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Spherical DG-functors

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Abstract. For two DG-categories \mathcal{A} and \mathcal{B} we define the notion of a *spherical* Morita quasi-functor $\mathcal{A} \rightarrow \mathcal{B}$. We construct its associated autoequivalences: the *twist* $T \in \text{Aut } D(\mathcal{B})$ and the *cotwist* $F \in \text{Aut } D(\mathcal{A})$. We give sufficiency criteria for a quasi-functor to be spherical and for the twists associated to a collection of spherical quasi-functors to braid. Using the framework of DG-enhanced triangulated categories, we translate all of the above to Fourier–Mukai transforms between the derived categories of algebraic varieties. This is a broad generalization of the results on spherical objects in [ST01] and on spherical functors in [Ann07]. In fact, this paper replaces [Ann07], which has a fatal gap in the proof of its main theorem. Though conceptually correct, the proof was impossible to fix within the framework of triangulated categories.

Keywords. Algebraic geometry, derived categories, DG-categories, autoequivalences, Fourier–Mukai transforms, spherical functors, braid group actions

1. Introduction

Let X be a smooth projective variety over an algebraically closed field k of characteristic 0. Let $D(X)$ be the bounded derived category of coherent sheaves on X . In [ST01] Seidel and Thomas introduced the notion of a *spherical object* in $D(X)$. These objects are defined in terms of certain cohomological properties and they are mirror-symmetric analogues of Lagrangian spheres on a symplectic manifold. Given a Lagrangian sphere we can associate to it a symplectic automorphism called the *generalized Dehn twist*. Correspondingly:

Theorem ([ST01]). *Let $E \in D(X)$. The twist functor T_E is a certain functorial cone of the natural transformation $E \otimes_k \mathbf{R}\text{Hom}_X(E, -) \xrightarrow{\text{eval}} \text{Id}_{D(X)}$. If E is spherical, then T_E is an autoequivalence of $D(X)$.*

Moreover, in [ST01, Theorem 2.17] Seidel and Thomas give simple criteria on a set E_1, \dots, E_n of spherical objects in $D(X)$ sufficient to ensure that the corresponding spher-

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ical twists T_1, \dots, T_n represent the braid group B_n , in other words, that we have

$$\begin{aligned} T_i T_j T_i &\simeq T_j T_i T_j, & |i - j| = 1, \\ T_i T_j &\simeq T_j T_i, & |i - j| \geq 2. \end{aligned}$$

Spherical objects and twists quickly became an essential tool in studying derived categories of algebraic varieties as well as more classical areas of algebraic geometry [Muk87], [Bri08], [Bri09], [IU05], [BP14]. For some time now it has been understood by specialists that the notion of a spherical object should generalize to the notion of a *spherical functor* $D(Z) \xrightarrow{s} D(X)$ where Z is some other variety. Such a functor should produce two auto-equivalences—the *twist* $t \in \text{Aut } D(X)$ and the *cotwist* $f \in \text{Aut } D(Z)$. More generally, there should be a notion of a spherical functor between two abstract triangulated categories. Limited special cases of this appear in [Hor05], [Rou04] [Sze04], [Tod07], [KT07], but general treatment was obstructed by well-known imperfections of the axioms of triangulated categories such as non-functoriality of the cone construction and non-uniqueness of the data supplied by the octahedral axiom.

In this paper, we are able, at last, to give a fully general and rigorous treatment of spherical functors and to prove an ideal statement about their associated autoequivalences. Due to increased prominence of spherical twist autoequivalences in studying derived categories of algebraic varieties, our results have been anticipated and made use of even as the paper was being written. The works which already apply the results of this paper include [Add16], [DW16], [HLS16], [BPP17], [DS15].

A previous attempt at this general treatment was made in [Ann07]. Conceptually sound, it was brought low by the octahedral axiom. The proof of its main theorem [Ann07, Prop.1] contained a fatal gap which is impossible to fix within the axioms of triangulated categories. Nonetheless, it was clear that its ideas could work if we had an extra level of control over what the octahedral axiom provides us with.

We gain this extra control by passing to differential graded (DG) categories. The axioms of triangulated categories were developed in [Ver96] to describe the derived categories of algebraic varieties, which are cohomological truncations of certain natural DG-categories. The imperfections of these axioms can now clearly be seen as artefacts of the truncation. Working in the original DG-category provides us with precisely the layer of control that was missing. This allows us not only to fix the results in [Ann07], but to significantly improve upon them. It allows us to do something more — to provide for a collection of spherical functors, as [ST01] did for spherical objects, a set of straightforward criteria sufficient for braid relations to hold between their twists. For some years now the first author has been well aware of what these criteria should be, but proving them on the level of triangulated categories was hopeless.

We first state our results in the language of triangulated categories. Let A and B be two Karoubi closed triangulated categories and let s be an exact functor $A \rightarrow B$ which has left and right adjoints l and r . Suppose that we can construct a preferred functorial exact triangle for each of the four adjunction units and counits involved. Use these triangles to define the *twist* t of s by the exact triangle

$$sr \xrightarrow{\text{adj.counit}} \text{Id}_B \rightarrow t, \tag{1.1}$$

the *dual twist* t' of s by the exact triangle

$$t' \rightarrow \mathrm{Id}_B \xrightarrow{\mathrm{adj.unit}} sl, \quad (1.2)$$

the *cotwist* f of s by the exact triangle

$$f \rightarrow \mathrm{Id}_A \xrightarrow{\mathrm{adj.unit}} rs, \quad (1.3)$$

and the *dual cotwist* f' of s by the exact triangle

$$ls \xrightarrow{\mathrm{adj.counit}} \mathrm{Id}_A \rightarrow f'. \quad (1.4)$$

Define also two natural transformations

$$lt[-1] \xrightarrow{(1.1)} lsr \xrightarrow{\mathrm{adj.counit}} r, \quad (1.5)$$

$$r \xrightarrow{\mathrm{adj.unit}} rsl \xrightarrow{(1.3)} fl[1]. \quad (1.6)$$

Definition 1.1. The functor s is *spherical* if all of the following hold:

- (1) t and t' are quasi-inverse autoequivalences of B ,
- (2) f and f' are quasi-inverse autoequivalences of A ,
- (3) $lt[-1] \xrightarrow{(1.5)} r$ is an isomorphism of functors (“the twist identifies the adjoints”),
- (4) $r \xrightarrow{(1.6)} fl[1]$ is an isomorphism of functors (“the cotwist identifies the adjoints”).

The main obstruction is the lack of canonical functorial exact triangles (1.1)–(1.4) defining t , t' , f and f' . What [Ann07] tried to do was to assume that *some* functorial exact triangles as above exist, define s to be spherical if (2) and (4) hold, and then prove that for any spherical s the condition (1) also holds. In this paper, as explained in more detail below, we assume that

- A and B admit DG-enhancements,
- s , r and l descend from DG-functors S , R and L between some enhancements of A and B ,

and prove that there is a canonical construction of the exact triangles (1.1)–(1.4) determined by a certain equivalence class of S such that any two of the conditions in Definition 1.1 imply that all four of them hold and s is spherical. This is the ideal statement mentioned above.

Let us be more precise. Let A be a triangulated category. Traditionally, a *DG-enhancement* of A is a DG-category \mathcal{A} together with an isomorphism $H^0(\mathcal{A}) \simeq A$. A more useful notion for us is that of a *Morita enhancement*, which is a DG-category \mathcal{A} together with an isomorphism $D_c(\mathcal{A}) \simeq A$. Here $D_c(\mathcal{A})$ is the full subcategory of the derived category $D(\mathcal{A})$ consisting of compact objects. A *Morita equivalence* is a DG-functor $\mathcal{A} \xrightarrow{f} \mathcal{B}$ whose induced functor $D_c(\mathcal{A}) \xrightarrow{\mathbf{L}f^*} D_c(\mathcal{B})$ is an equivalence of categories. This is the right notion of equivalence for Morita enhancements. Thus we are led to work

in the Morita homotopy category $\text{Mrt}(DG\text{-Cat})$, which is the localization of the category $DG\text{-Cat}$ of all DG-categories by Morita equivalences. The objects of $\text{Mrt}(DG\text{-Cat})$ should be thought of as enhanced Karoubi closed triangulated categories with a fixed equivalence class of enhancements. The morphisms in $\text{Mrt}(DG\text{-Cat})$ are called *Morita quasi-functors*. Each Morita quasi-functor $\mathcal{A} \rightarrow \mathcal{B}$ induces a genuine exact functor $D_c(\mathcal{A}) \rightarrow D_c(\mathcal{B})$.

Let \mathcal{A} and \mathcal{B} be Morita enhancements of triangulated categories A and B . A fundamental result of Toën [Toë07, Theorem 7.2] implies that the Morita quasi-functors $\mathcal{A} \rightarrow \mathcal{B}$ are in 1-to-1 correspondence with the isomorphism classes in $D(\mathcal{A}\text{-}\mathcal{B})$ of the $\mathcal{A}\text{-}\mathcal{B}$ -bimodules which are \mathcal{B} -perfect, i.e. ${}_aM \in D_c(\mathcal{B})$ for all $a \in \mathcal{A}$. Given M in $D^{\mathcal{B}\text{-Perfect}}(\mathcal{A}\text{-}\mathcal{B})$, the derived tensor product functor

$$(-) \overset{L}{\otimes}_{\mathcal{A}} M : D_c(\mathcal{A}) \rightarrow D_c(\mathcal{B})$$

is the exact functor underlying the corresponding Morita quasi-functor. Thus, we think of $D^{\mathcal{B}\text{-Perfect}}(\mathcal{A}\text{-}\mathcal{B})$ as a triangulated category structure on the set $\text{Hom}_{\text{Mrt}(DG\text{-Cat})}(\mathcal{A}, \mathcal{B})$ and of morphisms in it as morphisms of Morita quasi-functors. This packages up into a 2-category structure on $\text{Mrt}(DG\text{-Cat})$ with a functor to the 2-category of Karoubi closed triangulated categories. See Section 4 for a brief survey on DG-enhancements.

We now describe our results. In the body of the paper they are stated in a slightly more flexible language of DG-bimodules. Here we state them in the language of Morita quasi-functors, which gives a more intuitive picture. Let $\mathcal{A} \overset{S}{\rightarrow} \mathcal{B}$ be a Morita quasi-functor and let $A \overset{s}{\rightarrow} B$ be the underlying exact functor. Assume that s has left and right adjoints $B \overset{l,r}{\rightarrow} A$ which also descend from Morita quasi-functors. The derived \mathcal{A} - and \mathcal{B} -duals of S in $D(\mathcal{B}\text{-}\mathcal{A})$ are then \mathcal{A} -perfect and hence define Morita quasi-functors $\mathcal{B} \overset{L,R}{\rightarrow} \mathcal{A}$. In Section 2.2 we construct *derived trace* and *action* maps

$$SR \overset{\text{tr}}{\rightarrow} \text{Id}_{\mathcal{B}} \quad \text{and} \quad LS \overset{\text{tr}}{\rightarrow} \text{Id}_{\mathcal{A}}, \tag{1.7}$$

$$\text{Id}_{\mathcal{B}} \overset{\text{act}}{\rightarrow} SL \quad \text{and} \quad \text{Id}_{\mathcal{A}} \overset{\text{act}}{\rightarrow} RS, \tag{1.8}$$

and prove that the exact functors underlying L and R are precisely l and r , and that the derived trace and action maps above induce the units and counits of the adjunctions of s , l and r . Then, working in the DG-enhancements, we construct natural exact triangles of Morita quasi-functors

$$SR \overset{\text{tr}}{\rightarrow} \text{Id}_{\mathcal{B}} \rightarrow T, \tag{1.9}$$

$$T' \rightarrow \text{Id}_{\mathcal{B}} \overset{\text{act}}{\rightarrow} SL, \tag{1.10}$$

$$F \rightarrow \text{Id}_{\mathcal{A}} \overset{\text{act}}{\rightarrow} RS, \tag{1.11}$$

$$LS \overset{\text{tr}}{\rightarrow} \text{Id}_{\mathcal{A}} \rightarrow F', \tag{1.12}$$

which define the *twist* T , the *dual twist* T' , the *cotwist* F and the *dual cotwist* F' of S . Thus we obtain a natural choice of functorial exact triangles (1.1)–(1.4) defining t , t' , f and f' . We then prove that t' and f' are left adjoint to t and f , respectively. All the

above constructions are readily seen to be Morita invariant, i.e. they are preserved if we replace \mathcal{A} or \mathcal{B} by a Morita equivalent DG-category. Hence they only depend on Morita equivalence classes of \mathcal{A} and \mathcal{B} and on $S \in \text{Hom}_{\text{Mtr}(DG\text{-Cat})}(\mathcal{A}, \mathcal{B})$.

The following is the main result of this paper:

Theorem 1.1 (see Theorem 5.1). *Suppose any two of the following conditions hold:*

- (1) t is an autoequivalence of B (“the twist is an equivalence”).
- (2) f is an equivalence of A (“the cotwist is an equivalence”).
- (3) $lt[-1] \xrightarrow{(5.11)} r$ is an isomorphism of functors (“the twist identifies the adjoints”).
- (4) $r \xrightarrow{(5.12)} fl[1]$ is an isomorphism of functors (“the cotwist identifies the adjoints”).

Then all four hold and S is said to be a spherical quasi-functor.

Finally, we give the braiding criteria for spherical quasi-functors. These have a natural interpretation in geometrical context that is the subject of a future paper [AL]. An example of these criteria being satisfied can be seen in a construction by Khovanov and Thomas in [KT07].

Let A_1, \dots, A_n, B be triangulated categories with Morita enhancements $\mathcal{A}_1, \dots, \mathcal{A}_n, \mathcal{B}$. Let $\mathcal{A}_i \xrightarrow{S_i} \mathcal{B}$ be spherical Morita quasi-functors. For any $i \neq j$ the trace maps $S_i R_i \xrightarrow{\text{tr}} \text{Id}_{\mathcal{B}}$ and $S_j R_j \xrightarrow{\text{tr}} \text{Id}_{\mathcal{B}}$ define a map

$$S_i R_i S_j R_j \xrightarrow{S_i R_i \text{tr} \oplus \text{tr} S_j R_j} S_i R_i \oplus S_j R_j. \tag{1.13}$$

Next, for any $i \neq j$ define a Morita quasi-functor $\mathcal{A}_i \xrightarrow{O_i} \mathcal{A}_i$ by

$$O_{ij} = F_i \text{Cone}(L_i S_j R_j S_i \xrightarrow{\text{tr} \circ (L_i \text{tr} S_i)} \text{Id}_{\mathcal{A}_i}). \tag{1.14}$$

As S_i is spherical, we have $R_i[-1] \simeq F_i L_i$, so $S_i R_i \xrightarrow{\text{tr}} \text{Id}_{\mathcal{B}}$ and $S_j R_j \xrightarrow{\text{tr}} \text{Id}_{\mathcal{B}}$ define (cf. Section 6.2) a map

$$S_i O_{ij} R_i \rightarrow S_i R_i S_j R_j \oplus S_j R_j S_i R_i. \tag{1.15}$$

Theorem 1.2 (Theorems 6.1–6.2). *Suppose that for all $i, j \in 1, \dots, n$ the following hold:*

- (1) If $|i - j| > 1$, there exists an isomorphism

$$S_i R_i S_j R_j \simeq S_j R_j S_i R_i$$

which commutes with the maps (1.13).

- (2) If $|i - j| = 1$, there exists an isomorphism

$$S_i O_{ij} R_i \simeq S_j O_{ji} R_j$$

which commutes with the maps (1.15).

Then the twists T_1, \dots, T_n generate a categorical action of the braid group \mathcal{B}_n on B .

Finally, we interpret the above in the context of algebraic geometry. Let Z and X be separated schemes of finite type over k . Let $D_{\text{qc}}(Z)$ and $D_{\text{qc}}(X)$ be the derived categories of quasi-coherent sheaves, and let $D(Z)$ and $D(X)$ be the bounded derived categories of coherent sheaves on Z and X . Let \mathcal{A} and \mathcal{B} be the standard DG-enhancements of $D(Z)$ and $D(X)$. These are given by the DG-categories of h -injective complexes of sheaves on Z and X , respectively. In Example 4.3 we prove an analogue for the bounded coherent derived categories of the famous result of Toën [Toë07, Theorem 8.9] for the unbounded quasi-coherent ones. We prove that the exact functors $D(Z) \rightarrow D(X)$ which descend from the Morita quasi-functors $\mathcal{A} \rightarrow \mathcal{B}$ are precisely the Fourier–Mukai transforms. Given an object $E \in D(Z \times X)$, the Fourier–Mukai transform Φ_E is a priori a functor $D_{\text{qc}}(Z) \rightarrow D_{\text{qc}}(X)$. In Example 4.3 we identify $\text{Hom}_{\text{Mitt}(DG\text{-Cat})}(\mathcal{A}, \mathcal{B})$ with the full subcategory of $D(Z \times X)$ consisting of the objects E such that Φ_E restricts to $D(Z) \rightarrow D(X)$. Under this identification, each Morita quasi-functor $\mathcal{A} \xrightarrow{S} \mathcal{B}$ goes to an object $E \in D(Z \times X)$ such that $D(Z) \xrightarrow{\Phi_E} D(X)$ is the exact functor s underlying S .

The above results for Morita quasi-functors can then all be interpreted for Fourier–Mukai transforms. Let $E \in D(Z \times X)$ be such that Φ_E restricts to a functor $D(Z) \xrightarrow{s} D(X)$ and this restriction has a left adjoint which is also a Fourier–Mukai transform. For example, it is sufficient to assume that E is proper over Z and X and perfect over Z and X . Our results for Morita quasi-functors provide natural constructions at the level of Fourier–Mukai kernels of the right and left adjoints r and l and of all four adjunctions, units and counits involved. We conjecture that these coincide with the explicit formulas proved independently in [AL12] and [AL16]. Regardless of whether this holds or not, the functorial exact triangles (1.1)–(1.4) defining the twists and cotwists t, t', f and f' are well-defined and depend only on $E \in D(Z \times X)$. We say that E is *spherical over Z* if the four conditions of Definition 1.1 are satisfied. Our main theorem then applies to show that, in fact, it suffices to verify any two of these four conditions. The braiding criteria above translate similarly to the language of Fourier–Mukai kernels. It is worth noting that if we set $Z = \text{Spec } k$ then the natural isomorphism $Z \times X \simeq X$ identifies $D(Z \times X)$ with $D(X)$ and our results immediately imply the results in [ST01].

Finally, we also describe in Section 5.2 a variation on all of the above. It uses a slightly different enhancement framework which allows one to work with the unbounded derived categories $D_{\text{qc}}(Z)$ and $D_{\text{qc}}(X)$. The penalty is a strong smoothness condition: We can only work with $E \in D_{\text{qc}}(X \times Y)$ such that Φ_E has a left adjoint which is also a Fourier–Mukai transform and they both take compact objects to compact objects.

About the structure of this paper: In Section 2.1 we give an overview of the facts we need on DG-categories and DG-modules over them. In Section 2.2 we define the dualizing functors for DG-modules and DG-bimodules. We then construct and study *homotopy trace* and *action* maps and show them to be units and counits of homotopy adjunctions between an \mathcal{A} - \mathcal{B} -bimodule M and its \mathcal{A} - and \mathcal{B} -duals $M^{\mathcal{A}}$ and $M^{\mathcal{B}}$. In Section 3.1 we give an overview of twisted complexes over a DG-category and prove explicit formulas for taking a tensor product and for dualizing at the level of twisted complexes. Section 3.2 summarizes the facts we need about pretriangulated categories. In Section 3.3 we develop a theory of *twisted cubes*, which acts as a “higher” octahedral axiom for the world of pre-

triangulated categories. In Section 4 we explain the framework of DG-enhancements of triangulated categories and its applications to algebraic geometry. In Section 5.1 we construct twists and cotwists of a DG-bimodule, define a notion of a spherical DG-bimodule and prove our main theorem on the level of DG-bimodules. In Section 5.2 we interpret this for Fourier–Mukai transforms between the derived categories of algebraic varieties via the framework introduced in Section 4. In Section 6 we state and prove braiding criteria for spherical DG-bimodules. Finally, the Appendix contains some technical results we need in Section 6 on constructing homotopy equivalences between twisted complexes. There the authors have to resort to using A_∞ -categories, A_∞ -functors and the interpretation of DG quasi-functors as strictly unital A_∞ -functors between the corresponding DG-categories. It is something they quite happily avoided doing throughout the rest of the paper.

2. Preliminaries

Some proofs in this paper rely on explicit computations where matching up the signs becomes important. As there are different sign conventions present in the literature for the material in this section, we make our choices explicit at the cost of restating some very well-known definitions. We aim to enable our reader to verify all the computations which are “left to the reader”.

Notation. Throughout the paper all schemes are defined and all DG-categories are considered over the same base field we denote by k .

Let X be a scheme. We denote by $D_{\text{qc}}(X)$, resp. $D(X)$, the full subcategory of the derived category of $\mathcal{O}_X\text{-Mod}$ consisting of complexes with quasi-coherent, resp. bounded and coherent, cohomology.

2.1. DG-categories, modules and bimodules

Throughout this section, k is a commutative ring.

2.1.1. DG-categories. Let E and F be complexes of k -modules. Define $E \otimes F$ to be the complex of k -modules

$$(E \otimes F)_n = \bigoplus_{i+j=n} E_i \otimes F_j, \quad d(e \otimes f) = de \otimes f + (-1)^{\deg(e)} e \otimes df. \quad (2.1)$$

We have the standard sign-twisting isomorphism $E \otimes F \xrightarrow{\sim} F \otimes E$ given by

$$e \otimes f \mapsto (-1)^{\deg(e) \deg(f)} f \otimes e. \quad (2.2)$$

Define $\text{Hom}_k(E, F)$ to be the complex of k -modules

$$\text{Hom}_k^n(E, F) = \bigoplus_{j-i=n} \text{Hom}_k(E_i, F_j), \quad df = d_F \circ f - (-1)^{\deg(f)} f \circ d_E. \quad (2.3)$$

A *DG-category* over k is a category \mathcal{A} whose morphism spaces $\text{Hom}_{\mathcal{A}}(a, b)$ are complexes of k -modules and whose composition maps

$$\text{Hom}_{\mathcal{A}}(b, c) \otimes \text{Hom}_{\mathcal{A}}(a, b) \rightarrow \text{Hom}_{\mathcal{A}}(a, c)$$

are closed degree 0 maps of complexes of k -modules. The homotopy category $H^0(\mathcal{A})$ has the same objects as \mathcal{A} and its morphisms spaces are 0th cohomologies of their counterparts in \mathcal{A} . Let **Mod**- k be the DG-category of complexes of k -modules with morphism spaces defined by (2.3) and the composition $(f \circ g)(s) = f(g(s))$. See [Kel94, §§1.1–1.2] or [Toë11] for details.

Given a DG-category \mathcal{A} denote by \mathcal{A}^{opp} the *opposite DG-category* of \mathcal{A} . Its objects are the same as those of \mathcal{A} and for all $a, b \in \mathcal{A}^{\text{opp}}$ we have $\text{Hom}_{\mathcal{A}^{\text{opp}}}(a, b) = \text{Hom}_{\mathcal{A}}(b, a)$. The composition is defined by composing the sign-twisting isomorphism (2.2) with the composition map of \mathcal{A} . In other words, we set

$$\beta \circ_{\mathcal{A}^{\text{opp}}} \alpha = (-1)^{\text{deg}(\alpha) \text{deg}(\beta)} \alpha \circ_{\mathcal{A}} \beta$$

for all $\alpha \in \text{Hom}_{\mathcal{A}^{\text{opp}}}(a, b)$ and $\beta \in \text{Hom}_{\mathcal{A}^{\text{opp}}}(b, c)$.

Let \mathcal{A} and \mathcal{B} be DG-categories. A DG-*functor* $\mathcal{A} \rightarrow \mathcal{B}$ is a k -linear functor which preserves the grading and the differential on morphisms. Wherever the context permits we omit “DG-” and simply say “functor”. A *degree n natural transformation* of DG-functors $\Phi \xrightarrow{t} \Psi$ is a collection

$$\{t(a) \in \text{Hom}_{\mathcal{B}}^n(\Phi(a), \Psi(a))\}_{a \in \mathcal{A}}$$

where $t(a') \circ \Phi(\alpha) = (-1)^{nm} \Psi(\alpha) \circ t(a)$ for every $\alpha \in \text{Hom}_{\mathcal{A}}^m(a, a')$. Define the DG-category $\text{DGFun}(\mathcal{A}, \mathcal{B})$ as follows. Its objects are DG-functors $\mathcal{A} \rightarrow \mathcal{B}$. Its morphism complexes $\text{Hom}_{\text{DGFun}(\mathcal{A}, \mathcal{B})}(\Phi, \Psi)$ consist of all natural transformations $\Phi \xrightarrow{t} \Psi$ graded by degree and with differentials defined levelwise by those of \mathcal{B} , i.e. $dt(a) = d_{\mathcal{B}}(t(a))$ for each $a \in \mathcal{A}$. The composition maps are also defined levelwise by those of \mathcal{B} .

We denote by *DG-Cat* the category whose objects are all small DG-categories over k and whose morphisms are DG-functors between them.

2.1.2. Closed symmetrical monoidal structure on DG-Cat. Let \mathcal{A} and \mathcal{B} be DG-categories. We define $\mathcal{A} \otimes_k \mathcal{B}$ to be the DG-category whose objects are pairs (a, b) with $a \in \mathcal{A}, b \in \mathcal{B}$, whose morphism complexes are

$$\text{Hom}_{\mathcal{A} \otimes_k \mathcal{B}}(a \otimes b, a' \otimes b') = \text{Hom}_{\mathcal{A}}(a \otimes a') \otimes \text{Hom}_{\mathcal{B}}(b \otimes b')$$

and whose composition is defined by

$$(\alpha' \otimes \beta') \circ (\alpha \otimes \beta) = (-1)^{\text{deg}(\beta') \text{deg}(\alpha)} (\alpha' \circ \alpha) \otimes (\beta' \circ \beta).$$

This construction is bifunctorial in \mathcal{A} and \mathcal{B} and defines a monoidal structure on *DG-Cat* whose unit is k .

The monoidal structure (\otimes_k, k) is symmetric via the natural isomorphism

$$\mathcal{B} \otimes_k \mathcal{A} \xrightarrow{\sim} \mathcal{A} \otimes_k \mathcal{B} \tag{2.4}$$

defined on objects by $b \otimes a \mapsto a \otimes b$ and on morphisms by $\beta \otimes \alpha \mapsto (-1)^{\text{deg}(\alpha) \text{deg}(\beta)} \alpha \otimes \beta$.

The monoidal structure (\otimes_k, k) is, moreover, closed. The internal Hom is given by $\text{DGFun}(-, -)$. Explicitly, for any DG-categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$ we have the natural isomorphism

$$\text{DGFun}(\mathcal{A} \otimes_k \mathcal{B}, \mathcal{C}) \xrightarrow{\sim} \text{DGFun}(\mathcal{B}, \text{DGFun}(\mathcal{A}, \mathcal{C})) \tag{2.5}$$

which takes any $\mathcal{A} \otimes_k \mathcal{B} \xrightarrow{\Phi} \mathcal{C}$ to the functor

$$\forall b \in \mathcal{B} \quad b \mapsto \Phi(- \otimes b), \quad \forall \beta \in \text{Hom}_{\mathcal{B}}(b, b') \quad \beta \mapsto \Phi(\text{Id} \otimes \beta),$$

and any $\Phi \xrightarrow{f} \Psi$ to the natural transformation $\{\Phi(- \otimes b) \xrightarrow{f} \Psi(- \otimes b)\}_{b \in \mathcal{B}}$. The object set of $\text{DGFun}(-, -)$ is the set $\text{Hom}_{\text{DG-Cat}}(-, -)$, so the isomorphism (2.5) induces an adjunction isomorphism between $(-)\otimes_k \mathcal{A}$ and $\text{DGFun}(\mathcal{A}, -)$ which makes DGFun the internal Hom in $(\text{DG-Cat}, \otimes_k, k)$.

For any DG-categories \mathcal{A} and \mathcal{B} we have the tautological categorical isomorphisms

$$(\mathcal{A} \otimes_k \mathcal{B})^{\text{opp}} \simeq \mathcal{A}^{\text{opp}} \otimes_k \mathcal{B}^{\text{opp}}, \tag{2.6}$$

$$\text{DGFun}(\mathcal{A}, \mathcal{B})^{\text{opp}} \simeq \text{DGFun}(\mathcal{A}^{\text{opp}}, \mathcal{B}^{\text{opp}}). \tag{2.7}$$

The former isomorphism sends any pair of objects or morphisms in $\mathcal{A} \otimes_k \mathcal{B}$ to themselves considered as elements of $\mathcal{A}^{\text{opp}} \otimes_k \mathcal{B}^{\text{opp}}$. The latter sends any functor or natural transformation in $\text{DGFun}(\mathcal{A}, \mathcal{B})$ to itself considered as an element of $\text{DGFun}(\mathcal{A}^{\text{opp}}, \mathcal{B}^{\text{opp}})$.

Finally, for any DG-categories $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ we have the simultaneous evaluation functor

$$\text{DGFun}(\mathcal{A}, \mathcal{C}) \otimes_k \text{DGFun}(\mathcal{B}, \mathcal{D}) \rightarrow \text{DGFun}(\mathcal{A} \otimes_k \mathcal{B}, \mathcal{C} \otimes_k \mathcal{D}) \tag{2.8}$$

which sends any pair of functors $\mathcal{A} \rightarrow \mathcal{C}$ and $\mathcal{B} \rightarrow \mathcal{D}$ to the functor of simultaneously evaluating them on any pair of objects or morphisms in $\mathcal{A} \otimes_k \mathcal{B}$; similarly for natural transformations of such pairs of functors. Note that there are no sign twists involved.

2.1.3. DG-modules. A (right) \mathcal{A} -module is a functor from \mathcal{A}^{opp} to $\mathbf{Mod}\text{-}k$. Denote by $\mathbf{Mod}\text{-}\mathcal{A}$ the DG-category $\text{DGFun}(\mathcal{A}^{\text{opp}}, \mathbf{Mod}\text{-}k)$. For brevity and to mimic the notation used for DG-algebras, for any $E, F \in \mathbf{Mod}\text{-}\mathcal{A}$ we write $\text{Hom}_{\mathcal{A}}(E, F)$ for $\text{Hom}_{\mathbf{Mod}\text{-}\mathcal{A}}(E, F)$. The category $H^0(\mathbf{Mod}\text{-}\mathcal{A})$ admits a natural structure of triangulated category which is defined levelwise by the usual triangulated structure on $H^0(\mathbf{Mod}\text{-}k)$ [Kel94, §2.2].

For any $E \in \mathbf{Mod}\text{-}\mathcal{A}$ and $a \in \mathcal{A}$ we write E_a for the complex of k -modules $E(a)$. We write $v \in E$ if $v \in E_a$ for some $a \in \mathcal{A}$. The Yoneda embedding $\mathcal{A} \hookrightarrow \mathbf{Mod}\text{-}\mathcal{A}$ is the fully faithful functor defined on objects by

$$a \mapsto \text{Hom}_{\mathcal{A}}(-, a) \quad \forall a \in \mathcal{A}$$

and on morphisms by composition. For each $a \in \mathcal{A}$ denote by ${}_a\mathcal{A}$ its image under the Yoneda embedding; these are the representable objects of $\mathbf{Mod}\text{-}\mathcal{A}$. Note that for all $a, b \in \mathcal{A}$ we have ${}_a\mathcal{A}_b = \text{Hom}_{\mathcal{A}}(b, a)$. For any $E \in \mathbf{Mod}\text{-}\mathcal{A}$ trivially $\text{Hom}_{\mathcal{A}}({}_a\mathcal{A}, E) = E_a$. For each $s \in E_a$ and $\alpha \in {}_a\mathcal{A}_b$ we write $s \cdot \alpha$ for the element $(-1)^{\text{deg}(s)\text{deg}(\alpha)} E(\alpha)(s) \in E_b$. We have

$$(s \cdot \alpha) \cdot \beta = s \cdot (\alpha \circ \beta).$$

In other words, we can think of the data defining an \mathcal{A} -module E as a collection of fibers $E_a \in C(k)$ for each $a \in \mathcal{A}$ with a right action of (the Hom-spaces of) \mathcal{A} on them, such

that ${}_a\mathcal{A}_b$ acts on E_a and maps it to E_b . Similarly, a morphism of right \mathcal{A} -modules $E \xrightarrow{t} F$ can be thought of as a collection of maps $E_a \xrightarrow{t} F_a$ in $\mathbf{Mod}\text{-}k$ which commute with the \mathcal{A} -action: $t(s \cdot \alpha) = t(s) \cdot \alpha$ for any $s \in E_a$ and $\alpha \in {}_a\mathcal{A}$.

A *left \mathcal{A} -module* is a right \mathcal{A}^{opp} -module, i.e. a functor $\mathcal{A} \rightarrow \mathbf{Mod}\text{-}k$. To facilitate the treatment of bimodules, it is often useful to treat right \mathcal{A}^{opp} -modules as left \mathcal{A} -modules and employ for them the following notation. For any $F \in \mathbf{Mod}\text{-}\mathcal{A}^{\text{opp}}$ and $a \in \mathcal{A}$ we write ${}_aF$ (instead of F_a) for the complex $F(a)$. For each $a \in \mathcal{A}$ write \mathcal{A}_a for the image of a under the Yoneda embedding of \mathcal{A}^{opp} , i.e. for the functor $\text{Hom}_{\mathcal{A}}(a, -)$. Set $\alpha \cdot s = F(\alpha)(s)$ for each $s \in {}_aF$ and $\alpha \in \mathcal{A}_a$; it is a left action of \mathcal{A} on F . A morphism of left \mathcal{A} -modules $E \xrightarrow{t} F$ can be thought of as a collection of maps ${}_aE \xrightarrow{t} {}_aF$ which skew-commute with the \mathcal{A} -action: $t(\alpha \cdot s) = (-1)^{\text{deg}(t)\text{deg}(\alpha)} \alpha \cdot t(s)$ for any $s \in {}_aE$ and $\alpha \in {}_a\mathcal{A}$.

2.1.4. Tensor and Hom. Let \mathcal{A} be a DG-category and let E and F be a right and a left \mathcal{A} -module. Define the tensor product $E \otimes_{\mathcal{A}} F \in \mathbf{Mod}\text{-}k$ to be the quotient of $\bigoplus_{a \in \mathcal{A}} E_a \otimes_a F \in \mathbf{Mod}\text{-}k$ by the \mathcal{A} -action relations

$$(s \cdot \alpha) \otimes t = s \otimes (\alpha \cdot t) \quad \forall \alpha \in {}_b\mathcal{A}_a, s \in E_b, t \in {}_aF. \tag{2.9}$$

We extend this to a functor

$$(-) \otimes_{\mathcal{A}} (-): \mathbf{Mod}\text{-}\mathcal{A} \otimes_k \mathbf{Mod}\text{-}\mathcal{A}^{\text{opp}} \rightarrow \mathbf{Mod}\text{-}k \tag{2.10}$$

by defining the tensor product $\lambda \otimes \mu$ of two maps $E \xrightarrow{\lambda} E'$ and $F \xrightarrow{\mu} F'$ as follows. The map

$$\bigoplus_{a \in \mathcal{A}} E_a \otimes_a F \rightarrow \bigoplus_{a \in \mathcal{A}} E'_a \otimes_a F', \quad e \otimes f \mapsto (-1)^{\text{deg}(e)\text{deg}(\mu)} \lambda(e) \otimes \mu(f)$$

preserves \mathcal{A} -action relations and we define $\lambda \otimes \mu$ to be the induced map $E \otimes_{\mathcal{A}} F \rightarrow E' \otimes_{\mathcal{A}} F'$.

Similarly, we define the functor

$$\text{Hom}_{\mathcal{A}}(-, -): \mathbf{Mod}\text{-}\mathcal{A} \otimes_k (\mathbf{Mod}\text{-}\mathcal{A})^{\text{opp}} \rightarrow \mathbf{Mod}\text{-}k \tag{2.11}$$

on objects by $(E, F) \mapsto \text{Hom}_{\mathcal{A}}(F, E)$ and on morphisms as follows. For any pair of maps $E \xrightarrow{\lambda} E'$ and $F' \xrightarrow{\mu} F$ in $\mathbf{Mod}\text{-}\mathcal{A}$ we define the composition map

$$\text{Hom}_{\mathcal{A}}(F, E) \xrightarrow{\lambda \circ (-) \circ \mu} \text{Hom}_{\mathcal{A}}(F', E'), \quad \alpha \mapsto (-1)^{\text{deg}(\alpha)\text{deg}(\mu)} \lambda \circ \alpha \circ \mu.$$

2.1.5. DG-bimodules. An \mathcal{A} - \mathcal{B} -bimodule is an $\mathcal{A}^{\text{opp}} \otimes \mathcal{B}$ -module. We write $\mathcal{A}\text{-Mod}\text{-}\mathcal{B}$ for

$$\text{DGFun}(\mathcal{A} \otimes \mathcal{B}^{\text{opp}}, \mathbf{Mod}\text{-}k) \simeq \text{DGFun}(\mathcal{A}, \mathbf{Mod}\text{-}\mathcal{B}) \simeq \text{DGFun}(\mathcal{B}^{\text{opp}}, \mathbf{Mod}\text{-}\mathcal{A}^{\text{opp}})$$

considered as the DG-category of all \mathcal{A} - \mathcal{B} -bimodules. Let $M \in \mathcal{A}\text{-Mod}\text{-}\mathcal{B}$. For any $a \in \mathcal{A}, b \in \mathcal{B}$ we write ${}_aM_b$ for $M(a, b) \in \mathbf{Mod}\text{-}k$, write ${}_aM$ for the \mathcal{B} -module $M(a, -)$,

and M_b for the \mathcal{A}^{opp} -module $M(-, b)$. The functor $\mathcal{A} \rightarrow \mathbf{Mod}\text{-}\mathcal{B}$ which corresponds to M maps a to ${}_aM$. We can extend it to a functor

$$(-) \otimes_{\mathcal{A}} M : \mathbf{Mod}\text{-}\mathcal{A} \rightarrow \mathbf{Mod}\text{-}\mathcal{B},$$

where for any $E \in \mathbf{Mod}\text{-}\mathcal{A}$ and $b \in \mathcal{B}$ we set $(E \otimes_{\mathcal{A}} M)_b = E \otimes_{\mathcal{A}} M_b$ and have \mathcal{B} act via M_b . We can further extend this to a lift of the tensor bifunctor (2.10) from $\mathbf{Mod}\text{-}\mathcal{A}^{\text{opp}}$ to $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$ in the second argument. This admits a more general description.

Let \mathcal{A}, \mathcal{B} and \mathcal{C} be any DG-categories. Since $\mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{A}$ and $\mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$ are equivalent to $\text{DGFun}(\mathcal{C}, \mathbf{Mod}\text{-}\mathcal{A})$ and $\text{DGFun}(\mathcal{C}, \mathbf{Mod}\text{-}\mathcal{B})$, the composition functor

$$\text{DGFun}(\mathbf{Mod}\text{-}\mathcal{A}, \mathbf{Mod}\text{-}\mathcal{B}) \otimes_k \text{DGFun}(\mathcal{C}, \mathbf{Mod}\text{-}\mathcal{A}) \xrightarrow{(-) \circ (-)} \text{DGFun}(\mathcal{C}, \mathbf{Mod}\text{-}\mathcal{B})$$

induces via adjunction a functor

$$\text{DGFun}(\mathbf{Mod}\text{-}\mathcal{A}, \mathbf{Mod}\text{-}\mathcal{B}) \rightarrow \text{DGFun}(\mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{A}, \mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{B}) \tag{2.12}$$

best described as the functor of “defining fiberwise over \mathcal{C} ”. It takes a functor $\mathbf{Mod}\text{-}\mathcal{A} \xrightarrow{\Phi} \mathbf{Mod}\text{-}\mathcal{B}$ and defines a functor $\mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{A} \rightarrow \mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$ which takes any $\mathcal{C}\text{-}\mathcal{A}$ -bimodule E to the $\mathcal{C}\text{-}\mathcal{B}$ -bimodule whose fiber over each $c \in \mathcal{C}$ is $\Phi(c, E)$, and similarly for morphisms.

We can apply a similar procedure to functors whose domain is a tensor product of module categories via the simultaneous evaluation functor (2.8). We define the functor

$$(-) \otimes_{\mathcal{A}} (-) : \mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{A} \otimes_k \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B} \rightarrow \mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{B} \tag{2.13}$$

as the composition

$$\begin{array}{c} \text{DGFun}(\mathcal{C}, \mathbf{Mod}\text{-}\mathcal{A}) \otimes_k \text{DGFun}(\mathcal{B}^{\text{opp}}, \mathbf{Mod}\text{-}\mathcal{A}^{\text{opp}}) \\ \downarrow (2.8) \\ \text{DGFun}(\mathcal{C} \otimes_k \mathcal{B}^{\text{opp}}, \mathbf{Mod}\text{-}\mathcal{A} \otimes_k \mathbf{Mod}\text{-}\mathcal{A}^{\text{opp}}) \\ \downarrow (2.10) \circ (-) \\ \text{DGFun}(\mathcal{C} \otimes_k \mathcal{B}^{\text{opp}}, \mathbf{Mod}\text{-}k) \end{array}$$

Similarly, we use the $\mathbf{Mod}\text{-}k$ -valued Hom functor (2.11) to define the functors

$$\text{Hom}_{\mathcal{B}}(-, -) : \mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{B} \otimes_k (\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B})^{\text{opp}} \rightarrow \mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{A}, \tag{2.14}$$

$$\text{Hom}_{\mathcal{A}}(-, -) : \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{C} \otimes_k (\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B})^{\text{opp}} \rightarrow \mathcal{B}\text{-}\mathbf{Mod}\text{-}\mathcal{C}. \tag{2.15}$$

For any $\mathcal{A}\text{-}\mathcal{B}$ -bimodule M we have the usual Tensor-Hom adjunction: for any DG-category \mathcal{C} ,

$$(-) \otimes_{\mathcal{A}} M : \mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{A} \rightarrow \mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$$

is left adjoint to

$$\text{Hom}_{\mathcal{B}}(M, -) : \mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{B} \rightarrow \mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{A}.$$

Its adjunction counit

$$\mathrm{Hom}_{\mathcal{B}}(M, -) \otimes_{\mathcal{A}} M \rightarrow \mathrm{Id} \quad (2.16)$$

is given by the composition map

$$\mathrm{Hom}_{\mathcal{B}}(M, -) \otimes_{\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(\mathcal{B}, M) \rightarrow \mathrm{Hom}_{\mathcal{B}}(\mathcal{B}, -),$$

and its adjunction unit

$$\mathrm{Id} \rightarrow \mathrm{Hom}_{\mathcal{B}}(M, (-) \otimes_{\mathcal{A}} M) \quad (2.17)$$

is defined by

$$s \mapsto (\forall t \in {}_a M, t \mapsto s \otimes t) \quad \forall c \in \mathcal{C}, a \in \mathcal{A}, s \in c(-)_a.$$

Similarly,

$$M \otimes_{\mathcal{B}} (-): \mathcal{B}\text{-Mod-}\mathcal{C} \rightarrow \mathcal{A}\text{-Mod-}\mathcal{C}$$

is left adjoint to

$$\mathrm{Hom}_{\mathcal{A}^{\mathrm{opp}}}(M, -): \mathcal{A}\text{-Mod-}\mathcal{C} \rightarrow \mathcal{B}\text{-Mod-}\mathcal{C}$$

with analogous adjunction unit

$$\mathrm{Id} \rightarrow \mathrm{Hom}_{\mathcal{A}^{\mathrm{opp}}}(M, M \otimes_{\mathcal{B}} (-)) \quad (2.18)$$

and counit

$$M \otimes_{\mathcal{B}} \mathrm{Hom}_{\mathcal{A}^{\mathrm{opp}}}(M, -) \rightarrow \mathrm{Id}. \quad (2.19)$$

2.1.6. Derived category. A module $C \in \mathbf{Mod}\text{-}\mathcal{A}$ is *acyclic* if for each $a \in \mathcal{A}$ the complex of k -modules C_a is acyclic. A module $P \in \mathbf{Mod}\text{-}\mathcal{A}$ is *h -projective* if for every acyclic $C \in \mathbf{Mod}\text{-}\mathcal{A}$ we have $\mathrm{Hom}_{H^0(\mathbf{Mod}\text{-}\mathcal{A})}(P, C) = 0$. Denote by $\mathcal{P}(\mathcal{A})$ the corresponding full subcategory of $\mathbf{Mod}\text{-}\mathcal{A}$. A morphism $E \rightarrow F$ of \mathcal{A} -modules is a *quasi-isomorphism* if for each $a \in \mathcal{A}$ the induced morphism $E_a \rightarrow F_a$ is a quasi-isomorphism. Let $\mathcal{M}' \subset \mathbf{Mod}\text{-}\mathcal{A}$ be a full DG-subcategory. Then a *left* (resp. *right*) *resolution* of $E \in \mathcal{A}$ by $E' \in \mathcal{M}'$ is a quasi-isomorphism $E' \rightarrow E$ (resp. $E \rightarrow E'$). The *derived category* $D(\mathcal{A})$ is the localization of $H^0(\mathbf{Mod}\text{-}\mathcal{A})$ by the class of all quasi-isomorphisms. It can be understood explicitly as follows. By definition of acyclicity, $D(\mathcal{A})$ is the Verdier quotient of $H^0(\mathbf{Mod}\text{-}\mathcal{A})$ by $H^0(\mathcal{Ac}(\mathcal{A}))$. By definition of h -projectivity, $H^0(\mathcal{P}(\mathcal{A}))$ is left orthogonal to $H^0(\mathcal{Ac}(\mathcal{A}))$. Since left resolutions by h -projectives exist in $\mathbf{Mod}\text{-}\mathcal{A}$, we have in fact a semiorthogonal decomposition

$$H^0(\mathbf{Mod}\text{-}\mathcal{A}) = \langle H^0(\mathcal{Ac}(\mathcal{A})), H^0(\mathcal{P}(\mathcal{A})) \rangle.$$

This canonically identifies $D(\mathcal{A}) = H^0(\mathbf{Mod}\text{-}\mathcal{A})/H^0(\mathcal{Ac}(\mathcal{A}))$ with $H^0(\mathcal{P}(\mathcal{A}))$. In practice, we can use for resolutions a smaller full subcategory $\mathcal{SF}(\mathcal{A})$ of *semifree* modules in $\mathbf{Mod}\text{-}\mathcal{A}$. These are the modules $E \in \mathbf{Mod}\text{-}\mathcal{A}$ which admit an exhaustive filtration $0 = F_0 \subseteq F_1 \subseteq \dots \subseteq E$ whose quotients F_i/F_{i-1} are direct sums of shifts of representable modules. Any semifree module is h -projective and any \mathcal{A} -module can be resolved by a semifree module [Dri04, §C.8]. When k is a field, we have a functorial h -projective resolution of \mathcal{A} -modules provided by the *bar-resolution* $\bar{\mathcal{A}}$ of the diagonal \mathcal{A} - \mathcal{A} -bimodule \mathcal{A} (see e.g. [Kel94, §6.6]).

Another way to understand $D(\mathcal{A})$ is via either of the two natural model category structures induced on $\mathbf{Mod}\text{-}\mathcal{A}$ from $\mathbf{Mod}\text{-}k$. In particular, in the *projective* model category structure on $\mathbf{Mod}\text{-}k$ the weak equivalences are the quasi-isomorphisms and the fibrations are the termwise surjections of complexes. In the corresponding model category structure on $\mathbf{Mod}\text{-}\mathcal{A}$ we define the equivalences and the fibrations levelwise in $\mathbf{Mod}\text{-}k$, i.e. a morphism $A \rightarrow B$ is an equivalence (resp. fibration) if for every $a \in \mathcal{A}$ the morphism $A_a \rightarrow B_a$ is an equivalence (resp. fibration) in $\mathbf{Mod}\text{-}k$ [Toë07, §3]. It follows that every \mathcal{A} -module is fibrant, while the cofibrant modules are precisely the direct summands of semifree modules. We denote the full subcategory of $\mathbf{Mod}\text{-}\mathcal{A}$ consisting of cofibrant objects by $\mathcal{I}nt(\mathcal{A})$. It is the Karoubi completion of $\mathcal{S}\mathcal{F}(\mathcal{A})$.

Summarizing, we have a chain of full subcategories

$$\mathcal{S}\mathcal{F}(\mathcal{A}) \hookrightarrow \mathcal{I}nt(\mathcal{A}) \hookrightarrow \mathcal{P}(\mathcal{A}) \tag{2.20}$$

of $\mathbf{Mod}\text{-}\mathcal{A}$ which after applying H^0 becomes a chain of equivalent full triangulated subcategories

$$H^0(\mathcal{S}\mathcal{F}(\mathcal{A})) \xrightarrow{\sim} H^0(\mathcal{I}nt(\mathcal{A})) \xrightarrow{\sim} H^0(\mathcal{P}(\mathcal{A})) \tag{2.21}$$

of $H^0(\mathbf{Mod}\text{-}\mathcal{A})$. The natural functor $H^0(\mathbf{Mod}\text{-}\mathcal{A}) \rightarrow D(\mathcal{A})$ induces an equivalence of these with $D(\mathcal{A})$. In the language of Section 4, $\mathcal{S}\mathcal{F}(\mathcal{A})$, $\mathcal{I}nt(\mathcal{A})$ and $\mathcal{P}(\mathcal{A})$ are quasi-equivalent DG-enhancements of $D(\mathcal{A})$.

An \mathcal{A} -module E is *quasi-representable* if it is quasi-isomorphic to a representable module. We denote by $\mathcal{Q}r(\mathcal{A})$ and $\mathcal{P}^{qf}(\mathcal{A})$ the corresponding full subcategories of $\mathbf{Mod}\text{-}\mathcal{A}$ and of $\mathcal{P}(\mathcal{A})$. A semifree \mathcal{A} -module E is *finitely generated* if the filtration $F_0 \subseteq F_1 \subseteq \dots \subset E$ can be taken to be finite with quotients F_i/F_{i-1} finite direct sums of shifts of representables. Denote by $\mathcal{S}\mathcal{F}_{fg}(\mathcal{A})$ the corresponding full subcategory of $\mathcal{S}\mathcal{F}(\mathcal{A})$. Its homotopy category $H^0(\mathcal{S}\mathcal{F}_{fg}(\mathcal{A}))$ is the *triangulated hull* of $H^0(\mathcal{A})$ in $H^0(\mathbf{Mod}\text{-}\mathcal{A})$, i.e. the smallest full triangulated subcategory of $H^0(\mathbf{Mod}\text{-}\mathcal{A})$ containing $H^0(\mathcal{A})$. An \mathcal{A} -module E is *perfect* if its image in $D(\mathcal{A})$ lies in the full subcategory $D_c(\mathcal{A})$ of *compact* objects, i.e. $\text{Hom}_{D(\mathcal{A})}(E, -)$ commutes with infinite direct sums. We denote the full subcategories of perfect modules in $\mathbf{Mod}\text{-}\mathcal{A}$ and $\mathcal{P}(\mathcal{A})$ by $\mathcal{P}erf(\mathcal{A})$ and $\mathcal{P}^{Perf}(\mathcal{A})$, respectively.

In any category, an object E is a *retract* of an object F if there exist morphisms $E \rightarrow F \rightarrow E$ whose composition is the identity. For $E, F \in \mathbf{Mod}\text{-}\mathcal{A}$ we say that E is a *homotopy retract* of F if there exist $E \rightarrow F \rightarrow E$ whose composition is homotopic to the identity; in other words, E is a retract of F in $H^0(\mathbf{Mod}\text{-}\mathcal{A})$. In triangulated categories the notion of a retract is the same as that of a direct summand. The category $D_c(\mathcal{A})$ is the Karoubi completion of $H^0(\mathcal{S}\mathcal{F}_{fg}(\mathcal{A}))$ inside $D(\mathcal{A})$ [Kel94, §5]. Thus $\mathcal{P}^{Perf}(\mathcal{A})$ coincides with the full subcategory in $\mathbf{Mod}\text{-}\mathcal{A}$ of homotopy retracts of elements of $\mathcal{S}\mathcal{F}_{fg}(\mathcal{A})$.

Let \mathcal{A} and \mathcal{B} be DG-categories and let $M \in \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$. Let $(-)\overset{\mathbf{L}}{\otimes}_{\mathcal{A}} M$, $M\overset{\mathbf{L}}{\otimes}_{\mathcal{B}} (-)$, $\mathbf{R}\text{Hom}_{\mathcal{B}}(M, -)$ and $\mathbf{R}\text{Hom}_{\mathcal{A}^{opp}}(M, -)$ be the derived functors corresponding to the functors $(-)\otimes_{\mathcal{A}} M$, etc.

We say that an $\mathcal{A}\text{-}\mathcal{B}$ -bimodule M is

- \mathcal{A} -perfect if M_b is a perfect \mathcal{A}^{opp} -module for each $b \in \mathcal{B}$,
- \mathcal{B} -perfect if ${}_a M$ is a perfect \mathcal{B} -module for each $a \in \mathcal{A}$.

We similarly define the notions of \mathcal{A} - and \mathcal{B} -quasi-representability, h -projectivity, etc.

Since acyclicity is defined levelwise in $\mathbf{Mod}\text{-}k$, $\mathrm{Hom}_{\mathcal{B}}(M, -)$ takes acyclic modules to acyclic ones for any \mathcal{B} - h -projective M . The same is true for $M \otimes_{\mathcal{B}} (-)$, since it is trivially true for any \mathcal{B} -representable M . Thus to compute $\mathbf{R}\mathrm{Hom}_{\mathcal{B}}(M, -)$ and $M \overset{\mathbf{L}}{\otimes}_{\mathcal{B}} (-)$ it suffices to take a \mathcal{B} - h -projective resolution of M . Similarly, if M is \mathcal{A} - h -projective then $(-) \otimes_{\mathcal{A}} M$ and $\mathrm{Hom}_{\mathcal{A}^{\mathrm{opp}}}(M, -)$ compute $(-) \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} M$ and $\mathbf{R}\mathrm{Hom}_{\mathcal{A}^{\mathrm{opp}}}(M, -)$. If k is a field¹ then any h -projective \mathcal{A} - \mathcal{B} -bimodule is both \mathcal{A} - and \mathcal{B} - h -projective [Kel94, §6.1], and hence the derived functors above can be computed by taking an h -projective resolution of M .

It follows from the above that M is

- \mathcal{A} -perfect if and only if $M \overset{\mathbf{L}}{\otimes}_{\mathcal{B}} (-)$ restricts to $D_c(\mathcal{B}^{\mathrm{opp}}) \rightarrow D_c(\mathcal{A}^{\mathrm{opp}})$,
- \mathcal{B} -perfect if and only if $(-) \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} M$ restricts to $D_c(\mathcal{A}) \rightarrow D_c(\mathcal{B})$.

If k is a field we can be more precise. Let $M \in \mathcal{P}(\mathcal{A}\text{-}\mathcal{B})$. The functors $(-) \otimes_{\mathcal{A}} M$ and $M \otimes_{\mathcal{B}} (-)$ restrict to $\mathcal{A} \rightarrow \mathcal{P}(\mathcal{B})$ and $\mathcal{B}^{\mathrm{opp}} \rightarrow \mathcal{P}(\mathcal{A}^{\mathrm{opp}})$, and M is

- \mathcal{A} -perfect if and only if $M \otimes_{\mathcal{B}} (-)$ restricts to a functor $\mathcal{P}^{\mathrm{Perf}}(\mathcal{B}^{\mathrm{opp}}) \rightarrow \mathcal{P}^{\mathrm{Perf}}(\mathcal{A}^{\mathrm{opp}})$,
- \mathcal{B} -perfect if and only if $(-) \otimes_{\mathcal{A}} M$ restricts to a functor $\mathcal{P}^{\mathrm{Perf}}(\mathcal{A}) \rightarrow \mathcal{P}^{\mathrm{Perf}}(\mathcal{B})$.

2.2. Duals and adjoints

As before, let \mathcal{A} be a DG-category. Define the diagonal \mathcal{A} - \mathcal{A} -bimodule \mathcal{A} by setting ${}_a\mathcal{A}_b = \mathrm{Hom}_{\mathcal{A}}(b, a)$ for any $a, b \in \mathcal{A}$. Then ${}_a\mathcal{A}$ and \mathcal{A}_a are precisely the representable modules $\mathrm{Hom}_{\mathcal{A}}(-, a)$ and $\mathrm{Hom}_{\mathcal{A}}(a, -)$ in $\mathbf{Mod}\text{-}\mathcal{A}$ and $\mathbf{Mod}\text{-}\mathcal{A}^{\mathrm{opp}}$. This coincides with the notation introduced in §2.1.3.

The diagonal bimodule corresponds to the functor $\mathcal{A} \rightarrow \mathbf{Mod}\text{-}\mathcal{A}$ which sends a to ${}_a\mathcal{A}$. We have natural functorial isomorphisms

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, -) \simeq \mathrm{Id}_{\mathbf{Mod}\text{-}\mathcal{A}} \simeq (-) \otimes_{\mathcal{A}} \mathcal{A} \tag{2.22}$$

given for any \mathcal{A} -module M explicitly by

$$\begin{aligned} M &\rightarrow M \otimes_{\mathcal{A}} \mathcal{A}: s \mapsto s \otimes \mathrm{Id}_a, & a \in \mathcal{A}, s \in M_a, \\ M \otimes_{\mathcal{A}} \mathcal{A} &\rightarrow M: s \otimes \alpha \mapsto s.\alpha, & a, b \in \mathcal{A}, s \in M_b, \alpha \in {}_b\mathcal{A}_a, \\ M &\rightarrow \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, M): s \mapsto (\alpha \mapsto s.\alpha \ \forall b \in \mathcal{A}, \alpha \in {}_a\mathcal{A}_b), & a \in \mathcal{A}, s \in M_a, \\ \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, M) &\rightarrow M: \alpha \mapsto \alpha(\mathrm{Id}_a), & a \in \mathcal{A}, \alpha \in \mathrm{Hom}_{\mathcal{A}}({}_a\mathcal{A}, M). \end{aligned}$$

We use these isomorphisms implicitly throughout the paper.

On the other hand, $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{A})$ is the dualizing functor

$$(-)^{\mathcal{A}}: (\mathbf{Mod}\text{-}\mathcal{A})^{\mathrm{opp}} \rightarrow \mathbf{Mod}\text{-}\mathcal{A}^{\mathrm{opp}}.$$

Explicitly, for any $C \in \mathbf{Mod}\text{-}\mathcal{A}$ its dual module $C^{\mathcal{A}}$ is the $\mathcal{A}^{\mathrm{opp}}$ -module $a \mapsto \mathrm{Hom}_{\mathcal{A}}(C, {}_a\mathcal{A})$. For any morphism $C \xrightarrow{\alpha} D$ in $\mathbf{Mod}\text{-}\mathcal{A}$ the dual morphism $\alpha^{\mathcal{A}}$ is de-

¹ If it is not, one should take cofibrant replacements of \mathcal{A} and \mathcal{B} [Toë07, Prop. 3.3].

finned with a sign twist: for each $a \in \mathcal{A}$ define the requisite morphism $\text{Hom}_{\mathcal{A}}(D, {}_a\mathcal{A}) \rightarrow \text{Hom}_{\mathcal{A}}(C, {}_a\mathcal{A})$ by

$$\beta \mapsto (-1)^{\text{deg}(\beta)\text{deg}(\alpha)} \beta \circ \alpha.$$

Tautologically, $(-)^{\mathcal{A}}$ restricts to Id on the Yoneda embedded subcategories $\mathcal{A}^{\text{opp}} \hookrightarrow (\mathbf{Mod}\text{-}\mathcal{A})^{\text{opp}}$ and $\mathcal{A}^{\text{opp}} \hookrightarrow \mathbf{Mod}\text{-}\mathcal{A}^{\text{opp}}$. Therefore it induces an equivalence

$$\mathcal{SF}_{\text{fg}}(\mathcal{A})^{\text{opp}} \xrightarrow{\sim} \mathcal{SF}_{\text{fg}}(\mathcal{A}^{\text{opp}})$$

and a quasi-equivalence

$$\mathcal{P}^{\text{Perf}}(\mathcal{A})^{\text{opp}} \rightarrow \mathcal{P}^{\text{Perf}}(\mathcal{A}^{\text{opp}}),$$

whose induced maps on morphism complexes are homotopy equivalences. By abuse of notation, we also use $(-)^{\mathcal{A}}$ for the dualizing functor for \mathcal{A}^{opp} . The double dualizing functor $(-)^{\mathcal{A}\mathcal{A}}: \mathbf{Mod}\text{-}\mathcal{A} \rightarrow \mathbf{Mod}\text{-}\mathcal{A}$ is isomorphic to the identity on $\mathcal{SF}_{\text{fg}}(\mathcal{A})$ and homotopic to the identity on $\mathcal{P}^{\text{Perf}}(\mathcal{A})$. An analogous claim holds for $(-)^{\mathcal{A}\mathcal{A}}: \mathbf{Mod}\text{-}\mathcal{A}^{\text{opp}} \rightarrow \mathbf{Mod}\text{-}\mathcal{A}^{\text{opp}}$.

Let $C \in \mathbf{Mod}\text{-}\mathcal{B}$, $D \in \mathbf{Mod}\text{-}\mathcal{A}$ and let M be an $\mathcal{A}\text{-}\mathcal{B}$ -bimodule. There is a natural map of DG k -modules

$$D \otimes_{\mathcal{A}} \text{Hom}_{\mathcal{B}}(C, M) \rightarrow \text{Hom}_{\mathcal{B}}(C, D \otimes_{\mathcal{A}} M) \tag{2.23}$$

defined by setting, for any $a \in \mathcal{A}$,

$$s \otimes \gamma \mapsto (\forall t \in C, t \mapsto s \otimes \gamma(t)), \quad s \in D_a, \gamma \in \text{Hom}_{\mathcal{B}}(C, {}_aM).$$

This map is clearly an isomorphism when either C or D are representable. It follows that it is an isomorphism when either C or D lies in $\mathcal{SF}_{\text{fg}}(\mathcal{A})$, and a homotopy equivalence when either C or D lies in $\mathcal{P}^{\text{Perf}}(\mathcal{A})$.

If in (2.23) we set $\mathcal{B} = \mathcal{A}$ and let M be the diagonal bimodule \mathcal{A} , we obtain the evaluation map

$$D \otimes_{\mathcal{A}} C^{\mathcal{A}} \xrightarrow{\text{ev}} \text{Hom}_{\mathcal{A}}(C, D). \tag{2.24}$$

It is the same map of DG k -modules as the composition map

$$\text{Hom}_{\mathcal{A}}(\mathcal{A}, D) \otimes_{\mathcal{A}} \text{Hom}_{\mathcal{A}}(C, \mathcal{A}) \rightarrow \text{Hom}_{\mathcal{A}}(C, D).$$

Let M be an $\mathcal{A}\text{-}\mathcal{B}$ -bimodule. We define $M^{\mathcal{A}}$, the dual of M with respect to \mathcal{A} , to be the $\mathcal{B}\text{-}\mathcal{A}$ -bimodule $\text{Hom}_{\mathcal{A}^{\text{opp}}}(M, \mathcal{A})$. In other words, $M^{\mathcal{A}}$ corresponds to the functor $\mathcal{B} \rightarrow \mathbf{Mod}\text{-}\mathcal{A}$ which maps b to $(M_b)^{\mathcal{A}}$. Similarly, we define $M^{\mathcal{B}}$, the dual of M with respect to \mathcal{B} , to be the $\mathcal{B}\text{-}\mathcal{A}$ -bimodule $\text{Hom}_{\mathcal{B}}(M, \mathcal{B})$, which corresponds to the functor $\mathcal{A}^{\text{opp}} \rightarrow \mathbf{Mod}\text{-}\mathcal{B}^{\text{opp}}$ which maps a to $({}_aM)^{\mathcal{B}}$. More generally, define the functor $(-)^{\mathcal{A}}: (\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B})^{\text{opp}} \rightarrow \mathcal{B}\text{-}\mathbf{Mod}\text{-}\mathcal{A}$ fiberwise over \mathcal{B} by the dualizing functor of $\mathbf{Mod}\text{-}\mathcal{A}$, and define $(-)^{\mathcal{B}}$ similarly. Denote by $(-)^{\tilde{\mathcal{A}}}$ and $(-)^{\tilde{\mathcal{B}}}$ their derived functors $D(\mathcal{A}\text{-}\mathcal{B})^{\text{opp}} \rightarrow D(\mathcal{B}\text{-}\mathcal{A})$. Since $(-)^{\mathcal{A}}$ is defined fiberwise over \mathcal{B} , it sends \mathcal{A} - h -projective acyclic bimodules to acyclic ones. It follows that if M is \mathcal{A} - h -projective, then $M^{\tilde{\mathcal{A}}} \simeq (M)^{\mathcal{A}}$ in $D(\mathcal{B}\text{-}\mathcal{A})$. Similarly, if M is \mathcal{B} - h -projective, then $M^{\tilde{\mathcal{B}}} \simeq (M)^{\mathcal{B}}$ in $D(\mathcal{B}\text{-}\mathcal{A})$.

The evaluation map (2.24) induces a morphism of functors $\mathbf{Mod}\text{-}\mathcal{B} \rightarrow \mathbf{Mod}\text{-}\mathcal{A}$,

$$(-) \otimes_{\mathcal{B}} M^{\mathcal{B}} \rightarrow \mathrm{Hom}_{\mathcal{B}}(M, -). \tag{2.25}$$

It follows from the above that for any M the map (2.25) is an isomorphism on all of $\mathcal{S}\mathcal{F}_{\mathrm{fg}}(\mathcal{B})$, and a homotopy equivalence on all of $\mathcal{P}^{\mathrm{Perf}}(\mathcal{B})$. On the other hand, if M is \mathcal{B} - h -projective and \mathcal{B} -perfect, then for any $N \in \mathbf{Mod}\text{-}\mathcal{B}$ the morphism (2.25) is a quasi-isomorphism. This is because all ${}_a M$ lie in $\mathcal{P}^{\mathrm{Perf}}(\mathcal{B})$, and thus (2.25) is a homotopy equivalence levelwise in $\mathbf{Mod}\text{-}k$. Similarly, we obtain a morphism of functors

$$M^{\mathcal{A}} \otimes_{\mathcal{A}} (-) \rightarrow \mathrm{Hom}_{\mathcal{A}}(M, -), \tag{2.26}$$

which is a quasi-isomorphism on all of $\mathbf{Mod}\text{-}\mathcal{A}^{\mathrm{opp}}$ whenever M is \mathcal{A} - h -projective and \mathcal{A} -perfect.

Consider the map

$$M^{\mathcal{B}} \otimes_{\mathcal{A}} M \rightarrow \mathcal{B}$$

given by the Tensor-Hom adjunction counit (2.16) evaluated at the diagonal bimodule \mathcal{B} . Taking its right adjoint with respect to $M^{\mathcal{B}} \otimes_{\mathcal{A}} (-)$ yields a map $M \rightarrow M^{\mathcal{B}\mathcal{B}}$. The induced natural transformation

$$\mathrm{Id} \rightarrow (-)^{\mathcal{B}\mathcal{B}} \tag{2.27}$$

is a quasi-isomorphism for any \mathcal{B} - h -projective and \mathcal{B} -perfect M , since it is then a homotopy equivalence levelwise in $\mathbf{Mod}\text{-}\mathcal{B}$. We similarly define a natural transformation

$$\mathrm{Id} \rightarrow (-)^{\mathcal{A}\mathcal{A}}, \tag{2.28}$$

which is a quasi-isomorphism for any \mathcal{A} - h -projective and \mathcal{A} -perfect M .

The above properties of the natural transformations (2.25)–(2.28) imply the following:

Lemma 2.1. (1) For any $M \in D_{\mathcal{A}\text{-}\mathcal{P}\mathrm{erf}}(\mathcal{A}\text{-}\mathcal{B})$ we have an isomorphism of functors $D(\mathcal{A}^{\mathrm{opp}}) \rightarrow D(\mathcal{B}^{\mathrm{opp}})$:

$$M^{\tilde{\mathcal{A}}} \overset{\mathrm{L}}{\otimes}_{\mathcal{A}} (-) \simeq \mathbf{R}\mathrm{Hom}_{\mathcal{A}}(M, -). \tag{2.29}$$

(2) For any $M \in D_{\mathcal{B}\text{-}\mathcal{P}\mathrm{erf}}(\mathcal{A}\text{-}\mathcal{B})$ we have an isomorphism of functors $D(\mathcal{B}) \rightarrow D(\mathcal{A})$:

$$(-) \overset{\mathrm{L}}{\otimes}_{\mathcal{B}} M^{\tilde{\mathcal{B}}} \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{\mathcal{B}}(M, -). \tag{2.30}$$

(3) We have an isomorphism of endofunctors of $D_{\mathcal{A}\text{-}\mathcal{P}\mathrm{erf}}(\mathcal{A}\text{-}\mathcal{B})$:

$$\mathrm{Id} \xrightarrow{\sim} (-)^{\tilde{\mathcal{A}}\tilde{\mathcal{A}}}. \tag{2.31}$$

(4) We have an isomorphism of endofunctors of $D_{\mathcal{B}\text{-}\mathcal{P}\mathrm{erf}}(\mathcal{A}\text{-}\mathcal{B})$:

$$\mathrm{Id} \xrightarrow{\sim} (-)^{\tilde{\mathcal{B}}\tilde{\mathcal{B}}}. \tag{2.32}$$

In view of Tensor-Hom adjunction we then have:

Corollary 2.2. (1) For any $M \in D_{\mathcal{A}\text{-Perf}}(\mathcal{A}\text{-}\mathcal{B})$ the functor

$$(-) \overset{\mathbf{L}}{\otimes}_{\mathcal{B}} M^{\tilde{\mathcal{A}}}: D(\mathcal{B}) \rightarrow D(\mathcal{A})$$

is left adjoint to the functor

$$(-) \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} M: D(\mathcal{A}) \rightarrow D(\mathcal{B}).$$

(2) For any $M \in D_{\mathcal{B}\text{-Perf}}(\mathcal{A}\text{-}\mathcal{B})$ the functor

$$(-) \overset{\mathbf{L}}{\otimes}_{\mathcal{B}} M^{\tilde{\mathcal{B}}}: D(\mathcal{B}) \rightarrow D(\mathcal{A})$$

is right adjoint to the functor

$$(-) \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} M: D(\mathcal{A}) \rightarrow D(\mathcal{B}).$$

Proof. (1) By Lemma 2.1 we have the isomorphism of functors

$$(-) \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} M \xrightarrow{(2.31)} (-) \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} M^{\tilde{\mathcal{A}}\tilde{\mathcal{A}}} \xrightarrow{(2.29)} \mathbf{R}\mathrm{Hom}_{\mathcal{A}}(M^{\tilde{\mathcal{A}}}, -). \quad (2.33)$$

This isomorphism transforms the derived Tensor-Hom adjunction

$$(-) \overset{\mathbf{L}}{\otimes}_{\mathcal{B}} M^{\tilde{\mathcal{A}}} \leftrightarrow \mathbf{R}\mathrm{Hom}_{\mathcal{A}}(M^{\tilde{\mathcal{A}}}, -)$$

with its unit (2.17) and counit (2.16) into the desired adjunction

$$(-) \overset{\mathbf{L}}{\otimes}_{\mathcal{B}} M^{\tilde{\mathcal{A}}} \leftrightarrow (-) \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} M.$$

(2) Similarly, by Lemma 2.1 we have an isomorphism

$$(-) \overset{\mathbf{L}}{\otimes}_{\mathcal{B}} M^{\tilde{\mathcal{B}}} \xrightarrow{(2.30)} \mathbf{R}\mathrm{Hom}_{\mathcal{B}}(M, -) \quad (2.34)$$

which produces the desired adjunction out of the Tensor-Hom adjunction

$$(-) \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} M \leftrightarrow \mathbf{R}\mathrm{Hom}_{\mathcal{B}}(M^{\tilde{\mathcal{B}}}, -). \quad \square$$

It is helpful to have the units and counits of the adjunctions in Cor. 2.2 written down explicitly in terms of the maps between the corresponding bimodules:

Definition 2.3. Let $M \in \mathcal{A}\text{-Mod-}\mathcal{B}$. The \mathcal{B} -trace map

$$M^{\mathcal{B}} \otimes_{\mathcal{A}} M \xrightarrow{\mathrm{tr}} \mathcal{B} \quad (2.35)$$

in $\mathcal{B}\text{-Mod-}\mathcal{B}$ is the counit (2.16) of the Tensor-Hom adjunction evaluated at the diagonal bimodule \mathcal{B} . The *derived* \mathcal{B} -trace map is the induced map $M^{\mathcal{B}} \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} M \xrightarrow{\mathrm{tr}} \mathcal{B}$ in $D(\mathcal{B}\text{-}\mathcal{B})$.

The \mathcal{A} -trace map

$$M \otimes_{\mathcal{B}} M^{\mathcal{A}} \xrightarrow{\mathrm{tr}} \mathcal{A} \quad (2.36)$$

and its derived version are defined similarly.

For \mathcal{B} - and \mathcal{A} -perfect M the associativity of the composition map implies that the natural transformations

$$(-) \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} M \overset{\mathbf{L}}{\otimes}_{\mathcal{B}} M^{\tilde{\mathcal{A}}} \xrightarrow{\text{tr}} \text{Id}_{D(\mathcal{A})}, \quad (-) \overset{\mathbf{L}}{\otimes}_{\mathcal{B}} M^{\tilde{\mathcal{B}}} \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} M \xrightarrow{\text{tr}} \text{Id}_{D(\mathcal{B})}$$

coincide with the counits of the adjunctions in Cor. 2.2(1)–(2).

Definition 2.4. Let $M \in \mathcal{A}\text{-Mod-}\mathcal{B}$. The \mathcal{A} -action map

$$\mathcal{A} \xrightarrow{\text{act}} \text{Hom}_{\mathcal{B}}(M, M) \tag{2.37}$$

in $\mathcal{A}\text{-Mod-}\mathcal{A}$ is the unit (2.17) of the Tensor-Hom adjunction evaluated at \mathcal{A} . The *derived \mathcal{A} -action map* is the induced map $\mathcal{A} \xrightarrow{\text{act}} \mathbf{R}\text{Hom}_{\mathcal{B}}(M, M)$ in $D(\mathcal{A}\text{-}\mathcal{A})$. When M is \mathcal{B} -perfect we also use this term for the corresponding map $\mathcal{A} \xrightarrow{\text{act}} M \overset{\mathbf{L}}{\otimes}_{\mathcal{B}} M^{\tilde{\mathcal{B}}}$ obtained via the isomorphism (2.30).

The \mathcal{B} -action map

$$\mathcal{B} \xrightarrow{\text{act}} \text{Hom}_{\mathcal{A}^{\text{opp}}}(M, M) \tag{2.38}$$

and its derived versions are defined similarly.

For \mathcal{B} - and \mathcal{A} -perfect M the induced natural transformations

$$\text{Id}_{D(\mathcal{B})} \xrightarrow{\text{act}} (-) \overset{\mathbf{L}}{\otimes}_{\mathcal{B}} M^{\tilde{\mathcal{A}}} \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} M, \quad \text{Id}_{D(\mathcal{A})} \xrightarrow{\text{act}} (-) \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} M \overset{\mathbf{L}}{\otimes}_{\mathcal{B}} M^{\tilde{\mathcal{B}}}$$

coincide with the units the of adjunctions in Cor. 2.2(1)–(2). Showing this amounts to checking that

$$\begin{array}{ccc} (-) \otimes_{\mathcal{A}} \mathcal{A} & \xrightarrow{\text{Id} \otimes \text{act}} & (-) \otimes_{\mathcal{A}} \text{Hom}_{\mathcal{B}}(M, M) \\ \downarrow \simeq & & \downarrow (2.23) \\ (-) & \xrightarrow{(2.17)} & \text{Hom}_{\mathcal{B}}(M, (-) \otimes_{\mathcal{A}} M) \end{array}$$

commutes. It is a straightforward exercise we leave to the reader.

We would now like to lift these derived adjunctions to homotopy ones. That is, given an \mathcal{A} - and \mathcal{B} -perfect $M \in \mathcal{A}\text{-Mod-}\mathcal{B}$, we would like to write down h -projective resolutions of M , $M^{\tilde{\mathcal{A}}}$ and $M^{\tilde{\mathcal{B}}}$ and the four maps which induce in the homotopy category the units and counits of the two adjunctions in Cor. 2.2.

We use specific resolutions of M , $M^{\tilde{\mathcal{A}}}$ and $M^{\tilde{\mathcal{B}}}$ obtained via the bar-construction [Kel94, §6.6]. We briefly recall the essentials. Let $\tilde{\mathcal{A}} \rightarrow \mathcal{A}$ and $\tilde{\mathcal{B}} \rightarrow \mathcal{B}$ be the bar-resolutions of the diagonal bimodules in $\mathcal{A}\text{-Mod-}\mathcal{A}$ and $\mathcal{B}\text{-Mod-}\mathcal{B}$. These are quasi-isomorphisms with $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ h -projective. The induced natural transformations $(-) \otimes_{\mathcal{A}} \tilde{\mathcal{A}} \rightarrow \text{Id}_{\text{Mod-}\mathcal{A}}$ and $(-) \otimes_{\mathcal{B}} \tilde{\mathcal{B}} \rightarrow \text{Id}_{\text{Mod-}\mathcal{B}}$ are functorial h -projective resolutions for \mathcal{A} - and \mathcal{B} -modules, respectively. This can be seen via the following useful fact:

Proposition 2.5. *Let \mathcal{A} , \mathcal{B} and \mathcal{C} be DG-categories. Let M be an \mathcal{A} - \mathcal{B} -bimodule and N be a \mathcal{B} - \mathcal{C} -bimodule. If either*

- (1) *M is h -projective and N is a \mathcal{C} - h -projective, or*
- (2) *M is \mathcal{A} - h -projective and N is h -projective,*

then $M \otimes_B N$ is an h -projective \mathcal{A} - \mathcal{C} -bimodule.

Proof. Suppose M is h -projective and N is \mathcal{C} - h -projective, and let Q be any acyclic \mathcal{A} - \mathcal{C} -bimodule. By the adjunction of $(-)\otimes_B N$ and $\text{Hom}_{\mathcal{C}}(N, -)$ we have a natural isomorphism

$$\text{Hom}_{\mathcal{A}\text{-}\mathcal{C}}(M \otimes_B N, Q) \simeq \text{Hom}_{\mathcal{A}\text{-}\mathcal{B}}(M, \text{Hom}_{\mathcal{C}}(N, Q)).$$

Observe that for any $a \in \mathcal{A}$ and $b \in \mathcal{B}$, ${}_b N$ is an h -projective module and Q_a is an acyclic \mathcal{C} -module. It follows that $\text{Hom}_{\mathcal{C}}(N, Q)$ is an acyclic \mathcal{A} - \mathcal{B} -bimodule, and hence $\text{Hom}_{\mathcal{A}\text{-}\mathcal{B}}(M, \text{Hom}_{\mathcal{C}}(N, Q))$ is acyclic. We have now shown that $\text{Hom}_{\mathcal{A}\text{-}\mathcal{C}}(M \otimes_B N, Q)$ is acyclic for any acyclic Q , whence $M \otimes_B N$ is an h -projective \mathcal{A} - \mathcal{C} -bimodule.

The case of M being \mathcal{A} - h -projective and N being h -projective is treated similarly. \square

Similarly, for \mathcal{A} - \mathcal{B} -bimodules an h -projective resolution could be obtained by tensoring with $\overline{\mathcal{A}}^{\text{opp}} \otimes \overline{\mathcal{B}}$. However, there is another resolution more suited to our needs:

Corollary 2.6. *Let $M \in \mathcal{A}\text{-Mod-}\mathcal{B}$. Then $\overline{\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \overline{\mathcal{B}} \rightarrow M$ is an h -projective resolution of M .*

Proof. By Prop. 2.5 the bimodule $\overline{\mathcal{A}} \otimes_{\mathcal{A}} M$ is \mathcal{A} - h -projective, and then by Prop. 2.5 again, $(\overline{\mathcal{A}} \otimes_{\mathcal{A}} M) \otimes_{\mathcal{B}} \overline{\mathcal{B}}$ is h -projective. \square

Definition 2.7. Define $\overline{\mathcal{P}}(\mathcal{A}\text{-}\mathcal{B})$ to be the full subcategory of $\mathcal{P}(\mathcal{A}\text{-}\mathcal{B})$ consisting of all bimodules of the form $\overline{\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \overline{\mathcal{B}}$ for some $M \in \mathcal{A}\text{-Mod-}\mathcal{B}$.

Note that by Cor. 2.6 we have a canonical identification $H^0(\overline{\mathcal{P}}(\mathcal{A}\text{-}\mathcal{B})) \simeq D(\mathcal{A}\text{-}\mathcal{B})$.

Let N be any \mathcal{A} - \mathcal{B} -bimodule. The quasi-isomorphisms $\overline{\mathcal{A}} \rightarrow \mathcal{A}$ and $\overline{\mathcal{B}} \rightarrow \mathcal{B}$ and the functorial isomorphisms (2.22) yield functorial quasi-isomorphisms

$$\overline{\mathcal{A}} \otimes_{\mathcal{A}} N \xrightarrow{\sim} N \xleftarrow{\sim} N \otimes_{\mathcal{B}} \overline{\mathcal{B}}. \tag{2.39}$$

If N is in $\mathcal{P}(\mathcal{A}\text{-}\mathcal{B})$ then so are $\overline{\mathcal{A}} \otimes_{\mathcal{A}} N$ and $N \otimes_{\mathcal{B}} \overline{\mathcal{B}}$, and the two quasi-isomorphisms in (2.39) are actually homotopy equivalences. If moreover $N \in \overline{\mathcal{P}}(\mathcal{A}\text{-}\mathcal{B})$, we have canonical homotopy inverses of (2.39),

$$\overline{\mathcal{A}} \otimes_{\mathcal{A}} N \xleftarrow{\sim} N \xrightarrow{\sim} N \otimes_{\mathcal{B}} \overline{\mathcal{B}}, \tag{2.40}$$

induced by the comultiplication maps $\overline{\mathcal{A}} \rightarrow \overline{\mathcal{A}} \otimes_{\mathcal{A}} \overline{\mathcal{A}}$ and $\overline{\mathcal{B}} \rightarrow \overline{\mathcal{B}} \otimes_{\mathcal{B}} \overline{\mathcal{B}}$ defined in [Kel94, §6.6]. Moreover, these are genuine right inverses—the following compositions are not merely homotopic but equal to Id:

$$N \xrightarrow{(2.40)} \overline{\mathcal{A}} \otimes_{\mathcal{A}} N \xrightarrow{(2.39)} N, \quad N \xrightarrow{(2.40)} N \otimes_{\mathcal{B}} \overline{\mathcal{B}} \xrightarrow{(2.39)} N.$$

This is our main reason for introducing $\overline{\mathcal{P}}(\mathcal{A}\text{-}\mathcal{B})$: it makes a number of diagrams commute genuinely and not up to homotopy. Throughout the rest of the paper, where necessary, we implicitly identify any $N \in \overline{\mathcal{P}}(\mathcal{A}\text{-}\mathcal{B})$ with $\overline{\mathcal{A}} \otimes_{\mathcal{A}} N$ and $N \otimes_{\mathcal{B}} \overline{\mathcal{B}}$ via (2.39) and (2.40).

The dualization functors $(-)^{\mathcal{A}}$ and $(-)^{\mathcal{B}}$ do not restrict to functors $\bar{\mathcal{P}}(\mathcal{A}\text{-}\mathcal{B}) \rightarrow \bar{\mathcal{P}}(\mathcal{B}\text{-}\mathcal{A})$. We thus define:

Definition 2.8. Let $M \in \mathcal{A}\text{-Mod-}\mathcal{B}$. Define $M^{h\mathcal{A}}$ and $M^{h\mathcal{B}}$ to be the bimodules $\bar{\mathcal{B}} \otimes_{\mathcal{B}} M^{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}}$ and $\bar{\mathcal{B}} \otimes_{\mathcal{B}} M^{\mathcal{B}} \otimes_{\mathcal{A}} \bar{\mathcal{A}}$, respectively.

These are our chosen h -projective resolutions of the derived duals $M^{\bar{\mathcal{A}}}$ and $M^{\bar{\mathcal{B}}}$. We now proceed to define the unit and counit maps of our homotopy adjunctions.

Definition 2.9. Let $M \in \bar{\mathcal{P}}(\mathcal{A}\text{-}\mathcal{B})$. The homotopy \mathcal{A} -trace map $M \otimes_{\mathcal{B}} M^{h\mathcal{A}} \xrightarrow{\text{tr}} \bar{\mathcal{A}}$ is the composition

$$M \otimes_{\mathcal{B}} M^{h\mathcal{A}} \xrightarrow{\bar{\mathcal{B}} \rightarrow \mathcal{B}} M \otimes_{\mathcal{B}} M^{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\text{tr} \otimes \text{Id}} \bar{\mathcal{A}}. \tag{2.41}$$

Similarly, the homotopy \mathcal{B} -trace map $M^{h\mathcal{B}} \otimes_{\mathcal{A}} M \xrightarrow{\text{tr}} \bar{\mathcal{B}}$ is the composition

$$M^{h\mathcal{B}} \otimes_{\mathcal{A}} M \xrightarrow{\bar{\mathcal{A}} \rightarrow \mathcal{A}} \bar{\mathcal{B}} \otimes_{\mathcal{B}} M^{\mathcal{B}} \otimes_{\mathcal{A}} M \xrightarrow{\text{Id} \otimes \text{tr}} \bar{\mathcal{B}}. \tag{2.42}$$

Definition 2.10. Let $M \in \bar{\mathcal{P}}^{\mathcal{B}\text{-Perf}}(\mathcal{A}\text{-}\mathcal{B})$. The map

$$M \otimes_{\mathcal{B}} M^{h\mathcal{B}} \xrightarrow{\bar{\mathcal{B}} \rightarrow \mathcal{B}} M \otimes_{\mathcal{B}} M^{\mathcal{B}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{(2.25) \otimes \text{Id}} \text{Hom}_{\mathcal{B}}(M, M) \otimes_{\mathcal{A}} \bar{\mathcal{A}} \tag{2.43}$$

is a quasi-isomorphism since (2.25) is one. Thus there exists a homotopy lift of $\bar{\mathcal{A}} \xrightarrow{\text{act} \otimes \text{Id}} \text{Hom}_{\mathcal{B}}(M, M) \otimes_{\mathcal{A}} \bar{\mathcal{A}}$ along (2.43). Choose once and for all such a lift and call it the homotopy \mathcal{A} -action map

$$\bar{\mathcal{A}} \xrightarrow{\text{act}} M \otimes_{\mathcal{B}} M^{h\mathcal{B}}. \tag{2.44}$$

Let $M \in \bar{\mathcal{P}}^{\mathcal{A}\text{-Perf}}(\mathcal{A}\text{-}\mathcal{B})$. We define similarly the homotopy \mathcal{B} -action map

$$\bar{\mathcal{B}} \xrightarrow{\text{act}} M^{h\mathcal{A}} \otimes_{\mathcal{A}} M. \tag{2.45}$$

Proposition 2.11. Let \mathcal{A} and \mathcal{B} be DG-categories and $M \in \bar{\mathcal{P}}(\mathcal{A}\text{-}\mathcal{B})$.

If M is \mathcal{B} -perfect then $(-) \otimes_{\mathcal{B}} M^{h\mathcal{B}}$ is homotopy right adjoint to $(-) \otimes_{\mathcal{A}} M$ with the unit and the counit being the homotopy \mathcal{B} -action and \mathcal{B} -trace maps. That is, the compositions

$$M^{h\mathcal{B}} \xrightarrow{\text{Id} \otimes \text{act}} M^{h\mathcal{B}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{B}} \xrightarrow{\text{tr} \otimes \text{Id}} M^{h\mathcal{B}}, \tag{2.46}$$

$$M \xrightarrow{\text{act} \otimes \text{Id}} M \otimes_{\mathcal{B}} M^{h\mathcal{B}} \otimes_{\mathcal{A}} M \xrightarrow{\text{Id} \otimes \text{tr}} M \tag{2.47}$$

are homotopic to the identity maps.

If M is \mathcal{A} -perfect then $(-) \otimes_{\mathcal{B}} M^{h\mathcal{A}}$ is homotopy left adjoint to $(-) \otimes_{\mathcal{A}} M$ with the unit and the counit being the homotopy \mathcal{A} -action and \mathcal{A} -trace maps. That is, the compositions

$$M^{h\mathcal{A}} \xrightarrow{\text{act} \otimes \text{Id}} M^{h\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{A}} \xrightarrow{\text{Id} \otimes \text{tr}} M^{h\mathcal{A}}, \tag{2.48}$$

$$M \xrightarrow{\text{Id} \otimes \text{act}} M \otimes_{\mathcal{B}} M^{h\mathcal{A}} \otimes_{\mathcal{A}} M \xrightarrow{\text{tr} \otimes \text{Id}} M \tag{2.49}$$

are homotopic to the identity maps.

Proof. The compositions

$$\begin{aligned}
 M^{\tilde{\mathcal{B}}} &\xrightarrow{\text{Id} \otimes \text{act}} M^{\tilde{\mathcal{B}}} \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{\tilde{\mathcal{B}}} \xrightarrow{\text{tr} \otimes \text{Id}} M^{\tilde{\mathcal{B}}}, \\
 M &\xrightarrow{\text{act} \otimes \text{Id}} M \overset{\mathbf{L}}{\otimes}_{\mathcal{B}} M^{\tilde{\mathcal{B}}} \otimes_{\mathcal{A}} M \xrightarrow{\text{Id} \otimes \text{tr}} M
 \end{aligned}$$

are equal to $\text{Id}_{D(\mathcal{B}-\mathcal{A})}$ and $\text{Id}_{D(\mathcal{A}-\mathcal{B})}$. This is because the derived \mathcal{B} -action and \mathcal{B} -trace maps are the unit and the counit of a genuine adjunction between $(-)\overset{\mathbf{L}}{\otimes}_{\mathcal{B}} M^{\tilde{\mathcal{B}}}$ and $(-)\overset{\mathbf{L}}{\otimes}_{\mathcal{A}} M$.

By construction, the images of homotopy \mathcal{B} -trace and \mathcal{B} -action maps in $D(\mathcal{B}-\mathcal{B})$ are identified with their derived counterparts by the isomorphism $M^{\tilde{\mathcal{B}}} \simeq M^{h\mathcal{B}}$. It follows that the images of (2.46) and (2.47) in $D(\mathcal{B}-\mathcal{A})$ and $D(\mathcal{A}-\mathcal{B})$ are conjugate (and thus equal) to $\text{Id}_{D(\mathcal{B}-\mathcal{A})}$ and $\text{Id}_{D(\mathcal{A}-\mathcal{B})}$. Hence (2.46) and (2.47) themselves are homotopic to $\text{Id}_{\mathcal{B}\text{-Mod-}\mathcal{A}}$ and $\text{Id}_{\mathcal{A}\text{-Mod-}\mathcal{B}}$.

The second assertion is proved analogously. □

Let \mathcal{A}, \mathcal{B} and \mathcal{C} be DG-categories. Let $M \in \mathcal{A}\text{-Mod-}\mathcal{B}$ and $N \in \mathcal{B}\text{-Mod-}\mathcal{C}$. Consider the composition

$$\begin{aligned}
 N^{\mathcal{C}} \otimes_{\mathcal{B}} M^{\mathcal{B}} &\xrightarrow{\text{ev}} \text{Hom}_{\mathcal{B}}(M, N^{\mathcal{C}}) = \text{Hom}_{\mathcal{B}}(M, \text{Hom}_{\mathcal{C}}(N, \mathcal{C})) \\
 &\xrightarrow[\sim]{\text{adj.}} \text{Hom}_{\mathcal{B}}(M \otimes_{\mathcal{B}} N, \mathcal{C}) = (M \otimes_{\mathcal{B}} N)^{\mathcal{C}}. \tag{2.50}
 \end{aligned}$$

The first map is the evaluation map (2.24), it is a quasi-isomorphism if M is \mathcal{B} -perfect and \mathcal{B} - h -projective. The second map is the adjunction isomorphism for $(-)\otimes_{\mathcal{B}} N$ and $\text{Hom}_{\mathcal{C}}(N, -)$. Similarly,

$$N^{\mathcal{B}} \otimes_{\mathcal{B}} M^{\mathcal{A}} \xrightarrow{\text{ev}} \text{Hom}_{\mathcal{B}^{\text{opp}}}(N, \text{Hom}_{\mathcal{A}^{\text{opp}}}(M, \mathcal{A})) \xrightarrow[\sim]{\text{adj.}} (M \otimes_{\mathcal{B}} N)^{\mathcal{A}} \tag{2.51}$$

is a quasi-isomorphism if N is \mathcal{A} -perfect and \mathcal{A} - h -projective. We thus have:

Lemma 2.12. *Let \mathcal{A}, \mathcal{B} and \mathcal{C} be DG-categories. Let M and N be \mathcal{A} - \mathcal{B} - and \mathcal{B} - \mathcal{C} -bimodules.*

If M is \mathcal{B} -perfect we have an isomorphism in $D(\mathcal{C}-\mathcal{A})$:

$$N^{\tilde{\mathcal{C}}} \overset{\mathbf{L}}{\otimes}_{\mathcal{B}} M^{\tilde{\mathcal{B}}} \xrightarrow{(2.50)} (M \overset{\mathbf{L}}{\otimes}_{\mathcal{B}} N)^{\tilde{\mathcal{C}}}. \tag{2.52}$$

If N is \mathcal{B} -perfect we have an isomorphism in $D(\mathcal{C}-\mathcal{A})$:

$$N^{\tilde{\mathcal{B}}} \overset{\mathbf{L}}{\otimes}_{\mathcal{B}} M^{\tilde{\mathcal{A}}} \xrightarrow{(2.51)} (M \overset{\mathbf{L}}{\otimes}_{\mathcal{B}} N)^{\tilde{\mathcal{A}}}. \tag{2.53}$$

More is true:

Lemma 2.13. *Let \mathcal{A} and \mathcal{B} be DG-categories and $M \in D(\mathcal{A}-\mathcal{B})$ be \mathcal{A} - and \mathcal{B} -perfect. Then $\mathcal{B} \xrightarrow{\text{act}} M^{\tilde{\mathcal{A}}} \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} M$ is isomorphic in $D(\mathcal{B}-\mathcal{B})$ to*

$$(M^{\tilde{\mathcal{B}}} \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} M \xrightarrow{\text{tr}} \mathcal{B})^{l\tilde{\mathcal{B}}}.$$

Similarly, $A \xrightarrow{\text{act}} M \otimes_{\mathcal{B}}^{\mathbb{L}} M^{\tilde{\mathcal{B}}}$ is isomorphic in $D(\mathcal{A}\text{-}\mathcal{A})$ to

$$(M \otimes_{\mathcal{B}}^{\mathbb{L}} M^{\tilde{\mathcal{A}}} \xrightarrow{\text{tr}} \mathcal{A})^{r\tilde{\mathcal{A}}}.$$

Here by $(-)^{l\tilde{\mathcal{B}}}$ we mean dualizing a \mathcal{B} - \mathcal{B} -bimodule as a left \mathcal{B} -module. Similarly for $(-)^{r\tilde{\mathcal{A}}}$, etc.

Proof. We only prove the first assertion, the second is proved similarly. Replace M by an h -projective resolution. Then in $D(\mathcal{B}\text{-}\mathcal{B})$ the map $\mathcal{B} \xrightarrow{\text{act}} M^{\tilde{\mathcal{A}}} \otimes_{\mathcal{A}}^{\mathbb{L}} M$ is isomorphic to $\mathcal{B} \xrightarrow{\text{act}} \text{Hom}_{\mathcal{A}^{\text{opp}}}(M, M)$, and $M^{\tilde{\mathcal{B}}} \otimes_{\mathcal{A}}^{\mathbb{L}} M \xrightarrow{\text{tr}} \mathcal{B}$ is isomorphic to $M^{\mathcal{B}} \otimes_{\mathcal{A}} M \xrightarrow{\text{tr}} \mathcal{B}$. It now suffices to show that in $\mathcal{B}\text{-Mod-}\mathcal{B}$ the diagram

$$\begin{array}{ccc}
 \mathcal{B} & \xrightarrow{\text{act}} & \text{Hom}_{\mathcal{A}^{\text{opp}}}(M, M) \\
 \downarrow \text{act} & & \downarrow \text{Id} \otimes (2.27) \\
 & & \text{Hom}_{\mathcal{A}^{\text{opp}}}(M, M^{\mathcal{B}\mathcal{B}}) \\
 & & \downarrow \text{adjunction} \\
 \text{Hom}_{\mathcal{B}^{\text{opp}}}(\mathcal{B}, \mathcal{B}) & \xrightarrow{\text{tr}} & \text{Hom}_{\mathcal{B}^{\text{opp}}}(M^{\mathcal{B}} \otimes_{\mathcal{A}} M, \mathcal{B})
 \end{array} \tag{2.54}$$

is commutative, since its left column is an isomorphism and its right column a quasi-isomorphism. The diagram (2.54) commutes because both its halves can be readily seen to compose into the element of

$$\text{Hom}_{\mathcal{B}\text{-}\mathcal{B}}(\mathcal{B}, \text{Hom}_{\mathcal{B}^{\text{opp}}}(M^{\mathcal{B}} \otimes_{\mathcal{A}} M, \mathcal{B}))$$

which is adjoint to the trace map $M^{\mathcal{B}} \otimes_{\mathcal{A}} M \xrightarrow{\text{tr}} \mathcal{B}$ in

$$\text{Hom}_{\mathcal{B}\text{-}\mathcal{B}}(M^{\mathcal{B}} \otimes_{\mathcal{A}} M, \mathcal{B})$$

under the adjunction of $M^{\mathcal{B}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} (-)$ and $\text{Hom}_{\mathcal{B}^{\text{opp}}}(M^{\mathcal{B}} \otimes_{\mathcal{A}} M, -)$. □

Finally, we have the following analogue of Prop. 2.5 with h -projectivity replaced by perfection:

Proposition 2.14. *Let \mathcal{A} , \mathcal{B} and \mathcal{C} be DG-categories. Let M be a perfect \mathcal{A} - \mathcal{B} -bimodule and N be a \mathcal{C} -perfect \mathcal{B} - \mathcal{C} -bimodule. Then $M \otimes_{\mathcal{B}}^{\mathbb{L}} N$ is a perfect \mathcal{A} - \mathcal{C} -bimodule.*

Proof. Let $\bigoplus Q_i$ be an infinite direct sum of \mathcal{A} - \mathcal{C} -bimodules. We have a chain of natural isomorphisms:

$$\text{Hom}_{D(\mathcal{A}\text{-}\mathcal{C})}\left(M \otimes_{\mathcal{B}}^{\mathbb{L}} N, \bigoplus Q_i\right) \simeq \text{Hom}_{D(\mathcal{A}\text{-}\mathcal{B})}\left(M, \mathbf{R}\text{Hom}_{\mathcal{C}}\left(N, \bigoplus Q_i\right)\right), \tag{2.55}$$

$$\text{Hom}_{D(\mathcal{A}\text{-}\mathcal{B})}\left(M, \mathbf{R}\text{Hom}_{\mathcal{C}}\left(N, \bigoplus Q_i\right)\right) \simeq \text{Hom}_{D(\mathcal{A}\text{-}\mathcal{B})}\left(M, \bigoplus \mathbf{R}\text{Hom}_{\mathcal{C}}(N, Q_i)\right), \tag{2.56}$$

$$\mathrm{Hom}_{D(\mathcal{A}\text{-}\mathcal{B})}\left(M, \bigoplus \mathbf{R}\mathrm{Hom}_{\mathcal{C}}(N, Q_i)\right) \simeq \bigoplus \mathrm{Hom}_{D(\mathcal{A}\text{-}\mathcal{B})}(M, \mathbf{R}\mathrm{Hom}_{\mathcal{C}}(N, Q_i)), \quad (2.57)$$

$$\bigoplus \mathrm{Hom}_{D(\mathcal{A}\text{-}\mathcal{B})}(M, \mathbf{R}\mathrm{Hom}_{\mathcal{C}}(N, Q_i)) \simeq \bigoplus \mathrm{Hom}_{D(\mathcal{A}\text{-}\mathcal{C})}(M \overset{\mathbf{L}}{\otimes}_{\mathcal{B}} N, Q_i). \quad (2.58)$$

Here (2.55) and (2.58) are due to the adjunction of $(-)\overset{\mathbf{L}}{\otimes}_{\mathcal{B}} N$ and $\mathbf{R}\mathrm{Hom}_{\mathcal{C}}(N, -)$ done over \mathcal{A} , (2.56) is due to N being \mathcal{C} -perfect, and (2.57) is due to M being perfect.

Thus $\mathrm{Hom}_{D(\mathcal{A}\text{-}\mathcal{C})}(M \overset{\mathbf{L}}{\otimes}_{\mathcal{B}} N, -)$ commutes with infinite direct sums, i.e. $M \overset{\mathbf{L}}{\otimes}_{\mathcal{B}} N$ is perfect. \square

Recall that a DG-category \mathcal{A} is called *smooth* if the diagonal bimodule \mathcal{A} is a perfect \mathcal{A} - \mathcal{A} -bimodule.

Corollary 2.15. *Let \mathcal{A} be a smooth DG-category and \mathcal{B} be any DG-category. Then any \mathcal{B} -perfect \mathcal{A} - \mathcal{B} -bimodule N is perfect.*

Proof. By definition, \mathcal{A} being smooth means that \mathcal{A} is a perfect \mathcal{A} - \mathcal{A} -bimodule. We then apply Lemma 2.14 to conclude that $N \simeq \mathcal{A} \otimes_{\mathcal{A}} N$ is perfect. \square

3. Twisted complexes and twisted cubes

3.1. Twisted complexes

The notion of a twisted complex was introduced in [BK90]. There exist at present two different conventions for writing down twisted complexes: the original one introduced in [BK90] and a slightly different one introduced in [BLL04] where all the objects in a twisted complex are shifted so as to ensure that all the twisted maps have degree 1. Abstractly, this latter convention is more natural as these shifts are precisely what one has to do when taking the convolution of a twisted complex.

However, all the twisted complexes we work with in this paper are lifts of genuine complexes in the homotopy category, and hence they exist naturally in the convention of [BK90]. For this reason we are going to present the material in this section, such as the formulas for dualizing and tensoring twisted complexes, in the notation of [BK90]. The authors are well aware that the signs in these formulas are much simpler in the notation of [BLL04]. However, to actually apply any formula in the [BLL04] convention to the twisted complexes we work with throughout the paper, we would first have to shift everything to make all the twisted maps have degree 1, then apply the formula, and then shift everything back to relate the answer to what we are working with. This would introduce back all the complicated signs, and it is therefore better to write down the formulas in the [BK90] convention from the start.

The definitions in the published version of [BK90] contain sign errors. For the reader's convenience we give the corrected versions of these definitions:

Definition 3.1. Let \mathcal{A} be a DG-category. A *twisted complex* over \mathcal{A} is a collection

$$\{(E_i)_{i \in \mathbb{Z}}, \alpha_{ij} : E_i \rightarrow E_j\}$$

where E_i are objects in \mathcal{A} with $E_i = 0$ for all but a finite number of i , and α_{ij} are morphisms in \mathcal{A} of degree $i - j + 1$ satisfying the condition

$$(-1)^j d\alpha_{ij} + \sum_k \alpha_{kj} \circ \alpha_{ik} = 0.$$

A twisted complex is called *one-sided* if $\alpha_{ij} = 0$ for all $i \geq j$.

We adopt the following convention: to write a twisted complex we write down two expressions separated by a comma. The first expression is the i th graded part of the twisted complex. The second expression is the twisted map from i th to j th graded parts of the twisted complex. For example, (E_i, α_{ij}) is a twisted complex whose i th graded part is E_i and whose twisted map from E_i to E_j is α_{ij} .

To make twisted complexes over \mathcal{A} into a DG-category we define the Hom-complex from a twisted complex (E_i, α_{ij}) to a twisted complex (F_l, β_{lj}) to be the complex of k -modules whose degree p part is

$$\coprod_{p=q+l-k} \text{Hom}_{\mathcal{A}}^q(E_k, F_l)$$

with the differential defined by setting, for each $\gamma \in \text{Hom}_{\mathcal{A}}^q(E_k, F_l)$,

$$d\gamma = (-1)^l d_{\mathcal{A}}\gamma + \sum_{m \in \mathbb{Z}} (\beta_{lm} \circ \gamma - (-1)^{q+l-k} \gamma \circ \alpha_{mk}),$$

where $d_{\mathcal{A}}$ is the differential on morphisms in \mathcal{A} .

The signs and indices in the definitions above are set up precisely so as to ensure that the notion of *convolution*, defined below, extends naturally to a fully faithful functor from the DG-category of twisted complexes over \mathcal{A} to the DG-category $\mathbf{Mod}\text{-}\mathcal{A}$.

First we need to define the notion of a *shift* of an \mathcal{A} -module. We do it levelwise in $\mathbf{Mod}\text{-}k$ and, since we are dealing with right modules, we do not twist the \mathcal{A} -action. That is:

Definition 3.2. Let M be an \mathcal{A} -module. For any $n \in \mathbb{Z}$ define the \mathcal{A} -module $M[n]$ by setting

$$(M[n])_a = M_a[n] \quad \forall a \in \mathcal{A}$$

and having \mathcal{A} act via its action on M . That is, for any $\alpha \in {}_a\mathcal{A}_b$ and any $s \in (M[n])_a$ we set $s \cdot_{M[n]} \alpha \in (M[n])_b$ to be $s \cdot_M \alpha$.

Definition 3.3. Let \mathcal{A} be a DG-category and let (E_i, α_{ij}) be a twisted complex over \mathcal{A} . Let $\bigoplus_i E_i[-i]$ be the \mathcal{A} -module where we use the Yoneda embedding to embed each E_i into $\mathbf{Mod}\text{-}\mathcal{A}$. The *convolution* of (E_i, α_{ij}) is the \mathcal{A} -module obtained by taking $\bigoplus_i E_i[-i]$ and endowing it with a new differential $d + \sum_{i,j} \alpha_{ij}$, where d is the natural differential of $\bigoplus_i E_i[-i]$.

We use curly brackets to denote taking the convolution of the twisted complex, e.g. $\{E_i, \alpha_{ij}\}$.

The most time-consuming part of proving the results below is getting the signs to agree. Recall the definitions of the bimodule-valued tensor product and Hom functors (2.13)–(2.15). In particular, for any maps $E \xrightarrow{\alpha} E'$ in $\mathcal{B}\text{-Mod}\text{-}\mathcal{A}$ and $F \xrightarrow{\beta} F'$ in $\mathcal{A}\text{-Mod}\text{-}\mathcal{C}$ the

product map

$$E \otimes_{\mathcal{A}} F \xrightarrow{\alpha \otimes \beta} E' \otimes F'$$

in $\mathcal{B}\text{-Mod-}\mathcal{C}$ is given for every $(b, c) \in \mathcal{B} \otimes \mathcal{C}^{\text{opp}}$ and $a \in \mathcal{A}$ by

$$e \otimes f \mapsto (-1)^{\deg(e) \deg(\beta)} \alpha(e) \otimes \beta(f), \quad e \in {}_b E_a, f \in {}_a F_c.$$

Similarly, for any maps $E' \xrightarrow{\alpha} E$ in $\mathcal{A}\text{-Mod-}\mathcal{B}$ and $F \xrightarrow{\beta} F'$ in $\mathcal{A}\text{-Mod-}\mathcal{C}$ the composition map

$$\beta \circ (-) \circ \alpha: \text{Hom}_{\mathcal{A}}(E, F) \rightarrow \text{Hom}_{\mathcal{A}}(E', F')$$

in $\mathcal{B}\text{-Mod-}\mathcal{C}$ is given for every $(b, c) \in \mathcal{B} \otimes \mathcal{C}^{\text{opp}}$ by

$$f \mapsto (-1)^{\deg(f) \deg(\alpha)} \beta \circ f \circ \alpha, \quad f \in \text{Hom}_{\mathcal{A}}(E_b, F_c).$$

The formula for the other (“as right modules”) bimodule Hom functor (2.14) is identical.

Lemma 3.4. *Let \mathcal{A}, \mathcal{B} and \mathcal{C} be DG-categories and let (E_i, α_{ij}) be a twisted complex over $\mathcal{A}\text{-Mod-}\mathcal{B}$.*

(1) *Let (F_i, β_{ij}) be a twisted complex over $\mathcal{B}\text{-Mod-}\mathcal{C}$. Then*

$$\begin{aligned} & \{E_i, \alpha_{ij}\} \otimes_{\mathcal{B}} \{F_i, \beta_{ij}\} \\ & \simeq \left\{ \bigoplus_{k+l=i} E_k \otimes_{\mathcal{B}} F_l, \sum_{l+m=j} (-1)^{l(k-m+1)} \alpha_{km} \otimes \text{Id}_l + \sum_{k+n=j} (-1)^k \text{Id}_k \otimes \beta_{ln} \right\}. \end{aligned} \tag{3.1}$$

(2) *Let (F_i, β_{ij}) be a twisted complex over $\mathcal{C}\text{-Mod-}\mathcal{B}$. Then*

$$\begin{aligned} & \text{Hom}_{\mathcal{B}}(\{E_i, \alpha_{ij}\}, \{F_i, \beta_{ij}\}) \\ & \simeq \left\{ \bigoplus_{l-k=i} \text{Hom}_{\mathcal{B}}(E_k, F_l), \sum_{l-m=j} (-1)^{m(m-k)+l+1} (-) \circ \alpha_{mk} + \sum_{n-k=j} (-1)^{(l-n+1)k} \beta_{ln} \circ (-) \right\}. \end{aligned} \tag{3.2}$$

Similarly, if (F_i, β_{ij}) is a twisted complex over $\mathcal{A}\text{-Mod-}\mathcal{C}$ then

$$\begin{aligned} & \text{Hom}_{\mathcal{A}}(\{E_i, \alpha_{ij}\}, \{F_i, \beta_{ij}\}) \\ & \simeq \left\{ \bigoplus_{l-k=i} \text{Hom}_{\mathcal{A}}(E_k, F_l), \sum_{l-m=j} (-1)^{m(m-k)+l+1} (-) \circ \alpha_{mk} + \sum_{n-k=j} (-1)^{(l-n+1)k} \beta_{ln} \circ (-) \right\}. \end{aligned} \tag{3.3}$$

Proof. (1) An isomorphism of DG-modules is an isomorphism of the underlying graded modules which respects the differential. As a graded $\mathcal{A}\text{-}\mathcal{C}$ -bimodule, i.e. forgetting the differential, the LHS of (3.1) is isomorphic to

$$\left(\bigoplus_{k \in \mathbb{Z}} E_k[-k] \right) \otimes_{\mathcal{B}} \left(\bigoplus_{l \in \mathbb{Z}} F_l[-l] \right),$$

while the RHS is isomorphic to

$$\bigoplus_{k,l \in \mathbb{Z}} (E_k \otimes_{\mathcal{B}} F_l[-k-l]).$$

There is a tautological isomorphism between the two:

$$e \otimes f \mapsto (-1)^{k'l} e \otimes f, \\ a \in \mathcal{A}^{\text{opp}}, b \in B, c \in C; k, l, k', l' \in \mathbb{Z}; e \in ({}_a(E_k)_b)_{k'}; f \in ({}_b(E_l)_c)_{l'},$$

which needs its sign twist to respect the \mathcal{B} -action relations of the corresponding tensor products. This isomorphism can be readily seen to also respect the differentials

$$d(e \otimes f) = (-1)^k d_{E_k} e \otimes f + \sum_m \alpha_{km}(e) \otimes f + (-1)^{k+k'+l} e \otimes d_{F_l} f \\ + \sum_n (-1)^{k+k'} e \otimes \beta_{ln}(f), \\ d(e \otimes f) = (-1)^{k+l} d_{E_k} e \otimes f + (-1)^{k+l+k'} e \otimes d_{F_l} f + \sum_m (-1)^{l(k-m+1)} \alpha_{km}(e) \otimes f \\ + \sum_n (-1)^{k+(l-n+1)k'} e \otimes \beta_{ln}(f)$$

on the LHS and the RHS of (3.1).

(2) We only prove the first statement, the second is proved identically. As a graded \mathcal{C} - \mathcal{A} -bimodule, the LHS of (3.2) is isomorphic to

$$\text{Hom}_{\mathcal{B}}\left(\bigoplus_k E_k[-k], \bigoplus_l F_l[-l]\right), \tag{3.4}$$

while the RHS is isomorphic to

$$\bigoplus_{k,l} \text{Hom}_{\mathcal{B}}(E_k, F_l)[-(l-k)]. \tag{3.5}$$

Since all the direct sums are finite, the obvious natural map from (3.5) to (3.4) is an isomorphism of graded \mathcal{C} - \mathcal{A} -bimodules. It does not respect the differentials given for any $a \in \mathcal{A}, c \in \mathcal{C}$ and $f \in \text{Hom}_{\mathcal{B}}^q({}_a(E_k), {}_c(F_l))$ by

$$df = (-1)^l d_{\mathcal{B}} f + \sum_m (-1)^{q+l-k+1} f \circ \alpha_{mk} + \sum_n \beta_{ln} \tag{3.6}$$

on the LHS of (3.2), and by

$$df = (-1)^{l-k} d_{\mathcal{B}} f + \sum_m (-1)^{q(m-k+1)+m(m-k)+l+1} f \circ \alpha_{mk} + \sum_n (-1)^{(l-n+1)k} \beta_{ln} \tag{3.7}$$

on the RHS of (3.2). One can now readily check that the composition of the natural isomorphism above with the automorphism of (3.5) which multiplies each $\text{Hom}_{\mathcal{B}}^q(E_k, F_l)$ by $(-1)^{(q-1)k}$ respects the differentials and thus yields the desired isomorphism of DG-bimodules. \square

Lemma 3.5. *Let \mathcal{A} and \mathcal{B} be DG-categories and let (E_i, α_{ij}) be a twisted complex of \mathcal{A} - \mathcal{B} -bimodules. Let E be its convolution $\{E_i, \alpha_{ij}\}$.*

(1) *Then*

$$E^{\mathcal{B}} \simeq \{E_{-i}^{\mathcal{B}}, (-1)^{j^2+ij+1} \alpha_{(-j)(-i)}^{\mathcal{B}}\}, \tag{3.8}$$

$$E^{\mathcal{A}} \simeq \{E_{-i}^{\mathcal{A}}, (-1)^{j^2+ij+1} \alpha_{(-j)(-i)}^{\mathcal{A}}\} \tag{3.9}$$

in \mathcal{A} -Mod- \mathcal{B} .

(2) *The \mathcal{A} -trace map $E \otimes_{\mathcal{B}} E^{\mathcal{A}} \xrightarrow{\text{tr}} \mathcal{A}$ is isomorphic to the image in \mathcal{A} -Mod- \mathcal{A} of the map*

$$\left(\bigoplus_{k-l=i} E_k \otimes_{\mathcal{B}} E_l^{\mathcal{A}}, \sum_{m-l=j} (-1)^{l(k-m+1)} \alpha_{km} \otimes \text{Id} + \sum_{k-n=j} (-1)^{k+n^2+nl+1} \text{Id} \otimes \alpha_{nl}^{\mathcal{A}} \right) \rightarrow \mathcal{A} \tag{3.10}$$

which consists of a single degree 0 map $\bigoplus_k E_k \otimes_{\mathcal{B}} E_k^{\mathcal{A}} \xrightarrow{\sum \text{tr}} \mathcal{A}$.

The \mathcal{A} -action map $\mathcal{A} \xrightarrow{\text{act}} \text{Hom}_{\mathcal{B}}(E, E)$ is isomorphic to the image in \mathcal{A} -Mod- \mathcal{A} of the map

$$\mathcal{A} \rightarrow \left(\bigoplus_{l-k=i} \text{Hom}_{\mathcal{B}}(E_k, E_l), \sum_{l-m=j} (-1)^{m(m-k)+l+1} (-) \circ \alpha_{mk} + \sum_{n-k=j} (-1)^{(l-n+1)k} \alpha_{ln} \circ (-) \right) \tag{3.11}$$

which consists of a single degree 0 map $\mathcal{A} \xrightarrow{\sum \text{act}} \bigoplus_k \text{Hom}_{\mathcal{B}}(E_k, E_k)$.

Analogous statements hold for the \mathcal{B} -trace and \mathcal{B} -action maps.

(3) *Suppose each E_i is h -projective and \mathcal{B} -perfect.*

The map $E \otimes_{\mathcal{B}} E^{h\mathcal{A}} \xrightarrow{\text{tr}} \bar{\mathcal{A}}$ is homotopy equivalent to the image in \mathcal{A} -Mod- \mathcal{A} of the map

$$\left(\bigoplus_{k-l=i} E_k \otimes_{\mathcal{B}} E_l^{h\mathcal{A}}, \sum_{m-l=j} (-1)^{l(k-m+1)} \alpha_{km} \otimes \text{Id} + \sum_{k-n=j} (-1)^{k+n^2+nl+1} \text{Id} \otimes \alpha_{nl}^{\mathcal{A}} \right) \rightarrow \bar{\mathcal{A}} \tag{3.12}$$

which consists of a single degree 0 map $\bigoplus_k E_k \otimes_{\mathcal{B}} E_k^{h\mathcal{A}} \xrightarrow{\sum \text{tr}} \bar{\mathcal{A}}$.

The map $\bar{\mathcal{A}} \xrightarrow{\text{act}} E \otimes_{\mathcal{B}} E^{h\mathcal{B}}$ is homotopy equivalent to the image in \mathcal{A} -Mod- \mathcal{A} of the map

$$\bar{\mathcal{A}} \rightarrow \left(\bigoplus_{k-l=i} E_k \otimes_{\mathcal{B}} E_l^{h\mathcal{B}}, \sum_{m-l=j} (-1)^{-l(k-m+1)} \alpha_{km} \otimes \text{Id} + \sum_{k-n=j} (-1)^{k+n^2+nl+1} \text{Id} \otimes \alpha_{nl}^{\mathcal{B}} \right) \tag{3.13}$$

which consists of a single degree 0 map $\bar{\mathcal{A}} \xrightarrow{\sum \text{act}} \bigoplus_k E_k \otimes_{\mathcal{B}} E_k^{h\mathcal{B}}$.

Analogous statements hold for the \mathcal{B} -trace and \mathcal{B} -action maps when the E_i are h -projective and \mathcal{A} -perfect.

Proof. (1) Both statements follow immediately from Lemma 3.4(2) by setting the twisted complex $\{F_i, \beta_{ij}\}$ to be the corresponding diagonal bimodule concentrated in degree 0.

(2) For the \mathcal{A} -trace map claim, the isomorphisms (3.9) and (3.1) compose to an isomorphism from $E \otimes_{\mathcal{B}} E^{\mathcal{A}}$ to the convolution of the LHS of (3.10). We claim that this isomorphism composes with the image of (3.10) in $\mathcal{A}\text{-Mod-}\mathcal{A}$ to give $E \otimes_{\mathcal{B}} E^{\mathcal{A}} \xrightarrow{\text{tr}} \mathcal{A}$. When checking maps to be equal it suffices to only consider them as maps of graded modules. Thus we are reduced to checking that the trace map of a finite direct sum of graded modules equals the sum of the trace maps of the individual modules. This is straightforward.

For the \mathcal{A} -action map, we are similarly reduced to checking that the action map $\mathcal{A} \rightarrow \text{Hom}_{\mathcal{B}}(E, E)$ is also compatible with finite direct sums.

(3) We only prove the first claim. The natural maps $\bar{\mathcal{A}} \rightarrow \mathcal{A}$ and $\bar{\mathcal{B}} \rightarrow \mathcal{B}$ induce isomorphisms in $D(\mathcal{A}\text{-}\mathcal{A})$ between $E \otimes_{\mathcal{B}} E^{h\mathcal{A}} \xrightarrow{\text{tr}} \bar{\mathcal{A}}$ and $E \otimes_{\mathcal{B}} E^{\mathcal{A}} \xrightarrow{\text{tr}} \mathcal{A}$ and also between (3.12) and (3.10). It follows from (2) that $E \otimes_{\mathcal{B}} E^{h\mathcal{A}} \xrightarrow{\text{tr}} \bar{\mathcal{A}}$ and (3.12) are isomorphic in $D(\mathcal{A}\text{-}\mathcal{A})$. Since all the bimodules involved are h -projective, the two are furthermore isomorphic in $H^0(\mathbf{Mod}\text{-}\mathcal{A})$, as required. \square

3.2. Pretriangulated categories

Let \mathcal{A} and \mathcal{B} be DG-categories. A functor $\mathcal{A} \xrightarrow{f} \mathcal{B}$ is a *quasi-equivalence* if f induces quasi-isomorphisms on morphism complexes and $H^0(\mathcal{A}) \xrightarrow{H^0(f)} H^0(\mathcal{B})$ is an equivalence of categories.

A DG-category \mathcal{A} is *pretriangulated* if $H^0(\mathcal{A})$ is a triangulated subcategory of $H^0(\mathbf{Mod}\text{-}\mathcal{A})$ under the Yoneda embedding. The *pretriangulated hull* $\text{Pre-Tr}(\mathcal{A})$ of \mathcal{A} is the category of *one-sided* twisted complexes in \mathcal{A} —these are the twisted complexes (E_i, q_{ij}) where $q_{ij} = 0$ if $i \geq j$. The convolution functor gives a fully faithful embedding $\text{Pre-Tr}(\mathcal{A}) \hookrightarrow \mathbf{Mod}\text{-}\mathcal{A}$ whose composition with $\mathcal{A} \rightarrow \text{Pre-Tr}(\mathcal{A})$ is the Yoneda embedding and whose image in $\mathbf{Mod}\text{-}\mathcal{A}$ is equivalent to $\mathcal{S}\mathcal{F}_{\text{fg}}(\mathcal{A})$ [Dri04, §2.4]. Hence $H^0(\text{Pre-Tr}(\mathcal{A}))$ coincides with the triangulated hull of $H^0(\mathcal{A})$ in $H^0(\mathbf{Mod}\text{-}\mathcal{A})$. Therefore \mathcal{A} is pretriangulated if and only if the natural embedding $\mathcal{A} \hookrightarrow \text{Pre-Tr}(\mathcal{A})$ is a quasi-equivalence. We say that \mathcal{A} is *strongly pretriangulated* if $\mathcal{A} \hookrightarrow \text{Pre-Tr}(\mathcal{A})$ is in fact an equivalence; in other words, if it has a quasi-inverse $\text{Pre-Tr}(\mathcal{A}) \xrightarrow{T} \mathcal{A}$. Note that in that case the convolution functor filters through the Yoneda embedding, i.e.

$$\text{Pre-Tr}(\mathcal{A}) \xrightarrow{T} \mathcal{A} \hookrightarrow \mathbf{Mod}\text{-}\mathcal{A}$$

is the convolution functor. For strongly pretriangulated categories, by abuse of notation, we mainly use the term *convolution functor* to mean T .

Let \mathcal{A} be any DG-category. It is known that $\text{Pre-Tr}(\mathcal{A})$ is strongly pretriangulated [BK90]. Also $\text{DGFun}(\mathcal{A}, \mathcal{C})$ is strongly pretriangulated for any strongly pretriangulated \mathcal{C} since we can define convolutions of twisted complexes levelwise in \mathcal{C} . In particular, $\mathbf{Mod}\text{-}\mathcal{A}$ is strongly pretriangulated since $\mathbf{Mod}\text{-}k$ is. Finally, a full subcategory of $\mathbf{Mod}\text{-}\mathcal{A}$ (or any other strongly pretriangulated DG-category) which is itself pretriangulated, e.g.

it descends to a triangulated subcategory of $H^0(\mathbf{Mod}\text{-}\mathcal{A})$, and closed under homotopy equivalences, is strongly pretriangulated. Therefore $\mathcal{P}(\mathcal{A})$ and $\mathcal{P}^{\text{Perf}}(\mathcal{A})$ are strongly pretriangulated and, for any other DG-category \mathcal{B} , $\mathcal{P}^{\mathcal{A}\text{-Perf}}(\mathcal{A}\text{-}\mathcal{B})$ and $\mathcal{P}^{\mathcal{B}\text{-Perf}}(\mathcal{A}\text{-}\mathcal{B})$ are also strongly pretriangulated. If \mathcal{A} itself is pretriangulated, then $\mathcal{P}^{\text{qr}}(\mathcal{A})$ and $\mathcal{P}^{\mathcal{A}\text{-qr}}(\mathcal{A}\text{-}\mathcal{B})$ are strongly pretriangulated. If, on the other hand, \mathcal{B} is pretriangulated, then $\mathcal{P}^{\mathcal{B}\text{-qr}}(\mathcal{A}\text{-}\mathcal{B})$ is strongly pretriangulated.

3.3. Twisted cubes

One of the chief technical tools we employ in this paper is the notion of a twisted cube over a pretriangulated category. This seemingly trivial extension of the notion of a twisted complex has some far-reaching consequences that we exploit. To the authors' knowledge, the material below is original to this paper.

We employ the following notation: Let $I = \{-1, 0\}^n$ enumerate vertices of an n -cube.² For $\bar{i}, \bar{j} \in I$ with $\bar{i} = (i_1, \dots, i_n)$ and $\bar{j} = (j_1, \dots, j_n)$ we write $\bar{j} > \bar{i}$ if $j_m \geq i_m$ for all m and $\bar{i} \neq \bar{j}$. For any $\bar{i} \in I$ we denote by $|\bar{i}|$ its degree $\sum i_m$.

Let \mathcal{C} be a pretriangulated category. A *twisted n -cube over \mathcal{C}* is

- (1) a set $\{X_{\bar{i}}\}_{\bar{i} \in I}$ of objects of \mathcal{C} ,
- (2) a set $\{q_{\bar{i}\bar{j}}\}_{\bar{i}, \bar{j} \in I, \bar{i} < \bar{j}}$ of morphisms in \mathcal{C} such that $q_{\bar{i}\bar{j}}$ is a morphism $X_{\bar{i}} \rightarrow X_{\bar{j}}$ of degree $|\bar{i}| - |\bar{j}| + 1$ which satisfies the relation

$$(-1)^{|\bar{j}|} d q_{\bar{i}\bar{j}} + \sum_{\bar{i} < \bar{k} < \bar{j}} q_{\bar{k}\bar{j}} q_{\bar{i}\bar{k}} = 0. \tag{3.14}$$

The *total complex* $\text{tot}(X_{\bar{i}}, q_{\bar{i}\bar{j}})$ of a twisted n -cube $(X_{\bar{i}}, q_{\bar{i}\bar{j}})$ is the one-sided twisted complex

$$\left(\bigoplus_{\bar{i} \in I, |\bar{i}|=i} X_{\bar{i}}, \sum_{\bar{i}, \bar{j} \in I, |\bar{i}|=i, |\bar{j}|=j} q_{\bar{i}\bar{j}} \right)$$

over \mathcal{C} . Its convolution is an object of \mathcal{C} which we call the *convolution of the twisted cube* $(X_{\bar{i}}, q_{\bar{i}\bar{j}})$.

Lemma 3.6 (The Cube Lemma). *Let $X = (X_{\bar{i}}, q_{\bar{i}\bar{j}})$ be a twisted n -cube indexed by I over a pretriangulated category \mathcal{C} . Choose $0 \leq m \leq n$ and choose any m indices in $1, \dots, n$ to define a splitting $I = J \times K$ with $J = \{-1, 0\}^m$, $K = \{-1, 0\}^{n-m}$. Then:*

- (1) Fix $\bar{k} \in K$. Then

$$(X_{(\bar{i}, \bar{k})})_{\bar{i} \in J} \quad \text{and} \quad ((-1)^{|\bar{k}|} q_{(\bar{i}, \bar{k})(\bar{j}, \bar{k})})_{\bar{i}, \bar{j} \in J}$$

form a twisted m -cube indexed by J over \mathcal{C} . We denote it by $Y^{\bar{k}}$ and call it a sign-twisted subcube of X , to stress that the morphisms in $Y^{\bar{k}}$ and in X differ (possibly) by a sign.

² We use $\{-1, 0\}^n$ rather than $\{0, 1\}^n$ as our indexing set since we want the arrows in the cube to go from lower to higher degree vertices, and we want the terminal end of the cube to have degree 0. This ensures that for a 1-cube diagram, i.e. a single morphism, the corresponding twisted complex coincides naturally with the cone of this morphism, with no shifts involved.

(2) Fix $\bar{k}, \bar{l} \in K$. For any $0 \leq i < j \leq m$ let

$$p_{ij}^{\bar{k}\bar{l}} = \sum_{\bar{i}, \bar{j} \in J, |\bar{i}|=i, |\bar{j}|=j} q_{(\bar{i}, \bar{k})(\bar{j}, \bar{l})}$$

The collection $(p_{ij}^{\bar{k}\bar{l}})_{i,j}$ defines a morphism of twisted complexes

$$\text{tot}(Y^{\bar{k}}) \rightarrow \text{tot}(Y^{\bar{l}})$$

of degree $|\bar{k}| - |\bar{l}| + 1$. Denote it by $p^{\bar{k}\bar{l}}$.

- (3) The twisted complexes $\text{tot}(Y^{\bar{k}})$ and the morphisms $p^{\bar{k}\bar{l}}$ form a twisted $(n - m)$ -cube over $\text{Pre-Tr}(\mathcal{C})$ indexed by K . Let $Z \in \text{Pre-Tr}(\text{Pre-Tr}(\mathcal{C}))$ be its total complex.
- (4) The (double) convolution of Z is isomorphic in \mathcal{C} to the convolution of the original twisted cube X . In particular, it is independent of m and of the choice of $I = J \times K$.

Proof. A straightforward verification. □

Given a twisted n -cube \bar{X} over a pretriangulated category \mathcal{C} , its image in $H^0(\mathcal{C})$ is an ordinary n -cube shaped diagram X which commutes (up to isomorphism). Roughly, the point of the Cube Lemma is that X can be canonically extended in $H^0(\mathcal{C})$ to an n -cube X' of side 2 with the following properties:³

- The vertices of X' are convolutions of the faces of X .
- The rows and columns of X' are exact triangles in $H^0(\mathcal{C})$.
- X' commutes (up to isomorphism).

This is best understood by looking at some examples. Let \mathcal{C} be a pretriangulated category.

(1) A twisted 0-cube over \mathcal{C} is a single object of \mathcal{C} .

(2) A twisted 1-cube over \mathcal{C} is a pair of objects A and B of \mathcal{C} together with a closed morphism

$$A \xrightarrow{f_{AB}} B$$

of degree 0. Write

$$a \xrightarrow{f_{ab}} b$$

for its image in $H^0(\mathcal{C})$. Here we denote by a, b and f_{ab} the classes of A, B and f_{AB} in $H^0(\mathcal{C})$.

There are no non-trivial ways to split this up as a cube of cubes, so the Cube Lemma does not tell us anything new. However, the total complex of this cube is, trivially,

$$A \xrightarrow[\text{deg.0}]{f_{AB}} B$$

³ By an n -cube of side 2 we mean an n -dimensional cube whose sides are two edges long, i.e. its vertices are enumerated by $\{-1, 0, 1\}^n$ instead of $\{-1, 0\}^n$.

and its convolution fits into a diagram

$$A \xrightarrow{f_{AB}} B \rightarrow \left\{ A \xrightarrow{f_{AB}}_{\text{deg.0}} B \right\} \rightarrow A[1] \tag{3.15}$$

in \mathcal{C} where the two new morphisms are induced by the canonical morphisms of twisted complexes

$$\left(B \xrightarrow{\text{Id}_B} \left(A \xrightarrow{f_{AB}}_{\text{deg.0}} B \right) \right), \tag{3.16}$$

$$\left(A \xrightarrow{f_{AB}}_{\text{deg.0}} B \right) \xrightarrow{\text{Id}_A} \left(A \xrightarrow{\quad} \right). \tag{3.17}$$

Moreover, the image of (3.15) in $H^0(\mathcal{C})$ is precisely the exact triangle

$$a \xrightarrow{f_{ab}} b \rightarrow \text{Cone}(f_{ab}) \rightarrow \tag{3.18}$$

which was the original point of [BK90].

Note that we can also complete $A \xrightarrow{f_{AB}}_{\text{deg.0}} B$ to the diagram

$$A \xrightarrow{f_{AB}} B \rightarrow \left\{ A \xrightarrow{-f_{AB}}_{\text{deg.0}} B \right\} \rightarrow A[1] \tag{3.19}$$

whose image in $H^0(\mathcal{C})$ is canonically isomorphic to (3.18). The two new morphisms in (3.19) are defined exactly as in (3.16) and (3.17).

Thus, convolving a twisted 1-cube produces an exact triangle in $H^0(\mathcal{C})$. In the language above, the image of a twisted 1-cube in $H^0(\mathcal{C})$ is an ordinary 1-cube and we can canonically complete it to a 1-cube of side 2 whose single row is an exact triangle. It is this, together with repeated application of the Cube Lemma, that produces the desired phenomena for twisted cubes of higher dimension.

(3) A twisted 2-cube over \mathcal{C} is a diagram

$$\begin{array}{ccc} A & \xrightarrow{f_{AB}} & B \\ f_{AC} \downarrow & \searrow f_{AD} & \downarrow f_{BD} \\ C & \xrightarrow{f_{CD}} & D \end{array} \tag{3.20}$$

of objects and morphisms in \mathcal{C} , where $f_{AB}, f_{AC}, f_{BD}, f_{CD}$ are closed maps of degree 0 and f_{AD} is a map of degree -1 such that

$$-df_{AD} = f_{BD}f_{AB} + f_{CD}f_{AC}. \tag{3.21}$$

The image of (3.20) in $H^0(\mathcal{C})$ is the diagram

$$\begin{array}{ccc} a & \xrightarrow{f_{ab}} & b \\ f_{ac} \downarrow & & \downarrow f_{bd} \\ c & \xrightarrow{f_{cd}} & d \end{array} \tag{3.22}$$

In general, f_{AD} is not closed and thus does not define a morphism in $H^0(\mathcal{C})$. However, the condition (3.21) on f_{AD} ensures that $f_{bd}f_{ab} + f_{cd}f_{ac} = 0$ in $H^0(\mathcal{C})$, i.e. the diagram (3.22) commutes up to the isomorphism $(-1)\text{Id}_d$.

The Cube Lemma tells us that $(A \xrightarrow[-\text{deg.0}]{-f_{AB}} B)$ and $(C \xrightarrow[-\text{deg.0}]{f_{CD}} D)$ are twisted 1-cubes and that the maps (f_{AC}, f_{AD}, f_{BD}) define a closed morphism f_{ABCD} of degree 0 between their convolutions, producing a twisted 1-cube

$$\{A \xrightarrow[-\text{deg.0}]{-f_{AB}} B\} \xrightarrow{f_{ABCD}} \{C \xrightarrow[-\text{deg.0}]{f_{CD}} D\}. \tag{3.23}$$

Using the argument in the above discussion of twisted 1-cubes we complete (3.22) to

$$\begin{array}{ccccccc} a & \xrightarrow{f_{ab}} & b & \longrightarrow & \text{Cone}(f_{ab}) & \longrightarrow & \\ f_{ac} \downarrow & & \downarrow f_{bd} & & \downarrow f_{abcd} & & \\ c & \xrightarrow{f_{cd}} & d & \longrightarrow & \text{Cone}(f_{cd}) & \longrightarrow & \end{array} \tag{3.24}$$

We then check that each of the squares (including the third ‘wrap-around’ square) in this diagram commutes (up to isomorphism). We can do this since we have constructed (3.24) as the image in $H^0(\mathcal{C})$ of an explicit diagram of twisted complexes in $\text{Pre-Tr}(\mathcal{C})$ and we can check that, in fact, that diagram itself commutes up to isomorphism.

Similarly, the Cube Lemma tells us that $(A \xrightarrow[-\text{deg.0}]{-f_{AC}} C)$ and $(B \xrightarrow[-\text{deg.0}]{f_{BD}} D)$ are twisted 1-cubes and that the maps (f_{AB}, f_{AD}, f_{CD}) define a closed morphism f_{ACBD} of degree 0 between their convolutions producing a twisted 1-cube

$$\{A \xrightarrow[-\text{deg.0}]{-f_{AC}} C\} \xrightarrow{f_{ACBD}} \{B \xrightarrow[-\text{deg.0}]{f_{BD}} D\}. \tag{3.25}$$

We can therefore complete (3.22) to

$$\begin{array}{ccc} a & \xrightarrow{f_{ab}} & b \\ f_{ac} \downarrow & & \downarrow f_{bd} \\ c & \xrightarrow{f_{cd}} & d \\ \downarrow & & \downarrow \\ \text{Cone}(f_{ac}) & \xrightarrow{f_{abcd}} & \text{Cone}(f_{bd}) \\ \downarrow & & \downarrow \end{array} \tag{3.26}$$

and check that each of the squares in it commutes.

Finally, the Cube Lemma tells us that the convolutions of the twisted 1-cubes (3.23) and (3.25) are both isomorphic to the convolution T of the original twisted 2-cube (3.20).

We can therefore fit together diagrams (3.26) and (3.24) and then complete them to the 2-cube of side 2

$$\begin{array}{ccccccc}
 a & \xrightarrow{f_{ab}} & b & \longrightarrow & \text{Cone}(f_{ab}) & \longrightarrow & \\
 \downarrow f_{ac} & & \downarrow f_{bd} & & \downarrow f_{abcd} & & \\
 c & \xrightarrow{f_{cd}} & d & \longrightarrow & \text{Cone}(f_{cd}) & \longrightarrow & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 \text{Cone}(f_{ac}) & \xrightarrow{f_{acbd}} & \text{Cone}(f_{bd}) & \longrightarrow & t & \longrightarrow & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 & & & & & &
 \end{array} \tag{3.27}$$

where all rows and columns are exact and where

$$\text{Cone}(f_{acbd}) \simeq t \simeq \text{Cone}(f_{abcd}).$$

We then check as above that every square in this diagram (including the ‘wrap-around’ ones) commutes up to isomorphism.

Lemma 3.7 (The Cube Completion Lemma). *Let $I = \{-1, 0\}^n$ and let $X = (X_{\bar{i}}, q_{\bar{i}\bar{j}})$ be a twisted n -cube over \mathcal{C} indexed by I . There exists a uniquely defined “ n -cube of side 2”—a diagram $Z = \{Z_{\bar{m}}, r_{\bar{m}\bar{n}}\}$ in \mathcal{C} indexed by $M = \{-1, 0, 1\}^n$ with the following properties:*

- (1) **Objects of Z .** *Let \bar{m} be any vertex of M . Define the splitting $I = J \times K$ by choosing for J all the indices $\lambda \in \{1, \dots, n\}$ where \bar{m}_λ equals 1. Let \bar{m}' be the restriction of \bar{m} to K .*

The object $Z_{\bar{m}}$ is isomorphic to the convolution of the sign-twisted subcube $Y^{\bar{m}'}$ of X constructed by the Cube Lemma with respect to the vertex \bar{m} of K . This cube consists of all the objects $X_{\bar{i}}$ such that \bar{i} restricts to \bar{m}' in K and all the morphisms between these vertices in X multiplied by $(-1)^{|\bar{m}'|}$.

Since \bar{m} uniquely determines the twisted cube $Y^{\bar{m}'}$, we also refer to this cube simply as $Y^{\bar{m}}$.

- (2) **Morphisms of Z .** *Let $\bar{l} \rightarrow \bar{m} \rightarrow \bar{n}$ be any row of M , i.e. for some $k \in \{1, \dots, n\}$ we have*

$$\begin{cases} \bar{l}_i = -1, \bar{m}_i = 0, \bar{n}_i = 1, & i = k, \\ \bar{l}_i = \bar{m}_i = \bar{n}_i, & i \neq k. \end{cases}$$

Take the sign-twisted subcube $Y^{\bar{n}}$ of X and split its index set into $J' \times K'$ where we choose for J' all the indices where \bar{l} and \bar{m} equal 1, and for K' the single remaining index k . Apply the Cube Lemma to $Y^{\bar{n}}$ with respect to this splitting to construct the twisted 1-cube

$$\{Y^{\bar{l}}\} \xrightarrow{\alpha} \{Y^{\bar{m}}_{\text{deg},0}\}$$

whose convolution is $\{Y^{\bar{n}}\}$. Then

$$Z_{\bar{l}} \xrightarrow{r_{\bar{l}\bar{m}}} Z_{\bar{m}} \xrightarrow{r_{\bar{m}\bar{n}}} Z_{\bar{n}} \xrightarrow{r_{\bar{n}\bar{l}}} Z_{\bar{l}}[1] \quad (3.28)$$

is the image in \mathcal{C} of the diagram

$$Y^{\bar{l}} \xrightarrow{\alpha} Y^{\bar{m}} \rightarrow (Y^{\bar{l}} \xrightarrow[\text{deg.0}]{\alpha} Y^{\bar{m}}) \rightarrow Y^{\bar{l}}[1] \quad (3.29)$$

constructed as explained when discussing the completion for twisted 1-cubes (cf. (3.15)).

- (3) Any morphism in Z which does not occur in (3.28) for some row $\bar{l} \rightarrow \bar{m} \rightarrow \bar{n}$ of M is 0.
- (4) **Recursivity.** Let $I = J \times K$ be a splitting as in the Cube Lemma and let Y be the twisted cube of sign-twisted subcubes of X constructed by the Cube Lemma with respect to this splitting. Then the cube Z_Y of side 2 in \mathcal{C} defined by Y is naturally a subcube of Z .
- (5) **Commutativity.** The image of the diagram Z in $H^0(\mathcal{C})$ commutes (up to isomorphism).

Proof. The first three properties uniquely define the diagram $Z = \{Z_{\bar{m}}, r_{\bar{m}\bar{n}}\}$. The recursivity is a straightforward verification. To prove the commutativity of Z it suffices to prove that every 2-face of Z commutes. This reduces via the recursivity to the case of X being a 2-cube, where it is again a straightforward verification. See the discussion of the completion for twisted 2-cubes. \square

4. DG-enhancements

4.1. On DG-enhancements of triangulated categories

Let T be a triangulated category. An *enhancement* of T is a pretriangulated DG-category \mathcal{A} together with an exact equivalence $H^0(\mathcal{A}) \xrightarrow{\epsilon} T$. Two enhancements (\mathcal{A}, ϵ) and $(\mathcal{A}', \epsilon')$ are *equivalent* if there exists a quasi-equivalence $\mathcal{A} \xrightarrow{f} \mathcal{A}'$. They are *strongly equivalent* if f can be chosen so that $\epsilon' \circ H^0(f) = \epsilon$. If we want to use DG-categories as enhancements of triangulated ones, we are led to work in the localization of $DG\text{-Cat}$, the category of all small DG-categories, by quasi-equivalences. We denote this localization by $\text{Ho}(DG\text{-Cat})$. For any small DG-categories \mathcal{A} and \mathcal{B} denote by $[\mathcal{A}, \mathcal{B}]$ the set of morphisms between \mathcal{A} and \mathcal{B} in $\text{Ho}(DG\text{-Cat})$. The elements of $[\mathcal{A}, \mathcal{B}]$ are called *quasi-functors*.

Any category quasi-equivalent to a pretriangulated category is itself pretriangulated. We denote the full subcategory of $\text{Ho}(DG\text{-Cat})$ consisting of classes of pretriangulated categories by $\text{Ho}(DG\text{-Cat}^{\text{pretr}})$. We call the elements of $\text{Ho}(DG\text{-Cat}^{\text{pretr}})$ *enhanced triangulated categories* and think of them as small triangulated categories with a fixed quasi-equivalence class of DG-enhancements. Similarly, we can think of a quasi-functor between two enhanced triangulated categories as an exact functor between the triangulated categories and a fixed choice of an equivalence class of DG-functors between their

enhancements which all descend to this exact functor. In this sense, exact functors and quasi-functors are precisely analogous to morphisms between cohomologies of two complexes and morphisms between their classes in the derived category.

One way to understand the morphism set $[\mathcal{A}, \mathcal{B}]$ in $\mathrm{Ho}(DG\text{-Cat})$ is via the model category structure on $DG\text{-Cat}$ constructed in [Tab05]. The weak equivalences are the quasi-equivalences, and the fibrations are defined in such a way that every object is fibrant. Therefore, the elements of $[\mathcal{A}, \mathcal{B}]$ can be identified with the functors from a fixed cofibrant replacement of \mathcal{A} into \mathcal{B} , up to homotopy. Moreover, there exists a cofibrant replacement functor $Q: DG\text{-Cat} \rightarrow DG\text{-Cat}$ equipped with a natural transformation $Q \rightarrow \mathrm{Id}$ such that $Q\mathcal{A} \rightarrow \mathcal{A}$ is a quasi-equivalence which is the identity on the sets of objects [Toë07, Prop. 2.3].

The set $[\mathcal{A}, \mathcal{B}]$ can be naturally endowed with a structure of an element of $\mathrm{Ho}(DG\text{-Cat})$ as follows. The tensor product $\otimes = \otimes_k$ of elements of $DG\text{-Cat}$ can be derived into a bifunctor

$$\overset{\mathbf{L}}{\otimes}: \mathrm{Ho}(DG\text{-Cat}) \times \mathrm{Ho}(DG\text{-Cat}) \rightarrow \mathrm{Ho}(DG\text{-Cat})$$

giving a symmetric monoidal structure for $\mathrm{Ho}(DG\text{-Cat})$. We compute $\mathcal{A} \overset{\mathbf{L}}{\otimes} \mathcal{B}$ as either $Q\mathcal{A} \otimes \mathcal{B}$ or $\mathcal{A} \otimes Q\mathcal{B}$. If k is a field, every small DG -category is k -flat and $\mathcal{A} \overset{\mathbf{L}}{\otimes} \mathcal{B} = \mathcal{A} \otimes \mathcal{B}$. The monoidal structure defined by $\overset{\mathbf{L}}{\otimes}$ on $\mathrm{Ho}(DG\text{-Cat})$ is closed [Toë07, §4.2], i.e. for any \mathcal{A} and \mathcal{B} in $\mathrm{Ho}(DG\text{-Cat})$ the functor $[(-) \otimes \mathcal{A}, \mathcal{B}]$ is representable by an object of $\mathrm{Ho}(DG\text{-Cat})$, defined up to unique isomorphism. Denoted by $\mathbf{R}\mathrm{Hom}(\mathcal{A}, \mathcal{B})$, it is constructed as the class in $\mathrm{Ho}(DG\text{-Cat})$ of $\mathcal{P}^{\mathcal{B}\text{-qr}}(Q\mathcal{A}\text{-}\mathcal{B})$ [Toë07, Theorem 6.1]. These are the h -projective $Q\mathcal{A}$ - \mathcal{B} -bimodules M where for all $a \in Q\mathcal{A}$ the \mathcal{B} -module ${}_aM$ is quasi-isomorphic (and hence homotopic as ${}_aM$ is h -projective [Kel94, Lemma 6.1(c)]) to a representable one. By [Toë07, Cor. 4.8] the isomorphism classes of $H^0(\mathcal{P}^{\mathcal{B}\text{-qr}}(Q\mathcal{A}\text{-}\mathcal{B}))$ are in natural bijection with the elements of $[\mathcal{A}, \mathcal{B}]$. Explicitly, any element of $[\mathcal{A}, \mathcal{B}]$ can be represented by a functor $Q\mathcal{A} \rightarrow \mathcal{B}$. Composing this with the Yoneda embedding $\mathcal{B} \rightarrow \mathbf{Mod}\text{-}\mathcal{B}$ defines a $Q\mathcal{A}$ - \mathcal{B} -bimodule which is even \mathcal{B} -representable. Any h -projective resolution of it defines the desired isomorphism class in $H^0(\mathcal{P}^{\mathcal{B}\text{-qr}}(Q\mathcal{A}\text{-}\mathcal{B}))$. Getting from $M \in \mathcal{P}^{\mathcal{B}\text{-qr}}(Q\mathcal{A}\text{-}\mathcal{B})$ to the corresponding quasi-functor $f \in [\mathcal{A}, \mathcal{B}]$ is more subtle, but it is easy to pin down the underlying functor $H^0(\mathcal{A}) \rightarrow H^0(\mathcal{B})$. Indeed, M defines a functor $Q\mathcal{A} \rightarrow \mathbf{Mod}\text{-}\mathcal{B}$ which maps every element of $Q\mathcal{A}$ to something homotopic to a representable element of $\mathbf{Mod}\text{-}\mathcal{B}$. This defines, up to isomorphism, the requisite functor $H^0(Q\mathcal{A}) \simeq H^0(\mathcal{A}) \rightarrow H^0(\mathcal{B})$. Indeed, this also shows that any morphism between two elements of $H^0(\mathcal{P}^{\mathcal{B}\text{-qr}}(Q\mathcal{A}\text{-}\mathcal{B}))$ induces a natural transformation between the underlying functors of the corresponding quasi-functors in a way which is compatible with compositions.

In other words, $\mathbf{R}\mathrm{Hom}(\mathcal{A}, \mathcal{B}) = \mathcal{P}^{\mathcal{B}\text{-qr}}(Q\mathcal{A}\text{-}\mathcal{B})$ ⁴ is, in a sense, a DG -enhancement of the set $[\mathcal{A}, \mathcal{B}]$. Let us therefore enrich $\mathrm{Ho}(DG\text{-Cat})$ to a 2-category by setting the category of morphisms from \mathcal{A} to \mathcal{B} to be $H^0(\mathbf{R}\mathrm{Hom}(\mathcal{A}, \mathcal{B}))$. By the above, each 1-morphism in

⁴ If k is a field, then $\mathcal{P}^{\mathcal{B}\text{-qr}}(Q\mathcal{A}\text{-}\mathcal{B})$ is quasi-equivalent to $\mathcal{P}^{\mathcal{B}\text{-qr}}(\mathcal{A}\text{-}\mathcal{B})$ and we use the latter instead.

$\text{Ho}(DG\text{-Cat})$ corresponds naturally to a quasi-functor from \mathcal{A} to \mathcal{B} . By abuse of notation, we now refer to the elements of $H^0(\mathbf{R}\mathcal{H}om(\mathcal{A}, \mathcal{B}))$ also as “quasi-functors”. There is a natural functor

$$\Phi: H^0(\mathbf{R}\mathcal{H}om(\mathcal{A}, \mathcal{B})) \rightarrow \text{Fun}(H^0(\mathcal{A}), H^0(\mathcal{B})) \quad (4.1)$$

which sends each quasi-functor to its underlying functor. Defining Φ depends on a choice for each quasi-representable object in $\mathbf{Mod}\text{-}\mathcal{B}$ of a homotopy to a representable one. A different choice would produce a different functor canonically isomorphic to Φ . We therefore make a particular choice for each \mathcal{B} and consider all functors Φ fixed. Our functors Φ package up into a 2-functor

$$\Phi: \text{Ho}(DG\text{-Cat}) \rightarrow \text{Cat} \quad (4.2)$$

into a 2-category Cat whose objects are small categories, whose 1-morphisms are functors and whose 2-morphisms are natural transformations.

By the above, if \mathcal{A} and \mathcal{B} lie in $\text{Ho}(DG\text{-Cat}^{\text{pretr}})$ then so does $\mathbf{R}\mathcal{H}om(\mathcal{A}, \mathcal{B})$. Therefore, in the 2-category $\text{Ho}(DG\text{-Cat}^{\text{pretr}})$ the morphism categories are themselves enhanced triangulated categories. The 2-functor Φ sends the triangulated category $H^0(\mathbf{R}\mathcal{H}om(\mathcal{A}, \mathcal{B}))$ of quasi-functors to the full subcategory in $\text{Fun}(H^0(\mathcal{A}), H^0(\mathcal{B}))$ consisting of exact functors. Moreover, for any morphism of quasi-functors, Φ sends its cone to a functorial cone of the underlying morphism of exact functors. This is exactly the situation we want to be in. This paper adheres to the currently prevalent philosophy that instead of working with triangulated categories \mathcal{A} and \mathcal{B} and the (non-triangulated) category $\text{ExFun}(\mathcal{A}, \mathcal{B})$ of exact functors between them, one should work with enhancements \mathcal{A} and \mathcal{B} of \mathcal{A} and \mathcal{B} in $\text{Ho}(DG\text{-Cat})$ (which are often unique up to isomorphism, cf. [LO10]), the enhanced triangulated category $\mathbf{R}\mathcal{H}om(\mathcal{A}, \mathcal{B})$ and the functor $H^0(\mathbf{R}\mathcal{H}om(\mathcal{A}, \mathcal{B})) \xrightarrow{\Phi} \text{ExFun}(\mathcal{A}, \mathcal{B})$. For years now, this has been practiced implicitly by all who work with Fourier–Mukai kernels of the derived functors between algebraic varieties (cf. Examples 4.2 and 4.3 below).

4.2. Morita enhancements

The triangulated categories we want to enhance are the derived categories of quasi-coherent sheaves and the bounded derived categories of coherent sheaves on separated schemes of finite type over k . All these categories are Karoubi closed. It turns out that the full subcategory of $\text{Ho}(DG\text{-Cat}^{\text{pretr}})$ consisting of those enhanced triangulated categories whose underlying triangulated categories are Karoubi closed admits a more natural description.

Define a DG-category \mathcal{A} to be *kc-triangulated* if it is pretriangulated and $H^0(\mathcal{A})$ is Karoubi closed.⁵ It follows that \mathcal{A} is kc-triangulated if and only if the Yoneda embedding $\mathcal{A} \hookrightarrow \mathcal{P}^{\text{Perf}}(\mathcal{A})$ is a quasi-equivalence. Denote by $\text{Ho}(DG\text{-Cat}^{\text{kctr}})$ the full subcategory of $\text{Ho}(DG\text{-Cat})$ consisting of kc-triangulated categories. The following is explained in

⁵ Here “kc” stands for “Karoubi closed”. These are simply called “triangulated DG-categories” in the papers of Toën, but we feel that this does not reflect well their main difference from the established notion of pretriangulated DG-categories.

detail in [Toë11, §4.4]. Let $\mathcal{A} \xrightarrow{f} \mathcal{B}$ be a functor between DG-categories. The induced functor $f_*: \mathbf{Mod}\text{-}\mathcal{B} \rightarrow \mathbf{Mod}\text{-}\mathcal{A}$ preserves acyclicity. Its left adjoint $f^*: \mathbf{Mod}\text{-}\mathcal{A} \rightarrow \mathbf{Mod}\text{-}\mathcal{B}$ preserves, by adjunction, h -projectivity. We say that f is a *Morita equivalence* if $D(\mathcal{B}) \xrightarrow{f_*} D(\mathcal{A})$ is an exact equivalence, or equivalently $\mathcal{P}^{\text{Perf}}(\mathcal{A}) \xrightarrow{f^*} \mathcal{P}^{\text{Perf}}(\mathcal{B})$ is a quasi-equivalence. The functor

$$\mathcal{P}^{\text{Perf}}(-): \text{Ho}(DG\text{-Cat}) \rightarrow \text{Ho}(DG\text{-Cat}^{\text{kctr}})$$

is the left adjoint of the natural inclusion $\text{Ho}(DG\text{-Cat}^{\text{kctr}}) \hookrightarrow \text{Ho}(DG\text{-Cat})$ [Toë11, Prop. 6]. It follows, as explained in [Toë11, §4.4], that $\mathcal{P}^{\text{Perf}}(-)$ induces an equivalence $\text{Mrt}(DG\text{-Cat}) \xrightarrow{\sim} \text{Ho}(DG\text{-Cat}^{\text{kctr}})$, where $\text{Mrt}(DG\text{-Cat})$ is the localization of $DG\text{-Cat}$ by Morita equivalences. We use this to identify Morita equivalence classes of small DG-categories with the elements of $\text{Ho}(DG\text{-Cat}^{\text{kctr}})$. In other words, when speaking of the class of a small DG-category \mathcal{A} in $\text{Ho}(DG\text{-Cat}^{\text{kctr}})$ we mean $\mathcal{P}^{\text{Perf}}(\mathcal{A})$.

We call the morphisms in $\text{Mrt}(DG\text{-Cat})$ *Morita quasi-functors*. By the above, Morita quasi-functors $\mathcal{A} \rightarrow \mathcal{B}$ correspond to the ordinary quasi-functors $\mathcal{P}^{\text{Perf}}(\mathcal{A}) \rightarrow \mathcal{P}^{\text{Perf}}(\mathcal{B})$. It follows from [Toë07, Th. 7.2] that $\mathbf{R}Hom(\mathcal{P}^{\text{Perf}}(\mathcal{A}), \mathcal{P}^{\text{Perf}}(\mathcal{B}))$ is quasi-equivalent to $\mathcal{P}^{\mathcal{B}\text{-Perf}}(\mathcal{A}\text{-}\mathcal{B})$. This gives a more natural DG-enhancement of the set $\text{Hom}_{\text{Mrt}(DG\text{-Cat})}(\mathcal{A}, \mathcal{B})$. In particular, we think of the elements of $D^{\mathcal{B}\text{-Perf}}(\mathcal{A}\text{-}\mathcal{B})$ as Morita quasi-functors $\mathcal{A} \rightarrow \mathcal{B}$. Note that given $M \in D_{\mathbf{L}}^{\mathcal{B}\text{-Perf}}(\mathcal{A}\text{-}\mathcal{B})$, the exact functor underlying the corresponding Morita quasi-functor is $(-) \otimes_{\mathcal{A}} M$.

This leads to a slightly different notion of DG-enhancement. Define a *Morita enhancement* of a small triangulated category A to be a small DG-category \mathcal{A} together with an isomorphism $D_{\mathbf{c}}(\mathcal{A}) \xrightarrow{\sim} A$. Since $D_{\mathbf{c}}(\mathcal{A}) = H^0(\mathcal{P}^{\text{Perf}}(\mathcal{A}))$, \mathcal{A} is a Morita enhancement of A if and only if its class in $\text{Ho}(DG\text{-Cat}^{\text{kctr}})$ is the usual enhancement of A . Moreover, we can similarly use small DG-categories to enhance non-small triangulated categories (i.e. unbounded derived categories of quasi-coherent sheaves). Define a *large Morita enhancement* of a triangulated category A to be a small DG-category \mathcal{A} together with an isomorphism $D(\mathcal{A}) \xrightarrow{\sim} A$. An advantage of this Morita point of view is that we use much smaller DG-categories to define our enhancements. In fact, the derived categories of schemes can be Morita enhanced by DG-algebras (cf. Examples 4.2 and 4.3).

4.3. Examples

The following examples illustrate the notions introduced in the previous section and explain the framework to which the main definitions of Section 5 rightfully belong. First is the usual framework of DG-enhancements:

Example 4.1. Let \mathcal{A} and \mathcal{B} be elements of $\text{Ho}(DG\text{-Cat})$. As described in Section 4, $\mathbf{R}Hom(\mathcal{A}, \mathcal{B})$ is represented in $\text{Ho}(DG\text{-Cat})$ by the full subcategory $\mathcal{P}^{\mathcal{B}\text{-qr}}(\mathcal{A}\text{-}\mathcal{B})$ of $\mathcal{P}(\mathcal{A}\text{-}\mathcal{B})$ consisting of \mathcal{B} -quasi-representable bimodules. In particular, such bimodules are \mathcal{B} -perfect.

Let $M \in \mathcal{P}^{\mathcal{B}\text{-qr}}(\mathcal{A}\text{-}\mathcal{B})$. The functor $H^0(\mathcal{A}) \rightarrow H^0(\mathcal{B})$ defined by the corresponding quasi-functor is the restriction of $(-) \otimes_{\mathcal{A}} M$ from $D(\mathcal{A}) \rightarrow D(\mathcal{B})$ to $H^0(\mathcal{A}) \rightarrow H^0(\mathcal{B})$. It

follows from Section 2.2 that if M is also \mathcal{A} -perfect, then $(-)\overset{\mathbf{L}}{\otimes}_{\mathcal{A}}M$, as a functor $D(\mathcal{A}) \rightarrow D(\mathcal{B})$, has left and right adjoints $(-)\overset{\mathbf{L}}{\otimes}_{\mathcal{B}}M^{\mathcal{A}}$ and $(-)\overset{\mathbf{L}}{\otimes}_{\mathcal{B}}M^{\mathcal{B}}$. If moreover $M^{\mathcal{A}}$ and $M^{\mathcal{B}}$ are \mathcal{A} -quasi-representable, then these adjoints restrict to functors $H^0(\mathcal{B}) \rightarrow H^0(\mathcal{A})$. In other words, $M^{\mathcal{A}}$ and $M^{\mathcal{B}}$ define quasi-functors $\mathcal{B} \rightarrow \mathcal{A}$ whose induced functors $H^0(\mathcal{B}) \rightarrow H^0(\mathcal{A})$ are left and right adjoint to the functor $H^0(\mathcal{A}) \rightarrow H^0(\mathcal{B})$ defined by M .

Next we illustrate Morita enhancements. In the two examples below we explain how derived categories of algebraic varieties are Morita enhanced by DG-algebras and how quasi-functors between these enhancements may be represented as DG-bimodules for these algebras:

Example 4.2. Let X and Y be quasi-compact, quasi-separated schemes over k . By [BvdB03, Th. 3.1.1] there exist compact generators E_X and E_Y of $D_{\text{qc}}(X)$ and $D_{\text{qc}}(Y)$. We choose h -injective resolutions of E_X and E_Y and define \mathcal{A} and \mathcal{B} to be their DG-End-algebras. Then \mathcal{A} and \mathcal{B} are the standard large Morita enhancements of $D_{\text{qc}}(X)$ and $D_{\text{qc}}(Y)$, i.e. $\mathcal{P}(\mathcal{A})$ and $\mathcal{P}(\mathcal{B})$ are their standard enhancements in the usual sense.

By [Toë07, Th. 7.2] the pullback along the Yoneda embedding $\mathcal{A} \hookrightarrow \mathbf{Mod}\text{-}\mathcal{A}$ induces an isomorphism

$$\mathbf{R}Hom_{\text{cts}}(\mathcal{P}(\mathcal{A}), \mathcal{P}(\mathcal{B})) \xrightarrow{\sim} \mathbf{R}Hom(\mathcal{A}, \mathcal{P}(\mathcal{B}))$$

in $\text{Ho}(DG\text{-Cat})$. Here $\mathbf{R}Hom_{\text{cts}}$ stands for the full subcategory consisting of *continuous* quasi-functors, i.e. the quasi-functors $\mathcal{P}(\mathcal{A}) \rightarrow \mathcal{P}(\mathcal{B})$ whose underlying functors $D(\mathcal{A}) \rightarrow D(\mathcal{B})$ commute with infinite direct sums. The universal properties of $\mathbf{R}Hom$ and [Toë07, Lemma 6.2] imply that $\mathbf{R}Hom(\mathcal{A}, \mathcal{P}(\mathcal{B}))$ is represented in $\text{Ho}(DG\text{-Cat})$ by $\mathcal{P}(\mathcal{A}\text{-}\mathcal{B})$. Explicitly, after replacing \mathcal{A} by its cofibrant resolution any quasi-functor in $H^0(\mathbf{R}Hom(\mathcal{A}, \mathcal{P}(\mathcal{B})))$ can be represented by an actual functor $\mathcal{A} \rightarrow \mathcal{P}(\mathcal{B})$. Taking an h -projective resolution of the corresponding \mathcal{A} - \mathcal{B} -bimodule gives the desired homotopy class in $\mathcal{P}(\mathcal{A}\text{-}\mathcal{B})$.

Thus every continuous quasi-functor $\mathcal{P}(\mathcal{A}) \rightarrow \mathcal{P}(\mathcal{B})$ can be represented by an element $M \in \mathcal{P}(\mathcal{A}\text{-}\mathcal{B})$. The underlying functor $D_{\text{qc}}(X) \rightarrow D_{\text{qc}}(Y)$ is then precisely $(-)\overset{\mathbf{L}}{\otimes}_{\mathcal{A}}M$. It follows from Section 2.2 that if M is \mathcal{A} - and \mathcal{B} -perfect, then $M^{\mathcal{A}}$ and $M^{\mathcal{B}}$ define quasi-functors $\mathcal{P}(\mathcal{B}) \rightarrow \mathcal{P}(\mathcal{A})$ such that $(-)\overset{\mathbf{L}}{\otimes}_{\mathcal{B}}M^{\mathcal{A}}$ and $(-)\overset{\mathbf{L}}{\otimes}_{\mathcal{B}}M^{\mathcal{B}}$ are the left and right adjoints of $(-)\overset{\mathbf{L}}{\otimes}_{\mathcal{A}}M$.

It is also shown in [Toë07, Section 8.3] that $\mathcal{A}^{\text{opp}} \otimes \mathcal{B}$ is the standard large Morita enhancement of $D(X \times_k Y)$ via a natural identification of $D(X \times_k Y)$ with $D(\mathcal{A}\text{-}\mathcal{B})$. Together with the above, we obtain an identification of $D(X \times_k Y)$ with $H^0(\mathbf{R}Hom_{\text{cts}}(\mathcal{P}(\mathcal{A}), \mathcal{P}(\mathcal{B})))$ which sends each object $E \in D(X \times_k Y)$ to a quasi-functor $\mathcal{P}(\mathcal{A}) \rightarrow \mathcal{P}(\mathcal{B})$ whose underlying functor $D_{\text{qc}}(X) \rightarrow D_{\text{qc}}(Y)$ is isomorphic to the Fourier–Mukai transform defined by E .

Example 4.3. Let X and Y be separated schemes of finite type over k . By [Rou08, Th. 7.39] there exist strong generators F_X and F_Y of $D(X)$. Choose h -injective resolutions of F_X and F_Y and let \mathcal{A} and \mathcal{B} be their DG-End-algebras. Then \mathcal{A} and \mathcal{B} are the

standard Morita enhancements of $D(X)$ and $D(Y)$, i.e. $\mathcal{P}^{\text{Perf}}(\mathcal{A})$ and $\mathcal{P}^{\text{Perf}}(\mathcal{B})$ are their standard enhancements in the usual sense. It was, moreover, proved in [Lun10, Th. 6.3] that for any choice of generators F_X and F_Y the DG-algebras \mathcal{A} and \mathcal{B} are smooth.

By [Toë07, Th. 7.2] the pullback along the Yoneda embedding $\mathcal{A} \hookrightarrow \mathcal{P}^{\text{Perf}}(\mathcal{A})$ induces an isomorphism

$$\mathbf{R} \text{Hom}(\mathcal{P}^{\text{Perf}}(\mathcal{A}), \mathcal{P}^{\text{Perf}}(\mathcal{B})) \xrightarrow{\sim} \mathbf{R} \text{Hom}(\mathcal{A}, \mathcal{P}^{\text{Perf}}(\mathcal{B}))$$

in $\text{Ho}(\text{DG-Cat})$. Once again, the universal properties of $\mathbf{R} \text{Hom}$ and [Toë07, Lemma 6.2] imply that $\mathbf{R} \text{Hom}(\mathcal{A}, \mathcal{P}^{\text{Perf}}(\mathcal{B}))$ is represented in $\text{Ho}(\text{DG-Cat})$ by $\mathcal{P}^{\mathcal{B}\text{-Perf}}(\mathcal{A}\text{-}\mathcal{B})$, the full subcategory of $\mathcal{P}(\mathcal{A}\text{-}\mathcal{B})$ consisting of \mathcal{B} -perfect bimodules. Explicitly, after replacing \mathcal{A} by its cofibrant resolution, any quasi-functor in $H^0(\mathbf{R} \text{Hom}(\mathcal{A}, \mathcal{P}^{\text{Perf}}(\mathcal{B})))$ can be represented by an actual functor $\mathcal{A} \rightarrow \mathcal{P}^{\text{Perf}}(\mathcal{B})$. Taking any h -projective resolution of the corresponding \mathcal{B} -perfect $\mathcal{A}\text{-}\mathcal{B}$ -bimodule, we obtain the desired homotopy class in $\mathcal{P}^{\mathcal{B}\text{-Perf}}(\mathcal{A}\text{-}\mathcal{B})$.

Thus every quasi-functor $\mathcal{P}^{\text{Perf}}(\mathcal{A}) \rightarrow \mathcal{P}^{\text{Perf}}(\mathcal{B})$ can be represented by an element $M \in \mathcal{P}^{\mathcal{B}\text{-Perf}}(\mathcal{A}\text{-}\mathcal{B})$ and the underlying functor $D(X) \rightarrow D(Y)$ is then $(-) \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} M$. It follows again from Section 2.2 that $M^{\mathcal{B}}$ defines a quasi-functor $\mathcal{P}^{\text{Perf}}(\mathcal{B}) \rightarrow \mathcal{P}^{\text{Perf}}(\mathcal{A})$ whose underlying functor $(-) \overset{\mathbf{L}}{\otimes}_{\mathcal{B}} M^{\mathcal{B}}$ is the right adjoint of $(-) \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} M$. If M is also \mathcal{A} -perfect, then $M^{\mathcal{A}}$ defines a quasi-functor whose underlying functor $(-) \overset{\mathbf{L}}{\otimes}_{\mathcal{B}} M^{\mathcal{A}}$ is the left adjoint of $(-) \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} M$.

It also follows from [Lun10, Prop. 6.14] that $\mathcal{A}^{\text{opp}} \otimes \mathcal{B}$ is the standard Morita enhancement of $D(X \times Y)$. Since \mathcal{A} is smooth, we have by Cor. 2.15 a natural inclusion $\mathcal{P}^{\mathcal{B}\text{-Perf}}(\mathcal{A}\text{-}\mathcal{B}) \subset \mathcal{P}^{\text{Perf}}(\mathcal{A}\text{-}\mathcal{B})$. This identifies each quasi-functor $\mathcal{P}^{\text{Perf}}(\mathcal{A}) \rightarrow \mathcal{P}^{\text{Perf}}(\mathcal{B})$ with an object $E \in D(X \times Y)$ in such a way that its underlying functor $D(X) \rightarrow D(Y)$ corresponds to the Fourier–Mukai transform defined by E .

5. Spherical DG-functors

5.1. Spherical bimodules and spherical quasi-functors

Let \mathcal{A} and \mathcal{B} be small DG-categories and $S \in D(\mathcal{A}\text{-}\mathcal{B})$ be \mathcal{A} - and \mathcal{B} -perfect. Denote by R and L the derived duals $S^{\tilde{\mathcal{B}}}$ and $S^{\tilde{\mathcal{A}}}$ in $D(\mathcal{B}\text{-}\mathcal{A})$. Let

$$s : D(\mathcal{A}) \rightarrow D(\mathcal{B})$$

be the exact functor $(-) \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} S$ and

$$r, l : D(\mathcal{B}) \rightarrow D(\mathcal{A})$$

be the exact functors $(-) \overset{\mathbf{L}}{\otimes}_{\mathcal{B}} S^{\tilde{\mathcal{B}}}$ and $(-) \overset{\mathbf{L}}{\otimes}_{\mathcal{B}} S^{\tilde{\mathcal{A}}}$. By Cor. 2.2, r and l are right and left adjoint to s .

As per Section 4 the objects of e.g. $D(\mathcal{A}\text{-}\mathcal{B})$ represent continuous quasi-functors $\mathcal{P}(\mathcal{A}) \rightarrow \mathcal{P}(\mathcal{B})$. The functors s, r and l are the exact functors underlying the quasi-functors S, R and L . Accordingly, we introduce the following notation. Given e.g. $S \in D(\mathcal{A}\text{-}\mathcal{B})$ and $R \in D(\mathcal{B}\text{-}\mathcal{A})$, we write SR for the object $R \overset{L}{\otimes}_{\mathcal{A}} S \in D(\mathcal{B}\text{-}\mathcal{B})$. The exact functor underlying the quasi-functor SR is then sr .

Definition 5.1. Define:

- The *twist* T of S is $\text{Cone}(SR \xrightarrow{\text{tr}} \mathcal{B})$ in $D(\mathcal{B}\text{-}\mathcal{B})$.
- The *dual twist* T' of S is $\text{Cone}(\mathcal{B} \xrightarrow{\text{act}} SL)[-1]$ in $D(\mathcal{B}\text{-}\mathcal{B})$.
- The *cotwist* F of S is $\text{Cone}(\mathcal{A} \xrightarrow{\text{act}} RS)[-1]$ in $D(\mathcal{A}\text{-}\mathcal{A})$.
- The *dual cotwist* F' of S is $\text{Cone}(LS \xrightarrow{\text{tr}} \mathcal{A})$ in $D(\mathcal{A}\text{-}\mathcal{A})$.

Thus we have the following natural exact triangles in $D(\mathcal{B}\text{-}\mathcal{B})$ and $D(\mathcal{A}\text{-}\mathcal{A})$:

$$SR \xrightarrow{\text{tr}} \mathcal{B} \rightarrow T, \tag{5.1}$$

$$T' \rightarrow \mathcal{B} \xrightarrow{\text{act}} SL, \tag{5.2}$$

$$F \rightarrow \mathcal{A} \xrightarrow{\text{act}} RS, \tag{5.3}$$

$$LS \xrightarrow{\text{tr}} \mathcal{A} \rightarrow F'. \tag{5.4}$$

Let $t, t': D(\mathcal{B}) \rightarrow D(\mathcal{B})$ and $f, f': D(\mathcal{A}) \rightarrow D(\mathcal{A})$ be the corresponding exact functors. By Cor. 2.2 the functorial exact triangles induced by (5.1)–(5.4) are

$$sr \xrightarrow{\text{adj.counit}} \text{Id}_{D(\mathcal{B})} \rightarrow t, \tag{5.5}$$

$$t' \rightarrow \text{Id}_{D(\mathcal{B})} \xrightarrow{\text{adj.unit}} sl, \tag{5.6}$$

$$f \rightarrow \text{Id}_{D(\mathcal{A})} \xrightarrow{\text{adj.unit}} rs, \tag{5.7}$$

$$ls \xrightarrow{\text{adj.counit}} \text{Id}_{D(\mathcal{A})} \rightarrow f', \tag{5.8}$$

i.e. t and $f[1]$ are functorial cones of the counit and the unit of the adjoint pair (s, r) , while $t'[1]$ and f' are functorial cones of the unit and the counit of the adjoint pair (l, s) .

Finally, consider the compositions

$$LT[-1] \xrightarrow{(5.1)} LSR \xrightarrow{\text{tr}} R, \tag{5.9}$$

$$R \xrightarrow{\text{act}} RSL \xrightarrow{(5.3)} FL[1], \tag{5.10}$$

and the induced natural transformations

$$lt[-1] \xrightarrow{(5.5)} lsr \xrightarrow{\text{adj.counit}} r, \tag{5.11}$$

$$r \xrightarrow{\text{adj.unit}} rsl \xrightarrow{(5.7)} fl[1]. \tag{5.12}$$

Definition 5.2. An object $S \in D(\mathcal{A}\text{-}\mathcal{B})$ is *spherical* if it is \mathcal{A} - and \mathcal{B} -perfect and the following hold:

- (1) t and t' are quasi-inverse autoequivalences of $D(\mathcal{B})$.
- (2) f and f' are quasi-inverse autoequivalences of $D(\mathcal{A})$.
- (3) $lt[-1] \xrightarrow{(5.11)} r$ is an isomorphism of functors (“the twist identifies the adjoints”).
- (4) $r \xrightarrow{(5.12)} fl[1]$ is an isomorphism of functors (“the cotwist identifies the adjoints”).

We say that an \mathcal{A} - \mathcal{B} -bimodule is *spherical* if its image in $D(\mathcal{A}\text{-}\mathcal{B})$ is spherical.

The following is the main theorem of this section:

Theorem 5.1. *Let S be an \mathcal{A} - and \mathcal{B} -perfect object of $D(\mathcal{A}\text{-}\mathcal{B})$. Suppose any two of the following conditions hold:*

- (1) t is an autoequivalence of $D(\mathcal{B})$ (“the twist is an equivalence”).
- (2) f is an autoequivalence of $D(\mathcal{A})$ (“the cotwist is an equivalence”).
- (3) $lt[-1] \xrightarrow{(5.11)} r$ is an isomorphism of functors (“the twist identifies the adjoints”).
- (4) $r \xrightarrow{(5.12)} fl[1]$ is an isomorphism of functors (“the cotwist identifies the adjoints”).

Then all four hold and S is spherical.

To prove this result we lift everything to the DG-enhancements $\mathcal{P}(\mathcal{A}\text{-}\mathcal{A})$, $\mathcal{P}(\mathcal{B}\text{-}\mathcal{B})$, $\mathcal{P}(\mathcal{A}\text{-}\mathcal{B})$ and $\mathcal{P}(\mathcal{B}\text{-}\mathcal{A})$ and work with twisted complexes over them. As these DG-categories are strongly pretriangulated, the canonical convolution functors send twisted complexes over them to (the Yoneda embeddings of) these categories themselves. Given e.g. a twisted complex $E_0 \rightarrow \cdots \rightarrow E_n$ over $\mathcal{P}(\mathcal{A}\text{-}\mathcal{A})$, we write $\{E_0 \rightarrow \cdots \rightarrow E_n\}$ for its convolution in $\mathcal{P}(\mathcal{A}\text{-}\mathcal{A})$.

Recall that $\mathbf{R}Hom_{\text{cts}}(\mathcal{P}(\mathcal{A}), \mathcal{P}(\mathcal{B}))$ is represented in $\text{Ho}(DG\text{-}Cat_{\vee})$ by $\mathcal{P}(\mathcal{A}\text{-}\mathcal{B})$ (cf. Example 4.2). Similarly, Morita quasi-functors $\mathcal{A} \rightarrow \mathcal{B}$, the morphisms from \mathcal{A} to \mathcal{B} in $\text{Mrt}(DG\text{-}Cat)$, are in 1-to-1 correspondence with ordinary quasi-functors $\mathcal{P}^{\text{Perf}}(\mathcal{A}) \rightarrow \mathcal{P}^{\text{Perf}}(\mathcal{B})$, and $\mathbf{R}Hom(\mathcal{P}^{\text{Perf}}(\mathcal{A}), \mathcal{P}^{\text{Perf}}(\mathcal{B}))$ is represented in $\text{Ho}(DG\text{-}Cat)$ by $\mathcal{P}^{\mathcal{A}\text{-}\text{Perf}}(\mathcal{A}\text{-}\mathcal{B})$ (cf. Example 4.3). Define a quasi-functor $\mathcal{P}(\mathcal{A}) \rightarrow \mathcal{P}(\mathcal{B})$ or a Morita quasi-functor $\mathcal{A} \rightarrow \mathcal{B}$ to be *spherical* if the corresponding element of $D(\mathcal{A}\text{-}\mathcal{B})$ is spherical.

Let $M = \bar{\mathcal{A}} \otimes_{\mathcal{A}} S \otimes_{\mathcal{B}} \bar{\mathcal{B}}$, with S viewed as the corresponding bimodule in $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$. Then M is an h -projective resolution of S in $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$. We now make use of the homotopy adjunction theory set up in §2.2, and in particular of h -projective resolutions $M^{h\mathcal{A}}$ and $M^{h\mathcal{B}}$ of $M^{\bar{\mathcal{A}}}$ and $M^{\bar{\mathcal{B}}}$.

Below, we use the following shorthand: τ denotes a map which consists of applying all possible instances of the canonical maps $\bar{\mathcal{A}} \rightarrow \mathcal{A}$, $\bar{\mathcal{B}} \rightarrow \mathcal{B}$, e.g. $M^{h\mathcal{A}} \xrightarrow{\tau} M^{\mathcal{A}}$ or $M \otimes_{\mathcal{B}} M^{h\mathcal{B}} \xrightarrow{\tau} M \otimes_{\mathcal{B}} M^{\mathcal{B}}$.

In the diagrams below, maps of degree 0 are denoted by solid arrows and maps of degree -1 are denoted by dashed arrows.

By Def. 2.10 of the homotopy action maps, the following diagrams commute up to homotopy:

$$\begin{array}{ccc}
 \bar{\mathcal{B}} \xrightarrow{\text{act}} M^{h\mathcal{A}} \otimes_{\mathcal{A}} M & \bar{\mathcal{A}} \xrightarrow{\text{act}} M \otimes_{\mathcal{B}} M^{h\mathcal{B}} & \\
 \tau \downarrow & \tau \downarrow & \\
 \mathcal{B} \xrightarrow{\text{act}} \text{Hom}_{\mathcal{A}}(M, M) & \mathcal{A} \xrightarrow{\text{act}} \text{Hom}_{\mathcal{B}}(M, M) &
 \end{array} \quad (5.13)$$

Fix $\theta_{\mathcal{B}} \in \text{Hom}_{\mathcal{B}\text{-}\mathcal{B}}^{-1}(\bar{\mathcal{B}}, \text{Hom}_{\mathcal{A}}(M, M))$ and $\theta_{\mathcal{A}} \in \text{Hom}_{\mathcal{A}\text{-}\mathcal{A}}^{-1}(\bar{\mathcal{A}}, \text{Hom}_{\mathcal{B}}(M, M))$ such that

$$\begin{aligned}
 (2.26) \circ \tau \circ \text{act} &= \text{act} \circ \tau + d\theta_{\mathcal{B}}, \\
 (2.25) \circ \tau \circ \text{act} &= \text{act} \circ \tau + d\theta_{\mathcal{A}},
 \end{aligned}$$

i.e. the squares in (5.13) commute up to $d\theta_{\mathcal{B}}$ and $d\theta_{\mathcal{A}}$.

To establish our homotopy adjunctions we have proved in Prop. 2.11 that the four compositions (2.46)–(2.49) are homotopic to the identity. We can now make this more precise: Let

$$\begin{aligned}
 \chi_{\mathcal{A}} &= M \xrightarrow{\text{Id} \otimes \theta_{\mathcal{B}}} M \otimes_{\mathcal{B}} \text{Hom}_{\mathcal{A}}(M, M) \xrightarrow{\text{ev}} M \in \text{Hom}_{\mathcal{A}\text{-}\mathcal{B}}^{-1}(M, M), \\
 \chi_{\mathcal{B}} &= M \xrightarrow{\theta_{\mathcal{A}} \otimes \text{Id}} \text{Hom}_{\mathcal{B}}(M, M) \otimes_{\mathcal{A}} M \xrightarrow{\text{ev}} M \in \text{Hom}_{\mathcal{A}\text{-}\mathcal{B}}^{-1}(M, M), \\
 \xi_{\mathcal{A}} &= M^{h\mathcal{A}} \xrightarrow{\text{Id} \otimes \theta_{\mathcal{B}} \otimes \text{Id}} \bar{\mathcal{B}} \otimes_{\mathcal{B}} \text{Hom}_{\mathcal{A}}(M, M) \otimes_{\mathcal{B}} M^{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\text{Id} \otimes (- \circ -) \otimes \text{Id}} M^{h\mathcal{A}} \\
 &\hspace{15em} \in \text{Hom}_{\mathcal{B}\text{-}\mathcal{A}}^{-1}(M^{h\mathcal{A}}, M^{h\mathcal{A}}), \\
 \xi_{\mathcal{B}} &= M^{h\mathcal{B}} \xrightarrow{\text{Id} \otimes \theta_{\mathcal{A}} \otimes \text{Id}} \bar{\mathcal{B}} \otimes_{\mathcal{B}} M^{\mathcal{B}} \otimes_{\mathcal{A}} \text{Hom}_{\mathcal{B}}(M, M) \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\text{Id} \otimes (- \circ -) \otimes \text{Id}} M^{h\mathcal{B}} \\
 &\hspace{15em} \in \text{Hom}_{\mathcal{B}\text{-}\mathcal{A}}^{-1}(M^{h\mathcal{B}}, M^{h\mathcal{B}}).
 \end{aligned}$$

The compositions (2.46)–(2.49) equal $\text{Id} + d\xi_{\mathcal{B}}$, $\text{Id} + d\chi_{\mathcal{B}}$, $\text{Id} + d\xi_{\mathcal{A}}$ and $\text{Id} + d\chi_{\mathcal{A}}$, respectively.

By construction, the homotopy action and trace maps are isomorphic in $D(\mathcal{A}\text{-}\mathcal{A})$ and $D(\mathcal{B}\text{-}\mathcal{B})$ to their derived counterparts. Therefore

$$\begin{aligned}
 T &\simeq \left\{ M^{h\mathcal{B}} \otimes_{\mathcal{A}} M \xrightarrow[\text{deg.0}]{\text{tr}} \bar{\mathcal{B}} \right\} \quad \text{in } D(\mathcal{B}\text{-}\mathcal{B}), \\
 T' &\simeq \left\{ \bar{\mathcal{B}} \xrightarrow[\text{deg.0}]{\text{act}} M^{h\mathcal{A}} \otimes_{\mathcal{A}} M \right\} \quad \text{in } D(\mathcal{B}\text{-}\mathcal{B}), \\
 F &\simeq \left\{ \bar{\mathcal{A}} \xrightarrow[\text{deg.0}]{\text{act}} M \otimes_{\mathcal{B}} M^{h\mathcal{B}} \right\} \quad \text{in } D(\mathcal{A}\text{-}\mathcal{A}), \\
 F' &\simeq \left\{ M \otimes_{\mathcal{B}} M^{h\mathcal{A}} \xrightarrow[\text{deg.0}]{\text{tr}} \bar{\mathcal{A}} \right\} \quad \text{in } D(\mathcal{A}\text{-}\mathcal{A}).
 \end{aligned}$$

Proposition 5.3. *We have*

$$T^{\dagger} \tilde{\mathcal{B}} \simeq T' \quad \text{in } D(\mathcal{B}\text{-}\mathcal{B}), \quad (F')^r \tilde{\mathcal{A}} \simeq F \quad \text{in } D(\mathcal{A}\text{-}\mathcal{A}).$$

Consequently, t' is the left adjoint of $t: D(\mathcal{B}) \rightarrow D(\mathcal{B})$, and f' is the left adjoint of $f: D(\mathcal{A}) \rightarrow D(\mathcal{A})$.

Proof. By definitions of T' and T we have exact triangles

$$T' \rightarrow \mathcal{B} \xrightarrow{\text{act}} M^{\tilde{\mathcal{A}}} \overset{\mathbb{L}}{\otimes}_{\mathcal{A}} M, \quad M^{\tilde{\mathcal{B}}} \overset{\mathbb{L}}{\otimes}_{\mathcal{A}} M \xrightarrow{\text{tr}} \mathcal{B} \rightarrow T$$

in $D(\mathcal{B}\text{-}\mathcal{B})$. Applying the functor $(-)^{l\tilde{\mathcal{B}}}$ to the latter we obtain an exact triangle

$$T^{l\tilde{\mathcal{B}}} \rightarrow \mathcal{B} \xrightarrow{\text{tr}^{l\tilde{\mathcal{B}}}} (M^{\tilde{\mathcal{B}}} \overset{\mathbb{L}}{\otimes}_{\mathcal{A}} M)^{l\tilde{\mathcal{B}}}.$$

Lemma 2.13 produces an isomorphism $M^{\tilde{\mathcal{A}}} \overset{\mathbb{L}}{\otimes}_{\mathcal{A}} M \xrightarrow{\sim} (M^{\tilde{\mathcal{B}}} \overset{\mathbb{L}}{\otimes}_{\mathcal{A}} M)^{l\tilde{\mathcal{B}}}$ which makes the diagram

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\text{act}} & M^{\tilde{\mathcal{A}}} \overset{\mathbb{L}}{\otimes}_{\mathcal{A}} M \\ \parallel & & \downarrow \sim \\ \mathcal{B} & \xrightarrow{\text{tr}^{l\tilde{\mathcal{B}}}} & (M^{\tilde{\mathcal{B}}} \overset{\mathbb{L}}{\otimes}_{\mathcal{A}} M)^{l\tilde{\mathcal{B}}} \end{array}$$

commute. Thus there exists $T' \simeq T^{l\tilde{\mathcal{B}}}$ which completes the above to an isomorphism of exact triangles.

An identical argument produces an isomorphism $(F')^{r\tilde{\mathcal{A}}} \simeq F$ in $D(\mathcal{A}\text{-}\mathcal{A})$. The final assertion then follows since by Cor. 2.2 the functors $(-)\overset{\mathbb{L}}{\otimes}_{\mathcal{B}} T^{l\tilde{\mathcal{B}}}$ and $(-)\overset{\mathbb{L}}{\otimes}_{\mathcal{A}} (F')^{r\tilde{\mathcal{A}}}$ are left and right adjoint to t and f' , respectively. \square

Thus, if t is an autoequivalence of $D(\mathcal{B})$ then t' is always its quasi-inverse, and similarly for f and f' .

Denote by $\mathcal{B} \xrightarrow{\text{act}} TT'$ and $T'T \xrightarrow{\text{tr}} \mathcal{B}$ the maps in $D(\mathcal{B}\text{-}\mathcal{B})$ which the isomorphism $T' \simeq T^{l\tilde{\mathcal{B}}}$ of Prop. 5.3 identifies with the derived action and trace maps for T . By construction of the (t', t) adjunction these maps induce its unit and counit.

Proposition 5.4. *The maps $\mathcal{B} \xrightarrow{\text{act}} TT'$ and $T'T \xrightarrow{\text{tr}} \mathcal{B}$ are isomorphic in $D(\mathcal{B}\text{-}\mathcal{B})$ to the maps*

$$\begin{array}{c} \tilde{\mathcal{B}} \xrightarrow{\text{deg.0}} \\ \left(M^{h\mathcal{B}} \otimes_{\mathcal{A}} M \xrightarrow{\text{tr} \oplus (\text{act} \otimes \text{Id})} \tilde{\mathcal{B}} \oplus (M^{h\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{B}} \otimes_{\mathcal{A}} M) \xrightarrow{\text{act} \oplus (-\text{Id} \otimes \text{tr})} M^{h\mathcal{A}} \otimes_{\mathcal{A}} M \right), \end{array} \tag{5.14}$$

$$\begin{array}{c} \left(M^{h\mathcal{B}} \otimes_{\mathcal{A}} M \xrightarrow{\text{tr} \oplus (-\text{Id} \otimes \text{act})} \tilde{\mathcal{B}} \oplus (M^{h\mathcal{B}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{A}} \otimes_{\mathcal{A}} M) \xrightarrow{\text{act} \oplus (\text{tr} \otimes \text{Id})} M^{h\mathcal{A}} \otimes_{\mathcal{A}} M \right) \\ \rightarrow \tilde{\mathcal{B}} \xrightarrow{\text{deg.0}} \end{array} \tag{5.15}$$

of twisted complexes over $\mathcal{B}\text{-Mod-}\mathcal{B}$ given, respectively, by

$$\tilde{\mathcal{B}} \xrightarrow{\text{Id} \oplus \text{act}} \tilde{\mathcal{B}} \oplus (M^{h\mathcal{A}} \otimes_{\mathcal{A}} M) \xrightarrow{\text{Id} \oplus (\text{Id} \otimes \text{act} \otimes \text{Id})} \tilde{\mathcal{B}} \oplus (M^{h\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{B}} \otimes_{\mathcal{A}} M), \tag{5.16}$$

$$\tilde{\mathcal{B}} \xrightarrow{-(\text{Id} \otimes \chi_{\mathcal{B}}) \circ \text{act}} M^{h\mathcal{A}} \otimes_{\mathcal{A}} M, \tag{5.17}$$

and

$$\bar{\mathcal{B}} \oplus (M^{h\mathcal{B}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{A}} \otimes_{\mathcal{A}} M) \xrightarrow{\text{Id} \oplus (\text{Id} \otimes \text{tr} \otimes \text{Id})} \bar{\mathcal{B}} \oplus (M^{\mathcal{B}} \otimes_{\mathcal{A}} M) \xrightarrow{\text{Id} \oplus \text{tr}} \bar{\mathcal{B}}, \quad (5.18)$$

$$M^{h\mathcal{B}} \otimes_{\mathcal{A}} M \xrightarrow{-\text{tr} \circ (\text{Id} \otimes \chi_{\mathcal{A}})} \bar{\mathcal{B}}. \quad (5.19)$$

Proof. We treat the case of the adjunction unit, the case of the counit is treated identically. It suffices to show that (5.14) is isomorphic in $D(\mathcal{B}\text{-}\mathcal{B})$ to $\mathcal{B} \xrightarrow{\text{act}} TT^{l\bar{\mathcal{B}}}$. The latter is isomorphic to

$$\mathcal{B} \xrightarrow{\text{act}} \text{Hom}_{\mathcal{B}}(\{M^{\mathcal{B}} \otimes_{\mathcal{A}} M \xrightarrow{\text{tr}}_{\text{deg.0}} \mathcal{B}\}, \{M^{\mathcal{B}} \otimes_{\mathcal{A}} M \xrightarrow{\text{tr}}_{\text{deg.0}} \mathcal{B}\}) \quad (5.20)$$

since $\{M^{\mathcal{B}} \otimes_{\mathcal{A}} M \xrightarrow{\text{tr}}_{\text{deg.0}} \mathcal{B}\}$ is a left \mathcal{B} - h -projective bimodule homotopically equivalent to T .

By the commutativity of (2.54), the composition of the \mathcal{B} -action map with the quasi-isomorphism

$$\text{Hom}_{\mathcal{A}}(M, M) \rightarrow \text{Hom}_{\mathcal{A}}(M, M^{\mathcal{B}\mathcal{B}}) \xrightarrow{\text{adjunction}} (M^{\mathcal{B}} \otimes_{\mathcal{A}} M)^{l\mathcal{B}} \quad (5.21)$$

is the left dual of the \mathcal{B} -trace map. The following is a chain of quasi-isomorphisms of twisted complexes:

$$\begin{array}{ccccc} M^{h\mathcal{B}} \otimes_{\mathcal{A}} M & \xrightarrow{\text{tr} \oplus (\text{act} \otimes \text{Id})} & \bar{\mathcal{B}} \oplus (M^{h\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{B}} \otimes_{\mathcal{A}} M) & \xrightarrow{\text{act} \oplus (-\text{Id} \otimes \text{tr})} & M^{h\mathcal{A}} \otimes_{\mathcal{A}} M \\ \tau \downarrow & \dashrightarrow^{0 \oplus (\theta_{\mathcal{B}} \otimes \tau)} & \downarrow^{\text{deg.0}} & \dashrightarrow^{-\theta_{\mathcal{B}} \oplus 0} & \downarrow^{\text{ev} \circ \tau} \\ M^{\mathcal{B}} \otimes_{\mathcal{A}} M & \xrightarrow{\text{tr} \oplus (\text{act} \otimes \text{Id})} & \mathcal{B} \oplus (\text{Hom}_{\mathcal{A}}(M, M) \otimes_{\mathcal{B}} M^{\mathcal{B}} \otimes_{\mathcal{A}} M) & \xrightarrow{\text{act} \oplus (-\text{Id} \otimes \text{tr})} & \text{Hom}_{\mathcal{A}}(M, M) \\ \text{Id} \downarrow & & \downarrow^{\text{tr} \oplus (\text{ev} \otimes \text{Id}) \circ \tau} & & \downarrow^{\text{ev} \circ \tau} \\ M^{\mathcal{B}} \otimes_{\mathcal{A}} M & \xrightarrow{\text{tr} \oplus (\text{tr}^{l\mathcal{B}} \otimes \text{Id})} & \mathcal{B} \oplus ((M^{\mathcal{B}} \otimes_{\mathcal{A}} M)^{l\mathcal{B}} \otimes_{\mathcal{B}} M^{\mathcal{B}} \otimes_{\mathcal{A}} M) & \xrightarrow{\text{tr}^{l\mathcal{B}} \oplus (-\text{Id} \otimes \text{tr})} & (M^{\mathcal{B}} \otimes_{\mathcal{A}} M)^{l\mathcal{B}} \\ \downarrow & & \downarrow^{\text{Id} \oplus (5.21)} & & \downarrow^{(5.21)} \\ \text{Hom}_{\mathcal{B}}(\mathcal{B}, M^{\mathcal{B}} \otimes_{\mathcal{A}} M) & \xrightarrow{\text{tr} \oplus (-) \oplus (-) \circ \text{tr}} & \text{Hom}_{\mathcal{B}}(\mathcal{B}, \mathcal{B}) \oplus \text{Hom}_{\mathcal{B}}(M^{\mathcal{B}} \otimes_{\mathcal{A}} M, M^{\mathcal{B}} \otimes_{\mathcal{A}} M) & \xrightarrow{(-) \circ \text{tr} \oplus -\text{tr} \circ (-)} & \text{Hom}_{\mathcal{B}}(M^{\mathcal{B}} \otimes_{\mathcal{A}} M, \mathcal{B}) \end{array} \quad (5.22)$$

By Lemma 3.5(2) the map (5.20) is isomorphic to the map

$$\begin{array}{ccc} \mathcal{B} & & \\ \downarrow^{\text{act} \oplus \text{act}} & & \\ \text{Hom}_{\mathcal{B}}(\mathcal{B}, M^{\mathcal{B}} \otimes_{\mathcal{A}} M) & \xrightarrow{\text{tr} \oplus (-) \oplus (-) \circ \text{tr}} & \text{Hom}_{\mathcal{B}}(\mathcal{B}, \mathcal{B}) \oplus \text{Hom}_{\mathcal{B}}(M^{\mathcal{B}} \otimes_{\mathcal{A}} M, M^{\mathcal{B}} \otimes_{\mathcal{A}} M) \xrightarrow{(-) \circ \text{tr} \oplus -\text{tr} \circ (-)} \text{Hom}_{\mathcal{B}}(M^{\mathcal{B}} \otimes_{\mathcal{A}} M, \mathcal{B}) \end{array} \quad (5.23)$$

To show that (5.14) is isomorphic in $D(\mathcal{B}\text{-}\mathcal{B})$ to (5.23), it now suffices to show that (5.22) \circ (5.14) is homotopic to (5.23) \circ τ . It is a routine check of the kind we normally

leave to the reader, but we write it out in detail just this once to give the flavor of the computations involved.

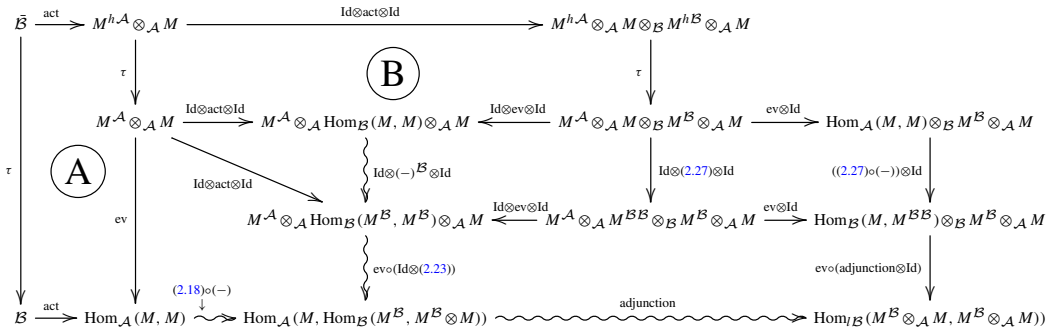
The composition of (5.14) with (5.22) is the map

$$\{ \text{Hom}_{\mathcal{B}}(\mathcal{B}, M^{\mathcal{B}} \otimes_{\mathcal{A}} M) \} \xrightarrow{\text{tr} \circ (-) \oplus (-) \circ \text{tr}} \text{Hom}_{\mathcal{B}}(\mathcal{B}, \mathcal{B}) \oplus \text{Hom}_{\mathcal{B}}(M^{\mathcal{B}} \otimes_{\mathcal{A}} M, M^{\mathcal{B}} \otimes_{\mathcal{A}} M) \xrightarrow{(-) \circ \text{tr} \oplus -\text{tr} \circ (-)} \text{Hom}_{\mathcal{B}}(M^{\mathcal{B}} \otimes_{\mathcal{A}} M, \mathcal{B}) \} \quad (5.24)$$

where

$$\alpha_1 = (\bar{\mathcal{B}} \xrightarrow{-\theta_{\mathcal{B}}} \text{Hom}_{\mathcal{A}}(M, M) \xrightarrow{(5.21)} \text{Hom}_{\mathcal{B}}(M^{\mathcal{B}} \otimes_{\mathcal{A}} M, \mathcal{B})) + (\bar{\mathcal{B}} \xrightarrow{\text{act}} M^{h\mathcal{A}} \otimes_{\mathcal{A}} M \xrightarrow{-\text{Id} \otimes \chi_{\mathcal{B}}} M^{h\mathcal{A}} \otimes_{\mathcal{A}} M \xrightarrow{(5.21) \circ \text{ev} \circ \tau} \text{Hom}_{\mathcal{B}}(M^{\mathcal{B}} \otimes_{\mathcal{A}} M, \mathcal{B}))$$

and the composition α_0 can be computed by considering the following diagram:



This diagram commutes except for the sections marked (A) and (B). These commute up to $d\theta_{\mathcal{B}}$ and $\tau \otimes d\theta_{\mathcal{A}} \otimes \text{Id}$, respectively. The upper right border of this diagram composes to α_0 , while its bottom line composes to $\mathcal{B} \xrightarrow{\text{act}} \text{Hom}_{\mathcal{B}}(M^{\mathcal{B}} \otimes_{\mathcal{A}} M, M^{\mathcal{B}} \otimes_{\mathcal{A}} M)$. It follows that

$$\alpha_0 = \text{act} \circ \tau + d(\beta_1 \circ \theta_{\mathcal{B}} + \beta_2 \circ (\tau \otimes \theta_{\mathcal{A}} \otimes \text{Id}) \circ \text{act})$$

where β_1 and β_2 are the corresponding compositions of the wavy arrows in the diagram.

Thus (5.24) is the sum of (5.23) $\circ \tau$ and the map

$$\{ \text{Hom}_{\mathcal{B}}(\mathcal{B}, M^{\mathcal{B}} \otimes_{\mathcal{A}} M) \} \xrightarrow{\text{tr} \circ (-) \oplus (-) \circ \text{tr}} \text{Hom}_{\mathcal{B}}(\mathcal{B}, \mathcal{B}) \oplus \text{Hom}_{\mathcal{B}}(M^{\mathcal{B}} \otimes_{\mathcal{A}} M, M^{\mathcal{B}} \otimes_{\mathcal{A}} M) \xrightarrow{(-) \circ \text{tr} \oplus -\text{tr} \circ (-)} \text{Hom}_{\mathcal{B}}(M^{\mathcal{B}} \otimes_{\mathcal{A}} M, \mathcal{B}) \} \quad (5.25)$$

and it remains to show that (5.25) is a boundary.

It suffices to show that

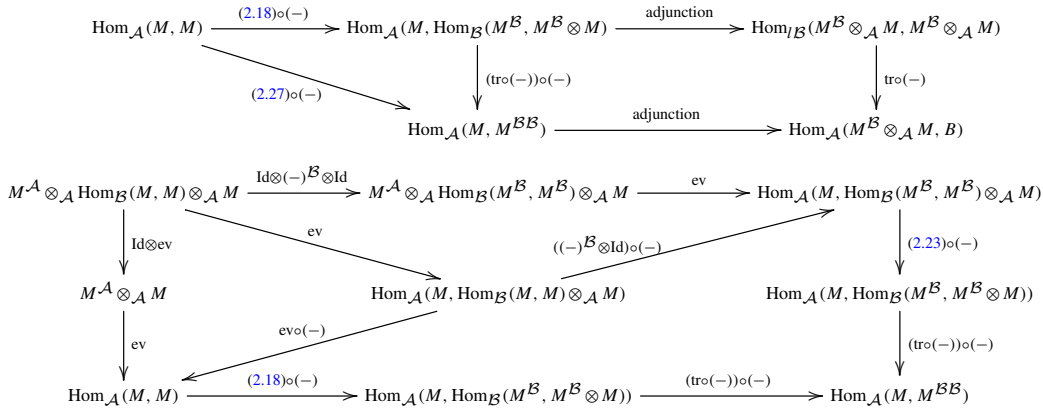
$$\alpha_1 = -(\text{tr} \circ (-)) \circ (\beta_1 \circ \theta_{\mathcal{B}} + \beta_2 \circ (\tau \otimes \theta_{\mathcal{A}} \otimes \text{Id}) \circ \text{act}).$$

By definition of α_1 and χ_B , this would follow from

$$(5.21) \circ \theta_B = (\text{tr} \circ (-)) \circ \beta_1 \circ \theta_B,$$

$$(5.21) \circ \text{ev} \circ (\text{Id} \otimes \text{ev}) \circ (\tau \otimes \theta_A \otimes \text{Id}) \circ \text{act} = (\text{tr} \circ (-)) \circ \beta_2 \circ (\tau \otimes \theta_A \otimes \text{Id}) \circ \text{act}.$$

In fact, a stronger statement is true: (5.21) = (tr ◦ (-)) ◦ β₁ and (tr ◦ (-)) ◦ β₁ ◦ ev ◦ (Id ⊗ ev) = (tr ◦ (-)) ◦ β₂. It is equivalent to the commutativity of the following two diagrams:



which is readily checked. □

Let $\mathcal{A} \xrightarrow{\text{act}} FF'$ and $F'F \xrightarrow{\text{tr}} \mathcal{A}$ be the maps in $D(\mathcal{A}\text{-}\mathcal{A})$ which the isomorphism $(F')^r \bar{\mathcal{A}} \simeq F$ of Prop. 5.3 identifies with the derived action and trace maps for F . The following proposition is proved in the same way:

Proposition 5.5. *The maps $\mathcal{A} \xrightarrow{\text{act}} FF'$ and $F'F \xrightarrow{\text{tr}} \mathcal{A}$ are isomorphic in $D(\mathcal{A}\text{-}\mathcal{A})$ to the maps*

$$\begin{aligned} \bar{\mathcal{A}}_{\text{deg.0}} &\rightarrow \\ \left(M \otimes_B M^{h\mathcal{A}} \xrightarrow{\text{tr} \oplus (-\text{Id} \otimes \text{act})} \bar{\mathcal{A}} \oplus (M \otimes_B M^{h\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_B M^{hB}) \xrightarrow{\text{act} \oplus (\text{tr} \otimes \text{Id})} M \otimes_B M^{hB} \right), \end{aligned} \tag{5.26}$$

$$\begin{aligned} \left(M \otimes_B M^{h\mathcal{A}} \xrightarrow{\text{tr} \oplus (\text{act} \otimes \text{Id})} \bar{\mathcal{A}} \oplus (M \otimes_B M^{hB} \otimes_{\mathcal{A}} M \otimes_B M^{h\mathcal{A}}) \xrightarrow{\text{act} \oplus (-\text{Id} \otimes \text{tr})} M \otimes_B M^{hB} \right) \\ \rightarrow \bar{\mathcal{A}}_{\text{deg.0}} \end{aligned} \tag{5.27}$$

of twisted complexes over $\mathcal{A}\text{-Mod-}\mathcal{A}$ given, respectively, by

$$\bar{\mathcal{A}} \xrightarrow{\text{Id} \oplus \text{act}} \bar{\mathcal{A}} \oplus (M \otimes_B M^{hB}) \xrightarrow{\text{Id} \oplus (-\text{Id} \otimes \text{act} \otimes \text{Id})} \bar{\mathcal{A}} \oplus (M \otimes_B M^{h\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_B M^{hB}), \tag{5.28}$$

$$\bar{\mathcal{A}} \xrightarrow{-(\chi_{\mathcal{A}} \otimes \text{Id}) \circ \text{act}} M \otimes_B M^{hB}, \tag{5.29}$$

and

$$\bar{A} \oplus (M \otimes_B M^{hB} \otimes_A M \otimes_B M^{hA}) \xrightarrow{\text{Id} \oplus -(\text{Id} \otimes \text{tr} \otimes \text{Id})} \bar{A} \oplus (M \otimes_B M^{hA}) \xrightarrow{\text{Id} \oplus \text{tr}} \bar{A}, \tag{5.30}$$

$$M \otimes_B M^{hA} \xrightarrow{-\text{tr} \circ (\chi_B \otimes \text{Id})} \bar{A}. \tag{5.31}$$

Consider the twisted 2-cube over $\mathcal{B}\text{-Mod-}\mathcal{A}$

$$\begin{array}{ccc} 0 & \longrightarrow & M^{hA} \\ \downarrow & & \downarrow \text{Id} \otimes \text{act} \\ M^{hB} & \xrightarrow{\text{act} \otimes \text{Id}} & M^{hA} \otimes_A M \otimes_B M^{hB} \end{array} \tag{5.32}$$

By the Cube Lemma (Lemma 3.6) the convolutions of the rows of (5.32) fit into a 1-cube (i.e. a single morphism) whose convolution is the convolution of the total complex of the 2-cube. And similarly for the convolutions of the columns of (5.32). This is formalized in the Cube Completion Lemma (Lemma 3.7) which constructs for us the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M^{hA} & \longrightarrow & M^{hA} & \xrightarrow{[1]} & \longrightarrow \\ \downarrow & & \downarrow \text{Id} \otimes \text{act} & & \downarrow & & \\ M^{hB} & \xrightarrow{\text{act} \otimes \text{Id}} & M^{hA} \otimes_A M \otimes_B M^{hB} & \longrightarrow & \{M^{hB} \rightarrow M^{hA} \otimes_{\text{deg.0}} M \otimes M^{hB}\} & \xrightarrow{[1]} & \longrightarrow \\ \downarrow & & \downarrow & & \downarrow & & \\ M^{hB} & \longrightarrow & \{M^{hA} \rightarrow M^{hA} \otimes_{\text{deg.0}} M \otimes M^{hB}\} & \longrightarrow & \{M^{hA} \oplus M^{hB} \rightarrow M^{hA} \otimes_{\text{deg.0}} M \otimes M^{hB}\} & \xrightarrow{[1]} & \longrightarrow \\ \downarrow [1] & & \downarrow [1] & & \downarrow [1] & & \\ & & & & & & \end{array} \tag{5.33}$$

in $\mathcal{B}\text{-Mod-}\mathcal{A}$. The morphisms marked [1] are morphisms of degree 1 which “wrap around” to the beginning of the corresponding row or column. We have not labeled all the maps within twisted complexes or the morphisms between their convolutions in (5.33), but the precise formulas can be found in Lemma 3.7.

Let now Q be the convolution of the 2-cube (5.32) shifted by one to the right, that is,

$$\begin{aligned} Q &:= \{M^{hA} \oplus_{\text{deg.0}} M^{hB} \xrightarrow{-\text{Id} \otimes \text{act} - \text{act} \otimes \text{Id}} M^{hA} \otimes_A M \otimes_B M^{hB}\} \\ &\simeq \text{Cone}(R \oplus L \xrightarrow{\text{Id} \otimes \text{act} + \text{act} \otimes \text{Id}} RSL)[-1] \quad \text{in } D(\mathcal{B}\text{-}\mathcal{A}). \end{aligned}$$

The diagram (5.33) descends to a commutative 3×3 diagram in $D(\mathcal{B}\text{-}\mathcal{A})$ whose rows and columns are exact:

$$\begin{array}{ccccc}
 0 & \longrightarrow & L & \xlongequal{\quad} & L \\
 \downarrow & & \downarrow (\text{act})L & & \downarrow \alpha' \\
 R & \xrightarrow{R(\text{act})} & RSL & \xrightarrow{\zeta} & RT'[1] \\
 \parallel & & \downarrow \eta & & \downarrow \\
 R & \xrightarrow{\alpha} & FL[1] & \longrightarrow & Q[1]
 \end{array} \tag{5.34}$$

The connecting morphisms for the exact triangles are the images of the morphisms labeled [1] in (5.33).

Lemma 5.6. *The following are equivalent:*

- $r \xrightarrow{(5.12)} fl[1]$ is an isomorphism (condition (4) of Theorem 5.1).
- $r \oplus l \xrightarrow{r(\text{unit}) \oplus \text{unit}} rsl$ is an isomorphism.
- $Q \simeq 0$ in $D(\mathcal{B}\text{-}\mathcal{A})$.
- α is an isomorphism in $D(\mathcal{B}\text{-}\mathcal{A})$.
- α' is an isomorphism in $D(\mathcal{B}\text{-}\mathcal{A})$.

Proof. Denote by q the functor $(-)\overset{L}{\otimes} Q$ from $D(\mathcal{B})$ to $D(\mathcal{A})$. The morphisms $R \rightarrow RSL$ and $L \rightarrow RSL$ in (5.34) induce the natural transformations $r \xrightarrow{r(\text{unit})} rsl$ and $l \xrightarrow{\text{unit}} rsl$. Hence $R \xrightarrow{\alpha} FL[1]$ induces the natural transformation (5.12). The functorial exact triangle $r \xrightarrow{(5.12)} fl[1] \rightarrow q[1]$ induced by the bottom row of (5.34) implies that $r \xrightarrow{(5.12)} fl[1]$ is an isomorphism if and only if q is the zero functor. Similarly, the exact triangle $R \oplus L \xrightarrow{\text{Id} \otimes \text{act} \oplus \text{act} \otimes \text{Id}} RSL \rightarrow Q[1]$ implies that $r \oplus l \xrightarrow{r(\text{unit}) \oplus \text{unit}} rsl$ is an isomorphism if and only if $q[1]$ is the zero functor.

Clearly $Q \simeq 0$ implies that q is the zero functor. On the other hand, if q is the zero functor then it sends all representable \mathcal{B} -modules to $0 \in D(\mathcal{A})$. Thus ${}_bQ$ is an acyclic \mathcal{A} -module for all $b \in \mathcal{B}$, and hence Q is acyclic. We conclude that q is the zero functor if and only if $Q \simeq 0$ in $D(\mathcal{B}\text{-}\mathcal{A})$.

Finally, $Q \simeq 0$ is equivalent to α (resp. α') being an isomorphism by exactness of the bottom row (resp. right column) of the diagram (5.34). \square

Now define

$$\begin{aligned}
 Q' &:= \{M^{h\mathcal{B}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{A}} \xrightarrow{-\text{tr} \otimes \text{Id} - \text{Id} \otimes \text{tr}} M^{h\mathcal{A}} \oplus_{\text{deg.0}} M^{h\mathcal{B}}\} \\
 &\simeq \text{Cone}(LSR \xrightarrow{L\text{tr} \oplus \text{tr}R} L \oplus R)[1] \quad \text{in } D(\mathcal{B}\text{-}\mathcal{A}).
 \end{aligned}$$

Then, in a similar way, the twisted 2-cube

$$\begin{array}{ccc}
 M^{h\mathcal{B}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{A}} & \xrightarrow{\text{tr} \otimes \text{Id}} & M^{h\mathcal{A}} \\
 \text{Id} \otimes \text{tr} \downarrow & & \downarrow \\
 M^{h\mathcal{B}} & \longrightarrow & 0
 \end{array} \tag{5.35}$$

produces the following 3×3 diagram in $D(\mathcal{B}\text{-}\mathcal{A})$ whose rows and columns are exact triangles:

$$\begin{array}{ccccc}
 Q'[-1] & \longrightarrow & F'R[-1] & \xrightarrow{\beta'} & L \\
 \downarrow & & \downarrow & & \parallel \\
 LT[-1] & \longrightarrow & LSR & \xrightarrow{L(\text{tr})} & L \\
 \downarrow \beta & & \downarrow (\text{tr})R & & \downarrow \\
 R & \xlongequal{\quad} & R & \longrightarrow & 0
 \end{array} \tag{5.36}$$

Arguing as in the proof of Lemma 5.6 we obtain:

Lemma 5.7. *The following are equivalent:*

- $lt[-1] \xrightarrow{(5.11)} r$ is an isomorphism (condition (3) of Theorem 5.1).
- $lsr \xrightarrow{l(\text{counit}) \oplus \text{counit}} l \oplus r$ is an isomorphism.
- $Q' \simeq 0$ in $D(\mathcal{B}\text{-}\mathcal{A})$.
- β is an isomorphism in $D(\mathcal{B}\text{-}\mathcal{A})$.
- β' is an isomorphism in $D(\mathcal{B}\text{-}\mathcal{A})$.

Consider now the twisted 2-cube over $\text{Pre-Tr}(\mathcal{B}\text{-}\mathbf{Mod}\text{-}\mathcal{B})$:

$$\begin{array}{ccc}
 M^{h\mathcal{A}} \otimes_{\mathcal{A}} M & \xrightarrow{\text{Id}} & M^{h\mathcal{A}} \otimes_{\mathcal{A}} M \\
 \downarrow 1 \rightarrow 1: \text{Id} \otimes \text{act} \otimes \text{Id} & \dashrightarrow 1 \rightarrow 1: -\text{Id} \otimes \chi_{\mathcal{B}} & \downarrow 1 \rightarrow 1: \text{Id} \\
 (M^{h\mathcal{B}} \otimes_{\mathcal{A}} M \xrightarrow{-\text{act} \otimes \text{Id}} M^{h\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{B}} \otimes_{\mathcal{A}} M) & \xrightarrow[1 \rightarrow 1: -\text{Id} \otimes \text{tr}]{0 \rightarrow 0: \text{tr}} & (\bar{\mathcal{B}} \xrightarrow{\text{act}} M^{h\mathcal{A}} \otimes_{\mathcal{A}} M)
 \end{array} \tag{5.37}$$

A priori the total complex of a face of a twisted cube over $\text{Pre-Tr}(\mathcal{B}\text{-}\mathbf{Mod}\text{-}\mathcal{B})$ is an object of $\text{Pre-Tr Pre-Tr}(\mathcal{B}\text{-}\mathbf{Mod}\text{-}\mathcal{B})$. However, there is a canonical equivalence

$$\text{Pre-Tr Pre-Tr}(\mathcal{B}\text{-}\mathbf{Mod}\text{-}\mathcal{B}) \xrightarrow{\sim} \text{Pre-Tr}(\mathcal{B}\text{-}\mathbf{Mod}\text{-}\mathcal{B})$$

(see [BK90, §2]). We implicitly use this equivalence wherever possible.

The Cube Completion Lemma constructs from the 2-cube (5.37) a 3×3 commutative diagram in $D(\mathcal{B}\text{-}\mathcal{B})$ whose rows and columns are exact. We now compute this diagram.

The left column of (5.37) is the image under $(-)\otimes_{\mathcal{A}} M[-1]$ of the first map in the right column of (5.33). It descends to the morphism $SL[-1] \xrightarrow{S\alpha'} SRT'$ in (5.34) in $D(\mathcal{B}\text{-}\mathcal{B})$ and its convolution is isomorphic to SQ .

The bimodule $\bar{\mathcal{B}}$ is homotopy equivalent to the total complex of the right column of (5.37):

$$(\bar{\mathcal{B}})_{\text{deg.0}} \xrightarrow{0 \rightarrow 0: \text{Id} \oplus - \text{act}} (\bar{\mathcal{B}} \oplus M^{h\mathcal{A}} \otimes_{\mathcal{A}} M)_{\text{deg.0}} \xrightarrow{\text{act} \oplus \text{Id}} M^{h\mathcal{A}} \otimes_{\mathcal{A}} M. \tag{5.38}$$

The total complex of the top row of (5.37) is the null-homotopic twisted complex $(M^{hA} \otimes_{\mathcal{A}} M \xrightarrow[\text{deg.0}]{\text{Id}} M^{hA} \otimes_{\mathcal{A}} M)$, while the total complex of the bottom row is the twisted complex

$$M^{hB} \otimes_{\mathcal{A}} M \xrightarrow{\text{tr} \oplus (\text{act} \otimes \text{Id})} \bar{\mathcal{B}} \oplus (M^{hA} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{hB} \otimes_{\mathcal{A}} M) \xrightarrow[\text{deg.0}]{\text{act} \oplus (-\text{Id} \otimes \text{tr})} M^{hA} \otimes_{\mathcal{A}} M \tag{5.39}$$

which we have shown in Prop. 5.4 to convolve to TT' .

By the Cube Lemma, the total complex of the whole 2-cube equals the total complex of the 1-cube constructed from its rows. It is then clear that the total complex of (5.37) is homotopy equivalent to (5.39):

$$\begin{array}{ccccc}
 (M^{hA} \otimes_{\mathcal{A}} M \xrightarrow[\text{deg.0}]{-\text{Id}} M^{hA} \otimes_{\mathcal{A}} M) & \xrightarrow[\text{deg.0}]{\begin{array}{l} 0 \rightarrow 0: 0 \oplus \text{Id} \otimes \text{act} \otimes \text{Id} \\ 0 \rightarrow 1: -\text{Id} \otimes \chi_{\mathcal{B}}, 1 \rightarrow 1: \text{Id} \end{array}} & (M^{hB} \otimes_{\mathcal{A}} M \xrightarrow[\text{deg.0}]{\text{tr} \oplus (\text{act} \otimes \text{Id})} \bar{\mathcal{B}} \oplus (M^{hA} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{hB} \otimes_{\mathcal{A}} M) & \xrightarrow[\text{deg.0}]{\text{act} \oplus (-\text{Id} \otimes \text{tr})} & M^{hA} \otimes_{\mathcal{A}} M) \\
 & \searrow \text{dashed} & & \downarrow \text{Id} & \\
 & & (M^{hB} \otimes_{\mathcal{A}} M \xrightarrow[\text{deg.0}]{\text{tr} \oplus (\text{act} \otimes \text{Id})} \bar{\mathcal{B}} \oplus (M^{hA} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{hB} \otimes_{\mathcal{A}} M) & \xrightarrow[\text{deg.0}]{\text{act} \oplus (-\text{Id} \otimes \text{tr})} & M^{hA} \otimes_{\mathcal{A}} M) \\
 & & \uparrow & & \uparrow \\
 & & (M^{hB} \otimes_{\mathcal{A}} M \xrightarrow[\text{deg.0}]{-\text{Id}} M^{hB} \otimes_{\mathcal{A}} M) & &
 \end{array}
 \tag{5.40}$$

Consider the map which the Cube Lemma constructs from the total complex of the right column of (5.37) to the total complex of the whole 2-cube. It composes with the homotopy equivalences (5.38) and (5.40) to give the map (5.14). The latter was proven in Prop. 5.4 to be isomorphic in $D(\mathcal{B}\text{-}\mathcal{B})$ to $\mathcal{B} \xrightarrow{\text{act}} TT'$.

Putting together all of the above, we see that the diagram constructed by the Cube Completion Lemma from (5.37) is isomorphic in $D(\mathcal{B}\text{-}\mathcal{B})$ to

$$\begin{array}{ccccc}
 SL[-1] & \xlongequal{\quad} & SL[-1] & \longrightarrow & 0 \\
 S(\alpha') \downarrow & & \downarrow & & \downarrow \\
 SRT' & \longrightarrow & T' & \longrightarrow & TT' \\
 \downarrow & & \downarrow & & \parallel \\
 SQ & \longrightarrow & \mathcal{B} & \xrightarrow{\text{act}} & TT'
 \end{array}
 \tag{5.41}$$

Similarly, the following twisted 2-cube over $\text{Pre-Tr}(\mathcal{B}\text{-}\mathbf{Mod}\text{-}\mathcal{B})$:

$$\begin{array}{ccccc}
 (M^{hB} \otimes_{\mathcal{A}} M \xrightarrow[\text{deg.0}]{\text{tr}} \bar{\mathcal{B}}) & \xrightarrow[\text{deg.0}]{\begin{array}{l} -1 \rightarrow -1: -\text{Id} \otimes \text{act} \\ 0 \rightarrow 0: \text{act} \end{array}} & (M^{hB} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{hA} \otimes_{\mathcal{A}} M) & \xrightarrow[\text{deg.0}]{-\text{tr} \otimes \text{Id}} & M^{hA} \otimes_{\mathcal{A}} M) \\
 -1 \rightarrow -1: \text{Id} \downarrow & \searrow \text{dashed} & & \downarrow & -1 \rightarrow -1: \text{Id} \otimes \text{tr} \otimes \text{Id} \\
 M^{hB} \otimes_{\mathcal{A}} M & \xrightarrow{\text{Id}} & M^{hB} \otimes_{\mathcal{A}} M & &
 \end{array}
 \tag{5.42}$$

produces the following diagram in $D(\mathcal{B}\text{-}\mathcal{B})$ with exact rows and columns:

$$\begin{array}{ccccc}
 T'T & \xrightarrow{\text{tr}} & \mathcal{B} & \longrightarrow & SQ' \\
 \parallel & & \downarrow & & \downarrow \\
 T'T & \longrightarrow & T & \longrightarrow & SLT \\
 \downarrow & & \downarrow & & \downarrow^{S(\beta)} \\
 0 & \longrightarrow & SR[1] & \equiv & SR[1]
 \end{array} \tag{5.43}$$

Similarly, we incorporate the maps $\mathcal{A} \xrightarrow{\text{act}} FF'$ and $F'F \xrightarrow{\text{tr}} \mathcal{A}$ into the following two 3×3 diagrams in $D(\mathcal{A}\text{-}\mathcal{A})$ with exact rows and columns:

$$\begin{array}{ccccc}
 RS[-1] & \equiv & RS[-1] & \longrightarrow & 0 \\
 (\alpha)S \downarrow & & \downarrow & & \downarrow \\
 FLS & \longrightarrow & F & \longrightarrow & FF' \\
 \downarrow & & \downarrow & & \parallel \\
 QS & \longrightarrow & \mathcal{A} & \xrightarrow{\text{act}} & FF'
 \end{array} \tag{5.44}$$

$$\begin{array}{ccccc}
 F'F & \xrightarrow{\text{tr}} & \mathcal{A} & \longrightarrow & Q'S \\
 \parallel & & \downarrow & & \downarrow \\
 F'F & \longrightarrow & F' & \longrightarrow & F'RS \\
 \downarrow & & \downarrow & & \downarrow^{(\beta')S} \\
 0 & \longrightarrow & LS[1] & \equiv & LS[1]
 \end{array} \tag{5.45}$$

We obtain immediately:

Proposition 5.8.

- (1) If the natural transformation $lt[-1] \xrightarrow{(5.11)} r$ is an isomorphism (condition (3) of Theorem 5.1) then the adjunction counits $t't \rightarrow \text{Id}$ and $f'f \rightarrow \text{Id}$ are isomorphisms.
- (2) If the natural transformation $r \xrightarrow{(5.12)} fl[1]$ is an isomorphism (condition (4) of Theorem 5.1) then the adjunction units $\text{Id} \rightarrow tt'$ and $\text{Id} \rightarrow ff'$ are isomorphisms.

Proof. We only prove the first claim. By Lemma 5.7 condition (3) of Theorem 5.1 is equivalent to $Q \simeq 0$ in $D(\mathcal{B}\text{-}\mathcal{A})$. Therefore $SQ \simeq 0$ in $D(\mathcal{B}\text{-}\mathcal{B})$, and since the bottom row of (5.41) is exact, $\mathcal{B} \xrightarrow{\text{act}} TT'$ is an isomorphism. Thus $\text{Id} \xrightarrow{\text{unit}} tt'$ is an isomorphism. Similarly, $QS \simeq 0$ in $D(\mathcal{A}\text{-}\mathcal{A})$, and by the exactness of the bottom row of (5.44) the map $\mathcal{A} \xrightarrow{\text{act}} FF'$ is an isomorphism. Hence $\text{Id} \xrightarrow{\text{unit}} ff'$ is also an isomorphism. \square

Lemma 5.9. *Let α, α', β and β' be as in diagrams (5.34) and (5.36). Then:*

- (1) *The composition $R \xrightarrow{(\text{act})R} FF'R \xrightarrow{F\beta'} FL[1]$ is the map α .*
- (2) *The composition $LT[-1] \xrightarrow{\alpha'T} RT'T \xrightarrow{R(\text{tr})} R$ is the map β .*
- (3) *The composition $L \xrightarrow{L(\text{act})} LTT' \xrightarrow{\beta T'} RT'[1]$ is the map α' .*
- (4) *The composition $F'R[-1] \xrightarrow{F'\alpha} F'FL \xrightarrow{(\text{tr})L} L$ is the map β' .*

Proof. We only prove the first claim, the other three are proved analogously. Note also that throughout the proof we omit labeling the internal twisted maps inside twisted complexes, since they are not relevant to our argument. The results we quote before stating each twisted complex identify these maps explicitly.

By construction of (5.34) the map $R \xrightarrow{\alpha} FL[1]$ in $D(\mathcal{B}\text{-}\mathcal{A})$ descends from the map of twisted complexes

$$\begin{array}{ccc}
 & & (M^{h\mathcal{B}}) \\
 & & \downarrow \text{act} \otimes \text{Id} \\
 (M^{h\mathcal{A}} & \longrightarrow & M^{h\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{B}} \\
 & & \text{deg.0}
 \end{array} \tag{5.46}$$

By Prop. 5.5 the map $R \xrightarrow{(\text{act})R} FF'R$ descends from the map of twisted complexes

$$\begin{array}{ccc}
 & & (M^{h\mathcal{B}}) \\
 & & \downarrow \text{Id} \oplus (\text{Id} \otimes \text{act}) \\
 & & M^{h\mathcal{B}} \oplus (M^{h\mathcal{B}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{B}}) \\
 & & \downarrow \text{Id} \oplus (-\text{Id} \otimes \text{act} \otimes \text{Id}) \\
 (M^{h\mathcal{B}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{A}} & \longrightarrow & M^{h\mathcal{B}} \oplus (M^{h\mathcal{B}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{B}}) \longrightarrow M^{h\mathcal{B}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{B}} \\
 & & \text{deg.0}
 \end{array} \tag{5.47}$$

Finally, $FF'R \xrightarrow{F\beta'} FL[1]$ descends from the map of twisted complexes which is computed as follows.

By construction of the diagram (5.36) the map $F'R \xrightarrow{\beta'} L[1]$ descends from

$$\begin{array}{ccc}
 (M^{h\mathcal{B}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{A}} & \longrightarrow & M^{h\mathcal{A}} \\
 \downarrow -\text{tr} \otimes \text{Id} & & \downarrow \\
 (M^{h\mathcal{A}}) & & \\
 \text{deg.-1} & &
 \end{array} \tag{5.48}$$

On the other hand, F is the convolution of $(\bar{\mathcal{A}} \xrightarrow[\text{deg.0}]{\text{act}} M \otimes_{\mathcal{B}} M^{h\mathcal{B}})$. Thus the map $FF'R \xrightarrow{F\beta'} FL[1]$ is

$$\{M^{h\mathcal{B}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{A}} \xrightarrow{\text{Id} \otimes \text{tr}} M^{h\mathcal{B}}\}_{\text{deg.0}} \otimes \{ \bar{\mathcal{A}} \xrightarrow[\text{deg.0}]{\text{act}} M \otimes_{\mathcal{B}} M^{h\mathcal{B}} \} \xrightarrow{(5.48) \otimes \text{Id}} M^{h\mathcal{A}}_{\text{deg.-1}} \otimes \{ \bar{\mathcal{A}} \xrightarrow[\text{deg.0}]{\text{act}} M \otimes_{\mathcal{B}} M^{h\mathcal{B}} \}.$$

Lemma 3.4 tells us how to take tensor product of twisted complexes in a way compatible with convolutions. It follows from it that $FF'R \xrightarrow{F\beta'} FL[1]$ descends from the map

$$\begin{array}{ccc} (M^{h\mathcal{B}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{A}} \xrightarrow{\text{Id} \otimes \text{tr}} M^{h\mathcal{B}} \oplus (M^{h\mathcal{B}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{B}}) \xrightarrow{\text{Id} \otimes \text{tr}} M^{h\mathcal{B}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{B}}) & \xrightarrow{\text{Id} \otimes \text{tr}} & M^{h\mathcal{B}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{B}} \\ \downarrow -\text{tr} \otimes \text{Id} & & \downarrow 0 \oplus (-\text{tr} \otimes \text{Id}) \\ (M^{h\mathcal{A}} \xrightarrow{\text{Id} \otimes \text{tr}} M^{h\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{B}}) & & \end{array} \quad (5.49)$$

It remains to prove that the composition of (5.47) and (5.49) is homotopic to (5.46). This is equivalent to the following diagram commuting up to homotopy:

$$\begin{array}{ccccc} M^{h\mathcal{B}} & \xrightarrow{\text{Id} \otimes \text{act}} & M^{h\mathcal{B}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{B}} & \xrightarrow{\text{Id} \otimes \text{act} \otimes \text{Id}} & (M^{h\mathcal{B}} \otimes_{\mathcal{A}} M) \otimes_{\mathcal{B}} M^{h\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{B}} \\ & \searrow \text{Id} & \downarrow \text{tr} \otimes \text{Id} & & \downarrow \text{tr} \otimes \text{Id} \\ & & M^{h\mathcal{B}} & \xrightarrow{\text{act} \otimes \text{Id}} & M^{h\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{B}}. \end{array} \quad (5.50)$$

This is clear: the square in (5.50) commutes up to homotopy by the functoriality of the tensor product, while the triangle commutes up to homotopy by Prop. 2.11. \square

Let $\gamma : F'L[-1] \rightarrow LT'[1]$ be the map induced by the following morphism of twisted complexes:

$$\begin{array}{ccc} (M^{h\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{A}} \xrightarrow[\text{deg.0}]{\text{Id} \otimes \text{tr}} M^{h\mathcal{A}}) & & \\ \downarrow \text{Id} & & \\ (M^{h\mathcal{A}} \xrightarrow{\text{act} \otimes \text{Id}} M^{h\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{A}}) & & \end{array} \quad (5.51)$$

Lemma 5.10. *The morphism (5.51) is a homotopy equivalence. Consequently, the map γ is an isomorphism.*

Proof. Consider the composition

$$M^{h\mathcal{A}} \otimes_{\mathcal{A}} (M \otimes_{\mathcal{B}} M^{h\mathcal{A}}) \xrightarrow{\text{Id} \otimes \text{tr}} M^{h\mathcal{A}} \xrightarrow{\text{act} \otimes \text{Id}} (M^{h\mathcal{A}} \otimes_{\mathcal{A}} M) \otimes_{\mathcal{B}} M^{h\mathcal{A}}. \quad (5.52)$$

We claim that the homotopy inverse of (5.51) is the morphism

$$\begin{array}{ccc}
 (M^{h\mathcal{A}} \xrightarrow{\text{act} \otimes \text{Id}} M^{h\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{A}}) & & \\
 \downarrow \xi_{\mathcal{A}}^2 & \text{Id} - (5.52) & \downarrow -\xi_{\mathcal{A}} \circ (\text{Id} \otimes \text{tr}) \\
 (M^{h\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{A}} \xrightarrow{\text{Id} \otimes \text{tr}} M^{h\mathcal{A}}) & \xrightarrow{\text{Id} \otimes \text{tr}} & M^{h\mathcal{A}}
 \end{array} \tag{5.53}$$

Indeed, the composition of (5.51) with (5.53) is the morphism of twisted complexes

$$\begin{array}{ccc}
 (M^{h\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{A}} \xrightarrow{\text{Id} \otimes \text{tr}} M^{h\mathcal{A}}) & & \\
 \downarrow \text{Id} - (5.52) & \dashrightarrow & \downarrow -\xi_{\mathcal{A}} \circ (\text{Id} \otimes \text{tr}) \\
 (M^{h\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{A}} \xrightarrow{\text{Id} \otimes \text{tr}} M^{h\mathcal{A}}) & \xrightarrow{\text{Id} \otimes \text{tr}} & M^{h\mathcal{A}}
 \end{array}$$

which differs from the identity morphism by

$$\begin{array}{ccc}
 (M^{h\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{A}} \xrightarrow{\text{Id} \otimes \text{tr}} M^{h\mathcal{A}}) & & \\
 \downarrow (5.52) & \dashrightarrow & \downarrow \xi_{\mathcal{A}} \circ (\text{Id} \otimes \text{tr}) \\
 (M^{h\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{A}} \xrightarrow{\text{Id} \otimes \text{tr}} M^{h\mathcal{A}}) & \xrightarrow{\text{Id} \otimes \text{tr}} & M^{h\mathcal{A}}
 \end{array}$$

This is null-homotopic because it is the differential of the following degree -1 morphism of twisted complexes:

$$\begin{array}{ccc}
 (M^{h\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{A}} \xrightarrow{\text{Id} \otimes \text{tr}} M^{h\mathcal{A}}) & & \\
 \swarrow \text{act} \otimes \text{Id} & & \downarrow \xi_{\mathcal{A}} \\
 (M^{h\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{A}} \xrightarrow{\text{Id} \otimes \text{tr}} M^{h\mathcal{A}}) & \xrightarrow{\text{Id} \otimes \text{tr}} & M^{h\mathcal{A}}
 \end{array}$$

Thus the composition of (5.51) with (5.53) is homotopic to Id.

The composition of (5.53) and (5.51) being homotopic to Id is proved similarly. \square

Lemma 5.11. *The composition $F'L[-1] \xrightarrow{F'\alpha'} F'RT' \xrightarrow{\beta'T'} LT'[1]$ equals the map γ .*

Proof. Arguing as in Lemma 5.9 we see that $F'L[-1] \xrightarrow{F'\alpha'} F'RT'$ descends from the twisted complex map

$$\begin{array}{ccc}
 (M^{hA} \otimes_A M \otimes_B M^{hA} \xrightarrow{\quad} M^{hA}) & & \\
 \downarrow 0 \oplus (\text{Id} \otimes \text{act} \otimes \text{Id} \otimes \text{Id}) & & \downarrow \text{Id} \otimes \text{act} \\
 (M^{hB} \otimes_A M \otimes_B M^{hA} \xrightarrow{\quad} M^{hB} \oplus (M^{hA} \otimes_A M \otimes_B M^{hB} \otimes_A M \otimes_B M^{hA}) \xrightarrow{\quad} M^{hA} \otimes_A M \otimes_B M^{hB}) & \xrightarrow{\quad} & M^{hA} \otimes_A M \otimes_B M^{hB}
 \end{array} \tag{5.54}$$

Once again we omit labeling the internal twisted maps inside twisted complexes since they are not relevant to our argument. Similarly, $F'RT' \xrightarrow{\beta'T'} LT'[1]$ descends from the twisted complex map

$$\begin{array}{ccc}
 (M^{hB} \otimes_A M \otimes_B M^{hA} \xrightarrow{\quad} M^{hB} \oplus (M^{hA} \otimes_A M \otimes_B M^{hB} \otimes_A M \otimes_B M^{hA}) \xrightarrow{\quad} M^{hA} \otimes_A M \otimes_B M^{hB}) & & \\
 \downarrow \text{tr} \otimes \text{Id} & & \downarrow 0 \oplus (\text{Id} \otimes \text{Id} \otimes \text{tr} \otimes \text{Id}) \\
 (M^{hA} \xrightarrow{\quad} M^{hA} \otimes_A M \otimes_B M^{hA}) & &
 \end{array} \tag{5.55}$$

Hence the composition $F'L[-1] \xrightarrow{F'\alpha'} F'RT' \xrightarrow{\beta'T'} LT'[1]$ descends from

$$\begin{array}{ccc}
 (M^{hA} \otimes_A M \otimes_B M^{hA} \xrightarrow{\quad} M^{hA}) & & \\
 \downarrow \text{Id} \otimes ((\text{Id} \otimes \text{tr}) \circ (\text{act} \otimes \text{Id})) \otimes \text{Id} & & \\
 (M^{hA} \xrightarrow{\quad} M^{hA} \otimes_A M \otimes_B M^{hA}) & &
 \end{array} \tag{5.56}$$

By Prop. 2.11 the composition

$$M \xrightarrow{\text{act} \otimes \text{Id}} M \otimes_B M^{hB} \otimes_A M \xrightarrow{\text{Id} \otimes \text{tr}} M$$

is homotopic to Id, and thus (5.56) is homotopic to the map γ . □

Lemma 5.12. *The following maps are equal:*

- $LT[-1] \xrightarrow{\alpha \circ \beta} FL[1]$.
- $LT[-1] \xrightarrow{(5.26)LT} FF'LT[-1] \xrightarrow{F\gamma T} FLT'T[1] \xrightarrow{FL(5.15)} FL[1]$.

Proof. By Lemma 5.9 the composition $LT[-1] \xrightarrow{\alpha \circ \beta} FL[1]$ equals the composition

$$LT[-1] \xrightarrow{\alpha'T} RT'T \xrightarrow{R(5.15)} R \xrightarrow{(5.26)R} FF'R \xrightarrow{F\beta'} FL[1]. \tag{5.57}$$

By functoriality of tensor product the composition (5.57) equals the composition

$$LT[-1] \xrightarrow{\alpha'T} RT'T \xrightarrow{(5.26)RT'T} FF'RT'T \xrightarrow{FF'R(5.15)} FF'R \xrightarrow{F\beta'} FL[1], \tag{5.58}$$

which by functoriality of tensor product again equals the composition

$$LT[-1] \xrightarrow{(5.26)LT} FF'LT[-1] \xrightarrow{F'F'\alpha'T} FF'RT'T \xrightarrow{F\beta'T'T} FLT'T[1] \xrightarrow{FL(5.15)} FL[1]. \tag{5.59}$$

The claim now follows by applying Lemma 5.11 to the two maps in the middle of (5.59). □

Similarly, let $\gamma' : RT[-1] \rightarrow FR[1]$ be the map induced by the following morphism of twisted complexes:

$$\begin{array}{ccc} (M^{h\mathcal{B}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{B}} \xrightarrow{\text{tr} \otimes \text{Id}} M^{h\mathcal{B}}) & & \\ \downarrow \text{Id} & & \\ (M^{h\mathcal{B}} \xrightarrow{\text{Id} \otimes \text{act}} M^{h\mathcal{B}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{h\mathcal{B}}) & & \end{array} \tag{5.60}$$

The following two results are proved identically to Lemmas 5.10 and 5.12:

Lemma 5.13. *The morphism (5.60) is a homotopy equivalence. Consequently, the map γ' is an isomorphism.*

Lemma 5.14. *The following maps are equal:*

- $F'R[-1] \xrightarrow{\alpha' \circ \beta'} RT'[1].$
- $F'R[-1] \xrightarrow{F'R(5.14)} F'RTT'[-1] \xrightarrow{F'\gamma'T'} F'FRT'[1] \xrightarrow{(5.27)RT'} RT'[1].$

Thus, if the adjunction maps (5.26) and (5.15) are isomorphisms, then the composition $\alpha \circ \beta$ is an isomorphism, and it filters through the canonical map $RSL \xrightarrow{\eta} FL[1].$

We are now in a position to prove the main theorem. Before we begin, recall that in a triangulated category all retracts are split. More precisely, let $Z \xrightarrow{e} Y$ be a retract in a triangulated category, that is, there exists $Y \xrightarrow{g} Z$ with $Z \xrightarrow{e} Y \xrightarrow{g} Z$ being the identity. Then for any completion of g to an exact triangle $X \xrightarrow{f} Y \xrightarrow{g} Z, X \oplus Z \xrightarrow{f \oplus e} Y$ is an isomorphism. Moreover, its inverse is of the form $Y \xrightarrow{h \oplus g} X \oplus Z$ for some morphism $Y \xrightarrow{h} X.$ This can be established using only the axioms of triangulated categories, though for enhanced triangulated categories one can see it very explicitly on the level of twisted complexes.

Proof of Theorem 5.1. (3)&(4) \Rightarrow (1)&(2): Suppose that the natural transformations $lt[-1] \xrightarrow{(5.11)} r$ and $r \xrightarrow{(5.12)} fl[1]$ are functorial isomorphisms. In other words, conditions (3) and (4) hold. Then by Proposition 5.8 the units and counits of both adjoint pairs (t', t) and (f', f) are isomorphisms. Hence (t', t) and (f', f) are pairs of mutually inverse equivalences, that is, conditions (1) and (2) hold.

(1)&(3) \Rightarrow (4), (1)&(4) \Rightarrow (3), (2)&(3) \Rightarrow (4), (2)&(4) \Rightarrow (3): We only prove the first assertion, the other three are proved similarly.

Assume that (1) and (3) hold. Condition (1) is (t', t) being mutually inverse equivalences. In particular, the adjunction unit $\text{Id} \rightarrow tt'$ is an isomorphism. Therefore the morphism $\mathcal{B} \xrightarrow{(5.14)} T'T$, which by Prop. 5.4 induces this adjunction unit, is also an isomorphism. On the other hand, by Lemma 5.7 condition (3) is equivalent to the map $LT \xrightarrow{\beta} R[1]$ in (5.36) being an isomorphism.

By Lemma 5.7 condition (4) is equivalent to the map $L \xrightarrow{\alpha'} RT'[1]$ in (5.34) being an isomorphism. By Lemma 5.9 the map $L \xrightarrow{\alpha'} RT'[1]$ decomposes as

$$L \xrightarrow{L(5.14)} LTT' \xrightarrow{\beta T'} RT'[1].$$

By the above, both the compositants are isomorphisms. Hence $L \xrightarrow{\alpha'} RT'[1]$ is also an isomorphism, as desired.

(1)&(2) \Rightarrow (4): Assume that (1) and (2) hold. Then $\text{Id} \xrightarrow{(5.26)} FF'$ and $T'T \xrightarrow{(5.15)} \text{Id}$ are isomorphisms. By Lemma 5.10 the map $F'L[-1] \xrightarrow{\gamma} LT'[1]$ induced by (5.51) is always an isomorphism. By Lemma 5.12 the map $LT[-1] \xrightarrow{\beta} R \xrightarrow{\alpha} FL[1]$ decomposes as

$$LT[-1] \xrightarrow{(5.26)LT} FF'LT[-1] \xrightarrow{F\gamma T} FLT'T[1] \xrightarrow{FL(5.15)} FL[1]$$

and is therefore an isomorphism.

This isomorphism $\alpha \circ \beta$ filters through the canonical map $RSL \xrightarrow{\eta} FL[1]$, thus $FL[1]$ is a retract of RSL . More specifically, denote by $\bar{\eta}$ the map $FL[1] \xrightarrow{(\alpha \circ \beta)^{-1}} LT[-1] \xrightarrow{\beta} R \xrightarrow{R_{\text{act}}} RSL$, so that

$$FL[1] \xrightarrow{\bar{\eta}} RSL \xrightarrow{\eta} FL[1]$$

is the identity map. Since all retracts in triangulated categories are split, and $L \xrightarrow{\text{act}L} RSL \xrightarrow{\eta} FL[1]$ is an exact triangle, it follows that there exists a map $RSL \xrightarrow{\overline{\text{act}L}} L$ such that

$$L \oplus FL[1] \xrightarrow{(\text{act}L) \oplus \bar{\eta}} RSL \xrightarrow{(\overline{\text{act}L}) \oplus \eta} L \oplus FL[1]$$

are mutually inverse isomorphisms. Similarly, since $F'F \xrightarrow{(5.27)} \text{Id}$ and $\text{Id} \xrightarrow{(5.14)} T'T'$ are isomorphisms, Lemmas 5.13 and 5.14 imply that the map $F'R[-1] \xrightarrow{\alpha' \circ \beta'} RT'[1]$ is an isomorphism. Let $\bar{\zeta}$ be the map $RT'[1] \xrightarrow{(\alpha' \circ \beta')^{-1}} F'R[-1] \xrightarrow{\beta'} L \xrightarrow{\text{act}L} RSL$. Then there exists a map $RSL \xrightarrow{\bar{R}_{\text{act}}} R$ such that

$$R \oplus RT'[1] \xrightarrow{(R_{\text{act}}) \oplus \bar{\zeta}} RSL \xrightarrow{(\bar{R}_{\text{act}}) \oplus \zeta} R \oplus RT'[1]$$

are mutually inverse isomorphisms.

Since $T'T \xrightarrow{(5.15)} \mathcal{B}$ is an isomorphism, it follows from the exactness of rows and columns in (5.43) that $SLT[-1] \xrightarrow{S\beta} SR$ is an isomorphism. So is $SLT[-1] \xrightarrow{S\alpha \circ S\beta}$

$SFL[1]$, and hence so must also be $SR \xrightarrow{S\alpha} SFL[1]$. Then $S(\alpha \circ \beta)^{-1} = (S\beta)^{-1} \circ (S\alpha)^{-1}$, and hence the following diagram commutes:

$$\begin{array}{ccc} SFL[1] & \xrightarrow{S\bar{\eta}} & SRSL \\ S\alpha \uparrow \sim & \nearrow SRact & \\ SR & & \end{array}$$

Consider now the map $SFL[1] \rightarrow T'[1]$ which is adjoint to $FL[1] \xrightarrow{\zeta \circ \bar{\eta}} RT'[1]$. It filters through

$$SFL[1] \xrightarrow{S(\zeta \circ \bar{\eta})} SRT'[1],$$

which we can rewrite as

$$SFL[1] \xrightarrow{(S\alpha)^{-1}} SR \xrightarrow{SRact} SRSL \xrightarrow{S\zeta} SRT'[1],$$

and $R \xrightarrow{Ract} RSL \xrightarrow{\zeta} RT'[1]$ is the zero map. We conclude that $FL[1] \xrightarrow{\zeta \circ \bar{\eta}} RT'[1]$ is adjoint to the zero map, and hence itself is the zero map.

Similarly, $S\alpha'$ and $S\beta'$ are isomorphisms and the following diagram commutes

$$\begin{array}{ccc} SRT'[1] & \xrightarrow{S\bar{\zeta}} & SRSL \\ S\alpha' \uparrow \sim & \nearrow SactL & \\ SL & & \end{array}$$

It follows similarly that $SL \xrightarrow{S(\overline{Ract} \circ actL)} SR$ is the zero map, and hence so is $L \xrightarrow{\overline{Ract} \circ actL} R$.

Observe now that the composition

$$L \oplus FL[1] \xrightarrow{(actL) \oplus \bar{\eta}} RSL \xrightarrow{(\overline{Ract}) \oplus \zeta} R \oplus RT'[1]$$

is an isomorphism and we have shown that the compositions

$$L \xrightarrow{actL} RSL \xrightarrow{\overline{Ract}} R \quad \text{and} \quad FL[1] \xrightarrow{\bar{\eta}} RSL \xrightarrow{\zeta} RT'[1]$$

are the zero maps. It follows that the compositions

$$L \xrightarrow{actL} RSL \xrightarrow{\zeta} RT'[1] \quad \text{and} \quad FL[1] \xrightarrow{\bar{\eta}} RSL \xrightarrow{\overline{Ract}} R$$

are isomorphisms. The former composition is, by definition, the map $L \xrightarrow{\alpha'} RT'[1]$. It follows by Lemma 5.7 that condition (4) holds, as desired. \square

5.2. Applications to algebraic geometry

In this section we interpret the results of Section 5.1 in the context of algebraic geometry.

Let Z and X be two separated schemes of finite type over k . Recall that for any $E \in D_{\text{qc}}(Z \times X)$ the Fourier–Mukai transform Φ_E is the functor $D_{\text{qc}}(Z) \rightarrow D_{\text{qc}}(X)$ defined by

$$\mathbf{R}\pi_{X*}(E \otimes^{\mathbf{L}} \pi_Z^*(-)),$$

where π_Z and π_X are the projections from $Z \times X$ to Z and X . Note that Φ_E does not a priori restrict to a functor $D(Z) \rightarrow D(X)$.

As explained in Example 4.3, we can Morita enhance $D(Z)$ and $D(X)$ by smooth DG-algebras \mathcal{A} and \mathcal{B} whose classes in $\text{Ho}(DG\text{-Cat}^{\text{kctr}})$ are the standard enhancements of $D(Z)$ and $D(X)$. Moreover, $D(Z \times X)$ is Morita enhanced by the DG-algebra $\mathcal{A}^{\text{opp}} \otimes \mathcal{B}$ and the following holds. Recall that Morita quasi-functors $\mathcal{A} \rightarrow \mathcal{B}$ are identified naturally with the elements of $D^{\mathcal{B}\text{-Perf}}(\mathcal{A}\text{-}\mathcal{B})$. Since \mathcal{A} is smooth, we have a natural inclusion $D^{\mathcal{B}\text{-Perf}}(\mathcal{A}\text{-}\mathcal{B}) \hookrightarrow D_c(\mathcal{A}\text{-}\mathcal{B})$. Thus to each Morita quasi-functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ corresponds an element in $D_c(\mathcal{A}\text{-}\mathcal{B})$, and so an element $E \in D(Z \times X)$. The Fourier–Mukai transform Φ_E restricts to a functor $D(Z) \xrightarrow{\Phi_E} D(X)$, and this functor is isomorphic to the exact functor $D(Z) \rightarrow D(X)$ underlying F .

Similarly, $X \times Z$, $Z \times Z$ and $X \times X$ are Morita enhanced by $\mathcal{B}^{\text{opp}} \otimes \mathcal{A}$, $\mathcal{A}^{\text{opp}} \otimes \mathcal{A}$ and $\mathcal{B}^{\text{opp}} \otimes \mathcal{B}$ with a similar correspondence between Morita quasi-functors and Fourier–Mukai transforms. We implicitly identify $X \times Z$ with $Z \times X$ using the canonical isomorphism between the two. For any object E in $D_c(\mathcal{A})$, $D_c(\mathcal{B})$, $D_c(\mathcal{A}\text{-}\mathcal{B})$, etc. let \bar{E} be the corresponding object in $D(Z)$, $D(X)$, $D(Z \times X)$, etc.

Let $\bar{S} \in D(Z \times X)$ be such that the corresponding $S \in D_c(\mathcal{A}\text{-}\mathcal{B})$ is \mathcal{A} - and \mathcal{B} -perfect. Let $L = S^{\bar{\mathcal{A}}}$ and $R = S^{\bar{\mathcal{B}}}$. These are \mathcal{A} -perfect and \mathcal{B} -perfect, respectively. Since \mathcal{A} and \mathcal{B} are smooth, L and R lie in $D_c(\mathcal{B}\text{-}\mathcal{A})$ by Cor. 2.15. The corresponding objects \bar{L} and \bar{R} in $D(X \times Z)$ define the Fourier–Mukai transforms $D(X) \xrightarrow{\Phi_{\bar{L}, \bar{R}}} D(Z)$ which are left and right adjoint to $D(Z) \xrightarrow{\Phi_{\bar{S}}} D(X)$. The adjunction counits and units are the natural transformations of the Fourier–Mukai transforms induced by the derived trace and action maps

$$SR \xrightarrow{\text{tr}} \mathcal{B} \quad \text{and} \quad LS \xrightarrow{\text{tr}} \mathcal{A}, \tag{5.61}$$

$$\mathcal{B} \xrightarrow{\text{act}} SL \quad \text{and} \quad \mathcal{A} \xrightarrow{\text{act}} RS. \tag{5.62}$$

The cotwists $F, F' \in D(\mathcal{A}\text{-}\mathcal{A})$ and the twists $T, T' \in D(\mathcal{B}\text{-}\mathcal{B})$ of S were defined in Section 5.1 as the cones and the (-1) -shifted cones of the derived trace and action maps above. It follows from Cor. 2.15 that they are all compact objects. Hence we can define the *cotwist* and the *dual cotwist* of \bar{S} to be the corresponding objects \bar{F} and $\bar{F}' \in D(Z \times Z)$ and the *twist* and the *dual twist* of \bar{S} to be \bar{T} and \bar{T}' in $D(X \times X)$. Finally, define

$$\Phi_{\bar{R}} \rightarrow \Phi_{\bar{F}} \Phi_{\bar{L}}[1], \tag{5.63}$$

$$\Phi_{\bar{L}} \Phi_{\bar{T}}[-1] \rightarrow \Phi_{\bar{R}} \tag{5.64}$$

to be the natural transformations of Fourier–Mukai transforms which correspond to the natural transformations $lt[-1] \xrightarrow{(5.11)} r$ and $r \xrightarrow{(5.12)} fl[1]$ constructed in Section 5.1.

The algebras \mathcal{A} and \mathcal{B} are constructed as DG-End-algebras of h -injective strong generators F_Z and F_X of $D(Z)$ and $D(X)$. It follows from [Lun10, Th. 6.3] that choosing a different generator F'_X of e.g. $D(X)$ produces a Morita equivalent DG-algebra \mathcal{B}' . More generally, it follows that a different choice of generators F'_Z and F'_X produces Morita equivalent DG-algebras \mathcal{A}' , \mathcal{B}' , $\mathcal{A}'\text{-}\mathcal{B}'$, $\mathcal{A}'\text{-}\mathcal{A}'$ and $\mathcal{B}'\text{-}\mathcal{B}'$.

All the constructions from Section 5.1 we have used so far were defined entirely in terms of the derived duals R and L of S and the derived trace and action maps. One can check that the derived dualizing functors and the derived trace and action maps are preserved under Morita equivalences. Thus the objects \bar{L} , \bar{R} , \bar{F} , \bar{F}' , \bar{T} , \bar{T}' and the natural transformations (5.63)–(5.64) defined above depend only on $\bar{S} \in D(Z \times X)$ itself, and not on our choice of the generators F_X and F_Y of $D(Z)$ and $D(X)$.

Though we have established that the above objects and maps are well-defined and are determined only by $\bar{S} \in D(Z \times X)$, to actually compute them in any practical scenario would require explicit formulas for \bar{L} , \bar{R} in terms of \bar{S} as well as explicit formulas for the maps in $D(X \times X)$ and $D(Z \times Z)$ which correspond to the derived trace and action maps. To this end we offer the following:

Conjecture 5.15. *Let $\bar{S} \in D(Z \times X)$ be such that the corresponding $S \in D_c(\mathcal{A}\text{-}\mathcal{B})$ is \mathcal{A} - and \mathcal{B} -perfect. Then*

$$\bar{L} \simeq \mathbf{R} \operatorname{Hom}_{Z \times X}(\bar{S}, \pi_Z^!(\mathcal{O}_Z)), \quad \bar{R} \simeq \mathbf{R} \operatorname{Hom}_{Z \times X}(\bar{S}, \pi_X^!(\mathcal{O}_X)),$$

and the maps in $D(Z \times Z)$ and $D(X \times X)$ which correspond to the derived trace and action maps (5.61)–(5.62) are isomorphic to the explicit maps written down in [AL12] and [AL16] which lift the adjunction counits and units of Fourier–Mukai transforms to the level of Fourier–Mukai kernels.

Finally, we need an intrinsic condition on $\bar{S} \in D(Z \times X)$ on the algebro-geometric side which ensures that the corresponding $S \in D_c(\mathcal{A}\text{-}\mathcal{B})$ is \mathcal{A} - and \mathcal{B} -perfect.

Lemma 5.16. *Let $\bar{S} \in D(Z \times X)$. The Fourier–Mukai transform $\Phi_{\bar{S}}$ restricts to $D(Z) \rightarrow D(X)$ and this restriction has a left adjoint which is also a Fourier–Mukai transform if and only if the corresponding object $S \in D_c(\mathcal{A}\text{-}\mathcal{B})$ is \mathcal{A} - and \mathcal{B} -perfect.*

Proof. As explained above, $S \in D_c(\mathcal{A}\text{-}\mathcal{B})$ is \mathcal{B} -perfect if and only if $\Phi_{\bar{S}}$ restricts to

$$D(Z) \rightarrow D(X). \text{ In that case } D_c(\mathcal{A}) \xrightarrow{(-) \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} S} D_c(\mathcal{B}) \text{ corresponds to } D(Z) \xrightarrow{\Phi_{\bar{S}}} D(X).$$

Suppose now S is also \mathcal{A} -perfect. By Cor. 2.2 the functor $(-) \overset{\mathbf{L}}{\otimes}_{\mathcal{B}} S^{\tilde{\mathcal{A}}}$ is left adjoint to $(-) \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} S$. Moreover, since S is \mathcal{A} -perfect, so is $S^{\tilde{\mathcal{A}}}$. Hence there exists an object in $D(X \times Z)$ which defines the Fourier–Mukai transform $D(X) \rightarrow D(Z)$ which corresponds to $(-) \overset{\mathbf{L}}{\otimes}_{\mathcal{B}} S^{\tilde{\mathcal{A}}}$. In particular, this Fourier–Mukai transform is left adjoint to $\Phi_{\bar{S}}$.

Conversely, if there exists a Fourier–Mukai transform $D(X) \rightarrow D(Z)$ which is the left adjoint to $\Phi_{\bar{S}}$, let L be the corresponding object of $D^{\mathcal{A}\text{-Perf}}(\mathcal{B}\text{-}\mathcal{A})$. Then $(-) \overset{\mathbf{L}}{\otimes}_{\mathcal{B}} L$ is left adjoint to $(-) \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} S$ as functors between $D_c(\mathcal{A})$ and $D_c(\mathcal{B})$. But since derived tensor

product commutes with infinite direct sums, these are, in fact, adjoint on the whole of $D(\mathcal{A})$ and $D(\mathcal{B})$.

Since L is \mathcal{A} -perfect, by Cor. 2.2 the functor $(-)\overset{\mathbf{L}}{\otimes}_{\mathcal{B}}L$ from $D(\mathcal{A})$ to $D(\mathcal{B})$ has a right adjoint $(-)\overset{\mathbf{L}}{\otimes}_{\mathcal{A}}L^{\tilde{\mathcal{A}}}$. By uniqueness of adjoints we conclude that the functors $(-)\overset{\mathbf{L}}{\otimes}_{\mathcal{A}}S$ and $(-)\overset{\mathbf{L}}{\otimes}_{\mathcal{A}}L^{\tilde{\mathcal{A}}}$ are isomorphic. Since L is \mathcal{A} -perfect, so is $L^{\tilde{\mathcal{A}}}$. Hence $(-)\overset{\mathbf{L}}{\otimes}_{\mathcal{A}}L^{\tilde{\mathcal{A}}}$ takes compact objects to compact objects, and hence so does $(-)\overset{\mathbf{L}}{\otimes}_{\mathcal{A}}S$. We conclude that S is also \mathcal{A} -perfect, as desired. \square

Theorem 5.1 immediately implies the following:

Theorem 5.2. *Let $\bar{S} \in D(Z \times X)$ be such that $\Phi_{\bar{S}}$ restricts to $D(Z) \rightarrow D(X)$ and this restriction has a left adjoint which is also a Fourier–Mukai transform.*

Suppose any two of the following conditions hold:

- (1) $\Phi_{\bar{T}}$ is an autoequivalence of $D(X)$ (“the twist is an equivalence”).
- (2) $\Phi_{\bar{F}}$ is an equivalence of $D(Z)$ (“the cotwist is an equivalence”).
- (3) $\Phi_{\bar{R}} \xrightarrow{(5.63)} \Phi_{\bar{F}}\Phi_{\bar{L}}[1]$ is an isomorphism of functors (“the twist identifies the adjoints”).
- (4) $\Phi_{\bar{L}}\Phi_{\bar{T}}[-1] \xrightarrow{(5.64)} \Phi_{\bar{R}}$ is an isomorphism of functors (“the cotwist identifies the adjoints”).

Then all four of them hold. If that happens, we say that \bar{S} is spherical over Z .

We can repeat all the arguments in this section using the framework of Example 4.2 rather than Example 4.3. Thus we would work with large Morita enhancements of $D_{\text{qc}}(Z)$ and $D_{\text{qc}}(X)$, rather than Morita enhancements of $D(Z)$ and $D(X)$. This yields a construction of *twists* and *cotwists* as functors $D_{\text{qc}}(X) \rightarrow D_{\text{qc}}(X)$ and $D_{\text{qc}}(Z) \rightarrow D_{\text{qc}}(Z)$ and an analogue of Theorem 5.2. However, we would have to impose the following condition on the objects of $\bar{S} \in D_{\text{qc}}(Z \times X)$ we work with: $\Phi_{\bar{S}}$ must have a left adjoint which is a Fourier–Mukai transform and they must both send compact objects to compact objects. This condition can also be stated equivalently as: $\Phi_{\bar{S}}$ must have left and right adjoints which are both Fourier–Mukai transforms.

6. Braiding criteria for spherical DG-functors

Let $\mathcal{A}_1, \mathcal{A}_2$ and \mathcal{B} be small DG-categories and let $S_1 \in D(\mathcal{A}_1\text{-}\mathcal{B})$ and $S_2 \in D(\mathcal{A}_2\text{-}\mathcal{B})$ be two spherical objects. We keep all the notation conventions of Section 5. For example, R_i denotes $S_i^{\tilde{\mathcal{B}}}$, $S_i R_i$ denotes $R_i \overset{\mathbf{L}}{\otimes}_{\mathcal{A}_i} S_i$, T_i denotes the cone of $S_i R_i \xrightarrow{\text{tr}} \mathcal{B}$, etc.

In particular, $M = \bar{\mathcal{A}}_1 \otimes_{\mathcal{A}_1} S_1 \otimes_{\mathcal{B}} \bar{\mathcal{B}}$ and $N = \bar{\mathcal{A}}_2 \otimes_{\mathcal{A}_2} S_2 \otimes_{\mathcal{B}} \bar{\mathcal{B}}$ are h -projective resolutions of S_1 and S_2 . In this section it has been possible to simplify a number of computations by replacing all homotopy trace maps $M^{h\mathcal{B}} \otimes_{\mathcal{A}} M \xrightarrow{\text{tr}} \bar{\mathcal{B}}$ and $N^{h\mathcal{B}} \otimes_{\mathcal{A}} N \xrightarrow{\text{tr}} \bar{\mathcal{B}}$ by their compositions with $\bar{\mathcal{B}} \xrightarrow{\tau} \mathcal{B}$. To keep the notation simple, we write $M^{h\mathcal{B}} \otimes_{\mathcal{A}} M \xrightarrow{\text{tr}} \mathcal{B}$ and $N^{h\mathcal{B}} \otimes_{\mathcal{A}} N \xrightarrow{\text{tr}} \mathcal{B}$ for these compositions throughout.

such that

$$f \text{ is a homotopy equivalence, } ds_1 = \alpha - \beta f, \quad ds_2 = \gamma s_1.$$

Since all the source bimodules are h -projective, the Hom^i -spaces above are isomorphic to the Ext^i -spaces between the same objects in $D(\mathcal{B}\text{-}\mathcal{B})$.

In particular, we can lift the isomorphism

$$S_1 R_1 S_2 R_2 \xrightarrow{\phi} S_2 R_2 S_1 R_1$$

in $D(\mathcal{B}\text{-}\mathcal{B})$ to some homotopy equivalence

$$f \in \text{Hom}_{\mathcal{B}\text{-}\mathcal{B}}^0(N^{h\mathcal{B}} \otimes_{\mathcal{A}_2} N \otimes_{\mathcal{B}} M^{h\mathcal{B}} \otimes_{\mathcal{A}_1} M, M^{h\mathcal{B}} \otimes_{\mathcal{A}_1} M \otimes_{\mathcal{B}} N^{h\mathcal{B}} \otimes_{\mathcal{A}_2} N).$$

The fact that ϕ makes (6.1) commute in $D(\mathcal{B}\text{-}\mathcal{B})$ implies that $\alpha - \beta f$ vanishes in

$$\text{Hom}_{D(\mathcal{B}\text{-}\mathcal{B})}(S_1 R_1 S_2 R_2, S_1 R_1 \oplus S_2 R_2).$$

Hence we can find some

$$s_1 \in \text{Hom}_{\mathcal{B}\text{-}\mathcal{B}}^{-1}(N^{h\mathcal{B}} \otimes_{\mathcal{A}_2} N \otimes_{\mathcal{B}} M^{h\mathcal{B}} \otimes_{\mathcal{A}_1} M, (M^{h\mathcal{B}} \otimes_{\mathcal{A}_1} M) \oplus (N^{h\mathcal{B}} \otimes_{\mathcal{A}_2} N))$$

with $ds_1 = \alpha - \beta f$. But there is no a priori reason for the class of γs_1 to vanish in $\text{Ext}_{D(\mathcal{B}\text{-}\mathcal{B})}^{-1}(S_1 R_1 S_2 R_2, \mathcal{B})$, which is what we need to warrant the existence of

$$s_2 \in \text{Hom}_{\mathcal{B}\text{-}\mathcal{B}}^{-2}(N^{h\mathcal{B}} \otimes_{\mathcal{A}_2} N \otimes_{\mathcal{B}} M^{h\mathcal{B}} \otimes_{\mathcal{A}_1} M, \bar{\mathcal{B}})$$

with $ds_2 = \gamma s_1$, whence as explained above the claim of this theorem would follow.

It suffices, however, to find

$$t_1 \in \text{Hom}_{\mathcal{B}\text{-}\mathcal{B}}^{-1}(N^{h\mathcal{B}} \otimes_{\mathcal{A}_2} N \otimes_{\mathcal{B}} M^{h\mathcal{B}} \otimes_{\mathcal{A}_1} M, (M^{h\mathcal{B}} \otimes_{\mathcal{A}_1} M) \oplus (N^{h\mathcal{B}} \otimes_{\mathcal{A}_2} N))$$

with $dt_1 = 0$ and $\gamma t_1 = \gamma s_1$ in $\text{Ext}_{D(\mathcal{B}\text{-}\mathcal{B})}^{-1}(S_1 R_1 S_2 R_2, \mathcal{B})$. For if we then replace s_1 with $s_1 - t_1$, the condition $ds_1 = \alpha - \beta f$ would still hold, but the class of γs_1 would now vanish in $\text{Ext}_{D(\mathcal{B}\text{-}\mathcal{B})}^{-1}(S_1 R_1 S_2 R_2, \mathcal{B})$ as required. Thus it remains to show that the class $[\gamma s_1]$ in $\text{Ext}_{D(\mathcal{B}\text{-}\mathcal{B})}^{-1}(S_1 R_1 S_2 R_2, \mathcal{B})$ lifts with respect to

$$\text{Ext}_{D(\mathcal{B}\text{-}\mathcal{B})}^{-1}(S_1 R_1 S_2 R_2, S_1 R_1 \oplus S_2 R_2) \xrightarrow{\gamma(-)} \text{Ext}_{D(\mathcal{B}\text{-}\mathcal{B})}^{-1}(S_1 R_1 S_2 R_2, \mathcal{B}) \quad (6.4)$$

to some class in $\text{Ext}_{D(\mathcal{B}\text{-}\mathcal{B})}^{-1}(S_1 R_1 S_2 R_2, S_1 R_1 \oplus S_2 R_2)$.

We claim that, in fact, (6.4) is surjective. Indeed, it follows from Prop. 2.11 via the usual adjunction-type argument that for any $N_1 \in D(\mathcal{A}_2\text{-}\mathcal{A}_1)$ and $N_2 \in D(\mathcal{B}\text{-}\mathcal{B})$ the map

$$\text{Ext}_{D(\mathcal{B}\text{-}\mathcal{B})}^{-1}(S_1 N_1 R_2, N_2) \rightarrow \text{Ext}_{D(\mathcal{B}\text{-}\mathcal{B})}^{-1}(N_1, R_1 N_2 S_2) \quad (6.5)$$

given by

$$\alpha \mapsto N_1 \xrightarrow{\text{act } N_1 \text{ act}} R_1 S_1 N_1 R_2 S_2 \xrightarrow{R_1 \alpha S_2} R_1 N_2 S_2$$

is a functorial isomorphism. We thus have a commutative diagram

$$\begin{CD}
 \text{Ext}_{D(\mathcal{B}-\mathcal{B})}^{-1}(S_1 R_1 S_2 R_2, S_1 R_1 \oplus S_2 R_2) @>(6.4)>> \text{Ext}_{D(\mathcal{B}-\mathcal{B})}^{-1}(S_1 R_1 S_2 R_2, \mathcal{B}) \\
 @V \sim \downarrow (6.5) VV @VV \sim \downarrow (6.5) V \\
 \text{Ext}_{D(\mathcal{A}_2-\mathcal{A}_1)}^{-1}(R_1 S_2, R_1 S_1 R_1 S_2 \oplus R_1 S_2 R_2 S_2) @>(R_1 \gamma S_2)(-)>> \text{Ext}_{D(\mathcal{A}_2-\mathcal{A}_1)}^{-1}(R_1 S_2, R_1 S_2)
 \end{CD} \tag{6.6}$$

The map $R_1 \gamma S_2$ is

$$R_1 S_1 R_1 S_2 \oplus R_1 S_2 R_2 S_2 \xrightarrow{R_1 \text{ tr } S_2 \oplus R_1 \text{ tr } S_2} R_1 S_2,$$

and by Prop. 2.11 the map

$$R_1 S_2 \xrightarrow{\frac{1}{2} \text{ act } R_1 S_2 \oplus \frac{1}{2} R_1 S_2 \text{ act}} R_1 S_1 R_1 S_2 \oplus R_1 S_2 R_2 S_2$$

is its left inverse in $D(\mathcal{A}_2-\mathcal{A}_1)$. Therefore

$$\text{Ext}_{D(\mathcal{A}_2-\mathcal{A}_1)}^{-1}(R_1 S_2, R_1 S_1 R_1 S_2 \oplus R_1 S_2 R_2 S_2) \xrightarrow{(R_1 \gamma S_2)(-)} \text{Ext}_{D(\mathcal{A}_2-\mathcal{A}_1)}^{-1}(R_1 S_2, R_1 S_2)$$

is surjective, and hence so is (6.4) as desired. \square

6.2. Braiding

Define

$$O_i = F_i \left\{ L_i S_j R_j S_i \xrightarrow[\text{deg.0}]{\text{tr} \circ (L_i \text{ tr } S_i)} \mathcal{A}_i \right\} \in D(\mathcal{A}_i-\mathcal{A}_i) \tag{6.7}$$

where $i, j \in \{1, 2\}$, $i \neq j$. For spherical S_1, S_2 the natural map $R_i[-1] \xrightarrow{\alpha} F_i L_i$ is an isomorphism and it identifies the map in (6.7) with the map

$$R_i S_j R_j S_i[-1] \xrightarrow{R_i \text{ tr } S_i} R_i S_i[-1] \rightarrow F_i$$

whose second compositant comes from the exact triangle $F_i \rightarrow \mathcal{A}_i \rightarrow R_i S_i$. Thus O_1 and O_2 are isomorphic to the convolutions of the twisted complexes

$$\begin{aligned}
 O_1 &:= \left((M \otimes_{\mathcal{B}} N^{h\mathcal{B}} \otimes_{\mathcal{A}_2} N \otimes_{\mathcal{B}} M^{h\mathcal{B}}) \oplus \bar{\mathcal{A}}_1 \xrightarrow[\text{deg.0}]{(\text{Id} \otimes \text{tr} \otimes \text{Id}) \oplus (-\text{act})} M \otimes_{\mathcal{B}} M^{h\mathcal{B}} \right), \\
 O_2 &:= \left((N \otimes_{\mathcal{B}} M^{h\mathcal{B}} \otimes_{\mathcal{A}_1} M \otimes_{\mathcal{B}} N^{h\mathcal{B}}) \oplus \bar{\mathcal{A}}_2 \xrightarrow[\text{deg.0}]{(\text{Id} \otimes \text{tr} \otimes \text{Id}) \oplus (-\text{act})} N \otimes_{\mathcal{B}} N^{h\mathcal{B}} \right).
 \end{aligned}$$

There are natural maps

$$S_1 O_1 R_1 \rightarrow S_1 R_1 S_2 R_2 \oplus S_2 R_2 S_1 R_1, \tag{6.8}$$

$$S_2 O_2 R_2 \rightarrow S_1 R_1 S_2 R_2 \oplus S_2 R_2 S_1 R_1, \tag{6.9}$$

where (6.8) is the map induced by

$$M^{h\mathcal{B}} \otimes_{\mathcal{A}_1} M \otimes_{\mathcal{B}} N^{h\mathcal{B}} \otimes_{\mathcal{A}_2} N \otimes_{\mathcal{B}} M^{h\mathcal{B}} \otimes_{\mathcal{A}_1} M \xrightarrow{\text{Id} \otimes \text{Id} \otimes \text{tr} \otimes \text{Id} \otimes \text{Id}} M^{h\mathcal{B}} \otimes_{\mathcal{A}_1} M \otimes_{\mathcal{B}} N^{h\mathcal{B}} \otimes_{\mathcal{A}_2} N, \tag{6.10}$$

$$M^{h\mathcal{B}} \otimes_{\mathcal{A}_1} M \otimes_{\mathcal{B}} N^{h\mathcal{B}} \otimes_{\mathcal{A}_2} N \otimes_{\mathcal{B}} M^{h\mathcal{B}} \otimes_{\mathcal{A}_1} M \xrightarrow{\text{tr} \otimes \text{Id} \otimes \text{Id} \otimes \text{Id} \otimes \text{Id}} N^{h\mathcal{B}} \otimes_{\mathcal{A}_2} N \otimes_{\mathcal{B}} M^{h\mathcal{B}} \otimes_{\mathcal{A}_1} M, \tag{6.11}$$

and (6.9) is defined analogously.

The main result of this section is:

Theorem 6.2. *Suppose there exists an isomorphism*

$$S_1 O_1 R_1 \xrightarrow{\phi} S_2 O_2 R_2$$

which commutes with the maps (6.8) and (6.9). Then

$$T_1 T_2 T_1 \simeq T_2 T_1 T_2.$$

Proof. $T_1 T_2 T_1$ is isomorphic by the Cube Completion Lemma 3.7 to the convolution of the twisted cube

$$\tag{6.12}$$

We now use the isomorphism $(M^{h\mathcal{B}} \otimes M) \oplus (M^{h\mathcal{B}} \otimes M) \xrightarrow{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}} (M^{h\mathcal{B}} \otimes M) \oplus (M^{h\mathcal{B}} \otimes M)$ to rewrite the total complex of (6.12) as

$$\tag{6.13}$$

Let X and Y be the full subcomplexes of (6.13) which comprise its left two columns and its right column, respectively. Since the right column has no outgoing arrows, its incoming arrows define a closed degree 0 morphism $X \xrightarrow{\rho} Y$ whose total complex is (6.13). Let

$$Y' = (M^{h\mathcal{B}} \otimes M \xrightarrow{-\frac{1}{2}\text{Id} \otimes \text{act} \otimes \text{Id}} M^{h\mathcal{B}} \otimes M \otimes M^{h\mathcal{B}} \otimes M)_{\text{deg.}-2}.$$

Lemma 5.10 yields a homotopy equivalence $Y \xrightarrow{\gamma'} Y'$. The total complex of $X \xrightarrow{\rho} Y$ is then homotopy equivalent to the total complex of $X \xrightarrow{\gamma' \circ \rho} Y'$. Thus (6.13) is homotopy equivalent to the twisted complex

$$\begin{array}{c}
 M^{h\mathcal{B}} \otimes M \otimes N^{h\mathcal{B}} \otimes N \otimes M^{h\mathcal{B}} \otimes M \oplus M^{h\mathcal{B}} \otimes M \\
 \begin{array}{l}
 \swarrow \text{Id} \otimes \text{tr} \quad \downarrow \text{tr} \otimes \text{Id} \quad \searrow -\text{Id} \otimes \text{tr} \otimes \text{Id} \\
 M^{h\mathcal{B}} \otimes M \otimes N^{h\mathcal{B}} \otimes N \oplus N^{h\mathcal{B}} \otimes N \otimes M^{h\mathcal{B}} \otimes M \oplus M^{h\mathcal{B}} \otimes M \otimes M^{h\mathcal{B}} \otimes M \\
 \begin{array}{l}
 \downarrow \text{tr} \otimes \text{Id} \quad \swarrow -\text{Id} \otimes \text{tr} \quad \downarrow \text{tr} \otimes \text{Id} \\
 N^{h\mathcal{B}} \otimes N \oplus M^{h\mathcal{B}} \otimes M \\
 \downarrow \text{tr} \\
 \mathcal{B}_{\text{deg.}0}
 \end{array}
 \end{array}
 \end{array}
 \tag{6.14}$$

Now observe that $M^{h\mathcal{B}} \otimes \mathcal{O}_1 \otimes M[-3]$ is homotopy equivalent to the following initial subcomplex of (6.14):

$$\begin{array}{c}
 M^{h\mathcal{B}} \otimes M \otimes N^{h\mathcal{B}} \otimes N \otimes M^{h\mathcal{B}} \otimes M \oplus M^{h\mathcal{B}} \otimes M \\
 \begin{array}{l}
 \searrow -\text{Id} \otimes \text{tr} \otimes \text{Id} \\
 M^{h\mathcal{B}} \otimes M \otimes M^{h\mathcal{B}} \otimes M_{\text{deg.}-2} \\
 \downarrow -\frac{1}{2}\text{Id} \otimes \text{act} \otimes \text{Id}
 \end{array}
 \end{array}
 \tag{6.15}$$

By the same argument as above, (6.14) is homotopy equivalent to the twisted complex

$$\begin{array}{l}
 M^{h\mathcal{B}} \otimes \mathcal{O}_1 \otimes M \xrightarrow{\alpha} (M^{h\mathcal{B}} \otimes M \otimes N^{h\mathcal{B}} \otimes N) \oplus (N^{h\mathcal{B}} \otimes N \otimes M^{h\mathcal{B}} \otimes M) \\
 \xrightarrow{\gamma'} (M^{h\mathcal{B}} \otimes M) \oplus (N^{h\mathcal{B}} \otimes N) \xrightarrow{\delta_{\text{deg.}0}} \mathcal{B}.
 \end{array}
 \tag{6.16}$$

Similarly, $T_2 T_1 T_2$ is isomorphic to the convolution of the twisted complex

$$\begin{array}{l}
 N^{h\mathcal{B}} \otimes \mathcal{O}_2 \otimes N \xrightarrow{\alpha} (M^{h\mathcal{B}} \otimes M \otimes N^{h\mathcal{B}} \otimes N) \oplus (N^{h\mathcal{B}} \otimes N \otimes M^{h\mathcal{B}} \otimes M) \\
 \xrightarrow{\gamma'} (M^{h\mathcal{B}} \otimes M) \oplus (N^{h\mathcal{B}} \otimes N) \xrightarrow{\delta_{\text{deg.}0}} \mathcal{B}.
 \end{array}
 \tag{6.17}$$

The complexes (6.16) and (6.17) descend to the following complexes of objects in $D(\mathcal{B}\text{-}\mathcal{B})$:

$$S_1 O_1 R_1 \xrightarrow{(6.8)} S_1 R_1 S_2 R_2 \oplus S_2 R_2 S_1 R_1 \xrightarrow{\begin{pmatrix} S_1 R_1 \operatorname{tr} & -\operatorname{tr} S_1 R_1 \\ -\operatorname{tr} S_2 R_2 & S_2 R_2 \operatorname{tr} \end{pmatrix}} S_1 R_1 \oplus S_2 R_2 \xrightarrow{\operatorname{tr} \oplus \operatorname{tr}} \mathcal{B}, \quad (6.18)$$

$$S_2 O_2 R_2 \xrightarrow{(6.9)} S_1 R_1 S_2 R_2 \oplus S_2 R_2 S_1 R_1 \xrightarrow{\begin{pmatrix} S_1 R_1 \operatorname{tr} & -\operatorname{tr} S_1 R_1 \\ -\operatorname{tr} S_2 R_2 & S_2 R_2 \operatorname{tr} \end{pmatrix}} S_1 R_1 \oplus S_2 R_2 \xrightarrow{\operatorname{tr} \oplus \operatorname{tr}} \mathcal{B}. \quad (6.19)$$

By Theorem A.1 to show that (6.16) and (6.17) are homotopy equivalent in $\operatorname{Pre}\text{-}\operatorname{Tr}(\mathcal{B}\text{-}\mathbf{Mod}\text{-}\mathcal{B})$, and hence that $T_1 T_2 T_1$ and $T_2 T_1 T_2$ are isomorphic in $D(\mathcal{B}\text{-}\mathcal{B})$, it suffices to exhibit

$$\begin{aligned} f &\in \operatorname{Hom}_{\mathcal{B}\text{-}\mathcal{B}}^0(M^{h\mathcal{B}} \otimes \mathcal{O}_1 \otimes M, N^{h\mathcal{B}} \otimes \mathcal{O}_2 \otimes N), \\ s_1 &\in \operatorname{Hom}_{\mathcal{B}\text{-}\mathcal{B}}^{-1}(M^{h\mathcal{B}} \otimes \mathcal{O}_1 \otimes M, (M^{h\mathcal{B}} \otimes M \otimes N^{h\mathcal{B}} \otimes N) \oplus (N^{h\mathcal{B}} \otimes N \otimes M^{h\mathcal{B}} \otimes M)), \\ s_2 &\in \operatorname{Hom}_{\mathcal{B}\text{-}\mathcal{B}}^{-2}(M^{h\mathcal{B}} \otimes \mathcal{O}_1 \otimes M, (M^{h\mathcal{B}} \otimes M) \oplus (N^{h\mathcal{B}} \otimes N)), \\ s_3 &\in \operatorname{Hom}_{\mathcal{B}\text{-}\mathcal{B}}^{-3}(M^{h\mathcal{B}} \otimes \mathcal{O}_1 \otimes M, \mathcal{B}) \end{aligned}$$

such that

$$f \text{ is a homotopy equivalence, } ds_1 = \alpha - \beta f, \quad ds_2 = \gamma s_1, \quad ds_3 = -\delta s_2.$$

As in the proof of Theorem 6.1 we can lift ϕ to some homotopy equivalence f , and the existence of

$$\tilde{s}_1 \in \operatorname{Hom}_{\mathcal{B}\text{-}\mathcal{B}}^{-1}(M^{h\mathcal{B}} \otimes \mathcal{O}_1 \otimes M, (M^{h\mathcal{B}} \otimes M \otimes N^{h\mathcal{B}} \otimes N) \oplus (N^{h\mathcal{B}} \otimes N \otimes M^{h\mathcal{B}} \otimes M))$$

with $d\tilde{s}_1 = \alpha - \beta f$ is guaranteed by the commutation of ϕ with (6.8)–(6.9). Since $\gamma\alpha = \gamma\beta = 0$ we deduce that $d(\gamma\tilde{s}_1) = 0$. Thus $\gamma\tilde{s}_1$ defines the class $[\gamma\tilde{s}_1] \in \operatorname{Ext}_{D(\mathcal{B}\text{-}\mathcal{B})}^i(S_1 O_1 R_1, S_1 R_1 \oplus S_2 R_2)$, and since $\delta\gamma = 0$ the composition $\delta[\gamma\tilde{s}_1]$ vanishes in $\operatorname{Ext}_{D(\mathcal{B}\text{-}\mathcal{B})}^i(S_1 O_1 R_1, \mathcal{B})$. By Cor. 6.2 below there exists

$$t_1 \in \operatorname{Hom}_{\mathcal{B}\text{-}\mathcal{B}}^{-1}(M^{h\mathcal{B}} \otimes \mathcal{O}_1 \otimes M, (M^{h\mathcal{B}} \otimes M \otimes N^{h\mathcal{B}} \otimes N) \oplus (N^{h\mathcal{B}} \otimes N \otimes M^{h\mathcal{B}} \otimes M))$$

such that $dt_1 = 0$ and $[\gamma\tilde{s}_1] = [\gamma t_1]$ in $\operatorname{Ext}_{D(\mathcal{B}\text{-}\mathcal{B})}^i(S_1 O_1 R_1, S_1 R_1 \oplus S_2 R_2)$. Set $s_1 = \tilde{s}_1 - t_1$. We still have $ds_1 = \alpha - \beta f$, but the class of γs_1 in $\operatorname{Ext}_{D(\mathcal{B}\text{-}\mathcal{B})}^i(S_1 O_1 R_1, S_1 R_1 \oplus S_2 R_2)$ is zero, so there exists

$$\tilde{s}_2 \in \operatorname{Hom}_{\mathcal{B}\text{-}\mathcal{B}}^{-2}(M^{h\mathcal{B}} \otimes \mathcal{O}_1 \otimes M, (M^{h\mathcal{B}} \otimes M) \oplus (N^{h\mathcal{B}} \otimes N))$$

with $d\tilde{s}_2 = \gamma s_1$. Since $\delta\gamma = 0$, we have $d(\delta\tilde{s}_2) = 0$. Again, by Cor. 6.2 there exists

$$t_2 \in \operatorname{Hom}_{\mathcal{B}\text{-}\mathcal{B}}^{-2}(M^{h\mathcal{B}} \otimes \mathcal{O}_1 \otimes M, (M^{h\mathcal{B}} \otimes M) \oplus (N^{h\mathcal{B}} \otimes N))$$

with $dt_2 = 0$ and $[\delta t_2] = [\delta \tilde{s}_2]$. Set $s_2 = \tilde{s}_2 - t_2$. We still have $ds_2 = \gamma s_1$, but the class of δs_2 in $\text{Ext}_{D(\mathcal{B}-\mathcal{B})}^i(S_1 O_1 R_1, \mathcal{B})$ is zero, so there exists

$$s_3 \in \text{Hom}_{\mathcal{B}-\mathcal{B}}^{-3}(M^{h\mathcal{B}} \otimes \mathcal{O}_1 \otimes M, \mathcal{B})$$

with $ds_3 = -\delta s_2$. □

Lemma 6.1. *There is a diagram of Ext groups in $D(\mathcal{B}-\mathcal{B})$*

$$\begin{array}{ccc}
 & \text{Ext}_{D(\mathcal{B}-\mathcal{B})}^i(*, S_1 R_1) & \\
 \begin{array}{c} \nearrow \kappa_1 \\ \searrow \eta_1 \end{array} & & \begin{array}{c} \longleftarrow \mu_1 \\ \longrightarrow \nu_1 \end{array} \\
 \text{Ext}_{D(\mathcal{B}-\mathcal{B})}^i(*, \mathcal{B}) & & \text{Ext}_{D(\mathcal{B}-\mathcal{B})}^i(*, S_1 R_1 S_2 R_2 \oplus S_2 R_2 S_1 R_1) \\
 \begin{array}{c} \longleftarrow \eta_2 \\ \searrow \kappa_2 \end{array} & & \begin{array}{c} \longleftarrow \nu_2 \\ \longrightarrow \mu_2 \end{array} \\
 & \text{Ext}_{D(\mathcal{B}-\mathcal{B})}^i(*, S_2 R_2) &
 \end{array} \tag{6.20}$$

where $*$ can mean $S_1 O_1 R_1$ or $S_2 O_2 R_2$ (since they are isomorphic in the derived category).

Moreover, $\eta_i \kappa_i = \text{Id}$ and $\nu_i \mu_i = \text{Id}$, while $\nu_2 \mu_1 = -\kappa_2 \eta_1$, $\nu_1 \mu_2 = -\kappa_1 \eta_2$ and $\eta_1 \nu_1 = -\eta_2 \nu_2$.

Proof. Let ν_1 be the map $S_1 R_1 S_2 R_2 \oplus S_2 R_2 S_1 R_1 \xrightarrow{-S_1 R_1 \text{tr} \oplus \text{tr} S_1 R_1} S_1 R_1$. Similarly, let ν_2 be the map $S_1 R_1 S_2 R_2 \oplus S_2 R_2 S_1 R_1 \xrightarrow{\text{tr} S_2 R_2 \oplus -S_2 R_2 \text{tr}} S_2 R_2$. Let η_1 and η_2 be the trace maps $S_1 R_1 \xrightarrow{\text{tr}} \mathcal{B}$ and $S_2 R_2 \xrightarrow{\text{tr}} \mathcal{B}$.

Let μ_1 be the composition

$$\begin{aligned}
 & \text{Ext}_{D(\mathcal{B}-\mathcal{B})}^i(S_2 O_2 R_2, S_1 R_1) \xrightarrow{S_2 R_2(-)S_2 R_2} \text{Ext}_{D(\mathcal{B}-\mathcal{B})}^i(S_2 R_2 S_2 O_2 R_2 S_2 R_2, S_2 R_2 S_1 R_1 S_2 R_2) \\
 & \xrightarrow{S_2 \text{ act } O_2 \text{ act } R_2} \text{Ext}_{D(\mathcal{B}-\mathcal{B})}^i(S_2 O_2 R_2, S_2 R_2 S_1 R_1 S_2 R_2) \\
 & \xrightarrow{-\frac{1}{2}(\text{tr } S_1 R_1 S_2 R_2) \oplus \frac{1}{2}(S_2 R_2 S_1 R_1 \text{tr})} \text{Ext}_{D(\mathcal{B}-\mathcal{B})}^i(S_2 O_2 R_2, S_1 R_1 S_2 R_2 \oplus S_2 R_2 S_1 R_1). \tag{6.21}
 \end{aligned}$$

Let κ_1 be the composition

$$\begin{aligned}
 & \text{Ext}_{D(\mathcal{B}-\mathcal{B})}^i(S_1 O_1 R_1, \mathcal{B}) \xrightarrow{S_1 R_1(-)S_1 R_1} \text{Ext}_{D(\mathcal{B}-\mathcal{B})}^i(S_1 R_1 S_1 O_1 R_1 S_1 R_1, S_1 R_1 S_1 R_1) \\
 & \xrightarrow{S_1 \text{ act } O_1 \text{ act } R_1} \text{Ext}_{D(\mathcal{B}-\mathcal{B})}^i(S_1 O_1 R_1, S_1 R_1 S_1 R_1) \xrightarrow{S_1 R_1 \text{tr}} \text{Ext}_{D(\mathcal{B}-\mathcal{B})}^i(S_1 O_1 R_1, S_1 R_1). \tag{6.22}
 \end{aligned}$$

The maps μ_2 and κ_2 are defined analogously.

We have $\eta_1 \nu_1 = -\eta_2 \nu_2$ by functoriality of the tensor product. The relations $\eta_i \kappa_i = \text{Id}$ and $\nu_i \mu_i = \text{Id}$ are verified directly using Prop. 2.11. Let us prove that $\nu_2 \mu_1 = -\kappa_2 \eta_1$.

Consider the composition

$$\begin{aligned} \text{Ext}_{D(\mathcal{B}-\mathcal{B})}^i(S_2 O_2 R_2, \mathcal{B}) &\xrightarrow{S_2 R_2(-)S_2 R_2} \text{Ext}_{D(\mathcal{B}-\mathcal{B})}^i(S_2 R_2 S_2 O_2 R_2 S_2 R_2, S_2 R_2 S_2 R_2) \\ &\xrightarrow{S_2 \text{ act } O_2 \text{ act } R_2} \text{Ext}_{D(\mathcal{B}-\mathcal{B})}^i(S_2 O_2 R_2, S_2 R_2 S_2 R_2) \\ &\xrightarrow{-\frac{1}{2}(\text{tr } S_2 R_2) \oplus \frac{1}{2}(S_2 R_2 \text{ tr})} \text{Ext}_{D(\mathcal{B}-\mathcal{B})}^i(S_2 O_2 R_2, S_2 R_2 \oplus S_2 R_2) \end{aligned} \quad (6.23)$$

and the map

$$\begin{aligned} \text{Ext}_{D(\mathcal{B}-\mathcal{B})}^i(S_2 O_2 R_2, S_1 R_1 S_2 R_2 \oplus S_2 R_2 S_1 R_1) \\ \xrightarrow{\text{tr } S_2 R_2 \oplus S_2 R_2 \text{ tr}} \text{Ext}_{D(\mathcal{B}-\mathcal{B})}^i(S_2 O_2 R_2, S_2 R_2 \oplus S_2 R_2). \end{aligned} \quad (6.24)$$

Applying the map $S_1 R_1 \xrightarrow{\text{tr}} \mathcal{B}$ to every component of (6.21) and using functoriality we see that the square

$$\begin{array}{ccc} \text{Ext}_{D(\mathcal{B}-\mathcal{B})}^i(S_2 O_2 R_2, S_1 R_1) & \xrightarrow{\eta_1} & \text{Ext}_{D(\mathcal{B}-\mathcal{B})}^i(S_2 O_2 R_2, \mathcal{B}) \\ \mu_1 \downarrow & & \downarrow (6.23) \\ \text{Ext}_{D(\mathcal{B}-\mathcal{B})}^i(S_2 O_2 R_2, S_1 R_1 S_2 R_2 \oplus S_2 R_2 S_1 R_1) & \xrightarrow{(6.24)} & \text{Ext}_{D(\mathcal{B}-\mathcal{B})}^i(S_2 O_2 R_2, S_2 R_2 \oplus S_2 R_2) \end{array}$$

commutes. By inspection, the composition of (6.23) with the map

$$\text{Ext}_{D(\mathcal{B}-\mathcal{B})}^i(S_2 O_2 R_2, S_2 R_2 \oplus S_2 R_2) \xrightarrow{\text{Id} \oplus -\text{Id}} \text{Ext}_{D(\mathcal{B}-\mathcal{B})}^i(S_2 O_2 R_2, S_2 R_2) \quad (6.25)$$

is $-\kappa_2$, while the composition of (6.24) with (6.25) is ν_2 . It follows that $\nu_2 \mu_1 = -\kappa_2 \eta_1$, as desired. \square

Corollary 6.2. *The sequence*

$$\begin{aligned} \text{Ext}_{D(\mathcal{B}-\mathcal{B})}^i(S_1 O_1 R_1, S_1 R_1 S_2 R_2 \oplus S_2 R_2 S_1 R_1) &\xrightarrow{\gamma} \text{Ext}_{D(\mathcal{B}-\mathcal{B})}^i(S_1 O_1 R_1, S_1 R_1 \oplus S_2 R_2) \\ &\xrightarrow{\delta} \text{Ext}_{D(\mathcal{B}-\mathcal{B})}^i(S_1 O_1 R_1, \mathcal{B}) \end{aligned}$$

is exact in its middle term and surjective onto its last term.

Appendix. On homotopy equivalences of twisted complexes

Let \mathcal{C} be a strongly pretriangulated DG-category. The example one wants to keep in mind is $\mathcal{P}(\mathcal{A})$ for some DG-category \mathcal{A} , so that $H^0(\mathcal{C}) = D(\mathcal{A})$. Let (E_i, q_{ij}) be a twisted complex over \mathcal{C} . The objects E_i and the degree 0 morphisms $q_{i(i+1)}$ form an ordinary differential complex over $H^0(\mathcal{C})$:

$$\cdots \xrightarrow{q_{(i-2)(i-1)}} E_{i-1} \xrightarrow{q_{(i-1)i}} E_i \xrightarrow{q_{i(i+1)}} E_{i+1} \xrightarrow{q_{(i+1)(i+2)}} \cdots$$

Let (E_i, q_{ij}) and (F_i, r_{ij}) be twisted complexes over \mathcal{C} . We would like to know when their convolutions $\{E_i, q_{ij}\}$ and $\{F_i, r_{ij}\}$ are isomorphic in $H^0(\mathcal{C})$. Since \mathcal{C} was assumed

to be strongly pretriangulated, constructing an isomorphism of $\{E_i, q_{ij}\}$ and $\{F_i, r_{ij}\}$ in $H^0(\mathcal{C})$ is the same thing as constructing a homotopy equivalence of (E_i, q_{ij}) and (F_i, r_{ij}) in $\text{Pre-Tr}(\mathcal{C})$.

Suppose that the underlying differential complexes of (E_i, q_{ij}) and (F_i, r_{ij}) are isomorphic, more specifically—that we have a set of isomorphisms $E_i \xrightarrow{f_i} F_i$ in $H^0(\mathcal{C})$ which gives an isomorphism of these differential complexes. This alone does not ensure that (E_i, q_{ij}) and (F_i, r_{ij}) are isomorphic in $H^0(\mathcal{C})$, since the same differential complex over $H^0(\mathcal{C})$ can, in general, be lifted to several non-homotopically equivalent twisted complexes over \mathcal{C} . Thus the question arises: *what are the sufficient conditions on f_i for us to be able to cook up a homotopy equivalence of (E_i, q_{ij}) and (F_i, r_{ij}) from them?*

When trying to construct this homotopy equivalence even in simplest cases, one encounters a number of conditions which, at first glance, seem unavoidable, but in fact are redundant:

Example A.1. Let $E \xrightarrow{q} G$ and $F \xrightarrow{r} G$ be twisted complexes over \mathcal{C} . Let $E \xrightarrow{f} F$ be a homotopy equivalence in \mathcal{C} such that the square

$$\begin{array}{ccc} E & \xrightarrow{q} & G \\ f \downarrow & & \downarrow \text{Id} \\ F & \xrightarrow{r} & G \end{array} \tag{A.1}$$

commutes in $H^0(\mathcal{C})$. Since $H^0(\mathcal{C})$ is triangulated, there exists an isomorphism $\text{Cone}(q) \rightarrow \text{Cone}(r)$ which extends this square in $H^0(\mathcal{C})$ to an isomorphism of exact triangles. It follows that we can extend $E \xrightarrow{f} F$ and $G \xrightarrow{\text{Id}} G$ to a homotopy equivalence in $\text{Pre-Tr}(\mathcal{C})$ of the twisted complexes $E \xrightarrow{q} G$ and $F \xrightarrow{r} G$.

If we actually try and construct this homotopy equivalence, we run into the following type of problems:

Claim. Let $g \in \text{Hom}_{\mathcal{C}}^0(F, E)$ be a homotopy inverse of f ; in other words, there exist $h \in \text{Hom}_{\mathcal{C}}^{-1}(E, E)$ and $h' \in \text{Hom}_{\mathcal{C}}^{-1}(F, F)$ such that $gf - \text{Id} = dh$ and $fg - \text{Id} = dh'$. Then there exist mutually inverse homotopy equivalences

$$\begin{array}{ccc} \begin{array}{ccc} E & \xrightarrow{q} & G \\ f \downarrow & \searrow t & \downarrow \text{Id} \\ F & \xrightarrow{r} & G \end{array} & \begin{array}{ccc} F & \xrightarrow{r} & G \\ g \downarrow & \searrow t' & \downarrow \text{Id} \\ E & \xrightarrow{q} & G \end{array} & \begin{array}{l} t \in \text{Hom}_{\mathcal{C}}^{-1}(E, G) \\ t' \in \text{Hom}_{\mathcal{C}}^{-1}(F, G) \end{array} \end{array} \tag{A.2}$$

of $E \xrightarrow{q} G$ and $F \xrightarrow{r} G$ if and only if h and h' can be chosen so that the following equivalent conditions hold:

- $r(fh - h'f) = ds$ for some $s \in \text{Hom}_{\mathcal{C}}^{-2}(E, G)$.
- $q(gh' - hg) = ds'$ for some $s' \in \text{Hom}_{\mathcal{C}}^{-2}(F, G)$.

Proof. Straightforward verification. □

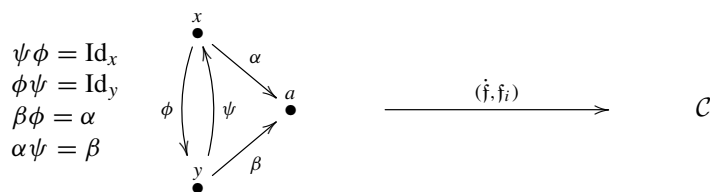
A priori, there is no reason to expect a class like $r(fh - h'f)$ in $\text{Hom}_{\mathcal{C}}^{-1}(E, G)$ to be null-homotopic. In fact, for general h and h' it would not be. So this may seem like a genuinely necessary condition.

However, it turns out that we can always choose h and h' so that even $fh - h'f$ and $gh' - hg$ are null-homotopic. Since $dq = dr = 0$, it would also imply the conditions above.

The explanation is: $fh - h'f$ and $gh' - hg$ are both killed by the differential, and thus define classes $\xi \in \text{Hom}_{H^0(\mathcal{C})}^{-1}(E, F)$ and $\xi' \in \text{Hom}_{H^0(\mathcal{C})}^{-1}(F, E)$, respectively. Since f and g are isomorphisms in $H^0(\mathcal{C})$, they identify both $\text{Hom}_{H^0(\mathcal{C})}^{-1}(E, F)$ and $\text{Hom}_{H^0(\mathcal{C})}^{-1}(F, E)$ with $\text{Hom}_{H^0(\mathcal{C})}^{-1}(E, E)$. A priori, neither ξ nor ξ' is zero, but one can check that ξ and $-\xi'$ give the same class in $\text{Hom}_{H^0(\mathcal{C})}^{-1}(E, E)$. We can therefore correct $h \in \text{Hom}_{\mathcal{C}}^{-1}(E, E)$ by this class and kill off both ξ and ξ' , as required.

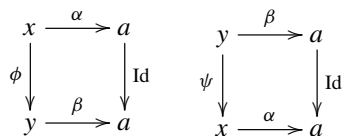
It is not a calculation one would want to try and write down in a larger, more complicated scenario. Fortunately, there turns out to be a more conceptual argument. It requires us to consider A_∞ -categories and A_∞ -functors—see [Kel01] and [LH03, §8] for the basics. In particular, we use the convention in [LH03, §8] for denoting A_∞ -functors as (\dot{f}, f_i) where \dot{f} is the object map, f_1 is the morphism map and $f_{i \geq 2}$ are the higher morphism maps.

A choice of h and h' as above and also of $j \in \text{Hom}_{\mathcal{C}}^{-2}(X, Y)$ and $j' \in \text{Hom}_{\mathcal{C}}^{-2}(Y, X)$ such that $fh - h'f = dj$ and $gh' - hg = dj'$ can readily be checked to be a part of precisely the data necessary to define a strictly unital A_∞ -functor



which sends x, y, a to E, F, G and $\phi, \psi, \alpha, \beta$ to f, g, q, r . Here, the quiver on the left defines an additive k -category whose objects are the vertices of the quiver and whose Hom-spaces are generated by the paths in the quiver, modulo the indicated relations. The trivial path from a vertex to itself correspond to its identity morphism. Denote this category by $\bar{\mathcal{B}}_1$; we think of it as of a DG-category concentrated in degree 0.

Conversely, any A_∞ -functor $\bar{\mathcal{B}}_1 \xrightarrow{(\dot{f}, f_i)} \mathcal{C}$ as above contains the data of homotopy equivalences (A.2). This is because (\dot{f}, f_i) extends naturally to an A_∞ -functor $\text{Pre-Tr}(\bar{\mathcal{B}}_1) \xrightarrow{(\dot{f}, f_i)} \text{Pre-Tr}(\mathcal{C})$. In $\text{Pre-Tr}(\bar{\mathcal{B}}_1)$ the twisted complexes $x \xrightarrow{\alpha} a$ and $y \xrightarrow{\beta} a$ are isomorphic. Specifically,



are mutually inverse isomorphisms. Their images under f_1 are the morphisms

$$\begin{array}{ccc}
 E & \xrightarrow{q} & G \\
 \downarrow f & \searrow f_2(\beta, \phi) & \downarrow \text{Id} \\
 F & \xrightarrow{r} & G
 \end{array}
 \qquad
 \begin{array}{ccc}
 F & \xrightarrow{r} & G \\
 \downarrow g & \searrow f_2(\alpha, \psi) & \downarrow \text{Id} \\
 E & \xrightarrow{q} & G
 \end{array}$$

in $\text{Pre-Tr}(\mathcal{C})$. Since $(\dot{f}, H^0(f_1))$ is an exact functor, these become mutually inverse isomorphisms in $H^0(\text{Pre-Tr}(\mathcal{C}))$. Thus, they are the mutually inverse homotopy equivalences (A.2) we want.

To construct a strictly unital A_∞ -functor $\bar{\mathcal{B}}_1 \xrightarrow{(\dot{f}, f_i)} \mathcal{C}$ it suffices to construct a strictly unital A_∞ -functor $\mathcal{B}_1 \xrightarrow{(\dot{g}, g_i)} \mathcal{C}$ where \mathcal{B}_1 is the category

$$\begin{array}{ccc}
 x & & \\
 \downarrow \phi & \searrow \alpha & \\
 y & & a \\
 & \nearrow \beta & \\
 & & \bullet
 \end{array}
 \qquad \beta\phi = \alpha$$

Roughly, this is because $\bar{\mathcal{B}}_1$ is the minimal A_∞ -structure of a certain DG-quotient of \mathcal{B}_1 whose universal properties ensure that $\mathcal{B}_1 \xrightarrow{(\dot{g}, g_i)} \mathcal{C}$ filters through some $\bar{\mathcal{B}}_1 \xrightarrow{(\dot{f}, f_i)} \mathcal{C}$. We will give the full argument in a greater generality later on in this section.

Thus we are reduced to constructing a strictly unital A_∞ -functor $\mathcal{B}_1 \xrightarrow{(\dot{g}, g_i)} \mathcal{C}$ which sends x, y, a to E, F, G and ϕ, α, β to f, q, r . The data of such a functor is simply the choice of $f_2(\beta, \phi) \in \text{Hom}_{\mathcal{C}}^{-1}(E, G)$ such that

$$q - rf = f_2(\beta, \phi).$$

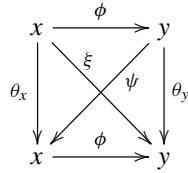
The existence of such a class in $\text{Hom}_{\mathcal{C}}^{-1}(E, G)$ is precisely the condition that (A.1) commutes in $H^0(\mathcal{C})$.

To sum up, a sufficient condition for the homotopy equivalence $E \xrightarrow{f} F$ to induce a homotopy equivalence

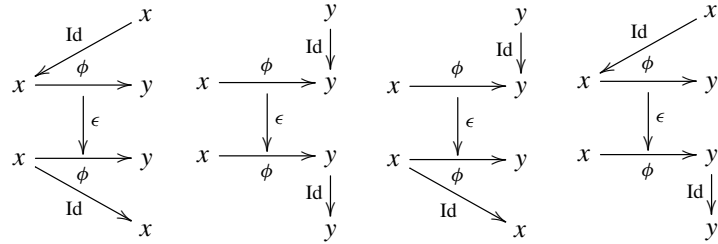
$$\{E \xrightarrow{q} G\} \rightarrow \{F \xrightarrow{r} G\} \tag{A.3}$$

is that f must commute with q and r in $H^0(\mathcal{C})$. This is also precisely the condition that a strictly unital A_∞ -functor $\mathcal{B}_1 \rightarrow \mathcal{C}$ exists which sends x, y, a to E, F, G and ϕ, α, β to f, q, r . All the other conditions which seemingly arise when one naively tries to construct the homotopy equivalence (A.3) are part of the data necessary to lift this functor to a functor $\bar{\mathcal{B}}_1 \rightarrow \mathcal{C}$. The latter gets done for us automatically by the universal properties of DG-quotients.

$\text{Pre-Tr}(\mathcal{B}_n)$. Define a functor from $\text{Pre-Tr}(\mathcal{B}_n)/(x \xrightarrow{\phi} y)$ to \mathcal{A} by sending ϵ to



Define a functor in the opposite direction by sending θ_x, θ_y, ψ and ξ to the compositions



in $\text{Pre-Tr}(\mathcal{B}_n)/(x \xrightarrow{\phi} y)$, respectively. One can readily check that these functors are mutually inverse. Hence $\text{Pre-Tr}(\mathcal{B}_n)/(x \xrightarrow{\phi} y)$ is isomorphic to \mathcal{A} , and the result follows. \square

Recall that an A_∞ -category is called *minimal* if it has $m_1 = 0$. Let \mathcal{A} be an A_∞ -category. The *minimal model* of \mathcal{A} is a minimal A_∞ -category \mathcal{A}' together with an A_∞ -quasi-isomorphism $\mathcal{A}' \rightarrow \mathcal{A}$. Such a model always exists and is unique up to A_∞ -isomorphism (see [LH03, §1.4.1] and [KS01, S6.4]).

Lemma A.4. *There exists a strictly unital A_∞ -quasi-isomorphism*

$$\bar{\mathcal{B}}_n \xrightarrow{(\hat{g}, g_i)} \mathcal{B}_n^f \tag{A.6}$$

which gives $\bar{\mathcal{B}}_n$ the structure of the minimal model of \mathcal{B}_n^f .

Proof. Recall that $\bar{\mathcal{B}}_n$ is an ordinary category considered as an A_∞ -category concentrated in degree 0. In particular, $\bar{\mathcal{B}}_n$ can be identified with its own graded homotopy category $H^\bullet(\bar{\mathcal{B}}_n)$.

The category $\bar{\mathcal{B}}_n$ is defined by the quiver (A.4), while Lemma A.3 identifies \mathcal{B}_n^f with the category defined by the DG-quiver (A.5). Forgetting the relations and identifying vertices and arrows which have the same labels gives the quiver (A.4) the structure of a subquiver of (A.5). This structure defines a map \hat{g} from the set of objects of $\bar{\mathcal{B}}_n$ to the set of objects of \mathcal{B}_n^f and a map g_1 of morphism spaces of $\bar{\mathcal{B}}_n$ into morphism spaces of \mathcal{B}_n^f . These are compatible with differentials, but are not compatible with compositions.

By inspection, (\hat{g}, g_1) does define an isomorphism

$$\bar{\mathcal{B}}_n \xrightarrow{\sim} H^\bullet(\mathcal{B}_n^f)$$

of graded homotopy categories. We can therefore apply the procedure described in [KS01, §6.4]. It can be readily checked that it constructs $\mathfrak{g}_{\geq 2}$ which extend \mathfrak{g} and \mathfrak{g}_1 to a strictly unital A_∞ -quasi-isomorphism $\bar{\mathcal{B}}_n \xrightarrow{(\dot{\mathfrak{g}}, \mathfrak{g}_i)} \mathcal{B}_n^f$, as required. \square

Before we proceed, we need to state the following well-known fact:

Lemma A.5. *Let \mathcal{A} be a DG-category, let $m \leq n$ be two integers and let A_m, \dots, A_n be objects of \mathcal{A} . The one-sided twisted complexes*

$$(E_i, q_{ij}) \in \text{Pre-Tr}(\mathcal{A}) \quad \text{with} \quad E_i = \begin{cases} A_i & \text{for } m \leq i \leq n, \\ 0 & \text{otherwise,} \end{cases}$$

are in 1-to-1 correspondence with the strictly unital A_∞ -functors

$$\gamma_i \gamma_{i+1} = 0 \quad \begin{matrix} a_m & & a_{m+1} & & a_{n-1} & & a_n \\ \bullet & \xrightarrow{\gamma_m} & \bullet & \xrightarrow{\gamma_{m+1}} & \dots & \xrightarrow{\gamma_{n-2}} & \bullet & \xrightarrow{\gamma_{n-1}} & \bullet \end{matrix} \xrightarrow{(\dot{\mathfrak{f}}, \mathfrak{f}_i)} \mathcal{C}$$

with $\dot{\mathfrak{f}}(a_i) = A_i$.

Proof. Mutually inverse maps between the two sets can be defined by setting

$$\dot{\mathfrak{f}}_k(\gamma_{i+k-1}, \gamma_{i+k-2}, \dots, \gamma_i) = (-1)^{i-1} q_{i(i+k)} \quad \forall i \in \{m, \dots, n\} \text{ and } k \in \{1, \dots, n-i\}$$

and vice versa. \square

Let \mathcal{C} be a strongly pretriangulated category and let (A_i, g_{ij}) be a one-sided twisted complex over \mathcal{C} concentrated in degrees $1, \dots, n$. Let (E_i, q_{ij}) and (F_i, r_{ij}) be one-sided twisted complexes over \mathcal{C} concentrated in degrees $0, \dots, n$ whose twisted subcomplexes supported in degrees $1, \dots, n$ are both equal to (A_i, g_{ij}) .

Let A denote the convolution of (A_i, g_{ij}) . Consider the closed degree 1 morphisms (q_{0j}) and (r_{0j}) from E_0 and F_0 to (A_i, g_{ij}) in $\text{Pre-Tr}(\mathcal{C})$. Denote by $E_0 \xrightarrow{q_0} A$ and $F_0 \xrightarrow{r_0} A$ the corresponding morphisms in \mathcal{C} .

Recall that \mathcal{B}_n is the category defined by

$$\begin{matrix} \beta\phi = \alpha \\ \gamma_1\alpha = \gamma_1\beta = 0 \\ \gamma_{i+1}\gamma_i = 0 \end{matrix} \quad \begin{matrix} x \\ \bullet \\ \downarrow \phi \\ y \\ \bullet \end{matrix} \quad \begin{matrix} \nearrow \alpha \\ \bullet \\ \nearrow \beta \end{matrix} \quad \begin{matrix} a_1 & \xrightarrow{\gamma_1} & a_2 & \xrightarrow{\gamma_2} & a_3 & \xrightarrow{\gamma_3} & \dots & \xrightarrow{\gamma_{n-2}} & a_{n-1} & \xrightarrow{\gamma_{n-1}} & a_n \end{matrix} \quad (A.7)$$

Proposition A.6. *There exists a strictly unital A_∞ -functor*

$$\mathcal{B}_n \xrightarrow{(\dot{\mathfrak{f}}, \mathfrak{f}_i)} \mathcal{C}$$

whose restrictions to the full subcategories of \mathcal{B}_n supported at x, a_1, \dots, a_n and y, a_1, \dots, a_n correspond to the twisted complexes (E_i, q_{ij}) and (F_i, r_{ij}) if and only if the following two equivalent conditions hold:

(1) There exist $f \in \text{Hom}_{\mathcal{C}}^0(E_0, F_0)$ and $s_i \in \text{Hom}_{\mathcal{C}}^{-k}(E_0, A_k)$ for $k \in \{1, \dots, n\}$ such that

$$q_{0k} - r_{0k}f = \sum_{1 \leq j \leq k-1} q_{jk}s_j + (-1)^k ds_k. \tag{A.8}$$

(2) There exists $f \in \text{Hom}_{H^0(\mathcal{C})}(E_0, F_0)$ such that

$$\begin{array}{ccc} E_0 & \xrightarrow{q_0} & A[1] \\ f \downarrow & \nearrow r_0 & \\ F_0 & & \end{array}$$

commutes in $H^0(\mathcal{C})$.

Proof. The existence of $(\check{f}, \check{f}_i) \Leftrightarrow (1)$. The condition that (\check{f}, \check{f}_i) restricts on x, a_1, \dots, a_n and y, a_1, \dots, a_n to the functors corresponding to (E_i, q_{ij}) and (F_i, r_{ij}) determines \check{f} and all the values of \check{f}_i other than

$$\check{f}_1(\phi), \check{f}_2(\beta, \phi), \check{f}_3(\gamma_1, \beta, \phi), \dots, \check{f}_{n+1}(\gamma_{n-1}, \dots, \gamma_1, \beta, \phi). \tag{A.9}$$

One can readily verify that if we set these to f, s_1, \dots, s_n , then the standard relations which (A.9) must satisfy according to the definition of an A_∞ -functor [Kel01, §3.4] become precisely the equations (A.8), and vice versa.

(1) \Leftrightarrow (2). Let $s_k \in \text{Hom}_{\mathcal{C}}^{-k}(E_0, A_k)$ for $k \in \{1, \dots, n\}$. Consider the degree 0 morphism $E_0 \xrightarrow{(s_k)} (A_i, g_{ij})$ in $\text{Pre-Tr}(\mathcal{C})$. It is a straightforward verification that $d(s_k)$ is the morphism $E_0 \rightarrow (A_i, g_{ij})$ whose component in $\text{Hom}_{\mathcal{C}}^{-k+1}(E_0, A_k)$ is precisely the RHS of (A.9).

On the other hand, for any $f \in \text{Hom}_{\mathcal{C}}^0(E_0, F_0)$ the LHS of (A.9) is the component in $\text{Hom}_{\mathcal{C}}^{-k+1}(E_0, A_k)$ of the morphism $E_0 \xrightarrow{(q_{0j})-(r_{0j})f} (A_i, g_{ij})$ in $\text{Pre-Tr}(\mathcal{C})$.

We conclude that (1) is equivalent to the existence of $f \in \text{Hom}_{\mathcal{C}}^0(E_0, F_0)$ and $s \in \text{Hom}_{\mathcal{C}}^0(E_0, A)$ such that $q_0 - r_0f = ds$. This is precisely the claim of (2). \square

The following is the main result of this section:

Theorem A.1. *Let $E_0 \xrightarrow{f} F_0$ be a homotopy equivalence satisfying the equivalent conditions of Prop. A.6. Then there exists a homotopy equivalence*

$$(E_i, q_{ij}) \xrightarrow{(f_{ij})} (F_i, r_{ij})$$

in $\text{Pre-Tr}(\mathcal{C})$.

Proof. By Prop. A.6 there exists a strictly unital A_∞ -functor

$$\mathcal{B}_n \xrightarrow{(\check{f}, \check{f}_i)} \mathcal{C}$$

with $\check{f}_1(\phi) = f$. It extends naturally to a strictly unital A_∞ -functor

$$\text{Pre-Tr}(\mathcal{B}_n) \xrightarrow{(\check{f}, \check{f}_i)} \text{Pre-Tr}(\mathcal{C}).$$

By [Kel06, §4.3] there exists a corresponding quasi-functor

$$\text{Pre-Tr}(\mathcal{B}_n) \xrightarrow{\Phi} \text{Pre-Tr}(\mathcal{C})$$

in $\text{Ho}(\text{DG-Cat})$ with $H\Phi \simeq H\mathfrak{f}$ as functors $H^0(\text{Pre-Tr}(\mathcal{B}_n)) \rightarrow H^0(\text{Pre-Tr}(\mathcal{C}))$. Since $H\mathfrak{f}(\phi) = f$ and since f is an isomorphism in $H^0(\mathcal{C})$, it follows that $H\Phi(\text{Cone}(\phi)) = 0$. By the universal property of DG-quotients [Dri04, Th. 1.6.2] the quasi-functor Φ lifts to a quasi-functor

$$\text{Pre-Tr}(\mathcal{B}_n)/\text{Cone}(\phi) \xrightarrow{\Phi'} \text{Pre-Tr}(\mathcal{C})$$

with $\Phi = \Phi'Q$ where Q is the quotient quasi-functor $\text{Pre-Tr}(\mathcal{B}_n) \rightarrow \text{Pre-Tr}(\mathcal{B}_n)/\text{Cone}(\phi)$. Denote by

$$\text{Pre-Tr}(\mathcal{B}_n)/\text{Cone}(\phi) \xrightarrow{(\dot{\mathfrak{f}}', \dot{\mathfrak{f}}'_i)} \text{Pre-Tr}(\mathcal{C})$$

the corresponding strictly unital A_∞ -functor. We have $(\dot{\mathfrak{f}}, \dot{\mathfrak{f}}_i) = (\dot{\mathfrak{f}}', \dot{\mathfrak{f}}'_i)Q$, and hence restricting to the full subcategory \mathcal{B}_n^f of $\text{Pre-Tr}(\mathcal{B}_n)/\text{Cone}(\phi)$ consisting of objects of \mathcal{B}_n we obtain a strictly unital A_∞ -functor

$$\mathcal{B}_n^f \xrightarrow{(\dot{\mathfrak{f}}', \dot{\mathfrak{f}}'_i)} \mathcal{C}.$$

Recall that in Lemma A.4 we have constructed a strictly unital A_∞ -quasi-isomorphism $\bar{\mathcal{B}}_n \xrightarrow{(\dot{\mathfrak{g}}, \dot{\mathfrak{g}}_i)} \mathcal{B}_n^f$ which gives $\bar{\mathcal{B}}_n$ the structure of the minimal model of \mathcal{B}_n^f . Taking the composition of $\bar{\mathcal{B}}_n \xrightarrow{(\dot{\mathfrak{g}}, \dot{\mathfrak{g}}_i)} \mathcal{B}_n^f \xrightarrow{(\dot{\mathfrak{f}}', \dot{\mathfrak{f}}'_i)} \mathcal{C}$ we obtain the strictly unital A_∞ -functor denoted

$$\bar{\mathcal{B}}_n \xrightarrow{(\dot{\mathfrak{h}}, \dot{\mathfrak{h}}_i)} \mathcal{C}.$$

We claim that $\bar{\mathcal{B}}_n \xrightarrow{(\dot{\mathfrak{h}}, \dot{\mathfrak{h}}_i)} \mathcal{C}$ restricts on the full subcategory $\mathcal{B}_n \hookrightarrow \bar{\mathcal{B}}_n$ to $\mathcal{B}_n \xrightarrow{(\dot{\mathfrak{f}}, \dot{\mathfrak{f}}_i)} \mathcal{C}$. As $(\dot{\mathfrak{f}}, \dot{\mathfrak{f}}_i) = (\dot{\mathfrak{f}}', \dot{\mathfrak{f}}'_i)Q$ this reduces to the following diagram being commutative:

$$\begin{array}{ccc} \mathcal{B}_n & & \\ \downarrow & \searrow Q & \\ \bar{\mathcal{B}}_n & \xrightarrow{(\dot{\mathfrak{g}}, \dot{\mathfrak{g}}_i)} & \mathcal{B}_n^f \end{array} \tag{A.10}$$

This is a straightforward check. On the one hand, in Lemma A.3 we have constructed an explicit isomorphism between \mathcal{B}_n^f and the category defined by (A.5). One can check that it identifies the DG-quotient functor $\mathcal{B}_n \xrightarrow{Q} \mathcal{B}_n^f$ with the functor induced by the inclusion of (A.7) into (A.5) as quivers with relations. On the other hand, in Lemma A.4 we have used the above isomorphism between \mathcal{B}_n^f and (A.5) to define $\dot{\mathfrak{g}}$ and $\dot{\mathfrak{g}}_1$ by the quiver inclusion of (A.4) into (A.5) which ignores relations. However, restricted from (A.4) to (A.7), this inclusion does respect the relations. Therefore $(\dot{\mathfrak{g}}, \dot{\mathfrak{g}}_1)$ restricted to \mathcal{B}_n is a genuine functor. One can check that this forces $\dot{\mathfrak{g}}_{\geq 3}$ constructed by the procedure in [KS01, §6.4] to be zero when restricted to \mathcal{B}_n . Thus $(\dot{\mathfrak{g}}, \dot{\mathfrak{g}}_i)$ restricted to \mathcal{B}_n is just the

functor $(\dot{\mathfrak{g}}, \mathfrak{g}_1)$, i.e. the functor defined by the inclusion of (A.7) into (A.5). The claim follows.

In $\text{Pre-Tr}(\bar{\mathcal{B}}_n)$ the twisted complexes $x \xrightarrow{\alpha} a_1 \rightarrow \dots \xrightarrow{\gamma_{n-1}} a_n$ and $y \xrightarrow{\beta} a_1 \rightarrow \dots \xrightarrow{\gamma_{n-1}} a_n$ are isomorphic, for instance the following

$$\begin{array}{ccccccccccc}
 \bullet & \xrightarrow{\alpha} & \bullet & \xrightarrow{\gamma_1} & \bullet & \xrightarrow{\gamma_2} & \dots & \xrightarrow{\gamma_{n-2}} & \bullet & \xrightarrow{\gamma_{n-1}} & \bullet \\
 \downarrow \phi & & \downarrow \text{Id} & & \downarrow \text{Id} & & & & \downarrow \text{Id} & & \downarrow \text{Id} \\
 \bullet & \xrightarrow{\beta} & \bullet & \xrightarrow{\gamma_1} & \bullet & \xrightarrow{\gamma_2} & \dots & \xrightarrow{\gamma_{n-2}} & \bullet & \xrightarrow{\gamma_{n-1}} & \bullet
 \end{array} \quad (\text{A.11})$$

is an isomorphism of twisted complexes. Hence the complexes are also isomorphic in $H^0(\text{Pre-Tr}(\bar{\mathcal{B}}_n))$, and hence their images under $(\dot{\mathfrak{h}}, H^0(\mathfrak{h}_1))$ are isomorphic in $H^0(\text{Pre-Tr}(\mathcal{C}))$. But by the claim above, $(\dot{\mathfrak{h}}, \mathfrak{h}_i)$ and $(\dot{\mathfrak{f}}, \mathfrak{f}_i)$ agree on the subcategory $\text{Pre-Tr}(\mathcal{B}_n)$ of $\text{Pre-Tr}(\bar{\mathcal{B}}_n)$. Hence $(\dot{\mathfrak{h}}, \mathfrak{h}_i)$ takes $x \xrightarrow{\alpha} a_1 \rightarrow \dots \xrightarrow{\gamma_{n-1}} a_n$ and $y \xrightarrow{\beta} a_1 \rightarrow \dots \xrightarrow{\gamma_{n-1}} a_n$ to (E_i, q_{ij}) and (F_i, r_{ij}) . The claim of the theorem follows. \square

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