{\Delta}^*([\xi^u\boxtimes\xi^v]),[\mathscr{O}_{X_w}]\rangle &= \langle[\xi^u\boxtimes\xi^v],\Delta_*[\mathscr{O}_{X_w}]\rangle\\ &= \Big\langle[\xi^u\boxtimes\xi^v],\sum_{u',v'} p^w_{u',v'}[\mathscr{O}_{X_{u'}}]\otimes[\mathscr{O}_{X_{v'}}]\Big\rangle \quad \text{by Proposition 4.1}\\ &= p^w_{u,v} \quad \text{by Proposition 3.6.}\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle\bar{\Delta}^*([\xi^u\boxtimes\xi^v]),[\mathscr{O}_{X_w}]\rangle &= \langle[\xi^u]\cdot[\xi^v],[\mathscr{O}_{X_w}]\rangle\\ &= d^w_{u,v} \quad \text{by Proposition 3.6 again.}\end{aligned}$$

□

Fix a large N and let

$$\mathbb{P} = (\mathbb{P}^N)^r \quad (r = \dim T).$$

For any $\mathbf{j} = (j_1, \dots, j_r) \in [N]^r$, where $[N] = \{0, 1, \dots, N\}$, set

$$\mathbb{P}^{\mathbf{j}} = \mathbb{P}^{N-j_1} \times \cdots \times \mathbb{P}^{N-j_r}.$$

We fix an identification $T \simeq (\mathbb{C}^*)^r$ *throughout the paper satisfying the condition that for any positive root* α, *the character* e^α (*under the identification*) *is given by* $z_1^{d_1(\alpha)} \dots z_r^{d_r(\alpha)}$ *for some* $d_i(\alpha) \geq 0$, *where* $(z_1, \dots, z_r)$ *are the standard coordinates on* $(\mathbb{C}^*)^r$. One such identification $T \simeq (\mathbb{C}^*)^r$ is given by $t \mapsto (e^{\alpha_1}(t), \dots, e^{\alpha_r}(t))$. *This will be our default choice.*

Let $E(T)_\mathbb{P} := (\mathbb{C}^{N+1} \setminus \{0\})^r$ be the total space of the standard principal T-bundle $E(T)_\mathbb{P} \to \mathbb{P}$. We can view $E(T)_\mathbb{P} \to \mathbb{P}$ as a finite-dimensional approximation of the classifying bundle for T. Let $\pi_X : X_\mathbb{P} := E(T)_\mathbb{P} \times^T X \to \mathbb{P}$ be the fibration with fiber $X = G/B$ associated to the principal T-bundle $E(T)_\mathbb{P} \to \mathbb{P}$, where we twist the standard action of T on X via

$$t \odot x = t^{-1}x. \tag{18}$$

For any T-subscheme $Y \subset X$, we denote $Y_\mathbb{P} := E(T)_\mathbb{P} \times^T Y \subset X_\mathbb{P}$.

The following theorem follows easily by using [CG, §5.2.14] together with [CG, Theorem 5.4.17] applied to the vector bundles $(BwB/B)_\mathbb{P} \to \mathbb{P}$.

Theorem 4.4. $K_0(X_\mathbb{P}) := \mathrm{Limit}_{n\to\infty} K_0((X_n)_\mathbb{P})$ *is a free module over the ring* $K_0(\mathbb{P}) = K^0(\mathbb{P})$ *with basis* $\{[\mathscr{O}_{(X_w)_\mathbb{P}}]\}_{w\in W}$, *where* K_0 (*resp.* K^0) *denotes the Grothendieck group associated to the semigroup of coherent sheaves* (*resp. locally free sheaves*). *Thus,* $K_0(X_\mathbb{P})$ *has a* $\mathbb{Z}$*-basis*

$$\{\pi_X^*([\mathscr{O}_{\mathbb{P}^\mathbf{j}}]) \cdot [\mathscr{O}_{(X_w)_\mathbb{P}}]\}_{\mathbf{j}\in[N]^r,\, w\in W},$$

where we view $[\mathscr{O}_{\mathbb{P}^\mathbf{j}}]$ *as an element of* $K_0(\mathbb{P}) = K^0(\mathbb{P})$. □

Let $Y := X \times X$. The diagonal map $\Delta : X \to Y$ gives rise to the embedding

$$\tilde{\Delta} : X_\mathbb{P} \to Y_\mathbb{P} = E(T)_\mathbb{P} \times^T Y \simeq X_\mathbb{P} \times_\mathbb{P} X_\mathbb{P}.$$

Thus, we get (denoting the projection $Y_\mathbb{P} \to \mathbb{P}$ by π_Y)

$$\tilde{\Delta}_*[\mathscr{O}_{(X_w)_\mathbb{P}}] = \sum_{\substack{u,v\in W\\ \mathbf{j}\in[N]^r}} c^w_{u,v}(\mathbf{j}) \pi_Y^*([\mathscr{O}_{\mathbb{P}^\mathbf{j}}]) \cdot [\mathscr{O}_{(X_u\times X_v)_\mathbb{P}}] \in K_0(Y_\mathbb{P}) \tag{19}$$

for some $c^w_{u,v}(\mathbf{j}) \in \mathbb{Z}$. Let

$$\mathbb{P}_\mathbf{j} = \mathbb{P}^{j_1} \times \cdots \times \mathbb{P}^{j_r},$$
$$\partial\mathbb{P}_\mathbf{j} = (\mathbb{P}^{j_1-1} \times \mathbb{P}^{j_2} \times \cdots \times \mathbb{P}^{j_r}) \cup \cdots \cup (\mathbb{P}^{j_1} \times \cdots \times \mathbb{P}^{j_{r-1}} \times \mathbb{P}^{j_r-1}),$$

where we interpret $\mathbb{P}^{-1} = \emptyset$. It is easy to see that, under the standard pairing on $K^0(\mathbb{P})$,

$$\langle [\mathscr{O}_{\mathbb{P}^\mathbf{j}}], [\mathscr{O}_{\mathbb{P}_{\mathbf{j}'}}(-\partial\mathbb{P}_{\mathbf{j}'})] \rangle = \delta_{\mathbf{j},\mathbf{j}'}. \tag{20}$$

Alternatively, it is a special case of [GK, Proposition 2.1 and §6.1].

Let $\bar{Y} = \bar{X} \times \bar{X}$ and $K^0(\bar{Y}_{\mathbb{P}})$ denote the Grothendieck group associated to the semi-group of coherent $\mathscr{O}_{\bar{Y}_{\mathbb{P}}}$-modules $\mathcal{S}$, i.e., those $\mathscr{O}_{\bar{Y}_{\mathbb{P}}}$-modules $\mathcal{S}$ such that $\mathcal{S}_{|(V^{S_1} \times V^{S_2})_{\mathbb{P}}}$ is a coherent $\mathscr{O}_{(V^{S_1} \times V^{S_2})_{\mathbb{P}}}$-module for all finite ideals $S_1, S_2 \subset W$. Also, let $\hat{\mathcal{L}}(\rho \boxtimes \rho)$ be the line bundle on $\bar{Y}_{\mathbb{P}}$ defined as

$$E(T)_{\mathbb{P}} \times^T e^{-2\rho}(\mathcal{L}(-\rho) \boxtimes \mathcal{L}(-\rho)) \to \bar{Y}_{\mathbb{P}},$$

where the action of T on the line bundle $e^{-2\rho}(\mathcal{L}(-\rho) \boxtimes \mathcal{L}(-\rho))$ over $\bar{Y}$ is also twisted the same way as in (18).

Lemma 4.5. *With the notation as above,*

$$c^w_{u,v}(\mathbf{j}) = \langle \pi^*_{\bar{Y}}[\mathscr{O}_{\mathbb{P}_\mathbf{j}}(-\partial \mathbb{P}_\mathbf{j})] \cdot [\widetilde{\xi^u \boxtimes \xi^v}], \tilde{\Delta}_*[\mathscr{O}_{(X_w)_{\mathbb{P}}}]\rangle,$$

where $\pi_{\bar{Y}} : \bar{Y}_{\mathbb{P}} \to \mathbb{P}$ is the projection, the coherent sheaf $\widetilde{\xi^u \boxtimes \xi^v}$ on $\bar{Y}_{\mathbb{P}}$ is defined as

$$\hat{\mathcal{L}}(\rho \boxtimes \rho) \otimes \mathscr{E}xt^{\ell(u)+\ell(v)}_{\mathscr{O}_{\bar{Y}_{\mathbb{P}}}}(\mathscr{O}_{(X^u \times X^v)_{\mathbb{P}}}, \mathscr{O}_{\bar{Y}_{\mathbb{P}}}),$$

and the pairing $\langle\, ,\, \rangle : K^0(\bar{Y}_{\mathbb{P}}) \otimes K_0(Y_{\mathbb{P}}) \to \mathbb{Z}$ is similar to the pairing defined earlier. Specifically,

$$\langle [\mathcal{S}], [\mathcal{F}] \rangle = \sum_i (-1)^i \chi(\bar{Y}_{\mathbb{P}}, \mathscr{T}or_i^{\mathscr{O}_{\bar{Y}_{\mathbb{P}}}}(\mathcal{S}, \mathcal{F})),$$

where χ is the Euler–Poincaré characteristic.

Proof. We have

$$\begin{aligned}
&\langle \pi^*_{\bar{Y}}[\mathscr{O}_{\mathbb{P}_\mathbf{j}}(-\partial \mathbb{P}_\mathbf{j})] \cdot [\widetilde{\xi^u \boxtimes \xi^v}], \tilde{\Delta}_*[\mathscr{O}_{(X_w)_{\mathbb{P}}}]\rangle \\
&= \Big\langle \pi^*_{\bar{Y}}[\mathscr{O}_{\mathbb{P}_\mathbf{j}}(-\partial \mathbb{P}_\mathbf{j})] \cdot [\widetilde{\xi^u \boxtimes \xi^v}], \sum_{\substack{u',v' \in W, \\ \mathbf{j}' \in [N]^r}} c^w_{u',v'}(\mathbf{j}')\, \pi^*_Y([\mathscr{O}_{\mathbb{P}_{\mathbf{j}'}}])[\mathscr{O}_{(X_{u'} \times X_{v'})_{\mathbb{P}}}] \Big\rangle \\
&= c^w_{u,v}(\mathbf{j}) \quad \text{by Proposition 3.6 and the identity (20).}
\end{aligned}$$

□

Definition 4.6 (Mixing group). Let T act on B via the inverse conjugation, i.e.,

$$t \cdot b = t^{-1} b t, \quad t \in T,\ b \in B.$$

Consider the ind-group scheme (over $\mathbb{P}$)

$$B_{\mathbb{P}} = E(T)_{\mathbb{P}} \times^T B \to \mathbb{P}.$$

Note that $B_{\mathbb{P}}$ is not a principal B-bundle since there is no right action of B on $B_{\mathbb{P}}$. Let Γ_0 be the group of global sections of the bundle $B_{\mathbb{P}}$ under pointwise multiplication. (Recall that Γ_0 can be identified with the set of regular maps $f : E(T)_{\mathbb{P}} \to B$ such that $f(e \cdot t) = t^{-1} \cdot f(e)$ for all $e \in E(T)_{\mathbb{P}}$ and $t \in T$.) Since $\mathrm{GL}(N+1)^r$ acts canonically on $B_{\mathbb{P}}$ compatible with its action on $\mathbb{P} = (\mathbb{P}^N)^r$, it also acts on Γ_0 via pull-back. Let Γ_B be the semidirect product $\Gamma_0 \rtimes \mathrm{GL}(N+1)^r$:

$$1 \to \Gamma_0 \to \Gamma_B \to \mathrm{GL}(N+1)^r \to 1.$$

Then Γ_B acts on $X_{\mathbb{P}}$ with orbits precisely equal to $\{(BwB/B)_{\mathbb{P}}\}_{w\in W}$, where the action of the subgroup Γ_0 is via the standard action of B on X. This follows from the following lemma.

Lemma 4.7. *For any $\bar{e} \in \mathbb{P}$ and any b in the fiber of $B_{\mathbb{P}}$ over $\bar{e}$, there exists a section $\gamma \in \Gamma_0$ such that $\gamma(\bar{e}) = b$.*

Proof. For a character λ of T, let $\mathscr{O}(\lambda)$ be the line bundle on $\mathbb{P}$ associated to the principal T-bundle $E(T)_{\mathbb{P}} \to \mathbb{P}$ via λ. For any positive real root α, let $U_\alpha \subset U$ be the corresponding one-parameter subgroup [K, §6.1.5(a)], where U is the unipotent radical of B. Then $B_{\mathbb{P}}$ contains the subbundle $H \times \mathscr{O}(-\alpha)$. By the assumption on the identification $T \simeq (\mathbb{C}^*)^r$, each $\mathscr{O}(-\alpha)$ is globally generated. Thus, $\Gamma_0(\bar{e}) \supset H \times U_\alpha$. Since Γ_0 is a group and by [K, Definition 6.2.7] the group U is generated by the subgroups $\{U_\alpha\}$, where α runs over the positive real roots, we get the lemma. □

Lemma 4.8. *Γ_B is connected.*

Proof. It suffices to show that Γ_0 is connected. But $\Gamma_0 \simeq H \times \Gamma(E(T)_{\mathbb{P}} \times^T U)$, where $\Gamma(E(T)_{\mathbb{P}} \times^T U)$ denotes the group of sections of the bundle $E(T)_{\mathbb{P}} \times^T U \to \mathbb{P}$. Thus, it suffices to show that the group of sections $\Gamma(E(T)_{\mathbb{P}} \times^T U)$ is connected. Using the T-equivariant contraction of U (in the analytic topology) given in [K, proof of Proposition 7.4.17], it is easy to see that the group of sections is contractible. In particular, it is connected. □

Similarly, we define $\Gamma_{B\times B}$ by replacing B by $B \times B$ and T by the diagonal $\Delta T \subset T \times T$ and we abbreviate it by Γ. Observe that Lemmas 4.7 and 4.8 remain true (by the same proof) for Γ_B replaced by Γ. (For the proof of Lemma 4.7, observe that the weights of $U_\alpha \times U_\beta$ under the ΔT-action are α, β. Similarly, for the proof of Lemma 4.8, observe that $U \times U$ is contractible under a $T \times T$ (in particular, ΔT)-equivariant contraction.)

Proposition 4.9. *For any coherent sheaf $\mathcal{S}$ on $\mathbb{P}$, and any $u, v \in W$,*

$$\pi^*[\mathcal{S}] \cdot [\widetilde{\xi^u \boxtimes \xi^v}] = [\pi^*(\mathcal{S}) \otimes_{\mathscr{O}_{\bar{Y}_{\mathbb{P}}}} (\widetilde{\xi^u \boxtimes \xi^v})] \in K^0(\bar{Y}_{\mathbb{P}}),$$

where we abbreviate $\pi_{\bar{Y}}$ by π and $\pi^*(\mathcal{S}) := \mathscr{O}_{\bar{Y}_{\mathbb{P}}} \otimes_{\mathscr{O}_{\mathbb{P}}} \mathcal{S}$. In particular,

$$\pi^*[\mathscr{O}_{\mathbb{P}_{\mathbf{j}}}(-\partial\mathbb{P}_{\mathbf{j}})] \cdot [\widetilde{\xi^u \boxtimes \xi^v}] = [\pi^*(\mathscr{O}_{\mathbb{P}_{\mathbf{j}}}(-\partial\mathbb{P}_{\mathbf{j}})) \otimes_{\mathscr{O}_{\bar{Y}_{\mathbb{P}}}} (\widetilde{\xi^u \boxtimes \xi^v})].$$

Proof. By definition,

$$\pi^*[\mathcal{S}] \cdot [\widetilde{\xi^u \boxtimes \xi^v}] = \sum_{i\geq 0} (-1)^i [\mathscr{T}\!or_i^{\mathscr{O}_{\bar{Y}_{\mathbb{P}}}}(\pi^*(\mathcal{S}), \widetilde{\xi^u \boxtimes \xi^v})].$$

Thus, it suffices to prove that

$$\mathscr{T}\!or_i^{\mathscr{O}_{\bar{Y}_{\mathbb{P}}}}(\pi^*\mathcal{S}, \widetilde{\xi^u \boxtimes \xi^v}) = 0, \qquad \forall i > 0.$$

Since the question is local in the base, we can assume that $\bar{Y}_{\mathbb{P}} \cong \mathbb{P} \times \bar{Y}$. Observe that, locally on the base,

$$\pi^* \mathcal{S} \simeq \mathcal{S} \boxtimes \mathcal{O}_{\bar{Y}} \quad \text{and} \quad \widetilde{\xi^u \boxtimes \xi^v} = \mathcal{O}_{\mathbb{P}} \boxtimes (\xi^u \boxtimes \xi^v),$$

where $\mathcal{S} \boxtimes \mathcal{O}_{\bar{Y}}$ means $\mathcal{S} \otimes_{\mathbb{C}} \mathcal{O}_{\bar{Y}}$ etc. Now, the result follows, since for algebras R and S over a field k and an R-module M and an S-module N,

$$\operatorname{Tor}_i^{R \boxtimes S}(M \boxtimes S, R \boxtimes N) = 0 \quad \text{for all } i > 0. \qquad \square$$

The following is our main technical result. The proof of its two parts are given in Sections 5 and 9 respectively.

Theorem 4.10. *For general $\gamma \in \Gamma = \Gamma_{B \times B}$, any $u, v, w \in W$, and $\mathbf{j} \in [N]^r$,*

(a) $\mathscr{T}or_i^{\mathcal{O}_{\bar{Y}_{\mathbb{P}}}}(\pi^*(\mathcal{O}_{\mathbb{P}_{\mathbf{j}}}(-\partial \mathbb{P}_{\mathbf{j}})) \otimes (\widetilde{\xi^u \boxtimes \xi^v}), \gamma_* \tilde{\Delta}_* \mathcal{O}_{(X_w)_{\mathbb{P}}}) = 0$ *for all $i > 0$, where we view any element $\gamma \in \Gamma$ as an automorphism of the scheme $\bar{Y}_{\mathbb{P}}$.*

(b) *Assume that $c^w_{u,v}(\mathbf{j}) \neq 0$, where $c^w_{u,v}(\mathbf{j})$ is defined by the identity* (19)*. Then*

$$H^p\left(\bar{Y}_{\mathbb{P}}, \pi^*(\mathcal{O}_{\mathbb{P}_{\mathbf{j}}}(-\partial \mathbb{P}_{\mathbf{j}})) \otimes (\widetilde{\xi^u \boxtimes \xi^v}) \otimes \gamma_* \tilde{\Delta}_* \mathcal{O}_{(X_w)_{\mathbb{P}}}\right) = 0$$

for all $p \neq |\mathbf{j}| + \ell(w) - (\ell(u) + \ell(v))$, where $|\mathbf{j}| := \sum_{i=1}^r j_i$.

Since Γ is connected, we get the following result as an immediate corollary of Lemma 4.5, Proposition 4.9 and Theorem 4.10.

Corollary 4.11. $(-1)^{\ell(w)-\ell(u)-\ell(v)+|\mathbf{j}|} c^w_{u,v}(\mathbf{j}) \in \mathbb{Z}_+$.

Recall the definition of the structure constants $p^w_{u,v} \in R(T)$ for the product in $K^{\text{top}}_T(X)$ from the beginning of this section. The following lemma follows easily from Proposition 4.1, identity (19) and [GK, Lemma 6.2] (see also [AGM, §3]).

Lemma 4.12. *For any $u, v, w \in W$, we can choose large enough N (depending upon u, v, w) and write (by* [GK, *Proposition* 2.2(c) *and Theorem* 5.1] *valid in the Kac–Moody case as well)*

$$p^w_{u,v} = \sum_{\mathbf{j} \in [N]^r} p^w_{u,v}(\mathbf{j})(e^{-\alpha_1} - 1)^{j_1} \dots (e^{-\alpha_r} - 1)^{j_r} \tag{21}$$

for some unique $p^w_{u,v}(\mathbf{j}) \in \mathbb{Z}$, where $\mathbf{j} = (j_1, \dots, j_r)$. Then

$$p^w_{u,v}(\mathbf{j}) = (-1)^{|\mathbf{j}|} c^w_{u,v}(\mathbf{j}). \tag{22}$$

As an immediate consequence of Corollary 4.11 and Lemma 4.12, we get the following main theorem of this paper, which was conjectured by Graham–Kumar [GK, Conjecture 3.1] in the finite case and proved in this case by Anderson–Griffeth–Miller [AGM, Corollary 5.2].

Theorem 4.13. *For any $u, v, w \in W$, and any symmetrizable Kac–Moody group G, the structure constants in $K_T^{\mathrm{top}}(X)$ satisfy*

$$(-1)^{\ell(u)+\ell(v)+\ell(w)}\, p_{u,v}^w \in \mathbb{Z}_+[(e^{-\alpha_1}-1), \dots, (e^{-\alpha_r}-1)]. \tag{23}$$

□

Recall [KK, Proposition 3.25] that

$$K^{\mathrm{top}}(X) \simeq \mathbb{Z} \otimes_{R(T)} K_T^{\mathrm{top}}(X), \tag{24}$$

where $\mathbb{Z}$ is considered as an $R(T)$-module via evaluation at 1. Express the product in $K^{\mathrm{top}}(X)$ in the 'basis' $\{\psi_o^u := 1 \otimes \psi^u\}_{u\in W}$:

$$\psi_o^u \cdot \psi_o^v = \sum_w a_{u,v}^w \psi_o^w \quad \text{for } a_{u,v}^w \in \mathbb{Z}.$$

Then, by the isomorphism (24),

$$a_{u,v}^w = p_{u,v}^w(1).$$

Thus, from Theorem 4.13, we immediately obtain the following result which was conjectured by A. S. Buch in the finite case and proved in this case by Brion [B].

Corollary 4.14. *For any $u, v, w \in W$,*

$$(-1)^{\ell(u)+\ell(v)+\ell(w)}\, a_{u,v}^w \in \mathbb{Z}_+.$$

Remark 4.15. We conjecture[2] that the analogue of Theorem 4.13 is true for the 'basis' ξ^u replaced by the structure sheaf 'basis' $\{\phi^u = [\mathcal{O}_{X^u}]\}_{u\in W}$ of $K_T^0(\bar{X})$. In the finite case, this was conjectured by Griffeth–Ram [GR] and proved in this case by Anderson–Griffeth–Miller [AGM, Corollary 5.3].

For the affine Kac–Moody group $G = \widehat{\mathrm{SL}_N}$ associated to SL_N, and its standard maximal parahoric subgroup P, let $\bar{\mathcal{X}} := G/P$ be the corresponding infinite Grassmannian. Then $K^0(\bar{\mathcal{X}})$ has the structure sheaf 'basis' $\{[\mathcal{O}_{X^u}]\}_{u\in W/W_o}$ over $\mathbb{Z}$, where W is the (affine) Weyl group of G and $W_o = S_N$ is the Weyl group of SL_N. Write, for any $u, v \in W/W_o$,

$$[\mathcal{O}_{X^u}] \cdot [\mathcal{O}_{X^v}] = \sum_{w\in W/W_o} b_{u,v}^w [\mathcal{O}_{X^w}] \quad \text{for some unique integers } b_{u,v}^w.$$

Now, the Lam–Schilling–Shimozono conjecture [LSS, Conjectures 7.20(2) and 7.21(3)] is the following:

$$(-1)^{\ell(u)+\ell(v)+\ell(w)}\, b_{u,v}^w \in \mathbb{Z}_+$$

if u, v, w are the minimal representatives in their cosets.

[2] This conjecture has now been proved by Baldwin–Kumar [BaK].

5. Study of some $\mathscr{E}xt$ and $\mathscr{T}or$ functors and proof of Theorem 4.10(a)

Proposition 5.1. *For any $j \in \mathbb{Z}$ and $u, w \in W$, as T-equivariant sheaves,*

$$\mathscr{T}or_j^{\mathscr{O}_{\bar{X}}}(\xi^u, \mathscr{O}_{X_w}) \simeq e^{-\rho}\mathcal{L}(-\rho) \otimes_{\mathscr{O}_{\bar{X}}} (\mathscr{E}xt_{\mathscr{O}_{\bar{X}}}^{\ell(u)-j}(\mathscr{O}_{X^u}, \mathscr{O}_{X_w})).$$

In particular, $\mathscr{E}xt^j_{\mathscr{O}_{\bar{X}}}(\mathscr{O}_{X^u}, \mathscr{O}_{X_w}) = 0$ for all $j > \ell(u)$.

Proof. By definition,

$$\xi^u = e^{-\rho}\mathcal{L}(-\rho)\, \mathscr{E}xt^{\ell(u)}_{\mathscr{O}_{\bar{X}}}(\mathscr{O}_{X^u}, \mathscr{O}_{\bar{X}}).$$

By Lemma 3.4, $\mathscr{O}_{X^u \cap V^v}$ admits a T-equivariant resolution (for any $v \in W$)

$$0 \to \mathcal{F}_n \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_0} \mathcal{F}_0 \to \mathscr{O}_{X^u \cap V^v} \to 0 \tag{25}$$

by T-equivariant free $\mathscr{O}_{V^v}$-modules of finite rank.

Since $\mathcal{M}_j := \mathscr{E}xt^j_{\mathscr{O}_{\bar{X}}}(\mathscr{O}_{X^u}, \mathscr{O}_{\bar{X}}) = 0$ for all $j \neq \ell(u)$ (see (5)), the dual complex

$$0 \leftarrow \mathcal{F}_n^* \xleftarrow{\delta^*_{n-1}} \mathcal{F}^*_{n-1} \leftarrow \cdots \leftarrow \mathcal{F}^*_{\ell(u)} \leftarrow \cdots \xleftarrow{\delta_0^*} \mathcal{F}_0^* \leftarrow 0 \tag{26}$$

gives rise to the resolution

$$0 \leftarrow \mathcal{M}_{\ell(u)} := \operatorname{Ker} \delta^*_{\ell(u)}/\operatorname{Im} \delta^*_{\ell(u)-1} \leftarrow \operatorname{Ker} \delta^*_{\ell(u)} \leftarrow \mathcal{F}^*_{\ell(u)-1} \leftarrow \cdots \leftarrow \mathcal{F}_0^* \leftarrow 0,$$

where $\mathcal{F}_i^* := \mathscr{H}om_{\mathscr{O}_{V^v}}(\mathcal{F}_i, \mathscr{O}_{V^v})$.

We next claim that $\operatorname{Ker} \delta_j^*$ is a direct summand $\mathscr{O}_{V^v}$-submodule of $\mathcal{F}_j^*$ for all $j \geq \ell(u)$:

We prove this by downward induction on j. Since (26) has cohomology only in degree $\ell(u)$, if $n > \ell(u)$ we have $\operatorname{Im} \delta^*_{n-1} = \mathcal{F}^*_n$ and hence $\operatorname{Ker} \delta^*_{n-1}$ is a direct summand $\mathscr{O}_{V^v}$-submodule of $\mathcal{F}^*_{n-1}$. Thus, $\operatorname{Ker} \delta^*_{n-2}$ is a direct summand of $\mathcal{F}^*_{n-2}$ if $n-2 \geq \ell(u)$. Continuing, we see that $\operatorname{Ker} \delta^*_{\ell(u)}$ is a direct summand $\mathscr{O}_{V^v}$-submodule of $\mathcal{F}^*_{\ell(u)}$.

Thus, we get a projective resolution

$$0 \to \mathcal{P}_{\ell(u)} \to \cdots \to \mathcal{P}_1 \to \mathcal{P}_0 \to \mathcal{M}_{\ell(u)} \to 0,$$

where $\mathcal{P}_0 := \operatorname{Ker} \delta^*_{\ell(u)}$ and $\mathcal{P}_i := \mathcal{F}^*_{\ell(u)-i}$ for $1 \leq i \leq \ell(u)$. Hence, restricted to the open subset V^v, $\mathscr{T}or_*^{\mathscr{O}_{\bar{X}}}(\xi^u, \mathscr{O}_{X_w})$ is the homology of the complex

$$0 \to (e^{-\rho}\mathcal{L}(-\rho)\,\mathcal{P}_{\ell(u)}) \otimes_{\mathscr{O}_{V^v}} \mathscr{O}_{X_w} \to \cdots \to (e^{-\rho}\mathcal{L}(-\rho)\,\mathcal{P}_0) \otimes_{\mathscr{O}_{V^v}} \mathscr{O}_{X_w} \to 0.$$

Now, we show that the j-th homology of the complex

$$\mathcal{C}: \quad 0 \to \mathcal{P}_{\ell(u)} \otimes_{\mathscr{O}_{V^v}} \mathscr{O}_{X_w} \to \cdots \to \mathcal{P}_0 \otimes_{\mathscr{O}_{V^v}} \mathscr{O}_{X_w} \to 0$$

is isomorphic to $\mathscr{E}xt^{\ell(u)-j}_{\mathscr{O}_{\bar{X}}}(\mathscr{O}_{X^u}, \mathscr{O}_{X_w})$:

Since

$$\mathcal{P}_i \otimes_{\mathcal{O}_{V^v}} \mathcal{O}_{X_w} \simeq \mathscr{H}om_{\mathcal{O}_{V^v}}(\mathcal{F}_{\ell(u)-i}, \mathcal{O}_{X_w}) \quad \text{for all } i \geq 1,$$

we get

$$\mathscr{H}_j(\mathcal{C}) \simeq \mathscr{E}xt^{\ell(u)-j}_{\mathcal{O}_{V^v}}(\mathcal{O}_{X^u}, \mathcal{O}_{X_w}) \quad \text{for all } j \geq 2. \tag{27}$$

Moreover, since $\mathcal{P}_0$ is a direct summand of $\mathcal{F}^*_{\ell(u)}$, we get

$$\mathscr{H}_1(\mathcal{C}) \simeq \mathscr{E}xt^{\ell(u)-1}_{\mathcal{O}_{V^v}}(\mathcal{O}_{X^u}, \mathcal{O}_{X_w}). \tag{28}$$

Now,

$$\mathscr{H}_0(\mathcal{C}) = \mathcal{P}_0 \otimes_{\mathcal{O}_{V^v}} \mathcal{O}_{X_w} / \operatorname{Im}(\mathcal{P}_1 \otimes_{\mathcal{O}_{V^v}} \mathcal{O}_{X_w}) \simeq \mathscr{E}xt^{\ell(u)}_{\mathcal{O}_{V^v}}(\mathcal{O}_{X^u}, \mathcal{O}_{X_w}), \tag{29}$$

since $\operatorname{Ker} \delta^*_{\ell(u)}$ is a direct summand of $\mathcal{F}^*_{\ell(u)}$, and $\operatorname{Ker} \delta^*_{\ell(u)+1} = \operatorname{Im} \delta^*_{\ell(u)}$ is a direct summand of $\mathcal{F}^*_{\ell(u)+1}$.

Finally,

$$\mathscr{E}xt^{j}_{\mathcal{O}_{V^v}}(\mathcal{O}_{X^u}, \mathcal{O}_{X_w}) = 0 \quad \text{for all } j > \ell(u). \tag{30}$$

To prove this, observe that, for $j > \ell(u)$,

$$0 \to \operatorname{Ker} \delta^*_j \to \mathcal{F}^*_j \xrightarrow{\delta^*_j} \operatorname{Im} \delta^*_j = \operatorname{Ker} \delta^*_{j+1} \to 0$$

is a split exact sequence since $\operatorname{Ker} \delta^*_{j+1}$ is projective. Thus,

$$0 \to \operatorname{Ker} \delta^*_j \otimes_{\mathcal{O}_{V^v}} \mathcal{O}_{X_w} \to \mathcal{F}^*_j \otimes_{\mathcal{O}_{V^v}} \mathcal{O}_{X_w} \to (\operatorname{Im} \delta^*_j) \otimes_{\mathcal{O}_{V^v}} \mathcal{O}_{X_w} \to 0$$

is exact. Moreover, $\operatorname{Im} \delta^*_j \hookrightarrow \mathcal{F}^*_{j+1}$ is a direct summand and hence

$$\operatorname{Im} \delta^*_j \otimes_{\mathcal{O}_{V^v}} \mathcal{O}_{X_w} \hookrightarrow \mathcal{F}^*_{j+1} \otimes_{\mathcal{O}_{V^v}} \mathcal{O}_{X_w}.$$

From this (30) follows.

Combining (27)–(30), we get the proposition. □

The following is a minor generalization of the 'acyclicity lemma' of Peskine–Szpiro [PS, Lemme 1.8].

Lemma 5.2. *Let R be a local noetherian CM domain and let*

$$0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to 0 \tag{$*$}$$

be a complex of finitely generated free R-modules. Fix a positive integer $d > 0$. Assume:

(a) *some irreducible component Z of the support of $M := \bigoplus_{i \geq 1} H_i(F_*)$ has codimension $\geq d$ in* Spec R, *and*
(b) *$F_i = 0$ for all $i > d$.*

Then $H_i(F_*) = 0$ for all $i > 0$.

Proof. Assume, if possible, that $M \neq 0$. Let $I \subset R$ be the annihilator of M and let $\mathfrak{p}$ be the (minimal) prime ideal containing I corresponding to Z. Then

$$M \otimes_R R_{\mathfrak{p}} \neq 0, \tag{31}$$

$$\operatorname{depth}(M \otimes_R R_{\mathfrak{p}}) = 0. \tag{32}$$

Next observe that

$$\begin{aligned}\operatorname{depth}(F_* \otimes_R R_{\mathfrak{p}}) &= \operatorname{depth} R_{\mathfrak{p}} := \operatorname{depth}_{\mathfrak{p}} R_{\mathfrak{p}} \\ &= \operatorname{codim}(\mathfrak{p} R_{\mathfrak{p}}) \quad \text{since } R_{\mathfrak{p}} \text{ is CM} \\ &= \operatorname{codim}(\mathfrak{p}) \geq d.\end{aligned}$$

Now, by applying the acyclicity lemma of Peskine–Szpiro [PS, Lemme 1.8] to the complex $F_* \otimes_R R_{\mathfrak{p}}$ and using the identities (31), (32), we get a contradiction.

Thus, $M = 0$, proving the lemma. □

Corollary 5.3. *Let Y be an irreducible CM variety and $d > 0$ a positive integer. Let*

$$0 \leftarrow \mathcal{G}^n \xleftarrow{\delta^{n-1}} \mathcal{G}^{n-1} \leftarrow \cdots \xleftarrow{\delta^0} \mathcal{G}^0 \leftarrow 0$$

be a complex of locally free $\mathcal{O}_Y$-modules of finite rank satisfying:

- *The support of the sheaf $\bigoplus_{i<d} \mathcal{H}^i(\mathcal{G}^*)$ has an irreducible component of codimension $\geq d$ in Y.*
- *$\mathcal{H}^j(\mathcal{G}^*) = 0$ for all $j > d$.*

Then $\mathcal{H}^j(\mathcal{G}^*) = 0$ for all $j < d$ as well.

Proof. We first claim by downward induction that $\operatorname{Ker} \delta^j$ is a direct summand of $\mathcal{G}^j$ for any $j \geq d$. The proof is similar to that given in the proof of Proposition 5.1. Thus, $H^*(\mathcal{G}^*) \simeq H^*(\mathcal{F}^*)$, where

$$\mathcal{F}^i = \mathcal{G}^i \quad \text{for all } i < d, \qquad \mathcal{F}^d = \operatorname{Ker} \delta^d, \qquad \mathcal{F}^i = 0 \quad \text{for } i > d.$$

Hence, we can assume that $\mathcal{G}^i = 0$ for all $i > d$. Now, we apply the last lemma to get the result. □

Proposition 5.4. *For any $u, w \in W$,*

$$\mathcal{E}xt^j_{\mathcal{O}_{\bar{X}}}(\mathcal{O}_{X^u}, \mathcal{O}_{X_w}) = 0 \quad \textit{for all } j < \ell(u).$$

Thus,

$$\mathcal{T}or_j^{\mathcal{O}_{\bar{X}}}(\xi^u, \mathcal{O}_{X_w}) = 0 \quad \textit{for all } j > 0.$$

Proof. We can of course replace $\bar{X}$ by V^v (for $v \in W$). Consider a locally $\mathcal{O}_{V^v}$-free resolution of finite rank

$$0 \to \mathcal{F}_n \to \mathcal{F}_{n-1} \to \cdots \to \mathcal{F}_0 \to \mathcal{O}_{X^u \cap V^v} \to 0.$$

Then, restricted to the open set V^v, $\mathscr{E}xt^j_{\mathscr{O}_{\bar{X}}}(\mathscr{O}_{X^u}, \mathscr{O}_{X_w})$ is the j-th cohomology of the complex

$$0 \leftarrow \mathscr{H}om_{\mathscr{O}_{\bar{X}}}(\mathcal{F}_n, \mathscr{O}_{X_w}) \leftarrow \cdots \leftarrow \mathscr{H}om_{\mathscr{O}_{\bar{X}}}(\mathcal{F}_0, \mathscr{O}_{X_w}) \leftarrow 0.$$

Since $\mathcal{F}_j$ is $\mathscr{O}_{\bar{X}}$-free,

$$\mathscr{H}om_{\mathscr{O}_{\bar{X}}}(\mathcal{F}_j, \mathscr{O}_{X_w}) \simeq \mathscr{H}om_{\mathscr{O}_{X_w}}(\mathcal{F}_j \otimes_{\mathscr{O}_{\bar{X}}} \mathscr{O}_{X_w}, \mathscr{O}_{X_w}).$$

Now, the first part of the proposition follows from Corollary 5.3 applied to $d = \ell(u)$ and from Proposition 5.1, by observing that the sheaf $\mathscr{E}xt^j_{\mathscr{O}_{\bar{X}}}(\mathscr{O}_{X^u}, \mathscr{O}_{X_w})$ has support in $X^u \cap X_w$, X_w is an irreducible CM variety [K, Theorem 8.2.2(c)], and for $u \leq w$, $\operatorname{codim}_{X_w}(X^u \cap X_w) = \ell(u)$ [K, Lemma 7.3.10].

The second assertion follows from the first and Proposition 5.1. □

As a consequence of Proposition 5.4, we prove Theorem 4.10(a).

Proof of Theorem 4.10(a). Since the assertion is local in $\mathbb{P}$, we can assume that $\bar{Y}_{\mathbb{P}} \simeq \mathbb{P} \times \bar{Y}$. Thus,

$$\pi^*\mathscr{O}_{\mathbb{P}_{\mathbf{j}}}(-\partial\mathbb{P}_{\mathbf{j}}) \simeq \mathscr{O}_{\mathbb{P}_{\mathbf{j}}}(-\partial\mathbb{P}_{\mathbf{j}}) \boxtimes \mathscr{O}_{\bar{Y}}, \tag{33}$$

$$\widetilde{\xi^u \boxtimes \xi^v} \cong \mathscr{O}_{\mathbb{P}} \boxtimes (\xi^u \boxtimes \xi^v), \tag{34}$$

$$\mathscr{O}_{(X_w \times X_w)_{\mathbb{P}}} \simeq \mathscr{O}_{\mathbb{P}} \boxtimes (\mathscr{O}_{X_w} \boxtimes \mathscr{O}_{X_w}). \tag{35}$$

We assert that for any $\mathscr{O}_{(Y_w)_{\mathbb{P}}}$-module $\mathcal{S}$ (where $(Y_w)_{\mathbb{P}} := (X_w \times X_w)_{\mathbb{P}}$),

$$\mathscr{T}or_i^{\mathscr{O}_{\bar{Y}_{\mathbb{P}}}}\big(\pi^*(\mathscr{O}_{\mathbb{P}_{\mathbf{j}}}(-\partial\mathbb{P}_{\mathbf{j}})) \otimes (\widetilde{\xi^u \boxtimes \xi^v}), \mathcal{S}\big) \simeq \mathscr{T}or_i^{\mathscr{O}_{(Y_w)_{\mathbb{P}}}}\Big(\mathscr{O}_{(Y_w)_{\mathbb{P}}} \otimes_{\mathscr{O}_{\bar{Y}_{\mathbb{P}}}} \big(\pi^*\mathscr{O}_{\mathbb{P}_{\mathbf{j}}}(-\partial\mathbb{P}_{\mathbf{j}}) \otimes (\widetilde{\xi^u \boxtimes \xi^v})\big), \mathcal{S}\Big). \tag{36}$$

To prove (36), from Proposition 5.4 and the isomorphisms (33)–(35), it suffices to observe the following (where we take $R = \mathscr{O}_{\bar{Y}_{\mathbb{P}}}$, $S = \mathscr{O}_{(Y_w)_{\mathbb{P}}}$, $M = \pi^*(\mathscr{O}_{\mathbb{P}_{\mathbf{j}}}(-\partial\mathbb{P}_{\mathbf{j}})) \otimes (\widetilde{\xi^u \boxtimes \xi^v})$ and $N = \mathcal{S}$).

Let R, S be commutative rings with ring homomorphism $R \to S$, M an R-module and N an S-module. Then $N \otimes_S (S \otimes_R M) \simeq N \otimes_R M$. This gives rise to the following isomorphism provided $\operatorname{Tor}_j^R(S, M) = 0$ for all $j > 0$:

$$\operatorname{Tor}_i^R(M, N) \simeq \operatorname{Tor}_i^S(S \otimes_R M, N). \tag{37}$$

Clearly,

$$\mathscr{T}or_i^{\mathscr{O}_{(Y_w)_{\mathbb{P}}}}\Big(\mathscr{O}_{(Y_w)_{\mathbb{P}}} \otimes_{\mathscr{O}_{\bar{Y}_{\mathbb{P}}}} \big(\pi^*\mathscr{O}_{\mathbb{P}_{\mathbf{j}}}(-\partial\mathbb{P}_{\mathbf{j}}) \otimes (\widetilde{\xi^u \boxtimes \xi^v})\big), \gamma_*\tilde{\Delta}_*\mathscr{O}_{(X_w)_{\mathbb{P}}}\Big) \simeq \mathscr{T}or_i^{\mathscr{O}_{(Y_w)_{\mathbb{P}}}}\Big((\gamma^{-1})_*\big(\mathscr{O}_{(Y_w)_{\mathbb{P}}} \otimes_{\mathscr{O}_{\bar{Y}_{\mathbb{P}}}} (\pi^*\mathscr{O}_{\mathbb{P}_{\mathbf{j}}}(-\partial\mathbb{P}_{\mathbf{j}}) \otimes (\widetilde{\xi^u \boxtimes \xi^v}))\big), \tilde{\Delta}_*\mathscr{O}_{(X_w)_{\mathbb{P}}}\Big).$$

By Lemma 4.7, the closures of Γ-orbits in $(Y_w)_{\mathbb{P}}$ are precisely $(X_x \times X_y)_{\mathbb{P}}$ for $x, y \leq w$. By Proposition 5.4 and the isomorphism (37) (applied to $R = \mathscr{O}_{\bar{X}}$, $S = \mathscr{O}_{X_w}$, $M = \xi^u$ and $N = \mathscr{O}_{X_x}$), we get

$$\mathscr{T}or_j^{\mathscr{O}_{X_w}}(\mathscr{O}_{X_w} \otimes_{\mathscr{O}_{\bar{X}}} \xi^u, \mathscr{O}_{X_x}) = 0, \qquad \forall x \leq w,\ j \geq 1. \tag{38}$$

Further, by the identities (33)–(35) and (38), $\mathcal{F} := \mathcal{O}_{(Y_w)_{\mathbb{P}}} \otimes_{\mathcal{O}_{\tilde{Y}_{\mathbb{P}}}} (\pi^* \mathcal{O}_{\mathbb{P}_{\mathbf{j}}}(-\partial\mathbb{P}_{\mathbf{j}}) \otimes (\widetilde{\xi^u \boxtimes \xi^v}))$ is homologically transverse to the Γ-orbit closures in $(Y_w)_{\mathbb{P}}$. Thus, applying [AGM, Theorem 2.3] (with their $G = \Gamma$, $X = (Y_w)_{\mathbb{P}}$, $\mathcal{E} = \tilde{\Delta}_* \mathcal{O}_{(X_w)_{\mathbb{P}}}$, and their $\mathcal{F}$ as the above $\mathcal{F}$) (a result originally due to Sierra [Si, Theorem 1.2]) we get the following identity:

$$\mathcal{T}or_i^{\mathcal{O}_{(Y_w)_{\mathbb{P}}}} \Big(\mathcal{O}_{(Y_w)_{\mathbb{P}}} \otimes_{\mathcal{O}_{\tilde{Y}_{\mathbb{P}}}} \big(\pi^* \mathcal{O}_{\mathbb{P}_{\mathbf{j}}}(-\partial\mathbb{P}_{\mathbf{j}}) \otimes (\widetilde{\xi^u \boxtimes \xi^v})\big), \gamma_* \tilde{\Delta}_* \mathcal{O}_{(X_w)_{\mathbb{P}}} \Big) = 0 \quad \text{for all } i > 0. \tag{39}$$

(Observe that even though Γ is infinite-dimensional, its action on $(Y_w)_{\mathbb{P}}$ factors through the action of a finite-dimensional quotient group $\bar{\Gamma}$ of Γ.)

Observe that $\gamma(\tilde{\Delta}(X_w)_{\mathbb{P}}) \subset (Y_w)_{\mathbb{P}}$, and thus by (36) and (39), we get

$$\mathcal{T}or_i^{\mathcal{O}_{\tilde{Y}_{\mathbb{P}}}} \big(\pi^*(\mathcal{O}_{\mathbb{P}_{\mathbf{j}}}(-\partial\mathbb{P}_{\mathbf{j}})) \otimes (\widetilde{\xi^u \boxtimes \xi^v}), \gamma_* \tilde{\Delta}_* \mathcal{O}_{(X_w)_{\mathbb{P}}}\big) = 0 \quad \text{for all } i > 0.$$

This proves Theorem 4.10(a). □

Lemma 5.5. *For any $u, w \in W$,*

$$\mathcal{T}or_j^{\mathcal{O}_{\bar{X}}}(\mathcal{O}_{X^u}, \mathcal{O}_{X_w}) = 0 \quad \textit{for all } j > 0.$$

Proof. We can of course replace $\bar{X}$ by the open set V^v (for $v \in W$) and consider the free resolution by $\mathcal{O}_{V^v}$-modules of finite rank:

$$0 \to \mathcal{F}_n \xrightarrow{\delta_{n-1}} \mathcal{F}_{n-1} \to \cdots \xrightarrow{\delta_0} \mathcal{F}_0 \to \mathcal{O}_{X^u \cap V^v} \to 0.$$

Since the assertion of the lemma is local in $\bar{X}$, we can (and do) replace V^v by suitable smaller open subsets in the following. By downward induction, we show that $\mathcal{D}_i := \operatorname{Im} \delta_i$ is a direct summand of $\mathcal{F}_i$ for all $i \geq \ell(u)$. Of course, the assertion holds for $i = n$. By induction, assume that $\mathcal{D}_{i+1}$ is a direct summand (where $i \geq \ell(u)$). Thus,

$$0 \to \mathcal{D}_{i+1}^{\perp} \xrightarrow{\delta_i} \mathcal{F}_i \to \cdots \to \mathcal{F}_0 \to \mathcal{O}_{X^u \cap V^v} \to 0 \tag{$\mathcal{C}_1$}$$

is a locally free resolution, where $\mathcal{D}_{i+1}^{\perp}$ is any $\mathcal{O}_{V^v}$-submodule of $\mathcal{F}_{i+1}$ such that $\mathcal{D}_{i+1} \oplus \mathcal{D}_{i+1}^{\perp} = \mathcal{F}_{i+1}$.

Consider the short exact sequence

$$0 \to \mathcal{D}_{i+1}^{\perp} \xrightarrow{\delta_i} \mathcal{F}_i \to \mathcal{F}_i/\delta_i(\mathcal{D}_{i+1}^{\perp}) \to 0. \tag{$\mathcal{C}_2$}$$

This gives rise to the exact sequence

$$0 \to \mathcal{H}om_{\mathcal{O}_{V^v}}(\mathcal{F}_i/\delta_i(\mathcal{D}_{i+1}^{\perp}), \mathcal{O}_{V^v}) \to \mathcal{H}om_{\mathcal{O}_{V^v}}(\mathcal{F}_i, \mathcal{O}_{V^v}) \xrightarrow{\delta_i^*} \mathcal{H}om_{\mathcal{O}_{V^v}}(\mathcal{D}_{i+1}^{\perp}, \mathcal{O}_{V^v}) \to \mathcal{E}xt^1_{\mathcal{O}_{V^v}}(\mathcal{F}_i/\delta_i(\mathcal{D}_{i+1}^{\perp}), \mathcal{O}_{V^v}) \to 0,$$

where the last zero is due to the fact that $\mathcal{F}_i$ is $\mathcal{O}_{V^v}$-free.

From the resolution ($\mathcal{C}_1$) and the identity (5) (since $i \geq \ell(u)$ by assumption), we see that the above map δ_i^* is surjective. Hence,

$$\mathscr{E}xt^1_{\mathscr{O}_{V^v}}(\mathcal{F}_i/\delta_i(\mathcal{D}_{i+1}^\perp), \mathscr{O}_{V^v}) = 0$$

and so

$$\mathscr{E}xt^1_{\mathscr{O}_{V^v}}(\mathcal{F}_i/\delta_i(\mathcal{D}_{i+1}^\perp), \mathcal{D}_{i+1}^\perp) = 0,$$

since $\mathcal{D}_{i+1}^\perp$ is a locally free $\mathscr{O}_{V^v}$-module.

Thus, the short exact sequence ($\mathcal{C}_2$) splits locally. In particular, $\mathcal{D}_i = \operatorname{Im} \delta_i$ is a direct summand locally. This completes the induction and hence we get a locally free resolution

$$0 \to \mathcal{D}_{\ell(u)}^\perp \to \mathcal{F}_{\ell(u)-1} \to \cdots \to \mathcal{F}_0 \to \mathscr{O}_{X^u \cap V^v} \to 0. \qquad (\mathcal{C}_3)$$

In particular,

$$\mathscr{T}or_j^{\mathscr{O}_{\bar{X}}}(\mathscr{O}_{X^u}, \mathscr{O}_{X_w}) = 0 \quad \text{for all } j > \ell(u).$$

Of course, $\mathscr{T}or_j^{\mathscr{O}_{\bar{X}}}(\mathscr{O}_{X^u}, \mathscr{O}_{X_w})$, restricted to V^v, is the j-th homology of the chain complex (of finitely generated locally free $\mathscr{O}_{X_w \cap V^v}$-modules)

$$0 \to \mathcal{D}_{\ell(u)}^\perp \otimes_{\mathscr{O}_{V^v}} \mathscr{O}_{X_w \cap V^v} \to \mathcal{F}_{\ell(u)-1} \otimes_{\mathscr{O}_{V^v}} \mathscr{O}_{X_w \cap V^v} \to \cdots \to \mathcal{F}_0 \otimes_{\mathscr{O}_{V^v}} \mathscr{O}_{X_w \cap V^v} \to 0. \qquad (\mathcal{C}_4)$$

Clearly, the support of the homology $\bigoplus_{i \geq 1} \mathscr{H}_i(\mathcal{C}_4)$ is contained in $X^u \cap X_w$. As observed in the proof of Proposition 5.4, $X^u \cap X_w$ is of codimension $\ell(u)$ in X_w.

Thus, by Lemma 5.2 with $d = \ell(u)$,

$$\mathscr{H}_i(\mathcal{C}_4) = 0 \quad \text{for all } i > 0. \qquad \square$$

Remark 5.6. As pointed out by the referee, the above lemma can also be deduced from Proposition 5.4 by using Theorem 10.4 and the long exact sequence for $\mathscr{T}or$.

As a consequence of Lemma 5.5, we get the following generalization.

Corollary 5.7. *For any finite union $Y = \bigcup_{i=1}^k X^{v_i}$ of opposite Schubert varieties, and any $w \in W$,*

(a) $\mathscr{T}or_j^{\mathscr{O}_{\bar{X}}}(\mathscr{O}_Y, \mathscr{O}_{X_w}) = 0$ *for all $j > 0$.*
(b) $H^j(X_n, \mathscr{O}_{Y \cap X_w}) = 0$ *for all $j > 0$, where n is any positive integer such that $X_n \supset X_w$.*

In particular, the lemma applies to $Y = \partial X^u$.

Proof. (a) We use double induction on the number of components k of Y and the dimension of $Y \cap X_w$ (i.e., the largest dimension of the irreducible components of $Y \cap X_w$; we declare the dimension of the empty space to be -1). If Y has one component, i.e., $k = 1$, then (a) follows from Lemma 5.5. If $\dim(Y \cap X_w) = -1$ (i.e., $Y \cap X_w$ is empty), then clearly

$$\mathscr{T}or_j^{\mathscr{O}_{\bar{X}}}(\mathscr{O}_Y, \mathscr{O}_{X_w}) = 0 \quad \text{for all } j \geq 0. \tag{40}$$

So, assume that $k \geq 2$ and $Y \cap X_w$ is nonempty. We can assume that v_1 is not larger than any v_i for $i \geq 2$ (for otherwise we can drop X^{v_1} from the union without changing Y). Let $Y_1 := X^{v_1}$ and $Y_2 := \bigcup_{i \geq 2} X^{v_i}$. Then, if $Y_1 \cap X_w$ is nonempty, $Y_1 \cap X_w = X_w^{v_1}$ properly contains $Y_1 \cap Y_2 \cap X_w$, since $v_1 \in X_w^{v_1}$ but $v_1 \notin Y_2 \cap X_w$. In particular, $X_w^{v_1}$ being irreducible (see Proposition 6.6 below),

$$\dim(Y \cap X_w) \geq \dim(Y_1 \cap X_w) > \dim(Y_1 \cap Y_2 \cap X_w). \tag{41}$$

The short exact sequence of sheaves

$$\mathscr{O}_Y \to \mathscr{O}_{Y_1} \oplus \mathscr{O}_{Y_2} \to \mathscr{O}_{Y_1 \cap Y_2} \to 0$$

yields the long exact sequence

$$\begin{aligned} \cdots \to \mathscr{T}or_{j+1}^{\mathscr{O}_{\bar{X}}}(\mathscr{O}_{Y_1 \cap Y_2}, \mathscr{O}_{X_w}) \to \mathscr{T}or_j^{\mathscr{O}_{\bar{X}}}(\mathscr{O}_Y, \mathscr{O}_{X_w}) \to \mathscr{T}or_j^{\mathscr{O}_{\bar{X}}}(\mathscr{O}_{Y_1} \oplus \mathscr{O}_{Y_2}, \mathscr{O}_{X_w}) \\ \to \mathscr{T}or_j^{\mathscr{O}_{\bar{X}}}(\mathscr{O}_{Y_1 \cap Y_2}, \mathscr{O}_{X_w}) \to \cdots . \end{aligned} \tag{42}$$

Now, since Y_2 has $k - 1$ components, induction on the number of components gives

$$\mathscr{T}or_j^{\mathscr{O}_{\bar{X}}}(\mathscr{O}_{Y_1} \oplus \mathscr{O}_{Y_2}, \mathscr{O}_{X_w}) = 0 \quad \text{for all } j > 0. \tag{43}$$

Since the scheme-theoretic intersection $Y_1 \cap Y_2$ is reduced (see §2) and it is a finite union of X^u's with $\dim(Y \cap X_w) > \dim(Y_1 \cap Y_2 \cap X_w)$ (by (41)), by induction we get

$$\mathscr{T}or_j^{\mathscr{O}_{\bar{X}}}(\mathscr{O}_{Y_1 \cap Y_2}, \mathscr{O}_{X_w}) = 0 \quad \text{for all } j > 0. \tag{44}$$

So, from (43)–(44) and the exact sequence (42), we get (a).

(b) We use the same induction as in (a). For $k = 1$, i.e., $Y \cap X_w = X_w^{v_1}$, the result is a particular case of [KuS, Corollary 3.2]. Now, take any $Y = \bigcup_{i=1}^k X^{v_i}$ and let Y_1, Y_2 be as in (a). By (a), we have the sheaf exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{O}_Y \otimes_{\mathscr{O}_{\bar{X}}} \mathscr{O}_{X_w} & \longrightarrow & (\mathscr{O}_{Y_1} \oplus \mathscr{O}_{Y_2}) \otimes_{\mathscr{O}_{\bar{X}}} \mathscr{O}_{X_w} & \longrightarrow & \mathscr{O}_{Y_1 \cap Y_2} \otimes_{\mathscr{O}_{\bar{X}}} \mathscr{O}_{X_w} & \longrightarrow & 0 \\ & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\ 0 & \longrightarrow & \mathscr{O}_{Y \cap X_w} & \longrightarrow & (\mathscr{O}_{Y_1 \cap X_w} \oplus \mathscr{O}_{Y_2 \cap X_w}) & \longrightarrow & \mathscr{O}_{Y_1 \cap Y_2 \cap X_w} & \longrightarrow & 0 \end{array}$$

The corresponding long exact cohomology sequence gives

$$\begin{aligned} \cdots \to H^{j-1}(X_n, \mathscr{O}_{Y_1 \cap X_w} \oplus \mathscr{O}_{Y_2 \cap X_w}) \to H^{j-1}(X_n, \mathscr{O}_{Y_1 \cap Y_2 \cap X_w}) \to H^j(X_n, \mathscr{O}_{Y \cap X_w}) \\ \to H^j(X_n, \mathscr{O}_{Y_1 \cap X_w} \oplus \mathscr{O}_{Y_2 \cap X_w}) \to \cdots . \end{aligned}$$

By induction,

$$H^j(X_n, \mathscr{O}_{Y_1\cap X_w} \oplus \mathscr{O}_{Y_2\cap X_w}) = 0, \quad \forall j > 0, \qquad H^{j-1}(X_n, \mathscr{O}_{Y_1\cap Y_2\cap X_w}) = 0, \quad \forall j > 1.$$

Thus, from the above long exact sequence,

$$H^j(X_n, \mathscr{O}_{Y\cap X_w}) = 0, \qquad \forall j > 1.$$

Write $Y_1 \cap Y_2 = \bigcup_{l=1}^d X^{u_l}$. Hence, $Y_1 \cap Y_2 \cap X_w = \bigcup_{l=1}^d X_w^{u_l}$. Thus, if nonempty, $Y_1 \cap Y_2 \cap X_w$ is connected as each $X_w^{u_j}$ contains w. This shows that

$$H^0(X_n, \mathscr{O}_{Y_1\cap X_w} \oplus \mathscr{O}_{Y_2\cap X_w}) \to H^0(X_n, \mathscr{O}_{Y_1\cap Y_2\cap X_w})$$

is surjective, which gives the vanishing of $H^1(X_n, \mathscr{O}_{Y\cap X_w})$. This proves (b). □

As a consequence of Lemma 5.5, we get the following.

Lemma 5.8. *For any $u, w \in W$ and any $j \geq 0$,*

$$\mathscr{E}xt^j_{\mathscr{O}_{X_w}}(\mathscr{O}_{X^u\cap X_w}, \mathscr{O}_{X_w}) = 0 \quad \textit{for } j \neq \ell(u). \tag{45}$$

Moreover,

$$\mathscr{E}xt^j_{\mathscr{O}_{\bar{X}}}(\mathscr{O}_{X^u}, \mathscr{O}_{\bar{X}}) \otimes_{\mathscr{O}_{\bar{X}}} \mathscr{O}_{X_w} \simeq \mathscr{E}xt^j_{\mathscr{O}_{X_w}}(\mathscr{O}_{X^u\cap X_w}, \mathscr{O}_{X_w}). \tag{46}$$

Proof. Again we can replace $\bar{X}$ by V^v (for $v \in W$). Consider an $\mathscr{O}_{V^v}$-locally free resolution (cf. the proof of Lemma 5.5, specifically $(\mathcal{C}_3)$) (possibly restricted to an open cover of V^v)

$$0 \to \mathcal{F}_{\ell(u)} \to \cdots \to \mathcal{F}_0 \to \mathscr{O}_{X^u\cap V^v} \to 0.$$

By Lemma 5.5, the following is a locally free $\mathscr{O}_{X_w\cap V^v}$-module resolution:

$$0 \to \mathcal{F}_{\ell(u)} \otimes_{\mathscr{O}_{V^v}} \mathscr{O}_{X_w\cap V^v} \to \cdots \to \mathcal{F}_0 \otimes_{\mathscr{O}_{V^v}} \mathscr{O}_{X_w\cap V^v} \to \mathscr{O}_{X^u\cap V^v} \otimes_{\mathscr{O}_{V^v}} \mathscr{O}_{X_w\cap V^v} \to 0. \tag{47}$$

Observe that $\mathscr{O}_{X^u\cap V^v} \otimes_{\mathscr{O}_{V^v}} \mathscr{O}_{X_w\cap V^v} \simeq \mathscr{O}_{X^u\cap X_w\cap V^v}$, being the definition of the scheme-theoretic intersection. Thus, $\mathscr{E}xt^j_{\mathscr{O}_{X_w}}(\mathscr{O}_{X^u\cap X_w}, \mathscr{O}_{X_w})$, restricted to the open set $X_w \cap V^v$, is the j-th cohomology of the cochain complex

$$0 \leftarrow \mathscr{H}om_{\mathscr{O}_{X_w\cap V^v}}(\mathcal{F}_{\ell(u)} \otimes_{\mathscr{O}_{V^v}} \mathscr{O}_{X_w\cap V^v}, \mathscr{O}_{X_w\cap V^v}) \leftarrow \cdots \leftarrow \mathscr{H}om_{\mathscr{O}_{X_w\cap V^v}}(\mathcal{F}_0 \otimes_{\mathscr{O}_{V^v}} \mathscr{O}_{X_w\cap V^v}, \mathscr{O}_{X_w\cap V^v}) \leftarrow 0.$$

Since $\mathscr{E}xt^j_{\mathscr{O}_{X_w}}(\mathscr{O}_{X^u\cap X_w}, \mathscr{O}_{X_w})$ has support in $X^u \cap X_w$ and $X^u \cap X_w$ has codimension $\ell(u)$ in X_w (see the proof of Proposition 5.4), by Lemma 5.2 we get $\mathscr{E}xt^j_{\mathscr{O}_{X_w}}(\mathscr{O}_{X^u\cap X_w}, \mathscr{O}_{X_w}) = 0$ for any $j \neq \ell(u)$. This proves (45).

For any i,

$$\mathscr{H}om_{\mathscr{O}_{X_w\cap V^v}}(\mathcal{F}_i \otimes_{\mathscr{O}_{V^v}} \mathscr{O}_{X_w\cap V^v}, \mathscr{O}_{X_w\cap V^v}) \simeq \mathscr{H}om_{\mathscr{O}_{V^v}}(\mathcal{F}_i, \mathscr{O}_{V^v}) \otimes_{\mathscr{O}_{V^v}} \mathscr{O}_{X_w\cap V^v}. \tag{48}$$

Further, by the identity (5),

$$0 \leftarrow \mathscr{E}xt^{\ell(u)}_{\mathcal{O}_{V^v}}(\mathcal{O}_{X^u \cap V^v}, \mathcal{O}_{V^v}) \leftarrow \mathscr{H}om_{\mathcal{O}_{V^v}}(\mathcal{F}_{\ell(u)}, \mathcal{O}_{V^v}) \leftarrow \cdots \leftarrow \mathscr{H}om_{\mathcal{O}_{V^v}}(\mathcal{F}_0, \mathcal{O}_{V^v}) \leftarrow 0$$

is a locally free $\mathcal{O}_{V^v}$-module resolution of $\mathscr{E}xt^{\ell(u)}_{\mathcal{O}_{V^v}}(\mathcal{O}_{X^u \cap V^v}, \mathcal{O}_{V^v})$. Hence, by the resolution (47) and the isomorphism (48), we get

$$\mathscr{E}xt^{\ell(u)-j}_{\mathcal{O}_{X_w}}(\mathcal{O}_{X^u \cap X_w}, \mathcal{O}_{X_w}) \simeq \mathscr{T}or_j^{\mathcal{O}_{\bar{X}}}(\mathscr{E}xt^{\ell(u)}_{\mathcal{O}_{\bar{X}}}(\mathcal{O}_{X^u}, \mathcal{O}_{\bar{X}}), \mathcal{O}_{X_w}) \quad \text{for all } j \geq 0. \tag{49}$$

Thus,

$$\mathscr{E}xt^{\ell(u)}_{\mathcal{O}_{X_w}}(\mathcal{O}_{X^u \cap X_w}, \mathcal{O}_{X_w}) \simeq \mathscr{E}xt^{\ell(u)}_{\mathcal{O}_{\bar{X}}}(\mathcal{O}_{X^u}, \mathcal{O}_{\bar{X}}) \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{O}_{X_w}.$$

This proves (46), by using the identity (5) and (45). □

Lemma 5.9. *For any $v \leq w$ and $u \in W$,*

$$\mathscr{T}or_i^{\mathcal{O}_{X_w}}(\mathcal{O}_{X^u \cap X_w}, \mathcal{O}_{X_v}) = 0 \quad \text{for all } i > 0.$$

Proof. We can replace $\bar{X}$ by V^θ (for $\theta \in W$). Take an $\mathcal{O}_{\bar{X}}$-locally free resolution (see $(\mathcal{C}_3)$ in the proof of Lemma 5.5)

$$0 \to \mathcal{F}_{\ell(u)} \to \cdots \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{O}_{X^u} \to 0.$$

By Lemma 5.5,

$$0 \to \mathcal{F}_{\ell(u)} \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{O}_{X_w} \to \cdots \to \mathcal{F}_0 \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{O}_{X_w} \to \mathcal{O}_{X^u \cap X_w} \to 0 \tag{$\mathcal{S}_1$}$$

is an $\mathcal{O}_{X_w}$-locally free resolution of $\mathcal{O}_{X^u \cap X_w}$. Thus, by base extension [L, Chap. XVI, §3], $\mathscr{T}or_i^{\mathcal{O}_{X_w}}(\mathcal{O}_{X^u \cap X_w}, \mathcal{O}_{X_v})$ is the i-th homology of the complex

$$0 \to \mathcal{F}_{\ell(u)} \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{O}_{X_v} \to \cdots \to \mathcal{F}_0 \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{O}_{X_v} \to 0.$$

From the exactness of $(\mathcal{S}_1)$ for w replaced by v, we get the lemma. □

6. Desingularization of Richardson varieties and flatness for the Γ-action

Let $S \subset W$ be a finite ideal and, as in Definition 3.1, let V^s be the corresponding B^--stable open subset $\bigcup_{w \in S}(w\ B^- \cdot x_o)$ of $\bar{X}$, where x_o is the base point $1.B$ of $\bar{X}$. It is a B^--stable subset, since by [KS, §2],

$$V^S = \bigcup_{w \in S} B^- w x_o.$$

Lemma 6.1. *For any $v \in W$ and any finite ideal $S \subset W$ containing v, there exists a closed normal subgroup N_S^- of B^- of finite codimension such that the quotient $Y^v(S) := N_S^- \backslash X^v(S)$ acquires a canonical structure of a B^--scheme of finite type over the base field $\mathbb{C}$ under the left multiplication action of B^- on $Y^v(S)$, so that the quotient $q : X^v(S) \to Y^v(S)$ is a principal N_S^--bundle, where $X^v(S) := X^v \cap V^S$. Of course, the map q is B^--equivariant.*

Proof. For any $u \in W$, the subgroup $U_u^- := U^- \cap uU^-u^{-1}$ acts freely and transitively on C^u via left multiplication, since $C^u = (U^- \cap uU^-u^{-1})uB/B$. Thus, $U_S^- := \bigcap_{u \in S} U_u^-$ acts freely on V^S. Clearly, $wB^- \cdot x_o$, for any $w \in S$, is stable under U_S^-, and further each orbit of U_S^- in the open subset $wB^- \cdot x_o$ of $\bar{X}$ is closed in $wB^- \cdot x_o$ (use [K, Lemma 6.1.3]). Thus, each orbit of U_S^- is closed in their union V^S. In fact, U_S^- acts properly on V^S. Hence, U_S^- acts freely and properly on $X^v(S)$. Take any closed normal subgroup N_S^- of B^- of finite codimension contained in U_S^-. Then $Y^v(S) := N_S^- \backslash X^v(S)$ acquires a canonical structure of a B^--scheme of finite type over $\mathbb{C}$ under the left multiplication action of B^- on $Y^v(S)$, so that the quotient map $q : X^v(S) \to Y^v(S)$ is a principal N_S^--bundle. □

Remark 6.2. (a) The above lemma allows us to define various local properties of X^v. In particular, a point $x \in X^v$ is called *normal* (resp. *CM*) if the corresponding point in the quotient $Y^v(S)$ has that property, where S is a finite ideal such that $x \in X^v(S)$. Clearly, the property does not depend upon the choice of S and N_S^-.

(b) It is possible that the scheme $Y^v(S)$ is not separated. However, as observed by M. Kashiwara, we can choose our closed normal subgroup N_S^- of B^- of finite codimension contained in U_S^- appropriately so that $Y^v(S)$ is indeed separated. In fact, we give the following more general result due to him.

Let $\mathbf{k}$ be a field and let $\{S_\lambda\}_{\lambda \in \Lambda}$ be a filtrant projective system of quasi-compact $\mathbf{k}$-schemes locally of finite type over $\mathbf{k}$. Assume that $f_{\lambda,\mu} : S_\mu \to S_\lambda$ is an affine morphism. Set $S = \text{Inv.lt.}_\lambda\, S_\lambda$ and let $p_\lambda : S \to S_\lambda$ be the canonical projection.

Lemma 6.3 (due to M. Kashiwara). *If S is separated, then S_λ is separated for some λ.*

Proof. Take a smallest element $\lambda_o \in \Lambda$. It is enough to show that for a pair of affine open subsets U_o, V_o of S_{λ_o}, $U_\lambda \cap V_\lambda \to U_\lambda \times V_\lambda$ is a closed embedding for some λ, where $U_\lambda := f_{\lambda_o,\lambda}^{-1}(U_o)$ and $V_\lambda := f_{\lambda_o,\lambda}^{-1}(V_o)$.

Note that $U_\lambda \cap V_\lambda$ is quasi-compact and of finite type over $\mathbf{k}$. Set $U = p_{\lambda_o}^{-1}(U_o)$ and $V = p_{\lambda_o}^{-1}(V_o)$. Since S is separated, $U \cap V \to U \times V$ is a closed embedding. In particular, $U \cap V$ is affine.

We have a projective system of schemes $\{U_\lambda \cap V_\lambda\}_{\lambda \in \Lambda}$ and $\{U_\lambda\}_{\lambda \in \Lambda}$, and a projective system of morphisms $\{U_\lambda \cap V_\lambda \to U_\lambda\}_{\lambda \in \Lambda}$. Since $\text{Inv.lt.}_\lambda\, (U_\lambda \cap V_\lambda) \simeq U \cap V$ is affine, the morphism $\text{Inv.lt.}_\lambda\, (U_\lambda \cap V_\lambda) \to \text{Inv.lt.}_\lambda\, (U_\lambda)$ is an affine morphism. Hence, $U_{\lambda_1} \cap V_{\lambda_1} \to U_{\lambda_1}$ is an affine morphism for some λ_1 by [GD, Théorème 8.10.5]. Hence, $U_{\lambda_1} \cap V_{\lambda_1}$ is affine. Now, by the assumption, $\mathcal{O}_S(U) \otimes \mathcal{O}_S(V) \to \mathcal{O}_S(U \cap V)$ is surjective. Since $U_o \cap V_o \to U_o$ is of finite type, $U \cap V \to U$ is of finite type. Hence, $\mathcal{O}_S(U \cap V)$ is an

$\mathscr{O}_S(U)$-algebra of finite type. Since $\mathscr{O}_S(U)\otimes\mathscr{O}_S(V)\simeq \text{Dir.lt.}_\lambda(\mathscr{O}_S(U)\otimes\mathscr{O}_{S_\lambda}(V_\lambda))$, there exists $\lambda_2\to\lambda_1$ such that $\mathscr{O}_S(U)\otimes\mathscr{O}_{S_{\lambda_2}}(V_{\lambda_2})\to\mathscr{O}_S(U\cap V)$ is surjective. This means that $U\cap V\to U\times V_{\lambda_2}$ is a closed embedding. Now, consider the projective system

$$U_\lambda\cap V_\lambda\to U_\lambda\times V_{\lambda_2}.$$

Its projective limit with respect to λ is isomorphic to $U\cap V\to U\times V_{\lambda_2}$, which is a closed embedding. Hence, again by loc. cit., $U_\lambda\cap V_{\lambda_3}\to U_{\lambda_3}\times V_{\lambda_2}$ is a closed embedding for some $\lambda_3\to\lambda_2$. Then $U_{\lambda_3}\cap V_{\lambda_3}\to U_{\lambda_3}\times V_{\lambda_3}$ is a closed embedding. □

Theorem 6.4. *For any $v\in W$ and any finite ideal $S\subset W$ containing v, there exists a smooth irreducible B^--scheme $Z^v(S)$ and a projective B^--equivariant morphism*

$$\pi_S^v: Z^v(S)\to X^v(S)$$

satisfying the following conditions:

(a) *The restriction $(\pi_S^v)^{-1}(C^v)\to C^v$ is an isomorphism.*
(b) *$\partial Z^v(S):=(\pi_S^v)^{-1}(\partial X^v(S))$ is a divisor with simple normal crossings, where $X^v(S):=X^v\cap V^S$ and $\partial X^v(S):=(\partial X^v)\cap V^S$.*

(*Here smoothness of $Z^v(S)$ means that there exists a closed subgroup N_S^- of B^- of finite codimension which acts freely and properly on $Z^v(S)$, such that the quotient is a smooth scheme of finite type over $\mathbb{C}$.*)

Proof. Observe that the action of B^- on $Y^v(S)$ factors through the action of the finite-dimensional algebraic group B^-/N_S^-, where $Y^v(S)$ is as defined in Lemma 6.1. Now, take a B^--equivariant desingularization $\theta:\bar{Z}^v(S)\to Y^v(S)$ such that θ is a projective morphism, $\theta^{-1}(N_S^-\backslash C^v)\to N_S^-\backslash C^v$ is an isomorphism and $\theta^{-1}(N_S^-\backslash(\partial X^v(S)))$ is a divisor with simple normal crossings [Ko, §3.3][3] (see also [Bi], [RY]). Now, taking the fiber product $Z^v(S)=\bar{Z}^v(S)\times_{Y^v(S)}X^v(S)$ clearly proves the theorem. □

For $w\in W$, take the ideal $S_w=\{u\le w\}$. Then, by [K, Lemma 7.1.22(b)],

$$X^v(S_w)\cap X_w=X_w^v.$$

Lemma 6.5. *The map*

$$\mu_w: U^-\times Z_w\to\bar{X},\quad (g,z)\mapsto g\cdot\theta_w(z),$$

is a smooth morphism, where $\theta_w: Z_w\to X_w$ is the B-equivariant BSDH desingularization corresponding to a fixed reduced decomposition $w=s_{i_1}\dots s_{i_n}$ (see proof of Proposition 4.1).

Proof. Consider the map

$$\bar{\mu}_w: G\times^B Z_w\to\bar{X},\quad [g,z]\mapsto g\theta_w(z).$$

[3] I thank Zinovy Reichstein for this reference.

Because of G-equivariance, it is a locally trivial fibration. Moreover, it has smooth fibers of finite type over $\mathbb{C}$ (isomorphic to $Z_{w^{-1}}$ for the decomposition $w^{-1} = s_{i_n} \dots s_{i_1}$):

To see this, let Z'_w be the fiber product

$$\begin{array}{ccc} Z'_w & \longrightarrow & Z_w \\ {\scriptstyle \theta'_w}\big\downarrow & \square & \big\downarrow{\scriptstyle \theta_w} \\ G & \longrightarrow & \bar{X} \end{array}$$

Then we have the fiber diagram

$$\begin{array}{ccc} G \times^B Z'_w & \longrightarrow & G \times^B Z_w \\ {\scriptstyle \hat{\mu}_w}\big\downarrow & \square & \big\downarrow{\scriptstyle \bar{\mu}_w} \\ G & \longrightarrow & \bar{X} \end{array}$$

In particular, the fibers of $\bar{\mu}_w$ are isomorphic to the fibers of $\hat{\mu}_w$. Now, it is easy to see that the map

$$Z'_{w^{-1}} \to G \times^B Z'_w, \quad z' \mapsto [\theta'_{w^{-1}}(z'), i(z')],$$

gives an isomorphism of $Z_{w^{-1}}$ with the fiber of $\hat{\mu}_w$ over 1, where $i : Z'_{w^{-1}} \to Z'_w$ is the isomorphism induced by the map $(p_n, \dots, p_1) \mapsto (p_1^{-1}, \dots, p_n^{-1})$.

In particular, $\bar{\mu}_w$ is a smooth morphism, and hence so is its restriction to the open subset $U^- \times Z_w$. □

Proposition 6.6. *For any symmetrizable Kac–Moody group G and any $v \le w \in W$, the Richardson variety $X_w^v := X_w \cap X^v \subset \bar{X}$ is irreducible, normal and CM (and of course of finite type over $\mathbb{C}$ since so is X_w). Moreover, $C_w \cap C^v$ is an open dense subset of X_w^v.*

Proof. Consider the multiplication map

$$\mu : G \times^B X_w \to \bar{X}, \quad [g, x] \mapsto gx.$$

Then, μ being G-equivariant, it is a fibration. Consider the pull-back fibration

$$\begin{array}{ccc} F_w^v & \xhookrightarrow{\hat{i}} & G \times^B X_w \\ \big\downarrow{\scriptstyle \hat{\mu}} & & \big\downarrow{\scriptstyle \mu} \\ X^v & \xhookrightarrow{i} & \bar{X} \end{array}$$

where i is the inclusion map.

Also, consider the projection map

$$\pi : G \times^B X_w \to \bar{X}, \quad [g, x] \mapsto gB.$$

Let $\hat{\pi}$ be the restriction $\hat{\pi} := \pi \circ \hat{i} : F_w^v \to \bar{X}$. Observe that since i is B^--equivariant (and μ is G-equivariant), $\hat{i}$ is B^--equivariant, and hence so is $\hat{\pi}$. In particular, $\hat{\pi}$ is a fibration over the open cell $B^-B/B \subset \bar{X}$. Moreover, since $\hat{\pi}$ is B^--equivariant (in particular, U^--equivariant) and U^- acts transitively on B^-B/B with trivial isotropy, $\hat{\pi}$ is a trivial fibration restricted to B^-B/B.

Now, by [KS, Propositions 3.2, 3.4], X^v is normal and CM (and of course irreducible). Also, $X_{w^{-1}}$ is normal, irreducible and CM [K, Theorem 8.2.2]. Thus, $\hat{\mu}$ being a fibration with fiber $X_{w^{-1}}$ (as can be seen by considering the embedding $X_{w^{-1}} \hookrightarrow G \times^B \tilde{X}_w,\ gB \mapsto [g, g^{-1}]$, where $\tilde{X}_w$ is the inverse image of X_w in G), F_w^v is irreducible, normal and CM, and hence so is its open subset $\hat{\pi}^{-1}(B^-B/B)$. But $\hat{\pi}$ is a trivial fibration restricted to B^-B/B with fiber over $1 \cdot B$ equal to $X_w^v = X_w \cap X^v$. Thus, X_w^v is irreducible, normal and CM under the scheme-theoretic intersection. Moreover, since X_w^v is Frobenius split in char. $p > 0$ [KuS, Proposition 5.3], we conclude that it is reduced.

Clearly, $C_w \cap C^v$ is an open subset of X_w^v. So, to prove that $C_w \cap C^v$ is dense in X_w^v, it suffices to show that it is nonempty, which follows from [K, proof of Lemma 7.3.10]. □

Remark 6.7. By the same proof as above, applying Corollary 10.5, we see that $X_w \cap \partial X^v$ is CM.

Theorem 6.8. *For any $v \le w$, consider the fiber product*

$$Z^v(S_w) \times_{\bar{X}} Z_w,$$

where Z_w is the BSDH (B-equivariant) desingularization of X_w (corresponding to a fixed reduced decomposition $w = s_{i_1} \dots s_{i_n}$ of w) and $\pi^v_{S_w} : Z^v(S_w) \to X^v(S_w)$ is a B^--equivariant desingularization of $X^v(S_w)$ as in Theorem 6.4. Then $Z^v(S_w) \times_{\bar{X}} Z_w$ is a smooth projective irreducible T-variety (of finite type over $\mathbb{C}$) with a canonical T-equivariant morphism

$$\pi_w^v : Z^v(S_w) \times_{\bar{X}} Z_w \to X_w^v.$$

Moreover, π_w^v is a T-equivariant desingularization which is an isomorphism restricted to the inverse image of the dense open subset $C^v \cap C_w$ of X_w^v. From now on, we abbreviate

$$Z_w^v := Z^v(S_w) \times_{\bar{X}} Z_w.$$

Proof. Consider the commutative diagram

$$\begin{array}{ccc} Z^v(S_w) \times_{\bar{X}} Z_w & \longrightarrow & X^v(S_w) \times_{\bar{X}} X_w \\ & \searrow^{\pi_w^v} & \| \\ & & X^v(S_w) \cap X_w \\ & & \| \\ & & X_w^v \end{array}$$

where the horizontal map is the fiber product of the two desingularizations, and π_w^v is the horizontal map under the above identification of $X^v(S_w) \times_{\bar{X}} X_w$ with X_w^v. Clearly, π_w^v is T-equivariant and it is an isomorphism restricted to the inverse image of the dense open subset $C^v \cap C_w$ of X_w^v. In particular, π_w^v is birational.

Define E_w^v as the fiber product

$$\begin{array}{ccc} E_w^v & \xrightarrow{\hat{\mu}_w^v} & Z^v(S_w) \\ {\scriptstyle f_w^v}\downarrow & \square & \downarrow{\scriptstyle \pi_{S_w}^v} \\ U^- \times Z_w & \xrightarrow{\mu_w} & \bar{X} \end{array}$$

where μ_w is as in Lemma 6.5. Since μ_w is a smooth morphism by Lemma 6.5, so is $\hat{\mu}_w^v$. But $Z^v(S_w)$ is a smooth scheme and hence so is E_w^v. Now, since both $U^- \times Z_w$ and $Z^v(S_w)$ are U^--schemes (with U^- acting on $U^- \times Z_w$ via left multiplication on the first factor) and the morphisms $\pi_{S_w}^v$ and μ_w are U^--equivariant, E_w^v is a U^--scheme (and f_w^v is U^--equivariant). Consider the composite morphism

$$E_w^v \xrightarrow{f_w^v} U^- \times Z_w \xrightarrow{\pi_1} U^-,$$

where π_1 is the projection on the first factor. It is U^--equivariant with respect to left multiplication on U^-. Let $\mathbb{F}$ be the fiber of $\pi_1 \circ f_w^v$ over 1. Define the isomorphism

$$\begin{array}{ccc} E_w^v & \overset{\sim}{\underset{\theta}{\longleftarrow}} & U^- \times \mathbb{F} \\ & {\scriptstyle \pi_1 \circ f_w^v}\searrow \quad \swarrow{\scriptstyle \bar{\pi}_1} & \\ & U^- & \end{array}$$

$$\theta(g, x) = g \cdot x, \quad \theta^{-1}(y) = \big((\pi_1 \circ f_w^v)(y), (\pi_1 \circ f_w^v(y))^{-1} y\big).$$

Since E_w^v is a smooth scheme, so is $\mathbb{F}$. But

$$\mathbb{F} = Z_w^v.$$

Now, $\pi_{S_w}^v$ is a projective morphism onto $X^v(S_w)$, and hence $\pi_{S_w}^v$ is a projective morphism considered as a map $Z^v(S_w) \to V^{S_w}$ (since $X^v(S_w) \subset V^{S_w}$ is closed). Also, μ_w has its image inside V^{S_w}, since $BuB/B \subset uB^-B/B$ for any $u \in W$.

Thus, f_w^v is a projective morphism, and hence

$$(f_w^v)^{-1}(1 \times Z_w) = Z_w^v$$

is a projective variety.

Now, as observed by D. Anderson and independently by M. Kashiwara, Z_w^v is irreducible:

Since $Z^v(S_w) \to X^v(S_w)$ is a proper desingularization, all its fibers are connected, and hence so are all the nonempty fibers of f_w^v. Now, $\mu_w^{-1}(\mathrm{Im}\, \pi_{S_w}^v) = U^- \times Y$, where

$Y \subset Z_w$ is the closed subvariety defined as the inverse image of the Richardson variety X_w^v under the BSDH desingularization $\theta_w : Z_w \to X_w$. Since X_w^v is irreducible, θ_w is proper, and all the fibers of θ_w are connected, $Y = \theta_w^{-1}(X_w^v)$ is connected and hence so is $\mu_w^{-1}(\operatorname{Im} \pi_{S_w}^v)$. Since the pull-back of a proper morphism is proper [H, Chap. II, Corollary 4.8], the surjective morphism $f_w^v : E_w^v \to U^- \times Y$ is proper. Now, as $U^- \times Y$ is connected and all the fibers of f_w^v over $U^- \times Y$ are nonempty and connected, we see that E_w^v is connected, and hence so is $\mathbb{F}$. Thus, $\mathbb{F}$ being smooth, it is irreducible. This proves the theorem. □

The action of B on Z_w factors through the action of a finite-dimensional quotient group $\bar{B} = B_w$ containing the maximal torus H. Let $\bar{U}$ be the image of U in $\bar{B}$.

Lemma 6.9. *For any $u \leq w$, the map $\bar{\mu} : \bar{U} \times Z_w^u \to Z_w$ is a smooth morphism, and hence so is $\bar{B} \times Z_w^u \to Z_w$, where $(b, z) \mapsto b \cdot \pi_2(z)$ for $b \in \bar{B}$ and $z \in Z_w^u$. (Here $\pi_2 : Z_w^u \to Z_w$ is the canonical projection map.)*

Proof. First of all, the map

$$\mu' : G \times^{B^-} Z^u(S_w) \to \bar{X}, \quad [g, z] \mapsto g\pi_{S_w}^u(z),$$

being G-equivariant, is a locally trivial fibration. (It is trivial over the open subset $U^- \subset \bar{X}$.)

We next claim that the following diagram is a Cartesian diagram:

$$\begin{array}{ccc} U \times Z_w^u & \longrightarrow & U \times Z^u(S_w) \\ {\scriptstyle \mu}\downarrow & \square & \downarrow{\scriptstyle \hat{\mu}'} \\ Z_w & \longrightarrow & \bar{X} \end{array} \qquad (\mathfrak{D})$$

where $\mu(u, z) = u \cdot \pi_2(z)$ and $\hat{\mu}'(u, z) = u \cdot \pi_{S_w}^u(z)$. Define

$$\theta : U \times Z_w^u \to (U \times Z^u(S_w)) \times_{\bar{X}} Z_w, \quad (u, z) \mapsto ((u, \pi_1(z)), u \cdot \pi_2(z)),$$

where $\pi_2 : Z_w^u \to Z_w$ and $\pi_1 : Z_w^u \to Z^u(S_w)$ are the canonical morphisms. Also define

$$\theta' : (U \times Z^u(S_w)) \times_{\bar{X}} Z_w \to U \times Z_w^u, \quad ((u, z_1), z_2) \mapsto (u, (z_1, u^{-1}z_2)).$$

Clearly θ and θ' are inverses to each other and hence θ is an isomorphism. Thus, $(\mathfrak{D})$ is a Cartesian diagram.

Now, consider the pull-back diagram:

$$\begin{array}{ccc} \mathbb{E} & \xrightarrow{\alpha} & G \times^{B^-} Z^u(S_w) \\ {\scriptstyle \beta}\downarrow & \square & \downarrow{\scriptstyle \mu'} \\ Z_w & \longrightarrow & \bar{X} \end{array}$$

Since μ' is a locally trivial fibration, so is the map β. Moreover, since $(\mathfrak{D})$ is a Cartesian diagram, $\alpha^{-1}(U \times Z^u(S_w)) \simeq U \times Z^u_w$ and $\beta_{|U \times Z^u_w} = \mu$. Thus, the differential of μ is surjective at the Zariski tangent spaces.

Since the morphism $\mu : U \times Z^u_w \to Z_w$ factors through a finite-dimensional quotient $\bar{\mu} : \bar{U} \times Z^u_w \to Z_w$, the differential of $\bar{\mu}$ continues to be surjective at the Zariski tangent spaces. Since $\bar{U}$, Z^u_w and Z_w are smooth varieties, we see that $\bar{\mu} : \bar{U} \times Z^u_w \to Z_w$ is a smooth morphism [H, Chap. III, Proposition 10.4].

To prove that the map $\bar{B} \times Z^u_w \to Z_w$ is a smooth morphism, it suffices to observe that $H \times Z_w \to Z_w$, $(h, z) \mapsto h \cdot z$, is a smooth morphism. This proves the lemma. □

Lemma 6.10. *The map $\bar{B} \times X^u_w \to X_w$, $(b, x) \mapsto b \cdot x$, is a flat morphism for any $u \leq w$.*

Proof. The map

$$\mu : G \times^{U^-} X^u(S_w) \to \bar{X}, \qquad [g, x] \mapsto g \cdot x,$$

being G-equivariant, is a fibration. In particular, it is a flat map, and hence its restriction (to an open subset) $\mu' : B \times X^u(S_w) \to \bar{X}$ is a flat map. Now, ${\mu'}^{-1}(X_w) = B \times X^u_w$. Thus, $\mu' : B \times X^u_w \to X_w$ is a flat map. Now, since $B \times X^u_w \to \bar{B} \times X^u_w$ is a locally trivial fibration (in particular, faithfully flat), the map $\bar{B} \times X^u_w \to X_w$ is flat [M, Chap. 3, §7]. This proves the lemma. □

The canonical action of $\Gamma = \Gamma_{B \times B}$ on $(Z^2_w)_{\mathbb{P}}$ descends to an action of a finite-dimensional quotient group $\bar{\Gamma} = \Gamma_w$:

$$\Gamma \twoheadrightarrow \bar{\Gamma} = \Gamma_w \twoheadrightarrow \mathrm{GL}(N+1)^r,$$

where $(Z^2_w)_{\mathbb{P}}$ and Γ are as in Section 4. In fact, we can (and do) take

$$\bar{\Gamma} = \bar{\Gamma}_0 \rtimes \mathrm{GL}(N+1)^r,$$

where $\bar{\Gamma}_0$ is the group of global sections of the bundle $E(T)_{\mathbb{P}} \times^T \bar{B}^2 \to \mathbb{P}$, where $\bar{B} = B_w$ is defined just above Lemma 6.9.

Lemma 6.11. *For any $\mathbf{j} = (j_1, \ldots, j_r) \in [N]^r$ and $u, v \leq w$, the map*

$$\tilde{m} : \bar{\Gamma} \times (Z^{u,v}_w)_{\mathbf{j}} \to (Z^2_w)_{\mathbb{P}}$$

is a smooth morphism, where $Z^{u,v}_w := Z^u_w \times Z^v_w$ under the diagonal action of T, $(Z^{u,v}_w)_{\mathbf{j}}$ is the inverse image of $\mathbb{P}_{\mathbf{j}}$ under the map $E(T)_{\mathbb{P}} \times^T Z^{u,v}_w \to \mathbb{P}$, and $\tilde{m}(\gamma, x) = \gamma \cdot \pi_2(x)$. (Here $\pi_2 : (Z^{u,v}_w)_{\mathbf{j}} \to (Z^2_w)_{\mathbb{P}}$ is the map induced from the canonical projection $p : Z^u_w \times Z^v_w \to Z^2_w$.)

Proof. Consider the following commutative diagram, where both the right horizontal maps are fibrations with leftmost spaces as fibers:

$$\begin{array}{ccccc}
\bar{\Gamma}_0 \times Z^{u,v}_w & \longrightarrow & \bar{\Gamma} \times (Z^{u,v}_w)_{\mathbf{j}} & \longrightarrow & \mathrm{GL}(N+1)^r \times \mathbb{P}_{\mathbf{j}} \\
\downarrow{\scriptstyle m'} & & \downarrow{\scriptstyle \tilde{m}} & & \downarrow{\scriptstyle m''} \\
Z^2_w & \longrightarrow & (Z^2_w)_{\mathbb{P}} & \longrightarrow & \mathbb{P} = (\mathbb{P}^N)^r
\end{array}$$

Here m' is the restriction of $\tilde{m}$, and m'' is the restriction of the standard map $\mathrm{GL}(N+1)^r \times \mathbb{P} \to \mathbb{P}$ induced from the action of $\mathrm{GL}(N+1)$ on $\mathbb{P}^N$. Thus, m' takes (γ, z) to $\gamma(*) \cdot p(z)$, where $*$ is the base point in $\mathbb{P}$. Clearly, m'' is a smooth morphism since it is $\mathrm{GL}(N+1)^r$-equivariant and $\mathrm{GL}(N+1)^r$ acts transitively on $\mathbb{P}$. We next claim that m' is a smooth morphism: By the analogue of Lemma 4.7 for Γ_B replaced by $\Gamma = \Gamma_{B\times B}$ (see the remark following Lemma 4.8), it suffices to show that

$$\bar{B}^2 \times Z_w^{u,v} \to Z_w^2$$

is a smooth morphism, which follows from Lemma 6.9 asserting that $\bar{B} \times Z_w^u \to Z_w$ is a smooth morphism. Since m' and m'' are smooth morphisms, so is $\tilde{m}$ by [H, Chap. III, Proposition 10.4]. □

Lemma 6.12. *Let $u, v \leq w$. The map $m : \bar{\Gamma} \times (X_w^{u,v})_{\mathbf{j}} \to (X_w^2)_{\mathbb{P}}$ is flat, where m is defined similarly to the map $\tilde{m} : \bar{\Gamma} \times (Z_w^{u,v})_{\mathbf{j}} \to (Z_w^2)_{\mathbb{P}}$ in Lemma 6.11.*

Similarly, its restriction $m' : \bar{\Gamma} \times \partial((X_w^{u,v})_{\mathbf{j}}) \to (X_w^2)_{\mathbb{P}}$ is flat, where $X_w^{u,v} := X_w^u \times X_w^u$,

$$\partial((X_w^{u,v})_{\mathbf{j}}) := ((\partial X^{u,v}) \cap (X_w^2))_{\mathbf{j}} \cup (X_w^{u,v})_{\partial\mathbb{P}_{\mathbf{j}}},$$

$(X_w^{u,v})_{\partial\mathbb{P}_{\mathbf{j}}}$ is the inverse image of $\partial\mathbb{P}_{\mathbf{j}}$ under the standard quotient map $E(T)_{\mathbb{P}} \times^T X_w^{u,v} \to \mathbb{P}$, and $\partial X^{u,v} := ((\partial X^u) \times X^v) \cup (X^u \times (\partial X^v))$.

Proof. Consider the following diagram where both the right horizontal maps are locally trivial fibrations with leftmost spaces as fibers:

$$\begin{array}{ccccc} \bar{\Gamma}_0 \times X_w^{u,v} & \longrightarrow & \bar{\Gamma} \times (X_w^{u,v})_{\mathbf{j}} & \longrightarrow & \mathrm{GL}(N+1)^r \times \mathbb{P}_{\mathbf{j}} \\ \downarrow{\scriptstyle \hat{m}'} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m''} \\ X_w^2 & \longrightarrow & (X_w^2)_{\mathbb{P}} & \longrightarrow & \mathbb{P} = (\mathbb{P}^N)^r \end{array}$$

Since the two horizontal maps are fibrations and m'' is a smooth morphism (see proof of Lemma 6.11), to prove that m is flat, it suffices to show that $\hat{m}' : \bar{\Gamma}_0 \times X_w^{u,v} \to X_w^2$ is a flat morphism. By the analogue of Lemma 4.7 for Γ, it suffices to show that

$$(\bar{B}^2) \times X_w^{u,v} \to X_w^2$$

is a flat morphism, which follows from Lemma 6.10.

Observe first that, by the same proof as that of Lemma 6.10, the morphism $\bar{B}^2 \times ((\partial X^{u,v}) \cap X_w^2) \to X_w^2$ is flat. Now, to prove that the map $\bar{\Gamma} \times \partial((X_w^{u,v})_{\mathbf{j}}) \to (X_w^2)_{\mathbb{P}}$ is flat, observe that (by the same proof as that of the first part) it is flat when restricted to the components $\Gamma_1 := \bar{\Gamma} \times ((\partial X^{u,v}) \cap X_w^2)_{\mathbf{j}}$ and $\Gamma_2 := \bar{\Gamma} \times (X_w^{u,v})_{\partial\mathbb{P}_{\mathbf{j}}}$ and also to $\Gamma_1 \cap \Gamma_2$. Thus, it is flat on $\Gamma_1 \cup \Gamma_2$, since for an affine scheme $Y = Y_1 \cup Y_2$, with closed subschemes Y_1, Y_2, and a morphism $f : Y \to X$ of schemes, the sequence

$$0 \to k[Y] \to k[Y_1] \oplus k[Y_2] \to k[Y_1 \cap Y_2] \to 0$$

is exact as a sequence of $k[X]$-modules. □

The following two lemmas are not used in the paper. However, we have included them for their potential usefulness. The first is used in the proof of the second.

Lemma 6.13. *For any $u \le w$,*

$$\mathcal{O}_{X^u}(-\partial X^u) \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{O}_{X_w}(-\partial X_w) \simeq \mathcal{O}_{X_w^u}\big(-((\partial X_w^u) \cup (X^u \cap \partial X_w))\big),$$

where recall that $\partial X_w^u := (\partial X^u) \cap X_w$ taken as the scheme-theoretic intersection inside $\bar{X}$.

Proof. First of all,

$$0 \to \mathcal{O}_{X^u}(-\partial X^u) \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{O}_{X_w} \to \mathcal{O}_{X^u} \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{O}_{X_w} = \mathcal{O}_{X_w^u} \to \mathcal{O}_{\partial X_w^u} \to 0$$

is exact since (by Corollary 5.7)

$$\mathcal{T}or_1^{\mathcal{O}_{\bar{X}}}(\mathcal{O}_{\partial X^u}, \mathcal{O}_{X_w}) = 0. \tag{50}$$

Thus,

$$\mathcal{O}_{X^u}(-\partial X^u) \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{O}_{X_w} \simeq \mathcal{O}_{X_w^u}(-\partial X_w^u). \tag{51}$$

Similarly,

$$\begin{aligned} 0 \to \mathcal{O}_{X^u}(-\partial X^u) \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{O}_{X_w}(-\partial X_w) &\to \mathcal{O}_{X^u}(-\partial X^u) \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{O}_{X_w} \\ &\to \mathcal{O}_{X^u}(-\partial X^u) \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{O}_{\partial X_w} \to 0 \end{aligned} \tag{52}$$

is exact since

$$\mathcal{T}or_1^{\mathcal{O}_{\bar{X}}}(\mathcal{O}_{X^u}(-\partial X^u), \mathcal{O}_{\partial X_w}) = 0. \tag{53}$$

To prove (53), observe that, by a proof similar to that of Corollary 5.7,

$$\mathcal{T}or_j^{\mathcal{O}_{\bar{X}}}(\mathcal{O}_{X^u}, \mathcal{O}_{\partial X_w}) = 0 \quad \text{and} \quad \mathcal{T}or_j^{\mathcal{O}_{\bar{X}}}(\mathcal{O}_{\partial X^u}, \mathcal{O}_{\partial X_w}) = 0, \quad \text{for all } j > 0. \tag{54}$$

Now, (51), (52) and (54) together prove the lemma. □

Lemma 6.14. *Let $u \le w$. As T-equivariant sheaves,*

$$\omega_{X_w^u} \simeq \mathcal{O}_{X_w^u}\big(-((\partial X_w^u) \cup (X^u \cap \partial X_w))\big),$$

where $X^u \cap \partial(X_w)$ is taken as the scheme-theoretic intersection inside $\bar{X}$.

Proof. Since X_w^u is CM by Proposition 6.6 (in particular, so is X_w) and the codimension of X_w^u in X_w is $\ell(u)$, the dualizing sheaf satisfies

$$\omega_{X_w^u} \simeq \mathcal{E}xt^{\ell(u)}_{\mathcal{O}_{X_w}}(\mathcal{O}_{X_w^u}, \omega_{X_w}) \tag{55}$$

(see [E, Theorem 21.15]). By the same proof as that of Lemma 5.8,

$$\mathcal{E}xt^{\ell(u)}_{\mathcal{O}_{X_w}}(\mathcal{O}_{X_w^u}, \omega_{X_w}) \simeq \mathcal{E}xt^{\ell(u)}_{\mathcal{O}_{\bar{X}}}(\mathcal{O}_{X^u}, \mathcal{O}_{\bar{X}}) \otimes_{\mathcal{O}_{\bar{X}}} \omega_{X_w}. \tag{56}$$

By [GK, Proposition 2.2], as T-equivariant sheaves,

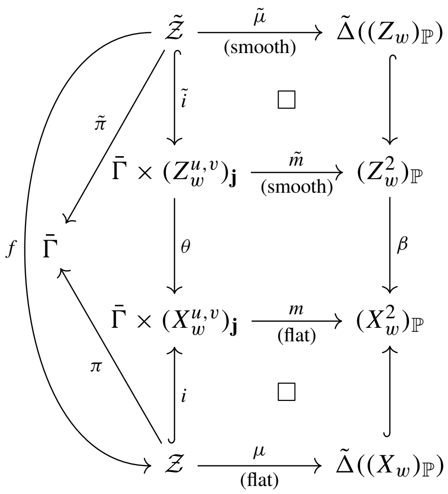
$$\omega_{X_w} \simeq e^{-\rho}\mathcal{L}(-\rho) \otimes \mathcal{O}_{X_w}(-\partial X_w). \tag{57}$$

(Even though in [GK] we assume that G is of finite type, the same proof works for a general Kac–Moody group.) Thus, the lemma follows by combining the isomorphisms (55)–(57) with Theorem 10.4 (due to Kashiwara) and Lemma 6.13. □

7. $\mathcal{Z}$ has rational singularities

Recall the definition of $\mathbb{P}$ and $\mathbb{P}_{\mathbf{j}}$ and the embedding $\tilde{\Delta}$ from Section 4. Fix $u, v \le w$ and $\mathbf{j}$. Also recall the definition of the quotient group $\bar{\Gamma} = \Gamma_w$ of Γ and the map $\tilde{m}$ from Lemma 6.11 and the map m from Lemma 6.12.

In the following commutative diagram, $\tilde{\mathcal{Z}}$ is defined as the fiber product $(\bar{\Gamma} \times (Z_w^{u,v})_{\mathbf{j}}) \times_{(Z_w^2)_{\mathbb{P}}} \tilde{\Delta}((Z_w)_{\mathbb{P}})$, and $\mathcal{Z}$ is defined as the fiber product $(\bar{\Gamma} \times (X_w^{u,v})_{\mathbf{j}}) \times_{(X_w^2)_{\mathbb{P}}} \tilde{\Delta}((X_w)_{\mathbb{P}})$. In particular, both $\mathcal{Z}$, $\tilde{\mathcal{Z}}$ are schemes of finite type over $\mathbb{C}$. The map f is the restriction of θ to $\tilde{\mathcal{Z}}$ (via $\tilde{i}$) with image inside $\mathcal{Z}$. The maps $\tilde{\pi}$ and π are obtained from the projections to the $\bar{\Gamma}$-factor via the maps $\tilde{i}$ and i respectively.



Lemma 7.1. Pic$(\bar{\Gamma})$ *is trivial.*

Proof. First of all, by the definition given above Lemma 6.11, $\bar{\Gamma}$ is the semidirect product of $\mathrm{GL}(N+1)^r$ with $\bar{\Gamma}_0 = \Gamma(E(T)_{\mathbb{P}} \times^T \bar{B}^2) \simeq H^2 \times \Gamma(E(T)_{\mathbb{P}} \times^T \bar{U}^2)$.

Since $\bar{U}^2$ is T-isomorphic to its Lie algebra, $\Gamma(E(T)_{\mathbb{P}} \times^T \bar{U}^2)$ is an affine space. Thus, as a variety, $\bar{\Gamma}$ (which is isomorphic to $\mathrm{GL}(N+1)^r \times H^2 \times \Gamma(E(T)_{\mathbb{P}} \times^T \bar{U}^2)$) is an open subset of an affine space $\mathbb{A}^N$. In particular, any prime divisor of $\bar{\Gamma}$ extends to a prime divisor of $\mathbb{A}^N$, and thus its ideal is principal. Hence, $\mathrm{Pic}(\bar{\Gamma}) = \{1\}$. □

The following result is a slight variant of [FP, Lemma, p. 108].

Lemma 7.2. *Let $f : W \to X$ be a flat morphism from a pure-dimensional CM scheme W of finite type over $\mathbb{C}$ to a CM irreducible variety X, and let Y be a closed CM subscheme of X of pure codimension d. Set $Z := f^{-1}(Y)$. If* $\mathrm{codim}_Z(W) \ge d$*, then equality holds and Z is CM.*

Proof (due to N. Mohan Kumar). The assertion and the assumptions of the lemma are clearly local, so we have a local map $A \to B$ of local rings with B flat over A. If $P \subset A$ is a prime ideal of codimension d with PB of pure codimension d, we only need to check that B/PB is CM. But A/P is CM, so we can pick a regular sequence $\{a_1, \dots, a_d\}$ mod P. By flatness of f, it remains a regular sequence in B/PB. □

We also need the original [FP, Lemma, p. 108].

Lemma 7.3. *Let $f : W \to X$ be a morphism from a pure-dimensional CM scheme W of finite type over $\mathbb{C}$ to a smooth irreducible variety X, and let Y be a closed CM subscheme of X of pure codimension d. Set $Z := f^{-1}(Y)$. If $\operatorname{codim}_Z(W) \geq d$, then equality holds and Z is CM.*

Proposition 7.4. *The schemes $\mathcal{Z}$ and $\tilde{\mathcal{Z}}$ are irreducible and the map $f : \tilde{\mathcal{Z}} \to \mathcal{Z}$ is a proper birational map. Thus, $\tilde{\mathcal{Z}}$ is a desingularization of $\mathcal{Z}$. Moreover, $\mathcal{Z}$ is CM with*

$$\dim(\mathcal{Z}) = |\mathbf{j}| + \ell(w) - \ell(u) - \ell(v) + \dim(\bar{\Gamma}), \tag{58}$$

where $|\mathbf{j}| := \sum_i j_i$ for $\mathbf{j} = (j_1, \dots, j_r)$.

Proof. We first show that $\tilde{\mathcal{Z}}$ and $\mathcal{Z}$ are pure-dimensional.

Since $\tilde{m}$ is a smooth (in particular, flat) morphism, $\operatorname{Im}\tilde{m}$ is an open subset of $(Z_w^2)_{\mathbb{P}}$ [H, Chap. III, Exercise 9.1]. Moreover, clearly $\operatorname{Im}\tilde{m} \supset (C_w^2)_{\mathbb{P}}$, thus $\operatorname{Im}\tilde{m}$ intersects $\tilde{\Delta}((Z_w)_{\mathbb{P}})$. Applying [H, Chap. III, Corollary 9.6] first to the morphism $\tilde{m} : \bar{\Gamma} \times (Z_w^{u,v})_{\mathbf{j}} \to \operatorname{Im}\tilde{m}$ and then to its restriction $\tilde{\mu}$ to $\tilde{\mathcal{Z}}$, we see that $\tilde{\mathcal{Z}}$ is pure-dimensional. Moreover,

$$\begin{aligned}\dim(\tilde{\mathcal{Z}}) &= \dim(\bar{\Gamma}) + |\mathbf{j}| + \dim(Z_w^{u,v}) - \dim((Z_w^2)_{\mathbb{P}}) + \dim(\tilde{\Delta}((Z_w)_{\mathbb{P}}))\\ &= \dim(\bar{\Gamma}) + |\mathbf{j}| + \ell(w) - \ell(u) - \ell(v).\end{aligned} \tag{59}$$

By the same argument, we see that $\mathcal{Z}$ is also pure-dimensional.

We now show that $\tilde{\mathcal{Z}}$ is irreducible:

The smooth morphism $\tilde{m} : \bar{\Gamma} \times (Z_w^{u,v})_{\mathbf{j}} \to (Z_w^2)_{\mathbb{P}}$ is $\bar{\Gamma}$-equivariant with respect to the left multiplication of $\bar{\Gamma}$ on the first factor of $\bar{\Gamma} \times (Z_w^{u,v})_{\mathbf{j}}$ and the standard action of $\bar{\Gamma}$ on $(Z_w^2)_{\mathbb{P}}$. Since $(C_w^2)_{\mathbb{P}}$ is a single $\bar{\Gamma}$-orbit (by the analogue of Lemma 4.7 for B replaced by $B \times B$), $\tilde{m}^{-1}((C_w^2)_{\mathbb{P}}) \to (C_w^2)_{\mathbb{P}}$ is a locally trivial fibration in the analytic topology. Further, since the fundamental group $\pi_1((C_w^2)_{\mathbb{P}}) = \{1\}$, and of course $\tilde{m}^{-1}((C_w^2)_{\mathbb{P}})$ is irreducible (in particular, connected), from the long exact homotopy sequence for the fibration $\tilde{m}^{-1}((C_w^2)_{\mathbb{P}}) \to (C_w^2)_{\mathbb{P}}$ we find that all its fibers are connected. Thus, the open subset $\tilde{\mathcal{Z}} \cap \tilde{m}^{-1}((C_w^2)_{\mathbb{P}})$ is connected as the fibers and the base are connected. Hence, it is irreducible (being smooth). Consider the closure $\tilde{\mathcal{Z}}_1 := \overline{\tilde{\mathcal{Z}} \cap \tilde{m}^{-1}((C_w^2)_{\mathbb{P}})}$. Then $\tilde{\mathcal{Z}}_1$ is an irreducible component of $\tilde{\mathcal{Z}}$. If possible, let $\tilde{\mathcal{Z}}_2$ be another irreducible component of $\tilde{\mathcal{Z}}$. Then $\tilde{\mu}(\tilde{\mathcal{Z}}_2) \subset \tilde{\Delta}((Z_w \setminus C_w)_{\mathbb{P}})$. Since $\dim(\tilde{\Delta}((Z_w \setminus C_w)_{\mathbb{P}})) < \dim(\tilde{\Delta}((Z_w)_{\mathbb{P}}))$ and each fiber of $\tilde{\mu}_{|\tilde{\mathcal{Z}}_2}$ is of dimension at most that of any fiber of $\tilde{\mu}$, we get $\dim(\tilde{\mathcal{Z}}_2) < \dim(\tilde{\mathcal{Z}}_1)$. This is a contradiction since $\tilde{\mathcal{Z}}$ is of pure dimension. Thus, $\tilde{\mathcal{Z}} = \tilde{\mathcal{Z}}_1$, and hence $\tilde{\mathcal{Z}}$ is irreducible.

The proof of the irreducibility of $\mathcal{Z}$ is similar. The only extra observation we need is that $\tilde{\mathcal{Z}} \cap \tilde{m}^{-1}((C_w^2)_{\mathbb{P}})$ maps surjectively onto $\mathcal{Z} \cap m^{-1}((C_w^2)_{\mathbb{P}})$ under f; in particular, $\mathcal{Z} \cap m^{-1}((C_w^2)_{\mathbb{P}})$ is irreducible.

The map f is clearly proper. Moreover, it is an isomorphism when restricted to the (nonempty) open subset

$$\tilde{\mathcal{Z}} \cap \big(\bar{\Gamma} \times ((C^u \cap C_w) \times (C^v \cap C_w))_{\mathbf{j}}\big)$$

onto its image (which is an open subset of $\mathcal{Z}$). (Here we have identified the inverse image $(\pi_w^u)^{-1}(C^u \cap C_w)$ inside Z_w^u with $C^u \cap C_w$ under the map π_w^u—see Theorem 6.8.)

The identity (58) follows from (59) since $\dim(\mathcal{Z}) = \dim(\tilde{\mathcal{Z}})$. Thus,

$$\operatorname{codim}_{\mathcal{Z}} (\bar{\Gamma} \times (X_w^{u,v})_{\mathbf{j}}) = \operatorname{codim}_{\tilde{\Delta}((X_w)_{\mathbb{P}})} ((X_w^2)_{\mathbb{P}}) = \ell(w).$$

Finally, $\mathcal{Z}$ is CM by Proposition 6.6 and Lemmas 6.12 and 7.2. This completes the proof of the proposition. □

Lemma 7.5. *The scheme $\mathcal{Z}$ is normal, irreducible and CM.*

Proof. By Proposition 7.4, $\mathcal{Z}$ is irreducible and CM.

As in the proof of Lemma 6.10, the map

$$\mu_o : G \times^{U^-} X^u(S'_u) \to \bar{X}, \quad [g, x] \mapsto g \cdot x,$$

being G-equivariant, is a locally trivial fibration, where $S'_u := \{v \in W : \ell(v) \leq \ell(u)+1\}$. Moreover, its fibers are clearly isomorphic to $F^u := \bigcup_{u \leq v:\, \ell(v) \leq \ell(u)+1} Bv^{-1}U^-/U^-$. Now, since X^u is normal [KS, Proposition 3.2] and any B^--orbit in $X^u(S'_u)$ is of codimension ≤ 1 in X^u, $X^u(S'_u)$ is smooth, and similarly so is F^u. (Here the smoothness of F^u means that there exists a closed normal subgroup B_1 of B of finite codimension such that B_1 acts freely and properly on F^u and the quotient $B_1 \backslash F^u$ is a smooth scheme of finite type over $\mathbb{C}$—see Lemma 6.1.) Thus, μ_o is a smooth morphism, and hence so is its restriction to the open subset $B \times X^u(S'_u) \to \bar{X}$. Let $\mu_o(w) : B \times (X^u(S'_u) \cap X_w) \to X_w$ be the restriction of the latter to the inverse image of X_w. The map $\mu_o(w)$ clearly factors through a smooth morphism $\bar{\mu}_o(w) : \bar{B} \times (X^u(S'_u) \cap X_w) \to X_w$, where $\bar{B}$ is a finite-dimensional quotient group of B. Hence, $\bar{\mu}_o(w)^{-1}(X_w^o) = \bar{B} \times (X^u(S'_u) \cap X_w^o)$ is a smooth variety, where $X_w^o := X_w \setminus \Sigma_w$ and Σ_w is the singular locus of X_w.

Following the same argument as in the proof of Lemma 6.12, we see that the restriction of the map $m : \bar{\Gamma} \times (X_w^{u,v})_{\mathbf{j}} \to (X_w^2)_{\mathbb{P}}$ to $\bar{m} : \bar{\Gamma} \times ((X^u(S'_u) \times X^v(S'_v)) \cap X_w^2)_{\mathbf{j}} \to (X_w^2)_{\mathbb{P}}$ is a smooth morphism (with open image Y), and hence so is its restricton $\hat{m} : \bar{m}^{-1}(\tilde{\Delta}((X_w)_{\mathbb{P}})) \to \tilde{\Delta}((X_w)_{\mathbb{P}})$. (Observe that Y does intersect $\tilde{\Delta}((X_w)_{\mathbb{P}})$, for otherwise $(\bar{\Gamma} \cdot \tilde{\Delta}((X_w)_{\mathbb{P}})) \cap Y = \emptyset$, which would imply that $(C_w^2)_{\mathbb{P}} \cap Y = \emptyset$, a contradiction.) Thus, $\hat{m}^{-1}(\tilde{\Delta}((X_w^o)_{\mathbb{P}}))$ is a smooth variety, which is open in $\mathcal{Z} = m^{-1}(\tilde{\Delta}((X_w)_{\mathbb{P}}))$. Let us denote the complement of $\bar{\Gamma} \times ((X^u(S'_u) \times X^v(S'_v)) \cap X_w^2)_{\mathbf{j}}$ in $\bar{\Gamma} \times (X_w^{u,v})_{\mathbf{j}}$ by F and denote $\hat{m}^{-1}(\tilde{\Delta}((\Sigma_w)_{\mathbb{P}}))$ by F'. Then F' is of codimension ≥ 2 in $\bar{m}^{-1}(\tilde{\Delta}((X_w)_{\mathbb{P}}))$, and hence in $\mathcal{Z}$. Clearly, F is of codimension ≥ 2 in $\bar{\Gamma} \times (X_w^{u,v})_{\mathbf{j}}$. Also, if F is nonempty, the restriction of the map m to F is again flat (by the same proof as that of Lemma 6.12) with image an open subset of $(X_w^2)_{\mathbb{P}}$ intersecting $\tilde{\Delta}((X_w)_{\mathbb{P}})$. Thus, the codimension of $F \cap \mathcal{Z}$ in $\mathcal{Z}$ is ≥ 2. This shows that the complement of the smooth locus of $\mathcal{Z}$ in $\mathcal{Z}$ is of codimension ≥ 2. Moreover, $\mathcal{Z}$ is CM by Proposition 7.4. Thus, by Serre's criterion [H, Chap. II, Theorem 8.22(A)], $\mathcal{Z}$ is normal. □

The following lemma and Proposition 7.7 are taken from our recent joint work with S. Baldwin [BaK]. Proposition 7.7 is used to give a shorter proof (than our original proof) of Theorem 8.5(b).

Lemma 7.6. *Let G be a group acting on a set X and let $Y \subset X$. Consider the action map $m : G \times Y \to X$. For $x \in X$ denote the orbit of x by $O(x)$ and the stabilizer by* $\mathrm{Stab}(x)$. *Then* $\mathrm{Stab}(x)$ *acts on the fiber $m^{-1}(x)$, and* $\mathrm{Stab}(x)\backslash m^{-1}(x) \simeq O(x) \cap Y$.

Proof. It is easy to check that

$$m^{-1}(x) = \{(g, h^{-1}x) : h \in G,\ h^{-1}x \in Y,\ g \in \mathrm{Stab}(x) \cdot h\}.$$

Thus, $\mathrm{Stab}(x)$ acts on $m^{-1}(x)$ by left multiplication on the left component. Since every element of $O(x) \cap Y$ is of the form $h^{-1}x$ for some $h \in G$, the second projection $m^{-1}(x) \to O(x) \cap Y$ is surjective. This map clearly factors through the quotient to give a map $\mathrm{Stab}(x)\backslash m^{-1}(x) \to O(x) \cap Y$. To show that this induced map is injective, note first that each class has a representative of the form $(h, h^{-1}x)$. Now, if $(h_1, h_1^{-1}x)$ and $(h_2, h_2^{-1}x)$ satisfy $h_1^{-1}x = h_2^{-1}x$ then $h_2h_1^{-1}x = x$, i.e., $h_2h_1^{-1} \in \mathrm{Stab}(x)$, i.e., $h_2 \in \mathrm{Stab}(x) \cdot h_1$, i.e., $(h_1, h_1^{-1}x)$ and $(h_2, h_2^{-1}x)$ belong to the same class. □

Proposition 7.7. *The scheme $\mathcal{Z}$ has rational singularities.*

Proof. Since μ is flat and $\tilde{\Delta}((X_w)_{\mathbb{P}})$ has rational singularities [K, Theorem 8.2.2(c)], by [El, Théorème 5] it is sufficient to show that the fibers of μ are disjoint unions of irreducible varieties with rational singularities.

Let $x \in \tilde{\Delta}((C_{w'})_{\mathbb{P}})$, where $w' \leq w$. Then, by Lemmas 7.6 and 4.7 (for $\Gamma_{B\times B}$), we have $\mathrm{Stab}(x)\backslash\mu^{-1}(x) \simeq (X^u \cap C_{w'} \times X^v \cap C_{w'})_{\mathbf{j}}$, where $\mathrm{Stab}(x)$ is taken with respect to the action of $\bar{\Gamma}$ on $(X^2_w)_{\mathbb{P}}$. By [Se, Proposition 3, §2.5], the quotient map $\bar{\Gamma} \to \mathrm{Stab}(x)\backslash\bar{\Gamma}$ is locally trivial in the étale topology.

Consider the pull-back diagram

$$\begin{array}{ccc} \mu^{-1}(x) & \subseteq & \bar{\Gamma} \times (X_w^{u,v})_{\mathbf{j}} \\ \downarrow & & \downarrow \\ \mathrm{Stab}(x)\backslash\mu^{-1}(x) & \subseteq & (\mathrm{Stab}(x)\backslash\bar{\Gamma}) \times (X_w^{u,v})_{\mathbf{j}} \end{array}$$

Since the right vertical map is a locally trivial fibration in the étale topology, the left vertical map is too. Now, $\mathrm{Stab}(x)\backslash\mu^{-1}(x) \simeq (X^u \cap C_{w'} \times X^v \cap C_{w'})_{\mathbf{j}}$ has rational singularities by [KuS, Theorem 3.1]. Further, $\mathrm{Stab}(x)$ being smooth and $\mu^{-1}(x) \to \mathrm{Stab}(x)\backslash\mu^{-1}(x)$ being locally trivial in the étale topology, we conclude that $\mu^{-1}(x)$ is a disjoint union of irreducible varieties with rational singularities by [KM, Corollary 5.11]. □

Proposition 7.8. *The scheme $\partial\mathcal{Z}$ is pure of codimension* 1 *in $\mathcal{Z}$ and it is CM, where the closed subscheme $\partial\mathcal{Z}$ of $\mathcal{Z}$ is defined as*

$$\partial\mathcal{Z} := (\bar{\Gamma} \times \partial((X_w^{u,v})_{\mathbf{j}})) \times_{(X^2_w)_{\mathbb{P}}} \tilde{\Delta}((X_w)_{\mathbb{P}}),$$

where $\partial((X_w^{u,v})_{\mathbf{j}})$ is defined in Lemma 6.12.

Proof. By Lemma 6.12, the map $\bar{\Gamma}\times\partial((X_w^{u,v})_{\mathbf{j}}) \xrightarrow{m'} (X_w^2)_{\mathbb{P}}$ is a flat morphism. Moreover, $\partial((X_w^{u,v})_{\mathbf{j}})$ is pure of codimension 1 in $(X_w^{u,v})_{\mathbf{j}}$. Further, $\operatorname{Im} m' = \operatorname{Im} m$ if $\partial\mathbb{P}_{\mathbf{j}} \neq \emptyset$. If $\partial\mathbb{P}_{\mathbf{j}} = \emptyset$, then

$$\operatorname{Im} m' \supset \Big(\Big(\Big(\bigcup_{u\to u'\leq\theta\leq w} C_\theta\Big)\times\Big(\bigcup_{v\leq\theta'\leq w} C_{\theta'}\Big)\Big)\cup\Big(\Big(\bigcup_{u\leq\theta\leq w} C_\theta\Big)\times\Big(\bigcup_{v\to v'\leq\theta'\leq w} C_{\theta'}\Big)\Big)\Big)_{\mathbb{P}}.$$

In particular, if nonempty, $\operatorname{Im} m'$ is open in $(X_w^2)_{\mathbb{P}}$ (since m' is flat) and intersects $\tilde{\Delta}((X_w)_{\mathbb{P}})$. Thus, by [H, Chap. III, Corollary 9.6], each fiber of m' (if nonempty) is pure of dimension

$$\dim(\bar{\Gamma}) + \dim((X_w^{u,v})_{\mathbf{j}}) - \dim((X_w^2)_{\mathbb{P}}) - 1.$$

Again applying [H, Chap. III, Corollary 9.6], we find that $\partial\mathcal{Z}$ is pure of dimension

$$\dim(\bar{\Gamma}) + \dim((X_w^{u,v})_{\mathbf{j}}) - \dim((X_w^2)_{\mathbb{P}}) - 1 + \dim(\tilde{\Delta}((X_w)_{\mathbb{P}})).$$

Hence, by the identity (58), $\partial\mathcal{Z}$ is pure of codimension 1 in $\mathcal{Z}$. Further, both $((\partial X^u)\cap X_w)\times X_w^v$ and $X_w^u\times((\partial X^v)\cap X_w)$ are CM by Proposition 6.6 and Remark 6.7, and so is their intersection. Moreover, their intersection is of pure codimension 1 in both of them. Hence, their union is CM (e.g. by [K, Theorem A.36]), and hence so is $((\partial X^{u,v})\cap X_w^2)_{\mathbf{j}}$. Also, $(X_w^{u,v})_{\partial\mathbb{P}_{\mathbf{j}}}$ and the intersection

$$((\partial X^{u,v})\cap X_w^2)_{\mathbf{j}}\cap (X_w^{u,v})_{\partial\mathbb{P}_{\mathbf{j}}} = ((\partial X^{u,v})\cap X_w^2)_{\partial\mathbb{P}_{\mathbf{j}}}$$

are CM since $\partial\mathbb{P}_{\mathbf{j}}$ is CM. Thus, their union $\partial((X_w^{u,v})_{\mathbf{j}})$ is CM since the intersection $((\partial X^{u,v})\cap X_w^2)_{\partial\mathbb{P}_{\mathbf{j}}}$ is CM of pure codimension 1 in both $((\partial X^{u,v})\cap X_w^2)_{\mathbf{j}}$ and $(X_w^{u,v})_{\partial\mathbb{P}_{\mathbf{j}}}$. Thus, $\partial\mathcal{Z}$ is CM by Lemma 7.2 applied to the morphism $\bar{\Gamma}\times\partial((X_w^{u,v})_{\mathbf{j}})\to(X_w^2)_{\mathbb{P}}$. □

As a consequence of Proposition 7.8 and Lemma 7.3, we get the following.

Corollary 7.9. *Assume that $c_{u,v}^w(\mathbf{j})\neq 0$, where $c_{u,v}^w(\mathbf{j})$ is defined by the identity* (19)*. Then, for general $\gamma\in\bar{\Gamma}$, the fiber $N_\gamma := \pi^{-1}(\gamma)\subset\mathcal{Z}$ is CM of pure dimension, where the morphism $\pi:\mathcal{Z}\to\bar{\Gamma}$ is defined at the beginning of this section. In fact, for any $\gamma\in\bar{\Gamma}$ such that N_γ is pure of dimension*

$$\dim(N_\gamma) = \dim(\mathcal{Z}) - \dim(\bar{\Gamma}) = |\mathbf{j}| + \ell(w) - \ell(u) - \ell(v), \tag{60}$$

N_γ is CM (and this condition is satisfied for general γ).

Similarly, if $|\mathbf{j}| + \ell(w) - \ell(u) - \ell(v) > 0$, then for general $\gamma\in\bar{\Gamma}$, the fiber $M_\gamma := {\pi_1}^{-1}(\gamma)\subset\partial\mathcal{Z}$ is CM of pure codimension 1 in N_γ, where π_1 is the restriction of the map π to $\partial\mathcal{Z}$. If $|\mathbf{j}| + \ell(w) - \ell(u) - \ell(v) = 0$, then for general $\gamma\in\bar{\Gamma}$, the fiber M_γ is empty.

In particular, for general $\gamma\in\bar{\Gamma}$,

$$\mathcal{E}xt^i_{\mathcal{O}_{N_\gamma}}(\mathcal{O}_{N_\gamma}(-M_\gamma),\omega_{N_\gamma}) = 0 \quad \textit{for all } i>0,$$

where $\mathcal{O}_{N_\gamma}(-M_\gamma)$ denotes the ideal sheaf of M_γ in N_γ, and ω_{N_γ} is the dualizing sheaf of N_γ.

Proof. We first show that π is a surjective morphism under the assumption that $c^w_{u,v}(\mathbf{j}) \neq 0$. By the definition,

$$\operatorname{Im}\pi = \{\gamma \in \bar{\Gamma} : \gamma((X^{u,v}_w)_{\mathbf{j}}) \cap \tilde{\Delta}((X_w)_{\mathbb{P}}) \neq \emptyset\}. \tag{61}$$

Since $\bar{\Gamma}$ is connected by Lemma 4.8, by the expression of $c^w_{u,v}(\mathbf{j})$ as in Lemma 4.5, $\gamma((X^{u,v})_{\mathbf{j}}) \cap \tilde{\Delta}((X_w)_{\mathbb{P}}) \neq \emptyset$ for any $\gamma \in \bar{\Gamma}$. But $\gamma((X^{u,v})_{\mathbf{j}}) \cap \tilde{\Delta}((X_w)_{\mathbb{P}}) = \gamma((X^{u,v}_w)_{\mathbf{j}}) \cap \tilde{\Delta}((X_w)_{\mathbb{P}})$ for any $\gamma \in \bar{\Gamma}$. Thus, π is surjective.

By Lemmas 7.3 and 7.5 applied to the morphism $\pi : \mathcal{Z} \to \bar{\Gamma}$, we see that if N_γ is pure and

$$\operatorname{codim}_{\mathcal{Z}}(N_\gamma) = \dim(\bar{\Gamma}), \tag{62}$$

then N_γ is CM.

Now the condition (62) is satisfied for γ in a dense open subset of $\bar{\Gamma}$ by [S, Chap. I, §6.3, Theorem 1.25]. Thus, N_γ is CM for general γ.

Similarly, we prove that M_γ is CM for general γ:

We first show that $\pi_1 : \partial\mathcal{Z} \to \bar{\Gamma}$ is surjective if $|\mathbf{j}| + \ell(w) - \ell(u) - \ell(v) > 0$. For if π_1 were not surjective, its image would be a proper closed subset of $\bar{\Gamma}$, since π_1 is a projective morphism. Hence, for general $\gamma \in \bar{\Gamma}$, $M_\gamma = \emptyset$, i.e., $N_\gamma \subset \mathcal{Z}\backslash\partial\mathcal{Z}$. But $\mathcal{Z}\backslash\partial\mathcal{Z}$ is an affine scheme, and N_γ is a projective scheme of positive dimension (because of the assumption $|\mathbf{j}| + \ell(w) - \ell(u) - \ell(v) > 0$). This is a contradiction, and hence π_1 is surjective. Thus, if $|\mathbf{j}| + \ell(w) - \ell(u) - \ell(v) > 0$, we deduce that for general $\gamma \in \bar{\Gamma}$, by [S, Chap. I, §6.3, Theorem 1.25] applied to the irreducible components of $\partial\mathcal{Z}$, M_γ is pure and

$$\operatorname{codim}_{\partial\mathcal{Z}}(M_\gamma) = \dim(\bar{\Gamma}). \tag{63}$$

Now, by the same argument as above, for general $\gamma \in \bar{\Gamma}$, M_γ is CM. Moreover, since $\partial\mathcal{Z}$ is of pure codimension 1 in $\mathcal{Z}$, we conclude (by (62)–(63)) that M_γ is of pure codimension 1 in N_γ (for general γ).

If $|\mathbf{j}| + \ell(w) - \ell(u) - \ell(v) = 0$, then $\dim(\partial\mathcal{Z}) < \dim(\bar{\Gamma})$. So, in this case, $\operatorname{Im}\pi_1$ is a proper closed subset of $\bar{\Gamma}$.

Since (for general γ) M_γ is of pure codimension 1 in N_γ and both are CM,

$$\mathcal{E}xt^i_{\mathcal{O}_{N_\gamma}}(\mathcal{O}_{N_\gamma}(-M_\gamma), \omega_{N_\gamma}) = 0 \quad \text{for all } i > 0.$$

To prove this, use the long exact $\mathcal{E}xt$ sequence associated to the sheaf exact sequence

$$0 \to \mathcal{O}_{N_\gamma}(-M_\gamma) \to \mathcal{O}_{N_\gamma} \to \mathcal{O}_{M_\gamma} \to 0$$

and the result that

$$\mathcal{E}xt^i_{\mathcal{O}_{N_\gamma}}(\mathcal{O}_{M_\gamma}, \omega_{N_\gamma}) = 0 \quad \text{unless} \quad i = 1$$

(see [I, Proposition 11.33 and Corollary 11.43]). □

8. Study of $R^p f_*(\omega_{\tilde{\mathcal{Z}}}(\partial\tilde{\mathcal{Z}}))$

From now on we assume that $c^w_{u,v}(\mathbf{j}) \neq 0$, where $c^w_{u,v}(\mathbf{j})$ is defined by the identity (19). We follow the notation from the big diagram in Section 7.

Lemma 8.1. *The line bundle $\mathcal{L}(\rho)_{|X^u}$ has a section with zero set precisely ∂X^u. In particular,*

$$\mathcal{L}(\rho)_{|X^u_w} \sim \sum_i b_i X_i \quad \textit{for some } b_i > 0,$$

where the X_i are the irreducible components of $(\partial X^u) \cap X_w$.

Proof. Consider the Borel–Weil isomorphism $\chi : L(\rho)^\vee \xrightarrow{\sim} H^0(\bar{X}, \mathcal{L}(\rho))$ given by $\chi(f)(gB) = [g, f(ge_\rho)]$, where e_ρ is a highest weight vector of the irreducible highest weight $G^{\min}$-module $L(\rho)$ with highest weight ρ, and $L(\rho)^\vee$ is the restricted dual of $L(\rho)$ [K, §8.1.21]. Then it is easy to see (using [K, Lemma 8.3.3]) that the section $\chi(e^*_{u\rho})_{|X^u}$ has zero set exactly ∂X^u, where $e_{u\rho}$ is the extremal weight vector of $L(\rho)$ with weight $u\rho$ and $e^*_{u\rho} \in L(\rho)^\vee$ is the linear form which takes value 1 on $e_{u\rho}$ and 0 on any weight vector of $L(\rho)$ of weight different from $u\rho$. This proves the lemma. ◻

A $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on an irreducible projective variety X is called *nef* (resp. *big*) if D has nonnegative intersection with every irreducible curve in X (resp. we have $\dim(H^0(X, \mathcal{O}_X(mD))) > cm^{\dim(X)}$ for some $c > 0$ and $m \gg 1$). If D is ample, it is nef and big [KM, Proposition 2.61].

Let $\pi : X \to Y$ be a proper morphism between schemes and let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Assume that X is irreducible. Then D is said to be π*-nef* (resp. π*-big*) if D has nonnegative intersection with every irreducible curve in X contracted by π (resp. $\operatorname{rank} \pi_*\mathcal{O}_X(mD) > cm^n$ for some $c > 0$ and $m \gg 1$, where n is the dimension of a general fiber of π).

Proposition 8.2. *There exists a nef and big line bundle $\mathcal{M}$ on $(Z^{u,v}_w)_\mathbf{j}$ with a section with support precisely equal to $\partial((Z^{u,v}_w)_\mathbf{j})$, where $\partial((Z^{u,v}_w)_\mathbf{j})$ is, by definition, the inverse image of $\partial((X^{u,v}_w)_\mathbf{j})$ under the canonical map $(Z^{u,v}_w)_\mathbf{j} \to (X^{u,v}_w)_\mathbf{j}$ induced by the T-equivariant map $\pi^{u,v}_w : Z^{u,v}_w := Z^u_w \times Z^v_w \to X^{u,v}_w := X^u_w \times X^v_w$, and $\partial((X^{u,v}_w)_\mathbf{j})$ is defined in Lemma 6.12. Moreover, $\mathcal{M}$ can be chosen to be the pull-back of an ample line bundle $\mathcal{M}'$ on $(X^{u,v}_w)_\mathbf{j}$.*

Proof. Take an ample line bundle $\mathcal{H}$ on $\mathbb{P}_\mathbf{j}$ with a section with support precisely equal to $\partial\mathbb{P}_\mathbf{j}$. Also, let $\mathcal{L}_{Z^{u,v}_w}(\rho \boxtimes \rho)$ be the pull-back of the line bundle $\mathcal{L}(\rho) \boxtimes \mathcal{L}(\rho)$ on $\bar{X} \times \bar{X}$ via the standard morphism

$$Z^{u,v}_w \to \bar{X} \times \bar{X}.$$

Since $e^{u\rho+v\rho}\mathcal{L}_{Z^{u,v}_w}(\rho \boxtimes \rho)$ is a T-equivariant line bundle, we get the line bundle $\tilde{\mathcal{L}}(-\rho \boxtimes -\rho) := E(T)_\mathbf{j} \times^T (e^{u\rho+v\rho}\mathcal{L}_{Z^{u,v}_w}(\rho \boxtimes \rho)) \to (Z^{u,v}_w)_\mathbf{j}$ over the base space $(Z^{u,v}_w)_\mathbf{j}$. Now, consider the line bundle (for some large enough $N > 0$)

$$\mathcal{M} := \tilde{\mathcal{L}}(-\rho \boxtimes -\rho) \otimes \pi^*(\mathcal{H}^N),$$

where $\pi : E(T)_{\mathbf{j}} \times^T Z_w^{u,v} \to \mathbb{P}_{\mathbf{j}}$ is the canonical projection. Take the section θ of $\tilde{\mathcal{L}}(-\rho \boxtimes -\rho)$ given by $[e, z] \mapsto [e, 1_{u\rho+v\rho} \otimes (\bar{\chi}(e^*_{u\rho}) \boxtimes \bar{\chi}(e^*_{v\rho}))(z)]$ for $e \in E(T)_{\mathbf{j}}$ and $z \in Z_w^{u,v}$, where $1_{u\rho+v\rho}$ denotes the constant section of the trivial line bundle over $Z_w^{u,v}$ with the H-action on the fiber given by the H-weight $u\rho + v\rho$, and $\bar{\chi} \boxtimes \bar{\chi}$ is the pull-back of the Borel–Weil isomorphism $\chi \boxtimes \chi : L(\rho)^\vee \otimes L(\rho)^\vee \simeq H^0(\bar{X}^2, \mathcal{L}(\rho) \boxtimes \mathcal{L}(\rho))$ to $Z_w^{u,v}$ (see proof of Lemma 8.1). Also, take any section σ of $\mathcal{H}^N$ with zero set precisely $\partial\mathbb{P}_{\mathbf{j}}$, and let $\hat{\sigma}$ be its pull-back to $(Z_w^{u,v})_{\mathbf{j}}$. Then the zero set of the tensor product of θ and $\hat{\sigma}$ is precisely $\partial((Z_w^{u,v})_{\mathbf{j}})$ (see proof of Lemma 8.1).

The line bundle $\mathcal{M}$ is the pull-back of the line bundle $\mathcal{M}' := \tilde{\mathcal{L}}'(-\rho \boxtimes -\rho) \otimes \pi_1^*(\mathcal{H}^N)$ on $E(T)_{\mathbf{j}} \times^T X_w^{u,v}$ via the standard morphism

$$E(T)_{\mathbf{j}} \times^T Z_w^{u,v} \to E(T)_{\mathbf{j}} \times^T X_w^{u,v},$$

where π_1 is the projection $E(T)_{\mathbf{j}} \times^T X_w^{u,v} \to \mathbb{P}_{\mathbf{j}}$ and $\tilde{\mathcal{L}}'(-\rho \boxtimes -\rho)$ is the line bundle

$$E(T)_{\mathbf{j}} \times^T \left(e^{u\rho+v\rho}(\mathcal{L}(\rho) \boxtimes \mathcal{L}(\rho))_{|X_w^{u,v}}\right).$$

Then, by [KM, Proposition 1.45 and Theorems 1.37 and 1.42], $\mathcal{M}'$ is ample on $(X_w^{u,v})_{\mathbf{j}}$ for large enough N. Since the pull-back of an ample line bundle via a birational morphism is nef and big [D, §1.29], $\mathcal{M}$ is nef and big. This proves the proposition. □

We recall the following 'relative Kawamata–Viehweg vanishing theorem' valid for proper morphisms [D, Exercise 2, p. 217]; replace Debarre's D by D' and take $D' := \mathcal{L} - D/N$.

Theorem 8.3. *Let $\tilde{\pi} : \tilde{Z} \to \bar{\Gamma}$ be a proper surjective morphism of irreducible varieties with $\tilde{Z}$ a smooth variety. Let $\mathcal{L}$ be a line bundle on $\tilde{Z}$ such that $\mathcal{L}^N(-D)$ is $\tilde{\pi}$-nef and $\tilde{\pi}$-big for a simple normal crossing divisor*

$$D = \sum_i a_i D_i, \quad \text{where } 0 < a_i < N \text{ for all } i.$$

Then $R^p\tilde{\pi}_*(\mathcal{L} \otimes \omega_{\tilde{Z}}) = 0$ for all $p > 0$. □

Proposition 8.4. *For the morphism $\tilde{\pi} : \tilde{\mathcal{Z}} \to \bar{\Gamma}$ (see the big diagram in Section 7),*

$$R^p\tilde{\pi}_*(\omega_{\tilde{\mathcal{Z}}}(\partial\tilde{\mathcal{Z}})) = 0 \quad \text{for all } p > 0,$$

where $\partial\tilde{\mathcal{Z}} := f^{-1}(\partial\mathcal{Z})$ ($\partial\mathcal{Z}$ being defined in Proposition 7.8 taken here with the reduced scheme structure) and $\omega_{\tilde{\mathcal{Z}}}(\partial\tilde{\mathcal{Z}})$ denotes the sheaf $\mathcal{H}om_{\mathcal{O}_{\tilde{\mathcal{Z}}}}(\mathcal{O}_{\tilde{\mathcal{Z}}}(-\partial\tilde{\mathcal{Z}}), \omega_{\tilde{\mathcal{Z}}})$.

(*Observe that f being a desingularization of a normal scheme $\mathcal{Z}$ and $\partial\mathcal{Z}$ being reduced, $\partial\tilde{\mathcal{Z}}$ is a reduced scheme.*)

Proof. Fix a nef and big line bundle $\mathcal{M}$ on $(Z_w^{u,v})_{\mathbf{j}}$ with its divisor $\sum_{i=1}^d b_i Z_i$ (with $b_i > 0$) supported precisely in $\partial((Z_w^{u,v})_{\mathbf{j}})$, which is the pull-back of an ample line bundle $\mathcal{M}'$ on $(X_w^{u,v})_{\mathbf{j}}$ (Proposition 8.2). Choose an integer $N > b_i$ for all i. Consider the line bundle $\mathcal{L}$ on the smooth scheme $\tilde{\mathcal{Z}}$ corresponding to the reduced divisor $\partial\tilde{\mathcal{Z}}$ (observe that $\partial\tilde{\mathcal{Z}}$ is a divisor of $\tilde{\mathcal{Z}}$, i.e., a pure scheme of codimension 1 in $\tilde{\mathcal{Z}}$, since it is the zero set of a line bundle on $\tilde{\mathcal{Z}}$ by using the definition of $\partial\mathcal{Z}$) and let D be the following divisor on $\tilde{\mathcal{Z}}$:

$$D = \sum_i (N - b_i)\tilde{Z}_i,$$

where

$$\tilde{Z}_i := (\bar{\Gamma} \times Z_i) \times_{(Z_w^2)_{\mathbb{P}}} \tilde{\Delta}((Z_w)_{\mathbb{P}}).$$

Observe that each $\tilde{Z}_i$ is a smooth irreducible divisor of $\tilde{\mathcal{Z}}$, and moreover for any collection $\tilde{Z}_{i_1}, \dots, \tilde{Z}_{i_q}$, $1 \le i_1 < \cdots < i_q \le d$, the intersection $\bigcap_{p=1}^{q} \tilde{Z}_{i_p}$ (if nonempty) is smooth of pure codimension q in $\tilde{\mathcal{Z}}$. (To prove this, use Theorem 6.4 and follow the proofs of Theorem 6.8, Lemmas 6.9 and 6.11 and Proposition 7.4.) In particular, $\tilde{Z}_i$'s are distinct. It is easy to see that

$$\partial\tilde{\mathcal{Z}} = \sum \tilde{Z}_i,$$

and hence it is a simple normal crossing divisor. Then

$$\mathcal{L}^N(-D) = \mathscr{O}_{\tilde{\mathcal{Z}}}\Big(\sum_i b_i \tilde{Z}_i\Big) \simeq \tilde{i}^*\Big(\mathscr{O}_{\bar{\Gamma}\times(Z_w^{u,v})_{\mathbf{j}}}\Big(\sum b_i(\bar{\Gamma} \times Z_i)\Big)\Big).$$

Moreover, since $\sum b_i Z_i$ is a nef divisor on $(Z_w^{u,v})_{\mathbf{j}}$ and $\tilde{i}$ is injective, $\mathcal{L}^N(-D)$ is $\tilde{\pi}$-nef [D, §1.6].

Observe further that, by definition, the line bundle $\mathcal{L}^N(-D)$ on $\tilde{\mathcal{Z}}$ is the pull-back of the line bundle $\mathcal{S} := i^*(\epsilon \boxtimes \mathcal{M}')$ on $\mathcal{Z}$ via f, where ϵ is the trivial line bundle on $\bar{\Gamma}$. Now, $\mathcal{M}'$ being an ample line bundle on $(X_w^{u,v})_{\mathbf{j}}$, $\mathcal{S}$ is π-big. But, f being birational, the general fibers of $\tilde{\pi}$ have the same dimension as the general fibers of π (use [S, Chap. I, §6.3, Theorem 1.25]). Hence, $\mathcal{L}^N(-D)$ is $\tilde{\pi}$-big.

The map f is surjective since it is proper and birational by Proposition 7.4. Also, the map $\tilde{\pi}$ is surjective since so is π (see the proof of Corollary 7.9). Thus, by Theorem 8.3, the proposition follows. □

Theorem 8.5. *For the morphism $f : \tilde{\mathcal{Z}} \to \mathcal{Z}$,*

(a) $R^p f_*(\omega_{\tilde{\mathcal{Z}}}(\partial\tilde{\mathcal{Z}})) = 0$ *for all* $p > 0$, *and*

(b) $f_*(\omega_{\tilde{\mathcal{Z}}}(\partial\tilde{\mathcal{Z}})) = \omega_{\mathcal{Z}}(\partial\mathcal{Z})$.

Proof. The map f is surjective as observed above. With the notation of the proof of Proposition 8.4, $\mathcal{L}^N(-D)$ is $\tilde{\pi}$-nef and $\tilde{\pi}$-big. Since the fibers of f are contained in the fibers of $\tilde{\pi}$, $\mathcal{L}^N(-D)$ is f-nef. Moreover, since f is birational, clearly $\mathcal{L}^N(-D)$ is f-big. Now, applying Theorem 8.3 to the morphism $f : \tilde{\mathcal{Z}} \to \mathcal{Z}$, we get (a).

(b) First, we claim

$$\mathscr{O}_{\tilde{\mathcal{Z}}}(\partial\tilde{\mathcal{Z}}) \simeq \mathscr{H}om_{\mathscr{O}_{\tilde{\mathcal{Z}}}}(f^*\mathscr{O}_{\mathcal{Z}}(-\partial\mathcal{Z}), \mathscr{O}_{\tilde{\mathcal{Z}}}), \tag{64}$$

where $\mathscr{O}_{\tilde{\mathcal{Z}}}(\partial\tilde{\mathcal{Z}}) := \mathscr{H}om_{\mathscr{O}_{\tilde{\mathcal{Z}}}}(\mathscr{O}_{\tilde{\mathcal{Z}}}(-\partial\tilde{\mathcal{Z}}), \mathscr{O}_{\tilde{\mathcal{Z}}})$. To see this, first note that by [Stacks, Tag 01HJ, Lemma 25.4.7], since $f^{-1}(\partial\mathcal{Z}) = \partial\tilde{\mathcal{Z}}$ is the scheme-theoretic inverse image, the natural morphism

$$f^*(\mathscr{O}_{\mathcal{Z}}(-\partial\mathcal{Z})) \to \mathscr{O}_{\tilde{\mathcal{Z}}}(-\partial\tilde{\mathcal{Z}})$$

is surjective. As f is a desingularization (Proposition 7.4), the kernel of this morphism is supported on a proper closed subset of $\tilde{\mathcal{Z}}$ and hence is a torsion sheaf. This implies that the dual map $\mathcal{O}_{\tilde{\mathcal{Z}}}(\partial\tilde{\mathcal{Z}}) \to \mathcal{H}om_{\mathcal{O}_{\tilde{\mathcal{Z}}}}(f^*(\mathcal{O}_{\mathcal{Z}}(-\partial\mathcal{Z})), \mathcal{O}_{\tilde{\mathcal{Z}}})$ is an isomorphism, proving (64).

To complete the proof of (b), we compute

$$\begin{aligned} f_*(\omega_{\tilde{\mathcal{Z}}}(\partial\tilde{\mathcal{Z}})) &= f_*(\omega_{\tilde{\mathcal{Z}}} \otimes \mathcal{H}om_{\mathcal{O}_{\tilde{\mathcal{Z}}}}(f^*\mathcal{O}_{\mathcal{Z}}(-\partial\mathcal{Z}), \mathcal{O}_{\tilde{\mathcal{Z}}})) \quad \text{by (64)} \\ &= f_*\mathcal{H}om_{\mathcal{O}_{\tilde{\mathcal{Z}}}}(f^*\mathcal{O}_{\mathcal{Z}}(-\partial\mathcal{Z}), \omega_{\tilde{\mathcal{Z}}}) \\ &= \mathcal{H}om_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{O}_{\mathcal{Z}}(-\partial\mathcal{Z}), f_*\omega_{\tilde{\mathcal{Z}}}) \quad \text{by adjunction [H, Chap. II, §5]} \\ &= \mathcal{H}om_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{O}_{\mathcal{Z}}(-\partial\mathcal{Z}), \omega_{\mathcal{Z}}) \quad \text{by Proposition 7.7 and [KM, Theorem 5.10]} \\ &= \omega_{\mathcal{Z}}(\partial\mathcal{Z}). \end{aligned}$$

□

As an immediate consequence of Proposition 8.4, Theorem 8.5 and the Grothendieck spectral sequence [J, Part I, Proposition 4.1], we get the following:

Corollary 8.6. *Let $\pi : \mathcal{Z} \to \bar{\Gamma}$ be the morphism as in the big diagram in Section 7. Then*

$$R^p\pi_*(\omega_{\mathcal{Z}}(\partial\mathcal{Z})) = 0 \quad \textit{for all } p > 0.$$

9. Proof of Theorem 4.10(b)

By using Kashiwara's result $\xi^u = \mathcal{O}_{X^u}(-\partial X^u)$ (Theorem 10.4) and the vanishing

$$\mathcal{T}or_1^{\mathcal{O}_{\tilde{Y}_{\mathbb{P}}}}(\gamma_*\tilde{\Delta}_*\mathcal{O}_{(X_w)_{\mathbb{P}}}, \mathcal{O}_{\partial(X^{u,v}_{\mathbf{j}})}) = 0 \quad \text{for general } \gamma \in \bar{\Gamma}$$

(which can be proved by an argument similar to the proof of Theorem 4.10(a) using Corollary 5.7), Theorem 4.10(b) is clearly equivalent to the following vanishing:

Theorem 9.1. *Assume that $c^w_{u,v}(\mathbf{j}) \neq 0$. For general $\gamma \in \bar{\Gamma}$,*

$$H^p\big(X^{u,v}_{\mathbf{j}} \cap \gamma\tilde{\Delta}((X_w)_{\mathbb{P}}), \mathcal{O}(-\bar{M}_\gamma)\big) = 0 \quad \textit{for all } p \neq |\mathbf{j}| + \ell(w) - \ell(u) - \ell(v),$$

where $\bar{M}_\gamma := M_{\gamma^{-1}}$ is the subscheme $(\partial(X^{u,v}_{\mathbf{j}})) \cap \gamma\tilde{\Delta}((X_w)_{\mathbb{P}})$ and (as earlier)

$$\partial(X^{u,v}_{\mathbf{j}}) := (\partial X^u \times X^v)_{\mathbf{j}} \cup (X^u \times \partial X^v)_{\mathbf{j}} \cup (X^u \times X^v)_{\partial\mathbb{P}_{\mathbf{j}}},$$

and $\mathcal{O}(-\bar{M}_\gamma)$ denotes the ideal sheaf of $\bar{M}_\gamma$ in $X^{u,v}_{\mathbf{j}} \cap \gamma\tilde{\Delta}((X_w)_{\mathbb{P}})$.

Proof. By Lemma 7.5 and Proposition 7.8, $\mathcal{Z}$ and $\partial\mathcal{Z}$ are CM and $\partial\mathcal{Z}$ is pure of codimension 1 in $\mathcal{Z}$. Thus, we get the vanishing (see the proof of Corollary 7.9)

$$\mathcal{E}xt^i_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{O}_{\mathcal{Z}}(-\partial\mathcal{Z}), \omega_{\mathcal{Z}}) = 0 \quad \text{for all } i \geq 1. \tag{65}$$

Also, by Corollary 7.9, for general $\gamma \in \bar{\Gamma}$,

$$\mathcal{E}xt^i_{\mathcal{O}_{\bar{N}_\gamma}}(\mathcal{O}_{\bar{N}_\gamma}(-\bar{M}_\gamma), \omega_{\bar{N}_\gamma}) = 0 \quad \text{for all } i > 0,$$

where $\bar{N}_\gamma := N_{\gamma^{-1}}$ is the subscheme $(X^{u,v}_{\mathbf{j}}) \cap \gamma\tilde{\Delta}((X_w)_{\mathbb{P}})$.

Hence, by the Serre duality [H, Chap. III, Theorem 7.6] applied to $\bar{N}_\gamma$ and the local-to-global Ext spectral sequence [Go, Chap. II, Théorème 7.3.3]) the theorem is equivalent to the vanishing (for general $\gamma \in \bar{\Gamma}$)

$$H^p\big(\bar{N}_\gamma, \mathscr{H}om_{\mathscr{O}_{\bar{N}_\gamma}}(\mathscr{O}_{\bar{N}_\gamma}(-\bar{M}_\gamma), \omega_{\bar{N}_\gamma})\big) = 0 \quad \text{for all } p > 0, \tag{66}$$

since (for general $\gamma \in \bar{\Gamma}$) $\bar{N}_\gamma$ is CM and $\dim(\bar{N}_\gamma) = |\mathbf{j}| + \ell(w) - \ell(u) - \ell(v)$ (Corollary 7.9).

For general $\gamma \in \bar{\Gamma}$,

$$\omega_{\mathcal{Z}}(\partial\mathcal{Z})_{|\pi^{-1}(\gamma^{-1})} \simeq \omega_{\pi^{-1}(\gamma^{-1})}(\partial\mathcal{Z} \cap \pi^{-1}(\gamma^{-1})) = \omega_{\bar{N}_\gamma}(\bar{M}_\gamma), \tag{67}$$

where $\omega_{\bar{N}_\gamma}(\bar{M}_\gamma) := \mathscr{H}om_{\mathscr{O}_{\bar{N}_\gamma}}(\mathscr{O}_{\bar{N}_\gamma}(-\bar{M}_\gamma), \omega_{\bar{N}_\gamma})$. To prove the above, observe first that by [S, Chap. I, §6.3, Theorem 1.25] and [H, Chap. III, Exercise 10.9] applied to π, there exists an open nonempty subset $\bar{\Gamma}_o \subset \bar{\Gamma}$ such that $\pi : \pi^{-1}(\bar{\Gamma}_o) \to \bar{\Gamma}_o$ is a flat morphism. (By the proof of Corollary 7.9, π is surjective.) Now, since $\bar{\Gamma}_o$ is smooth and $\mathcal{Z}$ and $\partial\mathcal{Z}$ are CM, and the assertion is local in $\bar{\Gamma}$, it suffices to observe (see [I, Corollary 11.35]) that for a nonzero function θ on $\bar{\Gamma}_o$, there is an isomorphism of sheaves of $\mathscr{O}_{\mathcal{Z}_\theta}$-modules

$$\mathcal{S}/\theta \cdot \mathcal{S} \simeq \mathscr{H}om_{\mathscr{O}_{\mathcal{Z}_\theta}}\big(\mathscr{O}_{\mathcal{Z}}(-\partial\mathcal{Z})/\theta \cdot \mathscr{O}_{\mathcal{Z}}(-\partial\mathcal{Z}), \omega_{\mathcal{Z}_\theta}\big),$$

where $\mathcal{Z}_\theta$ denotes the zero scheme of θ in $\mathcal{Z}$ and $\mathcal{S} := \mathscr{H}om_{\mathscr{O}_{\mathcal{Z}}}(\mathscr{O}_{\mathcal{Z}}(-\partial\mathcal{Z}), \omega_{\mathcal{Z}})$. Choosing θ to be in a local coordinate system, we can continue and get (67).

Now, the vanishing of $R^p\pi_*(\omega_{\mathcal{Z}}(\partial\mathcal{Z}))$ for $p > 0$ (Corollary 8.6) implies the following vanishing, for general $\gamma \in \bar{\Gamma}$:

$$H^p(\bar{N}_\gamma, \omega_{\bar{N}_\gamma}(\bar{M}_\gamma)) = 0 \quad \text{for all } p > 0. \tag{68}$$

To prove this, since $\mathcal{Z}$ and $\partial\mathcal{Z}$ are CM, $\bar{\Gamma}_o$ is smooth and $\pi : \pi^{-1}(\bar{\Gamma}_o) \to \bar{\Gamma}_o$ is flat, observe that $\omega_{\mathcal{Z}}(\partial\mathcal{Z})$ is flat over the base $\bar{\Gamma}_o$:

To show this, let $A = \mathscr{O}_{\bar{\Gamma}_o}$, $B = \mathscr{O}_{\pi^{-1}(\bar{\Gamma}_o)}$, and $M = \omega_{\mathcal{Z}}(\partial\mathcal{Z})|_{\pi^{-1}(\bar{\Gamma}_o)}$. By taking stalks, we immediately reduce to showing that for an embedding of local rings $A \subset B$ such that A is regular and B is flat over A, the module M is flat over A. Now, to prove this, let $\{x_1, \dots, x_d\}$ be a minimal set of generators of the maximal ideal of A. Let $K_\bullet = K_\bullet(x_1, \dots, x_d)$ be the Koszul complex of the x_i's over A. Then, recall that a finitely generated B-module N is flat over A iff $K_\bullet \otimes_A N$ is exact except at the extreme right, i.e., $H^i(K_\bullet \otimes_A N) = 0$ for $i < d$ [E, Theorem 6.8 and Corollary 17.5]. Thus, by hypothesis, $K_\bullet \otimes_A B$ is exact except at the extreme right, and hence the x_i's form a B-regular sequence by [E, Theorem 17.6]. Now, since $\mathscr{O}_{\mathcal{Z}}$ and $\mathscr{O}_{\partial\mathcal{Z}}$ are CM and $\partial\mathcal{Z}$ is pure of codimension 1 in $\mathcal{Z}$, we see that $\mathscr{O}_{\mathcal{Z}}(-\partial\mathcal{Z})$ is a CM $\mathscr{O}_{\mathcal{Z}}$-module. Thus, by [I, Proposition 11.33], M is a CM B-module of dimension equal to $\dim(B)$. Therefore, by [I, Exercise 11.36], the x_i's form a regular sequence on the B-module M. Hence, $(K_\bullet \otimes_A B) \otimes_B M \simeq K_\bullet \otimes_A M$ is exact except at the extreme right by [E, Corollary 17.5]. This proves that M is flat over A, as desired.

Hence, (68) folows from the semicontinuity theorem ([H, Chap. III, Theorem 12.8 and Corollary 12.9] or [Ke, Theorem 13.1]).

Thus, (66) (which is nothing but (68)) is established. Hence, the theorem follows, and thus Theorem 4.10(b) is established. □

10. Appendix (by Masaki Kashiwara): Determination of the dualizing sheaf of X^v

Let $v \in W$. Set $\mathfrak{C}^v := \bigcup_{y \in W,\, \ell(y) \leq \ell(v)+1} C^y$, where $C^y := B^- yB/B \subset \bar{X}$. (By definition, $\mathfrak{C}^v$ only depends upon $\ell(v)$.) Then $\mathfrak{C}^v$ is an open subset of $\bar{X}$. Moreover, $X^v \cap \mathfrak{C}^v$ is a smooth scheme, since X^v is normal [KS, Proposition 3.2] and any B^--orbit in $X^v \cap \mathfrak{C}^v$ is of codimension ≤ 1. Recall from Section 3 the definition of

$$\xi^v := e^{-\rho}\mathcal{L}(\rho)\omega_{X^v} = e^{-\rho}\mathcal{L}(-\rho)\mathcal{E}xt^{\ell(v)}_{\mathcal{O}_{\bar{X}}}(\mathcal{O}_{X^v}, \mathcal{O}_{\bar{X}}).$$

Since $\mathcal{O}_{X^v}$ is a CM ring [KS, Proposition 3.4], we see that ξ^v is a CM $\mathcal{O}_{X^v}$-module. Also, since $X^v \cap \mathfrak{C}^v$ is a smooth scheme, $\xi^v|_{\mathfrak{C}^v}$ is an invertible $\mathcal{O}_{X^v}|_{\mathfrak{C}^v}$-module.

For any $y \in W$, let $i_y : \{\text{pt}\} \to \bar{X}$ be the morphism given by $\text{pt} \mapsto yx_o$. Then (as H-modules)

$$i_y^*\mathcal{L}(\lambda) \simeq \mathbb{C}_{-y\lambda} \quad \text{for any character } \lambda \text{ of } H. \tag{69}$$

Let $\pi_i : \bar{X} \to \bar{X}_i$ be the projection, where $\bar{X}_i := G/P_i$, P_i being the minimal standard parabolic subgroup containing the simple root α_i for $1 \leq i \leq r$.

Lemma 10.1. *On some B^--stable neighborhood of C^v, we have a B^--equivariant isomorphism $\xi^v \simeq \mathcal{O}_{X^v}$.*

Proof. Since $\xi^v|_{\mathfrak{C}^v}$ is an invertible B^--equivariant $\mathcal{O}_{X^v}|_{\mathfrak{C}^v}$-module, it is enough to show that $i_v^*\xi^v \simeq \mathbb{C}$ as H-modules. This follows from $i_v^*(\mathcal{E}xt^{\ell(v)}_{\mathcal{O}_{\bar{X}}}(\mathcal{O}_{X^v}, \mathcal{O}_{\bar{X}})) \simeq \det(T_{vx_o}\bar{X}/T_{vx_o}X^v) \simeq \mathbb{C}_{\rho - v\rho}$ and $i_v^*\mathcal{L}(-\rho) \simeq \mathbb{C}_{v\rho}$ by (69). □

Set $A_v := \{y \in W : y > v \text{ and } \ell(y) = \ell(v)+1\}$. The above lemma implies that, as B^--equivariant $\mathcal{O}_{\bar{X}}$-modules,

$$\xi^v|_{\mathfrak{C}^v} \simeq \mathcal{O}_{X^v}\Big(\sum_{y \in A_v} m_y X^y\Big)\Big|_{\mathfrak{C}^v} \tag{70}$$

for some $m_y \in \mathbb{Z}$. Recall that $\partial X^v = \bigcup_{y \in A_v} X^y$.

Lemma 10.2. *We have $\xi^v|_{\mathfrak{C}^v} \simeq \mathcal{O}_{X^v}(-\partial X^v)|_{\mathfrak{C}^v}$, where $\mathcal{O}_{X^v}(-\partial X^v) \subset \mathcal{O}_{X^v}$ is the ideal sheaf of the reduced subscheme ∂X^v of X^v.*

Proof. The proof is similar to the one of Lemma 10.1. For $y \in A_v$, y is a smooth point of X^v (since $X^v \cap \mathfrak{C}^v$ is smooth). Hence,

$$\begin{aligned} i_y^*\big(\mathcal{E}xt^{\ell(v)}_{\mathcal{O}_{\bar{X}}}(\mathcal{O}_{X^v}, \mathcal{O}_{\bar{X}})\big) &\simeq \det(T_{yx_o}\bar{X}/T_{yx_o}X^v) \\ &\simeq \det(T_{yx_o}\bar{X}/T_{yx_o}X^y) \otimes \det(T_{yx_o}X^v/T_{yx_o}X^y)^{\otimes(-1)} \\ &\simeq \mathbb{C}_{\rho - y\rho} \otimes \det(T_{yx_o}X^v/T_{yx_o}X^y)^{\otimes(-1)}. \end{aligned}$$

Thus, $i_y^*\xi^v \simeq (T_{yx_o}X^v/T_{yx_o}X^y)^{\otimes(-1)}$ as H-modules by (69). On the other hand,

$$i_y^*\Big(\mathcal{O}_{X^v}\Big(\sum_{z\in A_v} m_z X^z\Big)\Big) \simeq (T_{yx_o}X^v/T_{yx_o}X^y)^{\otimes m_y} \quad \text{as } H\text{-modules.}$$

Hence, by (70), we have $m_y = -1$. Note that $T_{yx_o}X^v/T_{yx_o}X^y$ is not a trivial H-module by the next lemma. □

Lemma 10.3. *Let $v, y \in W$ satisfy $v < y$ and $\ell(y) = \ell(v) + 1$. Then*

$$T_{yx_o}X^v/T_{yx_o}X^y \simeq \mathbb{C}_\beta$$

as H-modules, where β is the positive real root such that $yv^{-1} = s_\beta$.

Proof. We use induction on $\ell(y)$. Take a simple reflection s_i such that $ys_i < y$.

(i) If $vs_i > v$, then $y = vs_i$. Thus, $T_{yx_o}X^v/T_{yx_o}X^y \simeq T_{yx_o}\pi_i^{-1}\pi_i(yx_o)$, and hence $T_{yx_o}X^v/T_{yx_o}X^y \simeq \mathbb{C}_{-y\alpha_i} \simeq \mathbb{C}_{v\alpha_i} = \mathbb{C}_\beta$.

(ii) If $vs_i < v$, then $\pi_i : X^v \to \bar{X}_i$ is a local embedding at yx_o since $C^v \cup C^y$ is open in X^v, $\pi_i|_{C^v\cup C^y}$ is an injective map onto an open subset of $\pi_i(X^v)$, and $\pi_i(X^v) = \pi_i(X^{vs_i})$ is normal (since $X^{vs_i} \to \pi_i(X^{vs_i})$ is a $\mathbb{P}^1$-fibration and X^{vs_i} is normal by [KS, Proposition 3.2]). Moreover, $\pi_i(X^v)$ is smooth at $\pi_i(yx_o)$ since the B^--orbit of $\pi_i(yx_o)$ is of codimension 1 in $\pi_i(X^v)$. Hence, $T_{yx_o}X^v/T_{yx_o}X^y \simeq T_{\pi_i(yx_o)}(\pi_i(X^v))/T_{\pi_i(yx_o)}(\pi_i(X^y)) \simeq T_{ys_ix_o}X^{vs_i}/T_{ys_ix_o}X^{ys_i}$. By the induction hypothesis, this is isomorphic to $\mathbb{C}_\beta$. □

Let $j : \mathfrak{C}^v \hookrightarrow \bar{X}$ be the open embedding.

Theorem 10.4. *For any $v \in W$, we have a B^--equivariant isomorphism*

$$\xi^v \simeq \mathcal{O}_{X^v}(-\partial X^v).$$

Hence, the dualizing sheaf ω_{X^v} of X^v is T-equivariantly isomorphic to

$$\mathbb{C}_\rho \otimes \mathcal{L}(-\rho) \otimes \mathcal{O}_{X^v}(-\partial X^v).$$

Proof. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{X^v}(-\partial X^v) & \longrightarrow & \mathcal{O}_{X^v} & \longrightarrow & \mathcal{O}_{\partial X^v} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \wr & & \downarrow & & \\
0 & \longrightarrow & j_*j^{-1}\mathcal{O}_{X^v}(-\partial X^v) & \longrightarrow & j_*j^{-1}\mathcal{O}_{X^v} & \longrightarrow & j_*j^{-1}\mathcal{O}_{\partial X^v} & &
\end{array}$$

where the middle vertical arrow is an isomorphism because X^v is normal and $X^v \setminus \mathfrak{C}^v$ is of codimension ≥ 2 in X^v, and the right vertical arrow is a monomorphism because the closure of $\partial X^v \cap \mathfrak{C}^v$ coincides with ∂X^v. Hence, $j_*j^{-1}\mathcal{O}_{X^v}(-\partial X^v) \simeq \mathcal{O}_{X^v}(-\partial X^v)$. On the other hand, since ξ^v is a CM $\mathcal{O}_{X^v}$-module, we have

$$\xi^v \simeq j_*j^{-1}\xi^v \simeq j_*j^{-1}\mathcal{O}_{X^v}(-\partial X^v) \simeq \mathcal{O}_{X^v}(-\partial X^v),$$

where the second isomorphism is due to Lemma 10.2. □

Corollary 10.5. *$\mathscr{O}_{X^v}(-\partial X^v)$ is a CM $\mathscr{O}_{X^v}$-module and $\mathscr{O}_{\partial X^v}$ is a CM ring.*

Proof. Since ξ^v is a CM $\mathscr{O}_{X^v}$-module, so is $\mathscr{O}_{X^v}(-\partial X^v)$ by the above theorem.

Applying the functor $\mathscr{H}om_{\mathscr{O}_{\bar{X}}}(\bullet, \mathscr{O}_{\bar{X}})$ to the exact sequence

$$0 \to \mathscr{O}_{X^v}(-\partial X^v) \to \mathscr{O}_{X^v} \to \mathscr{O}_{\partial X^v} \to 0,$$

we obtain $\mathscr{E}xt^k_{\mathscr{O}_{\bar{X}}}(\mathscr{O}_{\partial X^v}, \mathscr{O}_{\bar{X}}) = 0$ for $k \neq \ell(v), \ell(v)+1$. We also have an exact sequence

$$0 \to \mathscr{E}xt^{\ell(v)}_{\mathscr{O}_{\bar{X}}}(\mathscr{O}_{\partial X^v}, \mathscr{O}_{\bar{X}}) \to \mathscr{E}xt^{\ell(v)}_{\mathscr{O}_{\bar{X}}}(\mathscr{O}_{X^v}, \mathscr{O}_{\bar{X}}).$$

Since ∂X^v has codimension $\ell(v)+1$, we have $\mathscr{E}xt^{\ell(v)}_{\mathscr{O}_{\bar{X}}}(\mathscr{O}_{\partial X^v}, \mathscr{O}_{\bar{X}}) = 0$. Hence, $\mathscr{O}_{\partial X^v}$ is a CM ring. □

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