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Almost sure global well-posedness for the energy-critical defocusing nonlinear wave equation on \mathbb{R}^d , $d = 4$ and 5

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Abstract. We consider the energy-critical defocusing nonlinear wave equation (NLW) on \mathbb{R}^d , $d = 4, 5$. We prove almost sure global existence and uniqueness for NLW with rough random initial data in $H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$ with $0 < s \leq 1$ if $d = 4$, and $0 \leq s \leq 1$ if $d = 5$. The randomization we consider is naturally associated with the Wiener decomposition and with modulation spaces. The proof is based on a probabilistic perturbation theory. Under some additional assumptions, for $d = 4$, we also prove the probabilistic continuous dependence of the flow on the initial data (in the sense proposed by Burq and Tzvetkov [19]).

Keywords. Nonlinear wave equations, almost sure well-posedness, probabilistic continuous dependence, Wiener decomposition

1. Introduction

1.1. Energy-critical nonlinear wave equations

We consider the Cauchy problem for the energy-critical defocusing nonlinear wave equation (NLW) on \mathbb{R}^d , $d = 4$ or 5 :

$$\begin{cases} \partial_t^2 u - \Delta u + F(u) = 0, \\ (u, \partial_t u)|_{t=0} = (\phi_0, \phi_1), \end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d, \quad (1.1)$$

where $F(u) = |u|^{4/(d-2)}u$ and u is a real-valued function on $\mathbb{R} \times \mathbb{R}^d$. Our main focus in this paper is to study the global-in-time behavior of solutions with *random* and *rough* initial data.

The flow of equation (1.1) formally conserves the energy $E(u)$ defined by

$$E(u) = E(u, \partial_t u) := \int_{\mathbb{R}^d} \left(\frac{1}{2} (\partial_t u)^2 + \frac{1}{2} |\nabla u|^2 + \frac{d-2}{2d} |u|^{\frac{2d}{d-2}} \right) dx. \quad (1.2)$$

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We define the energy space $\mathcal{E}(\mathbb{R}^d)$ associated to (1.1) to be the space of pairs (f, g) of real-valued functions of finite energy,

$$\mathcal{E}(\mathbb{R}^d) := \left\{ (f, g) : E(f, g) := \int_{\mathbb{R}^d} \left(\frac{1}{2}g^2 + \frac{1}{2}|\nabla f|^2 + \frac{d-2}{2d}|f|^{\frac{2d}{d-2}} \right) dx < \infty \right\}.$$

Then, in view of the Sobolev embedding $\dot{H}^1(\mathbb{R}^d) \subset L^{\frac{2d}{d-2}}(\mathbb{R}^d)$, we immediately see that

$$\mathcal{E}(\mathbb{R}^d) = \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d).$$

It is well known that the NLW equation (1.1) on \mathbb{R}^d enjoys several symmetries. Of particular importance is the following scaling invariance:

$$u(t, x) \mapsto u_\lambda(t, x) := \lambda^{(d-2)/2} u(\lambda t, \lambda x). \quad (1.3)$$

Namely, if u is a solution of (1.1), then u_λ is also a solution of (1.1) with rescaled initial data. Notice that

$$\|(u_\lambda(0), \partial_t u_\lambda(0))\|_{\dot{H}^s(\mathbb{R}^d) \times \dot{H}^{s-1}(\mathbb{R}^d)} = \lambda^{s-1} \|(u(0), \partial_t u(0))\|_{\dot{H}^s(\mathbb{R}^d) \times \dot{H}^{s-1}(\mathbb{R}^d)}. \quad (1.4)$$

One then defines the so-called scaling critical Sobolev index $s_c := 1$ to be the index s for which the homogeneous $\dot{H}^s(\mathbb{R}^d) \times \dot{H}^{s-1}(\mathbb{R}^d)$ -norm of $(u(0), \partial_t u(0))$ is invariant under the scaling (1.3). We notice that the critical space $\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ under the scaling coincides with the energy space $\mathcal{E}(\mathbb{R}^d)$. Moreover, the energy $E(u)$ defined in (1.2) is also invariant under the scaling. Therefore, we refer to the NLW (1.1) on \mathbb{R}^d as *energy-critical*.

Heuristically speaking, for an energy-critical NLW, there is a delicate balance between the linear and nonlinear parts of the equation, which has made the analysis of such equations rather intricate. Nonetheless, after an intensive effort which materialized in many articles, it is now known that the energy-critical defocusing nonlinear wave equations on \mathbb{R}^d , $d \geq 3$, are globally well-posed in the energy space and that all solutions in the energy space scatter. The small energy data theory goes back to Strauss [60], Rauch [56], and Pecher [54]. The global regularity (referring to the fact that smooth initial data lead to smooth global solutions) was proved in the works of Struwe [61], Grillakis [30, 31], and Shatah and Struwe [58]. Regarding global well-posedness in the energy space, scattering, and global space-time bounds, we cite Shatah and Struwe [59], Kapitanski [35], Ginibre, Soffer, and Velo [29], Bahouri and Shatah [4], Bahouri and Gérard [3], Nakanishi [48, 49], and Tao [62].

On the other hand, there are ill-posedness results for the energy-critical NLW below the scaling critical regularity $s_c = 1$. When $d = 3$ and 4, Christ, Colliander, and Tao [21] proved that the solution map of the energy-critical NLW fails to be continuous at zero in the $H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$ -topology for $0 < s < 1$. See also [20, 1, 39, 40, 13, 33] for other ill-posedness results for nonlinear wave and Schrödinger equations.

In spite of the above deterministic ill-posedness results, in this paper we consider the Cauchy problem (1.1) with general initial data (ϕ_0, ϕ_1) that do not belong to the energy space, in a probabilistic manner. More precisely, for $d = 4$ or 5, given

$$(u_0, u_1) \in H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d) \setminus H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$$

for some $s \in (0, 1)$, we consider its randomization (u_0^ω, u_1^ω) defined in (1.13) below. This randomization has the same regularity as (u_0, u_1) and is not smoother in terms of differentiability, almost surely (Lemma 2.2). Our first task is to construct local-in-time solutions of (1.1) with initial data $(\phi_0, \phi_1) = (u_0^\omega, u_1^\omega)$ in a probabilistic manner. As in the work of Burq and Tzvetkov [17], the key point is the improved integrability properties of the randomization (u_0^ω, u_1^ω) . This allows us to obtain improved probabilistic local-in-time Strichartz estimates (Proposition 2.3). Then, we prove almost sure local existence (with uniqueness) by a simple fixed point argument in Strichartz spaces (Theorem 1.1). This almost sure local existence result is accompanied by a probabilistic small data global result (Theorem 1.2). In [5, 6], the author with Bényi and Oh considered the same problem for the cubic nonlinear Schrödinger equation (NLS) on \mathbb{R}^d , $d \geq 3$.

Once we construct local-in-time solutions almost surely, our next task is to extend them globally in time. Indeed, this probabilistic global-in-time argument is the main goal and novelty of this paper. Informally speaking, there have been two kinds of globalization arguments in the probabilistic setting: (i) invariant measure argument and (ii) certain probabilistic adaptations of deterministic globalization arguments.

Bourgain [7] proved global existence for NLS on \mathbb{T} almost surely with respect to the associated Gibbs measure. The basic idea behind his argument is to use (formal) *invariance* of this Gibbs measure in place of conservation laws. He made this rigorous by exploiting the invariance of the finite-dimensional Gibbs measures for the associated finite-dimensional approximations of NLS. This approach has been used in subsequent works; see for example [8, 9, 65, 16, 18, 50, 45, 26, 14, 24, 57, 12]. In our context, however, there is no apparent (formally) invariant measure and this approach is not available.

Recently, there have been several probabilistic global-in-time arguments in the absence of an invariant measure. Colliander and Oh [23] introduced a *probabilistic high-low decomposition method* to prove almost sure global well-posedness of cubic NLS on \mathbb{T} below $L^2(\mathbb{T})$. This is an adaptation of Bourgain's high-low decomposition method [10] in the probabilistic setting. This approach was also used by Lührmann and Mendelson [42] in the context of energy-subcritical NLW on \mathbb{R}^3 .

Burq and Tzvetkov [19] considered (energy-subcritical) cubic NLW on \mathbb{T}^3 and proved probabilistic global well-posedness for rough random data in $H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$, $0 \leq s < 1$. The main ingredient is to establish a *probabilistic a priori bound on the energy*.¹ This is in some sense a probabilistic analogue of the fact that a conservation law yields (subcritical) global well-posedness in the deterministic theory.

Probabilistic a priori bounds have also been combined with a *probabilistic compactness method*. Using this strategy, Burq, Thomann, and Tzvetkov [15] obtained almost sure existence of global solutions of the (energy-critical and energy-supercritical) cubic wave equation on \mathbb{T}^d , $d \geq 4$, with rough random data. As in the deterministic setting, the compactness method does not yield the uniqueness of such solutions. Note that this work was inspired by an earlier work of Nahmod, Pavlović, and Staffilani [46]. They considered the Navier–Stokes equations on \mathbb{T}^2 and \mathbb{T}^3 and proved almost sure existence of global weak solutions with rough random data. These weak solutions were shown to be unique on \mathbb{T}^2 .

¹ The argument for obtaining almost sure global existence when $s = 0$ is more involved.

In the following, we use a *probabilistic perturbation theory* to prove almost sure global well-posedness² for the energy-critical NLW (1.1) on \mathbb{R}^d with rough random data in $H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$, where $0 < s \leq 1$ if $d = 4$, and $0 \leq s \leq 1$ if $d = 5$. See Theorem 1.3 below. This is the first instance when a perturbation theory is successfully applied in the probabilistic setting to yield almost sure global existence. See also [6]. Perturbation theory has played an important role in the study of deterministic energy-critical NLW and NLS. Moreover, perturbation theory has been previously used to prove global well-posedness for various other equations in the deterministic setting. See for example [22, 37, 63, 38]. Tao, Viřan, and Zhang [63] used perturbation theory to prove (among other results) global well-posedness of NLS with a combined power nonlinearity, one of the powers being energy-critical, while the other is energy-subcritical. Such an equation can be viewed as a perturbation of the energy-critical NLS, with the smallness of the error coming from the subcritical nature of the other power nonlinearity.

Now, let us turn our attention to equation (1.1) with random initial data $(\phi_0, \phi_1) = (u_0^\omega, u_1^\omega)$. By denoting the linear³ and nonlinear parts of the solution u^ω of (1.1) by $z^\omega(t) = S(t)(u_0^\omega, u_1^\omega)$ and $v^\omega(t) := u^\omega(t) - z^\omega(t)$, equation (1.1) reduces to

$$\begin{cases} \partial_t^2 v^\omega - \Delta v^\omega + F(v^\omega + z^\omega) = 0, \\ (v^\omega, \partial_t v^\omega)|_{t=0} = (0, 0). \end{cases} \quad (1.5)$$

Namely, the nonlinear part v^ω satisfies the energy-critical NLW with a perturbation. The crucial point in our approach is the fact that the error $F(v^\omega + z^\omega) - F(v^\omega)$ can be made small (on short time intervals) thanks to the improved local-in-time Strichartz estimates satisfied by the *random* linear part z^ω .

Another essential ingredient for an actual implementation of the probabilistic perturbation theory is a probabilistic a priori bound on the energy of the nonlinear part v^ω of the solution u^ω on each finite time interval (Proposition 5.2). The probabilistic energy bound that we use here is the analogue of that obtained by Burq and Tzvetkov [19] for the cubic NLW on \mathbb{T}^3 . In [6], the author with Bényi and Oh applied a similar probabilistic perturbation theory in the context of the defocusing cubic NLS on \mathbb{R}^d , $d \geq 3$. The global-in-time result of [6], however, is conditional, even for the energy-critical cubic NLS on \mathbb{R}^4 . On \mathbb{R}^4 , this failure is due to the fact that we do not have a probabilistic a priori energy bound on the nonlinear part of a solution. The key advantage of NLW, in comparison to NLS, is the presence of the term $\int \frac{1}{2} (\partial_t v)^2 dx$ in the energy. We also emphasize the importance of the specific power nonlinearity in obtaining the probabilistic energy bound. It would be interesting to extend our almost sure global well-posedness result to the energy-critical quintic NLW on \mathbb{R}^3 due to its physical relevance. In view of Remark 1.4 below, such almost sure global existence would follow once we establish an analogous probabilistic energy bound on \mathbb{R}^3 .

² Here, almost sure global well-posedness means almost sure global existence, uniqueness, and a weak form of continuous dependence (see [8] and [23]). This is not to be confused with probabilistic Hadamard global well-posedness that we describe below.

³ See (1.16) for the precise definition of the linear propagator $S(t)$.

The almost sure global well-posedness described in Theorem 1.3 refers to almost sure global existence, uniqueness, and a weak form of continuous dependence (as in [8] and [23]; see Remark 1.2(ii) below). Recently, Burq and Tzvetkov [19] introduced the stronger notion of *probabilistic Hadamard global well-posedness*. This refers to almost sure global existence and uniqueness, together with *probabilistic continuity* of the flow with respect to the random initial data. In Theorem 1.4, we prove, for $d = 4$, that the flow of equation (1.1) is continuous in probability, under some extra assumptions. This allows us to establish probabilistic Hadamard global well-posedness of the energy-critical defocusing cubic NLW (1.1) on \mathbb{R}^4 in $H^s(\mathbb{R}^4) \times H^{s-1}(\mathbb{R}^4)$, $0 < s \leq 1$, in the sense of [19].

1.2. Randomization adapted to the Wiener decomposition and modulation spaces

Starting with the works of Bourgain [8] and Burq and Tzvetkov [17], there have been many results on probabilistic constructions of solutions of evolution equations with random initial data. Many of the probabilistic results in the literature are on compact manifolds M , where there is a countable basis $\{e_n\}_{n \in \mathbb{N}}$ of $L^2(M)$ consisting of eigenfunctions of the Laplace–Beltrami operator. This gives a natural way to introduce a randomization. Given $u_0(x) = \sum_{n=1}^\infty c_n e_n(x) \in H^s(M)$, one defines its randomization by

$$u_0^\omega(x) = \sum_{n=1}^\infty g_n(\omega) c_n e_n(x), \tag{1.6}$$

where $\{g_n\}_{n \in \mathbb{N}}$ is a sequence of independent random variables. On \mathbb{R}^d , however, there is no countable basis of $L^2(\mathbb{R}^d)$ consisting of eigenfunctions of the Laplacian. In order to overcome this difficulty, several approaches have been introduced. For example, randomizations analogous to (1.6) have been considered with respect to a countable basis of $L^2(\mathbb{R}^d)$ consisting of eigenfunctions of the Laplacian with a confining potential, such as the harmonic oscillator $-\Delta + |x|^2$. Some results for the NLS with a harmonic potential were then transferred to the usual NLS on \mathbb{R}^d via the lens transform (see [64, 14, 24, 55]). Another approach consisted in working on the unit sphere \mathbb{S}^3 , and then transferring results to \mathbb{R}^3 via the Penrose transform (see [25, 26, 27]).

In this paper, we use a simple randomization for functions on \mathbb{R}^d , naturally associated to the Wiener decomposition and modulation spaces. This seems to be quite canonical from the point of view of the time–frequency analysis [32]. Such a randomization was previously used in [42, 5, 6] (see also [68]).

Given $d \geq 1$, let Q_n be the unit cube $n + [-1/2, 1/2)^d$ centered at $n \in \mathbb{Z}^d$. Then, consider the uniform partition of the frequency space $\mathbb{R}^d = \bigcup_{n \in \mathbb{Z}^d} Q_n$, commonly referred to as the *Wiener decomposition* [66]. Noting that $\sum_{n \in \mathbb{Z}^d} \chi_{Q_n}(\xi) \equiv 1$, we have

$$u = \sum_{n \in \mathbb{Z}^d} \chi_{Q_n}(D)u = \sum_{n \in \mathbb{Z}^d} \chi_{Q_0}(D - n)u,$$

where $\chi_{Q_n}(D)u := \mathcal{F}^{-1}[\chi_{Q_n} \widehat{u}]$. In the following, we consider a smoothed version of this decomposition.

Let $\tilde{\psi} \in \mathcal{S}(\mathbb{R}^d)$ be a real-valued function such that $\tilde{\psi}(\xi) \geq 0$, $\tilde{\psi}(-\xi) = \tilde{\psi}(\xi)$ for all $\xi \in \mathbb{R}^d$ and

$$\tilde{\psi}(\xi) = \begin{cases} 1 & \text{if } \xi \in Q_0, \\ 0 & \text{if } \xi \notin [-1, 1]^d. \end{cases}$$

Set

$$\psi(\xi) := \frac{\tilde{\psi}(\xi)}{\sum_{n \in \mathbb{Z}^d} \tilde{\psi}(\xi - n)}.$$

Then $\psi \in \mathcal{S}(\mathbb{R}^d)$ is a real-valued function, $0 \leq \psi(\xi) \leq 1$ for all $\xi \in \mathbb{R}^d$, $\text{supp } \psi \subset [-1, 1]^d$,

$$\psi(-\xi) = \psi(\xi) \quad \text{for all } \xi \in \mathbb{R}^d, \tag{1.7}$$

and $\sum_{n \in \mathbb{Z}^d} \psi(\xi - n) \equiv 1$. Now, we define the Fourier multiplier

$$\psi(D - n)u(x) = \int_{\mathbb{R}^d} \psi(\xi - n)\widehat{u}(\xi)e^{2\pi i x \cdot \xi} d\xi. \tag{1.8}$$

Then any function u on \mathbb{R}^d can be written as

$$u = \sum_{n \in \mathbb{Z}^d} \psi(D - n)u. \tag{1.9}$$

From the symmetry condition (1.7) imposed on ψ , we have

$$\overline{\psi(D + n)u} = \psi(D - n)u \quad \text{for real-valued } u, \tag{1.10}$$

since $\overline{\widehat{u}(\xi)} = \widehat{u}(-\xi)$. In particular, $\psi(D)u$ is real-valued for real-valued u .

Remark 1.1. The modulation spaces introduced by Feichtinger [28] are naturally associated to the uniform decomposition (1.9). Indeed, the modulation space $M_s^{p,q}$, $0 < p, q \leq \infty$, $s \in \mathbb{R}$, consists of all tempered distributions $u \in \mathcal{S}'(\mathbb{R}^d)$ for which the (quasi) norm

$$\|u\|_{M_s^{p,q}(\mathbb{R}^d)} := \|\langle n \rangle^s \|\psi(D - n)u\|_{L_x^p(\mathbb{R}^d)}\|_{\ell_n^q(\mathbb{Z}^d)} \tag{1.11}$$

is finite. Compare (1.11) with the definition of the Besov spaces which are naturally associated to the decomposition of the frequency space into dyadic annuli.

In the following, we introduce the randomization adapted to the uniform decomposition (1.9). Let $\{g_{n,j}\}_{n \in \mathbb{Z}^d, j=0,1}$ be a sequence of mean zero complex-valued random variables on a probability space (Ω, \mathcal{F}, P) such that $g_{-n,j} = \overline{g_{n,j}}$ for all $n \in \mathbb{Z}^d, j = 0, 1$. Assume also that $\{g_{0,j}, \text{Re } g_{n,j}, \text{Im } g_{n,j}\}_{n \in \mathcal{I}, j=0,1}$ are independent and endowed with probability distributions⁴ $\mu_{0,j}, \mu_{n,j}^{(1)}$, and $\mu_{n,j}^{(2)}$. Here, the index set \mathcal{I} is given by

$$\mathcal{I} = \bigcup_{k=0}^{d-1} \mathbb{Z}^k \times \mathbb{Z}_+^* \times \{0\}^{d-k-1} \tag{1.12}$$

and is such that $\mathbb{Z}^d = \mathcal{I} \cup (-\mathcal{I}) \cup \{0\}$.

⁴ The probability distribution μ of a real random variable g is an induced probability measure defined by $\mu(A) := P(\{\omega \in \Omega : g(\omega) \in A\})$ for all measurable sets $A \subset \mathbb{R}$.

For functions u_0, u_1 on \mathbb{R}^d , we define the *Wiener randomization* (u_0^ω, u_1^ω) of (u_0, u_1) by

$$(u_0^\omega, u_1^\omega) := \left(\sum_{n \in \mathbb{Z}^d} g_{n,0}(\omega) \psi(D-n)u_0, \sum_{n \in \mathbb{Z}^d} g_{n,1}(\omega) \psi(D-n)u_1 \right). \tag{1.13}$$

Note that if u_0 and u_1 are real-valued, then their randomizations u_0^ω and u_1^ω are also real-valued. More precisely, by (1.10), we have

$$\begin{aligned} u_j^\omega &= g_{0,j} \psi(D)u_j + \sum_{n \in \mathcal{I}} g_{n,j} \psi(D-n)u_j + g_{-n,j} \psi(D+n)u_j \\ &= g_{0,j} \psi(D)u_j + 2 \operatorname{Re} \sum_{n \in \mathcal{I}} g_{n,j} \psi(D-n)u_j, \quad j = 0, 1. \end{aligned} \tag{1.14}$$

We make the following assumption: there exists $c > 0$ such that

$$\int_{\mathbb{R}} e^{\gamma x} d\mu_{0,j} \leq e^{c\gamma^2} \quad \text{and} \quad \int_{\mathbb{R}} e^{\gamma x} d\mu_{n,j}^{(k)}(x) \leq e^{c\gamma^2} \tag{1.15}$$

for all $\gamma \in \mathbb{R}$, $n \in \mathbb{Z}^d$, $j = 0, 1$, and $k = 1, 2$. Note that (1.15) is satisfied by standard complex-valued Gaussian random variables, standard Bernoulli random variables, and any random variables with compactly supported distributions.

It is easy to see that if $(u_0, u_1) \in H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$ for some $s \in \mathbb{R}$, then the randomized function (u_0^ω, u_1^ω) is almost surely in $H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$ (Lemma 2.2). One can also show that there is no smoothing upon randomization in terms of differentiability (see, for example, [17, Lemma B.1]). Instead, the main point of this randomization is its improved integrability: if $u_j \in L^2(\mathbb{R}^d)$, $j = 0, 1$, then the randomized function u_j^ω is almost surely in $L^p(\mathbb{R}^d)$ for any finite $p \geq 2$. Such results for random Fourier series are known as Paley–Zygmund’s theorem [53] (see also [34, 2, 5]).

1.3. Main results

In this subsection, we state the main results of the paper. In the following, we denote the linear propagator of the linear wave equation by $S(t)$. Namely, the solution of the linear wave equation with initial data $(u, \partial_t u)|_{t=0} = (u_0, u_1)$ is denoted by

$$S(t)(u_0, u_1) := \cos(t|\nabla|)u_0 + \frac{\sin(t|\nabla|)}{|\nabla|}u_1. \tag{1.16}$$

We first state an almost sure local well-posedness result.

Theorem 1.1 (Almost sure local well-posedness). *Let $d = 4$ or 5 and $0 \leq s \leq 1$. Given $(u_0, u_1) \in H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$, let (u_0^ω, u_1^ω) be the randomization defined in (1.13), satisfying (1.15).*

- (a) *The energy-critical defocusing NLW (1.1) on \mathbb{R}^d admits almost surely a unique local solution with initial data (u_0^ω, u_1^ω) at $t = 0$. More precisely, there exist $C, c, \gamma > 0$ such that for each T sufficiently small, there exists a set $\Omega_T \subset \Omega$ with the following properties:*

- (i) $P(\Omega_T^c) < C \exp(-c/T^\gamma)$.
- (ii) For each $\omega \in \Omega_T$, there exists a unique solution u^ω of (1.1) with $(u^\omega, \partial_t u^\omega)|_{t=0} = (u_0^\omega, u_1^\omega)$ such that

$$(u^\omega, \partial_t u^\omega) \in (S(t)(u_0^\omega, u_1^\omega), \partial_t S(t)(u_0^\omega, u_1^\omega)) + C([-T, T]; H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)) \subset C([-T, T]; H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)).$$

Here, uniqueness holds in a ball centered at $S(\cdot)(u_0^\omega, u_1^\omega)$ in

$$C([-T, T]; \dot{H}^1(\mathbb{R}^d)) \cap L^{\frac{d+2}{d-2}}([-T, T]; L^{\frac{2(d+2)}{d-2}}(\mathbb{R}^d)).$$

- (b) Let $(w_0, w_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. Then there is a positive $T' = T'(T, w_0, w_1) < T$ such that for all $\omega \in \Omega_T$, the energy-critical defocusing NLW on \mathbb{R}^d admits a unique solution with initial data

$$(u, \partial_t u)|_{t=t_*} = (S(t_*)(u_0^\omega, u_1^\omega), \partial_t S(t_*)(u_0^\omega, u_1^\omega)) + (w_0, w_1),$$

satisfying

$$(u^\omega, \partial_t u^\omega) \in C([t_* - T', t_* + T']; H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)),$$

as long as $[t_* - T', t_* + T'] \subset [-T, T]$. Here, uniqueness holds in a ball centered at

$$S(\cdot - t_*) \left((S(t_*)(u_0^\omega, u_1^\omega), \partial_t S(t_*)(u_0^\omega, u_1^\omega)) + (w_0, w_1) \right)$$

$$\text{in } C([t_* - T', t_* + T']; \dot{H}^1(\mathbb{R}^d)) \cap L^{\frac{d+2}{d-2}}([t_* - T', t_* + T']; L^{\frac{2(d+2)}{d-2}}(\mathbb{R}^d)).$$

We prove Theorem 1.1(a) by considering equation (1.5) satisfied by the nonlinear part v^ω of a solution u^ω , and by viewing the linear part $z^\omega(t) = S(t)(u_0^\omega, u_1^\omega)$ as a random forcing term. We then run a simple fixed point argument in Strichartz spaces. The improved local-in-time Strichartz estimates in Proposition 2.3 play an essential role. Part (b) is essentially a corollary of (a), stating that the almost sure local existence and uniqueness still hold for more general initial data.

We note that the proofs of the analogues of Theorem 1.1 for NLS on \mathbb{R}^d and \mathbb{T}^d in [47, 5, 6] are more intricate. Lastly, notice that Theorem 1.1 is of a local-in-time nature, and hence it also holds for the energy-critical focusing NLW on \mathbb{R}^d , $d = 4$ and 5 . The same comment applies to the probabilistic small data global theory (Theorem 1.2 below).

Remark 1.2. While Theorem 1.1 does not yield the continuous dependence of the flow of equation (1.1) on \mathbb{R}^d , $d = 4$ and 5 , in $H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$ with $0 \leq s < 1$, we can modify the proof of Theorem 1.1 to obtain a mild form of continuous dependence. More precisely, we first fix “good” initial data (u_0^ω, u_1^ω) such that Theorem 1.1 yields the corresponding solution of equation (1.1) on the time interval $[-T, T]$ for some $T > 0$. Next, we consider $(\tilde{u}_0, \tilde{u}_1) \in H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$ such that $\|(\tilde{u}_0, \tilde{u}_1) - (u_0^\omega, u_1^\omega)\|_{H^1 \times L^2} \ll 1$.

Then, by modifying the proof of Theorem 1.1, we can construct a solution \tilde{u} of (1.1) on $[-cT, cT]$ with $(\tilde{u}, \partial_t \tilde{u})|_{t=0} = (\tilde{u}_0, \tilde{u}_1)$. Moreover, we have

$$\|(\tilde{u}(t), \partial_t \tilde{u}(t)) - (u^\omega(t), \partial_t u^\omega(t))\|_{H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)} \leq C \|(\tilde{u}_0, \tilde{u}_1) - (u_0^\omega, u_1^\omega)\|_{H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)}$$

for $|t| \leq cT$. See the works of Bourgain [8] and Colliander and Oh [23] for related discussions.

For $d = 4$, if we replace the smooth cutoff ψ in the definition of the Wiener randomization (1.13) by the characteristic function χ_{Q_0} , we obtain more. Namely, Theorem 1.4 below yields the probabilistic continuous dependence of the flow of equation (1.1) on \mathbb{R}^4 in $H^s(\mathbb{R}^4) \times H^{s-1}(\mathbb{R}^4)$, $0 < s \leq 1$.

We now turn to the global-in-time behavior of the local solutions constructed above. The same nonlinear estimates as in the proof of Theorem 1.1 together with the improved global-in-time Strichartz estimate (Lemma 2.5) yield the following small data global result.

Theorem 1.2 (Probabilistic small data global theory). *Let $d = 4$ or 5 and $\frac{(d+1)(d-2)}{(d-1)(d+2)} \leq s \leq 1$. Given $(u_0, u_1) \in \dot{H}^s(\mathbb{R}^d) \times \dot{H}^{s-1}(\mathbb{R}^d)$, let (u_0^ω, u_1^ω) be the randomization defined in (1.13), satisfying (1.15). Then there exist $C, c > 0$ such that for each $0 < \varepsilon \ll 1$ there exists a set Ω_ε with the following properties:*

- (i) $P(\Omega_\varepsilon^c) \leq C \exp\left(-\frac{c}{\varepsilon^2(\|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}})^2}\right) \rightarrow 0$ as $\varepsilon \rightarrow 0$.
- (ii) For each $\omega \in \Omega_\varepsilon$, there exists a unique global solution u^ω of the energy-critical defocusing NLW (1.1) on \mathbb{R}^d with initial data

$$(u^\omega, \partial_t u^\omega)|_{t=0} = (\varepsilon u_0^\omega, \varepsilon u_1^\omega)$$

such that

$$(u^\omega, \partial_t u^\omega) \in (S(t)(\varepsilon u_0^\omega, \varepsilon u_1^\omega), \partial_t S(t)(\varepsilon u_0^\omega, \varepsilon u_1^\omega)) + C(\mathbb{R}; \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)).$$

- (iii) Scattering holds for each $\omega \in \Omega_\varepsilon$. More precisely, for each $\omega \in \Omega_\varepsilon$, there exists a pair $(v_{0,\pm}^\omega, v_{1,\pm}^\omega) \in \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ such that the norm

$$\|(u^\omega(t), \partial_t u^\omega(t)) - (S(t)(\varepsilon u_0^\omega + v_{0,\pm}^\omega, \varepsilon u_1^\omega + v_{1,\pm}^\omega), \partial_t S(t)(\varepsilon u_0^\omega + v_{0,\pm}^\omega, \varepsilon u_1^\omega + v_{1,\pm}^\omega))\|_{\dot{H}^1 \times L^2}$$

tends to zero as $t \rightarrow \pm\infty$.

We refer to [42, 6] for analogous probabilistic small data global results in the context of the quintic NLW on \mathbb{R}^3 and the cubic NLS on \mathbb{R}^d , $d \geq 3$, respectively.

Remark 1.3. In proving scattering in Theorem 1.2, we exploit the finiteness of the global-in-time Strichartz $L_t^{\frac{d+2}{d-2}}(\mathbb{R}; L_x^{\frac{2(d+2)}{d-2}})$ -norm of the nonlinear part v^ω of a solution. This space-time norm is finite almost surely due to the smallness of the initial data. In

the case of *large data*, the best global space-time bound one could expect (guided by the literature on the deterministic energy-critical NLW [3, 4, 62]) is

$$\|v\|_{L_t^{\frac{d+2}{d-2}}(\mathbb{R}; L_x^{\frac{2(d+2)}{d-2}}(\mathbb{R}^d))} \leq C(\|(v, \partial_t v)\|_{L_t^\infty(\mathbb{R}; \dot{H}_x^1(\mathbb{R}^d) \times L_x^2(\mathbb{R}^d))}), \tag{1.17}$$

where $C(\cdot)$ is a positive nondecreasing function. For large data, however, we do not have a uniform-in-time bound on $\|(v(t), \partial_t v(t))\|_{\dot{H}^1 \times L^2}$. More precisely, the probabilistic energy bound in Proposition 5.2 below grows in time, diverging as $t \rightarrow \infty$. Therefore, even if one could prove (1.17), we would still not have sufficient information on v for almost sure large data scattering.

Before stating the main result of this paper, we first recall the definition of the set \mathcal{M}_s of measures on $H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$ from [19].

Definition 1.1. Let $0 \leq s \leq 1$ and $(u_0, u_1) \in H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$. The map $\omega \mapsto (u_0^\omega, u_1^\omega)$ is measurable from (Ω, \mathcal{F}, P) into $H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$ endowed with the Borel σ -algebra, since its partial sums

$$\left(\sum_{|n| \leq N} g_{n,0}(\omega) \psi(D-n)u_0, \sum_{|n| \leq N} g_{n,1}(\omega) \psi(D-n)u_1 \right)$$

form a Cauchy sequence in $L^2(\Omega; H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d))$. Therefore, we define the induced probability measure

$$\mu_{(u_0, u_1)}(A) := P((u_0^\omega, u_1^\omega) \in A)$$

for all measurable $A \subset H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$. We then set

$$\mathcal{M}_s := \bigcup_{(u_0, u_1) \in H^s \times H^{s-1}} \{\mu_{(u_0, u_1)}\}.$$

For any $(u_0, u_1) \in H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$, the measure $\mu_{(u_0, u_1)}$ is supported on $H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$, and moreover $\mu_{(u_0, u_1)}(H^{s'}(\mathbb{R}^d) \times H^{s'-1}(\mathbb{R}^d)) = 0$ if $s' > s$ and $(u_0, u_1) \notin H^{s'}(\mathbb{R}^d) \times H^{s'-1}(\mathbb{R}^d)$. This is another way of saying that the randomization is not regularizing in terms of differentiability.

We are now ready to state the main result of this paper.

Theorem 1.3 (Almost sure global well-posedness). *Let $d = 4$ or 5 , $0 < s \leq 1$ if $d = 4$, and $0 \leq s \leq 1$ if $d = 5$. There exists a set $\Sigma \subset H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$ of full measure, i.e. $\mu(\Sigma) = 1$ for all $\mu \in \mathcal{M}_s$, such that for any $(\phi_0, \phi_1) \in \Sigma$, the energy-critical defocusing NLW (1.1) on \mathbb{R}^d with initial data $(u, \partial_t u)|_{t=0} = (\phi_0, \phi_1)$ admits a unique global solution with*

$$\begin{aligned} (u, \partial_t u) &\in (S(t)(\phi_0, \phi_1), \partial_t S(t)(\phi_0, \phi_1)) + C(\mathbb{R}; H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)) \\ &\subset C(\mathbb{R}; H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)). \end{aligned}$$

Moreover, if $\Phi(t) : (\phi_0, \phi_1) \mapsto u(t)$ denotes the solution map of equation (1.1), then $\Phi(t)(\Sigma)$ is a set of full measure for all $t \in \mathbb{R}$.

The proof of Theorem 1.3 is based on a *probabilistic perturbation theory*. More precisely, we combine the global space-time bounds of solutions of the energy-critical defocusing NLW from [3, 62] with a perturbation lemma, and design a “good” deterministic local well-posedness theory (Proposition 4.3) for a perturbed NLW of the form

$$\begin{cases} \partial_t^2 v - \Delta v + F(v + f) = 0, \\ (v, \partial_t v)|_{t=0} = (v_0, v_1) \in \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d), \end{cases} \tag{1.18}$$

where f satisfies some smallness assumption on small time intervals. Recall that the usual local well-posedness argument for the energy-critical NLW (1.1) with initial data $(v_0, v_1) \in \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ yields a local time of existence depending on the profile of (v_0, v_1) . The term “good” local well-posedness refers to the fact that the local time of existence depends only on the $\dot{H}^1 \times L^2$ -norm of the initial data (v_0, v_1) and on the perturbation f .

Given randomized initial data $(u_0^\omega, u_1^\omega) \in H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$, with $0 < s < 1$ if $d = 4$, and $0 \leq s < 1$ if $d = 5$, let u^ω be the corresponding solution of equation (1.1). Also, we denote the linear and nonlinear parts of u^ω by $z^\omega(t) = S(t)(u_0^\omega, u_1^\omega)$ and $v^\omega(t) := u^\omega(t) - z^\omega(t)$ as before. Then a crucial ingredient of the proof of Theorem 1.3 is a probabilistic energy bound for v^ω (Proposition 5.2). The bound follows from the improved local-in-time Strichartz estimates (Proposition 2.3) and (nonlinear) Gronwall’s inequality. These Strichartz estimates are also the key to showing that z^ω satisfies almost surely the smallness assumption in the “good” local well-posedness theory. Finally, noting that v^ω satisfies (1.18) with $f = z^\omega$, we deduce the almost sure global existence of v^ω by iterating the “good” local well-posedness. We remark that the nonlinear part v^ω satisfies $(v^\omega(t), \partial_t v^\omega(t)) \in (H^1(\mathbb{R}^d), L^2(\mathbb{R}^d))$ for all $t \in \mathbb{R}$, and in particular it has improved regularity in comparison to the linear part z^ω , which is merely in $H^s(\mathbb{R}^d)$ with $0 < s < 1$ if $d = 4$, and $0 \leq s < 1$ if $d = 5$. In [6], the author with Bényi and Oh considered the same problem for the energy-critical cubic NLS on \mathbb{R}^4 . In this case, they could only prove “conditional” almost sure global well-posedness, assuming a probabilistic energy bound on the nonlinear part of a solution (see Remark 1.4 below).

Theorem 1.3 does not cover the case $s = 0$ in dimension $d = 4$. This is due to the use of the Sobolev embedding $W^{s,r}(\mathbb{R}^4) \subset L^\infty(\mathbb{R}^4)$ for $sr > 4$, which requires $s > 0$ (see Propositions 2.3(iii) and 5.2). For $d = 5$, one does not need to use such a Sobolev embedding, and thus we can include the case $s = 0$.

Notice that the full measure set Σ of initial data in Theorem 1.3 is constructed in such a way that $\Phi(t)(\Sigma)$ remains of full measure for all $t \in \mathbb{R}$, in other words, the measure does not become smaller under the evolution of the flow of (1.1). See [19, 52] for related results.

In the definitions of the uniform decomposition (1.9) and of the Wiener randomization (1.13), we use a smooth cutoff function ψ . Theorems 1.1–1.3 still hold even when we replace ψ by the characteristic function χ_{Q_0} .

We present the proof of Theorem 1.3 in Section 5. In particular, Theorem 1.3 is a consequence of Theorem 5.3, Corollary 5.5, and Proposition 5.6. In addition, in Proposition 5.4, we prove space-time bounds for the nonlinear part v^ω of the solution on any given

finite time interval $[0, T]$. Unfortunately, these bounds diverge as $T \rightarrow \infty$, and hence the present paper does not provide global space-time bounds for v^ω . As a consequence, our arguments are not sufficient to obtain scattering of the global solution u in Theorem 1.3 to a linear solution (which is standardly proved using global space-time bounds). Therefore, almost sure large data scattering remains a challenging open problem and new ideas seem to be needed to tackle it. (See also Remark 1.3 above.)

Our last result concerns the probabilistic continuous dependence of the flow of equation (1.1) on \mathbb{R}^4 . As mentioned above, Christ, Colliander, and Tao [21] proved ill-posedness of (1.1) in $H^s(\mathbb{R}^4) \times H^{s-1}(\mathbb{R}^4)$ with $0 < s < 1$ in the deterministic theory, by showing that the solution map is not continuous at zero. In the following, we show that the solution map is, however, continuous *in probability* in $H^s(\mathbb{R}^4) \times H^{s-1}(\mathbb{R}^4)$ with $0 < s < 1$. The notion of probabilistic continuous dependence of the flow on the initial data used here was first introduced by Burq and Tzvetkov [19].

Theorem 1.4 (Probabilistic continuous dependence). *Let $d = 4$ and $0 < s \leq 1$. Assume that in the definition of the randomization (1.13), the smooth cutoff ψ is replaced by the characteristic function χ_{Q_0} of the unit cube $Q_0 = [-1/2, 1/2]^4$ centered at the origin. Assume also that the probability distributions $\mu_{0,j}, \mu_{n,j}^{(1)}, \mu_{n,j}^{(2)}, n \in \mathcal{I}, j = 0, 1$, are symmetric.⁵ Let $T > 0, \mu \in \mathcal{M}_s, R > 0$, and*

$$B_R := \{(w_0, w_1) \in H^s(\mathbb{R}^4) \times H^{s-1}(\mathbb{R}^4) : \|(w_0, w_1)\|_{H^s(\mathbb{R}^4) \times H^{s-1}(\mathbb{R}^4)} \leq R\}$$

be the closed ball of radius R centered at the origin in $H^s(\mathbb{R}^4) \times H^{s-1}(\mathbb{R}^4)$. If $\Phi(t) : (\phi_0, \phi_1) \mapsto u(t)$ is the solution map of the energy-critical defocusing cubic NLW (1.1) on \mathbb{R}^4 , defined μ -almost everywhere in Theorem 1.3, then for any $\delta, \eta > 0$,

$$\begin{aligned} &\mu \otimes \mu \left(((w_0, w_1), (w'_0, w'_1)) \in (H^s(\mathbb{R}^4) \times H^{s-1}(\mathbb{R}^4))^2 : \right. \\ &\|(\Phi(t)(w_0, w_1) - \Phi(t)(w'_0, w'_1)), \\ &\quad \left. \partial_t(\Phi(t)(w_0, w_1) - \Phi(t)(w'_0, w'_1))\right\|_{L^\infty([0,T]; H^s(\mathbb{R}^4) \times H^{s-1}(\mathbb{R}^4))} > \delta \mid \\ &\quad (w_0, w_1), (w'_0, w'_1) \in B_R \text{ and } \|(w_0, w_1) - (w'_0, w'_1)\|_{H^s(\mathbb{R}^4) \times H^{s-1}(\mathbb{R}^4)} < \eta \\ &\quad \leq g(\delta, \eta), \quad (1.19) \end{aligned}$$

where

$$\lim_{\eta \rightarrow 0} g(\delta, \eta) = 0 \quad \text{for all } \delta > 0.$$

In stating Theorem 1.4, we assume that the randomization (1.13) is implemented with the characteristic function χ_{Q_0} , instead of the smooth cutoff function ψ . In particular, the support of $\chi_{Q_0}(\cdot - n)$ and that of $\chi_{Q_0}(\cdot - m)$, namely the cubes $n + Q_0$ and $m + Q_0$, are disjoint for $n, m \in \mathbb{Z}^4, n \neq m$. This plays an important role in the proof. We believe that a statement analogous to Theorem 1.4 is true for a general smooth cutoff function ψ . Unfortunately, we do not know how to prove such a claim at this point.

⁵ A probability measure θ on \mathbb{R} is called *symmetric* if $\int_{\mathbb{R}} f(x) d\theta(x) = \int_{\mathbb{R}} f(-x) d\theta(x)$ for all $f \in L^1(d\theta)$.

One of the several key points in the proof of Theorem 1.4 is that the power of the nonlinearity needs to be at least three (see Step 3 of the proof, where we use the Strichartz $L_t^2 L_x^8(\mathbb{R}^4)$ norm). Therefore, our method does not apply to the energy-critical defocusing NLW on \mathbb{R}^5 , for which the power of the nonlinearity is $7/3$.

In order to prove Theorem 1.4, we first control the linear parts of the solutions $\Phi(t)(w_0, w_1)$ and $\Phi(t)(w'_0, w'_1)$, as well as their difference. The key element here is the improved local-in-time Strichartz estimates, while the context in which they are used is analogous to that in [19]. The novel element of the proof is the control of the Strichartz norms of the nonlinear parts of the solutions, which allows us to control the difference of these nonlinear parts. In [19], in the case of the energy-subcritical cubic NLW on \mathbb{T}^3 , a mere probabilistic energy bound was sufficient to control that difference.

Remark 1.4. In the spirit of [6], we can also prove “conditional” almost sure global well-posedness for the energy-critical defocusing quintic NLW on \mathbb{R}^3 , provided we assume the following Hypothesis,⁶ i.e. a probabilistic energy bound on the nonlinear part v^ω of the solution:

Hypothesis. Given $T, \varepsilon > 0$, there exists $C(T, \varepsilon)$ nondecreasing in T and nonincreasing in ε and there exists $\tilde{\Omega}_{T,\varepsilon}$ with $P(\tilde{\Omega}_{T,\varepsilon}^c) < \varepsilon$ such that, for all $\omega \in \tilde{\Omega}_{T,\varepsilon}$, the solution v^ω of (1.5) satisfies

$$\|v^\omega(t)\|_{L^\infty([0,T]; \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3))} \leq C(T, \varepsilon).$$

Furthermore, under the Hypothesis, one can also prove probabilistic continuous dependence of the flow of the energy-critical defocusing quintic NLW on \mathbb{R}^3 . The proofs of these results follow the same lines as those of Theorems 1.3 and 1.4.

1.4. Probabilistic well-posedness results regarding NLW

To the best on the author’s knowledge, Theorem 1.3 is the first result on almost sure global well-posedness for an *energy-critical* hyperbolic/dispersive PDE with large data below the energy space. In the following, we briefly mention some of the references in the literature regarding almost sure global well-posedness of NLW. All the results below concern defocusing NLW and we do not explicitly mention this in the following.

In what concerns NLW on \mathbb{T}^d , Burq and Tzvetkov [19] proved probabilistic global well-posedness for (energy-subcritical) cubic NLW on \mathbb{T}^3 for data in $H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$, $0 \leq s < 1$, while Burq, Thomann, and Tzvetkov [15] considered (energy-critical and supercritical) cubic NLW on \mathbb{T}^d , $d \geq 4$, and proved almost sure global existence, without uniqueness, for data in $H^s(\mathbb{T}^d) \times H^{s-1}(\mathbb{T}^d)$, $0 < s < 1$.

⁶ Very recently, the author with Tadahiro Oh [51] proved the almost sure global well-posedness of the energy-critical defocusing quintic nonlinear wave equation on \mathbb{R}^3 below the energy space. The main new ingredient in the proof is a uniform probabilistic energy bound for approximating random solutions. In particular, we proved almost sure global well-posedness without directly establishing the Hypothesis. Nevertheless, the Hypothesis follows as a byproduct of the construction in [51].

Burq and Tzvetkov [16] also considered subcubic NLW on the unit ball in \mathbb{R}^3 with the power nonlinearity $|u|^{p-1}u$, $1 < p < 3$, and proved almost sure global well-posedness of NLW for a large set of radially symmetric data in $\bigcap_{s < 1/2} H^s$. More precisely, they constructed the Gibbs measure for NLW on the unit ball and the global-in-time flow on the support of the Gibbs measure, proving also the invariance of the Gibbs measure under the flow. The almost sure global well-posedness part was extended by the same authors to $1 < p < 4$ in [18], and the invariance of the measure for $p = 3$ was proved by de Suzzoni [25]. Finally, Bourgain and Bulut [11, 12] extended the above results to all $1 < p < 5$. Notice that for $3 \leq p < 5$, the above almost sure global existence results are below the scaling critical regularity.

Let us now turn our attention to known results on \mathbb{R}^d . De Suzzoni [26] considered subquartic NLW on \mathbb{R}^3 , namely $3 \leq p < 4$, and proved global existence, uniqueness, and scattering for a set of full measure of radially symmetric data of low regularity, that do not belong to $L^2(\mathbb{R}^3)$. In [27], de Suzzoni proved an almost sure global existence, uniqueness, and scattering result for cubic NLW on \mathbb{R}^3 , without the radial symmetry assumption. The Penrose transform played an essential role in both articles. Recently, Lührmann and Mendelson [42] proved almost sure global well-posedness of energy-subcritical NLW on \mathbb{R}^3 with a power nonlinearity $|u|^{p-1}u$, $3 \leq p < 5$, with random initial data in $H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$, for some $s < 1$. For $p \in ((7 + \sqrt{73})/4, 5)$, the regularity of initial data can be taken below the critical regularity dictated by the scaling invariance. For the energy-critical NLW on \mathbb{R}^3 ($p = 5$), they obtained small data almost sure global well-posedness and scattering.

Finally, there are other classes of almost sure global well-posedness results on \mathbb{R}^d and other unbounded domains. They involve the construction of invariant Gibbs measures on such domains. In particular, on \mathbb{R}^d , the typical functions in the support of Gibbs measures do not decay at spatial infinity, and thus do not belong to the Lebesgue spaces $L^p(\mathbb{R}^d)$ or Sobolev spaces $H^s(\mathbb{R}^d)$. See, for example, the work of McKean and Vaninsky [43] on \mathbb{R} and the recent work of Xu [67] concerning cubic NLW on \mathbb{R}^3 with radial symmetry.

1.5. Notation

If u satisfies the wave equation

$$\partial_t^2 u - \Delta u + F(u) = 0$$

on the interval I containing t_0 and t , then the following Duhamel's formula holds:

$$u(t) = S(t - t_0)(u(t_0), \partial_t u(t_0)) - \int_{t_0}^t \frac{\sin((t - t')|\nabla|)}{|\nabla|} F(u(t')) dt'. \quad (1.20)$$

If $F(u) = |u|^{4/(d-2)}u$, we use the fundamental theorem of calculus to write

$$F(u) - F(\tilde{u}) = (u - \tilde{u}) \int_0^1 F'(\lambda u + (1 - \lambda)\tilde{u}) d\lambda,$$

and we immediately deduce that

$$|F(u) - F(\tilde{u})| \lesssim |u - \tilde{u}|(|u|^{4/(d-2)} + |\tilde{u}|^{4/(d-2)}). \quad (1.21)$$

Definition 1.2. Let $\gamma \in \mathbb{R}$ and $d \geq 2$. We say that (q, r) is an $\dot{H}^\gamma(\mathbb{R}^d)$ -wave admissible pair if $q \geq 2, 2 \leq r < \infty$,

$$\frac{1}{q} + \frac{d-1}{2r} \leq \frac{d-1}{4} \quad \text{and} \quad \frac{1}{q} + \frac{d}{r} = \frac{d}{2} - \gamma.$$

We recall some Strichartz estimates for wave equations on \mathbb{R}^d . For more details as well as other Strichartz estimates, see [29, 41, 36].

Proposition 1.5 (Strichartz estimates on \mathbb{R}^d). *Let $d \geq 2, \gamma > 0$, let (q, r) be an $\dot{H}^\gamma(\mathbb{R}^d)$ -wave admissible pair, and (\tilde{q}, \tilde{r}) be an $\dot{H}^{1-\gamma}(\mathbb{R}^d)$ -wave admissible pair. If u solves*

$$\partial_t^2 u - \Delta u + F = 0 \quad \text{with} \quad u(0) = u_0, \partial_t u(0) = u_1$$

on $I \times \mathbb{R}^d$, where $0 \in I$, then

$$\begin{aligned} \|u\|_{L_t^\infty(I; \dot{H}_x^\gamma)} + \|\partial_t u\|_{L_t^\infty(I; \dot{H}_x^{\gamma-1})} + \|u\|_{L_t^q(I; L_x^r)} \\ \lesssim \|u_0\|_{\dot{H}^\gamma} + \|u_1\|_{\dot{H}^{\gamma-1}} + \|F\|_{L_t^{\tilde{q}'}(I; L_x^{\tilde{r}'})}. \end{aligned} \quad (1.22)$$

For convenience, we often denote the space $L_t^q(I; L_x^r)$ by $L_I^q L_x^r$, or by $L_T^q L_x^r$ if $I = [0, T]$.

The wave admissible pairs for $d = 4$ and 5 that appear most often in this paper are:

- \dot{H}^1 -wave admissible: $(\frac{d+2}{d-2}, \frac{2(d+2)}{d-2})$ if $d = 4$ or 5 , and $(2, 8)$ if $d = 4$,
- \dot{H}^0 -wave admissible: $(\infty, 2)$.

In particular, the Strichartz space $L_t^{\frac{d+2}{d-2}}(I; L_x^{\frac{2(d+2)}{d-2}}(\mathbb{R}^d))$ will appear very often in our analysis, and therefore we fix the notation:

$$\|f\|_{X(I \times \mathbb{R}^d)} := \|f\|_{L_t^{\frac{d+2}{d-2}}(I; L_x^{\frac{2(d+2)}{d-2}}(\mathbb{R}^d))},$$

where $I \subset \mathbb{R}$ denotes a time interval.

This paper is organized as follows. In Section 2, we prove probabilistic Strichartz estimates. Section 3 is dedicated to the proof of Theorems 1.1 and 1.2. In Section 4, we design a “good” deterministic local well-posedness theory using a perturbation lemma. Section 5 contains the proof of Theorem 1.3 (reformulated as Theorem 5.3, Corollary 5.5, and Proposition 5.6). Finally, in Section 6 we prove Theorem 1.4.

2. Probabilistic estimates

In this section we recall some basic properties of randomized functions and then present some improved Strichartz estimates under randomization.

First recall the following probabilistic estimate.

Lemma 2.1. *Let $\{g_n\}_{n \in \mathbb{Z}^d}$ be a sequence of complex-valued, mean zero, random variables such that $g_{-n} = \overline{g_n}$ for all $n \in \mathbb{Z}^d$. Assume that $g_0, \operatorname{Re} g_n, \operatorname{Im} g_n, n \in \mathcal{I} := \bigcup_{k=0}^{d-1} \mathbb{Z}^k \times \mathbb{Z}_+^* \times \{0\}^{d-k-1}$, are independent and have respective distributions $\mu_0, \mu_n^{(1)}, \mu_n^{(2)}$. Assume that there exists $c > 0$ such that*

$$\int_{\mathbb{R}} e^{\gamma x} d\mu_0 \leq e^{c\gamma^2} \quad \text{and} \quad \int_{\mathbb{R}} e^{\gamma x} d\mu_n^{(k)} \leq e^{c\gamma^2} \tag{2.1}$$

for all $\gamma \in \mathbb{R}, n \in \mathbb{Z}^d, k = 1, 2$. Then there exists $C > 0$ such that

$$\left\| \sum_{n \in \mathbb{Z}^d} g_n(\omega) c_n \right\|_{L^p(\Omega)} \leq C \sqrt{p} \|c_n\|_{\ell_n^2(\mathbb{Z}^d)} \tag{2.2}$$

for any $p \geq 2$ and any sequence $\{c_n\}_{n \in \mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$ satisfying $c_{-n} = \overline{c_n}$ for all $n \in \mathbb{Z}^d$.

Proof. As in (1.14), we notice that

$$\sum_{n \in \mathbb{Z}^d} g_n c_n = g_0 c_0 + 2 \operatorname{Re} \sum_{n \in \mathcal{I}} g_n c_n = g_0 c_0 + 2 \sum_{n \in \mathcal{I}} (\operatorname{Re} g_n \operatorname{Re} c_n - \operatorname{Im} g_n \operatorname{Im} c_n).$$

Since $g_0 = \operatorname{Re} g_0, \operatorname{Re} g_n, \operatorname{Im} g_n$, with $n \in \mathcal{I}$, are mean zero, independent real random variables and their distributions satisfy (2.1), it is sufficient to apply [17, Lemma 3.1] to obtain the conclusion. \square

Secondly, recall that if $\phi \in H^s$, then the randomization ϕ^ω is in H^s almost surely as long as (2.1) is satisfied. More precisely, we have the following lemma, whose proof is based on Lemma 2.1. See [5] for the details.

Lemma 2.2. *Let $\{g_n\}_{n \in \mathbb{Z}^d}$ be a sequence of random variables satisfying the hypotheses of Lemma 2.1. Let $\phi \in H^s(\mathbb{R}^d)$ be a real-valued function and let ϕ^ω be its real-valued randomization defined by*

$$\phi^\omega := \sum_{n \in \mathbb{Z}^d} g_n(\omega) \psi(D - n)\phi.$$

Then

$$P(\|\phi^\omega\|_{H_x^s(\mathbb{R}^d)} > \lambda) \leq C \exp(-c\lambda^2 / \|\phi\|_{H^s}^2).$$

Before continuing further, we briefly recall the definitions of the smooth projections from Littlewood–Paley theory. Let φ be a smooth real-valued bump function supported on $\{\xi \in \mathbb{R}^d : |\xi| \leq 2\}$ and $\varphi \equiv 1$ on $\{\xi : |\xi| \leq 1\}$. If $N > 1$ is a dyadic number, we define the smooth projection $\mathbf{P}_{\leq N}$ onto frequencies $\{|\xi| \leq N\}$ by

$$\widehat{\mathbf{P}_{\leq N} f}(\xi) := \varphi(\xi/N) \widehat{f}(\xi).$$

Similarly, we can define the smooth projection \mathbf{P}_N onto frequencies $\{|\xi| \sim N\}$ by

$$\widehat{\mathbf{P}_N f}(\xi) := (\varphi(\xi/N) - \varphi(2\xi/N)) \widehat{f}(\xi).$$

We make the convention that $\mathbf{P}_{\leq 1} = \mathbf{P}_1$. Bernstein's inequality states that

$$\|\mathbf{P}_{\leq N} f\|_{L^q(\mathbb{R}^d)} \lesssim N^{d/p-d/q} \|\mathbf{P}_{\leq N} f\|_{L^p(\mathbb{R}^d)}, \quad 1 \leq p \leq q \leq \infty. \quad (2.3)$$

The same inequality holds if we replace $\mathbf{P}_{\leq N}$ by \mathbf{P}_N . As an immediate corollary, we have

$$\|\psi(D - n)\phi\|_{L^q(\mathbb{R}^d)} \lesssim \|\psi(D - n)\phi\|_{L^p(\mathbb{R}^d)}, \quad 1 \leq p \leq q \leq \infty, \quad (2.4)$$

for all $n \in \mathbb{Z}^d$. This follows by applying (2.3) to $\phi_n(x) := e^{2\pi i n \cdot x} \psi(D - n)\phi(x)$ and noting that $\text{supp } \widehat{\phi}_n \subset [-1, 1]^d$. The point of (2.4) is that once a function is (roughly) restricted to a cube in the Fourier space, we do not lose any regularity to go from the L^q -norm to the L^p -norm, $q \geq p$.

Proposition 2.3 (Improved local-in-time Strichartz estimates). *Let $d \geq 1$, let u_0, u_1 be two real-valued functions defined on \mathbb{R}^d , and let (u_0^ω, u_1^ω) be their randomization defined in (1.13), satisfying (1.15). Let $I = [a, b] \subset \mathbb{R}$ be a compact time interval.*

- (i) *If $u_0 \in L^2(\mathbb{R}^d)$ and $u_1 \in \dot{H}^{-1}(\mathbb{R}^d)$, then given $1 \leq q < \infty$ and $2 \leq r < \infty$, there exist $C, c > 0$ such that*

$$P(\|S(t)(u_0^\omega, u_1^\omega)\|_{L_t^q(I; L_x^r)} > \lambda) \leq C \exp\left(-c \frac{\lambda^2}{|I|^{2/q} (\|u_0\|_{L^2} + \|u_1\|_{\dot{H}^{-1}})^2}\right).$$

- (ii) *If $u_0 \in L^2(\mathbb{R}^d)$ and $u_1 \in H^{-1}(\mathbb{R}^d)$, then given $1 \leq q < \infty$, $2 \leq r < \infty$, there exist $C, c > 0$ such that*

$$\begin{aligned} &P(\|S(t)(u_0^\omega, u_1^\omega)\|_{L_t^q(I; L_x^r)} > \lambda) \\ &\leq C \exp\left(-c \frac{\lambda^2}{\max(1, |a|^2, |b|^2) |I|^{2/q} (\|u_0\|_{L^2} + \|u_1\|_{H^{-1}})^2}\right). \end{aligned}$$

- (iii) *If $u_0 \in H^s(\mathbb{R}^d)$ and $u_1 \in H^{s-1}(\mathbb{R}^d)$ for some $0 < s \leq 1$, then given $1 \leq q < \infty$, there exist $C, c > 0$ such that*

$$\begin{aligned} &P(\|S(t)(u_0^\omega, u_1^\omega)\|_{L_t^q(I; L_x^\infty)} > \lambda) \\ &\leq C \exp\left(-c \frac{\lambda^2}{\max(1, |a|^2, |b|^2) |I|^{2/q} (\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}})^2}\right). \end{aligned}$$

Proof. (i). Let $1 \leq q < \infty$, $2 \leq r < \infty$, and $p \geq \max(q, r)$. Then, using Minkowski’s integral inequality, (2.2), and Bernstein’s inequality (2.4), we obtain

$$\begin{aligned} (\mathbb{E} \|S(t)(u_0^\omega, u_1^\omega)\|_{L_t^q(I; L_x^r)}^p)^{1/p} &\leq \|S(t)(u_0^\omega, u_1^\omega)\|_{L^p(\Omega)} \|L_t^q L_x^r\| \\ &\lesssim \sqrt{p} \|\psi(D-n) \cos(t|\nabla|) u_0\|_{\ell_n^2} \|L_t^q L_x^r\| + \sqrt{p} \left\| \left\| \psi(D-n) \frac{\sin(t|\nabla|)}{|\nabla|} u_1 \right\|_{\ell_n^2} \right\|_{L_t^q L_x^r} \\ &\lesssim \sqrt{p} \|\psi(D-n) \cos(t|\nabla|) u_0\|_{L_x^r} \|L_t^q \ell_n^2\| + \sqrt{p} \left\| \left\| \psi(D-n) \frac{\sin(t|\nabla|)}{|\nabla|} u_1 \right\|_{L_x^r} \right\|_{L_t^q \ell_n^2} \\ &\lesssim \sqrt{p} \|\psi(D-n) \cos(t|\nabla|) u_0\|_{L_x^2} \|L_t^q \ell_n^2\| + \sqrt{p} \left\| \left\| \psi(D-n) \frac{\sin(t|\nabla|)}{|\nabla|} u_1 \right\|_{L_x^2} \right\|_{L_t^q \ell_n^2} \\ &\lesssim \sqrt{p} \|\psi(D-n) u_0\|_{L_x^2} \|L_t^q \ell_n^2\| + \sqrt{p} \|\psi(D-n) |\nabla|^{-1} u_1\|_{L_x^2} \|L_t^q \ell_n^2\| \\ &\lesssim \sqrt{p} |I|^{1/q} (\|u_0\|_{L^2} + \|u_1\|_{\dot{H}^{-1}}). \end{aligned}$$

Then by Chebyshev’s inequality we have

$$P(\|S(t)(u_0^\omega, u_1^\omega)\|_{L_t^q(I; L_x^r)} > \lambda) < \left(\frac{C|I|^{1/q} p^{1/2} (\|u_0\|_{L^2} + \|u_1\|_{\dot{H}^{-1}})}{\lambda} \right)^p$$

for $p \geq \max(q, r)$.

Let $p := \left(\frac{\lambda}{C|I|^{1/q} e(\|u_0\|_{L^2} + \|u_1\|_{\dot{H}^{-1}})} \right)^2$. If $p \geq \max(q, r)$, we have

$$\begin{aligned} P(\|S(t)(u_0^\omega, u_1^\omega)\|_{L_t^q(I; L_x^r)} > \lambda) &< \left(\frac{C|I|^{1/q} p^{1/2} (\|u_0\|_{L^2} + \|u_1\|_{\dot{H}^{-1}})}{\lambda} \right)^p \\ &= e^{-p} = \exp\left(-c \frac{\lambda^2}{|I|^{2/q} (\|u_0\|_{L^2} + \|u_1\|_{\dot{H}^{-1}})^2}\right). \end{aligned}$$

Otherwise, if $p = \left(\frac{\lambda}{C|I|^{1/q} e(\|u_0\|_{L^2} + \|u_1\|_{\dot{H}^{-1}})} \right)^2 \leq \max(q, r)$, we choose C such that $Ce^{-\max(q,r)} \geq 1$. We then have

$$\begin{aligned} P(\|S(t)(u_0^\omega, u_1^\omega)\|_{L_t^q(I; L_x^r)} > \lambda) &\leq 1 \leq Ce^{-\max(q,r)} \leq Ce^{-p} \\ &= C \exp\left(-c \frac{\lambda^2}{|I|^{2/q} (\|u_0\|_{L^2} + \|u_1\|_{\dot{H}^{-1}})^2}\right). \end{aligned}$$

(ii) We pay a particular attention to low frequencies since, for $u_1 \in H^{-1}(\mathbb{R}^d)$ and $n = 0$, $\psi(D-n)|\nabla|^{-1}u_1$ is not in $L^2(\mathbb{R}^d)$. For $|n| \geq 1$, we argue as in (i). Using Minkowski’s

integral inequality, (2.2), Bernstein’s inequality (2.4), and the fact that $|\frac{\sin(t|\xi|)}{t|\xi|}| \leq 1$ for all $\xi \neq 0$, we deduce for $p \geq \max(q, r)$ that

$$\begin{aligned} & \left(\mathbb{E} \left\| \frac{\sin(t|\nabla|)}{|\nabla|} u_1^\omega \right\|_{L_t^q(I; L_x^r)}^p \right)^{1/p} \leq \left\| \frac{\sin(t|\nabla|)}{|\nabla|} u_1^\omega \right\|_{L^p(\Omega)} \left\| \frac{\sin(t|\nabla|)}{|\nabla|} u_1^\omega \right\|_{L_t^q L_x^r} \\ & \lesssim \sqrt{p} \left\| \left\| \psi(D-n) \frac{\sin(t|\nabla|)}{|\nabla|} u_1 \right\|_{\ell_n^2} \right\|_{L_t^q L_x^r} \lesssim \sqrt{p} \left\| \left\| \psi(D-n) \frac{\sin(t|\nabla|)}{|\nabla|} u_1 \right\|_{L_x^r} \right\|_{L_t^q \ell_n^2} \\ & \lesssim \sqrt{p} \left\| \left\| \psi(D-n) \frac{\sin(t|\nabla|)}{|\nabla|} u_1 \right\|_{L_x^2} \right\|_{L_t^q \ell_n^2} \\ & \lesssim \sqrt{p} |I|^{1/q} \left(\sum_{|n| \geq 1} \|\psi(D-n)|\nabla|^{-1} u_1\|_{L_x^2}^2 + \sup_{t \in I} \left\| \psi(D) \frac{\sin(t|\nabla|)}{t|\nabla|} t u_1 \right\|_{L_x^2}^2 \right)^{1/2} \\ & \lesssim \sqrt{p} |I|^{1/q} \left(\sum_{|n| \geq 1} \|\psi(D-n)|\nabla|^{-1} u_1\|_{L_x^2}^2 + \max(|a|^2, |b|^2) \|\psi(D) u_1\|_{L_x^2}^2 \right)^{1/2} \\ & \lesssim \sqrt{p} \max(1, |a|, |b|) |I|^{1/q} \|u_1\|_{H^{-1}}. \end{aligned}$$

Arguing as in part (i) for $\cos(t|\nabla|)u_0^\omega$ yields the conclusion of (ii).

(iii) Take $r \gg 1$ such that $sr > d$. Then $W^{s,r}(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$, and thus

$$\|S(t)(u_0^\omega, u_1^\omega)\|_{L_t^q(I; L_x^\infty)} \lesssim \|S(t)(\langle \nabla \rangle^s u_0^\omega, \langle \nabla \rangle^s u_1^\omega)\|_{L_t^q(I; L_x^r)}.$$

Applying (ii) with (q, r) gives the conclusion of (iii). □

Corollary 2.4. *Let $d \geq 1$, $u_0 \in L^2(\mathbb{R}^d)$, $u_1 \in H^{-1}(\mathbb{R}^d)$, and (u_0^ω, u_1^ω) their randomization defined in (1.13), satisfying (1.15). Then, given $1 \leq q < \infty$, $2 \leq r < \infty$, $0 < \gamma < 1/q$, and $I \subset [a, b]$ a compact time interval, we have*

$$\|S(t)(u_0^\omega, u_1^\omega)\|_{L_t^q(I; L_x^r)} \leq |I|^\gamma (\|u_0\|_{L^2} + \|u_1\|_{H^{-1}}) \tag{2.5}$$

outside a set of probability at most $C \exp(-\frac{c|I|^{2(\gamma-1/q)}}{\max(1, |a|^2, |b|^2)})$.

Proof. The conclusion is obtained by taking $\lambda = |I|^\gamma (\|u_0\|_{L^2} + \|u_1\|_{H^{-1}})$ in Proposition 2.3(ii). □

We conclude this section with some improved global-in-time Strichartz estimates.

Proposition 2.5 (Improved global-in-time Strichartz estimates). *Let $d = 4$ or 5 and $\frac{(d+1)(d-2)}{(d-1)(d+2)} \leq s \leq 1$. Let $u_0 \in \dot{H}^s(\mathbb{R}^d)$ and $u_1 \in \dot{H}^{s-1}(\mathbb{R}^d)$, and let (u_0^ω, u_1^ω) be their randomization defined in (1.13), satisfying (1.15). Then, given $\frac{2(d+2)}{d-2} \leq r < \infty$, there exist $C, c > 0$ such that*

$$P\left(\|S(t)(u_0^\omega, u_1^\omega)\|_{L_t^{\frac{d+2}{d-2}}(\mathbb{R}; L_x^r(\mathbb{R}^d))} > \lambda\right) \leq C \exp\left(-c \frac{\lambda^2}{(\|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}})^2}\right).$$

Proof. As in the proof of Proposition 2.3(i), for $p \geq r$ we have

$$\begin{aligned} & \left(\mathbb{E} \|S(t)(u_0^\omega, u_1^\omega)\|_{L_t^{\frac{d+2}{d-2}} L_x^r(\mathbb{R}; L_x^r(\mathbb{R}^d))}^p \right)^{1/p} \leq \|S(t)(u_0^\omega, u_1^\omega)\|_{L^p(\Omega)} \|L_t^{\frac{d+2}{d-2}} L_x^r\| \\ & \lesssim \sqrt{p} \|\psi(D-n) \cos(t|\nabla|)u_0\|_{\ell_n^2} \|L_t^{\frac{d+2}{d-2}} L_x^r\| + \left\| \left\| \psi(D-n) \frac{\sin(t|\nabla|)}{|\nabla|} u_1 \right\|_{\ell_n^2} \right\|_{L_t^{\frac{d+2}{d-2}} L_x^r} \\ & \lesssim \sqrt{p} \|\psi(D-n) \cos(t|\nabla|)u_0\|_{L_t^{\frac{d+2}{d-2}} L_x^r} \|\ell_n^2\| + \sqrt{p} \left\| \left\| \psi(D-n) \frac{\sin(t|\nabla|)}{|\nabla|} u_1 \right\|_{L_t^{\frac{d+2}{d-2}} L_x^r} \right\|_{\ell_n^2}. \end{aligned}$$

Then, we choose $\tilde{r} \leq \frac{2(d+2)}{d-2}$ such that $(\frac{d+2}{d-2}, \tilde{r})$ is $\dot{H}^s(\mathbb{R}^d)$ -wave admissible. Note that the condition $\frac{d-2}{d+2} + \frac{d}{\tilde{r}} = \frac{d}{2} - s$ together with $\frac{(d+1)(d-2)}{(d-1)(d+2)} \leq s \leq 1$ yields

$$\frac{2(d-1)(d+2)}{d^2 - 3d + 6} \leq \tilde{r} \leq \frac{2(d+2)}{d-2}.$$

In turn, this shows that $\frac{d-2}{d+2} + \frac{d-1}{2\tilde{r}} \leq \frac{d-1}{4}$, and thus it is indeed possible to choose $(\frac{d+2}{d-2}, \tilde{r})$, with $\tilde{r} \leq \frac{2(d+2)}{d-2}$, to be $\dot{H}^s(\mathbb{R}^d)$ -wave admissible. Then, by Bernstein’s inequality (2.4) and Strichartz estimates (1.22), we have

$$\begin{aligned} & \left(\mathbb{E} \|S(t)(u_0^\omega, u_1^\omega)\|_{L_t^{\frac{d+2}{d-2}} L_x^r(\mathbb{R}; L_x^r(\mathbb{R}^d))}^p \right)^{1/p} \lesssim \sqrt{p} \|\psi(D-n) \cos(t|\nabla|)u_0\|_{L_t^{\frac{d+2}{d-2}} L_x^r} \|\ell_n^2\| \\ & \quad + \left\| \left\| \psi(D-n) \frac{\sin(t|\nabla|)}{|\nabla|} u_1 \right\|_{L_t^{\frac{d+2}{d-2}} L_x^{\tilde{r}}} \right\|_{\ell_n^2} \\ & \lesssim \sqrt{p} \|\psi(D-n)u_0\|_{\dot{H}^s} \|\ell_n^2\| + \|\psi(D-n)u_1\|_{\dot{H}^{s-1}} \|\ell_n^2\| \\ & \lesssim \sqrt{p} (\|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}}). \end{aligned}$$

The conclusion then follows as in the proof of Proposition 2.3(i). □

3. Almost sure local well-posedness. Probabilistic small data global theory

In this section we prove Theorem 1.1 concerning the local well-posedness of the energy-critical defocusing NLW (1.1) on \mathbb{R}^d , $d = 4$ and 5 , and Theorem 1.2 concerning the small data global theory.

Proof of Theorem 1.1. Since the linear part $z^\omega := S(t)(u_0^\omega, u_1^\omega)$ of the solution is well defined for all times, it suffices to prove almost sure local existence and uniqueness for equation (1.5) satisfied by the nonlinear part $v^\omega := u^\omega - z^\omega$. Also, by the time reversibility of (1.5), it is sufficient to work with $t \geq 0$.

Let $0 < T \leq 1$, to be fixed later, and $0 < \theta < \frac{d-2}{d+2}$. By the improved local-in-time Strichartz estimates (2.5), there exists a set Ω_T such that for all $\omega \in \Omega_T$ we have

$$\|z^\omega\|_{X([-T, T] \times \mathbb{R}^d)} \leq (2T)^\theta (\|u_0\|_{L^2} + \|u_1\|_{H^{-1}}) \tag{3.1}$$

and

$$P(\Omega_T^c) \leq C \exp(-cT^{2(\theta - \frac{d-2}{d+2})}) = C \exp(-c/T^\gamma),$$

with $\gamma = 2(\frac{d-2}{d+2} - \theta) > 0$.

By Duhamel's formula (1.20), v^ω is a solution of (1.5) if and only if

$$v^\omega(t) = - \int_0^t \frac{\sin((t-t')|\nabla|)}{|\nabla|} F(v^\omega + z^\omega)(t') dt'. \tag{3.2}$$

We define

$$\Gamma^\omega v(t) := - \int_0^t \frac{\sin((t-t')|\nabla|)}{|\nabla|} F(v + z^\omega)(t') dt'. \tag{3.3}$$

We prove that, for all $\omega \in \Omega_T$, Γ^ω is a contraction on the ball

$$B_a := \{v \in C([0, T]; \dot{H}^1(\mathbb{R}^d)) \cap X([0, T] \times \mathbb{R}^d) : \|v\|_{L_t^\infty([0, T]; \dot{H}_x^1(\mathbb{R}^d))} + \|v\|_{X([0, T] \times \mathbb{R}^d)} \leq a\},$$

with a to be chosen later. By the Banach fixed point theorem, this shows that the equation $\Gamma^\omega v = v$, and therefore (1.5), has a unique solution v^ω in B_a .

We first prove that Γ^ω maps B_a into itself. Using the Strichartz estimates (1.22) and (3.1), we deduce for $v \in B_a$ and $\omega \in \Omega_T$ that

$$\begin{aligned} \|\Gamma^\omega v\|_{L_t^\infty([0, T]; \dot{H}_x^1(\mathbb{R}^d))} + \|\Gamma^\omega v\|_{X([0, T] \times \mathbb{R}^d)} &\leq C \|F(v + z^\omega)\|_{L_T^1 L_x^2} \\ &\leq C_1 \|v\|_{X([0, T] \times \mathbb{R}^d)}^{\frac{d+2}{d-2}} + C_1 \|z^\omega\|_{X([0, T] \times \mathbb{R}^d)}^{\frac{d+2}{d-2}} \\ &\leq C_1 a^{\frac{d+2}{d-2}} + C_1 (2T)^{\frac{(d+2)\theta}{d-2}} (\|u_0\|_{L^2} + \|u_1\|_{H^{-1}})^{\frac{d+2}{d-2}}. \end{aligned} \tag{3.4}$$

Taking a such that $C_1 a^{4/(d-2)} < 1/2$ and T sufficiently small, we obtain

$$\|\Gamma^\omega v\|_{L_t^\infty([0, T]; \dot{H}_x^1(\mathbb{R}^d))} + \|\Gamma^\omega v\|_{X([0, T] \times \mathbb{R}^d)} \leq a,$$

and thus Γ^ω maps the ball B_a into itself for all $\omega \in \Omega_T$. Similarly, for $v_1, v_2 \in B_a$ and $\omega \in \Omega_T$ we have

$$\begin{aligned} &\|\Gamma^\omega v_1 - \Gamma^\omega v_2\|_{L_t^\infty([0, T]; \dot{H}_x^1(\mathbb{R}^d))} + \|\Gamma^\omega v_1 - \Gamma^\omega v_2\|_{X([0, T] \times \mathbb{R}^d)} \\ &\leq C_2 (\|v_1\|_{X([0, T] \times \mathbb{R}^d)}^{4/(d-2)} + \|v_2\|_{X([0, T] \times \mathbb{R}^d)}^{4/(d-2)} + (2T)^{4\theta/(d-2)} (\|u_0\|_{L^2} + \|u_1\|_{H^{-1}})^{4/(d-2)}) \\ &\quad \times \|v_1 - v_2\|_{X([0, T] \times \mathbb{R}^d)} \\ &\leq C_2 (2a^{4/(d-2)} + (2T)^{4\theta/(d-2)} (\|u_0\|_{L^2} + \|u_1\|_{H^{-1}})^{4/(d-2)}) \|v_1 - v_2\|_{X([0, T] \times \mathbb{R}^d)}. \end{aligned} \tag{3.5}$$

Taking a and T smaller if needed, we find that Γ^ω is a contraction on B_a for all $\omega \in \Omega_T$. This proves the existence of a unique solution v^ω of (1.5) in B_a .

Next, we show that $v^\omega \in L_t^\infty([0, T]; L_x^2(\mathbb{R}^d))$. By (3.2) and (2.5), we have

$$\begin{aligned} \|v^\omega(t)\|_{L_x^2(\mathbb{R}^d)} &\leq \int_0^t \left\| \frac{\sin((t-t')|\nabla|)}{|\nabla|} F(v^\omega + z^\omega)(t') \right\|_{L_x^2} dt' \\ &\leq \int_0^t (t-t') \|F(v^\omega + z^\omega)(t')\|_{L_x^2} dt' \\ &\leq C_3 T (\|v^\omega\|_{X([0,T]\times\mathbb{R}^d)}^{\frac{d+2}{d-2}} + \|z^\omega\|_{X([0,T]\times\mathbb{R}^d)}^{\frac{d+2}{d-2}}) \\ &\leq C_3 T (a^{\frac{d+2}{d-2}} + (2T)^{\frac{(d+2)\theta}{d-2}} (\|u_0\|_{L^2} + \|u_1\|_{H^{-1}})^{\frac{d+2}{d-2}}) \leq Ta \end{aligned}$$

for all $\omega \in \Omega_T$ if $C_3 a^{4/(d-2)} \leq 1/2$ and T is sufficiently small.

Finally, we prove that $\partial_t v^\omega \in L_t^\infty([0, T]; L_x^2(\mathbb{R}^d))$. By (1.22), the bound on $\|F(v + z^\omega)\|_{L_T^1 L_x^2}$ obtained in (3.4), and the choice of a and T , for $\omega \in \Omega_T$ we have

$$\begin{aligned} \|\partial_t v^\omega\|_{L_t^\infty([0,T]; L_x^2(\mathbb{R}^d))} &\leq C \|F(v^\omega + z^\omega)\|_{L_T^1 L_x^2} \\ &\leq C_1 a^{\frac{d+2}{d-2}} + C_1 (2T)^{\frac{(d+2)\theta}{d-2}} (\|u_0\|_{L^2} + \|u_1\|_{H^{-1}})^{\frac{d+2}{d-2}} \leq a. \end{aligned}$$

This concludes the proof of (a).

To prove (b), we decompose the solution u^ω into its linear and nonlinear parts, $u^\omega = \tilde{z}^\omega + \tilde{v}^\omega$, where

$$\begin{aligned} \tilde{z}^\omega(t) &= S(t - t_*) \left((S(t_*)(u_0^\omega, u_1^\omega), \partial_t S(t_*)(u_0^\omega, u_1^\omega)) + (w_0, w_1) \right) \\ &= S(t)(u_0^\omega, u_1^\omega) + S(t - t_*)(w_0, w_1) \end{aligned}$$

and the nonlinear part \tilde{v}^ω satisfies

$$\tilde{v}^\omega(t) = - \int_{t_*}^t \frac{\sin((t-t')|\nabla|)}{|\nabla|} F(\tilde{v}^\omega + \tilde{z}^\omega)(t') dt'.$$

As in (a), it is sufficient to design a fixed point argument to prove the local existence and uniqueness of \tilde{v}^ω on $[t_* - T', t_* + T']$ for all $\omega \in \Omega_T$. The key observation is that, for all $\omega \in \Omega_T$, the improved local-in-time Strichartz estimates hold uniformly on subintervals $[t_* - T', t_* + T'] \subset [-T, T]$. Therefore,

$$\begin{aligned} \|\tilde{z}^\omega\|_{X([t_*-T', t_*+T']\times\mathbb{R}^d)} &\leq \|S(t)(u_0^\omega, u_1^\omega)\|_{X([t_*-T', t_*+T']\times\mathbb{R}^d)} \\ &\quad + \|S(t - t_*)(w_0, w_1)\|_{X([t_*-T', t_*+T']\times\mathbb{R}^d)} \\ &\leq (2T)^\theta (\|u_0\|_{L^2} + \|u_1\|_{H^{-1}}) + \|S(t)(w_0, w_1)\|_{X([-T', T']\times\mathbb{R}^d)}. \end{aligned}$$

We then choose $T' < T$ sufficiently small depending on w_0 and w_1 such that the norm $\|S(t)(w_0, w_1)\|_{X([-T', T']\times\mathbb{R}^d)}$ is small. The rest of the proof follows that of (a). \square

We conclude this section with the proof of the probabilistic small data global theory of Theorem 1.2.

Proof of Theorem 1.2. It suffices to prove almost sure global existence and uniqueness for equation (1.5) satisfied by the nonlinear part $v^\omega := u^\omega - z^\omega$ of the solution.

Let $\eta > 0$ be sufficiently small such that

$$2C_1\eta^{4/(d-2)} \leq 1, \quad 3C_2\eta^{4/(d-2)} \leq 1/2,$$

where C_1, C_2 are the constants appearing in (3.4) and (3.5) above. Then, by Proposition 2.5, there exists a set Ω_ε with

$$P(\Omega_\varepsilon^c) \leq C \exp\left(-c \frac{\eta^2}{\varepsilon^2(\|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}})^2}\right)$$

such that, for all $\omega \in \Omega_\varepsilon$, we have

$$\|z^\omega\|_{X(\mathbb{R} \times \mathbb{R}^d)} = \|S(t)(\varepsilon u_0^\omega, \varepsilon u_1^\omega)\|_{L_t^{\frac{d+2}{d-2}}(\mathbb{R}; L_x^{\frac{2(d+2)}{d-2}}(\mathbb{R}^d))} \leq \eta.$$

We now prove that Γ^ω defined in (3.3) is a contraction on the ball

$$B_\eta := \{v \in C(\mathbb{R}; \dot{H}^1(\mathbb{R}^d)) \cap X(\mathbb{R} \times \mathbb{R}^d) : \|v\|_{L_t^\infty(\mathbb{R}; \dot{H}_x^1(\mathbb{R}^d))} + \|v\|_{X(\mathbb{R} \times \mathbb{R}^d)} \leq \eta\}.$$

Indeed, similarly to (3.4) and (3.5), for $\omega \in \Omega_\varepsilon$ and $v, v_1, v_2 \in B_\eta$ we have

$$\begin{aligned} \|\Gamma^\omega v\|_{L_t^\infty(\mathbb{R}; \dot{H}_x^1(\mathbb{R}^d))} + \|\Gamma^\omega v\|_{X(\mathbb{R} \times \mathbb{R}^d)} &\leq C_1 \|v\|_{X(\mathbb{R} \times \mathbb{R}^d)}^{\frac{d+2}{d-2}} + C_1 \|z^\omega\|_{X(\mathbb{R} \times \mathbb{R}^d)}^{\frac{d+2}{d-2}} \leq 2C_1 \eta^{\frac{d+2}{d-2}} \\ &\leq \eta \end{aligned}$$

and

$$\begin{aligned} \|\Gamma^\omega v_1 - \Gamma^\omega v_2\|_{L_t^\infty(\mathbb{R}; \dot{H}_x^1(\mathbb{R}^d))} + \|\Gamma^\omega v_1 - \Gamma^\omega v_2\|_{X(\mathbb{R} \times \mathbb{R}^d)} \\ \leq C_2 (\|v_1\|_{X(\mathbb{R} \times \mathbb{R}^d)}^{4/(d-2)} + \|v_2\|_{X(\mathbb{R} \times \mathbb{R}^d)}^{4/(d-2)} + \|z^\omega\|_{X(\mathbb{R} \times \mathbb{R}^d)}^{4/(d-2)}) \|v_1 - v_2\|_{X(\mathbb{R} \times \mathbb{R}^d)} \\ \leq 3C_2 \eta^{4/(d-2)} \|v_1 - v_2\|_{X(\mathbb{R} \times \mathbb{R}^d)} \leq \frac{1}{2} \|v_1 - v_2\|_{X(\mathbb{R} \times \mathbb{R}^d)}. \end{aligned}$$

Thus, Γ^ω is indeed a contraction on B_η as long as $\omega \in \Omega_\varepsilon$. Therefore, by the Banach fixed point theorem, for all $\omega \in \Omega_\varepsilon$, there exists a unique global solution $v^\omega \in B_\eta$ of (1.5). By the Strichartz estimates (1.22), we also have

$$\|\partial_t v^\omega\|_{L_t^\infty(\mathbb{R}; L_x^2(\mathbb{R}^d))} \lesssim \|v^\omega\|_{X(\mathbb{R} \times \mathbb{R}^d)}^{\frac{d+2}{d-2}} + \|z^\omega\|_{X(\mathbb{R} \times \mathbb{R}^d)}^{\frac{d+2}{d-2}} \lesssim \eta^{\frac{d+2}{d-2}}.$$

Since we have the global space-time bound

$$\|v^\omega\|_{X(\mathbb{R} \times \mathbb{R}^d)} \leq \eta < \infty,$$

a standard argument shows that, for all $\omega \in \Omega_\varepsilon$, $(v^\omega, \partial_t v^\omega)$ scatters in $\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ to a linear solution, both forward and backward in time. \square

4. Deterministic local well-posedness

This section is dedicated to the local well-posedness of the energy-critical defocusing nonlinear wave equation with a deterministic perturbation on \mathbb{R}^d , $d = 4$ and 5 . We start with a standard local well-posedness result and a blowup criterion. We then upgrade these to a “good” local well-posedness result in which the time of existence depends only on the $\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ -norm of the initial data and on the perturbation. This upgraded local well-posedness is one of the two main ingredients in proving Theorem 1.3, the other being a probabilistic energy bound.

Proposition 4.1 (Deterministic local well-posedness). *Let $d = 4$ or 5 and $(v_0, v_1) \in \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. Let $t_0 \in \mathbb{R}$ and let I be an interval containing t_0 . Then there exists $\delta > 0$ sufficiently small such that if*

$$\|f\|_{X(I \times \mathbb{R}^d)} \leq \delta^{\frac{d-2}{d+2}}$$

and

$$\|S(t - t_0)(v_0, v_1)\|_{X(I \times \mathbb{R}^d)} \leq \delta, \tag{4.1}$$

then the Cauchy problem

$$\begin{cases} \partial_t^2 v - \Delta v + F(v + f) = 0, \\ (v, \partial_t v)|_{t=t_0} = (v_0, v_1), \end{cases} \tag{4.2}$$

admits a unique solution v with $(v, \partial_t v) \in C(I; \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d))$. Here, v is unique in the ball $B_a(I)$ of $X(I \times \mathbb{R}^d)$ defined by

$$B_a(I) := \{v \in X(I \times \mathbb{R}^d) : \|v\|_{X(I \times \mathbb{R}^d)} \leq a\}, \tag{4.3}$$

where $a = C_0 \delta$ for some $C_0 > 0$.

The proof of Proposition 4.1 is standard, and therefore we omit it. It consists in using Duhamel’s formula (1.20) to design a fixed point argument in B_a . As a consequence of Proposition 4.1, we obtain the following blowup criterion.

Lemma 4.2 (Blowup criterion). *Let $d = 4$ or 5 and $(v_0, v_1) \in \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. Let $T_0 < T_1 < T_2$ and let f be a function with $\|f\|_{X([T_0, T_2] \times \mathbb{R}^d)} < \infty$. If v is a solution on $[T_0, T_1]$ of the Cauchy problem*

$$\begin{cases} \partial_t^2 v - \Delta v + F(v + f) = 0, \\ (v, \partial_t v)|_{t=T_0} = (v_0, v_1), \end{cases} \tag{4.4}$$

satisfying $\|v\|_{X([T_0, T_1] \times \mathbb{R}^d)} < \infty$, then there exists $\varepsilon_0 > 0$ such that the solution v can be uniquely extended to $[T_1, T_1 + \varepsilon_0]$.

Equivalently, if $T_1 < \infty$ is the maximal time of existence of the solution v of (4.4), then

$$\|v\|_{X([T_0, T_1] \times \mathbb{R}^d)} = \infty.$$

Condition (4.1) in Proposition 4.1 shows that the local time of existence of the solution of the perturbed NLW (4.2) depends on the profile of the initial data (v_0, v_1) . In the following, we upgrade the local well-posedness result in Proposition 4.1 to a “good” local well-posedness, in which the local time of existence depends only on the $\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ -norm of (v_0, v_1) and on the perturbation f .

Proposition 4.3 (“Good” local well-posedness). *Let $d = 4$ or 5 and $(v_0, v_1) \in \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. Let $K, \gamma > 0$ and $t_0 < T$, and let f be a real-valued function defined on $[t_0, T]$. Then there exists $\tau = \tau(\|(v_0, v_1)\|_{\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)}, K, \gamma)$ sufficiently small and nonincreasing in the first two arguments such that if*

$$\|f\|_{X([t_0, t_0 + \tau_*] \times \mathbb{R}^d)} \leq K \tau_*^\gamma \tag{4.5}$$

for some $0 < \tau_* \leq \tau$, then (4.2) admits a unique solution $(v, \partial_t v)$ in $C([t_0, t_0 + \tau_*]; \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d))$. Moreover,

$$\begin{aligned} \|(v, \partial_t v)\|_{L_t^\infty([t_0, t_0 + \tau_*]; \dot{H}_x^1(\mathbb{R}^d) \times L_x^2(\mathbb{R}^d))} + \|v\|_{L_t^q([t_0, t_0 + \tau_*]; L_x^r(\mathbb{R}^d))} \\ \leq C(\|(v_0, v_1)\|_{\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)}) \end{aligned} \tag{4.6}$$

for all $\dot{H}^1(\mathbb{R}^d)$ -wave admissible pairs (q, r) , where $C(\cdot)$ is a positive nondecreasing function.

Here uniqueness holds in the following sense. There exists a family $\{\mathcal{I}_n\}_{n \in \mathbb{N}}$ of disjoint intervals covering $[t_0, t_0 + \tau_*]$ such that v is unique in each ball $B_a(\mathcal{I}_n)$ of $X(\mathcal{I}_n)$ for all $n \in \mathbb{N}$, where $a > 0$ is a sufficiently small constant.

The key ingredients in the proof of Proposition 4.3 are the following perturbation lemmas.

Lemma 4.4 (Short-time perturbations). *Let $d = 4$ or 5 , $(v_0, v_1) \in \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, $I \subset \mathbb{R}$ be a compact time interval, and $t_0 \in I$. Let v be a solution defined on $I \times \mathbb{R}^d$ of the perturbed equation*

$$\partial_t^2 v - \Delta v + F(v) = e$$

with initial data $(v, \partial_t v)|_{t=t_0} = (v_0, v_1)$. Let $(w_0, w_1) \in \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ and let w be the solution of the energy-critical defocusing nonlinear wave equation on $I \times \mathbb{R}^d$ with initial data $(w, \partial_t w)|_{t=t_0} = (w_0, w_1)$. Then there exist $\delta > 0$ and $\varepsilon_0 > 0$ sufficiently small such that if $0 < \varepsilon < \varepsilon_0$ and

$$\|v\|_{X(I \times \mathbb{R}^d)} \leq \delta, \tag{4.7}$$

$$\|(v_0 - w_0, v_1 - w_1)\|_{\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)} \leq \varepsilon, \tag{4.8}$$

$$\|e\|_{L_t^1(I; L_x^2(\mathbb{R}^d))} \leq \varepsilon, \tag{4.9}$$

then there exists $C \geq 1$ such that

$$\sup_{t \in I} \|(v(t) - w(t), \partial_t v(t) - \partial_t w(t))\|_{\dot{H}_x^1(\mathbb{R}^d) \times L_x^2(\mathbb{R}^d)} + \|v - w\|_{L_t^q(I; L_x^r(\mathbb{R}^d))} \leq C\varepsilon$$

for all $\dot{H}^1(\mathbb{R}^d)$ -wave admissible pairs (q, r) .

Proof. Without loss of generality, we can assume that $t_0 = \inf I$. We set $V := w - v$. Then V satisfies the equation

$$\partial_t^2 V - \Delta V + F(V + v) - F(v) + e = 0.$$

By Duhamel’s formula, Strichartz estimates, (1.21), Hölder’s inequality, and (4.7)–(4.9), we find for $0 < \varepsilon < \varepsilon_0$ that

$$\begin{aligned} & \| (V, \partial_t V) \|_{L^\infty([t_0, t]; \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d))} + \| V \|_{L^q([t_0, t]; L_x^r(\mathbb{R}^d))} + \| V \|_{X([t_0, t] \times \mathbb{R}^d)} \\ & \lesssim \| (V(t_0), \partial_t V(t_0)) \|_{\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)} + \| V \|_{X([t_0, t] \times \mathbb{R}^d)}^{\frac{d+2}{d-2}} \\ & \quad + \| V \|_{X([t_0, t] \times \mathbb{R}^d)} \| v \|_{X([t_0, t] \times \mathbb{R}^d)}^{4/(d-2)} + \| e \|_{L^1([t_0, t]; L_x^2(\mathbb{R}^d))} \\ & \lesssim \varepsilon + \delta^{4/(d-2)} \| V \|_{X([t_0, t] \times \mathbb{R}^d)} + \| V \|_{X([t_0, t] \times \mathbb{R}^d)}^{\frac{d+2}{d-2}} \end{aligned}$$

for all $\dot{H}^1(\mathbb{R}^d)$ -wave admissible pairs (q, r) . If ε_0 and δ are sufficiently small, then a standard continuity argument yields $\| V \|_{X([t_0, t] \times \mathbb{R}^d)} \lesssim \varepsilon$ for all $t \in I$. We then obtain

$$\sup_{t \in I} \| (V, \partial_t V) \|_{\dot{H}_x^1(\mathbb{R}^d) \times L_x^2(\mathbb{R}^d)} + \| V \|_{L_t^q(I; L_x^r(\mathbb{R}^d))} \lesssim \varepsilon. \quad \square$$

Lemma 4.5 (Long-time perturbations). *Let $d = 4$ or 5 , $(v_0, v_1) \in \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, $I \subset \mathbb{R}$ be a compact time interval, $t_0 \in I$, and $M > 0$. Let v be a solution defined on $I \times \mathbb{R}^d$ of the perturbed equation*

$$\partial_t^2 v - \Delta v + F(v) = e$$

with initial data $(v, \partial_t v)|_{t=t_0} = (v_0, v_1)$, satisfying

$$\| v \|_{X(I \times \mathbb{R}^d)} \leq M. \tag{4.10}$$

Let $(w_0, w_1) \in \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ and let w be the solution of the energy-critical defocusing nonlinear wave equation on $I \times \mathbb{R}^d$ with initial data $(w, \partial_t w)|_{t=t_0} = (w_0, w_1)$. Then there exists $\tilde{\varepsilon}(M) > 0$ sufficiently small such that if $0 < \varepsilon < \tilde{\varepsilon}(M)$ and

$$\| (v_0 - w_0, v_1 - w_1) \|_{\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)} \leq \varepsilon, \tag{4.11}$$

$$\| e \|_{L_t^1(I; L_x^2(\mathbb{R}^d))} \leq \varepsilon, \tag{4.12}$$

then

$$\sup_{t \in I} \| (v(t) - w(t), \partial_t v(t) - \partial_t w(t)) \|_{\dot{H}_x^1(\mathbb{R}^d) \times L_x^2(\mathbb{R}^d)} + \| v - w \|_{L_t^q(I; L_x^r(\mathbb{R}^d))} \leq C(M)\varepsilon$$

for all $\dot{H}^1(\mathbb{R}^d)$ -wave admissible pairs (q, r) . Here $C(M) \geq 1$ is a nondecreasing function of M .

Proof. Let $\delta > 0$ be as in Lemma 4.4. Without loss of generality, we can assume $t_0 = \inf I$. The bound (4.10) allows us to divide the interval I into $J = J(M, \delta)$ subintervals $I_j = [t_j, t_{j+1}]$ such that

$$\|v\|_{X(I_j \times \mathbb{R}^d)} \sim \delta \tag{4.13}$$

for all $j = 0, \dots, J - 1$. By (4.11) and (4.12) with $0 < \varepsilon < \varepsilon_0$, Lemma 4.4 yields, on the first interval I_0 ,

$$\sup_{t \in I_0} \|(v(t) - w(t), \partial_t v(t) - \partial_t w(t))\|_{\dot{H}_x^1(\mathbb{R}^d) \times L_x^2(\mathbb{R}^d)} + \|v - w\|_{L_t^q(I_0; L_x^r(\mathbb{R}^d))} \leq C\varepsilon$$

for all $\dot{H}^1(\mathbb{R}^d)$ -wave admissible pairs (q, r) . In particular,

$$\|(v(t_1) - w(t_1), \partial_t v(t_1) - \partial_t w(t_1))\|_{\dot{H}_x^1(\mathbb{R}^d) \times L_x^2(\mathbb{R}^d)} \leq C\varepsilon.$$

If ε is so small that $C\varepsilon < \varepsilon_0$, we can apply Lemma 4.4 on the interval I_1 to obtain

$$\sup_{t \in I_1} \|(v(t) - w(t), \partial_t v(t) - \partial_t w(t))\|_{\dot{H}_x^1(\mathbb{R}^d) \times L_x^2(\mathbb{R}^d)} + \|v - w\|_{L_t^q(I_1; L_x^r(\mathbb{R}^d))} \leq C^2\varepsilon.$$

Arguing recursively, we obtain

$$\sup_{t \in I_j} \|(v(t) - w(t), \partial_t v(t) - \partial_t w(t))\|_{\dot{H}_x^1(\mathbb{R}^d) \times L_x^2(\mathbb{R}^d)} + \|v - w\|_{L_t^q(I_j; L_x^r(\mathbb{R}^d))} \leq C^{j+1}\varepsilon$$

for each $j = 0, 1, \dots, J - 1$, as long as $\max_{j=0, J-1} C^j \varepsilon < \varepsilon_0$. Since $J = J(M, \delta)$ is finite, the conclusion follows with $\tilde{\varepsilon}(M) := \varepsilon_0 / C^{J(M, \delta)}$. \square

Before proceeding to the proof of Proposition 4.3, we recall a global space-time bound for solutions of the energy-critical defocusing nonlinear wave equation on $\mathbb{R} \times \mathbb{R}^d$, $d = 4$ and 5.

Lemma 4.6 (Global space-time bound for energy-critical defocusing NLW on \mathbb{R}^d , $d = 4, 5$). *Let $d = 4$ or 5 and $(v_0, v_1) \in \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. Let w be the solution of the energy-critical defocusing nonlinear wave equation on $\mathbb{R} \times \mathbb{R}^d$:*

$$\begin{cases} \partial_t^2 w - \Delta w + F(w) = 0, \\ w(t_0) = v_0, \quad \partial_t w(t_0) = v_1. \end{cases}$$

Then

$$\|w\|_{X(\mathbb{R} \times \mathbb{R}^d)} < C(\|(v_0, v_1)\|_{\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)}), \tag{4.14}$$

where $C(\cdot)$ is a positive nondecreasing function.

Proof. By the work of Shatah and Struwe [59], $(w, \partial_t w) \in L_{\text{loc}}^q(\mathbb{R}, \dot{B}_q^{1/2} \times \dot{B}_q^{-1/2})$ with $q = \frac{2(d+1)}{d-1}$. Then, by [48, Proposition 4.5(ii)], we have

$$\lim_{t \rightarrow \infty} \int_{\mathbb{R}^d} |w(t, x)|^{\frac{2d}{d-2}} dx = 0. \tag{4.15}$$

As in [3, proof of Proposition 2.4] (see also [4]), a simple argument then shows that $w \in L_t^q(\mathbb{R}; L_x^r(\mathbb{R}^d))$ for any $\dot{H}^1(\mathbb{R}^d)$ -wave admissible pair (q, r) . In particular, $w \in X(\mathbb{R} \times \mathbb{R}^d)$.

Next, to prove the global space-time bound (4.14), we use a concentration-compactness argument adapted to dimensions 4 and 5. More precisely, we use a profile decomposition theorem for solutions of the energy-critical defocusing nonlinear wave equation on \mathbb{R}^d , $d = 4, 5$. This theorem states that if $\{(\phi_{0,n}, \phi_{1,n})\}_{n \in \mathbb{N}}$ is a bounded sequence in $\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, then for every $\ell \geq 1$, the solutions w_n of (1.1) with initial data $(w_n, \partial_t w_n)|_{t=0} = (\phi_{0,n}, \phi_{1,n})$ can be decomposed (on a subsequence) as

$$w_n(t, x) = \sum_{j=1}^{\ell} \frac{1}{(\lambda_n^{(j)})^{(d-2)/2}} U^{(j)}\left(\frac{t - t_n^{(j)}}{\lambda_n^{(j)}}, \frac{x - x_n^{(j)}}{\lambda_n^{(j)}}\right) + r_n^{(\ell)}(t, x), \tag{4.16}$$

where $U^{(j)}$ are some solutions of (1.1) with initial data in $\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, $\lambda_n^{(j)} > 0$, $t_n^{(j)} \in \mathbb{R}$, $x_n^{(j)} \in \mathbb{R}^d$, and $\limsup_{n \rightarrow \infty} \|r_n^{(\ell)}\|_{X(\mathbb{R} \times \mathbb{R}^d)} \rightarrow 0$ as $\ell \rightarrow \infty$.

For $d = 3$, the analogue of the above profile decomposition theorem was proved by Bahouri and Gérard [3]. We point out that the extension to dimensions 4 and 5 poses no difficulty and simply consists in changing the numerology, as dictated by the dimension-dependent Sobolev embeddings and Strichartz estimates. For $d = 5$, some additional care is needed since the degree of the nonlinearity $|w|^{4/3}w$ is not an integer.

The global space-time bound (4.14) then follows by a contradiction argument, as in [3, proof of Corollary 2]. Indeed, assume that there exists a sequence $\{w_n\}_{n \in \mathbb{N}}$ of solutions such that

$$\sup_{n \in \mathbb{N}} E(w_n) < \infty, \quad \lim_{n \rightarrow \infty} \|w_n\|_{X(\mathbb{R} \times \mathbb{R}^d)} = \infty.$$

Since $\sup_{n \in \mathbb{N}} E(w_n) < \infty$, we can apply the above profile decomposition theorem to the sequence $\{w_n\}_{n \in \mathbb{N}}$. In particular, (4.16) implies that $\{w_n\}_{n \in \mathbb{N}}$ is bounded in $X(\mathbb{R} \times \mathbb{R}^d)$. This is a contradiction, and hence (4.14) holds.

See also [37, Lemma 4.3, Corollary 4.5] for slightly different versions of the profile decomposition theorem and of the global space-time bound (4.14) for $d = 3, 4, 5$. \square

Proof of Proposition 4.3. We prove that any solution v of (4.2) defined on $[t_0, t_0 + \tau_*]$, if it exists, satisfies the a priori bound

$$\begin{aligned} \|(v, \partial_t v)\|_{L_t^\infty([t_0, t_0 + \tau_*]; \dot{H}_x^1(\mathbb{R}^d) \times L_x^2(\mathbb{R}^d))} + \|v\|_{L_t^q([t_0, t_0 + \tau_*]; L_x^r(\mathbb{R}^d))} \\ \leq C(\|(v_0, v_1)\|_{\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)}) \end{aligned} \tag{4.17}$$

for all $\dot{H}^1(\mathbb{R}^d)$ -wave admissible pairs (q, r) and for a positive nondecreasing function $C(\cdot)$, provided that $0 < \tau_* \leq \tau(\|(v_0, v_1)\|_{\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)}, K, \gamma)$ and f satisfies (4.5). This, together with Proposition 4.1 and Lemma 4.2, shows that there exists a unique solution $(v, \partial_t v) \in C([t_0, t_0 + \tau_*]; \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d))$ satisfying (4.17). Indeed, since

$$\|S(t - t_0)(v_0, v_1)\|_{X(\mathbb{R} \times \mathbb{R}^d)} \lesssim \|v_0\|_{\dot{H}^1(\mathbb{R}^d)} + \|v_1\|_{L^2(\mathbb{R}^d)} < \infty,$$

one can always find a small time interval $\mathcal{I}_0 = [t_0, t_1) \subset [t_0, t_0 + \tau_*]$ on which the conditions of Proposition 4.1 are satisfied. As a consequence, there exists a solution v of (4.2) with $(v, \partial_t v) \in C(\mathcal{I}_0; \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d))$, unique in the ball $B_a(\mathcal{I}_0)$ of $X(\mathcal{I}_0 \times \mathbb{R}^d)$, where a is sufficiently small. Furthermore, by the a priori bound (4.17) and Lemma 4.2, v can be extended to some interval $\mathcal{I}_1 = [t_1, t_2) \subset [t_0, t_0 + \tau_*]$ and the extension is unique in $B_a(\mathcal{I}_1)$. On the interval $\mathcal{I}_0 \cup \mathcal{I}_1$, v still satisfies (4.17) and hence it can be further extended. Arguing recursively, we can extend v as long as (4.17) is satisfied. Hence, we can define it on the whole interval $[t_0, t_0 + \tau_*]$. Moreover, there exist disjoint intervals $\{\mathcal{I}_n\}_{n \in \mathbb{N}}$ with $[t_0, t_0 + \tau_*] = \bigcup_{n \in \mathbb{N}} \mathcal{I}_n$ such that v is unique in $B_a(\mathcal{I}_n)$ for all $n \in \mathbb{N}$.

We now focus on finding $\tau = \tau(\|(v_0, v_1)\|_{\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)}, K, \gamma)$ such that the a priori bound (4.17) holds for $\tau_* = \tau$ provided (4.5) holds for this value of τ_* . It will be clear from the proof below that (4.5) also implies (4.17) for all $0 < \tau_* < \tau$. By Duhamel’s formula (1.20), Strichartz estimates (1.22), and (4.5), we have

$$\begin{aligned} & \|(v, \partial_t v)\|_{L_t^\infty([t_0, t_0 + \tau]; \dot{H}_x^1(\mathbb{R}^d) \times L_x^2(\mathbb{R}^d))} + \|v\|_{L_t^q([t_0, t_0 + \tau]; L_x^r(\mathbb{R}^d))} \\ & \lesssim \|(v_0, v_1)\|_{\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)} + \|F(v + f)\|_{L_t^1([t_0, t_0 + \tau]; L_x^2(\mathbb{R}^d))} \\ & \lesssim \|v_0\|_{\dot{H}^1(\mathbb{R}^d)} + \|v_1\|_{L^2(\mathbb{R}^d)} + \|v\|_{X([t_0, t_0 + \tau] \times \mathbb{R}^d)}^{\frac{d+2}{d-2}} + C(K\tau^\gamma)^{\frac{d+2}{d-2}} \end{aligned} \tag{4.18}$$

for all $\dot{H}^1(\mathbb{R}^d)$ -wave admissible pairs (q, r) . As a consequence, in order to obtain (4.17), it is sufficient to show that

$$\|v\|_{X([t_0, t_0 + \tau] \times \mathbb{R}^d)} \leq C(\|(v_0, v_1)\|_{\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)})$$

with $C(\cdot)$ a positive nondecreasing function.

Let w be the solution of the energy-critical defocusing nonlinear wave equation on $\mathbb{R} \times \mathbb{R}^d$ with the same initial conditions as v :

$$\begin{cases} \partial_t^2 w - \Delta w + F(w) = 0, \\ w(t_0) = v_0, \quad \partial_t w(t_0) = v_1. \end{cases}$$

By (4.14), we have $\|w\|_{X(\mathbb{R} \times \mathbb{R}^d)} < C(\|(v_0, v_1)\|_{\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)})$. Then we divide \mathbb{R} into $J = J(\|(v_0, v_1)\|_{\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)}, \eta)$ subintervals $I_j = [t_j, t_{j+1}]$ such that

$$\|w\|_{X(I_j \times \mathbb{R}^d)} \sim \eta$$

for some small $\eta > 0$ to be chosen later.

Fix $\tau > 0$ to be chosen later. We write $[t_0, t_0 + \tau] = \bigcup_{j=0}^{J'-1} ([t_0, t_0 + \tau] \cap I_j)$ for some $J' \leq J$, where $[t_0, t_0 + \tau] \cap I_j \neq \emptyset$ for $0 \leq j \leq J' - 1$.

Since the nonlinear evolution of w on each I_j is small, it follows that so is the linear evolution $S(t - t_j)(w(t_j), \partial_t w(t_j))$. Indeed, recall first Duhamel’s formula

$$w(t) = S(t - t_j)(w(t_j), \partial_t w(t_j)) - \int_{t_j}^t \frac{\sin((t - t')|\nabla|)}{|\nabla|} F(w(t')) dt' \quad \text{for } t \in I_j.$$

Then, by the Strichartz estimates (1.22),

$$\begin{aligned} \|S(t - t_j)(w(t_j), \partial_t w(t_j))\|_{X(I_j \times \mathbb{R}^d)} &\leq \|w\|_{X(I_j \times \mathbb{R}^d)} + C \|F(w)\|_{L_t^1(I_j; L_x^2(\mathbb{R}^d))} \\ &\leq \|w\|_{X(I_j \times \mathbb{R}^d)} + C \|w\|_{X(I_j \times \mathbb{R}^d)}^{\frac{d+2}{d-2}} \leq \eta + C \eta^{\frac{d+2}{d-2}} \leq 2\eta \end{aligned} \quad (4.19)$$

for $j = 0, 1, \dots, J' - 1$ and η sufficiently small.

We now use the Perturbation Lemma 4.5 to show that, on each interval I_j , $v - w$ is small in the $L_t^\infty(I_j; \dot{H}_x^1(\mathbb{R}^d) \times L_x^2(\mathbb{R}^d))$ -norm, as well as in Strichartz norms. We first estimate v on I_0 . Arguing as before and using $(v(t_0), \partial_t v(t_0)) = (w(t_0), \partial_t w(t_0))$, we obtain

$$\begin{aligned} \|v\|_{X(I_0 \times \mathbb{R}^d)} &\leq \|S(t - t_0)(w(t_0), \partial_t w(t_0))\|_{X(I_0 \times \mathbb{R}^d)} + C \|v\|_{X(I_0 \times \mathbb{R}^d)}^{\frac{d+2}{d-2}} + C \|f\|_{X(I_0 \times \mathbb{R}^d)}^{\frac{d+2}{d-2}} \\ &\leq 2\eta + C \|v\|_{X(I_0 \times \mathbb{R}^d)}^{\frac{d+2}{d-2}} + C(K\tau^\gamma)^{\frac{d+2}{d-2}}. \end{aligned} \quad (4.20)$$

By taking $\eta \ll 1$ sufficiently small and $K\tau^\gamma \ll 1$, it follows by a standard continuity argument that $\|v\|_{X(I_0 \times \mathbb{R}^d)} \leq 3\eta + CK\tau^\gamma$. Furthermore, by taking $\tau = \tau(K, \gamma, \eta)$ so small that

$$CK\tau^\gamma \leq \eta, \quad (4.21)$$

we obtain

$$\|v\|_{X(I_0 \times \mathbb{R}^d)} \leq 4\eta.$$

Thus, condition (4.10) in Lemma 4.5 is satisfied on I_0 with $M = 4\eta$. We are thus left with estimating the error $e := F(v + f) - F(v)$. First, consider ε_0 to be chosen later such that $0 < \varepsilon_0 < \tilde{\varepsilon}(4\eta)$, where $\tilde{\varepsilon}(4\eta)$ is as in Lemma 4.5. As above, we have

$$\begin{aligned} \|e\|_{L_t^1(I_0; L_x^2(\mathbb{R}^d))} &\leq C \|v\|_{X(I_0 \times \mathbb{R}^d)}^{4/(d-2)} \|f\|_{X(I_0 \times \mathbb{R}^d)} + C \|f\|_{X(I_0 \times \mathbb{R}^d)}^{\frac{d+2}{d-2}} \\ &\leq C\eta^{4/(d-2)} K\tau^\gamma + C(K\tau^\gamma)^{\frac{d+2}{d-2}} \leq CK\tau \leq \varepsilon_0, \end{aligned}$$

provided we choose $\tau = \tau(K, \gamma, \eta, \varepsilon_0)$ so small that $CK\tau \leq \varepsilon_0$. Thus, condition (4.12) is satisfied on I_0 .

Applying the Perturbation Lemma 4.5 on the interval I_0 , we then deduce that

$$\sup_{t \in I_0} \|(v - w, \partial_t v - \partial_t w)\|_{\dot{H}_x^1(\mathbb{R}^d) \times L_x^2(\mathbb{R}^d)} \leq C(4\eta)\varepsilon_0.$$

In particular,

$$\|(v(t_1) - w(t_1), \partial_t v(t_1) - \partial_t w(t_1))\|_{\dot{H}_x^1(\mathbb{R}^d) \times L_x^2(\mathbb{R}^d)} \leq C(4\eta)\varepsilon_0 =: \varepsilon_1. \quad (4.22)$$

Then, proceeding as in (4.20) and using (4.19) and (4.21), we obtain

$$\begin{aligned} \|v\|_{X(I_1 \times \mathbb{R}^d)} &\leq \|S(t - t_1)(w(t_1), \partial_t w(t_1))\|_{X(I_1 \times \mathbb{R}^d)} \\ &\quad + \|S(t - t_1)((v - w)(t_1), (\partial_t v - \partial_t w)(t_1))\|_{X(I_1 \times \mathbb{R}^d)} \\ &\quad + C\|v\|_{X(I_1 \times \mathbb{R}^d)}^{\frac{d+2}{d-2}} + C\|f\|_{X(I_1 \times \mathbb{R}^d)}^{\frac{d+2}{d-2}} \\ &\leq 2\eta + \varepsilon_1 + C\|v\|_{X(I_1 \times \mathbb{R}^d)}^{\frac{d+2}{d-2}} + C(K\tau^\gamma)^{\frac{d+2}{d-2}} \\ &\leq 2\eta + \varepsilon_1 + C\|v\|_{X(I_1 \times \mathbb{R}^d)}^{\frac{d+2}{d-2}} + \eta^{\frac{d+2}{d-2}}. \end{aligned}$$

Therefore, imposing that $\varepsilon_1 = \varepsilon_1(\eta) < \eta$ and fixing $\eta \ll 1$ sufficiently small, by a standard continuity argument we have

$$\|v\|_{X(I_1 \times \mathbb{R}^d)} \leq 4\eta.$$

Thus, condition (4.10) in Lemma 4.5 is satisfied on I_1 with $M = 4\eta$. Then, by (4.22), condition (4.11) is satisfied on I_1 if we choose ε_0 so small that $\varepsilon_1 = C(4\eta)\varepsilon_0 < \tilde{\varepsilon}(4\eta)$. As for the error, we have as above

$$\begin{aligned} \|e\|_{L^1_t(I_1; L^2_x(\mathbb{R}^d))} &\leq C\|v\|_{X(I_1 \times \mathbb{R}^d)}^{4/(d-2)} \|f\|_{X(I_1 \times \mathbb{R}^d)} + C\|f\|_{X(I_1 \times \mathbb{R}^d)}^{\frac{d+2}{d-2}} \\ &\leq C\eta^{4/(d-2)} K\tau^\gamma + C(K\tau^\gamma)^{\frac{d+2}{d-2}} \leq CK\tau \leq \varepsilon_1 \end{aligned}$$

for $\tau = \tau(K, \gamma, \eta, \varepsilon_0, \varepsilon_1)$ sufficiently small. Therefore, condition (4.12) is also satisfied on I_1 and we can apply the Perturbation Lemma 4.5 on this interval to obtain

$$\sup_{t \in I_1} \|(v - w, \partial_t v - \partial_t w)\|_{\dot{H}^1_x(\mathbb{R}^d) \times L^2_x(\mathbb{R}^d)} \leq C(4\eta)^2 \varepsilon_0.$$

We proceed similarly for the intervals $I_2, \dots, I_{J'-1}$. On each I_j , $j = 1, \dots, J' - 1$, we impose that

$$\varepsilon_j := C(4\eta)^j \varepsilon_0$$

satisfies $\varepsilon_j < \tilde{\varepsilon}(4\eta)$ and $\varepsilon_j < \eta$. In order to satisfy all these conditions, it is enough to fix

$$\varepsilon_0(\|(v_0, v_1)\|_{\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)}, \eta) = \frac{1}{2} \frac{\min(\eta, \tilde{\varepsilon}(4\eta))}{C(4\eta)^{J(\|(v_0, v_1)\|_{\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)}, \eta)}}.$$

Furthermore, for all $j = 0, 1, \dots, J' - 1$, we impose the condition

$$\|e\|_{L^1_t(I_j; L^2_x(\mathbb{R}^d))} \leq CK\tau \leq \varepsilon_j.$$

To satisfy this, we fix $\tau = \tau(\|(v_0, v_1)\|_{\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)}, K, \gamma)$ such that

$$CK\tau \leq \min\{\eta, \varepsilon_0, \varepsilon_1, \dots, \varepsilon_{J'-1}\} = \min\{\eta, \varepsilon_0(\|(v_0, v_1)\|_{\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)}, \eta)\}.$$

Since $J(\|(v_0, v_1)\|_{\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)}, \eta)$ is nondecreasing in $\|(v_0, v_1)\|_{\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)}$, we see easily that τ can be chosen to be nonincreasing in both $\|(v_0, v_1)\|_{\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)}$ and K .

Applying the Perturbation Lemma 4.5 recursively on the intervals I_j , we conclude that any solution v defined on $[t_0, t_0 + \tau] \times \mathbb{R}^d$ satisfies the following a priori estimates:

$$\begin{aligned} \|v\|_{X([t_0, t_0+\tau] \times \mathbb{R}^d)} &\leq 4\eta J'(\|(v_0, v_1)\|_{\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)}) \leq C(\|(v_0, v_1)\|_{\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)}), \\ \sup_{t \in [t_0, t_0+\tau]} \|((v-w)(t), (\partial_t v - \partial_t w)(t))\|_{\dot{H}_x^1(\mathbb{R}^d) \times L_x^2(\mathbb{R}^d)} + \|v-w\|_{L_t^q([t_0, t_0+\tau]; L_x^r(\mathbb{R}^d))} &\leq \tilde{\varepsilon}(4\eta), \end{aligned}$$

where $C(\cdot)$ is a positive nondecreasing function. Combining this with (4.18) and (4.21) yields the estimate (4.17). \square

5. Almost sure global existence and uniqueness

The goal of this section is to prove the main result of the paper, namely Theorem 1.3. We start by stating and proving a probabilistic energy bound. The conclusion of Theorem 1.3 then follows from Theorem 5.3, Corollary 5.5, and Proposition 5.6 below.

We first recall a nonlinear Gronwall’s inequality that will be useful in proving the probabilistic energy bound for $d = 5$. See, for example, [44, Theorem 1, p. 360] for more details.

Lemma 5.1 (A nonlinear Gronwall’s inequality). *Fix $T > 0$, $c \geq 0$, and $0 \leq \alpha < 1$. Let u and b be nonnegative continuous functions defined on $[0, T]$ and satisfying*

$$u(t) \leq c + \int_0^t b(s)u(s)^\alpha ds, \quad \text{for all } t \in [0, T].$$

Then

$$u(t) \leq \left(c^{1-\alpha} + (1-\alpha) \int_0^t b(s) ds \right)^{1/(1-\alpha)} \quad \text{for all } t \in [0, T].$$

Following the same lines as in [19, Proposition 2.2], we show that the energy $E(v^\omega)$ of the nonlinear part v^ω of the solution u^ω is almost surely bounded for solutions v^ω of (1.5).

Proposition 5.2 (Probabilistic energy bound). *Let $d = 4$ or 5 and $0 < \varepsilon \ll 1$. Let $(u_0, u_1) \in H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$ with $0 < s \leq 1$ if $d = 4$, and $0 \leq s \leq 1$ if $d = 5$, and let (u_0^ω, u_1^ω) be the randomization defined in (1.13), satisfying (1.15). Given $1 \leq T < \infty$, let v^ω be a solution of the Cauchy problem (1.5) on $[0, T]$. Then there exists a set $\tilde{\Omega}_{T,\varepsilon} \subset \Omega$ with $P(\tilde{\Omega}_{T,\varepsilon}^c) < \varepsilon/2$ such that for all $t \in [0, T]$ and all $\omega \in \tilde{\Omega}_{T,\varepsilon}$, we have*

$$E(v^\omega(t)) \leq C(T, \varepsilon, \|(u_0, u_1)\|_{H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)}),$$

and thus also

$$\|(v^\omega, \partial_t v^\omega)\|_{L_t^\infty([0, T]; \dot{H}_x^1(\mathbb{R}^d) \times L_x^2(\mathbb{R}^d))} \leq C(T, \varepsilon, \|(u_0, u_1)\|_{H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)}), \quad (5.1)$$

where $C(T, \varepsilon, \|(u_0, u_1)\|_{H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)})$ is a constant depending only on T , ε , and $\|(u_0, u_1)\|_{H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)}$.

Proof. Taking the time derivative of the energy, we obtain

$$\begin{aligned} \frac{d}{dt} E(v^\omega(t)) &= \int_{\mathbb{R}^d} (\partial_t v^\omega \partial_t^2 v^\omega + \nabla \partial_t v^\omega \cdot \nabla v^\omega + F(v^\omega) \partial_t v^\omega) dx \\ &= \int_{\mathbb{R}^d} \partial_t v^\omega (\partial_t^2 v^\omega - \Delta v^\omega + F(v^\omega)) dx \\ &= \int_{\mathbb{R}^d} \partial_t v^\omega [F(v^\omega) - F(z^\omega + v^\omega)] dx. \end{aligned}$$

Using the Cauchy–Schwarz and Hölder inequalities, it then follows that

$$\begin{aligned} \left| \frac{d}{dt} E(v^\omega(t)) \right| &\leq C(E(v^\omega(t)))^{1/2} \|F(v^\omega) - F(z^\omega + v^\omega)\|_{L_x^2(\mathbb{R}^d)} \\ &\leq C(E(v^\omega(t)))^{1/2} (\|z^\omega\|_{L_x^{\frac{d+2}{d-2}}(\mathbb{R}^d)}^{\frac{d+2}{d-2}} + \| |z^\omega| |v^\omega|^{4/(d-2)} \|_{L_x^2(\mathbb{R}^d)}). \end{aligned} \quad (5.2)$$

We first consider the case $d = 4$. By (5.2), we have

$$\left| \frac{d}{dt} E(v^\omega(t)) \right| \leq C(E(v^\omega(t)))^{1/2} (\|z^\omega\|_{L_x^6(\mathbb{R}^4)}^3 + \|z^\omega\|_{L_x^\infty(\mathbb{R}^4)} \|v^\omega\|_{L_x^4(\mathbb{R}^4)}^2),$$

and therefore

$$\left| \frac{d}{dt} (E(v^\omega(t)))^{1/2} \right| \leq C \|z^\omega\|_{L_x^6(\mathbb{R}^4)}^3 + C \|z^\omega\|_{L_x^\infty(\mathbb{R}^4)} (E(v^\omega(t)))^{1/2}. \quad (5.3)$$

Integrating this from $t = 0$ to $t \leq T$, we then obtain

$$(E(v^\omega(t)))^{1/2} \leq C \|z^\omega\|_{X([0,T] \times \mathbb{R}^4)}^3 + C \int_0^t \|z^\omega(s)\|_{L_x^\infty(\mathbb{R}^4)} (E(v^\omega(s)))^{1/2} ds.$$

Then, by Gronwall’s inequality,

$$(E(v^\omega(t)))^{1/2} \leq C \|z^\omega\|_{X([0,T] \times \mathbb{R}^4)}^3 e^{C \|z^\omega\|_{L_t^1([0,T]; L_x^\infty(\mathbb{R}^4))}}. \quad (5.4)$$

Next, we consider $K_1, K_2 > 0$ such that $C \exp(-cK_1^2) + C \exp(-cK_2^2) < \varepsilon/2$, where $C, c > 0$ are such that both the estimates in Proposition 2.3(ii) for $(q, r) = (3, 6)$ and in Proposition 2.3(iii) for $(1, \infty)$ hold. Then, for $0 < s \leq 1$, by Proposition 2.3(ii)&(iii) with $\lambda = K_1 T^{4/3} (\|u_0\|_{L^2} + \|u_1\|_{H^{-1}})$ and $\lambda = K_2 T^2 (\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}})$ respectively, there exists $\tilde{\Omega}_{T,\varepsilon}(\mathbb{R}^4) \subset \Omega$ with $P(\tilde{\Omega}_{T,\varepsilon}^c(\mathbb{R}^4)) < \varepsilon/2$ such that

$$\begin{aligned} \|z^\omega\|_{X([0,T] \times \mathbb{R}^4)} &\leq K_1 T^{4/3} (\|u_0\|_{L^2(\mathbb{R}^4)} + \|u_1\|_{H^{-1}(\mathbb{R}^4)}), \\ \|z^\omega\|_{L_t^1([0,T]; L_x^\infty(\mathbb{R}^4))} &\leq K_2 T^2 (\|u_0\|_{H^s(\mathbb{R}^4)} + \|u_1\|_{H^{s-1}(\mathbb{R}^4)}) \end{aligned}$$

for any $\omega \in \tilde{\Omega}_{T,\varepsilon}(\mathbb{R}^4)$. Therefore, by (5.4), we conclude for $d = 4$ that

$$\begin{aligned} E(v^\omega(t)) &\leq (K_1^3 T^4 (\|u_0\|_{L^2(\mathbb{R}^4)} + \|u_1\|_{H^{-1}(\mathbb{R}^4)})^3 e^{CK_2 T^2 (\|u_0\|_{H^s(\mathbb{R}^4)} + \|u_1\|_{H^{s-1}(\mathbb{R}^4)})^2} \\ &\leq C(T, \varepsilon, \|(u_0, u_1)\|_{H^s(\mathbb{R}^4) \times H^{s-1}(\mathbb{R}^4)}) \end{aligned}$$

for all $t \in [0, T]$ and all $\omega \in \tilde{\Omega}_{T,\varepsilon}(\mathbb{R}^4)$. Notice that, for $d = 4$, we have used Proposition 2.3(iii), which requires $s > 0$.

We now turn to the case $d = 5$. Using (5.2), we find that

$$\left| \frac{d}{dt} E(v^\omega(t)) \right| \leq C(E(v^\omega(t)))^{1/2} (\|z^\omega\|_{L_x^{14/3}(\mathbb{R}^5)}^{7/3} + \|z^\omega\|_{L_x^{10}(\mathbb{R}^5)} \|v^\omega\|_{L_x^{10/3}(\mathbb{R}^5)}^{4/3}),$$

and therefore

$$\left| \frac{d}{dt} (E(v^\omega(t)))^{1/2} \right| \leq C \|z^\omega\|_{L_x^{14/3}(\mathbb{R}^5)}^{7/3} + C \|z^\omega\|_{L_x^{10}(\mathbb{R}^5)} ((E(v^\omega(t)))^{1/2})^{4/5}. \tag{5.5}$$

Integrating from $t = 0$ to $t \leq T$, we obtain

$$(E(v^\omega(t)))^{1/2} \leq C \|z^\omega\|_{X([0,T] \times \mathbb{R}^5)}^{7/3} + C \int_0^t \|z^\omega(s)\|_{L_x^{10}(\mathbb{R}^5)} ((E(v^\omega(s)))^{1/2})^{4/5} ds.$$

Then, by the nonlinear Gronwall’s inequality in Lemma 5.1 with $\alpha = 4/5$,

$$(E(v^\omega(t)))^{1/2} \leq C \|z^\omega\|_{X([0,T] \times \mathbb{R}^5)}^{7/3} + C \|z^\omega\|_{L_t^1([0,T]; L_x^{10}(\mathbb{R}^5))}^5.$$

Applying Proposition 2.3(ii) as above, we deduce that there exists a set $\tilde{\Omega}_{T,\varepsilon}(\mathbb{R}^5) \subset \Omega$ with $P(\tilde{\Omega}_{T,\varepsilon}^c(\mathbb{R}^5)) = 0$ such that

$$\begin{aligned} \|z^\omega\|_{X([0,T] \times \mathbb{R}^5)} &\leq K_1 T^{10/7} (\|u_0\|_{L^2(\mathbb{R}^5)} + \|u_1\|_{H^{-1}(\mathbb{R}^5)}), \\ \|z^\omega\|_{L_t^1([0,T], L_x^{10}(\mathbb{R}^5))} &\leq K_2 T^2 (\|u_0\|_{L^2(\mathbb{R}^5)} + \|u_1\|_{H^{-1}(\mathbb{R}^5)}) \end{aligned}$$

for any $\omega \in \tilde{\Omega}_{T,\varepsilon}(\mathbb{R}^5)$ and for some K_1 and K_2 depending on ε . Therefore, for $d = 5$,

$$\begin{aligned} E(v(t)) &\leq (K_1 T^{10/7} (\|u_0\|_{L^2(\mathbb{R}^5)} + \|u_1\|_{H^{-1}(\mathbb{R}^5)}))^{14/3} \\ &\quad + (K_2 T^2 (\|u_0\|_{L^2(\mathbb{R}^5)} + \|u_1\|_{H^{-1}(\mathbb{R}^5)}))^{10} \\ &\leq C(T, \varepsilon, \|(u_0, u_1)\|_{L^2(\mathbb{R}^5) \times H^{-1}(\mathbb{R}^5)}) \end{aligned}$$

for all $t \in [0, T]$ and all $\omega \in \tilde{\Omega}_{T,\varepsilon}(\mathbb{R}^5)$. Notice that, for $d = 5$, we have only applied Proposition 2.3(ii), which allows us to consider $(u_0, u_1) \in H^s(\mathbb{R}^5) \times H^{s-1}(\mathbb{R}^5)$ with $0 \leq s \leq 1$, thus also including $s = 0$.

Finally, (5.1) follows by noticing that

$$\|(v^\omega(t), \partial_t v^\omega(t))\|_{\dot{H}_x^1(\mathbb{R}^d) \times L_x^2(\mathbb{R}^d)}^2 \leq 2E(v^\omega(t)) \leq C(T, \varepsilon, \|(u_0, u_1)\|_{H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)})$$

and by using

$$\begin{aligned} \|v^\omega(t)\|_{L^2_x(\mathbb{R}^d)} &= \left\| \int_0^t \partial_t v^\omega(s) ds \right\|_{L^2_x(\mathbb{R}^d)} \leq \int_0^t \|\partial_t v^\omega(s)\|_{L^2_x(\mathbb{R}^d)} ds \\ &\leq T \|\partial_t v^\omega\|_{L^{\infty}_t([0,T]; L^2_x(\mathbb{R}^d))} \\ &\leq C(T, \varepsilon, \|(u_0, u_1)\|_{H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)}). \end{aligned}$$

This concludes the proof. □

Remark 5.1. The argument in the proof of Proposition 5.2 cannot be used to prove a probabilistic energy bound for the analogous problem on \mathbb{R}^3 . Indeed, from (5.2) we see that in the proof of Proposition 5.2 we require the control of $\|z^\omega |v^\omega|^{4/(d-2)}\|_{L^2(\mathbb{R}^d)}$ in terms of the energy $E(v^\omega)$. We notice that

$$\|z^\omega |v^\omega|^{4/(d-2)}\|_{L^2(\mathbb{R}^d)} \leq \|z^\omega\|_{L^\infty(\mathbb{R}^d)} \|v^\omega\|_{L^{8/(d-2)}(\mathbb{R}^d)}^{4/(d-2)}. \tag{5.6}$$

The energy $E(v^\omega)$ only controls the $L^p(\mathbb{R}^d)$ -norms of v^ω for $2 \leq p \leq \frac{2d}{d-2}$. Therefore, in order to control the right-hand side of (5.6) in terms of $E(v^\omega)$, one needs $\frac{8}{d-2} \leq \frac{2d}{d-2}$, that is, $d \geq 4$. This shows that for $d = 3$ a more intricate analysis is needed to prove a probabilistic energy bound.

The main result of this section is the following theorem concerning the almost sure global existence and uniqueness for the energy-critical defocusing NLW in $H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$ with $0 < s \leq 1$ if $d = 4$, and $0 \leq s \leq 1$ if $d = 5$.

Theorem 5.3 (Almost sure global existence and uniqueness). *Let $d = 4$ or 5 . Let $(u_0, u_1) \in H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$, with $0 < s \leq 1$ if $d = 4$, and $0 \leq s \leq 1$ if $d = 5$, and let (u_0^ω, u_1^ω) be the randomization defined in (1.13), satisfying (1.15). Then the defocusing energy-critical NLW (1.1) on \mathbb{R}^d admits almost surely a unique global solution with initial data (u_0^ω, u_1^ω) at $t = 0$. More precisely, there exists a set $\tilde{\Omega} \subset \Omega$ with $P(\tilde{\Omega}) = 1$ such that, for each $\omega \in \tilde{\Omega}$, there exists a unique global solution u^ω of equation (1.1) with $(u^\omega, \partial_t u^\omega)|_{t=0} = (u_0^\omega, u_1^\omega)$ and such that*

$$\begin{aligned} (u^\omega, \partial_t u^\omega) &\in (S(t)(u_0^\omega, u_1^\omega), \partial_t S(t)(u_0^\omega, u_1^\omega)) + C(\mathbb{R}; H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)) \\ &\subset C(\mathbb{R}; H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)). \end{aligned}$$

Uniqueness here holds in the following sense. The set $\tilde{\Omega}$ can be written as $\tilde{\Omega} = \bigcup_{\varepsilon > 0} \Omega_\varepsilon$ with $P(\Omega_\varepsilon^c) < \varepsilon$ and for any $\varepsilon > 0$, $\omega \in \Omega_\varepsilon$, and $0 < T < \infty$, there exists a family $\{\mathcal{I}_n\}_{n \in \mathbb{N}}$ of disjoint intervals covering $[-T, T]$ such that the nonlinear part of the solution, $v^\omega = u^\omega - S(\cdot)(u_0^\omega, u_1^\omega)$, is unique in the ball $B_a(\mathcal{I}_n)$ of $X(\mathcal{I}_n \times \mathbb{R}^d)$ for all $n \in \mathbb{N}$, where $a > 0$ is a small constant.

As in [7, 23], the following proposition stating “almost” almost sure global existence and uniqueness for (1.1) readily implies Theorem 5.3.

Proposition 5.4 (“Almost” almost sure global existence and uniqueness). *Let $d = 4$ or 5 . Let $(u_0, u_1) \in H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$ with $0 < s \leq 1$ if $d = 4$, and $0 \leq s \leq 1$ if $d = 5$, and let (u_0^ω, u_1^ω) be the randomization defined in (1.13), satisfying (1.15). Then for any $0 < \varepsilon \ll 1$ and $T \geq 1$, there exists $\Omega_{T,\varepsilon} \subset \Omega$ with $P(\Omega_{T,\varepsilon}^c) < \varepsilon$ such that for any $\omega \in \Omega_{T,\varepsilon}$ there exists a unique solution u^ω of equation (1.1) on $[0, T] \times \mathbb{R}^d$ with initial data $(u^\omega, \partial_t u^\omega)|_{t=0} = (u_0^\omega, u_1^\omega)$ and such that*

$$(u^\omega, \partial_t u^\omega) \in (S(t)(u_0^\omega, u_1^\omega), \partial_t S(t)(u_0^\omega, u_1^\omega)) + C([0, T]; H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)) \subset C([0, T]; H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)).$$

Moreover, the nonlinear part of the solution $v^\omega = u^\omega - S(\cdot)(u_0^\omega, u_1^\omega)$ satisfies the bounds

$$\begin{aligned} & \|v^\omega\|_{L_t^q([0,T]; L_x^r(\mathbb{R}^d))} \\ & \leq \tilde{F}\left(\sup_{t \in [0,T]} \|(v^\omega(t), \partial_t v^\omega(t))\|_{\dot{H}_x^1(\mathbb{R}^d) \times L_x^2(\mathbb{R}^d)}, \|(u_0, u_1)\|_{H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)}\right) T^7 \left(\log \frac{T}{\varepsilon}\right)^3 \\ & \leq C(T, \varepsilon, \|(u_0, u_1)\|_{H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)}) \end{aligned}$$

for all $\dot{H}^1(\mathbb{R}^d)$ -wave admissible pairs (q, r) . Here $\tilde{F} : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ is a nondecreasing function in both variables satisfying $F(0, \cdot) \equiv 0$.

Uniqueness here holds in the following sense. There exists a family $\{\mathcal{I}_n\}_{n \in \mathbb{N}}$ of disjoint intervals covering $[0, T]$ such that the nonlinear part v^ω of the solution is unique in each ball $B_a(\mathcal{I}_n)$ of $X(\mathcal{I}_n \times \mathbb{R}^d)$ for all $n \in \mathbb{N}$, where $a > 0$ is a small constant.

Before proving Proposition 5.4, we first show how Theorem 5.3 follows from it.

Proof of Theorem 5.3. By the time reversibility of the equation, it is sufficient to prove the almost sure existence of unique solutions defined on the time interval $[0, \infty)$. For fixed $\varepsilon > 0$, we consider $T_j = 2^j$ and $\varepsilon_j = 2^{-j}\varepsilon$, and using Proposition 5.4, we obtain $\Omega_{T_j, \varepsilon_j}$. Considering now $\Omega_\varepsilon := \bigcap_{j=1}^\infty \Omega_{T_j, \varepsilon_j}$, we find that $P(\Omega_\varepsilon^c) < \varepsilon$ and (1.1) admits a unique solution on $[0, \infty)$ for all $\omega \in \Omega_\varepsilon$. Finally, defining $\tilde{\Omega} := \bigcup_{\varepsilon > 0} \Omega_\varepsilon$, we conclude that $P(\tilde{\Omega}^c) = 0$ and (1.1) admits a unique solution on $[0, \infty)$ for all $\omega \in \tilde{\Omega}$. \square

We continue with the proof of Proposition 5.4. The main ingredients of the proof are Proposition 5.2 giving the probabilistic energy bound and Proposition 4.3 containing the “good” deterministic local well-posedness.

Proof of Proposition 5.4. By Proposition 5.2, there exists $\tilde{\Omega}_{T,\varepsilon}$ with $P(\tilde{\Omega}_{T,\varepsilon}^c) < \varepsilon/2$ such that for any $\omega \in \tilde{\Omega}_{T,\varepsilon}$, the solution of (1.5) satisfies the a priori bound

$$\begin{aligned} A & := \sup_{t \in [0,T]} \|(v^\omega(t), \partial_t v^\omega(t))\|_{\dot{H}_x^1(\mathbb{R}^d) \times L_x^2(\mathbb{R}^d)} \\ & \leq C(T, \varepsilon, \|(u_0, u_1)\|_{H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)}). \end{aligned} \tag{5.8}$$

We set

$$K := C(\|u_0\|_{L^2(\mathbb{R}^d)} + \|u_1\|_{H^{-1}(\mathbb{R}^d)}) \quad \text{and} \quad \gamma < \frac{d-2}{d+2}. \tag{5.9}$$

Consider $\tau = \tau(A, K, \gamma)$ defined in Proposition 4.3. Fix $\tau_* \leq \tau$ to be chosen later. We cover the interval $[0, T]$ by $[T/\tau_*]$ or $[T/\tau_*] + 1$ subintervals $\mathfrak{J}_k := [k\tau_*, (k + 1)\tau_*] \cap [0, T]$, $k = 0, 1, \dots$. By Corollary 2.4, for each k , there exists $\Omega_k \subset \Omega$ with

$$P(\Omega_k^c) \leq C \exp\left(-\frac{c}{T^2 \tau_*^{2(\frac{d-2}{d+2}-\gamma)}}\right)$$

such that for any $\omega \in \Omega_k$ we have $\|z^\omega\|_{X(\mathfrak{J}_k \times \mathbb{R}^d)} \leq K \tau_*^\gamma$. Consider $\widehat{\Omega}_{T,\varepsilon} := \bigcap_k \Omega_k \subset \Omega$. Then, for any $\omega \in \widehat{\Omega}_{T,\varepsilon}$ and for all $k = 0, 1, \dots$, we have

$$\|z^\omega\|_{X(\mathfrak{J}_k \times \mathbb{R}^d)} \leq K \tau_*^\gamma$$

and

$$\begin{aligned} P(\widehat{\Omega}_{T,\varepsilon}^c) &\leq \left(\left[\frac{T}{\tau_*}\right] + 1\right) C \exp\left(-\frac{c}{T^2 \tau_*^{2(\frac{d-2}{d+2}-\gamma)}}\right) \\ &\leq \frac{T}{\tau_*} \tau_* \exp\left(-\frac{c}{2T^2 \tau_*^{2(\frac{d-2}{d+2}-\gamma)}}\right) = T \exp\left(-\frac{c}{2T^2 \tau_*^{2(\frac{d-2}{d+2}-\gamma)}}\right) \end{aligned}$$

if τ_* is sufficiently small. Fixing

$$\tau_*(A, K, \gamma, T, \varepsilon) = \min\left\{\tau(A, K, \gamma), \frac{1}{2}\left(\frac{c}{2T^2 \log(2T/\varepsilon)}\right)^{\frac{d+2}{2(d-2-\gamma(d+2))}}\right\}, \quad (5.10)$$

we obtain $P(\widehat{\Omega}_{T,\varepsilon}^c) < \varepsilon/2$.

We now define $\Omega_{T,\varepsilon} := \widetilde{\Omega}_{T,\varepsilon} \cap \widehat{\Omega}_{T,\varepsilon}$. Notice that $P(\Omega_{T,\varepsilon}^c) < \varepsilon$. For any $\omega \in \Omega_{T,\varepsilon}$, the conditions of Proposition 4.3 are satisfied on each subinterval \mathfrak{J}_k with $f = z^\omega$, K and γ defined above in (5.9), and $\|(v(k\tau_*), \partial_t v(k\tau_*))\|_{\dot{H}_x^1(\mathbb{R}^d) \times L_x^2(\mathbb{R}^d)} \leq A$ with A defined above in (5.8). Applying successively Proposition 4.3 on each \mathfrak{J}_k , $k = 0, 1, \dots$, we obtain for all $\omega \in \Omega_{T,\varepsilon}$ a unique solution v^ω of (1.5) such that $(v^\omega, \partial_t v^\omega)$ is in $C([0, T]; \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d))$. By (5.1), $(v^\omega, \partial_t v^\omega)$ also belongs to this class. The uniqueness is in the sense of Proposition 4.3.

Moreover, with the choice of τ_* in (5.10) and fixing $\gamma = \frac{5d-14}{6(d+2)} < \frac{d-2}{d+2}$, we see from (4.6) that this solution satisfies

$$\begin{aligned} \|v^\omega\|_{L_t^q([0,T]; L_x^r(\mathbb{R}^d))} &\leq C(A) \frac{T}{\tau_*} \leq C(A) T \max\left\{\frac{1}{\tau(A, K)}, \left(T^2 \log \frac{T}{\varepsilon}\right)^3\right\} \\ &\leq \tilde{F}(A, K) T^7 \left(\log \frac{T}{\varepsilon}\right)^3 \end{aligned}$$

for all $\dot{H}^1(\mathbb{R}^d)$ -wave admissible pairs (q, r) and $A > 0$. Since $C(A)$ is a nondecreasing function of A , and $\tau(A, K)$ is nonincreasing in both A and K , it follows that \tilde{F} can be chosen to be nondecreasing in both variables. Then, by (5.8), we conclude that

$$\|v^\omega\|_{L_t^q([0,T]; L_x^r(\mathbb{R}^d))} \leq C(T, \varepsilon, \|(u_0, u_1)\|_{H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)}). \quad \square$$

The following corollary shows that almost sure global existence and uniqueness for the energy-critical defocusing NLW on \mathbb{R}^d , $d = 4$ and 5 , can also be proved for more general initial data than the ones considered in Theorem 5.3.

Corollary 5.5 (Enhanced almost sure global existence and uniqueness). *Let $d = 4$ or 5 . Let $(u_0, u_1) \in H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$ with $0 < s \leq 1$ if $d = 4$, and $0 \leq s \leq 1$ if $d = 5$, and let (u_0^ω, u_1^ω) be the randomization defined in (1.13), satisfying (1.15). Let $t_* \in \mathbb{R}$ and $(v_0, v_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. Then there exists a set $\Omega' \subset \Omega$ with $P(\Omega') = 1$ such that, for each $\omega \in \Omega'$, the energy-critical defocusing NLW on \mathbb{R}^d with initial data*

$$(u^\omega, \partial_t u^\omega)|_{t=t_*} = (S(t_*)(u_0^\omega, u_1^\omega), \partial_t S(t_*)(u_0^\omega, u_1^\omega)) + (v_0, v_1)$$

admits a unique global solution u^ω with

$$(u^\omega, \partial_t u^\omega) \in (S(t)(u_0^\omega, u_1^\omega), \partial_t S(t)(u_0^\omega, u_1^\omega)) + C(\mathbb{R}; H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)) \\ \subset C(\mathbb{R}; H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)).$$

Uniqueness here holds in the following sense. The set Ω' can be written as $\Omega' = \bigcup_{\varepsilon > 0} \Omega'_\varepsilon$ with $P((\Omega'_\varepsilon)^c) < \varepsilon$ and for any $\varepsilon > 0$, $\omega \in \Omega'_\varepsilon$, and $0 < T < \infty$, there exists a family $\{\mathcal{I}_n\}_{n \in \mathbb{N}}$ of disjoint intervals covering $[t_* - T, t_* + T]$ such that the nonlinear part of the solution, $v^\omega = u^\omega - S(\cdot)(u_0^\omega, u_1^\omega)$, is unique in the ball $B_a(\mathcal{I}_n)$ of $X(\mathcal{I}_n \times \mathbb{R}^d)$ for all $n \in \mathbb{N}$, where $a > 0$ is a small constant.

Proof. Let $0 < \varepsilon \ll 1$ and $2 \leq T < \infty$. Then $\max(|t_*|, |t_* + T|) \geq 1$. We look for a solution of the energy-critical defocusing NLW on \mathbb{R}^d of the form

$$u^\omega(t) = S(t)(u_0^\omega, u_1^\omega) + v^\omega(t),$$

where the nonlinear part v^ω satisfies

$$\begin{cases} \partial_t^2 v^\omega - \Delta v^\omega + F(v^\omega + z^\omega) = 0, \\ (v^\omega, \partial_t v^\omega)|_{t=t_*} = (v_0, v_1) \end{cases} \tag{5.11}$$

with $z^\omega = S(t)(u_0^\omega, u_1^\omega)$.

We first prove a probabilistic energy bound for v^ω . By the Sobolev embedding $\dot{H}^1(\mathbb{R}^d) \subset L^{2d/(d-2)}(\mathbb{R}^d)$, we have

$$E(v^\omega(t)) \leq \frac{1}{2} \|(v^\omega(t), \partial_t v^\omega(t))\|_{\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)}^2 + C \|v^\omega(t)\|_{\dot{H}^1(\mathbb{R}^d)}^{2d/(d-2)}. \tag{5.12}$$

We first consider the case $d = 4$. The same computations as in Proposition 5.2 show that (5.3) holds. Integrating (5.3) from t_* to t , where $t_* \leq t \leq t_* + T$, and using $(v^\omega, \partial_t v^\omega)|_{t=t_*} = (v_0, v_1)$ and (5.12), we find that

$$(E(v^\omega(t)))^{1/2} \leq C(\|v_0\|_{\dot{H}^1(\mathbb{R}^4)} + \|v_1\|_{L^2(\mathbb{R}^4)} + \|v_0\|_{\dot{H}^1(\mathbb{R}^4)}^2) + C \|z^\omega\|_{X([t_*, t_*+T] \times \mathbb{R}^4)}^3 \\ + C \int_{t_*}^t \|z^\omega(s)\|_{L_x^\infty(\mathbb{R}^4)} (E(v^\omega(s)))^{1/2} ds.$$

Then, by Gronwall's inequality, for all $t \in [t_*, t_* + T]$,

$$\begin{aligned} (E(v^\omega(t)))^{1/2} &\leq C(\|v_0\|_{\dot{H}^1(\mathbb{R}^4)} + \|v_1\|_{L^2(\mathbb{R}^4)} + \|v_0\|_{\dot{H}^1(\mathbb{R}^4)}^2 + \|z^\omega\|_{X([t_*, t_*+T] \times \mathbb{R}^4)}^3) \\ &\quad \times e^{C\|z^\omega\|_{L_t^1([t_*, t_*+T]; L_x^\infty(\mathbb{R}^4))}}. \end{aligned} \tag{5.13}$$

Next, we consider $K_1, K_2 > 0$ such that $C \exp(-cK_1^2) + C \exp(-cK_2^2) < \varepsilon/2$, where $C, c > 0$ are such that both the estimates in Proposition 2.3(ii) for $(q, r) = (3, 6)$ and in Proposition 2.3(iii) for $(1, \infty)$ hold. Then, by Proposition 2.3(ii)&(iii) with

$$\lambda = K_1 T^{1/3} \max(|t_*|, |t_* + T|)(\|u_0\|_{L^2(\mathbb{R}^4)} + \|u_1\|_{H^{-1}(\mathbb{R}^4)})$$

and

$$\lambda = K_2 T \max(|t_*|, |t_* + T|)(\|u_0\|_{H^s(\mathbb{R}^4)} + \|u_1\|_{H^{s-1}(\mathbb{R}^4)})$$

respectively, there exists $\tilde{\Omega}_{t_*, T, \varepsilon}(\mathbb{R}^4) \subset \Omega$ with $P(\tilde{\Omega}_{t_*, T, \varepsilon}^c(\mathbb{R}^4)) < \varepsilon/2$ such that

$$\begin{aligned} \|z^\omega\|_{X([t_*, t_*+T] \times \mathbb{R}^4)} &\leq K_1 T^{1/3} \max(|t_*|, |t_* + T|)(\|u_0\|_{L^2(\mathbb{R}^4)} + \|u_1\|_{H^{-1}(\mathbb{R}^4)}), \\ \|z^\omega\|_{L_t^1([t_*, t_*+T]; L_x^\infty(\mathbb{R}^4))} &\leq K_2 T \max(|t_*|, |t_* + T|)(\|u_0\|_{H^s(\mathbb{R}^4)} + \|u_1\|_{H^{s-1}(\mathbb{R}^4)}) \end{aligned}$$

for any $\omega \in \tilde{\Omega}_{t_*, T, \varepsilon}(\mathbb{R}^4)$ and $0 < s \leq 1$. Combining these with (5.13) yields, for all $\omega \in \tilde{\Omega}_{t_*, T, \varepsilon}$,

$$\begin{aligned} \|(v^\omega, \partial_t v^\omega)\|_{L_t^\infty([t_*, t_*+T]; \dot{H}_x^1(\mathbb{R}^4) \times L_x^2(\mathbb{R}^4))} &\leq C(\|v_0\|_{\dot{H}^1(\mathbb{R}^4)} + \|v_1\|_{L^2(\mathbb{R}^4)} + \|v_0\|_{\dot{H}^1(\mathbb{R}^4)}^2 \\ &\quad + K_1^3 T \max(|t_*|, |t_* + T|)^3 (\|u_0\|_{L^2(\mathbb{R}^4)} + \|u_1\|_{H^{-1}(\mathbb{R}^4)})^3) \\ &\quad \times e^{CK_2 T \max(|t_*|, |t_*+T|)(\|u_0\|_{H^s(\mathbb{R}^4)} + \|u_1\|_{H^{s-1}(\mathbb{R}^4)})} \\ &\leq C(t_*, T, \varepsilon, \|(v_0, v_1)\|_{\dot{H}^1(\mathbb{R}^4) \times L^2(\mathbb{R}^4)}, \|(u_0, u_1)\|_{H^s(\mathbb{R}^4) \times H^{s-1}(\mathbb{R}^4)}) =: \tilde{A}(\mathbb{R}^4). \end{aligned} \tag{5.14}$$

We now turn to the case $d = 5$. The same computations as in Proposition 5.2 show that (5.5) holds. Integrating (5.5) from t_* to t , where $t_* \leq t \leq t_* + T$, and using $(v^\omega, \partial_t v^\omega)|_{t=t_*} = (v_0, v_1)$ and (5.12), we obtain

$$\begin{aligned} (E(v^\omega(t)))^{1/2} &\leq C(\|v_0\|_{\dot{H}^1(\mathbb{R}^5)} + \|v_1\|_{L^2(\mathbb{R}^5)} + \|v_0\|_{\dot{H}^1(\mathbb{R}^5)}^{5/3}) + C\|z^\omega\|_{X([t_*, t_*+T] \times \mathbb{R}^5)}^{7/3} \\ &\quad + C \int_{t_*}^t \|z^\omega(s)\|_{L_x^{10}(\mathbb{R}^5)} ((E(v^\omega(s)))^{1/2})^{4/5} ds. \end{aligned}$$

Then, by the nonlinear Gronwall's inequality in Lemma 5.1 with $\alpha = 4/5$,

$$\begin{aligned} (E(v^\omega(t)))^{1/2} &\leq C(\|v_0\|_{\dot{H}^1(\mathbb{R}^5)} + \|v_1\|_{L^2(\mathbb{R}^5)} + \|v_0\|_{\dot{H}^1(\mathbb{R}^5)}^{5/3}) + C\|z^\omega\|_{X([t_*, t_*+T] \times \mathbb{R}^5)}^{7/3} \\ &\quad + C\|z^\omega\|_{L_t^1([t_*, t_*+T]; L_x^{10}(\mathbb{R}^5))}^5 \end{aligned}$$

for all $t \in [t_*, t_* + T]$. Applying Proposition 2.3(ii) as above, we obtain the existence of a set $\tilde{\Omega}_{t_*, T, \varepsilon}(\mathbb{R}^5) \subset \Omega$ such that $P(\tilde{\Omega}_{t_*, T, \varepsilon}^c(\mathbb{R}^5)) < \varepsilon/2$ and, for all $\omega \in \tilde{\Omega}_{t_*, T, \varepsilon}(\mathbb{R}^5)$,

$$\begin{aligned} \|(v^\omega, \partial_t v^\omega)\|_{L^\infty([t_*, t_*+T]; \dot{H}_x^1(\mathbb{R}^5) \times L_x^2(\mathbb{R}^5))} &\leq C(\|v_0\|_{\dot{H}^1(\mathbb{R}^5)} + \|v_1\|_{L^2(\mathbb{R}^5)} + \|v_0\|_{\dot{H}^1(\mathbb{R}^5)}^{5/3}) \\ &\quad + (K_1 T^{3/7} \max(|t_*|, |t_* + T|)(\|u_0\|_{L^2(\mathbb{R}^5)} + \|u_1\|_{H^{-1}(\mathbb{R}^5)})^{7/3} \\ &\quad + (K_2 T \max(|t_*|, |t_* + T|)(\|u_0\|_{L^2(\mathbb{R}^5)} + \|u_1\|_{H^{-1}(\mathbb{R}^5)})^5) \\ &\leq C(t_*, T, \varepsilon, \|(v_0, v_1)\|_{\dot{H}^1(\mathbb{R}^5) \times L^2(\mathbb{R}^5)}, \|(u_0, u_1)\|_{L^2(\mathbb{R}^5) \times H^{-1}(\mathbb{R}^5)}) =: \tilde{A}(\mathbb{R}^5). \end{aligned} \tag{5.15}$$

We now return to general dimension $d = 4$ or 5 . For \tilde{A} defined in (5.14) for $d = 4$, and in (5.15) for $d = 5$, and for K and γ as in (5.9), we consider $\tau(\tilde{A}, K, \gamma)$ defined in Proposition 4.3. Following the proof of Proposition 5.4 with $[0, T]$ replaced by $[t_*, t_*+T]$, $\tau(A, K, \gamma)$ replaced by $\tau(\tilde{A}, K, \gamma)$,

$$\begin{aligned} \tau_*(\tilde{A}, K, \gamma, t_*, T, \varepsilon) \\ := \min \left\{ \tau(\tilde{A}, K, \gamma), \frac{1}{2} \left(\frac{c}{2 \max(|t_*|^2, |t_* + T|^2) \log(2T/\varepsilon)} \right)^{\frac{d+2}{2(d-2-\gamma(d+2))}} \right\}, \end{aligned}$$

and initial data (v_0, v_1) at $t = t_*$ for the first interval $[t_*, t_* + \tau_*]$ on which we apply Proposition 4.3, we obtain the existence of a set $\Omega'_{t_*, T, \varepsilon} \subset \Omega$ with $P((\Omega'_{t_*, T, \varepsilon})^c) < \varepsilon$ such that, for any $\omega \in \Omega'_{t_*, T, \varepsilon}$, (5.11) admits a unique solution on $[t_*, t_* + T]$. Corollary 5.5 then follows exactly the same way Theorem 5.3 follows from Proposition 5.4. \square

Theorem 5.3 above essentially states that there exists a set $\Theta \subset H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$ of initial data with $0 < s \leq 1$ if $d = 4$, and $0 \leq s \leq 1$ if $d = 5$, of full measure, such that for each $(\phi_0, \phi_1) \in \Theta$, equation (1.1) admits a unique global solution with (ϕ_0, ϕ_1) as initial data at $t = 0$. We recall the notation $\Phi(t) : (\phi_0, \phi_1) \mapsto u(t)$ for the solution map of (1.1). Even though at time $t = 0$ we have a set Θ of initial data of full measure, this does not a priori prevent $\Phi(t)(\Theta)$ from being of small measure for $t \neq 0$. Proposition 5.6 below ensures that there exists a set Σ on which the flow is globally defined and $\Phi(t)(\Sigma)$ is of full measure for all $t \in \mathbb{R}$. See [19, Proposition 3.1] and [52, Theorem 1.2] for related results concerning cubic NLW on \mathbb{T}^3 and cubic NLS on \mathbb{T} , respectively.

Proposition 5.6 (Existence of invariant sets of full measure). *Let $d = 4$ or 5 , $0 < s \leq 1$ if $d = 4$, and $0 \leq s \leq 1$ if $d = 5$. Then, for any countable subgroup \mathcal{T} of $(\mathbb{R}, +)$, there exists $\Sigma_{\mathcal{T}} \subset H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$ of full measure, i.e. $\mu_{(w_0, w_1)}(\Sigma_{\mathcal{T}}) = 1$ for all $\mu_{(w_0, w_1)} \in \mathcal{M}_s$, such that for every $(\phi_0, \phi_1) \in \Sigma_{\mathcal{T}}$, equation (1.1) admits a unique global solution u with initial data $(u, \partial_t u)|_{t=0} = (\phi_0, \phi_1)$, and $\Phi(t)(\Sigma_{\mathcal{T}}) = \Sigma_{\mathcal{T}}$ for all $t \in \mathcal{T}$.*

As a consequence, there exists a set $\Sigma \subset H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$ such that the flow of the energy-critical defocusing NLW (1.1) is globally defined on Σ and $\Phi(t)(\Sigma)$ is of full measure for all $t \in \mathbb{R}$.

Proof. Let $t_* \in \mathbb{R}$. We first find full measure sets Θ and $\Theta_n(t_*)$ of initial data in $H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$ with $0 < s \leq 1$ if $d = 4$, and $0 \leq s \leq 1$ if $d = 5$, that give rise to unique global

solutions of the energy-critical defocusing NLW on \mathbb{R}^d . This is merely a reformulation of Theorem 5.3 and Corollary 5.5. These sets will then be used to construct a set Σ of full measure such that the flow of equation (1.1) is globally defined on Σ and $\Phi(t)(\Sigma)$ is of full measure for all $t \in \mathbb{R}$.

In the following, we present the proof for the case $d = 4$. Let $0 < \gamma < 1/3$. Given $A, K > 0$, we recall $\tau(A, K, \gamma)$ defined in Proposition 4.3. For a finite time interval $I = [a, b]$ and $\tau_* > 0$, we denote by $\mathcal{I}_k(I) = [a + k\tau_*, a + (k + 1)\tau_*]$, $k = 0, 1, \dots$, the $\lceil |I|/\tau_* \rceil$ or $\lceil |I|/\tau_* \rceil + 1$ subintervals of length τ_* covering I . For $0 < s \leq 1$, we then consider the set

$$\Theta := \left\{ (\phi_0, \phi_1) \in H^s(\mathbb{R}^4) \times H^{s-1}(\mathbb{R}^4) : S(t)(\phi_0, \phi_1) \in L^3_{t,\text{loc}} L^6_x \cap L^1_{t,\text{loc}} L^\infty_x, \right. \\ \left. \begin{aligned} &\text{there exist } C, K > 0 \text{ such that } \|S(t)(\phi_0, \phi_1)\|_{X(\mathcal{I}_k(I) \times \mathbb{R}^4)} \leq K \tau_*^\gamma \\ &\text{for all } I = [-T, T], 0 < T < \infty, k = 0, 1, \dots, \text{ and some} \\ &0 < \tau_*(I) \leq \tau \left(C \|S(t)(\phi_0, \phi_1)\|_{X(I \times \mathbb{R}^4)}^3 e^{C \|S(t)(\phi_0, \phi_1)\|_{L^1_t L^\infty_x}}, K, \gamma \right) \right\}. \end{aligned} \right.$$

We have seen in the proofs of Proposition 5.2, Proposition 5.4, and Theorem 5.3 that $\mu_{(u_0, u_1)}(\Theta) = 1$ for any $\mu_{(u_0, u_1)} \in \mathcal{M}_s$. Moreover, for any $(\phi_0, \phi_1) \in \Theta$, there exists a unique global solution of equation (1.1) with initial data $(u(0), \partial_t u(0)) = (\phi_0, \phi_1)$.

Similarly, for $0 < s \leq 1$, $t_* \in \mathbb{R}$, and $n \in \mathbb{N}$, we define

$$\Theta_n(t_*) := \left\{ (\phi_0, \phi_1) \in H^s(\mathbb{R}^4) \times H^{s-1}(\mathbb{R}^4) : S(t)(\phi_0, \phi_1) \in L^3_{t,\text{loc}} L^6_x \cap L^1_{t,\text{loc}} L^\infty_x, \right. \\ \left. \begin{aligned} &\text{there exist } C, K > 0 \text{ such that } \|S(t)(\phi_0, \phi_1)\|_{X(\mathcal{I}_k(I) \times \mathbb{R}^4)} \leq K \tau_*^\gamma \\ &\text{for all } I = [t_* - T, t_* + T], 0 < T < \infty, k = 0, 1, \dots, \text{ and some} \\ &0 < \tau_*(I) \leq \tau \left(C(n + n^2 + \|S(t)(\phi_0, \phi_1)\|_{X(I \times \mathbb{R}^4)}^3) e^{C \|S(t)(\phi_0, \phi_1)\|_{L^1_t L^\infty_x}}, K, \gamma \right) \right\}. \end{aligned} \right.$$

Notice that $\Theta = \Theta_0(0)$. By Proposition 4.3, $\tau(A, K, \gamma)$ is nonincreasing in A . As a consequence, $\Theta_n(t_*) \subset \Theta_m(t_*)$ for all $n \geq m$ and $t_* \in \mathbb{R}$. By Corollary 5.5 and its proof based on Propositions 5.2 and Proposition 5.4, we have that $\mu_{(u_0, u_1)}(\Theta_n(t_*)) = 1$ for all $\mu_{(u_0, u_1)} \in \mathcal{M}_s$, $n \in \mathbb{N}$, and $t_* \in \mathbb{R}$. Moreover, for any $t_* \in \mathbb{R}$, $(v_0, v_1) \in H^1(\mathbb{R}^4) \times L^2(\mathbb{R}^4)$ with $\|(v_0, v_1)\|_{H^1 \times L^2} \leq n$, and any $(\phi_0, \phi_1) \in \Theta_n(t_*)$, the defocusing cubic NLW on \mathbb{R}^4 with initial data

$$(u, \partial_t u)|_{t=t_*} = S(t_*)(\phi_0, \phi_1) + (v_0, v_1) \tag{5.16}$$

admits a unique global solution u with

$$(u, \partial_t u) \in (S(t)(\phi_0, \phi_1), \partial_t S(t)(\phi_0, \phi_1)) + C(\mathbb{R}; H^1(\mathbb{R}^4) \times L^2(\mathbb{R}^4)). \tag{5.17}$$

Next, we show that for any $t_0 \in \mathbb{R}$, $n \in \mathbb{N}$, and $(\phi_0, \phi_1) \in \Theta_n(t_*)$, we have

$$(\psi_0, \psi_1) := (S(t_0)(\phi_0, \phi_1), \partial_t S(t_0)(\phi_0, \phi_1)) \in \Theta_n(t_* - t_0). \tag{5.18}$$

It follows easily that

$$S(t)(\psi_0, \psi_1) = S(t + t_0)(\phi_0, \phi_1) \in L^3_{t,\text{loc}}L^6_x \cap L^1_{t,\text{loc}}L^\infty_x.$$

Let $I = [t_* - T, t_* + T]$ for some $0 < T < \infty$. By the second condition in the definition of $\Theta_n(t_*)$ applied to (ϕ_0, ϕ_1) on the interval $[t_* - T, t_* + T]$, it follows that

$$\|S(t - t_0)(\psi_0, \psi_1)\|_{X(\mathcal{I}_k([t_* - T, t_* + T]) \times \mathbb{R}^4)} = \|S(t)(\phi_0, \phi_1)\|_{X(\mathcal{I}_k([t_* - T, t_* + T]) \times \mathbb{R}^4)} \leq K \tau_*^\gamma$$

for all k , and for some positive τ_* satisfying

$$\tau_* \leq \tau \left(C(n + n^2 + \|S(t - t_0)(\psi_0, \psi_1)\|_{X([t_* - T, t_* + T] \times \mathbb{R}^4)}^3 \times e^{C\|S(t - t_0)(\psi_0, \psi_1)\|_{L^1_{[t_* - T, t_* + T]L^\infty_x}}, K, \gamma} \right).$$

Therefore, a simple change of variables shows that (ψ_0, ψ_1) satisfies the second condition in the definition of $\Theta_n(t_* - t_0)$ on $[t_* - t_0 - T, t_* - t_0 + T]$. Hence, $(\psi_0, \psi_1) \in \Theta_n(t_* - t_0)$ and (5.18) is proved.

Let \mathcal{T} be a countable subgroup of $(\mathbb{R}, +)$. Define

$$\tilde{\Theta}_{\mathcal{T}} := \bigcap_{t_* \in \mathcal{T}} \bigcap_{n \in \mathbb{N}} \Theta_n(t_*).$$

Notice that $\mu_{(u_0, u_1)}(\tilde{\Theta}_{\mathcal{T}}) = 1$ for all $\mu_{(u_0, u_1)} \in \mathcal{M}_s$, since $\tilde{\Theta}_{\mathcal{T}}$ is a countable intersection of full measure sets $\Theta_n(t_*)$. By (5.18) and since $\mathcal{T} - t = \mathcal{T}$ for any $t \in \mathcal{T}$, it follows that for all $t \in \mathcal{T}$ and $(\phi_0, \phi_1) \in \tilde{\Theta}_{\mathcal{T}}$ we have

$$(S(t)(\phi_0, \phi_1), \partial_t S(t)(\phi_0, \phi_1)) \in \tilde{\Theta}_{\mathcal{T}}. \quad (5.19)$$

Using the fact that $0 \in \mathcal{T}$ and thus $\tilde{\Theta}_{\mathcal{T}} \subset \bigcap_{n \in \mathbb{N}} \Theta_n(0)$, we deduce from the above discussion regarding the properties of $\Theta_n(t_*)$ that equation (1.1) admits a unique global solution with initial data at $t = 0$ of the form $(\phi_0, \phi_1) + (v_0, v_1)$, where $(\phi_0, \phi_1) \in \tilde{\Theta}_{\mathcal{T}}$ and $(v_0, v_1) \in H^1(\mathbb{R}^4) \times L^2(\mathbb{R}^4)$. Moreover, by (5.17) and (5.19),

$$(u(t), \partial_t u(t)) \in \tilde{\Theta}_{\mathcal{T}} + H^1(\mathbb{R}^4) \times L^2(\mathbb{R}^4) \quad \text{for all } t \in \mathcal{T}.$$

In other words, if we set $\Sigma_{\mathcal{T}} := \tilde{\Theta}_{\mathcal{T}} + H^1(\mathbb{R}^4) \times L^2(\mathbb{R}^4)$, the flow of equation (1.1) is defined globally in time on $\Sigma_{\mathcal{T}}$ and

$$\Phi(t)(\Sigma_{\mathcal{T}}) \subset \Sigma_{\mathcal{T}} \quad \text{for all } t \in \mathcal{T}.$$

Using the time reversibility of the equation and the fact that $-t \in \mathcal{T}$ for all $t \in \mathcal{T}$, we then conclude that $\Phi(t)(\Sigma_{\mathcal{T}}) = \Sigma_{\mathcal{T}}$ for all $t \in \mathcal{T}$. Moreover, $\mu_{(u_0, u_1)}(\Sigma_{\mathcal{T}}) = 1$ for all $\mu_{(u_0, u_1)} \in \mathcal{M}_s$, since $\tilde{\Theta}_{\mathcal{T}}$ is of full measure.

Lastly, noticing that for any $t \in \mathbb{R}$, $t\mathbb{Z}$ is a countable subgroup of $(\mathbb{R}, +)$ containing t , we define

$$\Sigma := \bigcup_{t \in \mathbb{R}} \Sigma_{t\mathbb{Z}}.$$

It easily follows that Σ is of full measure and the flow of equation (1.1) is globally defined on Σ . Moreover,

$$\Sigma_{t\mathbb{Z}} = \Phi(t)(\Sigma_{t\mathbb{Z}}) \subset \Phi(t)(\Sigma).$$

Therefore,

$$\mu_{(u_0, u_1)}(\Phi(t)(\Sigma)) = 1$$

for all $t \in \mathbb{R}$ and $\mu_{(u_0, u_1)} \in \mathcal{M}_s$. This completes the proof of Proposition 5.6 in the case $d = 4$.

The proof for $d = 5$ is completely analogous. The only difference is the definition of $\Theta_n(t_*)$ (and thus also of $\Theta = \Theta_0(0)$) which, for $d = 5$, becomes

$$\Theta_n(t_*, \mathbb{R}^5) := \left\{ (\phi_0, \phi_1) \in H^s(\mathbb{R}^5) \times H^{s-1}(\mathbb{R}^5) : S(t)(\phi_0, \phi_1) \in L_{t, \text{loc}}^{7/3} L_x^{14/3} \cap L_{t, \text{loc}}^1 L_x^{10}, \right. \\ \left. \text{there exist } C, K > 0 \text{ such that } \|S(t)(\phi_0, \phi_1)\|_{X(\mathcal{I}_k(I) \times \mathbb{R}^5)} \leq K \tau_*^\gamma \right. \\ \left. \text{for all } I = [t_* - T, t_* + T], 0 < T < \infty, k = 0, 1, \dots, \text{ and some} \right. \\ \left. 0 < \tau_*(I) \leq \tau \left(C(n + n^{5/3} + \|S(t)(\phi_0, \phi_1)\|_{X(I \times \mathbb{R}^5)}^{7/3} + \|S(t)(\phi_0, \phi_1)\|_{L_t^1 L_x^{10}}^5), K, \gamma \right) \right\},$$

where $0 \leq s \leq 1$ and $0 < \gamma < 3/7$. □

6. Probabilistic continuous dependence of the flow

In this section we prove the probabilistic continuity of the flow of the energy-critical defocusing cubic NLW (1.1) on \mathbb{R}^4 in $H^s(\mathbb{R}^4) \times H^{s-1}(\mathbb{R}^4)$, $0 < s \leq 1$, with respect to the initial data. The notion of probabilistic continuity that we use here was first proposed by Burq and Tzvetkov [19]. For the readers' convenience, we first recall two lemmas from [19, Appendix A.2], to be used in the proof of Proposition 6.3 below.

Lemma 6.1 ([19, Lemma A.9]). *If h is a Bernoulli random variable independent of a real random variable g with symmetric distribution θ , then hg also has the distribution θ .*

Lemma 6.2 ([19, Lemma A.8]). *Let Y_j , $j = 1, 2$, be Banach spaces endowed with measures μ_j . Let $f : Y_1 \times Y_2 \rightarrow \mathbb{C}$ and $g_1, g_2 : Y_2 \rightarrow \mathbb{C}$ be measurable functions. Then*

$$\mu_1 \otimes \mu_2 \left((x_1, x_2) \in Y_1 \times Y_2 : |f(x_1, x_2)| > \lambda \mid |g_1(x_2)| \leq \varepsilon, |g_2(x_2)| \leq R \right) \\ \leq \sup_{x_2 \in Y_2, |g_1(x_2)| \leq \varepsilon, |g_2(x_2)| \leq R} \mu_1(x_1 \in Y_1 : |f(x_1, x_2)| > \lambda).$$

Building upon the improved local-in-time Strichartz estimates in Proposition 2.3, we will use the strategy developed by Burq and Tzvetkov in [19, Appendix A.2] to obtain the following result.

Proposition 6.3. *Let $d = 4$. Assume that in the definition of the randomization (1.13), the smooth cutoff ψ is replaced by the characteristic function χ_{Q_0} of the unit cube Q_0 centered at the origin. Assume also that the probability distributions $\mu_{0,j}, \mu_{n,j}^{(1)}, \mu_{n,j}^{(2)}$, $n \in \mathcal{I}, j = 0, 1$, are symmetric. Let $0 < s \leq 1, T > 0$, and $\mu \in \mathcal{M}_s$. Then, given $1 \leq q < \infty$ and $2 \leq r \leq \infty$, there exist constants $C, c > 0$ such that for every $\varepsilon, \lambda, \Lambda, R > 0$, we have*

$$\begin{aligned} &\mu \otimes \mu \left(((w_0, w_1), (w'_0, w'_1)) \in (H^s(\mathbb{R}^4) \times H^{s-1}(\mathbb{R}^4))^2 : \right. \\ &\|S(t)(w_0 - w'_0, w_1 - w'_1)\|_{L^q_t([0,T];L^r_x)} > \lambda \text{ or } \|S(t)(w_0 + w'_0, w_1 + w'_1)\|_{L^q_t([0,T];L^r_x)} > \Lambda \\ &\quad \left. | \| (w_0 - w'_0, w_1 - w'_1) \|_{H^s \times H^{s-1}} \leq \varepsilon \text{ and } \| (w_0 + w'_0, w_1 + w'_1) \|_{H^s \times H^{s-1}} \leq R \right) \\ &\leq C \left(\exp \left(-c \frac{\lambda^2}{\max(1, T^2) T^{2/q} \varepsilon^2} \right) + \exp \left(-c \frac{\Lambda^2}{\max(1, T^2) T^{2/q} R^2} \right) \right). \end{aligned} \tag{6.1}$$

Proof. For simplicity of notation, we assume $T \geq 1$. The case $0 < T < 1$ is completely analogous, with the only change coming from Proposition 2.3(ii)&(iii).

We consider $Y := \mathbb{R} \times \mathbb{R}^{\mathcal{I}}$ equipped with the Banach space structure induced by the ℓ^∞ -norm, where \mathcal{I} is the index set in (1.12) for $d = 4$. If $\mathcal{B}(\mathbb{R})$ denotes the Borel σ -algebra of \mathbb{R} , we endow Y with the σ -algebra $\mathcal{B}(\mathbb{R})^{\otimes(0) \cup \mathcal{I}}$ generated by

$$\left\{ \prod_{n \in \{0\} \cup \mathcal{I}} A_n : A_n \in \mathcal{B}(\mathbb{R}) \text{ and } A_n = \mathbb{R} \text{ except for finitely many } n \right\}.$$

Let $\{k_n\}_{n \in \{0\} \cup \mathcal{I}}$ be a system of independent Bernoulli variables. Then the map

$$\omega \mapsto \{k_n(\omega)\}_{n \in \{0\} \cup \mathcal{I}}$$

is measurable and we endow Y with the probability measure induced by this map:

$$\mu_0(A) := P(\{k_n(\omega)\}_{n \in \{0\} \cup \mathcal{I}} \in A) \quad \text{for any } A \in \mathcal{B}(\mathbb{R})^{\otimes(0) \cup \mathcal{I}}.$$

For $\{h_n\}_{n \in \{0\} \cup \mathcal{I}}$, we set $h_{-n} = h_n$, which defines h_n for all $n \in \mathbb{Z}^4$. Then, for $\phi \in L^2(\mathbb{R}^4)$ real-valued, we set

$$h \odot \phi := \sum_{n \in \mathbb{Z}^4} h_n \chi_{Q_0}(D - n) \phi = h_0 \chi_{Q_0}(D) \phi + 2 \operatorname{Re} \sum_{n \in \mathcal{I}} h_n \chi_{Q_0}(D - n) \phi, \tag{6.2}$$

where χ_{Q_0} is the characteristic function of the unit cube $Q_0 = [-1/2, 1/2]^4$ centered at the origin. Since

$$(n + Q_0) \cap (m + Q_0) = \emptyset \quad \text{for any } n, m \in \mathbb{Z}^4, n \neq m, \tag{6.3}$$

we have

$$\begin{aligned} h \odot \phi^\omega &= \sum_{n \in \mathbb{Z}^4} h_n \chi_{Q_0}(D - n) \phi^\omega = \sum_{n \in \mathbb{Z}^4} h_n \chi_{Q_0}(D - n) \sum_{m \in \mathbb{Z}^4} g_m(\omega) \chi_{Q_0}(D - m) \phi \\ &= \sum_{n \in \mathbb{Z}^4} h_n g_n(\omega) \chi_{Q_0}(D - n) \phi = h_0 g_0(\omega) \chi_{Q_0}(D) \phi + \operatorname{Re} \sum_{n \in \mathcal{I}} h_n g_n(\omega) \chi_{Q_0}(D - n) \phi \\ &= h_0 g_0(\omega) \chi_{Q_0}(D) \phi \\ &\quad + \sum_{n \in \mathcal{I}} (h_n \operatorname{Re} g_n(\omega) \operatorname{Re} \chi_{Q_0}(D - n) \phi - h_n \operatorname{Im} g_n(\omega) \operatorname{Im} \chi_{Q_0}(D - n) \phi). \end{aligned} \tag{6.4}$$

Let $\{h_{n,j}\}_{n \in \{0\} \cup \mathcal{I}}$, $j = 0, 1$, be two systems of independent Bernoulli random variables such that $\{h_{0,j}, g_{0,j}, h_{n,j}, \operatorname{Re} g_{n,j}, \operatorname{Im} g_{n,j}\}_{n \in \mathcal{I}, j=0,1}$ are independent. For $j = 0, 1$, we then consider the following random variables with values in $\mathbb{R}^{\mathbb{Z}^4}$ endowed with the σ -algebra $\mathcal{B}(\mathbb{R})^{\otimes \mathbb{Z}^4}$:

$$g_j := \left(\begin{array}{c} g_{0,j} \\ \operatorname{Re} g_{n,j} \\ \operatorname{Im} g_{n,j} \end{array} \right)_{n \in \mathcal{I}}, \quad h_j g_j := \left(\begin{array}{c} h_{0,j} g_{0,j} \\ h_{n,j} \operatorname{Re} g_{n,j} \\ h_{n,j} \operatorname{Im} g_{n,j} \end{array} \right)_{n \in \mathcal{I}} : \Omega \rightarrow \mathbb{R} \times \mathbb{R}^{\mathcal{I}} \times \mathbb{R}^{\mathcal{I}} = \mathbb{R}^{\mathbb{Z}^4}.$$

By the independence of $\{h_{0,j}, g_{0,j}, h_{n,j}, \operatorname{Re} g_{n,j}, \operatorname{Im} g_{n,j}\}_{n \in \mathcal{I}, j=0,1}$ and the fact that $g_{0,j}$, $\operatorname{Re} g_{n,j}$, and $\operatorname{Im} g_{n,j}$ are symmetric, it then follows using Lemma 6.1 that the probability distributions $\theta_{h_j g_j}$ and θ_{g_j} of $h_j g_j$ and g_j respectively coincide. As a consequence, if $\mu = \mu_{(u_0, u_1)}$ for some $(u_0, u_1) \in H^s(\mathbb{R}^4) \times H^{s-1}(\mathbb{R}^4)$, then for all $\lambda > 0$,

$$\begin{aligned} & \mu \otimes \mu_0 \otimes \mu_0((w_0, w_1) \in H^s \times H^{s-1}, (h_0, h_1) \in Y \times Y : \\ & \qquad \|S(t)(h_0 \odot w_0, h_1 \odot w_1)\|_{L_t^q L_x^r} > \lambda) \\ &= P(\|S(t)(h_0(\omega)g_0(\omega) \odot u_0, h_1(\omega)g_1(\omega) \odot u_1)\|_{L_t^q L_x^r} > \lambda) \\ &= \int_{\mathbb{R}^{\mathbb{Z}^4} \times \mathbb{R}^{\mathbb{Z}^4}} \mathbf{1}_{\{\|S(t)(y \odot u_0, y' \odot u_1)\|_{L_t^q L_x^r} > \lambda\}} d\theta_{h_0 g_0}(y) d\theta_{h_1 g_1}(y') \\ &= \int_{\mathbb{R}^{\mathbb{Z}^4} \times \mathbb{R}^{\mathbb{Z}^4}} \mathbf{1}_{\{\|S(t)(y \odot u_0, y' \odot u_1)\|_{L_t^q L_x^r} > \lambda\}} d\theta_{g_0}(y) d\theta_{g_1}(y') \\ &= P(\|S(t)(g_0(\omega) \odot u_0, g_1(\omega) \odot u_1)\|_{L_t^q L_x^r} > \lambda) \\ &= \mu((w_0, w_1) \in H^s(\mathbb{R}^4) \times H^{s-1}(\mathbb{R}^4) : \|S(t)(w_0, w_1)\|_{L_t^q L_x^r} > \lambda). \end{aligned}$$

Arguing analogously, we find that

$$\begin{aligned} M_1 &:= \mu \otimes \mu((w_0, w_1), (w'_0, w'_1)) \in (H^s(\mathbb{R}^4) \times H^{s-1}(\mathbb{R}^4))^2 : \\ & \quad \|S(t)(w_0 - w'_0, w_1 - w'_1)\|_{L_t^q([0, T]; L_x^r)} > \lambda \mid \\ & \quad \|(w_0 - w'_0, w_1 - w'_1)\|_{H^s \times H^{s-1}} \leq \varepsilon, \|(w_0 + w'_0, w_1 + w'_1)\|_{H^s \times H^{s-1}} \leq R) \\ &= \mu \otimes \mu \otimes \mu_0 \otimes \mu_0((w_0, w_1), (w'_0, w'_1)) \in (H^s(\mathbb{R}^4) \times H^{s-1}(\mathbb{R}^4))^2, (h_0, h_1) \in Y \times Y : \\ & \quad \|S(t)(h_0 \odot (w_0 - w'_0), h_1 \odot (w_1 - w'_1))\|_{L_t^q([0, T]; L_x^r)} > \lambda \mid \\ & \quad \|(h_0 \odot (w_0 - w'_0), h_1 \odot (w_1 - w'_1))\|_{H^s \times H^{s-1}} \leq \varepsilon \\ & \quad \text{and } \|(h_0 \odot (w_0 + w'_0), h_1 \odot (w_1 + w'_1))\|_{H^s \times H^{s-1}} \leq R). \end{aligned}$$

Noticing by (6.2) and (6.3) that $\|h \odot \phi\|_{H^\sigma} = \|\phi\|_{H^\sigma}$ for any $\phi \in H^\sigma$, $\sigma \in \mathbb{R}$, and any Bernoulli random variable h , it follows that

$$\begin{aligned} M_1 &= \mu \otimes \mu \otimes \mu_0 \otimes \mu_0((w_0, w_1), (w'_0, w'_1)) \in (H^s(\mathbb{R}^4) \times H^{s-1}(\mathbb{R}^4))^2, (h_0, h_1) \in Y \times Y : \\ & \quad \|S(t)(h_0 \odot (w_0 - w'_0), h_1 \odot (w_1 - w'_1))\|_{L_t^q([0, T]; L_x^r)} > \lambda \mid \\ & \quad \|(w_0 - w'_0, w_1 - w'_1)\|_{H^s \times H^{s-1}} \leq \varepsilon, \|(w_0 + w'_0, w_1 + w'_1)\|_{H^s \times H^{s-1}} \leq R). \end{aligned}$$

Then, by Lemma 6.2,

$$M_1 \leq \sup_{\|(w_0-w'_0, w_1-w'_1)\|_{H^s \times H^{s-1}} \leq \varepsilon} \mu_0 \otimes \mu_0((h_0, h_1) \in Y \times Y : \|S(t)(h_0 \odot (w_0 - w'_0), h_1 \odot (w_1 - w'_1))\|_{L^q_t([0, T]; L^r_x)} > \lambda).$$

By the improved local-in-time Strichartz estimates in Proposition 2.3(ii)&(iii) with Bernoulli random variables, it then follows that

$$M_1 \leq C \exp\left(-c \frac{\lambda^2}{T^{2+2/q} \varepsilon^2}\right).$$

Similarly, we obtain

$$\begin{aligned} M_2 &:= \mu \otimes \mu((w_0, w_1), (w'_0, w'_1)) \in (H^s(\mathbb{R}^4) \times H^{s-1}(\mathbb{R}^4))^2 : \\ &\quad \|S(t)(w_0 + w'_0, w_1 + w'_1)\|_{L^q_t([0, T]; L^r_x)} > \Lambda \mid \\ &\quad \|(w_0 - w'_0, w_1 - w'_1)\|_{H^s \times H^{s-1}} \leq \varepsilon \text{ and } \|(w_0 + w'_0, w_1 + w'_1)\|_{H^s \times H^{s-1}} \leq R \\ &\leq \sup_{\|(w_0+w'_0, w_1+w'_1)\|_{H^s \times H^{s-1}} \leq R} \mu_0 \otimes \mu_0((h_0, h_1) \in Y \times Y : \\ &\quad \|S(t)(h_0 \odot (w_0 + w'_0), h_1 \odot (w_1 + w'_1))\|_{L^q_t([0, T]; L^r_x)} > \Lambda) \\ &\leq C \exp\left(-c \frac{\Lambda^2}{T^{2+2/q} R^2}\right). \end{aligned}$$

Therefore, (6.1) follows. □

In the remainder of this section, we prove the probabilistic continuity of the flow with respect to the initial data in Theorem 1.4.

Proof of Theorem 1.4. For simplicity of notation, we assume $T \geq 1$. The case $0 < T < 1$ is completely analogous.

Step 1 (Control of the linear parts of the solutions and of their difference). Let $\{\eta_k\}_{k \in \mathbb{N}}$ with $\eta_k \in (0, 1)$ and $\eta_k \searrow 0$ as $k \rightarrow \infty$. By Proposition 6.3, for any $(q, r) \in \{(3, 6), (3, \infty), (2, 8)\}$, $\mu \in \mathcal{M}_s$, $\alpha \in (0, 1/2)$, and $L(\eta_k, T, R, \alpha) = L(\eta_k)$ to be chosen later such that $L(\eta_k) \rightarrow \infty$ as $k \rightarrow \infty$, we have

$$\begin{aligned} &\mu \otimes \mu((w_0, w_1), (w'_0, w'_1)) \in (H^s(\mathbb{R}^4) \times H^{s-1}(\mathbb{R}^4))^2 : \\ &\quad \|S(t)(w_0 - w'_0, w_1 - w'_1)\|_{L^q_T L^r_x} > \eta_k^{1-\alpha} \text{ or } \|S(t)(w_0, w_1)\|_{L^q_T L^r_x} > L(\eta_k) \\ &\quad \text{or } \|S(t)(w'_0, w'_1)\|_{L^q_T L^r_x} > L(\eta_k) \mid \\ &\quad \|(w_0 - w'_0, w_1 - w'_1)\|_{H^s(\mathbb{R}^4) \times H^{s-1}(\mathbb{R}^4)} \leq \eta_k, (w_0, w_1), (w'_0, w'_1) \in B_R \\ &\leq C \left(\exp\left(-\frac{c}{T^3 \eta_k^{2\alpha}}\right) + \exp\left(-c \frac{L(\eta_k)^2}{T^3 R^2}\right) \right) (1 + o(1)) \rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned}$$

Therefore, when we estimate the conditional probability in Theorem 1.4, we can assume that

$$\|z - z'\|_{L_T^3 L_x^r} \leq \eta_k^{1-\alpha}, \quad \|z\|_{L_T^3 L_x^r} \leq L(\eta_k), \quad \|z'\|_{L_T^3 L_x^r} \leq L(\eta_k) \quad \text{for } r = 6, \infty, \quad (6.5)$$

$$\|z - z'\|_{L_T^2 L_x^8} \leq \eta_k^{1-\alpha}, \quad \|z\|_{L_T^2 L_x^8} \leq L(\eta_k), \quad \|z'\|_{L_T^2 L_x^8} \leq L(\eta_k), \quad (6.6)$$

where we set $z(t) := S(t)(w_0, w_1)$ and $z'(t) := S(t)(w'_0, w'_1)$. We also denote by $v(t) := \Phi(t)(w_0, w_1) - z(t)$ and $v'(t) := \Phi(t)(w'_0, w'_1) - z'(t)$ the nonlinear parts of the solutions $\Phi(t)(w_0, w_1)$ and $\Phi(t)(w'_0, w'_1)$, respectively.

Step 2 (Control of Strichartz norms of the nonlinear parts of the solutions). We now prove that there exists $G(\eta_k) > 0$ such that

$$\begin{aligned} M(\eta_k) := \mu \otimes \mu & \left((w_0, w_1), (w'_0, w'_1) \in (H^s(\mathbb{R}^4) \times H^{s-1}(\mathbb{R}^4))^2 : \right. \\ & \|v\|_{L_T^3 L_x^6} + \|v\|_{L_T^2 L_x^8} > G(\eta_k) \text{ or } \|v'\|_{L_T^3 L_x^6} + \|v'\|_{L_T^2 L_x^8} > G(\eta_k); \\ & \left. \|z\|_{L_T^3 L_x^r} \leq L(\eta_k) \text{ and } \|z'\|_{L_T^3 L_x^r} \leq L(\eta_k) \text{ for } r = 6, \infty \mid \right. \\ & \left. \|(w_0 - w'_0, w_1 - w'_1)\|_{H^s(\mathbb{R}^4) \times H^{s-1}(\mathbb{R}^4)} \leq \eta_k \right. \\ & \left. \text{and } (w_0, w_1), (w'_0, w'_1) \in B_R \right) \rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned} \quad (6.7)$$

As a consequence, in addition to (6.5) and (6.6), we can assume that

$$\|v\|_{L_T^3 L_x^6} + \|v\|_{L_T^2 L_x^8} \leq G(\eta_k), \quad (6.8)$$

$$\|v'\|_{L_T^3 L_x^6} + \|v'\|_{L_T^2 L_x^8} \leq G(\eta_k), \quad (6.9)$$

when we estimate the conditional probability in Theorem 1.4.

Arguing as in the proof of Proposition 6.3, we get

$$\begin{aligned} M(\eta_k) \leq \sup & \left\{ \mu_0 \otimes \mu_0 \left((h_0, h_1) \in Y \times Y : \|\tilde{v}\|_{L_T^3 L_x^6} + \|\tilde{v}\|_{L_T^2 L_x^8} > G(\eta_k) \text{ or } \right. \right. \\ & \left. \|\tilde{v}'\|_{L_T^3 L_x^6} + \|\tilde{v}'\|_{L_T^2 L_x^8} > G(\eta_k); \right. \\ & \left. \|\tilde{z}\|_{L_T^3 L_x^r} \leq L(\eta_k) \text{ and } \|\tilde{z}'\|_{L_T^3 L_x^r} \leq L(\eta_k) \text{ for } r = 6, \infty \right) : \\ & \left. \|(w_0 - w'_0, w_1 - w'_1)\|_{H^s(\mathbb{R}^4) \times H^{s-1}(\mathbb{R}^4)} \leq \eta_k, (w_0, w_1), (w'_0, w'_1) \in B_R \right\}, \end{aligned}$$

where we have denoted by \tilde{z} and \tilde{v} the linear and nonlinear parts of the solution with initial data $(h_0 \odot w_0, h_1 \odot w_1)$, and by \tilde{z}' and \tilde{v}' the linear and nonlinear parts of the

solution with initial data $(h_0 \odot w'_0, h_1 \odot w'_1)$. Then we can upper bound $M(\eta_k)$ by

$$\begin{aligned}
 M(\eta_k) &\leq \sup_{(w_0, w_1) \in B_R} \mu_0 \otimes \mu_0 \left((h_0, h_1) \in Y \times Y : \|\tilde{v}\|_{L_T^3 L_x^6} + \|\tilde{v}\|_{L_T^2 L_x^8} > G(\eta_k), \right. \\
 &\quad \left. \|\tilde{z}\|_{L_T^3 L_x^r} \leq L(\eta_k) \text{ for } r = 6, \infty \right) \\
 &+ \sup_{(w'_0, w'_1) \in B_R} \mu_0 \otimes \mu_0 \left((h_0, h_1) \in Y \times Y : \|\tilde{v}'\|_{L_T^3 L_x^6} + \|\tilde{v}'\|_{L_T^2 L_x^8} > G(\eta_k), \right. \\
 &\quad \left. \|\tilde{z}'\|_{L_T^3 L_x^r} \leq L(\eta_k) \text{ for } r = 6, \infty \right) \\
 &= 2 \sup_{(w_0, w_1) \in B_R} P \left(\omega \in \tilde{\Omega}(w_0, w_1) : \|\tilde{v}^\omega\|_{L_T^3 L_x^6} + \|\tilde{v}^\omega\|_{L_T^2 L_x^8} > G(\eta_k), \right. \\
 &\quad \left. \|\tilde{z}^\omega\|_{L_T^3 L_x^r} \leq L(\eta_k) \text{ for } r = 6, \infty \right) \\
 &=: 2 \sup_{(w_0, w_1) \in B_R} M_{(w_0, w_1)}(\eta_k). \tag{6.10}
 \end{aligned}$$

Here, the set $\tilde{\Omega}(w_0, w_1) \subset \Omega$ with $P(\tilde{\Omega}(w_0, w_1)) = 1$ was defined in Theorem 5.3 and has the property that for all $\omega \in \tilde{\Omega}(w_0, w_1)$, (1.1) admits a unique global solution \tilde{u}^ω with initial data $(\tilde{u}^\omega(0), \partial_t \tilde{u}^\omega(0)) = (h_0(\omega) \odot w_0, h_1(\omega) \odot w_1)$. We also denoted by \tilde{z}^ω and \tilde{v}^ω the linear and nonlinear parts of the solution \tilde{u}^ω .

We choose $\varepsilon_k = T/\log(1/\eta_k)$ and consider the set

$$\begin{aligned}
 \tilde{\Omega}_1(w_0, w_1, \eta_k) &:= \{ \omega \in \tilde{\Omega}_1(w_0, w_1) \cap \Omega_{T, \varepsilon_k}(w_0, w_1) : \|\tilde{v}^\omega\|_{L_T^3 L_x^6} + \|\tilde{v}^\omega\|_{L_T^2 L_x^8} > G(\eta_k) \\
 &\quad \text{and } \|\tilde{z}^\omega\|_{L_T^3 L_x^r} \leq L(\eta_k) \text{ for } r = 6, \infty \},
 \end{aligned}$$

where $\Omega_{T, \varepsilon_k}(w_0, w_1)$ was defined in Proposition 5.4 and has $P(\Omega_{T, \varepsilon_k}^c(w_0, w_1)) < \varepsilon_k$. Then

$$M_{(w_0, w_1)}(\eta_k) \leq P(\tilde{\Omega}_1(w_0, w_1, \eta_k)) + \varepsilon_k.$$

Next, we show that there exists $G(\eta_k) = G(\eta_k, L(\eta_k), T, R)$ such that if $(w_0, w_1) \in B_R$, then $\tilde{\Omega}_1(w_0, w_1, \eta_k) = \emptyset$. In particular, this shows that

$$M_{(w_0, w_1)}(\eta_k) \leq \frac{T}{\log(1/\eta_k)} \rightarrow 0 \quad \text{as } k \rightarrow \infty. \tag{6.11}$$

First, for all $\omega \in \tilde{\Omega}_1(w_0, w_1, \eta_k)$, we have $\|\tilde{z}^\omega\|_{L_T^3(\mathbb{R}; L_x^r)} \leq L(\eta_k)$ for $r = 6, \infty$, and, as in (5.4), we obtain

$$\begin{aligned}
 E(\tilde{v}^\omega(t)) &\leq C \|\tilde{z}^\omega\|_{L_T^3 L_x^6}^6 e^{C \|\tilde{z}^\omega\|_{L_T^1 L_x^\infty}} \leq C \|\tilde{z}^\omega\|_{L_T^3 L_x^6}^6 e^{CT^{2/3} \|\tilde{z}^\omega\|_{L_T^3 L_x^\infty}} \\
 &\leq CL(\eta_k)^6 e^{CT^{2/3} L(\eta_k)} \leq e^{CT^{2/3} L(\eta_k)}.
 \end{aligned}$$

Then, for $(w_0, w_1) \in B_R$, by (5.7) we have

$$\begin{aligned}
 \|\tilde{v}^\omega\|_{L_T^3 L_x^6} + \|\tilde{v}^\omega\|_{L_T^2 L_x^8} &\leq \tilde{F}(e^{C_1 T^{2/3} L(\eta_k)}, R) T^7 (\log(T/\varepsilon))^3 \\
 &= F_R(e^{C_1 T^{2/3} L(\eta_k)}) T^7 (\log \log(1/\eta_k))^3 =: G(\eta_k), \tag{6.12}
 \end{aligned}$$

where $C_1 \geq 1$ is an absolute constant. In the second to last equality we have used the definition $F_R(x) := \tilde{F}(x, R)$ with \tilde{F} as in (5.7). This fixes $G(\eta_k)$ and shows that, with this choice, $\tilde{\Omega}_1(w_0, w_1, \eta_k) = \emptyset$. Thus, (6.11) holds for all $(w_0, w_1) \in B_R$. Combining (6.11) with (6.10), we deduce that $M(\eta_k) \rightarrow 0$ as $k \rightarrow \infty$, which proves (6.7).

Recall that \tilde{F} defined in (5.7) is a nondecreasing function in both its variables. By increasing $F_R = F(\cdot, R)$ if needed, we can choose it to be strictly increasing and to satisfy $F_R(x) \geq x$ for all $x > 0$. In particular,

$$F_R(e^{C_1 T^{2/3} L(\eta_k)}) \geq e^{C_1 T^{2/3} L(\eta_k)} \geq C_1 T^{2/3} L(\eta_k). \tag{6.13}$$

Then, by (6.13) and (6.12), and since $T \geq 1$, we have

$$C_1 L(\eta_k) < G(\eta_k). \tag{6.14}$$

Step 3 (Control of the difference of the nonlinear parts of the solutions). For the remainder of the proof, we assume that the bounds on the linear and nonlinear parts of solutions given in (6.5), (6.6), (6.8), and (6.9) hold.

Using the equations satisfied by v and v' , we obtain

$$\begin{aligned} & \frac{d}{dt} \|(v(t) - v'(t), \partial_t v(t) - \partial_t v'(t))\|_{\dot{H}_x^1 \times L_x^2}^2 \\ & \leq 2 \left| \int_{\mathbb{R}^4} \partial_t(v(t) - v'(t))(\partial_t^2 - \Delta)(v(t) - v'(t)) dx \right| \\ & \leq 2 \|(v(t) - v'(t), \partial_t v(t) - \partial_t v'(t))\|_{\dot{H}_x^1 \times L_x^2} \|(v + z)^3 - (v' + z')^3\|_{L_x^2}. \end{aligned}$$

Then, by the Sobolev embedding $\dot{H}^1(\mathbb{R}^4) \subset L^4(\mathbb{R}^4)$ and Hölder's inequality, it follows that

$$\begin{aligned} & \frac{d}{dt} \|(v(t) - v'(t), \partial_t v(t) - \partial_t v'(t))\|_{\dot{H}_x^1(\mathbb{R}^4) \times L_x^2(\mathbb{R}^4)} \\ & \leq C \|(v(t) - v'(t), \partial_t v(t) - \partial_t v'(t))\|_{\dot{H}_x^1 \times L_x^2} (\|v(t)\|_{L_x^8}^2 + \|v'(t)\|_{L_x^8}^2 + \|z(t)\|_{L_x^8}^2 + \|z'(t)\|_{L_x^8}^2) \\ & \quad + C \|z(t) - z'(t)\|_{L_x^6} (\|v(t)\|_{L_x^6}^2 + \|v'(t)\|_{L_x^6}^2 + \|z(t)\|_{L_x^6}^2 + \|z'(t)\|_{L_x^6}^2). \end{aligned}$$

Integrating in time from 0 to $t \leq T$ and using Hölder's inequality gives

$$\begin{aligned} & \|(v(t) - v'(t), \partial_t v(t) - \partial_t v'(t))\|_{\dot{H}_x^1(\mathbb{R}^4) \times L_x^2(\mathbb{R}^4)} \\ & \leq C \|z - z'\|_{L^3([0,t]; L_x^6)} (\|v\|_{L^3([0,t]; L_x^6)}^2 + \|v'\|_{L^3([0,t]; L_x^6)}^2 + \|z\|_{L^3([0,t]; L_x^6)}^2 + \|z'\|_{L^3([0,t]; L_x^6)}^2) \\ & \quad + C \int_0^t \|(v(t') - v'(t'), \partial_t v(t') - \partial_t v'(t'))\|_{\dot{H}_x^1(\mathbb{R}^4) \times L_x^2(\mathbb{R}^4)} \\ & \quad \quad \times (\|v(t')\|_{L_x^8}^2 + \|v'(t')\|_{L_x^8}^2 + \|z(t')\|_{L_x^8}^2 + \|z'(t')\|_{L_x^8}^2) dt'. \end{aligned}$$

Then, Gronwall’s inequality yields, for any $t \in [0, T]$,

$$\begin{aligned} & \| (v(t) - v'(t), \partial_t v(t) - \partial_t v'(t)) \|_{\dot{H}_x^1(\mathbb{R}^4) \times L_x^2(\mathbb{R}^4)} \\ & \leq C \| z - z' \|_{L^3([0,t]; L_x^6)} \left(\| v \|_{L^3([0,t]; L_x^6)}^2 + \| v' \|_{L^3([0,t]; L_x^6)}^2 + \| z \|_{L^3([0,t]; L_x^6)}^2 + \| z' \|_{L^3([0,t]; L_x^6)}^2 \right) \\ & \quad \times \exp \left(C \left(\| v \|_{L^2([0,t]; L_x^8)}^2 + \| v' \|_{L^2([0,t]; L_x^8)}^2 + \| z \|_{L^2([0,t]; L_x^8)}^2 + \| z' \|_{L^2([0,t]; L_x^8)}^2 \right) \right). \end{aligned}$$

By (6.5), (6.6), (6.8), and (6.9), it then follows for $t \in [0, T]$ that

$$\begin{aligned} & \| (v(t) - v'(t), \partial_t v(t) - \partial_t v'(t)) \|_{\dot{H}_x^1(\mathbb{R}^4) \times L_x^2(\mathbb{R}^4)} \\ & \leq C \eta_k^{1-\alpha} (L(\eta_k)^2 + G(\eta_k)^2) e^{C(G(\eta_k)^2 + L(\eta_k)^2)}. \end{aligned}$$

Then, by (6.14), it follows easily for $t \in [0, T]$ that

$$\| (v(t) - v'(t), \partial_t v(t) - \partial_t v'(t)) \|_{\dot{H}_x^1(\mathbb{R}^4) \times L_x^2(\mathbb{R}^4)} \leq C_2 \eta_k^{1-\alpha} e^{C_3 G(\eta_k)^2},$$

where C_2 and C_3 are absolute constants. Furthermore,

$$\begin{aligned} \| v(t) - v'(t) \|_{L_x^2(\mathbb{R}^4)} & \leq \int_0^t \| \partial_t v(t') - \partial_t v'(t') \|_{L_x^2(\mathbb{R}^4)} dt' \leq t \| \partial_t v(t') - \partial_t v'(t') \|_{L_t^\infty([0,T]; L_x^2)} \\ & \leq C_2 T \eta_k^{1-\alpha} e^{C_3 G(\eta_k)^2}. \end{aligned}$$

Hence,

$$\| (v - v', \partial_t v - \partial_t v') \|_{L_t^\infty([0,T]; H_x^1(\mathbb{R}^4) \times L_x^2(\mathbb{R}^4))} \leq 2C_2 T \eta_k^{1-\alpha} e^{C_3 G(\eta_k)^2}. \tag{6.15}$$

We now discuss the choice of $L(\eta_k)$. The two conditions that we need to impose on $L(\eta_k)$ are $L(\eta_k) \rightarrow \infty$ as $k \rightarrow \infty$, which is crucial in Step 1, and that the right-hand side of (6.15) tends to zero as $k \rightarrow \infty$.

Recall that $F_R : [0, \infty) \rightarrow [0, \infty)$ is a strictly increasing function satisfying $F_R(0) = 0$ and $\lim_{A \rightarrow \infty} F_R(A) = \infty$. In particular, F_R has at most countably many discontinuities. These are jump discontinuities that we denote by

$$0 \leq x_1 < x_2 < \dots .$$

We claim that, given a sequence $\{y_k\}_{k \in \mathbb{N}} \subset (0, \infty)$ with $y_k \nearrow \infty$, there exists another sequence $\{y'_k\}_{k \in \mathbb{N}} \subset [0, \infty)$ such that $y'_k \in \text{Ran } F_R$, $y'_k \leq y_k$ for all k , and $y'_k \rightarrow \infty$ as $k \rightarrow \infty$.

Indeed, if $y_k \in \text{Ran } F_R$, then we choose $y'_k := y_k$. Otherwise, if $y_k \notin \text{Ran } F_R$, it follows that $y_k \in [F_R(x_{n_k}^-), F_R(x_{n_k}^+)]$ for some $n_k \in \mathbb{N}$. In this case, if $n_k \geq 2$, we choose $y'_k := F_R(x_{n_{k-1}})$, otherwise we choose $y'_k = 0$. Clearly, $y'_k \leq y_k$ and $y'_k \in \text{Ran } F_R$ for all k . We then denote by $\{y'_{k_1}\}_{k_1 \in \mathbb{N}}$ and $\{y'_{k_2}\}_{k_2 \in \mathbb{N}}$ the subsequences of $\{y'_k\}_{k \in \mathbb{N}}$ corresponding to y_k in $\text{Ran } F_R$, respectively corresponding to y_k in $(\text{Ran } F_R)^c$. One of these two subsequences is necessarily infinite. Clearly, either $y'_{k_1} \nearrow \infty$ or $\{y'_{k_1}\}_{k_1 \in \mathbb{N}}$ is a finite set. Also, $\{y'_{k_2}\}_{k_2 \in \mathbb{N}}$ is either a nondecreasing sequence converging to infinity or a finite set. This shows that $y'_k \rightarrow \infty$ as $k \rightarrow \infty$.

We apply the above reasoning to the sequence

$$y_k := \sqrt{\frac{1 - 2\alpha}{2C_3 T^2}} \left(\log \frac{1}{\eta_k} \right)^{1/4} \nearrow \infty \quad \text{as } k \rightarrow \infty.$$

As a consequence, there exists $\eta'_k \in (0, 1)$ with $\eta'_k \geq \eta_k$ such that

$$\sqrt{\frac{1 - 2\alpha}{2C_3 T^2}} \left(\log \frac{1}{\eta'_k} \right)^{1/4} \in \text{Ran } F_R \quad \text{for all } k \in \mathbb{N} \quad \text{and} \quad \eta'_k \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

The function F_R being invertible on its range, we choose $L(\eta_k)$ as

$$L(\eta_k) := \frac{1}{C_1 T^{2/3}} \log \left[F_R^{-1} \left(\sqrt{\frac{1 - 2\alpha}{2C_3 T^{14}}} \left(\log \frac{1}{\eta'_k} \right)^{1/4} \right) \right]. \tag{6.16}$$

This guarantees that $L(\eta_k) \rightarrow \infty$ as $k \rightarrow \infty$, since $\eta'_k \rightarrow 0$ and $\lim_{y \in \text{Ran } F_R, y \rightarrow \infty} F_R^{-1}(y) = \infty$.

Moreover, by (6.15), the choice of $G(\eta_k)$ and $L(\eta_k)$ in (6.12) and (6.16), and $\eta'_k > \eta_k$, we have

$$\begin{aligned} \|(v - v', \partial_t v - \partial_t v')\|_{L_t^\infty([0, T]; H_x^1(\mathbb{R}^4) \times L_x^2(\mathbb{R}^4))} \\ \leq 2C_2 T \eta_k^{1-\alpha} e^{(1/2-\alpha)(\log(1/\eta'_k))^{1/2} (\log \log(1/\eta_k))^6} \\ \leq 2C_2 T \eta_k^{1-\alpha} e^{(1/2-\alpha) \log(1/\eta_k)} \leq 2C_2 T \eta_k^{1/2} \end{aligned} \tag{6.17}$$

for k sufficiently large.

On the other hand, notice that the condition $\|(w_0, w_1) - (w'_0, w'_1)\|_{H^s \times H^{s-1}} < \eta_k$ immediately implies the control on the difference of the linear parts of solutions:

$$\|(z - z', \partial_t z - \partial_t z')\|_{L_t^\infty([0, T]; H_x^s(\mathbb{R}^4) \times H_x^{s-1}(\mathbb{R}^4))} \leq T \eta_k.$$

Hence,

$$\begin{aligned} \|(\Phi(t)(w_0, w_1) - \Phi(t)(w'_0, w'_1), \\ \partial_t \Phi(t)(w_0, w_1) - \partial_t \Phi(t)(w'_0, w'_1))\|_{L_t^\infty([0, T]; H_x^s(\mathbb{R}^4) \times H_x^{s-1}(\mathbb{R}^4))} \leq 3C_2 T \eta_k^{1/2}. \end{aligned}$$

Therefore, for a fixed $\delta > 0$, the $\mu \otimes \mu$ -measure of $((w_0, w_1), (w'_0, w'_1)) \in B_R \times B_R$ such that

$$\begin{aligned} \|(\Phi(t)(w_0, w_1) - \Phi(t)(w'_0, w'_1), \\ \partial_t \Phi(t)(w_0, w_1) - \partial_t \Phi(t)(w'_0, w'_1))\|_{L_t^\infty([0, T]; H_x^s \times H_x^{s-1})} > \delta \end{aligned}$$

under the constraints $\|(w_0, w_1) - (w'_0, w'_1)\|_{H^s(\mathbb{R}^4) \times H^{s-1}(\mathbb{R}^4)} < \eta_k$ and (6.5)–(6.9) is zero if η_k is sufficiently small. This shows that the right-hand side of (1.19) converges indeed to zero as $\eta \rightarrow 0$. □

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