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# A sharply 2-transitive group without a non-trivial abelian normal subgroup

**Abstract.** We show that any group $G$ is contained in some sharply 2-transitive group $\mathcal{G}$ without a non-trivial abelian normal subgroup. This answers a long-standing open question. The involutions in the groups $\mathcal{G}$ that we construct have no fixed points.

**Keywords.** Sharply 2-transitive, free product, HNN extension, malnormal

## 1. Introduction

The *finite* sharply 2-transitive groups were classified by Zassenhaus [Z] in 1936 and it is known that any finite sharply 2-transitive group contains a non-trivial abelian normal subgroup.

In the infinite situation no classification is known (see [MK, Problem 11.52, p. 52]). It was a long-standing open problem whether every infinite sharply 2-transitive group contains a non-trivial abelian normal subgroup. Tits [Ti] proved that this holds for locally compact connected sharply 2-transitive groups. Several other papers showed that under certain special conditions the assertion holds [BN, GMS, GlGu, M, T2, Tu, W]. The reader may wish to consult the Appendix for more detail, and for a description of our main results using permutation-group-theoretic language.

An equivalent formulation of the above problem is whether every near-domain is a near-field (see [Hall, K, SSS] and the Appendix below).

We here show that this is not the case. We construct a sharply 2-transitive infinite group without a non-trivial abelian normal subgroup. In fact, the construction is similar in flavor to the free completion of partial generalized polygons [T1].

Recall that a proper subgroup $A$ of a group $G$ is *malnormal* in $G$ if $A \cap g^{-1}Ag = 1$ for all $g \in G \smallsetminus A$.

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**Theorem 1.1.** *Let $G$ be a group with a malnormal subgroup $A$ and an involution $t \in G \setminus A$ such that $A$ contains no involutions. Then for any $u, v \in G$ with $Au \neq Av$ there exist*

(a) *an extension $G \leq G_1$;*
(b) *a malnormal subgroup $A_1$ of $G_1$ such that $A_1$ does not contain involutions and satisfies $A_1 \cap G = A$;*
(c) *an element $f \in G_1$ such that $A_1 f = A_1 u$ and $A_1 t f = A_1 v$.*

**Remark 1.2.** It is easy to see (see §2) that in Theorem 1.1 we may assume that $u = 1$, $v \notin AtA$ and that either (1) $v^{-1} \notin AvA$, or (2) $v$ is an involution. If case (1) holds we take $G_1 = G * \langle f \rangle$ to be the free product of $G$ with an infinite cyclic group generated by $f$, and $A_1 = \langle A, f, tfv^{-1} \rangle$. If case (2) holds we take $G_1 = \langle G, f \mid f^{-1}tf = s \rangle$ an HNN extension and $A_1 = \langle A, f \rangle$.

As a corollary to Theorem 1.1 we get the following.

**Theorem 1.3.** *Let $G$ be a group with a malnormal subgroup $A$ such that $A$ contains no involutions. Assume further that $G$ is* `not` *sharply 2-transitive on the set $A\backslash G$ of right cosets. Then $G$ is contained in a group $\mathcal{G}$ having a malnormal subgroup $\mathcal{A}$ such that*

(1) *$\mathcal{A} \cap G = A$;*
(2) *$\mathcal{G}$ is sharply 2-transitive on the set $X := \mathcal{A}\backslash\mathcal{G}$ of right cosets;*
(3) *$\mathcal{A}$ contains no involutions (i.e. $\mathcal{G}$ is of permutational characteristic 2);*
(4) *$\mathcal{G}$ does not contain a non-trivial abelian normal subgroup;*
(5) *if $G$ is infinite then $G$ and $\mathcal{G}$ have the same cardinality (similarly for $X$ and $A\backslash G$).*

As an immediate consequence of Theorem 1.3 we have

**Theorem 1.4.** *Any group $G$ is contained in a group $\mathcal{G}$ acting sharply 2-transitively on a set $X$ such that each involution in $\mathcal{G}$ has no fixed point in $X$, and such that $\mathcal{G}$ does not contain a non-trivial abelian normal subgroup.*

*Proof.* For $|G| = 1, 2$ this is obvious. Otherwise take $A = 1$ in Theorem 1.3.    □

In fact there are many other ways to obtain a group $\mathcal{G}$ having a malnormal subgroup $\mathcal{A}$ and satisfying (2)–(4) of Theorem 1.3, e.g., take $G = \langle t \rangle * A$, where $t$ is an involution, and $A$ a non-trivial group without involutions, and apply Theorem 1.3. (Here the free product guarantees that $A$ is malnormal in $G$.)

Theorem 1.4 shows that there exists a sharply 2-transitive group $\mathcal{G}$ of *characteristic* 2 (see Definition A.2 in the Appendix) such that $\mathcal{G}$ does not contain a non-trivial abelian normal subgroup. Further, as noted in the Appendix, if $G$ is sharply 2-transitive of characteristic 3, then $G$ contains a non-trivial abelian normal subgroup. The cases where char($G$) is distinct from 2 and 3 remain open.

Finally, we mention that the hypothesis that $A$ does not contain involutions in Theorem 1.1 is used only in the case where we take $G_1$ to be an HNN extension of $G$, and then it is used only in the proof of the malnormality of $A_1$ in $G_1$.

## 2. Some preliminaries regarding Theorem 1.1

The following observations and remarks are here in order to explain to the reader the way we intend to prove Theorem 1.1, and to explain the main division between the two cases we deal with in §3 and §4.

In fact, Lemmas 2.1(3) and 2.2 below, together with Remark 2.3, show that we may assume throughout this paper that Hypothesis 2.4 holds; and that hypothesis naturally leads to the division into the two cases dealt with in §3 and §4.

**Lemma 2.1.** *Let $A$ be a malnormal subgroup of a group $G$ and let $g \in G \smallsetminus A$. Then*

(1) $C_G(a) \leq A$ *for all* $a \in A$, $a \neq 1$;
(2) $\langle g \rangle \cap A = 1$;
(3) $AgA$ *contains an involution iff* $g^{-1} \in AgA$.

*Proof.* (1) Let $a \in A$ with $a \neq 1$, and let $h \in C_G(a)$. Then $a \in A \cap A^h$. So $h \in A$, since $A$ is malnormal in $G$.

(2) Since $g \in C_G(g^k)$ for all integers $k$, part (2) follows from (1).

(3) If $g^{-1} \notin AgA$, then clearly $AgA$ does not contain an involution. Conversely, assume that $g^{-1} \in AgA$. Then $g^{-1} = agb$ for some $a, b \in A$, so $(ag)^2 = ab^{-1} \in A$. Then, by (2), either $(ag)^2 = 1$ or $ag \in A$. But $g \notin A$, so $ag \notin A$, and we have $(ag)^2 = 1$. Hence $AgA$ contains the involution $ag$. $\square$

We now make the following observation (and introduce the following notation):

**Lemma 2.2.** *Let $G$ be a group with a malnormal subgroup $A$ and an involution $t \in G \smallsetminus A$. Let $G_1$ be an extension of $G$ such that $G_1$ contains a malnormal subgroup $A_1$ with $A_1 \cap G = A$. Let $r, s \in G$ be such that $Ar \neq As$. Then*

(1) *there is at most one element* $f' \in G_1$ *with* $A_1 f' = A_1 r$ *and* $A_1 t f' = A_1 s$, *which we denote by* $f' = f_{r,s}$ *(if it exists).*

*The convention in* (2)–(4) *below is that the left side exists if and only if the right side does, and then they are equal:*

(2) $f_{r,s} g = f_{rg,sg}$ *for any* $g \in G$.
(3) $t f_{r,s} = f_{s,r}$.
(4) $f_{a_1 r, a_2 s} = f_{r,s}$ *for all* $a_1, a_2 \in A$.

*Proof.* (1) Let $f_1, f_2 \in G_1$ be such that $A_1 f_1 = A_1 f_2 = A_1 r$ and $A_1 t f_1 = A_1 t f_2 = A_1 s$. Then $f_1 f_2^{-1} \in A_1$ and $t f_1 f_2^{-1} t \in A_1$. Since $t \in G_1 \smallsetminus A_1$, and since $A_1$ is malnormal in $G_1$, we see that $f_1 f_2^{-1} = 1$, so $f_1 = f_2$.

(2) $A_1 f_{r,s} g = A_1 r g = A_1 f_{rg,sg}$ and $A_1 t f_{r,s} g = A_1 s g = A_1 f_{rg,sg}$. So, by (1), $f_{rg,sg} = f_{r,s} g$.

(3) $A_1 t f_{r,s} = A_1 s$, and $A_1 t t f_{r,s} = A_1 f_{r,s} = A_1 r$. So, by (1), $t f_{r,s} = f_{s,r}$.

(4) $A_1 f_{a_1 r, a_2 s} = A_1 a_1 r = A_1 r$, and $A_1 t f_{a_1 r, a_2 s} = A_1 a_2 s = A_1 s$. So, by (1), $f_{a_1 r, a_2 s} = f_{r,s}$. $\square$

**Remark 2.3.** Let the notation be as in Theorem 1.1. Notice that if there is an $f \in G$ such that $Af = Au$ and $Atf = Av$, we can just take $G_1 = G$ and $A_1 = A$ and there is nothing to prove in Theorem 1.1.

Hence we may assume throughout this paper that this is not the case. In view of (2) and (4) of Lemma 2.2, $f_{u,v} = f_{1,vu^{-1}}u$, and $f_{1,a'va} = f_{a^{-1},a'v}a = f_{1,v}a$, for $a, a' \in A$. Hence we may assume that $u = 1$ (and hence $v \notin A$) and replace $v$ by any element of the double coset $AvA$. By Lemma 2.1(3), we may assume that either $v^{-1} \notin AvA$, or $v$ is an involution. Further, since $f_{1,t} = 1$ and since $t$ is an involution, we may assume that $v \notin AtA$ and $v^{-1} \notin AtA$.

Hence it suffices to prove Theorem 1.1 under the following hypothesis which we assume for the rest of the paper.

**Hypothesis 2.4.** In the setting of Theorem 1.1, assume $u = 1$, $v, v^{-1} \notin AtA$ and either $v^{-1} \notin AvA$ or $v$ is an involution.

## 3. The case where $v^{-1} \notin AvA$

The purpose of this section is to prove Theorem 1.1 of the introduction in the case where $v^{-1} \notin AvA$. We refer the reader to Hypothesis 2.4 and to its explanation in §2. Thus, throughout this section we assume that $v^{-1} \notin AvA$. Also, throughout this section we use the notation and hypotheses of Theorem 1.1.

Let $\langle f_1 \rangle$ be an infinite cyclic group. We let

$$G_1 = G * \langle f_1 \rangle, \quad f_2 = tf_1v^{-1}, \quad A_1 = \langle A, f_1, f_2 \rangle.$$

In this section we will prove the following theorem.

**Theorem 3.1.** *We have*

(1) $A_1 = A * \langle f_1 \rangle * \langle f_2 \rangle$ *with $f_1, f_2$ of infinite order;*
(2) $A_1$ *is malnormal in $G_1$.*

Suppose Theorem 3.1 is proved. We now prove Theorem 1.1 in the case where $v^{-1} \notin AvA$.

*Proof of Theorem 1.1 in the case where $v^{-1} \notin AvA$.* Let $f := f_1$. Then $A_1f = A_1f_1 = A_1$, and

$$A_1tf = A_1tf_1 = A_1tf_1v^{-1}v = A_1f_2v = A_1v.$$

By Theorem 3.1(2), $A_1$ is malnormal in $G_1$. By Theorem 3.1(1), $A_1 \cap G = A$ and $f_2$ is of infinite order. Since $A_1 = A * \langle f_1 \rangle * \langle f_2 \rangle$ and $A$ does not contain involutions, $A_1$ does not contain involutions either. □

**Proposition 3.2.** $f_2$ *is of infinite order in $G_1$, and $A_1 = A * \langle f_1 \rangle * \langle f_2 \rangle$.*

*Proof.* We first show that $f_2$ is of infinite order. Indeed let $h := f_2^n$ for some $n \in \mathbb{Z}$, and write $h$ in terms of $f_1$ and elements of $G$. If $n > 0$, then $h$ starts with $t$ and ends with $v^{-1}$, while if $n < 0$, then $h$ starts with $v$ and ends with $t$. In particular $f_2$ has infinite order.

Next let $F := \langle f_1, f_2 \rangle$. Then any element of $F$ is a product of alternating powers of $f_1$ and $f_2$. As we saw in the previous paragraph, any non-zero power of $f_2$ starts with $t$ or $v$ and ends with $t$ or $v^{-1}$. Since $G_1 = G * \langle f_1 \rangle$, there will be no cancellation between powers of $f_1$ and powers of $f_2$. It follows that $F$ is a free group.

Now consider an element in $A_1 = \langle A, F \rangle$. It is an alternating product of elements of $A$ and elements of $F$. When we express it as an element of $G_1 = G * \langle f_1 \rangle$, $f_2$ is written as $tf_1v^{-1}$ and $f_2^{-1}$ is written as $vf_1^{-1}t$. Accordingly, an element $1 \neq a \in A$ in this alternating product is multiplied by $1, v^{-1}$ or $t$ on the left, and by $1, t$ or $v$ on the right. The possibilities are:

- $v^{-1}a, ta, at, av$: all are distinct from 1 since $t$ and $v$ are not in $A$.
- $tat, v^{-1}av$: all are distinct from 1 since they are conjugate to $a$.
- $tav, v^{-1}at$: all are distinct from 1 since $v \notin AtA$. $\qquad\square$

**Proposition 3.3.** $A_1$ *is a malnormal subgroup of* $G_1$.

*Proof.* We will show that the existence of $a, b \in A_1$ and $g \in G_1 \smallsetminus A_1$ such that $a \neq 1$ and $g^{-1}ag = b$ leads to a contradiction.

Let

$$a = a_1 f_{\delta_1}^{\epsilon_1} a_2 f_{\delta_2}^{\epsilon_2} \cdots a_n f_{\delta_n}^{\epsilon_n} a_{n+1}, \quad a \neq 1,$$
$$b = b_1 f_{\gamma_1}^{\mu_1} b_2 f_{\gamma_2}^{\mu_2} \cdots b_\ell f_{\gamma_\ell}^{\mu_\ell} b_{\ell+1},$$

where $a_i, b_j \in A$, $\epsilon_i, \mu_j = \pm 1$, $\delta_i, \gamma_j \in \{1, 2\}$, and if $\delta_i = \delta_{i-1}$ and $\epsilon_i = -\epsilon_{i-1}$ then $a_i \neq 1$ (i.e. there are no $f_i$-cancellations in $a$), and similarly there are no $f_i$-cancellations in $b$. Write

$$g = g_1 f_1^{\lambda_1} g_2 f_1^{\lambda_2} \cdots g_m f_1^{\lambda_m} g_{m+1} \in G_1 \smallsetminus A_1,$$

where $g_i \in G$, $\lambda_i = \pm 1$, and there are no $f_1$-cancellations in $g$.

Assume that $m$ is the least possible. We have the picture as in Figure 1 below.

**Case 1:** $m = n = 0$. In this case $b = g^{-1}a_1 g \in A_1 \cap G$. By Proposition 3.2, $A_1 \cap G = A$, so $b \in A$, and we get a contradiction to the malnormality of $A$ in $G$.

**Case 2:** $m = 0$ and $n > 0$. Since $G_1 = G * \langle f_1 \rangle$, we must have $n = \ell$. Consider Figure 1. By an analysis of the normal form in the free product $G * \langle f_1 \rangle$ we see that the only way we can get the equality $g_1^{-1}ag_1 = b$ is when both $\epsilon_1 = \mu_1$ and $\epsilon_n = \mu_n$. We distinguish a number of subcases:

(i) $\delta_1 = \gamma_1$ or $\delta_n = \gamma_n$.
(ii) $\delta_1 \neq \gamma_1$ and $\delta_n \neq \gamma_n$.

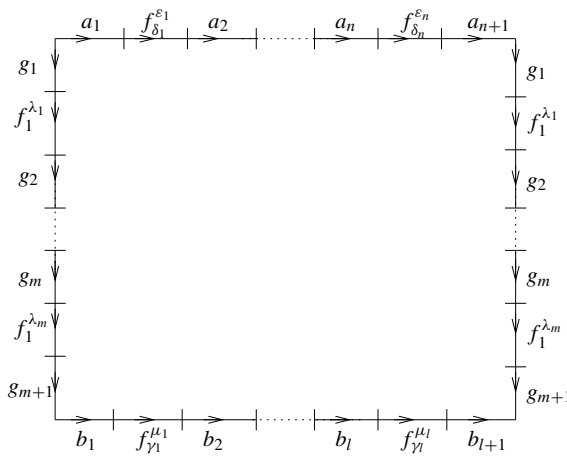    (a) $n = 1$.
    (b) $n > 1$.

**Fig. 1**

**Subcase (i).** By symmetry we may consider only the case where $\delta_1 = \gamma_1$. In this case, regardless of whether $\epsilon_1 = 1$ or $-1$ and whether $\delta_1 = 1$ or $2$, we get $g_1 = a_1 b_1^{-1} \in A$, a contradiction.

**Subcase (iia).** By symmetry we may assume that $\delta_1 = 1$ and $\gamma_1 = 2$.

Suppose first that $\epsilon_1 = \mu_1 = 1$. Then from the left side of Figure 1 we get $a_1^{-1} g_1 b_1 t = 1$, and from the right side we get $a_2 g_1 b_2^{-1} v = 1$. This implies that $t \in A g_1 A$ and $v^{-1} \in A g_1 A$. But then $v^{-1} \in A t A$, a contradiction.

Suppose next that $\epsilon_1 = \mu_1 = -1$. Then from the left side of Figure 1 we get $a_1^{-1} g_1 b_1 v = 1$, and from the right side we get $a_2 g_1 b_2^{-1} t = 1$. Again this implies that $v^{-1} \in A t A$, a contradiction.

**Subcase (iib).** By symmetry, we may assume without loss of generality that

$$\delta_1 = 1 \quad \text{and} \quad \gamma_1 = 2.$$

Suppose first that

$$\epsilon_1 = \mu_1 = 1.$$

We may further assume that

$$a_1^{-1} g_1 b_1 t = 1 \quad \text{and} \quad \epsilon_2 = \mu_2.$$

We now discuss the following cases separately:

- $\delta_2 = \gamma_2$. In this case, regardless of the sign of $\epsilon_2 = \mu_2$ and whether $\delta_2 = \gamma_2 = 1$ or $2$, we get $a_2^{-1} v^{-1} b_2 = 1$, which is false since $v \notin A$.
- $\epsilon_2 = \mu_2 = 1$, $\delta_2 = 1$, $\gamma_2 = 2$. We get $a_2^{-1} v^{-1} b_2 t = 1$, contradicting $v \notin A t A$.
- $\epsilon_2 = \mu_2 = -1$, $\delta_2 = 1$, $\gamma_2 = 2$. We get $a_2^{-1} v^{-1} b_2 v = 1$ with $b_2 \neq 1$. But this contradicts the malnormality of $A$ in $G$.

- $\epsilon_2 = \mu_2 = 1$, $\delta_2 = 2$, $\gamma_2 = 1$. We get $t a_2^{-1} v^{-1} b_2 = 1$, contrary to $v^{-1} \notin AtA$.
- $\epsilon_2 = \mu_2 = -1$, $\delta_2 = 2$, $\gamma_2 = 1$. We get $v^{-1} a_2^{-1} v^{-1} b_2 = 1$. This implies that $v^{-1} \in AvA$, contrary to our hypotheses.

Suppose next that

$$\epsilon_1 = \mu_1 = -1.$$

We may further assume that

$$a_1^{-1} g_1 b_1 v = 1 \quad \text{and} \quad \epsilon_2 = \mu_2.$$

Again we discuss the following cases separately:

- $\delta_2 = \gamma_2$. In this case, regardless of the sign of $\epsilon_2 = \mu_2$ and whether $\delta_2 = \gamma_2 = 1$ or 2, we get $a_2^{-1} t b_2 = 1$, which is false since $t \notin A$.
- $\epsilon_2 = \mu_2 = 1$, $\delta_2 = 1$, $\gamma_2 = 2$. We get $a_2^{-1} t b_2 t = 1$, and $b_2 \neq 1$. This contradicts the malnormality of $A$ in $G$.
- $\epsilon_2 = \mu_2 = -1$, $\delta_2 = 1$, $\gamma_2 = 2$. We get $a_2^{-1} t b_2 v = 1$, impossible, as above.
- $\epsilon_2 = \mu_2 = 1$, $\delta_2 = 2$, $\gamma_2 = 1$. We get $t a_2^{-1} t b_2 = 1$. This case forces $a_2 = b_2 = 1$ (because $A$ is malnormal in $G$). If $n = 2$ we get $v^{-1} a_3 g_1 b_3^{-1} = 1$. But this together with $a_1^{-1} g_1 b_1 v = 1$ implies that $v^{-1} \in AvA$, contrary to our hypotheses. Thus $n \geq 3$. But now we must have $\epsilon_3 = \mu_3$, and arguing exactly as in the previous cases, for all choices of $\epsilon_3 = \mu_3$, $\delta_3$ and $\gamma_3$, we get a contradiction as in one of the cases above.
- $\epsilon_2 = \mu_2 = -1$, $\delta_2 = 2$, $\gamma_2 = 1$. We get $v^{-1} a_2^{-1} t b_2 = 1$, impossible, as above.

**Case 3:** $n = 0 = \ell$ and $m > 0$. Notice that in this case there will be no cancellations in Figure 1, since otherwise we must have either $g_1^{-1} a_1 g_1 = 1$, or $g_{m+1}^{-1} b_1^{-1} g_{m+1} = 1$, which is false.

Hence we may assume that either $n > 0$ or $\ell > 0$ or both. By symmetry we may consider the following case:

**Case 4:** $m > 0$ and $n > 0$. Notice that $f_i$-cancellations have to occur in the product $g^{-1} a g b^{-1}$, since it is equal to 1. Now $f_i$-cancellations can occur only if one of the following occurs:

(i) The product $f_1^{-\lambda_1} g_1^{-1} a_1 f_{\delta_1}^{\epsilon_1}$ equals 1, $v^{-1}$, or $t$.
(ii) The product $f_{\delta_n}^{\epsilon_n} a_{n+1} g_1 f_1^{\lambda_1}$ equals 1, $t$ or $v$.
(iii) The product $f_1^{\lambda_m} g_{m+1} b_1 f_{\gamma_1}^{\mu_1}$ equals 1, $v^{-1}$ or $t$.
(iv) The product $f_{\gamma_\ell}^{\mu_\ell} b_{\ell+1} g_{m+1}^{-1} f_1^{-\lambda_m}$ equals 1, $t$ or $v$.

By symmetry, we may consider only (i). If $f_1^{-\lambda_1} g_1^{-1} a_1 f_{\delta_1}^{\epsilon_1} = 1$, then $g_1 f_1^{\lambda_1} = a_1 f_{\delta_1}^{\epsilon_1}$. Let $h := g_2 f_1^{\lambda_2} \cdots f_1^{\lambda_m} g_{m+1}$ and

$$a' = a_2 f_{\delta_2}^{\epsilon_2} \cdots f_{\delta_n}^{\epsilon_n} a_{n+1} g_1 f_1^{\lambda_1} = a_2 f_{\delta_2}^{\epsilon_2} \cdots f_{\delta_n}^{\epsilon_n} a_{n+1} a_1 f_{\delta_1}^{\epsilon_1} \in A_1.$$

Notice that $a'$ is conjugate to $a$, so $a' \neq 1$. Also $h = f_1^{-\lambda_1} g_1^{-1} g$, and $h \notin A_1$, since $f_1^{-\lambda_1} g_1^{-1} \in A_1$, while $g \notin A_1$. We get (see Figure 1) $g^{-1} a g = h^{-1} a' h \in A_1$, contradicting the minimality of $m$.

If $f_1^{-\lambda_1} g_1^{-1} a_1 f_{\delta_1}^{\epsilon_1} = v^{-1}$, then $g_1 f_1^{\lambda_1} = a_1 f_{\delta_1}^{\epsilon_1} v$. Let $h := v g_2 f_1^{\lambda_2} \cdots f_1^{\lambda_m} g_{m+1}$ and

$$a' = a_2 f_{\delta_2}^{\epsilon_2} \cdots f_{\delta_n}^{\epsilon_n} a_{n+1} g_1 f_1^{\lambda_1} v^{-1} = a_2 f_{\delta_2}^{\epsilon_2} \cdots f_{\delta_n}^{\epsilon_n} a_{n+1} a_1 f_{\delta_1}^{\epsilon_1} \in A_1.$$

As above, $1 \neq a' \in A_1$, and if $h \in A_1$, then $g = g_1 f_1^{\lambda_1} v^{-1} h = a_1 f_{\delta_1}^{\epsilon_1} h \in A_1$, which is false. We again get $g^{-1} a g = h^{-1} a' h \in A_1$, which contradicts the minimality of $m$.

Finally, if $f_1^{-\lambda_1} g_1^{-1} a_1 f_{\delta_1}^{\epsilon_1} = t$, then $g_1 f_1^{\lambda_1} = a_1 f_{\delta_1}^{\epsilon_1} t$. Let $h := t g_2 f_1^{\lambda_2} \cdots f_1^{\lambda_m} g_{m+1}$ and

$$a' = a_2 f_{\delta_2}^{\epsilon_2} \cdots f_{\delta_n}^{\epsilon_n} a_{n+1} g_1 f_1^{\lambda_1} t = a_2 f_{\delta_2}^{\epsilon_2} \cdots f_{\delta_n}^{\epsilon_n} a_{n+1} a_1 f_{\delta_1}^{\epsilon_1} \in A_1.$$

As above we get $1 \neq a' \in A_1$ and $h \notin A_1$, and again we get the same contradiction.

Note that if $\ell = 0$, then no cancellation of the type (iii) or (iv) above can occur.  □

*Proof of Theorem 3.1.* Part (1) holds by Proposition 3.2, and (2) by Proposition 3.3.  □

## 4. The case where $v$ is an involution and $v \notin A t A$

The purpose of this section is to prove Theorem 1.1 of the introduction in the case where $v$ is an involution. We refer the reader to Hypothesis 2.4 and to its explanation in §2. Thus, throughout this section we assume that $v$ is an involution and $v \notin A t A$. Throughout this section we use the notation and hypotheses of Theorem 1.1.

Let $\langle f \rangle$ be an infinite cyclic group. We define an HNN extension

$$G_1 = \langle G, f \mid f^{-1} t f = v \rangle, \qquad A_1 = \langle A, f \rangle.$$

In this section we will prove the following theorem.

**Theorem 4.1.** *We have*

(1) $A_1 = A * \langle f \rangle$;
(2) $A_1$ *is malnormal in* $G_1$.

Suppose Theorem 4.1 is proved. We now use it to prove Theorem 1.1 in the case where $v$ is an involution.

*Proof of Theorem 1.1 in the case where $v$ is an involution.* We have $A_1 f = A_1$ and $A_1 t f = A_1 f v = A_1 v$. By Theorem 4.1(2), $A_1$ is malnormal in $G_1$. By Theorem 4.1(1), $A_1 \cap G = A$. Also $A_1$ does not contain involutions, since $A_1 = A * \langle f \rangle$ and $A$ does not contain involutions.  □

**Remark 4.2.** Any element of $G_1$ has the form

$$g = g_1 f^{\delta_1} g_2 \cdots g_m f^{\delta_m} g_{m+1},$$

where $g_i \in G$, $i = 1, \ldots, m+1$, $\delta_i = \pm 1$, $i = 1, \ldots, m$. According to Britton's Lemma we say that there are *no $f$-cancellations in $g$* if the equality $\delta_i = -\delta_{i-1}$ implies that if $\delta_i = 1$, then $g_i \neq 1, t$, while if $\delta_i = -1$, then $g_i \neq 1, v$.
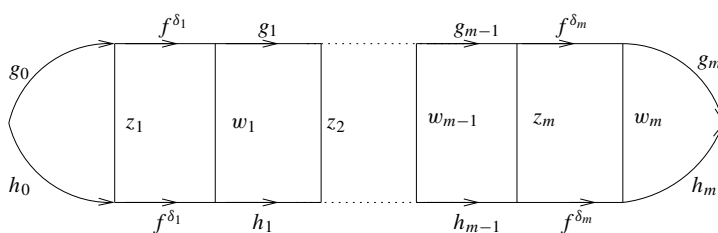
**Fig. 2**

Further let $g$ be as above, let $h \in G_1$, and write

$$h = h_1 f^{\eta_1} h_2 \cdots h_k f^{\eta_k} h_{k+1},$$

where $h_j \in G$, $j = 1, \ldots, k+1$, $\eta_j = \pm 1$, $j = 1, \ldots, k$, and there are no $f$-cancellations in $g$ and $h$.

Then $g = h$ if and only if $m = k$, $\delta_i = \eta_i$, $i = 1, \ldots, m$, and there are elements $w_0, z_1, w_1, z_2, w_2, \ldots, z_m, w_m, z_{m+1}$ such that for every oriented loop in Figure 2 the product of edges is 1, that is:

(a) $h_i = w_{i-1} g_i z_i$, $i = 1, \ldots, m+1$;
(b) $w_0 = 1$, $z_{m+1} = 1$;
(c) if $\delta_i = 1$, then either $z_i = 1$, $w_i = 1$, or $z_i = t$, $w_i = v$;
(d) if $\delta_i = -1$, then either $z_i = 1$, $w_i = 1$, or $z_i = v$, $w_i = t$.

**Lemma 4.3.** $A_1 = A * \langle f \rangle$.

*Proof.* Suppose that

$$g_1 f^{\delta_1} g_2 \cdots g_m f^{\delta_m} g_{m+1} = h_1 f^{\delta_1} h_2 \cdots h_m f^{\delta_m} h_{m+1},$$

and $h_i, g_i \in A$, $i = 1, \ldots, m+1$. By Remark 4.2, $h_1 = g_1 z_1$, hence, by Remark 4.2(a)–(d), since $t, v \notin A$, we have $z_1 = 1$, so $h_1 = g_1$, and then, by Remark 4.2(c)–(d), $w_1 = 1$.

Assume $w_i = 1$. Then $h_{i+1} = w_i g_{i+1} z_{i+1} = g_{i+1} z_{i+1}$. Since $t, v \notin A$, this implies $z_{i+1} = 1$, and then $w_{i+1} = 1$. So $g_i = h_i$ for $i = 1, \ldots, m+1$. Hence $A_1 = A * \langle f \rangle$. $\square$

**Proposition 4.4.** $A_1$ *is malnormal in* $G_1$.

*Proof.* We will show that the existence of $a, b \in A_1$ and $g \in G_1 \smallsetminus A_1$ such that $a \neq 1$ and $g^{-1} a g = b$ leads to a contradiction. Let

$$a = a_1 f^{\alpha_1} a_2 \cdots a_m f^{\alpha_m} a_{m+1}, \qquad b = b_1 f^{\beta_1} b_2 \cdots b_n f^{\beta_n} b_{n+1},$$

where $a_i, b_i \in A$, $\alpha_i, \beta_i = \pm 1$, and if $\alpha_i = -\alpha_{i-1}$, then $a_i \neq 1$, while if $\beta_i = -\beta_{i-1}$, then $b_i \neq 1$. Recall that by Lemma 4.3, $A_1 = A * \langle f \rangle$, and therefore in the above expressions for $a$ and $b$ there are no $f$-cancellations. We also have

$$g = g_1 f^{\delta_1} g_2 \cdots g_k f^{\delta_k} g_{k+1},$$

where $g_i \in G$, $\delta_i = \pm 1$, and $\delta_i = -\delta_{i-1}$ implies that if $\delta_i = 1$, then $g_i \neq 1, t$, while if $\delta_i = -1$, then $g_i \neq 1, v$.

We assume that $k$ is the least possible.

**Case 1:** $k = 0$. Then $g = g_1$, so we have

$$g_1^{-1} a_1 f^{\alpha_1} a_2 \cdots a_m f^{\alpha_m} a_{m+1} g_1 = b_1 f^{\beta_1} b_2 \cdots b_n f^{\beta_n} b_{n+1}.$$

We conclude that $n = m$ and $\alpha_i = \beta_i$ for $i = 1, \ldots, m$. If $m = n = 0$, then $a = a_1 \neq 1$, $b = b_1$, so $g_1^{-1} a_1 g_1 = b_1$, which is impossible because $A$ is malnormal in $G$.

Let $m = n > 0$. We obtain Figure 3 below, where

$$\begin{aligned}
&\text{if } \alpha_i = 1, &&\text{then either } p_i = q_i = 1, &&\text{or } p_i = t, \ q_i = v, \\
&\text{if } \alpha_i = -1, &&\text{then either } p_i = q_i = 1, &&\text{or } p_i = v, \ q_i = t.
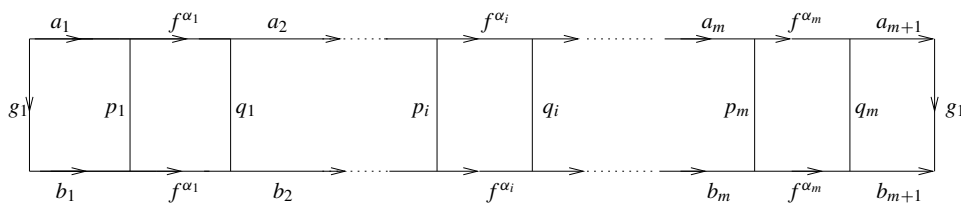\end{aligned} \tag{4.1}$$



**Fig. 3**

We have $p_1 = a_1^{-1} g_1 b_1 \notin A$ since $g_1 \notin A$. Now assume $p_i \notin A$. Then $q_i \notin A$ by (4.1), and by Britton's Lemma $p_{i+1} = a_{i+1}^{-1} q_i b_{i+1}$ is not in $A$ either. In particular $p_i, q_i \neq 1$ for all $i \leq m$.
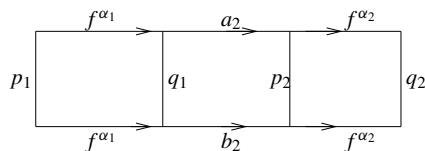
If $m = n \geq 2$, consider Figure 4:



**Fig. 4**

We now use (4.1). If $\alpha_1 = 1$, $\alpha_2 = 1$, then $q_1 = v$, $p_2 = t$, so $v = a_2 t b_2^{-1} \in AtA$, a contradiction.

If $\alpha_1 = 1$, $\alpha_2 = -1$, then $a_2 \neq 1$, $q_1 = v$, $p_2 = v$. Then $v a_2 v = b_2$, contradicting the malnormality of $A$ in $G$.

If $\alpha_1 = -1$, $\alpha_2 = 1$, then $a_2 \neq 1$, $q_1 = t$, $p_2 = t$, and $t a_2 t = b_2$, again contradicting the malnormality of $A$ in $G$.

If $\alpha_1 = -1$, $\alpha_2 = -1$, then $q_1 = t$, $p_2 = v$, and $v = a_2^{-1} t b_2 \in AtA$, a contradiction.

So we are left with the possibility $m = n = 1$. In Figure 3 above, after cutting and pasting we obtain the following Figure 5:
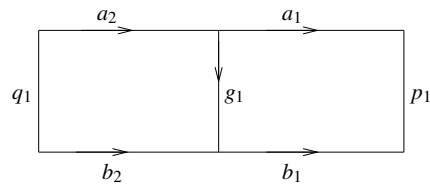
**Fig. 5**

If $\alpha_1 = 1$, then $p_1 = t$, $q_1 = v$, while if $\alpha_1 = -1$, then $p_1 = v$, $q_1 = t$. In both cases $v \in AtA$, contrary to the choice of $v$.

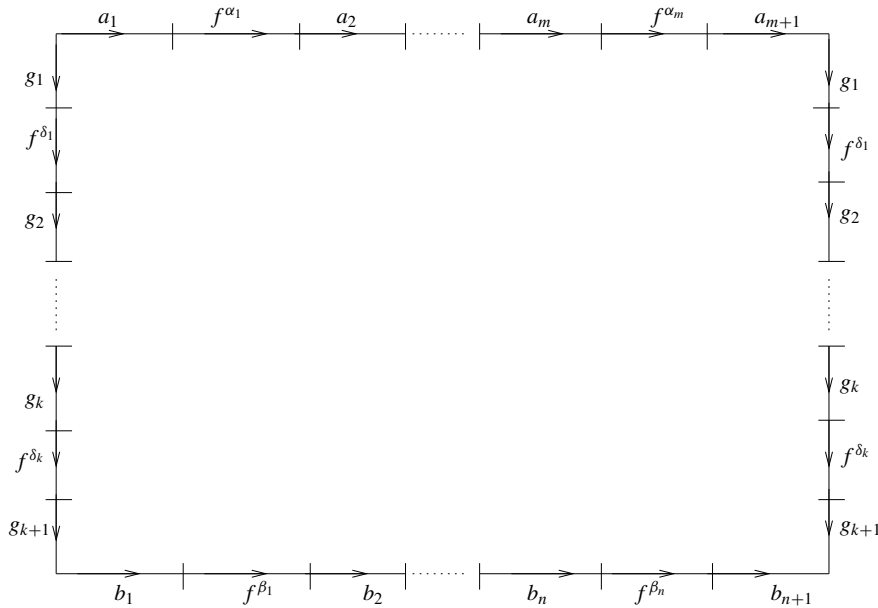**Case 2:** $k > 0$. Consider Figure 6:



**Fig. 6**

Notice that $f$-cancellations have to occur in the product $g^{-1}agb^{-1}$, since it is equal to 1. Therefore, at least one of the following cases must happen:

(1) $m = 0$, $a = a_1$, and $f^{-\delta_1}$ cancels with $f^{\delta_1}$ in $f^{-\delta_1}g_1^{-1}a_1g_1f^{\delta_1}$;
(2) $n = 0$, $b = b_1$ and $f^{\delta_k}$ cancels with $f^{-\delta_k}$ in $f^{\delta_k}g_{k+1}b_1g_{k+1}^{-1}f^{-\delta_k}$;
(3) $m > 0$, and $f^{-\delta_1}$ cancels with $f^{\alpha_1}$ in $f^{-\delta_1}g_1^{-1}a_1f^{\alpha_1}$;
(4) $m > 0$, and $f^{\alpha_m}$ cancels with $f^{\delta_1}$ in $f^{\alpha_m}a_{m+1}g_1f^{\delta_1}$;
(5) $n > 0$, and $f^{\delta_k}$ cancels with $f^{\beta_1}$ in $f^{\delta_k}g_{k+1}b_1f^{\beta_1}$;
(6) $n > 0$, and $f^{\beta_n}$ cancels with $f^{-\delta_k}$ in $f^{\beta_n}b_{n+1}g_{k+1}^{-1}f^{-\delta_k}$.

In case (1), $a = a_1 \neq 1$, so $g_1^{-1} a_1 g_1 = t$ or $v$. Hence $a_1$ is conjugate to an involution, which is impossible, as $A$ does not contain involutions.

Similarly, in case (2) we have $b = b_1 \neq 1$, so $g_{k+1} b_1 g_{k+1}^{-1} = t$ or $v$, again a contradiction.

In case (3) we have Figure 7 below, where $p, q \in \{1, t, v\}$ by Britton's Lemma. We define

$$a' = a_2 \cdots a_m f^{\alpha_m} a_{m+1} a_1 f^{\alpha_1} \quad \text{and} \quad h = q g_2 \cdots g_k f^{\delta_k} g_{k+1}.$$
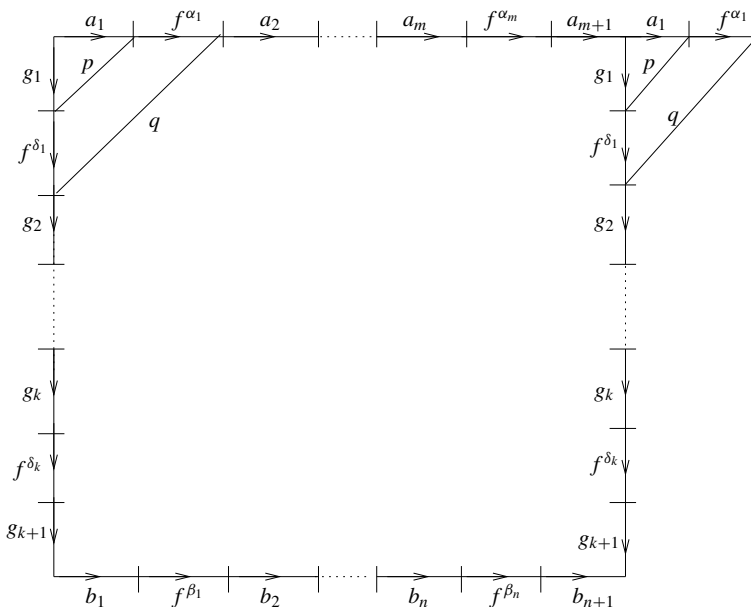


**Fig. 7**

We have $h^{-1} a' h = b$, where $a'$ is conjugate to $a$. So $a \neq 1$ implies $a' \neq 1$. Also the $f$-length of $h$ is $k - 1$. Notice that $h = f^{-\alpha_1} a_1^{-1} g$, and $h \notin A_1$ since $f^{-\alpha_1} a_1^{-1} \in A_1$ and $g \notin A_1$. We have obtained a contradiction to the minimality of $k$.

The remaining cases are handled in entirely the same way. $\qquad\square$

*Proof of Theorem 4.1.* Part (1) holds by Lemma 4.3, and (2) by Proposition 4.4. $\qquad\square$

## 5. The proof of Theorem 1.3

In this section we show how Theorem 1.3 of the introduction follows from Theorem 1.1.

Let $G$ be a group with a malnormal subgroup $A$ such that $A$ contains no involutions. Assume that $G$ is *not* 2-transitive on the set of right cosets $A \backslash G$. If there exists an involution $t \in G \setminus A$, set $G_0 := G$, $A_0 := A$. Otherwise, let $G_0 := G * \langle t \rangle$, where $t$ is an involution, and let $A_0 = A$. Then, by [MaKS, Corollary 4.1.5], $G$ is malnormal in $G_0$, and then since $A$ is malnormal in $G$, it is malnormal in $G_0$.

We now construct a sequence of groups $G_i$ and of subgroups $A_i \leq G_i$, $i = 0, 1, \ldots$, having the following properties for all $i \geq 0$:

(1) $G_i \leq G_{i+1}$ and $A_i \leq A_{i+1}$;
(2) $A_i$ is malnormal in $G_i$ and $t \in G_i \setminus A_i$;
(3) $A_i$ does not contain involutions;
(4) $A_{i+1} \cap G_i = A_i$;
(5) for each $v \in G_i \setminus A_i$ there exists $f_v \in A_{i+1}$ such that $A_{i+1} t f_v = A_{i+1} v$.

In order to construct $G_{i+1}, A_{i+1}$ from $G_i, A_i$ we enumerate the set $G_i \setminus A_i = \{v_\alpha : \alpha < \rho\}$ for some ordinal $\rho$. For each ordinal $\alpha < \rho$ we construct the pair $G_i^\alpha, A_i^\alpha$ and the element $f_{v_\alpha} \in A_i^\alpha$ having the following properties:

(i) $G_i^\beta \leq G_i^\alpha$ for all ordinals $\beta < \alpha$;
(ii) $A_i^\alpha$ is malnormal in $G_i^\alpha$ and $t \in G_i^\alpha \setminus A_i^\alpha$;
(iii) $A_i^\alpha$ contains no involutions;
(iv) $A_i^\alpha \cap G_i^\beta = A_i^\beta$ for all $\beta < \alpha$;
(v) $f_{v_\alpha} \in A_i^\alpha$ and $A_i^\alpha t f_{v_\alpha} = A_i^\alpha v_\alpha$.

We let $G_i^0 = G_i$ and $A_i^0 = A_i$. If $\alpha = \beta + 1$, we construct $(G_i^\alpha, A_i^\alpha, f_{v_\alpha})$ from $(G_i^\beta, A_i^\beta)$ as follows: If there is some $f \in A_i^\beta$ with $A_i^\beta t f = A_i^\beta v_\alpha$ we let $G_i^\alpha = G_i^\beta$, $A_i^\alpha = A_i^\beta$ and $f_{v_\alpha} = f$. Otherwise apply Theorem 1.1 to $G_i^\beta, A_i^\beta$ with $u = 1$ and $v = v_\alpha$ to obtain $G_i^\alpha, A_i^\alpha$ and $f_{v_\alpha} \in A_i^\alpha$. Of course, by construction, $A_i^\alpha$ contains no involutions and $A_i^\alpha \cap G_i^\beta = A_i^\beta$. So (i)–(v) hold.

For a limit ordinal $\alpha$ we set $G_i^{(\alpha,1)} = \bigcup_{\beta < \alpha} G_i^\beta$, $A_i^{(\alpha,1)} = \bigcup_{\beta < \alpha} A_i^\beta$. We now show that when $\alpha$ is a limit ordinal, $A_i^{(\alpha,1)}$ is malnormal in $G_i^{(\alpha,1)}$. Notice that for each ordinal $\beta < \alpha$ and each $g \in G_i^\beta \setminus A_i^\beta$, we have $g \in G_i^{(\alpha,1)} \setminus A_i^{(\alpha,1)}$. Indeed, else take the minimal $\gamma < \alpha$ such that $g \in A_i^\gamma$. Then, by definition, $\gamma$ is not a limit ordinal, and $g \in G_i^{\gamma-1} \setminus A_i^{\gamma-1}$. So $g \in A_i^\gamma \cap G_i^{\gamma-1} = A_i^{\gamma-1}$, a contradiction. This means that $A_i^{(\alpha,1)} \cap G_i^\beta = A_i^\beta$ for all ordinals $\beta < \alpha$.

Suppose now that $g^{-1} a g = b$ with $g \in G_i^{(\alpha,1)} \setminus A_i^{(\alpha,1)}$ and $a, b \in A_i^{(\alpha,1)}$. Then, by the previous paragraph, there exists $\beta < \alpha$ such that $a, b \in A_i^\beta$ and $g \in G_i^\beta \setminus A_i^\beta$, and then we get a contradiction to the malnormality of $A_i^\beta$ in $G_i^\beta$. Clearly $A_i^{(\alpha,1)}$ contains no involutions. Next if there exists $f \in A_i^{(\alpha,1)}$ such that $A_i^{(\alpha,1)} t f = A_i^{(\alpha,1)} u_\alpha$ then we let $G_i^\alpha = G_i^{(\alpha,1)}$, $A_i^\alpha = A_i^{(\alpha,1)}$ and $f_{v_\alpha} = f$. Else we construct $G_i^\alpha, A_i^\alpha$ and $f_{v_\alpha}$ from $G_i^{(\alpha,1)}$, $A_i^{(\alpha,1)}$ using Theorem 1.1 with $u = 1$ and $v = v_\alpha$ (just as in the construction above in the case of a non-limit ordinal). Again we see that (i)–(v) hold.

Finally, set

$$G_{i+1} = \bigcup_{\alpha < \rho} G_i^\alpha, \quad A_{i+1} = \bigcup_{\alpha < \rho} A_i^\alpha,$$

$$\mathcal{G} = \bigcup_{i < \omega} G_i, \quad \mathcal{A} = \bigcup_{i < \omega} A_i \quad \text{and} \quad X = \mathcal{A} \backslash \mathcal{G}.$$

As in the construction of $G_i^{(\alpha,1)}$, $A_i^{(\alpha,1)}$ in the case where $\alpha$ is a limit ordinal, we see that $\mathcal{A}$ is malnormal in $\mathcal{G}$, and $\mathcal{A} \cap G_i = A_i$ for each $i < \omega$. To see that the action of $G$ on $X$ is 2-transitive, just note that any $v \in \mathcal{G} \smallsetminus \mathcal{A}$ is contained in some $G_i$ so that there is some $f_v \in A_{i+1} \subseteq \mathcal{A}$ with $A_{i+1}tf_v = A_{i+1}v$. Since $A_{i+1} \leq \mathcal{A}$, we see that $\mathcal{A}tf_v = \mathcal{A}v$ as required. Since $\mathcal{A}$ is malnormal in $\mathcal{G}$, the action of $\mathcal{G}$ on $X$ is sharply 2-transitive. By construction, $\mathcal{A}$ contains no involutions.

Finally, as is well known, if $\mathcal{G}$ contains a non-trivial abelian normal subgroup, then necessarily all involutions in $\mathcal{G}$ commute with each other (see, e.g., [GMS, Remark 4.4]). But, by our construction, this is not the case in $\mathcal{G}$. Indeed, if $G_1 = G_0 * \langle f_1 \rangle$ is a free product, then $t$ does not commute with $f_1^{-1}tf_1$. Suppose that $G_1 = \langle G, f \mid f^{-1}tf = v \rangle$ is an HNN extension. Let $s \in G$ be an involution distinct from $t$ (notice that $t$ is not in the center of $G$ since $A$ is malnormal in $G$, so such an $s$ exists). Then $sf^{-1}sf$ and $f^{-1}sfs$ are in canonical form, so they are distinct, and the involutions $s$ and $f^{-1}sf$ do not commute.[1] This completes the proof of Theorem 1.3.

## Appendix. Some background and a permutation-group-theoretic point of view

Recall that a permutation group $G$ on a set $X$ is *regular* if it is transitive and no non-trivial element of $G$ fixes a point; $G$ is a *Frobenius group* on $X$ if $G$ is transitive on $X$, no non-trivial element in $G$ fixes more than one point, and some non-trivial elements of $G$ fix a point; and $G$ is *sharply 2-transitive* if $G$ is transitive on $X$, and for any two ordered pairs $(x_1, x_2)$, $(x_1', x_2') \in X \times X$ of distinct points in $X$, there exists a unique element $g \in G$ such that $x_i g = x_i'$, $i = 1, 2$.

**Remarks A.1.** Let $G$ be a group and let $A$ be a subgroup of $G$. Let $X := A \backslash G$ be the set of right cosets of $A$ in $G$. Then the following are equivalent:

(1)  $A$ is malnormal in $G$.
(2)  Either

    (a)  $A = 1$ and $G$ is regular on $X$, or
    (b)  $G$ is a Frobenius group on $X$ (so $A \neq 1$).

If a sharply 2-transitive group $G$ on $X$ contains a non-trivial normal abelian subgroup $B$, then $B$ is necessarily regular on $X$ and $G = HB$ with $H \cap B = 1$, where $H$ is the stabilizer in $G$ of some point in $X$. In this case we say that $G$ *splits,* otherwise we say that $G$ is *non-split.*

The primary example of sharply 2-transitive groups are the 1-*dimensional affine groups*. Given a field $F$, the 1-dimensional affine group over $F$ is the group $G := \{x \mapsto ax + b \mid a, b \in F, a \neq 0\}$ of functions on $X = F$. So $G$ is Frobenius on $X$.

If $G$ is a 1-dimensional affine group over $F$, then $G$ splits. Indeed, if we let $B = \{x \mapsto x + b \mid b \in F\}$ and $H = \{ax \mid a \in F, a \neq 0\}$, the stabilizer of 0 in $G$, then $B$ is

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[1] Note that we could start with a group $G_0$ which already contains an involution that does not commute with $t$. Then it would immediately follow that $\mathcal{G}$ does not split. We thank Uri Bader for pointing this out.

an abelian normal subgroup of $G$ and $G = BH$. In fact, in [K, §6] it is shown that sharply 2-transitive groups can be completely characterized by means of "1-dimensional affine" transformations $x \mapsto ax + b$ on an algebraic structure called a *near-domain* defined in [K, Definition, p. 21]. Further, the notion of a *near-field* is defined in [K, below the Definition on p. 21]. And in [K, Thm. 7.1, p. 25] it is shown that the assertion that every sharply 2-transitive group splits is equivalent to the assertion that *every near-domain is a near field* (see also [Hall, Subsection 20.7, p. 382], [SSS, Chapter 3]).

However, for an infinite sharply 2-transitive group $G$ it was a long-standing problem whether or not $G$ splits. It is known that a sharply 2-transitive group splits in the following cases:

- $G$ is locally compact connected [Ti];
- $G$ is locally finite [W];
- $G$ is definable in an o-minimal structure [T2];
- $G$ is linear (with certain additional restrictions) [GlGu];
- $G$ is locally linear (with some additional restrictions) [GMS];
- further splitting results can be found in [BN] and [SSS].

To state some additional splitting results we need to introduce some more definitions. So let $G$ be an infinite sharply 2-transitive group on a set $X$. Then $G$ contains "many" involutions. Let $I \subset G$ be the set of involutions in $G$. Then $I$ is a conjugacy class in $G$. If $i \in I$ has no fixed points in $X$ we say that $G$ is of characteristic 2 and we write $\mathrm{char}(G) = 2$. Otherwise each $i \in I$ fixes a unique point. In this case the set of all products of distinct involutions, $I^2 \smallsetminus \{1\}$, forms a conjugacy class in $G$, and a non-trivial power of an element in $I^2 \smallsetminus \{1\}$ belongs to $I^2 \smallsetminus \{1\}$. It follows that the elements in $I^2 \smallsetminus \{1\}$ either have an odd prime order $p$, or are of infinite order. In the former case we say that the characteristic of $G$ is $p$, and in the latter case we say that the characteristic of $G$ is 0. Hence we have the following definition.

**Definition A.2.** Let $G$ be a sharply 2-transitive group on a set $X$, and let $I$ be the set of involutions in $G$. Let $I^2 = \{ts \mid t, s \in I\}$. We define the *characteristic* of $G$, denoted $\mathrm{char}(G)$, as follows:

- $\mathrm{char}(G) = 2$ if $i \in I$ has no fixed point in $X$;
- $\mathrm{char}(G) = 0$ if each $g \in I^2 \smallsetminus \{1\}$ is of infinite order;
- $\mathrm{char}(G) = p$, where $p$ is an odd prime, if the order of each $g \in I^2 \smallsetminus \{1\}$ is $p$.

- In [K, Thm. 9.5, p. 42] and in [Tu] it was shown that if $\mathrm{char}(G) = 3$, then $G$ splits.
- In [M] it was shown that if the exponent of the point stabilizer is 3 or 6, then $G$ splits.

Using the above terminology, we can now rephrase Theorem 1.3 as follows.

**Theorem A.3.** *Every Frobenius or regular permutation group which is* not *sharply 2-transitive, and whose involutions do not have any fixed point, has a non-split sharply 2-transitive extension of characteristic* 2.

Here, by an "extension" we mean an extension of both the given set and the given permutation group.

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