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Representation homology, Lie algebra cohomology and the derived Harish-Chandra homomorphism

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Abstract. We study the derived representation scheme $\mathrm{DRep}_n(A)$ parametrizing the n -dimensional representations of an associative algebra A over a field of characteristic zero. We show that the homology of $\mathrm{DRep}_n(A)$ is isomorphic to the Chevalley–Eilenberg homology of the current Lie coalgebra $\mathfrak{gl}_n^*(\bar{C})$ defined over a Koszul dual coalgebra of A . This gives a conceptual explanation to some of the main results of [BKR] and [BR1], relating them (via Koszul duality) to classical theorems on (co)homology of current Lie algebras $\mathfrak{gl}_n(A)$. We extend the above isomorphism to representation schemes of Lie algebras: for a finite-dimensional reductive Lie algebra \mathfrak{g} , we define the derived affine scheme $\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})$ parametrizing the representations (in \mathfrak{g}) of a Lie algebra \mathfrak{a} ; we show that the homology of $\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})$ is isomorphic to the Chevalley–Eilenberg homology of the Lie coalgebra $\mathfrak{g}^*(\bar{C})$, where C is a cocommutative DG coalgebra Koszul dual to the Lie algebra \mathfrak{a} . We construct a canonical DG algebra map $\Phi_{\mathfrak{g}}(\mathfrak{a}) : \mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})^G \rightarrow \mathrm{DRep}_{\mathfrak{h}}(\mathfrak{a})^W$, relating the G -invariant part of representation homology of a Lie algebra \mathfrak{a} in \mathfrak{g} to the W -invariant part of representation homology of \mathfrak{a} in a Cartan subalgebra of \mathfrak{g} . We call this map the derived Harish-Chandra homomorphism as it is a natural homological extension of the classical Harish-Chandra restriction map.

We conjecture that, for a two-dimensional abelian Lie algebra \mathfrak{a} , the derived Harish-Chandra homomorphism is a quasi-isomorphism. We provide some evidence for this conjecture, including proofs for \mathfrak{gl}_2 and \mathfrak{sl}_2 as well as for \mathfrak{gl}_n , \mathfrak{sl}_n , \mathfrak{so}_n and \mathfrak{sp}_{2n} in the inductive limit as $n \rightarrow \infty$. For any complex reductive Lie algebra \mathfrak{g} , we compute the Euler characteristic of $\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})^G$ in terms of matrix integrals over G and compare it to the Euler characteristic of $\mathrm{DRep}_{\mathfrak{h}}(\mathfrak{a})^W$. This yields an interesting combinatorial identity, which we prove for \mathfrak{gl}_n and \mathfrak{sl}_n (for all n). Our identity is analogous to the classical Macdonald identity, and our quasi-isomorphism conjecture is analogous to the strong Macdonald conjecture proposed in [H1, F] and proved in [FGT]. We explain this analogy by giving a new homological interpretation of Macdonald’s conjectures in terms of derived representation schemes, parallel to our Harish-Chandra quasi-isomorphism conjecture.

Keywords. Derived representation scheme, Lie algebra cohomology, Chevalley–Eilenberg complex, Harish-Chandra homomorphism, Koszul duality, Macdonald identity

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1. Introduction

This paper is a sequel to [BKR] and [BR1] (see also [BFR]), where we study the derived representation scheme $\mathrm{DRep}_n(A)$ parametrizing the n -dimensional representations of an associative algebra A over a field k of characteristic 0. This scheme is constructed in [BKR] in an abstract way by extending the classical representation functor $\mathrm{Rep}_n(-)$ to the category of differential graded (DG) algebras and deriving it in the sense of non-abelian homological algebra [Q1, DS]. The derived scheme $\mathrm{DRep}_n(A)$ is represented by a commutative DG algebra which (to simplify the notation) we also denote by $\mathrm{DRep}_n(A)$. The DG algebra $\mathrm{DRep}_n(A)$ is well defined up to homotopy; its homology $H_\bullet[\mathrm{DRep}_n(A)]$ is an invariant of A , which we call the (n -dimensional) *representation homology*.

From the very beginning, it was clear that representation homology must be somehow related (dual) to the Chevalley–Eilenberg homology of matrix Lie algebras but the precise form of this relation has been elusive. The first goal of the present paper is to clarify the relation between representation homology and Lie algebra (co)homology and offer a simple explanation of the formalism developed in [BKR, BR1]. Our starting point is a basic principle of homological algebra called *Koszul duality*. In concrete terms, it can be stated as follows. Associated to a (nonunital or augmented) algebra A is a coassociative cofree DG coalgebra $\mathbf{B}A$ called the *bar construction* [EM]. Any natural construction $\mathcal{C}(A)$ on algebras can be formally dualized (by reversing the arrows) to give the corresponding construction for coalgebras. When applied to $\mathbf{B}A$, this dual coalgebra construction gives a natural homological construction for A which we call the Koszul dual of $\mathcal{C}(A)$. Many interesting complexes and homological structures related to associative algebras arise in this way. For example, the Connes–Tsygan complex $\mathrm{CC}(A)$ defining the cyclic homology of an algebra A can be identified (up to shift in degree) with the cocommutator subspace of $\mathbf{B}A$, which is the Koszul dual construction of the universal algebra trace $A \twoheadrightarrow A/[A, A]$. The cyclic bicomplex (originally introduced in [T] and [LQ] to explain the periodicity properties of cyclic homology) can be interpreted as the Koszul dual of a noncommutative de Rham complex $X(A) = [\cdots \rightarrow A \rightarrow \Omega_{A, \natural}^1 \rightarrow A \rightarrow \Omega_{A, \natural}^1 \rightarrow \cdots]$ called the periodic X -complex of A (see [Q3]). Another example (unrelated to cyclic

homology) is Stasheff’s construction of the Gerstenhaber bracket on Hochschild cohomology $\mathrm{HH}^\bullet(A, A)$ of an algebra A : this bracket turns out to be the Koszul dual of the usual Lie bracket on the space $\mathrm{Der}(A)$ of derivations of A (see [St]).

The main observation of the present paper is that $\mathrm{DRep}_n(A)$ is the Koszul dual of the classical Chevalley–Eilenberg complex $\mathcal{C}(\mathfrak{gl}_n(A); k)$ computing homology of the current Lie algebra $\mathfrak{gl}_n(A)$. Recall that, for any Lie algebra \mathfrak{g} , the Chevalley–Eilenberg complex $\mathcal{C}(\mathfrak{g}; k)$ has a natural structure of a cocommutative DG coalgebra. The dual construction—the Chevalley–Eilenberg complex $\mathcal{C}^c(\mathfrak{G}; k)$ of a Lie coalgebra \mathfrak{G} —has therefore the structure of a commutative DG algebra. We show (see Theorem 3.2) that, for any augmented DG algebra A , there is a natural isomorphism of commutative DG algebras

$$\mathrm{DRep}_n(A) \cong \mathcal{C}^c(\mathfrak{gl}_n^*(\bar{B}A); k), \tag{1.1}$$

where $\bar{B}A$ is the reduced bar construction of A and \mathfrak{gl}_n^* is the Lie coalgebra (linearly) dual to the matrix Lie algebra \mathfrak{gl}_n . Furthermore, under (1.1), the GL_n -invariant part of $\mathrm{DRep}_n(A)$ (see Section 2.4.2 for a precise definition) corresponds to the *relative* Chevalley–Eilenberg complex of the natural Lie coalgebra map $\mathfrak{gl}_n^*(BA) \rightarrow \mathfrak{gl}_n^*(k)$:

$$\mathrm{DRep}_n(A)^{\mathrm{GL}} \cong \mathcal{C}^c(\mathfrak{gl}_n^*(BA), \mathfrak{gl}_n^*(k); k), \tag{1.2}$$

which is Koszul dual to the relative Chevalley–Eilenberg complex $\mathcal{C}(\mathfrak{gl}_n(A), \mathfrak{gl}_n(k); k)$ of the Lie algebra inclusion $\mathfrak{gl}_n(k) \subset \mathfrak{gl}_n(A)$.

As a consequence of (1.1), the representation homology of an algebra A is isomorphic to the Chevalley–Eilenberg homology of the matrix Lie coalgebra $\mathfrak{gl}_n^*(\bar{B}A)$. This gives a conceptual explanation to many results of [BKR] and [BR1]. For example, the derived character maps $\mathrm{Tr}_n(A)_\bullet : \mathrm{HC}_\bullet(A) \rightarrow \mathrm{H}_\bullet[\mathrm{DRep}_n(A)]$ constructed in [BKR] are Koszul dual to the natural trace maps $\mathrm{H}_\bullet(\mathfrak{gl}_n(A); k) \rightarrow \mathrm{HC}_{\bullet-1}(A)$ relating homology of matrix Lie algebras to cyclic homology, the degree shift in cyclic homology being explained by the fact that $\mathrm{HC}_\bullet(BA) \cong \mathrm{HC}_{\bullet-1}(A)$. The stabilization theorem for representation homology proved in [BR1] is Koszul dual to the classical theorem of Tsygan [T] and Loday–Quillen [LQ], although one result does not automatically follow from the other (see Section 3.3).

It is important to note that, in (1.1) and (1.2), we can replace BA by any DG coalgebra C which is Koszul dual to the algebra A (see Section 2.3). In fact, the bar construction BA is the universal Koszul dual coalgebra of A but it is often convenient to work with other coalgebras. For example, if A is a quadratic *Koszul algebra*, we can replace BA by the coalgebra $(A^!)^*$ (the linear dual of the graded Koszul dual algebra of A), which is known to be a minimal model for A (see [LV]). In this case, we have natural isomorphisms (see Section 3.4)

$$\mathrm{DRep}_n(A) \cong \mathcal{C}^{-\bullet}(\mathfrak{gl}_n(A^!); k), \quad \mathrm{DRep}_n(A)^{\mathrm{GL}} \cong \mathcal{C}^{-\bullet}(\mathfrak{gl}_n(A^!), \mathfrak{gl}(k); k),$$

where $\mathcal{C}^{-\bullet}$ denotes the (absolute and relative) *cochain* complexes of the Lie algebra $\mathfrak{gl}_n(A^!)$ equipped with homological grading.

Another natural way to generalize (1.1) and (1.2) is to replace \mathfrak{gl}_n by an arbitrary finite-dimensional reductive Lie algebra \mathfrak{g} . In place of the derived representation

scheme $\text{DRep}_n(A)$, one should consider its Lie analogue: the derived scheme $\text{DRep}_{\mathfrak{g}}(\mathfrak{a})$ parametrizing the representations of a given Lie algebra \mathfrak{a} in \mathfrak{g} . By a representation of \mathfrak{a} in \mathfrak{g} we simply mean a Lie algebra homomorphism $\mathfrak{a} \rightarrow \mathfrak{g}$, and $\text{DRep}_{\mathfrak{g}}(\mathfrak{a})$ is defined by extending the representation functor $\text{Rep}_{\mathfrak{g}}(-)$ to the category of DG Lie algebras DGLA_k and deriving it using a natural model structure on DGLA_k . For an arbitrary \mathfrak{g} , the isomorphisms (1.1) and (1.2) then become (see Theorem 6.7)¹

$$\text{DRep}_{\mathfrak{g}}(\mathfrak{a}) \cong \mathcal{C}^c(\mathfrak{g}^*(\bar{C}); k), \quad \text{DRep}_{\mathfrak{g}}(\mathfrak{a})^G \cong \mathcal{C}^c(\mathfrak{g}^*(C), \mathfrak{g}^*; k), \tag{1.3}$$

where C is a cocommutative DG coalgebra Koszul dual to the Lie algebra \mathfrak{a} .

Now, let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} , and W the corresponding Weyl group. The natural inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ gives a morphism of Lie coalgebras $\mathfrak{g}^* \rightarrow \mathfrak{h}^*$, which, in turn, extends to a map of commutative DG algebras $\mathcal{C}^c(\mathfrak{g}^*(C), \mathfrak{g}^*; k) \rightarrow \mathcal{C}^c(\mathfrak{h}^*(C), \mathfrak{h}^*; k)$. The image of this last map consists of chains that are invariant under the action of W (cf. Proposition 7.1): thus, in combination with (1.3), we get a natural map

$$\Phi_{\mathfrak{g}}(\mathfrak{a}) : \text{DRep}_{\mathfrak{g}}(\mathfrak{a})^G \rightarrow \mathcal{C}^c(\mathfrak{h}^*(C), \mathfrak{h}^*; k)^W. \tag{1.4}$$

More formally, $\Phi_{\mathfrak{g}}(\mathfrak{a})$ can be defined as $\text{DRep}_{\mathfrak{g}}(\mathfrak{a})^G \rightarrow \text{DRep}_{\mathfrak{h}}(\mathfrak{a})^W$, which is a functorial derived extension of the restriction map $k[\text{Rep}_{\mathfrak{g}}(\mathfrak{a})]^G \rightarrow k[\text{Rep}_{\mathfrak{h}}(\mathfrak{a})]^W$. In the simplest case, when \mathfrak{a} is a one-dimensional Lie algebra, $\text{DRep}_{\mathfrak{g}}(\mathfrak{a}) \cong k[\mathfrak{g}]$, and (1.4) becomes $k[\mathfrak{g}]^G \rightarrow k[\mathfrak{h}]^W$, which is the classical Harish-Chandra homomorphism² (see Example 7.2). In general, we will refer to (1.4) as the *derived Harish-Chandra homomorphism*. By a well-known theorem of Chevalley [C], the classical Harish-Chandra homomorphism is actually an isomorphism:

$$k[\mathfrak{g}]^G \xrightarrow{\sim} k[\mathfrak{h}]^W. \tag{1.5}$$

It is therefore a natural question to ask if the map (1.4) is a quasi-isomorphism in general.

In the present paper, we address this question for finite-dimensional *abelian* Lie algebras. Note that if \mathfrak{a} is abelian, a choice of linear basis in \mathfrak{a} identifies $\text{Rep}_{\mathfrak{g}}(\mathfrak{a})$ with the commuting scheme of the reductive Lie algebra \mathfrak{g} (cf. [R]). Hence, in this case, $\text{DRep}_{\mathfrak{g}}(\mathfrak{a})$ should be thought of as the *derived commuting scheme* of \mathfrak{g} . We show that (1.4) cannot be a quasi-isomorphism (for all \mathfrak{g}) if $\dim_k(\mathfrak{a}) \geq 3$ (cf. Section 5.2); on the positive side, we expect that the following is true (see Conjecture 8.1):

$$\text{If } \mathfrak{a} \text{ is abelian and } \dim_k(\mathfrak{a}) = 2, \text{ then } \Phi_{\mathfrak{g}}(\mathfrak{a}) \text{ is a quasi-isomorphism.} \tag{1.6}$$

In the case of \mathfrak{gl}_n , this implies (see Conjecture 4.6)

$$\text{DRep}_n(k[x, y])^{\text{GL}} \cong k[x_1, \dots, x_n, y_1, \dots, y_n, \theta_1, \dots, \theta_n]^{S_n},$$

¹ The derived schemes $\text{DRep}_n(A)$ and $\text{DRep}_{\mathfrak{g}}(\mathfrak{a})$ are examples of a general operadic construction that we sketch in the Appendix. The isomorphisms (1.1)–(1.3) are special cases of Theorem A.4 proved in the Appendix.

² Harish-Chandra actually defined a homomorphism $\mathcal{D}(\mathfrak{g})^G \rightarrow \mathcal{D}(\mathfrak{h})^W$ of rings of invariant differential operators that reduces to $k[\mathfrak{g}]^G \rightarrow k[\mathfrak{h}]^W$ on the zero order differential operators (see [HC]).

where the polynomial ring on the right is homologically graded so that the variables x_1, \dots, x_n and y_1, \dots, y_n have degree 0, the variables $\theta_1, \dots, \theta_n$ have degree 1, and the differential is identically zero. The symmetric group S_n acts on this polynomial ring diagonally by permuting the triples (x_i, y_i, θ_i) .

Conjecture (1.6) can be restated in elementary terms, without using the language of derived schemes. To this end, consider the graded commutative algebra $k[\mathfrak{g} \times \mathfrak{g}] \otimes \wedge \mathfrak{g}^*$, where $k[\mathfrak{g} \times \mathfrak{g}]$ is the ring of polynomial functions on $\mathfrak{g} \times \mathfrak{g}$ assigned homological degree 0, and $\wedge \mathfrak{g}^*$ is the exterior algebra of the dual Lie algebra \mathfrak{g}^* assigned homological degree 1. The differential on $k[\mathfrak{g} \times \mathfrak{g}] \otimes \wedge \mathfrak{g}^*$ is defined by

$$d\varphi(\xi, \eta) := \varphi([\xi, \eta]), \quad \forall (\xi, \eta) \in \mathfrak{g} \times \mathfrak{g}, \forall \varphi \in \mathfrak{g}^*.$$

The DG algebra $(k[\mathfrak{g} \times \mathfrak{g}] \otimes \wedge \mathfrak{g}^*, d)$ represents the derived scheme $\text{DRep}_{\mathfrak{g}}(\mathfrak{a})$ in the homotopy category of commutative DG algebras, and the derived Harish-Chandra homomorphism $\Phi_{\mathfrak{g}}(\mathfrak{a})$ is given in this case by the natural restriction map (see Proposition 8.2)

$$(k[\mathfrak{g} \times \mathfrak{g}] \otimes \wedge \mathfrak{g}^*)^G \rightarrow (k[\mathfrak{h} \times \mathfrak{h}] \otimes \wedge \mathfrak{h}^*)^W. \tag{1.7}$$

Conjecture (1.6) is thus equivalent to the claim that (1.7) is a quasi-isomorphism.

Our second goal in this paper is to provide evidence for conjecture (1.6) and discuss some of its implications. First, in the case of \mathfrak{gl}_n , we prove that (1.6) holds for $n = 2$ and $n = \infty$ (see Theorem 4.7 and Theorem 4.10 respectively); we also prove that the map $H_{\bullet}(\Phi_{\mathfrak{gl}_n})$ induced by (1.4) on homology is surjective for all n (see Theorem 4.10). Second, we show (see Theorem 8.8) that our conjecture for \mathfrak{sl}_n is equivalent to that for \mathfrak{gl}_n (and hence holds for \mathfrak{sl}_2). Using a version of stabilization theorem of [BR1], we also verify (1.6) for the orthogonal and symplectic Lie algebras, \mathfrak{so}_n and \mathfrak{sp}_{2n} , in the inductive limit as $n \rightarrow \infty$ (see Section 8.5). Finally, we compute the weighted Euler characteristics of both sides of (1.4) and show that (1.6) implies the following constant term identity (see Conjecture 8.5):

$$\frac{(1-qt)^l}{(1-q)^l(1-t)^l} \text{CT} \left\{ \prod_{\alpha \in R} \frac{(1-qte^{\alpha})(1-e^{\alpha})}{(1-qe^{\alpha})(1-te^{\alpha})} \right\} = \sum_{w \in W} \frac{\det(1-qtw)}{\det(1-qw)\det(1-tw)}. \tag{1.8}$$

Here R is a system of roots of the Lie algebra \mathfrak{g} , $l := \dim_k(\mathfrak{h})$ is its rank and $\text{CT} : \mathbb{Z}[Q] \rightarrow \mathbb{Z}$ is the constant term map defined on the group ring of the root lattice of R (see (8.4)). The determinants on the right are taken in the natural (reflection) representation of W on \mathfrak{h} . One of our main results (Theorem 5.5) is that the identity (1.8) holds for \mathfrak{gl}_n and \mathfrak{sl}_n for all n , in which cases it can be described in purely combinatorial terms (see (5.2) and also Remark 3 after Theorem 5.5).

If $k = \mathbb{C}$, the identity (1.8) can be written in a more symmetric, integral form:

$$\int_G \frac{\det(1-qt \text{Ad } g)}{\det(1-q \text{Ad } g)\det(1-t \text{Ad } g)} dg = \frac{1}{|W|} \sum_{w \in W} \frac{\det(1-qtw)}{\det(1-qw)\det(1-tw)}. \tag{1.9}$$

Here the integration is taken over a real compact form of the complex Lie group G , which is equipped with the invariant Haar measure dg normalized so that $\int_G dg = 1$. Notice that if we specialize $t = 0$, (1.9) becomes the well-known identity

$$\int_G \frac{dg}{\det(1 - q \operatorname{Ad} g)} = \prod_{i=1}^l \frac{1}{1 - q^{d_i}}, \quad (1.10)$$

which exhibits the equality of the Poincaré series of both sides of the Chevalley isomorphism (1.5). The Chevalley isomorphism (1.5) has a natural ‘odd’ analogue: the Hopf–Koszul–Samelson isomorphism $(\wedge \mathfrak{g})^G \cong \wedge(\operatorname{Prim} \mathfrak{g})$, identifying the space of invariants in the exterior algebra of \mathfrak{g} with the exterior algebra of its subspace of primitive elements (see, e.g., [Me, Chap. 10]). At the level of Poincaré series, the Hopf–Koszul–Samelson isomorphism gives the identity

$$\int_G \det(1 + q \operatorname{Ad} g) dg = \prod_{i=1}^l (1 + q^{2d_i-1}), \quad (1.11)$$

which may be viewed as an ‘odd’ analogue of (1.10). In his original paper [M] on Macdonald conjectures, I. Macdonald observed that (1.11) arises as a specialization of his constant term identity³

$$\frac{1}{|W|} \operatorname{CT} \left\{ \prod_{n \geq 0} \prod_{\alpha \in R} \frac{1 - q^n e^\alpha}{1 - q^n t e^\alpha} \right\} = \prod_{n \geq 0} \prod_{i=1}^l \frac{(1 - q^n t)(1 - q^{n+1} t^{d_i-1})}{(1 - q^{n+1})(1 - q^n t^{d_i})}, \quad (1.12)$$

and he asked (cf. [M, Remark 2, p. 997]) whether (1.10) admits a (q, t) -generalization analogous to (1.12). It seems that (1.8) is an answer to Macdonald’s question.

The above analogy raises the question if the Macdonald identity (1.12) has a homological origin similar to that of (1.8). Building on [M], Hanlon [H1, H2] (and independently Feigin [F]) gave an interpretation of (1.12) in terms of cohomology of certain nilpotent Lie algebras: they made a precise conjecture (cf. [H1, Conjecture 1.5]) on the structure of this cohomology that entails (1.12). The Hanlon–Feigin conjecture (also known as the strong Macdonald conjecture) was proved in full generality by Fishel, Grojnowski and Teleman [FGT]. In the present paper, we will give a different interpretation of the Macdonald identity that clarifies its relation to our identity (1.8).

We begin with a general remark. Working with derived representation schemes $\operatorname{DRep}_{\mathfrak{g}}(\mathfrak{a})$, it is natural to put on \mathfrak{a} a (homological) grading: indeed, even when $\operatorname{Rep}_{\mathfrak{g}}(\mathfrak{a})$ is trivial (for example, when the grading on \mathfrak{a} does not allow any homomorphisms $\mathfrak{a} \rightarrow \mathfrak{g}$ other than zero), the derived scheme $\operatorname{DRep}_{\mathfrak{g}}(\mathfrak{a})$ may have an interesting (and quite non-trivial) structure. If \mathfrak{a} is abelian and $\dim_k(\mathfrak{a}) = 2$, putting a grading on \mathfrak{a} amounts to splitting it into the sum of two one-dimensional subspaces of homological degrees p and r (in this case we will write $\mathfrak{a} = \mathfrak{a}_{p,r}$ to emphasize the grading). It turns out that the structure of $\operatorname{DRep}_{\mathfrak{g}}(\mathfrak{a}_{p,r})$ essentially depends only on the parities of p and r (see Proposition 9.6),

³ The constant term identity (1.12) as well as its generalization (the so-called inner product identity) was proved for an arbitrary root system by I. Cherednik (see [Ch] and also [Kir]).

and therefore there are three possibilities, which we refer to as the *even*, *mixed* and *odd* cases (depending on whether p and r are both even, have opposite parities or are both odd).

In the even case, we show that conjecture (1.6) holds for $\mathfrak{a}_{p,r}$ if and only if it holds for \mathfrak{a} with trivial grading (i.e., $p = r = 0$); thus, in this case, we expect the Harish-Chandra homomorphism (1.4) to be a quasi-isomorphism, and the resulting constant term identity is (1.8).

In the mixed case, the situation seems quite different: the derived Harish-Chandra homomorphism is no longer a quasi-isomorphism, and we need to construct a new map. To explain the construction we return for a moment to the general situation. Drinfeld [Dr] introduced a natural functor on the category of Lie algebras that associates to a Lie algebra \mathfrak{a} the universal invariant bilinear form

$$\lambda(\mathfrak{a}) = \text{Sym}_k^2(\mathfrak{a}) / \langle [x, y] \cdot z - x \cdot [y, z] : x, y, z \in \mathfrak{a} \rangle.$$

Following a suggestion of Kontsevich [K], Getzler and Kapranov [GK] defined cyclic homology for Lie algebras (and more generally, for algebras over an arbitrary cyclic operad) as the nonabelian derived functor of the functor λ . Our starting point is a natural extension of the Drinfeld–Getzler–Kapranov construction: for an integer $d \geq 1$, we consider the functor $\lambda^{(d)} : \text{DGLA}_k \rightarrow \text{Com}_k$ assigning to a Lie algebra \mathfrak{a} (the target of) the universal invariant multilinear form on \mathfrak{a} of degree d (so that $\lambda^{(2)} = \lambda$). We prove (see Theorem 7.3) that, for any d , this functor has a left derived functor $\mathbb{L}\lambda^{(d)} : \text{Ho}(\text{DGLA}_k) \rightarrow \text{Ho}(\text{Com}_k)$ defined on the homotopy category of DG Lie algebras, and we let $\text{HC}_\bullet^{(d)}(\text{Lie}, \mathfrak{a})$ denote the homology of $\mathbb{L}\lambda^{(d)}(\mathfrak{a})$. The meaning of this construction is clarified by Theorem 7.4, which asserts that the (reduced) cyclic homology of the universal enveloping algebra $\mathcal{U}(\mathfrak{a})$ of any Lie algebra \mathfrak{a} has a canonical Hodge-type decomposition⁴

$$\overline{\text{HC}}_\bullet(\mathcal{U}\mathfrak{a}) = \bigoplus_{d \geq 1} \text{HC}_\bullet^{(d)}(\text{Lie}, \mathfrak{a}). \tag{1.13}$$

The decomposition (1.13) may be viewed as a Koszul dual of the classical Hodge decomposition of the cyclic homology of commutative algebras. In particular, as in the case of commutative algebras (cf. [BV]), there are Adams operations ψ^p on $\overline{\text{HC}}_\bullet(\mathcal{U}\mathfrak{a})$, whose (graded) eigenspaces are precisely $\text{HC}_\bullet^{(d)}(\text{Lie}, \mathfrak{a})$ (see Section 7.2.2). Next, for any homogeneous invariant polynomial on \mathfrak{g} of degree d , we construct natural trace maps (see Section 7.3)

$$\text{Tr}_\mathfrak{g}^{(d)}(\mathfrak{a}) : \mathbb{L}\lambda^{(d)}(\mathfrak{a}) \rightarrow \text{DRep}_\mathfrak{g}(\mathfrak{a})^G,$$

that are analogues of the derived character maps of [BKR] for Lie algebras. Letting d run over the set $\{d_1, \dots, d_l\}$ of fundamental degrees of \mathfrak{g} , we then define the homomorphism of commutative DG algebras

$$\text{Sym}_k \left[\bigoplus_{i=1}^l \text{Tr}_\mathfrak{g}^{(d_i)}(\mathfrak{a}) \right] : \text{Sym}_k \left[\bigoplus_{i=1}^l \mathbb{L}\lambda^{(d_i)}(\mathfrak{a}) \right] \rightarrow \text{DRep}_\mathfrak{g}(\mathfrak{a})^G, \tag{1.14}$$

⁴ This provides a (partial) answer to a question of V. Ginzburg about the existence of Hodge decomposition for cyclic homology of noncommutative algebras.

which we call the *Drinfeld trace map*. In the simplest case when \mathfrak{a} is a one-dimensional Lie algebra, the Drinfeld trace map coincides with the inverse of the Chevalley isomorphism (1.5) (see Example 7.10).

Returning to the derived commuting schemes, we may now state our last main result (see Theorem 9.2): for the two-dimensional abelian Lie algebra $\mathfrak{a} = \mathfrak{a}_{p,r}$ graded in such a way that p and r have opposite parities, the Drinfeld trace map (1.14) is a quasi-isomorphism, and at the level of Euler characteristics, it gives precisely the Macdonald identity (1.12). Unfortunately, our proof of Theorem 9.2 is not entirely self-contained: apart from results proved in this paper, it relies on one of the main theorems of [FGT]. Still, we believe that our interpretation of the Macdonald identity in terms of representation homology is, in some respects, more natural than the classical one in terms of Lie cohomology, and at the very least, it clarifies the relation between (1.12) and (1.8). We would also like to mention an interesting recent paper [Kh] which gives yet another homological interpretation of Macdonald's theory in terms of representation theory of current Lie algebras. It seems that our approach is related to that of [Kh] in a natural way (via Koszul duality at the level of derived module categories); it would be interesting to study this relation, especially with a view towards understanding (1.8).

Finally, we have to mention that, in the odd case (when p and r are both odd), the structure of the derived commuting scheme $\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a}_{p,r})^G$ remains mysterious to us. We do not know whether there exists a numerical identity analogous to (1.8) and (1.12) in this case.

The paper is organized as follows. Section 2 is preliminary: here, we recall basic facts of differential homological algebra and review the construction of derived representation schemes from [BKR] and [BR1]. In Section 3, we prove our first main result, Theorem 3.2, which relates representation homology to Lie (co)homology, and discuss its implications. In Section 4, we construct the derived Harish-Chandra homomorphism for representation schemes of associative algebras and state our main conjecture for \mathfrak{gl}_n (see Conjecture 4.6). The key results of this section are Theorem 4.7 (proof of Conjecture 4.6 for $n = 2$), Theorem 4.10 (surjectivity of the Harish-Chandra homomorphism on homology for all n and bijectivity for $n = \infty$) and Theorem 4.13 (proof of the analog of Conjecture 4.6 for q -polynomial algebras). Next, in Section 5, we compute Euler characteristics and deduce our constant term identity in the case of \mathfrak{gl}_n (see (5.2)). The main result of this section is Theorem 5.5, which proves the constant term identity for \mathfrak{gl}_n for all n . In Section 6, we define representation homology for an arbitrary (reductive) Lie algebra \mathfrak{g} . The main result of this section, Theorem 6.7, is a natural generalization of Theorem 3.2. In Section 7, we construct the derived Harish-Chandra homomorphism and the Drinfeld trace maps for representation schemes of Lie algebras. The main result is Theorem 7.4 that gives a Hodge decomposition for the cyclic homology of the universal enveloping algebra of any DG Lie algebra. In Section 8, we consider the derived commuting variety of a reductive Lie algebra; we state our quasi-isomorphism conjecture in full generality (Conjecture 8.1) and deduce the corresponding constant term identity (Conjecture 8.5); we also provide some evidence in favor of Conjecture 8.1, including its verification for orthogonal and symplectic Lie algebras in the limit $n \rightarrow \infty$ (Theorem 8.15). In Section 9, we explain the relation of our conjectures to the classical Macdonald conjecture [M] and

the strong Macdonald conjecture of [H1] and [FGT]. The main result of this section is Theorem 9.2 showing that the Drinfeld trace map is a quasi-isomorphism in the mixed case. Finally, in the Appendix, we generalize our construction of derived representation schemes to algebras over an arbitrary binary quadratic operad. This puts some of the main results of present paper in a proper perspective.

2. Preliminaries

In this section, we introduce notation and recall some basic results from the literature. In particular, we review the construction of derived representation schemes and derived character maps from [BKR].

2.1. Notation and conventions

Throughout this paper, k denotes a base field of characteristic zero. An unadorned tensor product \otimes stands for the tensor product \otimes_k over k . An algebra means an associative k -algebra with 1; the category of such algebras is denoted \mathbf{Alg}_k . Unless stated otherwise, all differential graded (DG) objects are equipped with differentials of degree -1 , and the Koszul sign rule is systematically used. The homological degree of a homogeneous element x in a graded vector space V will often be denoted by $|x|$. If V is a graded k -vector space, we denote by $T_k V$ its tensor algebra and by $\mathbf{Sym}_k(V)$ its graded symmetric algebra. Thus, $\mathbf{Sym}_k(V) = \mathbf{Sym}_k(V_{\text{ev}}) \otimes \wedge_k(V_{\text{odd}})$, where V_{ev} and V_{odd} are the even and odd components of V respectively. $\mathbf{Sym}^d(V)$ will denote the d -th symmetric power of V . Thus, $\mathbf{Sym}^d(V) = \bigoplus_{p+q=d} \mathbf{Sym}^p(V_{\text{ev}}) \otimes \wedge^q(V_{\text{odd}})$ and $\mathbf{Sym}_k(V) = \bigoplus_{d=0}^{\infty} \mathbf{Sym}^d(V)$. The graded symmetric coalgebra on V will be denoted by $\mathbf{Sym}^c(V)$. For $V, W \in \mathbf{Com}_k$, we define $\mathbf{Hom}(V, W)$ to be the complex whose space of p -chains is

$$\mathbf{Hom}(V, W)_p := \prod_n \mathbf{Hom}_k(V_n, W_{n+p})$$

and the differential is given by $df := d_W \circ f - (-1)^p f \circ d_V$, where $f \in \mathbf{Hom}(V, W)_p$.

2.2. The bar/cobar construction

In this and the next section, we briefly recall some classical results from differential homological algebra which are needed for the present paper. An excellent modern reference for this material is [LV].

2.2.1. DG algebras and coalgebras. Let \mathbf{DGA}_k denote the category of associative unital DG algebras over k with differential of degree -1 . Recall that $A \in \mathbf{DGA}_k$ is *augmented* if it is given together with a DG algebra map $\varepsilon : A \rightarrow k$. A morphism of augmented algebras $(A, \varepsilon_A) \rightarrow (B, \varepsilon_B)$ is a morphism $f : A \rightarrow B$ in \mathbf{DGA}_k satisfying $\varepsilon_B \circ f = \varepsilon_A$. We denote the category of augmented DG algebras by $\mathbf{DGA}_{k/k}$. This category is known to be equivalent to the category \mathbf{DGA} of nonunital DG algebras: the mutually inverse functors are given by

$$B' \leftarrow B, \quad \mathbf{DGA}_{k/k} \rightleftarrows \mathbf{DGA}, \quad A \mapsto \bar{A}$$

where $\bar{A} := \text{Ker}(\varepsilon)$ is the kernel of the augmentation map of A and $B' := k \oplus B$ is the unitalization of B equipped with the canonical projection $\varepsilon : B' \rightarrow k$. Similarly, we define the (equivalent) categories $\text{DGCA}_{k/k}$ and DGCA of commutative DG algebras: augmented and nonunital, respectively.

Let DGC (resp., DGC_k) denote the category of coassociative (resp., coassociative counital) DG coalgebras over k . We shall often work with augmented coassociative counital DG coalgebras C which are *conilpotent* in the sense that

$$\bar{C} = \bigcup_{n \geq 2} \text{Ker}[C \xrightarrow{\Delta^{(n)}} C^{\otimes n} \twoheadrightarrow \bar{C}^{\otimes n}] \tag{2.1}$$

where $\Delta^{(n)}$ denotes the n -th iteration of the comultiplication map $\Delta_C : C \rightarrow C \otimes C$ and \bar{C} is the cokernel of the augmentation map $\varepsilon_C : k \rightarrow C$. We denote the category of such coalgebras by $\text{DGC}_{k/k}$. Similarly, DGCC (resp., DGCC_k) will denote the category of cocommutative (resp., cocommutative counital) DG coalgebras over k , and $\text{DGCC}_{k/k}$ will denote the category of coaugmented conilpotent cocommutative DG coalgebras over k .

2.2.2. Twisting cochains. Given an algebra $R \in \text{DGA}_{k/k}$ and a coalgebra $C \in \text{DGC}_{k/k}$, we define a *twisting cochain* $\tau : C \rightarrow R$ to be a linear map of degree -1 satisfying

$$d_R \tau + \tau d_C + m_R(\tau \otimes \tau)\Delta_C = 0, \quad \tau \circ \varepsilon_C = 0, \quad \varepsilon_R \circ \tau = 0,$$

where d_R and d_C are the differentials on R and C , and m_R is the multiplication map on R . We write $\text{Tw}(C, R)$ for the set of all twisting cochains from C to R . It is easy to show that, for a fixed algebra R , the functor

$$\text{Tw}(-, R) : \text{DGC}_{k/k} \rightarrow \text{Sets}, \quad C \mapsto \text{Tw}(C, R),$$

is representable; the corresponding coalgebra $\mathbf{B}(R) \in \text{DGC}_{k/k}$ is called the *bar construction* of R : it is defined as the tensor coalgebra $T_k(R[1])$ with differential lifting d_R and m_R . Dually, for a fixed coalgebra C , the functor

$$\text{Tw}(C, -) : \text{DGA}_{k/k} \rightarrow \text{Sets}, \quad R \mapsto \text{Tw}(C, R),$$

is corepresentable; the corresponding algebra $\mathbf{\Omega}(C) \in \text{DGA}_{k/k}$ is called the *cobar construction* of C : it is defined as the tensor algebra $T_k(C[-1])$ with differential lifting d_C and Δ_C . Thus, we have canonical isomorphisms

$$\text{Hom}_{\text{DGA}_{k/k}}(\mathbf{\Omega}(C), R) = \text{Tw}(C, R) = \text{Hom}_{\text{DGC}_{k/k}}(C, \mathbf{B}(R)) \tag{2.2}$$

showing that $\mathbf{\Omega} : \text{DGC}_{k/k} \rightleftarrows \text{DGA}_{k/k} : \mathbf{B}$ are adjoint functors.

2.2.3. Koszul–Moore equivalence. The categories DGA_k and DGCA_k carry natural model structures (see [Hi]). The weak equivalences in these model categories are the quasi-isomorphisms and the fibrations are the degreewise surjective maps. The cofibrations are characterized in abstract terms: as morphisms satisfying the left lifting property with respect to the acyclic fibrations. The model structures on DGA_k and DGCA_k naturally induce

model structures on the corresponding categories of augmented algebras (cf. [DS, 3.10]). In particular, a morphism $f : A \rightarrow B$ in $\text{DGA}_{k/k}$ is a weak equivalence (resp., fibration or cofibration) iff $f : A \rightarrow B$ is a weak equivalence (resp., fibration or cofibration) in DGA_k . All objects in DGA_k and $\text{DGA}_{k/k}$ are fibrant. The cofibrant objects in $\text{DGA}_{k/k}$ can be described more explicitly than in DGA_k : every cofibrant $A \in \text{DGA}_{k/k}$ is isomorphic to a retract of $\Omega(C)$, where $\Omega(C)$ is the cobar construction of an augmented conilpotent coassociative DG coalgebra C (see [Ke, Theorem 4.3]).

There is a dual model structure on $\text{DGC}_{k/k}$, where the weak equivalences are the morphisms f such that $\Omega(f)$ is a quasi-isomorphism. A well-known result, due to J. C. Moore, asserts that the model categories $\text{DGA}_{k/k}$ and $\text{DGC}_{k/k}$ are Quillen equivalent: more precisely,

Theorem 2.1 ([Mo]). *The pair of adjoint functors*

$$\Omega : \text{DGC}_{k/k} \rightleftarrows \text{DGA}_{k/k} : \mathbf{B} \tag{2.3}$$

is a Quillen equivalence. In particular, the functors (2.3) induce mutually inverse equivalences between the homotopy categories

$$\mathbb{L}\Omega : \text{Ho}(\text{DGC}_{k/k}) \cong \text{Ho}(\text{DGA}_{k/k}) : \mathbb{R}\mathbf{B}.$$

The proof of this theorem can be found, for example, in [LV].

2.3. Koszul duality

Let $C \in \text{DGC}_{k/k}$ and let $R \in \text{DGA}_{k/k}$. Let $\text{Mod}(R)$ denote the category of right DG modules over R , and dually let $\text{CoMod}(C)$ denote the category of right DG comodules over C which are conilpotent in a sense similar to (2.1). Given a twisting cochain $\tau \in \text{Tw}(C, R)$ one can define the functors

$$- \otimes_{\tau} C : \text{Mod}(R) \rightleftarrows \text{CoMod}(C) : - \otimes_{\tau} R$$

called the *twisted tensor products*. Specifically, if $M \in \text{Mod}(R)$, then $M \otimes_{\tau} C$ is defined to be the DG C -comodule whose underlying graded comodule is $M \otimes_k C$ and whose differential is given by

$$d = d_M \otimes \text{Id} + \text{Id} \otimes d_C + (m \otimes \text{Id})(\text{Id} \otimes \tau \otimes \text{Id})(\text{Id} \otimes \Delta).$$

Similarly, for a DG comodule $N \in \text{CoMod}(C)$, one defines a DG R -module $N \otimes_{\tau} R$. In the same fashion, for a DG bicomodule $\mathcal{N} \in \text{Bicomod}(C)$, one defines a DG R -bimodule $R_{\tau} \otimes \mathcal{N} \otimes_{\tau} R$.

Next, recall that the *derived category* $\mathcal{D}(R)$ of DG modules is obtained by localizing $\text{Mod}(R)$ at the class of all quasi-isomorphisms. To introduce the dual notion for DG comodules one has to replace the quasi-isomorphisms by a more restricted class of morphisms in $\text{CoMod}(C)$. We call a morphism f in $\text{CoMod}(C)$ a *weak equivalence* if $f \otimes_{\tau_C} \Omega(C)$ is a quasi-isomorphism in $\text{Mod}\Omega(C)$, where $\tau_C : C \rightarrow \Omega(C)$ is the universal twisting cochain corresponding to the identity map under (2.2). The *coderived category* $\mathcal{D}^c(C)$ of DG comodules is then defined by localizing $\text{CoMod}(C)$ at the class of weak

equivalences. It is easy to check that the twisted tensor products induce a pair of adjoint functors

$$-\otimes_{\tau} C : \mathcal{D}(R) \rightleftarrows \mathcal{D}^c(C) : -\otimes_{\tau} R. \quad (2.4)$$

The following theorem characterizes the class of twisting cochains for which (2.4) are equivalences.

Theorem 2.2 (see [LV, Theorem 2.3.1]). *For $\tau \in \text{Tw}(C, R)$, the following are equivalent:*

- (i) *the functors (2.4) are mutually inverse equivalences of categories;*
- (ii) *the complex $C \otimes_{\tau} R$ is acyclic;*
- (iii) *the complex $R \otimes_{\tau} C$ is acyclic;*
- (iv) *the natural morphism $R \otimes_{\tau} C \otimes_{\tau} R \xrightarrow{\sim} R$ is a quasi-isomorphism;*
- (v) *the morphism $\Omega(C) \xrightarrow{\sim} R$ corresponding to τ under (2.2) is a quasi-isomorphism in $\text{DGA}_{k/k}$;*
- (vi) *the morphism $C \xrightarrow{\sim} \mathbf{B}(R)$ corresponding to τ under (2.2) is a weak equivalence in $\text{DGC}_{k/k}$.*

If conditions (i)–(vi) hold, the DG algebra R is determined by C up to isomorphism in $\text{Ho}(\text{DGA}_{k/k})$ and the DG coalgebra C is determined by R up to isomorphism in $\text{Ho}(\text{DGC}_{k/k})$.

A twisting cochain $\tau \in \text{Tw}(C, R)$ satisfying the conditions of Theorem 2.2 is called *acyclic*. In this case, the DG coalgebra C is called *Koszul dual* to the DG algebra R and R is called *Koszul dual* to C .

2.4. Derived representation schemes

Derived representation schemes were originally discussed in [CK] as part of a general program of deriving *Quot* schemes and other moduli spaces in algebraic geometry. A different, more algebraic approach was developed in [BKR], where it was shown, among other things, that the representation functor is a (left) Quillen functor on the model category of associative DG algebras. In this section, we briefly review the basic construction of [BKR].

2.4.1. The representation functor. For an integer $n \geq 1$, let $\mathcal{M}_n(k)$ denote the algebra of $n \times n$ matrices with entries in k . If $B \in \text{DGA}_k$, we write $\mathcal{M}_n(B) := \mathcal{M}_n(k) \otimes B$; this gives a functor on the category of DG algebras: $\mathcal{M}_n(-) : \text{DGA}_k \rightarrow \text{DGA}_k$. Next, we define

$$\sqrt[n]{-} : \text{DGA}_k \rightarrow \text{DGA}_k, \quad A \mapsto [A *_k \mathcal{M}_n(k)]^{\mathcal{M}_n(k)}, \quad (2.5)$$

where $A *_k \mathcal{M}_n(k)$ is the free product (coproduct) in DGA_k and $[\dots]^{\mathcal{M}_n(k)}$ denotes the (graded) centralizer of $\mathcal{M}_n(k)$ as the subalgebra in $A *_k \mathcal{M}_n(k)$.

The following proposition is a generalization (to DG algebras) of a classical result of G. Bergman (see [BKR, Proposition 2.1]).

Proposition 2.3. *The functor $\sqrt[n]{-}$ is left adjoint to $\mathcal{M}_n(-)$.*

Now, let DGCA_k be the category of commutative DG algebras. The natural inclusion functor $\text{DGCA}_k \rightarrow \text{DGA}_k$ has an obvious left adjoint given by

$$(-)_{\text{ab}} : \text{DGA}_k \rightarrow \text{DGCA}_k, \quad A \mapsto A_{\text{ab}} := A/\langle [A, A] \rangle, \tag{2.6}$$

where $\langle [A, A] \rangle$ is the two-sided DG ideal of A generated by the graded commutators. Combining (2.5) and (2.6), we define

$$(-)_n : \text{DGA}_k \rightarrow \text{DGCA}_k, \quad A \mapsto A_n := (\sqrt[n]{A})_{\text{ab}}. \tag{2.7}$$

Then, as a consequence of Proposition 2.3, we get

Theorem 2.4. *The functor $(-)_n$ is left adjoint to $\mathcal{M}_n(-)$ on the category of commutative DG algebras. Thus, for any $A \in \text{DGA}_k$, the DG algebra A_n (co)represents the functor of points of the affine DG scheme*

$$\text{Rep}_n(A) : \text{DGCA}_k \rightarrow \text{Sets}, \quad B \mapsto \text{Hom}_{\text{DGA}_k}(A, \mathcal{M}_n(B)), \tag{2.8}$$

parametrizing the n -dimensional representations of A .

Theorem 2.4 implies that there is a natural bijection

$$\text{Hom}_{\text{DGA}_k}(A, \mathcal{M}_n(B)) \cong \text{Hom}_{\text{DGCA}_k}(A_n, B), \tag{2.9}$$

functorial in $A \in \text{DGA}_k$ and $B \in \text{DGCA}_k$. Accordingly, we refer to (2.7) as the functor of n -dimensional representations, or simply the n -th representation functor on DGA_k . Letting $B = A_n$ in (2.9), we have a canonical DG algebra map $\pi_n : A \rightarrow \mathcal{M}_n(A_n)$, called the universal n -dimensional representation of A .

The algebra A_n has the following canonical presentation described in [BKR, Section 2.4]. Let $\{e_{ij}\}_{i,j=1}^n$ be the basis of elementary matrices in $\mathcal{M}_n(k)$. For each $a \in A$, define the ‘matrix’ elements of a in $A *_k \mathcal{M}_n(k)$ by

$$a_{ij} := \sum_{k=1}^n e_{ki} a e_{jk}.$$

Then $a_{ij} \in \sqrt[n]{A}$ for all $i, j = 1, \dots, n$, and we also write a_{ij} for the corresponding elements in A_n .

Lemma 2.5. *The algebra A_n is generated by the elements $\{a_{ij} : a \in A\}$ satisfying the relations*

$$(a + b)_{ij} = a_{ij} + b_{ij}, \quad (ab)_{ij} = \sum_{k=1}^n a_{ik} b_{kj}, \quad \lambda_{ij} = \delta_{ij} \lambda, \quad \forall a, b \in A, \lambda \in k.$$

The differential on A_n is determined by the formula $d(a_{ij}) = (da)_{ij}$. The universal n -dimensional representation of A is given by

$$\pi_n : A \rightarrow \mathcal{M}_n(A_n), \quad a \mapsto \|a_{ij}\|.$$

In this paper, we will work with augmented DG algebras. Note that if A is augmented, then A_n has a natural augmentation $\varepsilon_n : A_n \rightarrow k$ coming from (2.7) applied to the augmentation map of A . This defines a functor $\text{DGA}_{k/k} \rightarrow \text{DGCA}_{k/k}$, which we again denote by $(-)_n$. On the other hand, the matrix algebra functor can be modified in the following way:

$$\mathcal{M}'_n(-) : \text{DGCA}_{k/k} \rightarrow \text{DGA}_{k/k}, \quad \mathcal{M}'_n(B) := k \oplus \mathcal{M}_n(\bar{B}). \tag{2.10}$$

With this modification, Theorem 2.4 holds for augmented DG algebras. Moreover, as a special case of [BKR, Theorem 2.2], we have

Theorem 2.6. (a) *The adjoint functors $(-)_n : \text{DGA}_{k/k} \rightleftarrows \text{DGCA}_{k/k} : \mathcal{M}'_n(-)$ form a Quillen pair.*

(b) *The functor $(-)_n : \text{DGA}_{k/k} \rightarrow \text{DGCA}_{k/k}$ has a total left derived functor defined by*

$$\mathbb{L}(-)_n : \text{Ho}(\text{DGA}_{k/k}) \rightarrow \text{Ho}(\text{DGCA}_{k/k}), \quad A \mapsto (QA)_n,$$

where $QA \xrightarrow{\sim} A$ is any cofibrant resolution of A in $\text{DGA}_{k/k}$.

(c) *For any A in $\text{DGA}_{k/k}$ and B in $\text{DGCA}_{k/k}$, there is a canonical isomorphism*

$$\text{Hom}_{\text{Ho}(\text{DGCA}_{k/k})}(\mathbb{L}(A)_n, B) \cong \text{Hom}_{\text{Ho}(\text{DGA}_{k/k})}(A, \mathcal{M}'_n(B)).$$

Definition. Given $A \in \text{Alg}_{k/k}$, we define $\text{DRep}_n(A) := \mathbb{L}(QA)_n$, where $QA \xrightarrow{\sim} A$ is a cofibrant resolution of A in $\text{DGA}_{k/k}$. The homology of $\text{DRep}_n(A)$ is an augmented (graded) commutative algebra, which is independent of the choice of resolution (by Theorem 2.6). We set

$$\mathbf{H}_\bullet(A, n) := \mathbf{H}_\bullet[\text{DRep}_n(A)] \tag{2.11}$$

and call (2.11) the n -dimensional representation homology of A .

By [BKR, Theorem 2.5], for any $A \in \text{Alg}_k$, there is a natural isomorphism of algebras

$$\mathbf{H}_0(A, n) \cong A_n.$$

Hence $\text{DRep}_n(A)$ may indeed be viewed as a ‘higher’ derived version of the representation functor (2.7).

2.4.2. GL-invariants. The group $\text{GL}_n(k)$ acts naturally on A_n by DG algebra automorphisms. Precisely, each $g \in \text{GL}_n(k)$ defines a unique automorphism of A_n corresponding under (2.9) to the composite map

$$A \xrightarrow{\pi_n} \mathcal{M}_n(A_n) \xrightarrow{\text{Ad}(g)} \mathcal{M}_n(A_n). \tag{2.12}$$

This action is natural in A and thus defines the functor

$$(-)_n^{\text{GL}} : \text{DGA}_{k/k} \rightarrow \text{DGCA}_{k/k}, \quad A \mapsto A_n^{\text{GL}}, \tag{2.13}$$

which is a subfunctor of the representation functor $(-)_n$. On the other hand, there is a natural action of $\text{GL}_n(k)$ on the n -th representation homology of A , so we can form the invariant subalgebra $\mathbf{H}_\bullet(A, n)^{\text{GL}_n}$. The next theorem, which is a consequence of [BKR, Theorem 2.6], shows that these two constructions agree.

Theorem 2.7. (a) *The functor $(-)_n^{\text{GL}}$ has a total left derived functor*

$$\mathbb{L}[-]_n^{\text{GL}} : \text{Ho}(\text{DGA}_{k/k}) \rightarrow \text{Ho}(\text{DGCA}_{k/k}), \quad A \mapsto (QA)_n^{\text{GL}}.$$

(b) *For any $A \in \text{DGA}_{k/k}$, there is a natural isomorphism of graded algebras*

$$H_\bullet[\mathbb{L}(A)_n^{\text{GL}}] \cong H_\bullet(A, n)^{\text{GL}_n}.$$

Abusing notation we will often write $\text{DRep}_n(A)^{\text{GL}}$ instead of $\mathbb{L}(A)_n^{\text{GL}}$ for any DG algebra A .

2.4.3. *Trace maps.* Recall that, for an augmented DG algebra $R \in \text{DGA}_{k/k}$, we denote by $\bar{R} \subset R$ the kernel of the augmentation map of R . Now, we set

$$R_{\natural} := \bar{R}/[\bar{R}, \bar{R}] \cong R/(k + [R, R]),$$

where $[\bar{R}, \bar{R}]$ is the subcomplex of \bar{R} spanned by the commutators in \bar{R} . This defines the functor

$$(-)_{\natural} : \text{DGA}_{k/k} \rightarrow \text{Com}_k, \quad R \mapsto R_{\natural}, \tag{2.14}$$

which we call the (reduced) *cyclic functor*.

The next theorem, which is a well-known result due to Feigin and Tsygan, justifies our terminology.

Theorem 2.8 ([FT1]). (a) *The functor (2.14) has a total left derived functor*

$$\mathbb{L}(-)_{\natural} : \text{Ho}(\text{DGA}_{k/k}) \rightarrow \text{Ho}(\text{Com}_k), \quad A \mapsto (QA)_{\natural},$$

where QA is a(ny) cofibrant resolution of A in $\text{DGA}_{k/k}$.

(b) *For $A \in \text{Alg}_{k/k}$, there is a natural isomorphism of graded vector spaces*

$$H_\bullet[\mathbb{L}(A)_{\natural}] \cong \overline{\text{HC}}_\bullet(A),$$

where $\overline{\text{HC}}_\bullet(A)$ denotes the (reduced) cyclic homology of A .

For a conceptual proof of Theorem 2.8 we refer to [BKR, Section 3].

Now, fix $n \geq 1$ and, for $R \in \text{DGA}_k$, consider the composite map

$$R \xrightarrow{\pi_n} \mathcal{M}_n(R_n) \xrightarrow{\text{Tr}} R_n$$

where π_n is the universal representation of R and Tr is the usual matrix trace. This map factors through R_{\natural} , and its image lies in R_n^{GL} . Hence, we get a morphism of complexes

$$\text{Tr}_n(R)_\bullet : R/[R, R] \rightarrow R_n^{\text{GL}}, \tag{2.15}$$

which extends by multiplicativity to a map of graded commutative algebras

$$\mathbf{Sym} \text{Tr}_n(R)_\bullet : \mathbf{Sym}(R/[R, R]) \rightarrow R_n^{\text{GL}}. \tag{2.16}$$

If $R \in \text{DGA}_{k/k}$ is augmented, the natural inclusion $\bar{R} \hookrightarrow R$ induces a morphism of complexes $R_{\natural} \rightarrow R/[R, R]$. Composed with (2.15), this defines a morphism of functors that extends to a morphism of the derived functors from $\text{Ho}(\text{DGA}_{k/k})$ to $\text{Ho}(\text{Com}k)$:

$$\mathbb{L} \text{Tr}_n : \mathbb{L}(-)_{\natural} \rightarrow \mathbb{L}(-)_n^{\text{GL}}. \tag{2.17}$$

Now, for an ordinary k -algebra $A \in \text{Alg}_{k/k}$, applying (2.17) to a cofibrant resolution $R = QA$ of A in $\text{DGA}_{k/k}$, taking homology and using the identification of Theorem 2.8(b), we get natural maps

$$\text{Tr}_n(A)_{\bullet} : \overline{\text{HC}}_{\bullet}(A) \rightarrow \mathbf{H}_{\bullet}(A, n)^{\text{GL}_n}, \quad \forall n \geq 0. \tag{2.18}$$

In degree zero, $\text{Tr}_n(A)_0$ is induced by the obvious linear map $A \rightarrow k[\text{Rep}_n(A)]^{\text{GL}_n}$ defined by taking characters of representations. Thus, the higher components of (2.18) may be thought of as derived (or higher) characters of n -dimensional representations of A . For each $n \geq 1$, these characters assemble to a single homomorphism of graded commutative algebras which we denote

$$\mathbf{Sym} \text{Tr}_n(A)_{\bullet} : \mathbf{Sym}[\overline{\text{HC}}_{\bullet}(A)] \rightarrow \mathbf{H}_{\bullet}(A, n)^{\text{GL}}.$$

An explicit formula evaluating $\text{Tr}_n(A)_{\bullet}$ on cyclic chains is given in [BKR, Section 4.3].

3. Representation homology vs Lie (co)homology

The main result of this section (Theorem 3.2) identifies the representation homology of a DG algebra A with homology of the Lie coalgebra $\mathfrak{gl}_n^*(C)$ defined over a Koszul dual DG coalgebra C . This crucial observation is the starting point for the present paper.

3.1. Chevalley–Eilenberg complexes

First, we recall the definition of the classical Chevalley–Eilenberg complex (cf. [Q2, Appendix B]). If \mathfrak{g} is a DG Lie algebra, the *Chevalley–Eilenberg complex* of \mathfrak{g} with trivial coefficients is the (coaugmented, conilpotent) cocommutative DG coalgebra

$$\mathcal{C}(\mathfrak{g}; k) := (\mathbf{Sym}^c(\mathfrak{g}[1]), d_1 + d_2),$$

where d_1 is induced by the differential on \mathfrak{g} , and d_2 is the coderivation whose corestriction to $\mathfrak{g}[1]$ is given by the composite map

$$\mathbf{Sym}^2(\mathfrak{g}[1]) \cong k[1] \otimes k[1] \otimes \wedge^2 \mathfrak{g} \xrightarrow{\mu_1 \otimes [-, -]} k[1] \otimes \mathfrak{g} \cong \mathfrak{g}[1].$$

Here, $\mu_1 : k[1] \otimes k[1] \rightarrow k[1]$ is the natural map of degree -1 and $[-, -]$ is the Lie bracket on \mathfrak{g} . If $\mathfrak{h} \subset \mathfrak{g}$ is a DG Lie subalgebra, the relative Chevalley–Eilenberg complex $\mathcal{C}(\mathfrak{g}, \mathfrak{h}; k)$ is the DG coalgebra $(\mathbf{Sym}^c[(\mathfrak{g}/\mathfrak{h})[1]], d_1 + d_2)_{\mathfrak{h}}$, where $(-)_{\mathfrak{h}}$ denotes the subcomplex of \mathfrak{h} -coinvariants in $\mathbf{Sym}^c[(\mathfrak{g}/\mathfrak{h})[1]]$. We denote the homology of the complex $\mathcal{C}(\mathfrak{g}; k)$ (resp., $\mathcal{C}(\mathfrak{g}, \mathfrak{h}; k)$) by $\mathbf{H}_{\bullet}(\mathfrak{g}; k)$ (resp., $\mathbf{H}_{\bullet}(\mathfrak{g}, \mathfrak{h}; k)$).

Now, dually, if \mathfrak{G} is a DG Lie coalgebra with Lie cobracket $]-[: \mathfrak{G} \rightarrow \wedge^2 \mathfrak{G}$, the Chevalley–Eilenberg complex of \mathfrak{G} is defined to be the (augmented) commutative DG algebra

$$\mathcal{C}^c(\mathfrak{G}; k) := (\mathbf{Sym}(\mathfrak{G}[-1]), d_1 + d_2),$$

where d_1 is induced by the differential on \mathfrak{G} , and $d_2|_{\mathfrak{G}[-1]}$ is given by the composite map

$$\mathfrak{G}[-1] \cong k[-1] \otimes \mathfrak{G} \xrightarrow{\Delta_{-1} \otimes]-[:} k[-1] \otimes k[-1] \otimes \wedge^2 \mathfrak{G} \cong \mathbf{Sym}^2(\mathfrak{G}[-1]). \tag{3.1}$$

Here, $\Delta_{-1} : k[-1] \rightarrow k[-1] \otimes k[-1]$ takes $1_{k[-1]}$ to $-1_{k[-1]} \otimes 1_{k[-1]}$. When one has a surjection $\mathfrak{G} \twoheadrightarrow \mathfrak{H}$ of DG Lie coalgebras, the commutative DG algebra $(\mathbf{Sym}(\mathbf{Ker}(\mathfrak{G} \rightarrow \mathfrak{H})[-1]), d_1 + d_2)$ comes equipped with a coaction of \mathfrak{H} . The Chevalley–Eilenberg complex $\mathcal{C}^c(\mathfrak{G}, \mathfrak{H}; k)$ of the pair $(\mathfrak{G}, \mathfrak{H})$ is the DG subalgebra of \mathfrak{H} -invariants of $(\mathbf{Sym}(\mathbf{Ker}(\mathfrak{G} \rightarrow \mathfrak{H})[-1]), d_1 + d_2)$. Note that if \mathfrak{h} is a finite-dimensional DG Lie algebra and $\mathfrak{H} = \mathfrak{h}^*$,

$$\mathcal{C}^c(\mathfrak{G}, \mathfrak{H}; k) \cong (\mathbf{Sym}(\mathbf{Ker}(\mathfrak{G} \rightarrow \mathfrak{H})[-1]), d_1 + d_2)^{\mathfrak{h}}.$$

We denote the homology of the complex $\mathcal{C}^c(\mathfrak{G}; k)$ (resp., $\mathcal{C}^c(\mathfrak{G}, \mathfrak{H}; k)$) by $H_{\bullet}(\mathfrak{G}; k)$ (resp., $H_{\bullet}(\mathfrak{G}, \mathfrak{H}; k)$). Note that if \mathfrak{G} is concentrated in homological degree 0, then

$$H_{-1}(\mathfrak{G}; k) \cong \mathbf{Ker}]-[: \mathfrak{G} \rightarrow \wedge^2 \mathfrak{G}.$$

This is dual to the basic fact that $H_1(\mathfrak{g}; k) \cong \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ for a Lie algebra \mathfrak{g} concentrated in degree 0.

We note that the Chevalley–Eilenberg complex $\mathcal{C}(\mathfrak{g}; k)$ of a DG Lie algebra \mathfrak{g} is the analogue of the bar construction of a DG algebra, while the Chevalley–Eilenberg complex $\mathcal{C}^c(\mathfrak{G}; k)$ of a DG Lie coalgebra is the analog of the cobar construction of a DG coalgebra (see Section 6.2). We write $\mathcal{C}(\mathfrak{g}; k) = \mathbf{B}_{\mathbf{Lie}}(\mathfrak{g})$ and $\mathcal{C}^c(\mathfrak{G}; k) = \mathbf{\Omega}_{\mathbf{Lie}}(\mathfrak{G})$ when we want to emphasize these facts.

3.2. Derived representation schemes and homology of Lie coalgebras

For a DG coalgebra M and an augmented DG coalgebra $C \in \mathbf{DGC}_{k/k}$, we define an augmented tensor product $\bar{\otimes}$ by

$$M \bar{\otimes} C := k \oplus (M \otimes \bar{C}),$$

so that $\overline{M \bar{\otimes} C} = M \otimes \bar{C}$. The next proposition relates the (noncommutative) representation functor $\sqrt[n]{-}$ defined in (2.5) to the classical cobar construction introduced in Section 2.2.2.

Proposition 3.1. *There is a natural isomorphism of functors from $\mathbf{DGC}_{k/k}$ to $\mathbf{DGA}_{k/k}$:*

$$\sqrt[n]{-} \circ \mathbf{\Omega}(-) \cong \mathbf{\Omega}(\mathcal{M}_n^*(k) \bar{\otimes} -). \tag{3.2}$$

In particular, for any $C \in \mathbf{DGC}_{k/k}$, there is a natural isomorphism of DG algebras

$$\sqrt[n]{\overline{\mathbf{\Omega}(C)}} \cong \mathbf{\Omega}(\mathcal{M}_n^*(k) \bar{\otimes} C).$$

Proof. Indeed, for any $B \in \text{DGA}_{k/k}$, we have

$$\begin{aligned} \text{Hom}_{\text{DGA}_{k/k}}(\Omega(\mathcal{M}_n(k)^* \bar{\otimes} C), B) &\cong \text{Tw}(\mathcal{M}_n^*(k) \bar{\otimes} C, B) \\ &\stackrel{\text{def}}{=} \mathbf{MC}(\mathbf{Hom}(\mathcal{M}_n^*(k) \otimes \bar{C}, \bar{B})) \cong \mathbf{MC}(\mathbf{Hom}(\bar{C}, \mathbf{Hom}(\mathcal{M}_n^*(k), \bar{B})) \\ &= \text{Tw}(C, \mathbf{Hom}(\mathcal{M}_n^*(k), \bar{B})') = \text{Tw}(C, \mathcal{M}'_n(B)) \\ &\cong \text{Hom}_{\text{DGA}_{k/k}}(\Omega(C), \mathcal{M}'_n(B)) \cong \text{Hom}_{\text{DGA}_{k/k}}(\sqrt[n]{\Omega(C)}, B), \end{aligned}$$

where $\mathcal{M}'_n(B)$ is the unitalization of the DG algebra $\mathcal{M}_n(\bar{B})$ defined in (2.10). The result of Proposition 3.1 follows now from the Yoneda Lemma. \square

Remark. Proposition 3.1 implies an isomorphism of functors on $\text{Ho}(\text{DGA}_{k/k})$:

$$\mathbb{L}\sqrt[n]{-} \cong \Omega \circ \mathcal{M}_n^*(-) \circ \bar{B},$$

where \bar{B} is the reduced bar construction.

The following theorem makes precise the statement made in the Introduction that representation homology is Koszul dual to Lie algebra homology. This is one of the key results of the present paper. While it is an easy corollary of Proposition 3.1, we state it as a theorem in order to highlight its importance.

Theorem 3.2. *Let $A \in \text{DGA}_{k/k}$ and let $C \in \text{DGC}_{k/k}$ be a Koszul dual coalgebra of A . Then, for any $n \geq 1$, there are isomorphisms in $\text{Ho}(\text{DGCA}_{k/k})$:*

$$\text{DRep}_n(A) \cong \mathcal{C}^c(\mathfrak{gl}_n^*(\bar{C}); k), \quad \text{DRep}_n(A)^{\text{GL}} \cong \mathcal{C}^c(\mathfrak{gl}_n^*(C), \mathfrak{gl}_n^*(k); k). \quad (3.3)$$

Consequently,

$$\mathbf{H}_\bullet(A, n) \cong \mathbf{H}_\bullet(\mathfrak{gl}_n^*(\bar{C}); k), \quad \mathbf{H}_\bullet(A, n)^{\text{GL}} \cong \mathbf{H}_\bullet(\mathfrak{gl}_n^*(C), \mathfrak{gl}_n^*(k); k). \quad (3.4)$$

Proof. Note that $\text{DRep}_n(A) \cong \Omega(C)_n = [\sqrt[n]{\Omega(C)}]_{\text{ab}}$. By Proposition 3.1,

$$[\sqrt[n]{\Omega(C)}]_{\text{ab}} \cong \Omega(\mathcal{M}_n^*(k) \bar{\otimes} C)_{\text{ab}}.$$

By Theorem 6.2 (in particular, see (6.6)),

$$\Omega(\mathcal{M}_n(k) \bar{\otimes} C)_{\text{ab}} \cong \mathcal{C}^c(\mathcal{L}ie^c(\mathcal{M}_n^*(k) \bar{\otimes} C); k) = \mathcal{C}^c(\mathfrak{gl}_n^*(\bar{C}); k).$$

This proves that $\text{DRep}_n(A) \cong \mathcal{C}^c(\mathfrak{gl}_n^*(\bar{C}); k)$. Now, the $\text{GL}_n(k)$ -invariant subalgebra of $\mathcal{C}^c(\mathfrak{gl}_n^*(\bar{C}); k)$ is indeed $\mathcal{C}^c(\mathfrak{gl}_n^*(C), \mathfrak{gl}_n^*(k); k)$. This completes the proof. \square

Corollary 3.3. *Let A, C be as in Theorem 3.2. Assume, in addition, that C is finite-dimensional in each degree, and let $E := C^*$ be the (linear) dual DG k -algebra. Then*

$$\mathbf{H}_\bullet(A, n) \cong \mathbf{H}^{-\bullet}(\mathfrak{gl}_n(\bar{E}); k), \quad \mathbf{H}_\bullet(A, n)^{\text{GL}} \cong \mathbf{H}^{-\bullet}(\mathfrak{gl}_n(E), \mathfrak{gl}_n(k); k), \quad (3.5)$$

where $\mathbf{H}^{-\bullet}$ denotes (continuous)⁵ Lie algebra cohomology with negative grading.

⁵ See Theorem 6.3 and the subsequent remarks for a precise definition.

Proof. If C is finite-dimensional in each degree and $E := C^*$, then $H_\bullet(\mathfrak{gl}_n^*(\bar{C}); k) \cong H^{-\bullet}(\mathfrak{gl}_n(E); k)$ by (6.8). Hence, $H_\bullet(\mathfrak{gl}_n^*(C), \mathfrak{gl}_n^*(k); k) \cong H^{-\bullet}(\mathfrak{gl}_n(E), \mathfrak{gl}_n(k); k)$. The result is now immediate from Theorem 3.2. \square

Remark. Theorem 3.2 gives a natural interpretation to the Chevalley–Eilenberg Lie homology of the matrix Lie coalgebra $\mathfrak{gl}_n^*(\bar{C})$. Note that, in the first isomorphisms in (3.3) and (3.4), \mathfrak{gl}_n^* is defined over $\bar{C} = \text{Coker}(\varepsilon_C)$, not C itself. It is natural to ask whether the complex $\mathcal{C}^c(\mathfrak{gl}_n^*(C); k)$ for an arbitrary (but not necessarily augmented) coalgebra C can be identified with the DRep_n of some DG algebra A . The answer is ‘yes’: the corresponding A is given by the *extended cobar construction* $\Omega^{\text{ext}}(C)$ introduced in [AJ]. Precisely, for any counital coalgebra C , Anel and Joyal (see [AJ, Sect. 5.3]) define $\Omega^{\text{ext}}(C)$ as the Sweedler product⁶ $C \triangleright \text{MC}$ of C with the Maurer–Cartan algebra MC , which is the free DG algebra generated by one element u of degree -1 with differential $du = -u^2$. The argument of Proposition 3.1 combined with results of [AJ] gives a natural isomorphism of functors $\text{DGC}_k \rightarrow \text{DGA}_k$:

$$\sqrt[n]{-} \circ \Omega^{\text{ext}}(-) \cong \Omega^{\text{ext}}(\mathcal{M}_n^*(k) \otimes -),$$

and the result of Theorem 3.2 can thus be extended to arbitrary DG coalgebras as

$$\mathcal{C}^c(\mathfrak{gl}_n^*(C); k) \cong \text{DRep}_n[\Omega^{\text{ext}}(C)].$$

3.2.1. Representation homology of bimodules. In [BKR, Section 5], we defined representation homology of a DG algebra A with coefficients in an arbitrary DG bimodule over A . Proposition 3.4 below gives an interpretation of this construction in terms of homology of Lie coalgebras.

Let $\text{Bimod}(A)$ denote the category of DG bimodules over A . For a fixed $n \geq 1$, the universal representation $\pi_n : A \rightarrow \mathcal{M}_n(A_n)$ makes $\mathcal{M}_n(A_n)$ a DG A -bimodule, which also has a natural DG A_n -module structure compatible with the action of A . This allows one to define the functor

$$(-)_n^{\text{ab}} : \text{Bimod}(A) \rightarrow \text{Mod}(A_n), \quad M \mapsto M \otimes_{A^e} \mathcal{M}_n(A_n),$$

which we call the *van den Bergh functor* (as it first appeared in [VdB]). In [BFR, Section 5], it is shown that $(-)_n^{\text{ab}}$ is the abelianization of the nonadditive representation functor (2.7) in the sense of model categories (cf. [Q1, Section II.5]), whence its notation.

The derived functor of the van den Bergh functor is computed by the formula (see [BKR, Theorem 5.1])

$$\mathbb{L}(M)_n^{\text{ab}} = [F(R, M)]_n^{\text{ab}},$$

where $R \xrightarrow{\sim} A$ is a cofibrant resolution of A in DGA_k and $F(R, M)$ is a semifree resolution of M in $\text{Bimod}(R)$. This is independent of the choice of resolutions and allows one to define the n -th representation homology of a DG bimodule $M \in \text{Bimod}(A)$ by

$$H_\bullet(M, n) := H_\bullet[F(R, M)]_n^{\text{ab}}.$$

Now, assume that $A \in \text{DGA}_{k/k}$ and let $C \in \text{DGC}_{k/k}$ be a Koszul dual coalgebra to A .

⁶ See Appendix, (A.3), for the definition of the Sweedler product.

Proposition 3.4. *Let $M \in \text{Bimod}(A)$, and let N be a DG C -bicomodule such that there is a quasi-isomorphism $\Omega(C) \otimes_{\tau_C} N \otimes_{\tau_C} \Omega(C) \xrightarrow{\sim} M$ in $\text{Bimod}[\Omega(C)]$, where $\tau_C : C \rightarrow \Omega(C)$ is the universal twisting cochain associated to C . Then there is an isomorphism*

$$H_\bullet(M, n) \cong H_\bullet(\mathfrak{gl}_n^*(\bar{C}); \mathfrak{gl}_n^*(N)),$$

where $\mathfrak{gl}_n^*(N)$ denotes $\mathcal{M}_n^*(N)$ viewed as a DG Lie comodule over $\mathfrak{gl}_n^*(\bar{C})$ via the coadjoint coaction.

Proof. Indeed, $\Omega(C) \xrightarrow{\sim} A$ is a cofibrant resolution of A in $\text{DGA}_{k/k}$ and $\Omega(C) \otimes_{\tau_C} N \otimes_{\tau_C} \Omega(C) \xrightarrow{\sim} M$ is a semifree resolution M in $\text{Bimod}[\Omega(C)]$. Hence,

$$\mathbb{L}M_n^{\text{ab}} \cong [\Omega(C) \otimes_{\tau_C} N \otimes_{\tau_C} \Omega(C)]_n^{\text{ab}}.$$

Since $\Omega(C) \otimes_{\tau_C} N \otimes_{\tau_C} \Omega(C) \cong N \otimes \Omega(C)^e$ as graded $\Omega(C)$ -bimodules, $[\Omega(C) \otimes_{\tau_C} N \otimes_{\tau_C} \Omega(C)]_n^{\text{ab}}$ is generated as a graded $\Omega(C)_n$ -module by $\mathcal{M}_n^*(N)$. By Theorem 3.2, $\Omega(C)_n \cong \mathcal{C}^c(\mathfrak{gl}_n^*(\bar{C}); k)$. A direct computation then verifies that the differential on elements of $\mathcal{M}_n^*(N)$ in $[\Omega(C) \otimes_{\tau_C} N \otimes_{\tau_C} \Omega(C)]_n^{\text{ab}}$ is the sum of a term induced by the coadjoint coaction of $\mathfrak{gl}_n^*(\bar{C})$ on $\mathcal{M}_n^*(N)$ and another term induced by the differential of N . This proves that

$$[\Omega(C) \otimes_{\tau_C} N \otimes_{\tau_C} \Omega(C)]_n^{\text{ab}} \cong \mathcal{C}^c(\mathfrak{gl}_n^*(\bar{C}); \mathfrak{gl}_n^*(N)).$$

The desired result is now immediate. □

3.3. The Tsygan–Loday–Quillen theorem for coalgebras

For a coalgebra $C \in \text{DGC}_{k/k}$, let $\overline{\text{CC}}(C)$ denote the complex of reduced cyclic chains on C (the definition of $\overline{\text{CC}}(C)$ is formally dual to the definition of the reduced cyclic complex $\overline{\text{CC}}(A)$ of an augmented DG algebra A —see [L, 2.1.4 and 2.2.13]). For each $n \geq 0$, there is a generalized cotrace map (cf. [L, 2.2.10])

$$\text{coTr}_n(C) : \overline{\text{CC}}(C) \rightarrow \overline{\text{CC}}[\mathcal{M}_n^*(k) \bar{\otimes} C],$$

where $\mathcal{M}_n^*(k) := \mathcal{M}_n(k)^*$ is the matrix coalgebra dual to $\mathcal{M}_n(k)$. Furthermore, dual to the construction of [L, 10.2.3], one has a natural morphism of complexes

$$\theta : \overline{\text{CC}}(\mathcal{M}_n^*(C))[-1] \rightarrow \mathcal{C}^c(\mathfrak{gl}_n^*(C), \mathfrak{gl}_n^*(k); k),$$

which is induced by the projection

$$(\alpha_0, \dots, \alpha_n) \mapsto \alpha_0 \wedge \dots \wedge \alpha_n.$$

Abusing notation, we denote the composite map $\theta \circ \text{coTr}_n(C)$ by $\text{coTr}_n(C)$. Note that for any n , there is a canonical epimorphism in $\text{DGCA}_{k/k}$:

$$\mu_{n+1,n} : \mathcal{C}^c(\mathfrak{gl}_{n+1}^*(C), \mathfrak{gl}_{n+1}^*(k); k) \twoheadrightarrow \mathcal{C}^c(\mathfrak{gl}_n^*(C), \mathfrak{gl}_n^*(k); k).$$

Therefore, we can pass to the projective limit

$$\mathcal{C}^c(\mathfrak{gl}_\infty^*(C), \mathfrak{gl}_\infty^*(k); k) := \varprojlim_n \mathcal{C}^c(\mathfrak{gl}_n^*(C), \mathfrak{gl}_n^*(k); k).$$

It is easy to check that $\mu_{n+1,n} \circ \text{coTr}_{n+1}(C) = \text{coTr}_n(C)$. Hence we have a morphism of complexes

$$\text{coTr}_\infty(C) : \overline{\text{CC}}(C)[-1] \rightarrow \mathcal{C}^c(\mathfrak{gl}_\infty^*(C), \mathfrak{gl}_\infty^*(k); k).$$

This extends by multiplicativity to a morphism of DG algebras

$$\mathbf{Sym}(\text{coTr}_\infty) : \mathbf{Sym}_k(\overline{\text{CC}}(C)[-1]) \rightarrow \mathcal{C}^c(\mathfrak{gl}_\infty^*(C), \mathfrak{gl}_\infty^*(k); k). \tag{3.6}$$

Now, by a result of Quillen [Q3], there is a natural isomorphism

$$\Omega(C)_{\natural} \cong \overline{\text{CC}}(C)[-1],$$

where $(-)_\natural$ is the cyclic functor defined by (2.14). On the other hand, by Theorem 3.2, we have an isomorphism $\Omega(C)_n^{\text{GL}} \cong \mathcal{C}^c(\mathfrak{gl}_n^*(C), \mathfrak{gl}_n^*(k); k)$. The following lemma is verified by an easy computation which we leave to the reader.

Lemma 3.5. *For any $n \geq 0$, the following diagram commutes:*

$$\begin{array}{ccc} \Omega(C)_{\natural} & \xrightarrow[\text{Quillen}]{\cong} & \overline{\text{CC}}(C)[-1] \\ \text{Tr}_n \downarrow & & \downarrow \text{coTr}_n \\ \Omega(C)_n^{\text{GL}} & \xrightarrow[\text{Theorem 3.2}]{\cong} & \mathcal{C}^c(\mathfrak{gl}_n^*(C), \mathfrak{gl}_n^*(k); k) \end{array}$$

As a consequence, we get an isomorphism of (topological) DG algebras

$$\mathcal{C}^c(\mathfrak{gl}_\infty^*(C), \mathfrak{gl}_\infty^*(k); k) \cong [\Omega(C)]_\infty^{\text{GL}}. \tag{3.7}$$

Proposition 3.6. *Under (3.7), the morphism (3.6) is identified with $\mathbf{Sym}(\text{Tr}_\infty) : \mathbf{Sym}(C)_{\natural} \rightarrow [\Omega(C)]_\infty^{\text{GL}}$.*

Proposition 3.6 combined with [BR1, Lemma 4.1] implies that the image of (3.6) is a dense DG subalgebra of $\mathcal{C}^c(\mathfrak{gl}_\infty^*(C), \mathfrak{gl}_\infty^*(k); k)$, which, following [BR1], we denote by $\mathcal{C}^{\text{Tr}}(\mathfrak{gl}_\infty^*(C), \mathfrak{gl}_\infty^*(k); k)$. The main theorem of [BR1] (precisely, [BR1, Theorem 4.4]) then implies the following result, which is a (relative variant) of the Tsygan–Loday–Quillen theorem for coalgebras.⁷

Theorem 3.7. *For any $C \in \text{DGC}_{k/k}$, there are natural isomorphisms of DG algebras*

$$\mathbf{Sym}_k(\overline{\text{CC}}(C)[-1]) \cong \mathbf{Sym}_k[\Omega(C)_{\natural}] \cong \mathcal{C}^{\text{Tr}}(\mathfrak{gl}_\infty^*(C), \mathfrak{gl}_\infty^*(k); k).$$

Remarks. 1. Theorem 3.7 can be also proven by dualizing the invariant-theoretic arguments in [L, Ch. 10]. With Theorem 3.2, one can then presumably give a different proof of [BR1, Theorem 4.4]. This, however, does not seem to answer the questions of [BR1] regarding Koszul duality (see [BR1, Question 5.2.1]).

⁷ A nonrelative version of this result has been proven in [Ka] using a direct invariant-theoretic approach.

2. If the DG coalgebra C is bigraded (see Section 3.4 below), then the DG algebra $C^{\text{Tr}}(\mathfrak{gl}_\infty^*(C), \mathfrak{gl}_\infty^*(k); k)$ coincides with $\mathcal{C}^c(\mathfrak{gl}_\infty^*(C), \mathfrak{gl}_\infty^*(k); k)$. The bigraded dual of Theorem 3.7 yields (see (6.9)) the Tsygan–Loday–Quillen theorem for the bigraded DG algebra $A := C^*$. Hence, [BR1, Proposition 7.5] implies a version of the traditional Tsygan–Loday–Quillen theorem.

3.4. Bigraded algebras

Recall from [BR1, Section 7] that a *bigraded* DGA is a DG algebra equipped with homological as well as weight (polynomial) grading that is finite-dimensional in each weight degree.⁸ We remark that unlike in [BR1], we allow our DGA to have nonzero components in negative homological degree as well. The differential is required to obey the graded Leibniz rule with respect to the homological grading. Additionally, we require that a bigraded DGA be concentrated in nonnegative weight degrees and that the component in weight degree 0 be isomorphic to k . A bigraded DGA is therefore, augmented as well. Following [BR1, Section 7], one can verify that the category BiDGA_k of bigraded DGAs is a model category whose weak equivalences are the quasi-isomorphisms and whose fibrations are the surjective morphisms. The category BiDGCA_k of bigraded commutative DGAs has an analogous model structure,

Similarly, a coalgebra $C \in \text{DGC}_{k/k}$ will be called *bigraded* if \bar{C} has a homological as well as weight grading, is concentrated in positive weight degree and is finite-dimensional in each weight degree. Note that in this case, $\Omega(C)_{\text{ab}}$ is a bigraded commutative DGA. In the bigraded setting, we have the following result, the first part of which is Theorem 3.2 in the bigraded setting.

Theorem 3.8. *Let $A \in \text{BiDGA}_k$, and let C be a Koszul dual coalgebra to A . Assume that C is bigraded, and the quasi-isomorphism $\Omega(C) \xrightarrow{\sim} A$ is in BiDGA_k . Then:*

(a) *In $\text{Ho}(\text{BiDGCA}_{k/k})$, there are isomorphisms*

$$\text{DRep}_n(A) \cong \mathcal{C}^c(\mathfrak{gl}_n^*(\bar{C}); k), \quad \text{DRep}_n(A)^{\text{GL}} \cong \mathcal{C}^c(\mathfrak{gl}_n^*(C), \mathfrak{gl}_n^*(k); k).$$

(b) *If $E := C^*$ is bigraded dual of C , then*

$$H_\bullet(A, n) \cong H^{-\bullet}(\mathfrak{gl}_n(\bar{E}); k), \quad H_\bullet(A, n)^{\text{GL}} \cong H^{-\bullet}(\mathfrak{gl}_n(E), \mathfrak{gl}_n(k); k),$$

where $H^{-\bullet}$ denotes the classical Lie algebra cohomology⁹ equipped with negative (homological) grading.

Remark. For $A \in \text{BiDGA}_k$, at least one E as in Theorem 3.8(b) exists: namely, the bigraded dual of BA .

Proof of Theorem 3.8. The proof of Theorem 3.2 goes through word for word in the bigraded setting: this gives (a). Part (b) follows immediately from (a) and (6.8). \square

⁸ Bigraded DGAs are also referred to as weight-graded DGAs or WDGAs. See [LV, Sec. 1.5.11].

⁹ See remarks after Theorem 6.3.

We conclude this section with one application of Theorem 3.8. Let $A = A(V, R)$ be a quadratic associative algebra defined by a (finite-dimensional) quadratic data $(V, R) = \{R \subseteq V \otimes V\}$ (see [LV, Chap. 3]). Let $A^\dagger := A(s^{-1}V^*, s^{-2}R^\perp)$ denote its *graded*¹⁰ Koszul dual algebra. Recall that A is a Koszul algebra if (and only if) the canonical map $(A^\dagger)^* \rightarrow BA$ is a quasi-isomorphism (cf. [LV, Theorem 3.4.6]). Theorem 3.8 then implies

Corollary 3.9. *If A is a Koszul quadratic algebra then there are natural isomorphisms*

$$H_\bullet(A, n) \cong H^{-\bullet}(\mathfrak{gl}_n(\bar{A}^\dagger); k), \quad H_\bullet(A, n)^{\text{GL}} \cong H^{-\bullet}(\mathfrak{gl}_n(A^\dagger), \mathfrak{gl}_n(k); k).$$

4. Derived Harish-Chandra homomorphism

The classical Harish-Chandra homomorphism is defined by restricting the characters of representations (viewed as functions on a representation variety) to the subvariety of diagonal representations. In this section, we construct a derived version of this homomorphism.

4.1. Basic construction

Recall that, for a fixed DG algebra $A \in \text{DGA}_k$ and an integer $n \geq 1$, the affine DG scheme of n -dimensional representations of A is defined by the functor $\text{Rep}_n(A)$ which is represented by the commutative DG algebra A_n (see Section 2.4.1). Now, we introduce the functor of *diagonal representations*

$$\text{Diag}_n(A) : \text{DGCA}_k \rightarrow \text{Sets}, \quad B \mapsto \text{Hom}_{\text{DGA}_k}(A, B^{\times n}), \tag{4.1}$$

where $B^{\times n}$ denotes the (direct) product of n copies of B in the category of commutative DG algebras. There are natural isomorphisms of sets

$$\begin{aligned} \text{Hom}_{\text{DGA}_k}(A, B^{\times n}) &\cong \text{Hom}_{\text{DGA}_k}(A, B)^{\times n} \cong \text{Hom}_{\text{DGCA}_k}(A_{\text{ab}}, B)^{\times n} \\ &\cong \text{Hom}_{\text{DGCA}_k}((A_{\text{ab}})^{\otimes n}, B), \end{aligned}$$

which show that (4.1) is (co)represented by the commutative DG algebra $(A_{\text{ab}})^{\otimes n}$, the n -th tensor power of the abelianization of A .

Next, observe that the functor $\text{Diag}_n(A)$ comes together with a natural transformation

$$\text{Diag}_n(A) \rightarrow \text{Rep}_n(A), \tag{4.2}$$

defined by the algebra maps $B^{\times n} \hookrightarrow \mathcal{M}_n(B)$ identifying $B^{\times n}$ with the subalgebra of diagonal matrices in $\mathcal{M}_n(B)$. By Yoneda’s Lemma, this natural transformation gives a homomorphism of commutative DG algebras

$$\Phi_n(A) : A_n \rightarrow (A_{\text{ab}})^{\otimes n}, \tag{4.3}$$

¹⁰ We warn the reader that our definition of A^\dagger differs from that of [LV] by a choice of grading: our A^\dagger equals A^{i*} in the notation of [LV].

which is obviously functorial in $A \in \text{DGA}_k$. In terms of the generators of Lemma 2.5, the homomorphism $\Phi_n(A)$ is given by

$$a_{ij} \mapsto \delta_{ij} a_i, \tag{4.4}$$

where $a_i = 1 \otimes \dots \otimes a \otimes \dots \otimes 1$ is the image in $(A_{\text{ab}})^{\otimes n}$ of an element $a \in A$ sitting in the i -th tensor factor.

Now, for a commutative DG algebra $B \in \text{DGCA}_k$, let $S^n(B)$ denote the n -th symmetric power of B , i.e. $S^n(B) := [B^{\otimes n}]^{S_n}$. By definition, $S^n(B)$ is a subalgebra of $B^{\otimes n}$; since k has characteristic 0, we may identify $S^n(B)$ with the image of the symmetrization map $\text{Sym}^n(B) \hookrightarrow B^{\otimes n}$.

Lemma 4.1. *The map (4.3) restricts to a homomorphism of commutative DG algebras*

$$\Phi_n(A) : A_n^{\text{GL}} \rightarrow S^n(A_{\text{ab}}). \tag{4.5}$$

Proof. Consider the (right) action of S_n on $\mathcal{M}_n(k)$ by permuting the columns of matrices, i.e., if $M = \|M_1, \dots, M_n\| \in \mathcal{M}_n(k)$ then $M^\sigma := \|M_{\sigma^{-1}(1)}, \dots, M_{\sigma^{-1}(n)}\|$ for $\sigma \in S_n$. Define $\iota : S_n \rightarrow \text{GL}_n(k)$ by $\iota(\sigma) = I_n^\sigma$, where I_n is the identity matrix in $\mathcal{M}_n(k)$. Then, for any $M \in \mathcal{M}_n(k)$, we have $M^\sigma = M\iota(\sigma)$. Hence ι is an injective group homomorphism. We claim that (4.3) is an S_n -equivariant map provided S_n acts on A_n by restricting the canonical GL_n -action via ι . To see this recall that the GL_n -action on A_n comes from the adjoint action on $\mathcal{M}_n(k)$ (see (2.12)). Hence it suffices to check that the following diagram commutes for all $\sigma \in S_n$:

$$\begin{array}{ccc} \mathcal{M}_n(A_n) & \xrightarrow{\text{Id} \otimes \Phi_n} & \mathcal{M}_n[(A_{\text{ab}})^{\otimes n}] \\ \text{Ad}_{\iota(\sigma)} \downarrow & & \downarrow \text{Id} \otimes \sigma \\ \mathcal{M}_n(A_n) & \xrightarrow{\text{Id} \otimes \Phi_n} & \mathcal{M}_n[(A_{\text{ab}})^{\otimes n}] \end{array} \tag{4.6}$$

In terms of generators of A_n , the left vertical map in (4.6) is given by

$$\text{Ad}_{\iota(\sigma)} \|a_{ij}\| = \iota(\sigma) \|a_{ij}\| \iota(\sigma)^{-1} = \iota(\sigma) \|a_{ij}\| \iota(\sigma^{-1}) = I_n^\sigma \|a_{ij}\| I_n^{\sigma^{-1}} = \|a_{\sigma(i)\sigma(j)}\|.$$

On the other hand, by (4.4), the horizontal maps are $\|a_{ij}\| \mapsto \text{diag}(a_1, \dots, a_n)$. The commutativity of (4.6) is therefore obvious. Since $\Phi_n(A)$ is an equivariant map, it takes invariants to invariants. \square

The functor $S^n(-)_{\text{ab}} : \text{DGA}_k \rightarrow \text{DGCA}_k$ is not a homotopy functor in the sense that it does not preserve weak equivalences and hence does not descend to homotopy categories. However, just as the representation functor (2.7), $S^n(-)_{\text{ab}}$ admits a left derived functor that gives a functorial approximation to the induced functor at the level of homotopy categories.

Proposition 4.2. (a) *The functor $S^n(-)_{\text{ab}}$ has a total left derived functor given by*

$$\mathbb{L}S^n(-)_{\text{ab}} : \text{Ho}(\text{DGA}_k) \rightarrow \text{Ho}(\text{DGCA}_k), \quad A \mapsto S^n[(QA)_{\text{ab}}],$$

where QA is a cofibrant resolution of A .

(b) *For any $A \in \text{DGA}_k$, there is a natural isomorphism of graded commutative algebras*

$$\mathbf{H}_\bullet[\mathbb{L}S^n(A)_{\text{ab}}] \cong S^n[\mathbf{H}_\bullet(A, 1)].$$

Proof. (a) The abelianization $(-)_{\text{ab}} : \text{DGA}_k \rightarrow \text{DGCA}_k$ is a left Quillen functor adjoint to the inclusion $\text{DGCA}_k \hookrightarrow \text{DGA}_k$. Hence it maps acyclic cofibrations in DGA_k to weak equivalences. On the other hand, the functor S^n commutes with taking homology (see, e.g., [Q2, Appendix B, Prop. 2.1]), hence it maps weak equivalences (= quasi-isomorphisms) in DGCA_k to weak equivalences. It follows that $S^n(-)_{\text{ab}} = S^n \circ (-)_{\text{ab}}$ maps acyclic cofibrations to weak equivalences. Hence $S^n(-)_{\text{ab}}$ has a total left derived functor by Brown’s Lemma (see, e.g., [BKR, Lemma A.2]).

(b) follows from the obvious isomorphism $(-)_{\text{ab}} \cong (-)_1$ and the fact that $S^n \circ H_\bullet \cong H_\bullet \circ S^n$. □

Notation. We will write the derived functor $\mathbb{L}S^n(-)_{\text{ab}}$ as $S^n\text{DRep}_1(-)$. This is justified by Proposition 4.2: indeed, for $A \in \text{DGA}_k$, we have $\text{DRep}_1(A) \cong \mathbb{L}(A)_{\text{ab}}$, so $S^n\text{DRep}_1(A)$ is to be computed by the same formula as $\mathbb{L}S^n(A)_{\text{ab}}$.

Now, by Lemma 4.1, we have a natural transformation of functors $\Phi_n : (-)_n^{\text{GL}} \rightarrow S^n(-)_{\text{ab}}$. By Theorem 2.7 and Proposition 4.2, both $(-)_n^{\text{GL}}$ and $S^n(-)_{\text{ab}}$ have left derived functors. Hence, Φ_n induces a (unique) natural transformation at the level of homotopy categories:

$$\mathbb{L}\Phi_n : \mathbb{L}(-)_n^{\text{GL}} \rightarrow \mathbb{L}S^n(-)_{\text{ab}}. \tag{4.7}$$

For a fixed $A \in \text{DGA}_k$, this gives a canonical morphism in $\text{Ho}(\text{DGCA}_k)$:

$$\Phi_n(A) : \text{DRep}_n(A)^{\text{GL}} \rightarrow S^n\text{DRep}_1(A), \tag{4.8}$$

which we call the *derived Harish-Chandra homomorphism*.¹¹

Next, we introduce natural maps combining (4.8) with the trace maps constructed in Section 2.4.3. To this end, we note that the functors $(-)_n^{\text{GL}}$, $S^n(-)_{\text{ab}}$ and the natural transformation Φ_n are well-defined on the category of augmented DG algebras (cf. Theorem 2.6). Hence we can regard (4.7) as a morphism of functors from $\text{Ho}(\text{DGA}_{k/k})$ to $\text{Ho}(\text{DGCA}_{k/k})$. Composing this morphism with the trace (2.17), we get a morphism of functors from $\text{Ho}(\text{DGA}_{k/k})$ to $\text{Ho}(\text{Com}_k)$:

$$\mathbb{L}\overline{\text{Tr}}_n : \mathbb{L}(-)_{\natural} \rightarrow \mathbb{L}S^n(-)_{\text{ab}}. \tag{4.9}$$

The next proposition shows that $\mathbb{L}\overline{\text{Tr}}_n$ is essentially determined by $\mathbb{L}\overline{\text{Tr}}_1$.

Proposition 4.3. *For each $n \geq 1$, the morphism $\mathbb{L}\overline{\text{Tr}}_n$ factors as*

$$\mathbb{L}\overline{\text{Tr}}_n \cong S^n \circ \mathbb{L}\overline{\text{Tr}}_1,$$

where S^n is a symmetrization morphism (see (4.10))

Proof. It suffices to check this on cofibrant objects. If $A \in \text{DGA}_{k/k}$ is cofibrant, then $\mathbb{L}\overline{\text{Tr}}_n(A) = \overline{\text{Tr}}_n(A)$, and by Lemma 2.5, the composite map $\overline{\text{Tr}}_n(A) : A_{\natural} \rightarrow A_n^{\text{GL}} \rightarrow S^n(A_{\text{ab}})$ is given by

$$\bar{a} \mapsto \sum_{i=1}^n a_{ii} \mapsto \sum_{i=1}^n a_i,$$

¹¹ This terminology will be justified in the next section where we reinterpret the construction of $\Phi_n(A)$ in Lie-algebraic terms.

where $a \in A, \bar{a} \in A_{\natural}$ and $a_i = 1 \otimes \cdots \otimes a \otimes \cdots \otimes 1 \in (A_{\text{ab}})^{\otimes n}$ (with $a \in A_{\text{ab}}$ in the i -th position). Hence $\overline{\text{Tr}}_n(A)$ factors through the canonical map $A_{\natural} \rightarrow A_{\text{ab}}$, which is nothing but $\overline{\text{Tr}}_1(A)$. The induced map is precisely the symmetrization:

$$S^n : A_{\text{ab}} \rightarrow S^n(A)_{\text{ab}}, \quad a \mapsto \sum_{i=1}^n a_i. \tag{4.10}$$

This finishes the proof of Proposition 4.3. □

For any $A \in \text{DGA}_{k/k}$, the morphism (4.9) induces on homology natural maps

$$\overline{\text{Tr}}_n(A)_{\bullet} : \overline{\text{HC}}_{\bullet}(A) \rightarrow [\text{H}_{\bullet}(A, 1)^{\otimes n}]^{S_n}. \tag{4.11}$$

We call (4.11) the *reduced* trace maps. By Proposition 4.3, it suffices to compute these maps for $n = 1$. We give a general formula for $\overline{\text{Tr}}_1(A)_{\bullet}$ in our subsequent paper (see [BF⁺, Theorem 4.1]).

4.2. Interpretation in terms of Lie (co)homology

We now reinterpret the derived Harish-Chandra homomorphism (4.8) in Koszul dual terms of Lie coalgebras. Let $\mathfrak{h}_n(k)$ denote the Cartan subalgebra of $\mathfrak{gl}_n(k)$, which is the subalgebra $\mathcal{D}_n(k)$ of diagonal matrices in $\mathcal{M}_n(k)$ viewed as a Lie algebra. Dually, one has a surjection of coalgebras $\mathcal{M}_n^*(k) \twoheadrightarrow \mathcal{D}_n^*(k)$ where $\mathcal{D}_n^*(k)$ denotes the k -linear dual of $\mathcal{D}_n(k)$. Hence, for any $C \in \text{DGC}_{k/k}$, one has a morphism $\mathcal{M}_n^*(C) \twoheadrightarrow \mathcal{D}_n^*(C)$ of DG coalgebras that induces the natural map $\mathcal{M}_n^*(k) \twoheadrightarrow \mathcal{D}_n^*(k)$. Viewing $\mathcal{M}_n^*(C)$ and $\mathcal{D}_n^*(C)$ as DG Lie coalgebras, one obtains a morphism $\mathfrak{gl}_n^*(C) \rightarrow \mathfrak{h}_n^*(C)$ of DG Lie coalgebras that induces the natural morphism $\mathfrak{gl}_n^*(k) \rightarrow \mathfrak{h}_n^*(k)$ of Lie coalgebras. As a result, by functoriality, we get a morphism of commutative DG algebras

$$\Phi_n(C) : \mathcal{C}^c(\mathfrak{gl}_n^*(C), \mathfrak{gl}_n^*(k); k) \rightarrow \mathcal{C}^c(\mathfrak{h}_n^*(C), \mathfrak{h}_n^*(k); k). \tag{4.12}$$

Now, let $A = \Omega(C)$ be the cobar construction of C , which is a cofibrant DG algebra in $\text{DGA}_{k/k}$. Since \mathfrak{h}_n is abelian and $\dim \mathfrak{h}_n = n$, we have canonical isomorphisms in $\text{DGCA}_{k/k}$:

$$\mathcal{C}^c(\mathfrak{h}_n^*(C), \mathfrak{h}_n^*(k); k) \cong [\mathcal{C}^c(\mathfrak{gl}_1^*(\bar{C})]^{\otimes n} \cong (A_{\text{ab}})^{\otimes n}. \tag{4.13}$$

On the other hand, by Theorem 3.2,

$$\mathcal{C}^c(\mathfrak{gl}_n^*(C), \mathfrak{gl}_n^*(k); k) \cong A_n^{\text{GL}}. \tag{4.14}$$

With these isomorphisms we can compare (4.12) with the Harish-Chandra homomorphism (4.5). The following proposition is a direct consequence of Lemma 4.1.

Proposition 4.4. *With the identifications (4.13) and (4.14), the map $\Phi_n(C)$ agrees with $\Phi_n(A)$. The image of $\Phi_n(C)$ is contained in $\mathcal{C}^c(\mathfrak{h}_n^*(C), \mathfrak{h}_n^*(k); k)^{S_n}$, where the action of S_n on the Chevalley–Eilenberg complex comes from the natural action on \mathfrak{h}_n .*

In Section 7, we will construct the derived Harish-Chandra homomorphism for an arbitrary reductive Lie algebra \mathfrak{g} , generalizing the above construction for \mathfrak{gl}_n . Proposition 4.4 then shows that the Harish-Chandra homomorphism for associative algebras (when C is cocommutative) is a special case of the one for Lie algebras.

4.3. Symmetric algebras

In this subsection, we take A to be the symmetric algebra $\text{Sym}(V)$, where V is a vector space of dimension $r \geq 1$ concentrated in homological degree 0. This is a quadratic Koszul algebra defined by the quadratic data (V, S) with $S \subset V \otimes V$ spanned by the vectors of the form $v \otimes u - u \otimes v$ (cf. [LV, Example 3.2.4.2]).

4.3.1. The minimal resolution. Recall that A has a canonical (minimal) semifree resolution $R = \Omega(A^i)$ given by the cobar construction of the Koszul dual coalgebra $A^i = C(sV, s^2S)$ (see [LV, Sect. 3.2.1]). The algebra R is the tensor algebra generated by the vector space $s^{-1} \wedge(sV)$, whose elements of degree $k - 1$ we denote by

$$\lambda(v_1, \dots, v_k) := s^{-1}(sv_1 \wedge \dots \wedge sv_k) \in s^{-1} \wedge^k(sV). \tag{4.15}$$

With this notation, the differential on R satisfies

$$d\lambda(v_1, v_2) = -[v_1, v_2], \tag{4.16}$$

$$d\lambda(v_1, v_2, v_3) = -[v_1, \lambda(v_2, v_3)] - [v_2, \lambda(v_3, v_1)] - [v_3, \lambda(v_1, v_2)]. \tag{4.17}$$

In general, we have

Lemma 4.5. *The differential d on the minimal resolution R of $A = \text{Sym}(V)$ is defined by*

$$d\lambda(v_1, \dots, v_n) = \sum_{\substack{p+q=n \\ 1 \leq p \leq q}} (-1)^p \sum_{\sigma \in \text{Sh}(p,q)} (-1)^\sigma [\lambda(v_{\sigma(1)}, \dots, v_{\sigma(p)}), \lambda(v_{\sigma(p+1)}, \dots, v_{\sigma(p+q)})],$$

where $\text{Sh}(p, q)$ denotes the set of (p, q) -shuffles, i.e. $\sigma \in S_{p+q}$ of the form $\sigma = (i_1, \dots, i_p; i_{p+1}, \dots, i_{p+q})$ with $i_1 < \dots < i_p$ and $i_{p+1} < \dots < i_{p+q}$.

Proof. The proof is by direct computation using the definition of the differential on the cobar construction. We leave the details of this computation to the interested reader. \square

It follows from Lemma 4.5 that R_{ab} is isomorphic to is the graded symmetric algebra of $s^{-1} \wedge(sV)$ equipped with zero differential. Explicitly (omitting the shifts), we can write

$$R_{\text{ab}} = \mathbf{Sym}(\wedge^1 V \oplus \wedge^2 V \oplus \dots \oplus \wedge^r V) = \text{Sym}(V) \otimes \mathbf{Sym}(\wedge^2 V \oplus \dots \oplus \wedge^r V) \tag{4.18}$$

with the understanding that the elements of $\wedge^k V$ have (homological) degree $k - 1$.

4.3.2. Main conjecture for \mathfrak{gl}_n . For $A = \text{Sym}(V)$, the homomorphism (4.8) is given by

$$\Phi_n : \text{DRep}_n(A)^{\text{GL}} \rightarrow \mathbf{Sym}[\mathfrak{h}_n^* \otimes (\wedge^1 V \oplus \dots \oplus \wedge^r V)]^{\mathcal{S}_n},$$

where $\text{DRep}_n(A) \cong \mathbf{Sym}[\mathcal{M}_n^*(k) \otimes (\wedge^1 V \oplus \dots \oplus \wedge^r V)]$ as a graded commutative algebra. Explicitly, Φ_n is the restriction to $\text{DRep}_n(A)^{\text{GL}}$ of the map of commutative DG-algebras taking each generator of the form $e^{ij} \otimes \alpha$ to $\delta_{ij} e^i \otimes \alpha$. Here, the e^{ij} 's are a basis of $\mathcal{M}_n^*(k)$ dual to the elementary matrices, and the e^i 's form the basis of \mathfrak{h}_n^* dual to the standard basis, and α is any element in $\wedge^1 V \oplus \dots \oplus \wedge^r V$.

On the 0-th homologies, Φ_n induces an algebra map

$$H_0(\Phi_n) : k[\text{Rep}_n(A)]^{\text{GL}} \rightarrow \text{Sym}(\mathfrak{h}_n^* \otimes V)^{S_n}.$$

By [Do, Theorem 4.1] (see also [Val, Theorem 3]), this map is known to be an isomorphism for all V and all $n \geq 1$. It is therefore tempting to conjecture that Φ_n is actually a quasi-isomorphism, i.e. induces isomorphisms on homology in *all* homological degrees. This is indeed the case when $\dim(V) = 1$ (since $\text{Sym}(V)$ is a cofibrant DG algebra when $\dim(V) = 1$, and hence $\text{DRep}_n(A) \cong A_n$ has no higher homology). On the other hand, by evaluating Euler characteristics, we will show in Section 5.2 that Φ_n cannot be a quasi-isomorphism (for all n) when $\dim(V) \geq 3$. In the case $\dim(V) = 2$, we believe that the conjecture is still true.

To state our conjecture in more explicit terms, we choose a basis $\{x, y\}$ in V and identify $\text{Sym}(V) = k[x, y]$. Denote by θ the degree 1 element $\lambda(x, y) \in \wedge^2 V$ and write x_i (resp., y_i, θ_i) for the elements $e^i \otimes x$ (resp., $e^i \otimes y, e^i \otimes \theta$) in $\mathfrak{h}_n^* \otimes (\wedge^1 V \oplus \wedge^2 V)$. Then $\text{Sym}[\mathfrak{h}_n^* \otimes (\wedge^1 V \oplus \wedge^2 V)] \cong k[x_1, \dots, x_n, y_1, \dots, y_n, \theta_1, \dots, \theta_n]$, and we have

Conjecture 4.6. *For any $n \geq 1$, the map*

$$\Phi_n : \text{DRep}_n(k[x, y])^{\text{GL}} \rightarrow k[x_1, \dots, x_n, y_1, \dots, y_n, \theta_1, \dots, \theta_n]^{S_n} \quad (4.19)$$

is a quasi-isomorphism. Consequently, there is an isomorphism of graded commutative algebras

$$H_\bullet(k[x, y], n)^{\text{GL}} \cong k[x_1, \dots, x_n, y_1, \dots, y_n, \theta_1, \dots, \theta_n]^{S_n}.$$

Remark. When $\dim(V) = 2$, the map $H_0(\Phi_n)$ coincides with the usual Harish-Chandra homomorphism for \mathfrak{gl}_n as defined, for example, in [J]; it is therefore natural to call Φ_n the derived Harish-Chandra homomorphism.

As a first evidence for Conjecture 4.6, we recall a vanishing theorem for the representation homology of $A = k[x, y]$ proved in [BFR, Theorem 27]: for all $n \geq 1$,

$$H_i(k[x, y], n) = 0, \quad \forall i > n.$$

This implies that $H_i(k[x, y], n)^{\text{GL}} = 0$ for $i > n$, and hence Φ_n induce isomorphisms (which are actually the zero maps) in homological degrees $i > n$.

For $n = 1$, Conjecture 4.6 is obvious. Furthermore, we have

Theorem 4.7. *Conjecture 4.6 is true for $n = 2$.*

Proof. We will explicitly construct the inverse map to Φ at the level of homology. To simplify notation we set $A = k[x, y]$, and using [BKR, Theorem 2.5], identify $H_0(A, 2) = A_2$, where A_2 is the coordinate ring of the commuting scheme of 2×2 matrices:

$$A_2 = k[x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22}]/I,$$

with I generated by the relations

$$\begin{aligned} x_{12}y_{21} - y_{12}x_{21} &= 0, \\ x_{11}y_{12} + x_{12}y_{22} - y_{11}x_{12} - y_{12}x_{22} &= 0, \\ x_{21}y_{11} + x_{22}y_{21} - y_{21}x_{11} - y_{22}x_{21} &= 0. \end{aligned} \tag{4.20}$$

With this identification, we define an algebra homomorphism

$$\Psi_0 : k[x_1, x_2, y_1, y_2] \rightarrow \mathcal{M}_2(A_2)$$

by

$$x_1 \mapsto X, \quad x_2 \mapsto X^*, \quad y_1 \mapsto Y, \quad y_2 \mapsto Y^*,$$

where

$$X := \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \quad X^* := \begin{pmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{pmatrix}$$

and similarly for Y and Y^* . Note that X^* and Y^* are the classical adjoints of X and Y , and the relations (4.20) imply that $[X, Y] = 0$. It follows that the matrices X, Y, X^*, Y^* pairwise commute, and the map Ψ_0 is thus well-defined.

We claim that Ψ_0 restricts to an algebra *isomorphism*

$$\Psi_0 : k[x_1, x_2, y_1, y_2]^{S_2} \xrightarrow{\sim} A_2^{\text{GL}_2}, \tag{4.21}$$

where $A_2^{\text{GL}} \subset A_2$ is identified with a subalgebra of scalar matrices in $\mathcal{M}_2(A_2)$. Indeed, the invariant subalgebra $k[x_1, x_2, y_1, y_2]^{S_2}$ is generated by the five elements: $x_1 + x_2, y_1 + y_2, x_1x_2, y_1y_2$ and $x_1y_1 + x_2y_2$, which are mapped by (4.21) to $\text{Tr}(X), \text{Tr}(Y), \det(X), \det(Y)$ and $\text{Tr}(XY)$, respectively. It is immediate to see that $H_0(\Phi) \circ \Psi_0 = \text{Id}$. On the other hand, as mentioned above, the map $H_0(\Phi)$ is known to be an isomorphism.¹² The map Ψ_0 is thus the inverse of $H_0(\Phi)$, and hence an isomorphism as well. We will use the following notation for the generators of $k[x_1, x_2, y_1, y_2]^{S_2}$:

$$\begin{aligned} T(x) &:= x_1 + x_2, & T(y) &:= y_1 + y_2, & D(x) &:= x_1x_2, & D(y) &:= y_1y_2, \\ T(xy) &:= x_1y_1 + x_2y_2. \end{aligned}$$

Now, it is easy to check that the graded algebra $k[x_1, x_2, y_1, y_2, \theta_1, \theta_2]^{S_2}$ is generated by its degree zero subalgebra $k[x_1, x_2, y_1, y_2]^{S_2}$ and three extra elements of degree 1, which we denote by

$$T(\theta) := \theta_1 + \theta_2, \quad T(x\theta) := x_1\theta_1 + x_2\theta_2, \quad T(y\theta) := y_1\theta_1 + y_2\theta_2.$$

These elements satisfy the following relations:

$$\begin{aligned} T(y) \cdot T(x\theta) \cdot T(\theta) - T(x) \cdot T(y\theta) \cdot T(\theta) &= 2T(x\theta) \cdot T(y\theta), \\ (T(x)^2 - 4D(x)) \cdot T(y\theta) - (2T(xy) - T(x)T(y)) \cdot T(x\theta) \\ &= (T(x)^2T(y) - 2D(x)T(y) - T(x)T(xy)) \cdot T(\theta), \\ (2T(xy) - T(x)T(y)) \cdot T(y\theta) - (T(y)^2 - 4D(y)) \cdot T(x\theta) \\ &= (T(y)^2T(x) - 2D(y)T(x) - T(y)T(xy)) \cdot T(\theta). \end{aligned}$$

¹² In fact, the map $H_0(\Phi)$ coincides with the isomorphism Δ' in [Val, Theorem 3].

On the other hand, by [BKR, Example 4.2], the full representation homology algebra $H_\bullet(A, 2)$ is generated by $H_0(A, 2) = A_2$ and three invariant elements τ, ξ, η of degree 1, satisfying

$$\begin{aligned} x_{12}\eta - y_{12}\xi &= (x_{12}y_{11} - y_{12}x_{11})\tau, \\ x_{21}\eta - y_{21}\xi &= (x_{21}y_{22} - y_{21}x_{22})\tau, \\ (x_{11} - x_{22})\eta - (y_{11} - y_{22})\xi &= (x_{11}y_{22} - y_{11}x_{22})\tau, \\ \xi\eta &= y_{11}(\xi\tau) - x_{11}(\eta\tau) = y_{22}(\xi\tau) - x_{22}(\eta\tau). \end{aligned}$$

We now extend the map (4.21) to the map

$$\Psi_\bullet : k[x_1, x_2, y_1, y_2, \theta_1, \theta_2]^{S_2} \rightarrow H_\bullet(A, 2)$$

by sending

$$T(\theta) \mapsto \tau, \quad T(x\theta) \mapsto \xi, \quad T(y\theta) \mapsto \eta.$$

A straightforward (but tedious) calculation, using the above relations, shows that Ψ_\bullet is a well-defined algebra map. Its image coincides with $H_\bullet(A, 2)^{GL}$, since $H_\bullet(A, 2)$ is generated over $H_0(A, 2)$ by GL -invariant elements, and we already know that Ψ_0 is an isomorphism onto $H_0(A, 2)^{GL}$. On the other hand, it is again easy to see that $H_\bullet(\Phi) \circ \Psi_\bullet = \text{Id}$. Hence Ψ_\bullet is injective and is thus the inverse of $H_\bullet(\Phi)$. \square

4.4. Conjecture 4.6 in the limit

In this section, we prove that Conjecture 4.6 holds in the limit $n \rightarrow \infty$.

4.4.1. *Supersymmetric polynomials and power sums.* Let x_1, \dots, x_d be variables of homological degree 0. Let the symmetric group S_d act on the x_i 's by permutations. Consider the power sums $P_i := \sum_{j=1}^d x_j^i$. It is a classical fact that the symmetric polynomials in x_1, \dots, x_d can be rewritten as polynomials of degree $\leq d$ in the power sums $P_i, i \geq 1$. Further, equip the variables x_1, \dots, x_d with weight 1, making $k[x_1, \dots, x_d]$ a weight graded (commutative) algebra. Let $k[x_1, \dots, x_d, \dots]^{S_\infty} := \varprojlim_d k[x_1, \dots, x_d]^{S_d}$ where the projective limit is taken in the category of weight graded algebras. Then the homomorphism

$$k[q_1, \dots, q_d, \dots] \rightarrow k[x_1, \dots, x_d, \dots]^{S_\infty}, \quad q_i \mapsto \sum_{j=1}^\infty x_j^i,$$

is an isomorphism.

We now generalize this classical fact. Consider the set of Nd variables $\{x_{\alpha,i} : 1 \leq \alpha \leq N, 1 \leq i \leq d\}$. Assume that the variables $x_{1,i}, \dots, x_{m,i}, 1 \leq i \leq d$, have odd homological degree, with the remaining variables having even homological degree. In particular, in $k[x_{\alpha,i} : 1 \leq \alpha \leq N, 1 \leq i \leq d]$, we have $x_{\alpha,i}^2 = 0$ for $\alpha \leq m$. Let S_d act on these variables, with a permutation $\sigma \in S_d$ taking $x_{\alpha,i}$ to $x_{\alpha,\sigma(i)}$. Let $k[x_{\alpha,i} : \alpha \leq N, i \in \mathbb{N}]^{S_\infty} := \varprojlim_d k[x_{\alpha,i} : 1 \leq \alpha \leq N, 1 \leq i \leq d]^{S_d}$ where the projective limit is taken in the category of bigraded DG algebras and the variables $x_{\alpha,i}$ have positive weight d_α . Consider the power sums $P_{\mathbf{a}} := \sum_{i \geq 1} \prod_{\alpha=1}^N x_{\alpha,i}^{a_\alpha}$ where $\mathbf{a} := (a_1, \dots, a_N)$ runs over $\{0, 1\}^m \times \mathbb{Z}_{\geq 0}^{N-m}$.

Proposition 4.8. *Let W be the k -vector space spanned by variables $\{q_{\mathbf{a}}\}$, where $\mathbf{a} \in \{0, 1\}^m \times \mathbb{Z}_{\geq 0}^{N-m}$.*

(i) *The homomorphism of bigraded (commutative) DG algebras*

$$\mathbf{Sym}_k W \rightarrow k[x_{\alpha,i} : 1 \leq \alpha \leq N, 1 \leq i \leq d]^{S_d}, \quad q_{\mathbf{a}} \mapsto P_{\mathbf{a}}, \tag{4.22}$$

restricts to an isomorphism of bigraded k -vector spaces

$$\mathbf{Sym}_k^{\leq d} W \cong k[x_{\alpha,i} : 1 \leq \alpha \leq N, 1 \leq i \leq d]^{S_d}.$$

(ii) *The homomorphisms (4.22) induce an isomorphism of bigraded (commutative) DG algebras*

$$\mathbf{Sym}_k W \cong k[x_{\alpha,i} : \alpha \leq N, i \in \mathbb{N}]^{S_{\infty}}.$$

Proof. Clearly, (i) implies (ii). We therefore show (i). Any element of $k[x_{\alpha,i} : 1 \leq \alpha \leq N, 1 \leq i \leq d]^{S_d}$ is a k -linear combination of orbit sums of the form

$$O_{\mu} := \sum_{\sigma \in S_d} \sigma \left(\prod_{\alpha=1}^N \prod_{j=1}^p x_{\alpha,j}^{\mu_{\alpha,j}} \right)$$

where $p \leq d$ and no column of the $N \times p$ -matrix $\mu := (\mu_{\alpha,j})_{1 \leq \alpha \leq N, 1 \leq j \leq p}$ is identically zero. If $p = 1$, then O_{μ} is precisely the power sum $P_{\mathbf{a}}$ where $\mathbf{a} = (\mu_{\alpha,1}, \dots, \mu_{\alpha,N})$. Note that

$$O_{\mu} = \pm \prod_{j=1}^p (P_{(\mu_{1,j}, \dots, \mu_{N,j})}) + \sum_{\nu} c_{\nu} O_{\nu}$$

where ν runs over matrices with less than p columns (and where $c_{\nu} \in k$). Hence, by induction on p , we see that O_{μ} is represented as an element in the image of $\mathbf{Sym}_k^{\leq p} W$ under the homomorphism (4.22). Clearly, this representation is unique. This proves (i). \square

4.4.2. Proof of Conjecture 4.6 as $n \rightarrow \infty$. The minimal free resolution of $A = k[x, y]$ is given by $R = k\langle x, y, \theta \rangle$ with differential $d\theta = [x, y]$. Then $R_{\text{ab}} \cong k[x, y, \theta]$ with $d\theta = 0$ and

$$S^n(R_{\text{ab}}) \cong k[x_1, \dots, x_n, y_1, \dots, y_n, \theta_1, \dots, \theta_n]^{S_n}, \quad n \geq 1.$$

By Theorem 2.8, $H_{\bullet}(R_{\text{ab}}) \cong \overline{\text{HC}}_{\bullet}(A)$; on the other hand, by [L, Theorem 3.4.12], we can identify $\overline{\text{HC}}_{\bullet}(A) \cong \Omega^{\bullet}(A)/d\Omega^{\bullet-1}(A)$, where $\Omega^{\bullet}(A)$ is the algebraic de Rham complex of A . With these identifications, the reduced trace maps (4.11) are given by the following formulas (see [BF⁺, Section 4.4.1])

$$\begin{aligned} \overline{\text{Tr}}_n(A)_0 : A &\rightarrow S^n(R_{\text{ab}}), & P(x, y) &\mapsto \sum_{i=1}^n P(x_i, y_i), \\ \overline{\text{Tr}}_n(A)_1 : \Omega^1(A)/dA &\rightarrow S^n(R_{\text{ab}}), \\ & [P(x, y)dx + Q(x, y)dy] &\mapsto \sum_{i=1}^n (P_y(x_i, y_i) - Q_x(x_i, y_i))\theta_i. \end{aligned}$$

Using these formulas, it is easy to prove the following fact.

Lemma 4.9. *The image of $\overline{\text{Tr}}_n(A)_\bullet$ in $S^n(R_{\text{ab}})$ is spanned by the power sums*

$$P_{abc} := \sum_{i=1}^n x_i^a y_i^b \theta_i^c, \quad (a, b, c) \in \mathbb{Z}_{\geq 0}^2 \times \{0, 1\}.$$

Endow R with the structure of a bigraded DG algebra, where $x, y \in R$ have weight 1 and homological degree 0 and $\theta \in R$ has weight 2 and homological degree 1. Note that there is a natural (surjective) homomorphism $S^n(R_{\text{ab}}) \rightarrow S^{n-1}(R_{\text{ab}})$ for any $n \geq 2$. Let $S^\infty(R_{\text{ab}}) := \varprojlim_n S^n(R_{\text{ab}})$ where the projective limit is taken in the category of bigraded commutative (DG) algebras. Using Proposition 4.8 and Lemma 4.9, we can now establish the following result, which can be viewed as a further evidence in favor of Conjecture 4.6.

Theorem 4.10. *Let Φ_n be the derived Harish-Chandra homomorphism for $A = k[x, y]$ (see (4.19)).*

- (i) *For any $n \geq 1$, the map $H_\bullet(\Phi_n)$ is degreewise surjective.*
- (ii) *The map $H_\bullet(\Phi_\infty)$ is an isomorphism of bigraded commutative algebras.*

Proof. Note first that we have a commutative diagram

$$\begin{array}{ccc} \text{Sym}_k(\overline{\text{HC}}_\bullet(A)) & & \\ \downarrow & \searrow^{\text{Sym } \overline{\text{Tr}}_n(A)_\bullet} & \\ \text{Sym } \text{Tr}_n(A)_\bullet & & \\ \downarrow & & \\ H_\bullet(A, n)^{\text{GL}} & \xrightarrow{H_\bullet(\Phi_n)} & S^n(R_{\text{ab}}) \end{array}$$

The map Φ_n being surjective follows from Lemma 4.9 and Proposition 4.8(i). The map Φ_∞ being an isomorphism follows from Proposition 4.8 (ii), which says that $\text{Sym } \overline{\text{Tr}}_\infty(A)_\bullet$ is an isomorphism, and from [BR1, Theorem 4.4], which says that $\text{Sym } \text{Tr}_\infty(A)_\bullet$ is an isomorphism. □

Note that part (i) of Theorem 4.10 is the surjectivity part of Conjecture 4.6; thus, to prove Conjecture 4.6 it suffices to prove that the map $H_\bullet(\Phi_n)$ is injective. As a corollary of Theorem 4.10, we obtain (in the \mathfrak{gl}_n case) a classical result of A. Joseph [J, Theorem 2.9].

Corollary 4.11. *The restriction map*

$$\text{Sym}_k[\mathfrak{gl}_n(k) \oplus \mathfrak{gl}_n(k)]^{\text{GL}} \rightarrow \text{Sym}_k[\mathfrak{h}_n(k) \oplus \mathfrak{h}_n(k)]^{S_n} \cong k[x_1, \dots, x_n, y_1, \dots, y_n]^{S_n} \tag{4.23}$$

is (graded) surjective.

Proof. Note that (4.23) is precisely the composite map

$$k[\text{Rep}_n(k(x, y))]^{\text{GL}} \twoheadrightarrow k[\text{Rep}_n(k[x, y])]^{\text{GL}} \xrightarrow{H_0(\Phi_n)} S^n(k[x, y]).$$

The surjectivity of (4.23) thus follows from that of $H_0(\Phi_n)$. □

4.4.3. The case of three variables. One might expect that Conjecture 4.6 extends to polynomial algebras of more than two variables. We now show that this is not the case. In fact, Conjecture 4.6 fails already for polynomials of three variables. Let $A = k[x, y, z]$.

As in [BFR, Section 6.3.2], we write the minimal resolution of A in the form $R = k\langle x, y, z, \xi, \theta, \lambda, t \rangle$, where $\deg x = \deg y = \deg z = 0$, $\deg \xi = \deg \theta = \deg \lambda = 1$ and $\deg t = 2$. The differential on R is defined by

$$d\xi = [y, z], \quad d\theta = [z, x], \quad d\lambda = [x, y], \quad dt = [x, \xi] + [y, \theta] + [z, \lambda].$$

Again, we equip R with the structure of a bigraded DG algebra, with x, y, z having weight 1, ξ, θ, λ having weight 2 and t having weight 3.

As in Section 4.4.2, we can define $S^\infty(R_{\text{ab}})$ as a projective limit in the category of bigraded commutative (DG) algebras. Note that Proposition 4.8(ii) implies that $S^\infty(R_{\text{ab}})$ can be naturally identified with the free bigraded commutative (DG) algebra generated by the power sums

$$P_{\mathbf{a}} := \sum_{i=1}^{\infty} \xi_i^{a_1} \lambda_i^{a_2} \theta_i^{a_3} t_i^{a_4} x_i^{a_5} y_i^{a_6} z_i^{a_7}, \tag{4.24}$$

where the multi-index $\mathbf{a} := (a_1, \dots, a_7)$ runs over $\{0, 1\}^3 \times \mathbb{Z}_{\geq 0}^4$.

Now, by Theorem 2.8 and [L, Theorem 3.4.12], we can identify

$$H_\bullet(R_i) \cong \overline{\text{HC}}_\bullet(A) \cong \Omega^\bullet(A)/d\Omega^{\bullet-1}(A)$$

where $\Omega^\bullet(A)$ is the algebraic de Rham complex of A . This shows, in particular, that $\overline{\text{HC}}_i(A)$ vanish for $i \geq 3$. Representing the cyclic classes in $\overline{\text{HC}}_0(A)$, $\overline{\text{HC}}_1(A)$ and $\overline{\text{HC}}_2(A)$ by differential forms

$$\omega_0 = P, \quad \omega_1 = Pdx + Qdy + Rdz, \quad \omega_2 = Pdx \wedge dy + Qdy \wedge dz + Rdz \wedge dx,$$

we have the following formulas for the (reduced) trace maps (4.11) derived in [BF⁺, Section 4.4.2]:

$$\begin{aligned} \overline{\text{Tr}}_\infty(A)_0[\omega_0] &= \sum_{i=1}^{\infty} P^i, \\ \overline{\text{Tr}}_\infty(A)_1[\omega_1] &= \sum_{i=1}^{\infty} ((P_y^i - Q_x^i)\lambda_i + (Q_z^i - R_y^i)\xi_i + (R_x^i - P_z^i)\theta_i), \\ \overline{\text{Tr}}_\infty(A)_2[\omega_2] &= \sum_{i=1}^{\infty} ((P_{xz}^i + Q_{xx}^i + R_{xy}^i)\theta_i\lambda_i + (P_{yz}^i + Q_{xy}^i + R_{yy}^i)\lambda_i\xi_i \\ &\quad + (P_{zz}^i + Q_{xz}^i + R_{yz}^i)\xi_i\theta_i) \\ &\quad + \sum_{i=1}^{\infty} (P_z^i + Q_x^i + R_y^i)t_i. \end{aligned}$$

Here, $P_x, P_y, P_z, P_{xy}, \dots$ denote the derivatives of a polynomial $P \in k[x, y, z]$, and P^i stand for the corresponding elements $P(x_i, y_i, z_i) \in S^\infty(R_{\text{ab}})$.

The above formulas show that the image of the map $\overline{\text{Tr}}_\infty(A)_\bullet$ is spanned by power sums (4.24). However, this map is *not* surjective: for example, it is easy to see that no power sum $P_{\mathbf{a}}$ with $a_4 \geq 2$ appears in the image of $\overline{\text{Tr}}_\infty(A)_\bullet$. By [BR1, Theorem 4.4], we conclude

Proposition 4.12. *The map $H_\bullet(\Phi_\infty) : H_\bullet(A, \infty)^{\text{GL}} \rightarrow S^\infty(R_{\text{ab}})$ is injective but not surjective.*

Proposition 4.12 implies that if $\dim(V) = 3$, the derived Harish-Chandra homomorphism $\Phi_n(A)$ for $A = \text{Sym}(V)$ cannot be a quasi-isomorphism for any n . More generally, in Section 5.2, we will show that $\Phi_n(A)$ is not a quasi-isomorphism when $\dim(V) \geq 3$. Thus Conjecture 4.6 does not extend to higher dimensions. There is, however, another natural way to generalize this conjecture in which case we can actually prove that it holds.

4.5. Quantum polynomial algebras

In this section, we will show that (the analogue of) Conjecture 4.6 holds for the q -polynomial algebra

$$A_q := k\langle x, y \rangle / (xy - qyx), \quad q \in k^*.$$

Precisely, we will prove

Theorem 4.13. *If $q \in k^*$ is not a root of unity, the derived Harish-Chandra homomorphism*

$$\Phi_n(A_q) : \text{DRep}_n(A_q)^{\text{GL}} \rightarrow S^n \text{DRep}_1(A_q)$$

is a quasi-isomorphism for all $n \geq 1$.

We expect that the claim of Theorem 4.13 holds for an arbitrary value of q (except possibly for $q = 0$); however, our proof relies on the following vanishing result that requires the restriction on q .

Lemma 4.14 ([BFR]). *If $q \in k^*$ is not a root of unity, then, for all $n \geq 1$,*

$$H_i(A_q, n) = 0, \quad i > 0.$$

For the proof of Lemma 4.14 we refer the reader to [BFR, Section 6.2.2].

Proof of Theorem 4.13. Let us restate the claim of the theorem in explicit terms. The minimal cofibrant resolution of A_q is given by the free DG algebra $R = \mathbb{C}\langle x, y, \theta \rangle$ with x, y of degree 0, θ of degree 1, and the differential defined by $d\theta = xy - qyx$. Hence, $\text{DRep}_n(A_q)$ and $S^n \text{DRep}_1(A_q)$ are represented respectively by the following DG algebras:

$$B_n := k[x_{ij}, y_{ij}, \theta_{ij} : i, j = 1, \dots, n]^{\text{GL}}, \quad d\theta_{ij} = \sum_{k=1}^n (x_{ik}y_{kj} - qy_{ik}x_{kj}), \quad (4.25)$$

$$E_n := k[x_i, y_i, \theta_i : i = 1, \dots, n]^{S_n}, \quad d\theta_i = (1 - q)x_i y_i. \quad (4.26)$$

The derived Harish-Chandra homomorphism (4.8) is explicitly given by

$$\Phi_n : B_n \rightarrow E_n, \quad x_{ij} \mapsto \delta_{ij}x_i, \quad y_{ij} \mapsto \delta_{ij}y_i, \quad \theta_{ij} \mapsto \delta_{ij}\theta_i. \quad (4.27)$$

We need to show that (4.27) is a quasi-isomorphism.

First, we put an augmentation on the algebra A_q letting $\varepsilon(x) = \varepsilon(y) = 0$. The DG algebra R , and hence B_n and E_n , then becomes augmented in a natural way, and the map (4.27) preserves the augmentations. Next, using the notation of Section 2.4.3, we set $R_{\natural} := \bar{R}/[\bar{R}, \bar{R}]$ and define W to be $\mathbf{Sym}(R_{\natural})$. Note that W is filtered by the graded vector spaces $W^{\leq n} := \mathbf{Sym}^{\leq n}(R_{\natural})$, each of which carries a differential induced from R . We may think of R_{\natural} as the space of cyclic words in letters x, y, θ ; then $W^{\leq n}$ is spanned by products of at most n such words.

Now, consider the composition of morphisms of complexes

$$W^{\leq n} \xrightarrow{\text{Tr}_n} B_n \xrightarrow{\Phi_n} E_n. \tag{4.28}$$

The first map in (4.28) is defined by the trace morphism (2.17): explicitly, it sends a cyclic word to the corresponding trace expression (e.g., $xy\theta \mapsto \sum_{i,j,k} x_{ij}y_{jk}\theta_{ki}$). The second map is the Harish-Chandra homomorphism (4.27).

We claim that each arrow in (4.28) is actually a quasi-isomorphism. To see this we first show that the composition is a quasi-isomorphism. By Theorem 2.8, the homology of R_{\natural} can be identified with the reduced cyclic homology $\overline{\text{HC}}_{\bullet}(A_q)$. When q is not a root of unity, the latter is known to vanish in all positive degrees while spanned by the cyclic words $\{x^p\}$ and $\{y^p\}$ in degree 0 (see [Wa, Théorème 2.1]), i.e.,

$$\overline{\text{HC}}_{\bullet}(A_q) = \overline{\text{HC}}_0(A_q) \cong \bigoplus_{p \geq 1} (kx^p \oplus ky^p).$$

This implies

$$\begin{aligned} H_{\bullet}(W^{\leq n}) &= H_0(W^{\leq n}) \cong \text{Sym}^{\leq n}[H_0(R_{\natural})] \cong \text{Sym}^{\leq n}[\overline{\text{HC}}_0(A_q)] \\ &\cong k[X_p, Y_p : p = 1, 2, \dots]^{\leq n}, \end{aligned} \tag{4.29}$$

where the variables X_p and Y_p correspond to the cyclic words x^p and y^p .

Now, by Proposition 4.8(i), we can identify

$$E_n \cong \mathbf{Sym}^{\leq n}(\bar{R}_{\text{ab}}),$$

where $R_{\text{ab}} \cong k[x, y, \theta]$ with differential $d\theta = (1 - q)xy$. In this case, $H_{\bullet}(R_{\text{ab}})$ is concentrated in degree 0 and $H_0(R_{\text{ab}})$ is spanned (as a vector space) by the classes of x^p and y^p with $p \geq 1$. Hence the natural projection $R_{\natural} \twoheadrightarrow \bar{R}_{\text{ab}}$ induces an isomorphism of graded vector spaces $H_{\bullet}(R_{\natural}) \xrightarrow{\sim} H_{\bullet}(\bar{R}_{\text{ab}})$, which gives

$$H_{\bullet}(W^{\leq n}) \xrightarrow{\sim} H_{\bullet}(E_n). \tag{4.30}$$

It is easy to see that the isomorphism (4.30) is actually induced by the composite map (4.28); thus (4.28) is a quasi-isomorphism. Now, by [BR1, Theorem 3.1], we know that the trace map $\text{Tr}_n : W^{\leq n} \twoheadrightarrow R_n^{\text{GL}}$ is (degreewise) surjective. Since both $W^{\leq n}$ and R_n^{GL} are nonnegatively graded, the map induced by Tr_n on the zero homology is also surjective:

$$H_0(\text{Tr}_n) : H_0(W^{\leq n}) \twoheadrightarrow H_0(B_n) \tag{4.31}$$

On the other hand, by Lemma 4.14, the homology of R_n is concentrated in degree 0. Since $\text{GL}_n(k)$ is reductive, this implies

$$H_{\bullet}(B_n) \cong H_{\bullet}(R_n)^{\text{GL}} \cong H_0(R_n)^{\text{GL}} \cong H_0(B_n). \tag{4.32}$$

Combining (4.30)–(4.32), we now see that (4.28) induces an isomorphism

$$H_{\bullet}(W^{\leq n}) \twoheadrightarrow H_{\bullet}(B_n) \rightarrow H_{\bullet}(E_n), \tag{4.33}$$

where the first arrow is also surjective. It follows that both maps in (4.33) are isomorphisms. In particular, Φ_n is a quasi-isomorphism. \square

Remark. Note that (4.29) gives isomorphisms

$$H_{\bullet}(B_n) \cong H_{\bullet}(E_n) \cong k[X_p, Y_p : p = 1, 2, \dots]^{S_n}, \tag{4.34}$$

where the variables X_p and Y_p have homological degree 0.

5. Euler characteristics and constant term identities

In this section, we compare the Euler characteristics of both sides of Conjecture 4.6 and prove the resulting identities for all n . Throughout, we assume that $k = \mathbb{C}$.

5.1. A constant term identity for \mathfrak{gl}_n

The target of the Harish-Chandra homomorphism (4.19) is the graded commutative algebra $E_n := \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n, \theta_1, \dots, \theta_n]^{S_n}$ with x_i, y_i of homological degree 0 and θ_i of degree 1. It has an additional \mathbb{Z}^2 grading, which we call weight, such that $\text{wt}(x_i) = (1, 0)$, $\text{wt}(y_i) = (0, 1)$, $\text{wt}(\theta_i) = (1, 1)$.

Let $G_n(q, t, s)$ be the generating function

$$G_n(q, t, s) = \sum \dim((E_n)_{d,a,b}) s^d q^a t^b$$

of the dimensions of the homogeneous components $(E_n)_{d,a,b}$ of degree d and weight (a, b) .

Lemma 5.1. $G_n(q, t, s)$ is equal to the coefficient of v^n in the Taylor expansion at $v = 0$ of

$$\prod_{a,b \geq 0} \frac{1 + q^{a+1} t^{b+1} s v}{1 - q^a t^b v}.$$

Proof. A generating set is given by the S_n orbit sums of monomials $\theta^\nu x^\lambda y^\mu$ with exponents $\lambda, \mu \in \mathbb{Z}_{\geq 0}^n, \nu \in \{0, 1\}^n$. Such an orbit sum contributes $q^{|\lambda|+|\nu|} t^{|\mu|+|\nu|} s^{|\nu|}$ to the generating function, where $|\lambda| = \sum \lambda_i$. To get a basis we take the subset of such orbit sums whose exponents obey (i) ν is a partition: $\nu_i \geq \nu_{i+1}$ for all $i = 1, \dots, n - 1$, (ii) $\lambda_i \geq \lambda_{i+1}$ whenever $\nu_i = \nu_{i+1}$, (iii) $\mu_i \geq \mu_{i+1}$ whenever $\nu_i = \nu_{i+1} = 0$ and $\lambda_i = \lambda_{i+1}$ and (iv) $\mu_i > \mu_{i+1}$ whenever $\nu_i = \nu_{i+1} = 1$ and $\lambda_i = \lambda_{i+1}$. Such data are in one-to-one correspondence with families (m_{dab}) of nonnegative integers labeled by triples $(d, a, b) \in \{0, 1\} \times \mathbb{Z}_{\geq 0}^2$ such that $|m| = \sum_{d,a,b} m_{dab} = n$ and $m_{1ab} \leq 1$: for (ν, λ, μ) obeying (i)–(iv), m_{dab} is the number of parts of size b of the partition $(\mu_j)_{j \in J}$ where $j \in J$ iff $\nu_j = d$ and $\lambda_j = a$. This gives

$$\begin{aligned} \sum_{n=0}^{\infty} G_n(q, t, s) v^n &= \prod_{a,b \geq 0} \sum_{m=0}^{\infty} q^{ma} t^{mb} v^m \prod_{a,b \geq 0} \sum_{m=0}^1 q^{m(a+1)} t^{m(b+1)} s^m v^m \\ &= \prod_{a,b \geq 0} \frac{1 + q^{a+1} t^{b+1} s v}{1 - q^a t^b v}. \quad \square \end{aligned}$$

The (weighted) Euler characteristic of E_n is the value of G_n at $s = -1$, where the formula simplifies considerably. Let us use the standard notation $(v; q)_\infty := \prod_{j=0}^\infty (1 - q^j v)$ and $(v; q)_n := (v; q)_\infty / (q^n v; q)_\infty$.

Corollary 5.2. *The Euler characteristic $\chi(E_n, q, t) = G_n(q, t, s = -1)$ is the coefficient of v^n in the Taylor expansion at $v = 0$ of the function*

$$\frac{1 - v}{(v; q)_\infty (v; t)_\infty}.$$

More explicitly,

$$\chi(E_n, q, t) = \sum_{j=0}^n \frac{t^j}{(q; q)_{n-j} (t; t)_j}.$$

Proof. The first statement is immediate from Lemma 5.1; the second follows from the well-known Rothe identity

$$\frac{1}{(v; q)_\infty} = \sum_{n=0}^\infty \frac{v^n}{(q; q)_n}. \quad \square$$

Let us compare this Euler characteristic with the Euler characteristic of the graded commutative algebra $B_n := R_n^{\text{GL}}$ underlying the invariant part of the complex computing representation homology. Here

$$R_n = \mathbb{C}[x_{ij}, y_{ij}, \theta_{ij} : i, j = 1, \dots, n].$$

Again, we assign degree 0, 0, 1 and weight (1, 0), (0, 1) and (1, 1) to x_{ij} , y_{ij} and θ_{ij} , respectively. The weighted Euler characteristic of B_n can be extracted from the character valued weighted Euler characteristic

$$\chi(R_n, q, t, u) = \sum \text{trace}(u | (R_n)_{d,a,b}) (-1)^d q^a t^b, \quad u = \text{diag}(u_1, \dots, u_n) \in \text{GL}_n,$$

of the GL_n -module R_n .

Lemma 5.3.

$$\chi(R_n, q, t, u) = \prod_{1 \leq i, j \leq n} \frac{1 - qtu_i/u_j}{(1 - qu_i/u_j)(1 - tu_i/u_j)}.$$

Proof. We use the basis of monomials, noticing that u acts on $x_{i,j}$, $y_{i,j}$, $\theta_{i,j}$ by multiplication by u_i/u_j . □

To find the weighted Euler characteristic of B_n we need to select the coefficient of 1 in the expansion of the coefficients of the character valued Euler characteristic in Schur polynomials. This is obtained by taking the $1/n!$ times the constant term of the product with the Weyl denominator:

Corollary 5.4. *Let $\text{CT} : \mathbb{Z}[u_1^{\pm 1}, \dots, u_n^{\pm 1}][[q, t]] \rightarrow \mathbb{Z}[[q, t]]$ be the constant term map. Then*

$$\chi(B_n, q, t) = \frac{1}{n!} \frac{(1 - qt)^n}{(1 - q)^n (1 - t)^n} \text{CT} \prod_{1 \leq i \neq j \leq n} \frac{(1 - qtu_i/u_j)(1 - u_i/u_j)}{(1 - qu_i/u_j)(1 - tu_i/u_j)}.$$

Now, Conjecture 4.6 implies the equality

$$\chi(E_n, q, t) = \chi(B_n, q, t), \tag{5.1}$$

which, using Corollaries 5.2 and 5.4, we can write as the identity

$$\frac{1}{n!} \frac{(1-qt)^n}{(1-q)^n(1-t)^n} \text{CT} \prod_{1 \leq i \neq j \leq n} \frac{(1-qtu_i/u_j)(1-u_i/u_j)}{(1-qu_i/u_j)(1-tu_i/u_j)} = \sum_{j=0}^n \frac{t^j}{(q; q)_{n-j}(t; t)_j}. \tag{5.2}$$

The main result of this section is

Theorem 5.5. *The identity (5.2) holds for all $n \geq 1$.*

Proof. This is an immediate consequence of Theorem 4.13. Indeed, if we forget differentials, the commutative algebras B_n and E_n introduced in this section coincide with the algebras (4.25) and (4.26) introduced in Section 4.5. The differentials in (4.25) and (4.26) respect the weight gradings, and so does the Harish-Chandra homomorphism (4.27). Theorem 4.13 thus implies the equality (5.1), which is equivalent to (5.2). \square

Remarks. 1. The last isomorphism in (4.34) allows one to compute the right-hand side of the identity (5.2) directly, without using Lemma 5.1. Indeed, $\mathbb{C}[X_p, Y_p : p = 1, 2, \dots]^{S_n}$ consists of polynomials in $X_1, Y_1, X_2, Y_2, \dots$ of degree $\leq n$. The number of monomials of degree exactly n and of bidegree (a, b) is the coefficient of $v^n q^a t^b$ in the generating function $\prod_{k \geq 1} (1 - vq^k)^{-1} (1 - vt^k)^{-1}$. The number of polynomials of degree $\leq n$ can be obtained by multiplying this function by $1/(1-v)$. The result of Corollary 5.2 follows then easily from (4.34).

2. The identity (5.2) can be rewritten equivalently in the form

$$\int_{U(n)} \frac{\det(1-qt \text{Ad}(g))}{\det(1-q \text{Ad}(g)) \det(1-t \text{Ad}(g))} dg = \frac{1}{n!} \sum_{\sigma \in S_n} \frac{\det(1-qt\sigma)}{\det(1-q\sigma) \det(1-t\sigma)},$$

where the integration on the left is taken with respect to the normalized Haar measure over the n -th unitary group $U(n)$ and the determinants on the right are taken with respect to the natural action of S_n on \mathbb{C}^n . In this form, the identity (5.2) extends to an arbitrary reductive Lie algebra (see Section 8.2).

3. The left-hand side of (5.2) is the expansion at $q = t = 0$ of an integral over the product of unit circles $|u_i| = 1$ defined for $|q|, |t| < 1$ and can be computed by iterated residues, with the following result: let $Z_n(q, t)$ be the left-hand side of (5.2) and $Z(v, q, t) = 1 + \sum_{n=1}^{\infty} Z_n(q, t)v^n$ the generating function. Then

$$Z(v, q, t) = \exp\left(\sum_{\lambda} \frac{v^{|\lambda|}}{|\lambda|} W_{\lambda}(q, t)\right).$$

The sum is over all partitions (nonincreasing integer sequences converging to 0) $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq 0)$ of positive size $|\lambda| = \sum \lambda_i$. The coefficient $W_{\lambda}(q, t)$ is a regularized product over the boxes (i, j) of the Young diagram

$$Y(\lambda) = \{(i, j) \in \mathbb{Z}^2 : 1 \leq j \leq \lambda_i\} = \{(i, j) \in \mathbb{Z}^2 : 1 \leq i \leq \lambda'_j\}$$

of the partition λ , with conjugate partition λ' :

$$W_\lambda(q, t) = \prod'_{(i,j) \in Y(\lambda)} \frac{(1 - q^j t^i)(1 - q^{-j+1} t^{-i+1})}{(1 - q^{\lambda_i - j + 1} t^{-\lambda'_j + i})(1 - q^{-\lambda_i + j} t^{\lambda'_j - i + 1})}$$

In the regularized product \prod' we omit the factor $1 - q^0 t^0$ appearing at $(i, j) = (1, 1)$. Using the product formula of Corollary 5.2 for the generating function of the right-hand side of (5.2) we see that the identity (5.2) is equivalent to

$$\sum_{|\lambda|=n} W_\lambda(q, t) = \frac{1 - q^n t^n}{(1 - q^n)(1 - t^n)}, \quad n = 1, 2, \dots$$

This calculation is a trigonometric version of a similar calculation occurring in supersymmetric gauge theory [N]: the Nekrasov instanton partition function of $\mathcal{N} = 2$ super Yang–Mills theory on \mathbb{R}^4 with $U(N)$ gauge group has both an integral representation and an expression as a sum over collections of partitions (see [N, (3.10) and (1.6)], respectively). They are related by iterated residues.

5.2. Other examples

It is known that the derived Harish-Chandra homomorphism (4.8) induces an isomorphism on the 0-th homology for any commutative algebra.¹³ In Section 4.4.3, we have shown that this isomorphism cannot be extended to higher homologies for $A = k[x, y, z]$. We now give a general argument showing that Φ_n is not a quasi-isomorphism for $A = \text{Sym}(V)$ when $\dim(V) \geq 3$. We will also look at the algebra of dual numbers, in which case the analogue of Conjecture 4.6 also fails.

5.2.1. Euler characteristics for arbitrary symmetric algebras. Our aim is to prove

Proposition 5.6. *The derived Harish-Chandra homomorphism*

$$\Phi_n : \text{DRep}_n(\text{Sym}(V))^{\text{GL}} \rightarrow \mathbf{Sym}(\mathfrak{h}_n^* \otimes (\wedge^1 V \oplus \dots \oplus \wedge^r V))^{S_n}$$

is not a quasi-isomorphism when the dimension r of V is at least 3 (and when $n \geq 2$).

Proof. Suppose that $r := \dim_k(V) = 3$. Choose a basis $\{x, y, z\}$ of V . Note that both $\text{DRep}_n(\text{Sym}(V))^{\text{GL}}$ and $\mathbf{Sym}(\mathfrak{h}_n^* \otimes (\wedge^1 V \oplus \wedge^2 V \oplus \wedge^3 V))^{S_n}$ are \mathbb{Z}^3 -graded, with x having weight $(1, 0, 0)$, y having weight $(0, 1, 0)$ and z having weight $(0, 0, 1)$ (these weights uniquely determine the weight of each generator of $\text{DRep}_n(\text{Sym}(V))$ as well as the weight of each generator of $\mathbf{Sym}(\mathfrak{h}_n^* \otimes (\wedge^1 V \oplus \wedge^2 V \oplus \wedge^3 V))$). Further note that by its very construction, the derived Harish-Chandra homomorphism is weight preserving. We then claim that

Lemma 5.7. *The subcomplexes of weight $(1, 1, 1)$ in $\text{DRep}_n(\text{Sym}(V))^{\text{GL}}$ and $\mathbf{Sym}(\mathfrak{h}_n^* \otimes (\wedge^1 V \oplus \wedge^2 V \oplus \wedge^3 V))^{S_n}$ have different Euler characteristics.*

¹³ This follows from a theorem of Vaccarino [Va2, Theorem 4.1], which asserts that the so-called norm map $\mathbf{det} : S^n(A) \rightarrow A_n^{\text{GL}}$ is an isomorphism for any commutative k -algebra A and any $n \geq 1$. A direct calculation shows that $H_0(\Phi_n)$ coincides with the inverse of \mathbf{det} .

This shows that Φ_n cannot be a quasi-isomorphism when $r = 3$ and $n \geq 2$. Further, when $r > 3$, one can choose a basis $\{x_1, \dots, x_r\}$ of V , and equip both $\text{DRep}_n(\text{Sym}(V))^{\text{GL}}$ and $\mathbf{Sym}(\mathfrak{h}_n^* \otimes (\wedge^1 V \oplus \dots \oplus \wedge^r V))^{S_n}$ with the \mathbb{Z}^r -grading determined by fixing the weight of x_i to be $(0, \dots, 0, 1, 0, \dots, 0)$ (with i -th coordinate 1). Again, the derived Harish-Chandra homomorphism is weight preserving by its very construction. Further, it is easy to see that the subcomplexes of weight $(1, 1, 1, 0, \dots, 0)$ in $\text{DRep}_n(\text{Sym}(V))^{\text{GL}}$ and $\mathbf{Sym}(\mathfrak{h}_n^* \otimes (\wedge^1 V \oplus \wedge^2 V \oplus \wedge^3 V \oplus \dots \oplus \wedge^r V))^{S_n}$ are isomorphic to their counterparts for the case when $r = 3$. Thus, Lemma 5.7 implies the desired proposition in general.

It remains to verify Lemma 5.7. For notational brevity, we write $\zeta := -\lambda(x, y)$, $\eta := -\lambda(z, x)$, $\xi := -\lambda(y, z)$ and $t := -\lambda(x, y, z)$. Then the minimal resolution R is generated by $x, y, z, \zeta, \eta, \xi, t$ with $d\zeta = [x, y]$, $d\eta = [z, x]$, $d\xi = [y, z]$ and $dt = [\xi, x] + [\eta, y] + [\zeta, z]$. Thus, $\text{DRep}_n(k[x, y, z])$ is generated by the variables $x_{ij}, y_{ij}, z_{ij}, \zeta_{ij}, \eta_{ij}, \xi_{ij}, t_{ij}$ for $1 \leq i, j \leq n$ where $x_{ij} := e^{ij} \otimes x$, etc. The subcomplex of $\text{DRep}_n(k[x, y, z])^{\text{GL}}$ of weight $(1, 1, 1)$ is of the form $0 \rightarrow C_2 \rightarrow C_1 \rightarrow C_0 \rightarrow 0$ where

$$\begin{aligned} C_2 &:= \text{Span}\{\text{Tr}(t)\}, \\ C_1 &:= \text{Span}\{\text{Tr}(\xi x), \text{Tr}(\eta y), \text{Tr}(\zeta z), \text{Tr}(\xi) \text{Tr}(x), \text{Tr}(\eta) \text{Tr}(y), \text{Tr}(\zeta) \text{Tr}(z)\}, \\ C_0 &:= \text{Span}\{\text{Tr}(xyz), \text{Tr}(xzy), \text{Tr}(xy) \text{Tr}(z), \text{Tr}(yz) \text{Tr}(x), \\ &\quad \text{Tr}(zx) \text{Tr}(y), \text{Tr}(x) \text{Tr}(y) \text{Tr}(z)\}. \end{aligned}$$

For $n \geq 3$ the above elements are linearly independent. When $n = 2$, there is one relation:

$$\text{Tr}(xyz) + \text{Tr}(xzy) - \text{Tr}(xy) \text{Tr}(z) - \text{Tr}(yz) \text{Tr}(x) - \text{Tr}(zx) \text{Tr}(y) + \text{Tr}(x) \text{Tr}(y) \text{Tr}(z) = 0.$$

Hence, $\chi(\text{DRep}_n(k[x, y, z])_{(1,1,1)}^{\text{GL}}) = 1$ when $n \geq 3$ and 0 for $n = 2$.

On the other hand, writing $x_i := e^i \otimes x$, etc., we identify $\mathbf{Sym}(\mathfrak{h}_n^* \otimes (\wedge^1 V \oplus \wedge^2 V \oplus \wedge^3 V))^{S_n}$ with

$$A :=$$

$$k[x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_n, \xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n, \zeta_1, \dots, \zeta_n, t_1, \dots, t_n]^{S_n}.$$

In homological degree 0, we have

$$A_0(1, 1, 1) = \text{Span}\left(\sum_{(a,b,c) \in O} x_a y_b z_c\right)$$

where O runs over the orbits of S_n in $\{1, \dots, n\}^3$. For $n \geq 3$ there are five such orbits: those of $(1, 1, 1)$, $(1, 1, 2)$, $(1, 2, 1)$, $(2, 1, 1)$ and $(1, 2, 3)$. In homological degree 1,

$$A_1(1, 1, 1) = \text{Span}\left(\sum_{(a,b) \in O} \xi_a x_b, \sum_{(a,b) \in O} \eta_a y_b, \sum_{(a,b) \in O} \zeta_a z_b\right)$$

where O runs over the orbits of S_n in $\{1, \dots, n\}^2$. There are two such orbits: those of $(1, 1)$ and $(1, 2)$. Finally,

$$A_2(1, 1, 1) = \text{Span}(t_1 + \dots + t_n).$$

Thus, the Euler characteristic of $A(1, 1, 1)$ is 0 when $n \geq 3$ (and -1 when $n = 2$). This verifies Lemma 5.7, thereby completing the proof of Proposition 5.6. \square

5.2.2. *Dual numbers.* Let $A := \mathbb{C}[x]/(x^2)$, the algebra of *dual numbers*. In this case, $C = T^c(V)$ where V is a 1-dimensional vector space in homological degree 1. Explicitly, $\Omega(C)$ is the free DG algebra $R := \mathbb{C}\langle x, t_1, t_2, \dots \rangle$ with x having homological degree 0 and t_i having homological degree i for all $i \in \mathbb{N}$. The differential on $\Omega(C)$ is given by

$$dt_p = xt_{p-1} - t_1t_{p-2} + \dots + (-1)^{p-1}t_{p-1}x, \quad p \geq 1.$$

Note that as DG algebras, both A and R are *weight graded* as well, with the weight of x being 1 and that of t_p being $p + 1$ for all $p \geq 1$. Note that the derived Harish-Chandra homomorphism is weight preserving by its very construction. Also, all the complexes involved have finite total dimension in each weight degree. To see that the derived Harish-Chandra homomorphism is not a quasi-isomorphism in this case, it suffices to check that the weighted Euler characteristics $\chi(\mathrm{DRep}_2(A)^{\mathrm{GL}}, q)$ and $\chi((\mathrm{DRep}_1(A)^{\otimes 2})^{S_2}, q)$ differ. Here, for V a weight graded complex of \mathbb{C} -vector spaces that has finite total dimension in each weight degree, we define the weighted Euler characteristic of V by

$$\chi(V, q) := \sum_{i,j} (-1)^i \dim_{\mathbb{C}}(V_i(j))q^j$$

where $V_i(j)$ is the component of V_i of weight j .

Note that

$$\chi(S^2\mathrm{DRep}_1(A), q) = \frac{1}{2}[\chi(\mathrm{DRep}_1(A), q)^2 + \chi(\mathrm{DRep}_1(A), q^2)].$$

Since $\chi(\mathrm{DRep}_1(A), q) = \prod_{k=1}^{\infty} (1 - q^{2k+2})(1 - q^{2k+1})^{-1} = \sum_{j=0}^{\infty} q^{j(j+1)/2}$,

$$\chi(S^2\mathrm{DRep}_1(A), q) = 1 + q + q^2 + q^3 + q^4 + 2q^6 + \dots \tag{5.3}$$

On the other hand (see [BR1, Section 7.6]),

$$\chi(\mathrm{DRep}_2(A)^{\mathrm{GL}}, q) = \int_{U(2)} \prod_{i=1}^{\infty} \det(1 - q^i \mathrm{Ad}(g))^{(-1)^i} dg.$$

The above integral is taken over the unitary group $U(2) \subset \mathrm{GL}_2(\mathbb{C})$, where the Haar measure dg is normalized so that the volume of $U(2)$ is 1. The determinant is taken in the adjoint representation of $\mathrm{GL}_2(\mathbb{C})$ on $\mathcal{M}_2(\mathbb{C})$. The above integral can be computed directly, giving

$$\chi(\mathrm{DRep}_2(A)^{\mathrm{GL}}, q) = \frac{1}{1 - q}. \tag{5.4}$$

Comparing (5.3) and (5.4), one sees that the derived Harish-Chandra homomorphism

$$\Phi_2 : \mathrm{DRep}_2(\mathbb{C}[x]/(x^2))^{\mathrm{GL}_2} \rightarrow S^2\mathrm{DRep}_1(\mathbb{C}[x]/(x^2))$$

is not an isomorphism in $\mathrm{Ho}(\mathrm{DGCA}_{k/k})$.

6. Derived representation schemes of Lie algebras

In this section, we construct a (derived) representation functor on the category of DG Lie algebras. We prove a basic result (Theorem 6.7) relating the derived representation functor of Lie algebra to the homology of a Lie coalgebra defined over a Koszul dual cocommutative coalgebra.

6.1. Classical representation schemes

Let \mathfrak{g} be a finite-dimensional Lie algebra. For any Lie algebra \mathfrak{a} , there is an affine scheme $\text{Rep}_{\mathfrak{g}}(\mathfrak{a})$ parametrizing the representations of \mathfrak{a} in \mathfrak{g} . More precisely, the functor

$$\mathfrak{g}(-) : \text{Comm Alg}_k \rightarrow \text{LieAlg}_k, \quad B \mapsto \mathfrak{g}(B) := \mathfrak{g} \otimes B,$$

has a left adjoint

$$(-)_{\mathfrak{g}} : \text{LieAlg}_k \rightarrow \text{Comm Alg}_k, \quad \mathfrak{a} \mapsto \mathfrak{a}_{\mathfrak{g}}.$$

In particular, for a fixed Lie algebra \mathfrak{a} , the commutative algebra $\mathfrak{a}_{\mathfrak{g}}$ represents the functor¹⁴

$$\text{Rep}_{\mathfrak{g}}(\mathfrak{a}) : \text{Comm Alg}_k \rightarrow \text{Sets}, \quad B \mapsto \text{Hom}_{\text{LieAlg}_k}(\mathfrak{a}, \mathfrak{g}(B)).$$

By definition, $\mathfrak{a}_{\mathfrak{g}}$ is the coordinate ring of the affine scheme parametrizing the representations of \mathfrak{a} in \mathfrak{g} . For example, if \mathfrak{a} is the abelian two-dimensional Lie algebra over k , and \mathfrak{g} is reductive, then $\text{Rep}_{\mathfrak{g}}(\mathfrak{a})$ is the classical commuting scheme of the Lie algebra \mathfrak{g} .

In this section, we extend the functor $(-)_{\mathfrak{g}}$ to the category DGLA_k of DG Lie algebras and derive it using the natural model structure on DGLA_k . We then define the representation homology $H_{\bullet}(\mathfrak{a}, \mathfrak{g})$ as the homology of the corresponding derived functor.

6.2. Quillen equivalences for Lie (co)algebras

We begin with a brief review of the bar/cobar formalism in the Lie setting.

6.2.1. *Basic functors.* Recall that $\text{DGCA}_{k/k}$ denotes the category of commutative DG algebras augmented over k . Let DGLA_k denote the category of DG Lie algebras over k . Let $\text{DGCC}_{k/k}$ denote the category of DG cocommutative conilpotent coalgebras coaugmented over k . Similarly, let DGLC_k denote the category of conilpotent DG Lie coalgebras.

There is a pair of adjoint functors

$$\Omega_{\text{Comm}} : \text{DGCC}_{k/k} \rightleftarrows \text{DGLA}_k : \mathbf{B}_{\text{Lie}}, \tag{6.1}$$

where \mathbf{B}_{Lie} is defined by the classical Chevalley–Eilenberg complex of a DG Lie algebra (see Section 3.1) and Ω_{Comm} is the functor assigning to a cocommutative coalgebra C the free graded Lie algebra on the vector space $\bar{C}[-1]$. The differential on $\Omega_{\text{Comm}}(C)$ is given by $d_1 + d_2$, where d_1 is induced by the inner differential on C , and d_2 is the lift of the linear map

$$k[-1] \otimes \bar{C} \xrightarrow{\Delta_{-1} \otimes \Delta} k[-1] \otimes k[-1] \otimes \mathbf{Sym}^2(\bar{C}) \cong \wedge^2(\bar{C}[-1]),$$

where $\Delta_{-1} : k[-1] \rightarrow k[-1] \otimes k[-1]$ takes $1_{k[-1]}$ to $-1_{k[-1]} \otimes 1_{k[-1]}$, and Δ is the coproduct on \bar{C} .

Dually, there is a pair of adjoint functors

$$\Omega_{\text{Lie}} : \text{DGLC}_k \rightleftarrows \text{DGCA}_{k/k} : \mathbf{B}_{\text{Comm}}, \tag{6.2}$$

¹⁴ This functor in a more general operadic setting is briefly discussed in [G, Section 6].

where Ω_{Lie} is defined by the Chevalley–Eilenberg complex of a DG Lie coalgebra (see Section 3.1), and \mathbf{B}_{Comm} is the functor taking $R \in \text{DGCA}_{k/k}$ to the cofree DG Lie coalgebra $\mathfrak{L}(\bar{R}[1])$ equipped with (co)differential $d_1 + d_2$ where d_1 is induced by the differential on R and d_2 is determined by the linear map

$$\wedge^2(\bar{R}[1]) \cong k[1] \otimes k[1] \otimes \mathbf{Sym}^2(\bar{R}) \xrightarrow{\mu_1 \otimes \mu} k[1] \otimes \bar{R} \cong \bar{R}[1]$$

(Here, μ_1 identifies $1_{k[1]} \otimes 1_{k[1]}$ with $1_{k[1]}$ and $\mu : \mathbf{Sym}^2(\bar{R}) \rightarrow \bar{R}$ is induced by the multiplication map on R .)

Notation. If there is no danger of confusion, we will use the notation $\mathcal{C} := \mathbf{B}_{\text{Lie}}$ and $\mathcal{C}^c := \Omega_{\text{Lie}}$ for the Chevalley–Eilenberg functors on Lie algebras and Lie coalgebras, respectively.

6.2.2. *Model structures and Quillen theorems.* The following theorem collects basic facts about the model structures and Quillen equivalences for Lie (co)algebras: part (i) is well known (essentially due to Quillen [Q2]); for parts (ii) and (iii), see, for example, [Hi, Theorems 3.1 and 3.2] and [SW, Corollary 4.15].

- Theorem 6.1.** (i) *The categories $\text{DGCA}_{k/k}$ and DGLA_k have model structures where the weak equivalences are the quasi-isomorphisms and the fibrations are the degreewise surjective maps.*
- (ii) *The category DGLC_k (resp., $\text{DGCC}_{k/k}$) admits a model structure, where the weak equivalences are the maps f such that $\Omega_{\text{Comm}}(f)$ (resp., $\Omega_{\text{Lie}}(f)$) is a quasi-isomorphism and the cofibrations are degreewise monomorphisms.*
- (iii) *For the above model structures, the pairs of functors (6.1) and (6.2) are Quillen equivalences.*

Note that part (iii) says that the functors (6.1) and (6.2) induce derived equivalences

$$\begin{aligned} \mathbb{L}\Omega_{\text{Comm}} : \text{Ho}(\text{DGCC}_{k/k}) &\rightleftarrows \text{Ho}(\text{DGLA}_k) : \mathbb{R}\mathbf{B}_{\text{Lie}}, \\ \mathbb{L}\Omega_{\text{Lie}} : \text{Ho}(\text{DGLC}_k) &\rightleftarrows \text{Ho}(\text{DGCA}_{k/k}) : \mathbb{R}\mathbf{B}_{\text{Comm}}. \end{aligned}$$

6.2.3. *Relation to DG algebras and linear duality.* We now introduce the following functors on coalgebras: the coabelianization functor $(-)^{\text{ab}} : \text{DGC}_{k/k} \rightarrow \text{DGCC}_{k/k}$ assigning to a DG coalgebra C its maximal cocommutative DG subcoalgebra $C^{\text{ab}} \subseteq C$ (this functor is dual to the abelianization functor (2.6)); the universal coenveloping coalgebra functor $\mathcal{U}^c : \text{DGLC}_k \rightarrow \text{DGC}_{k/k}$ dual to the universal enveloping algebra functor $\mathcal{U} : \text{DGLA}_k \rightarrow \text{DGA}_{k/k}$; the Lie coalgebra functor $\mathcal{L}ie^c : \text{DGC}_{k/k} \rightarrow \text{DGLC}_k$ assigning to each C the coaugmentation coideal \bar{C} viewed as a DG Lie coalgebra (this functor is dual to the Lie algebra functor $\mathcal{L}ie : \text{DGA}_{k/k} \rightarrow \text{DGLA}_k$ assigning to $A \in \text{DGA}_{k/k}$ the augmentation ideal \bar{A} viewed as a DG Lie algebra). Their relationship to Quillen equivalences (6.1) and (6.2) is summarized by the following theorem.

Theorem 6.2. *In each of the following diagrams the square subdiagrams obtained by starting at any corner and mapping to the opposite corner commute (up to isomorphism).*

$$\begin{array}{ccc}
 \text{DGC}_{k/k} & \xrightleftharpoons[\mathbf{B}]{\Omega} & \text{DGA}_{k/k} \\
 \uparrow \text{in} & & \uparrow \mathcal{U} \\
 \text{DGC}_{k/k} & \xrightleftharpoons[\mathbf{B}_{\text{Lie}}]{\Omega_{\text{Comm}}} & \text{DGLA}_k \\
 \downarrow (-)^{\text{ab}} & & \downarrow \mathcal{L}ie
 \end{array}
 \qquad
 \begin{array}{ccc}
 \text{DGC}_{k/k} & \xrightleftharpoons[\mathbf{B}]{\Omega} & \text{DGA}_{k/k} \\
 \uparrow \mathcal{L}ie^c & & \uparrow (-)^{\text{ab}} \\
 \text{DGLC}_k & \xrightleftharpoons[\mathbf{B}_{\text{Comm}}]{\Omega_{\text{Lie}}} & \text{DGC}_{k/k} \\
 \downarrow \mathcal{U}^c & & \downarrow \text{in}
 \end{array}
 \tag{6.3}$$

Proof. Let $C \in \text{DGCC}_{k/k}$. Let $\mathcal{L}V$ denote the free Lie algebra generated by a graded vector space V . Since $T_k(V) \cong \mathcal{U}(\mathcal{L}V)$ as graded k -algebras, we have an isomorphism of graded algebras

$$\Omega(C) \cong \mathcal{U}[\Omega_{\text{Comm}}(C)].$$

The fact that this isomorphism commutes with differentials follows from the fact that the coalgebra C is cocommutative. Hence, on the category of cocommutative DG coalgebras, we have an isomorphism of functors

$$\Omega \cong \mathcal{U} \circ \Omega_{\text{Comm}}. \tag{6.4}$$

By adjunction, this gives an isomorphism

$$\mathbf{B}_{\text{Lie}} \circ \mathcal{L}ie \cong (-)^{\text{ab}} \circ \mathbf{B}, \tag{6.5}$$

which proves the result for the first diagram in (6.3). A similar argument shows that

$$(-)^{\text{ab}} \circ \Omega \cong \Omega_{\text{Lie}} \circ \mathcal{L}ie^c, \tag{6.6}$$

which by adjunction (together with (6.6)) gives an isomorphism of functors on commutative DG algebras

$$\mathbf{B} \cong \mathcal{U}^c \circ \mathbf{B}_{\text{Comm}}. \tag{6.7}$$

This establishes the desired result for the second diagram in (6.3). □

The following theorem, an immediate consequence of a simple generalization of [SW, Theorem 4.17], explains the canonical linear dualities relating Lie algebraic and coalgebraic Quillen functors.

Theorem 6.3. (i) *Let $\mathfrak{g} \in \text{DGLA}_k$ be finite-dimensional in each degree. There is a natural isomorphism*

$$\Omega_{\text{Lie}}(\mathfrak{g}^*) \cong \mathbf{B}_{\text{Lie}}(\mathfrak{g})^*. \tag{6.8}$$

(ii) *For $\mathfrak{G} \in \text{DGLC}_k$ finite-dimensional in each degree, we have a natural isomorphism*

$$\mathbf{B}_{\text{Lie}}(\mathfrak{G}^*) \cong \Omega_{\text{Lie}}(\mathfrak{G})^*. \tag{6.9}$$

Here, $(-)^*$ stands for the restricted dual on the right-hand side.

Note that the (full) graded k -linear dual of $\mathbf{B}_{\text{Lie}}(\mathfrak{g})$ is precisely the complex of Lie cochains of \mathfrak{g} with trivial coefficients. The restricted dual $\mathbf{B}_{\text{Lie}}(\mathfrak{g})^*$ is therefore the complex of continuous Lie cochains of \mathfrak{g} with coefficients in k , where k is given the discrete topology and $\mathcal{C}(\mathfrak{g}; k)$ is given the topology induced by the decreasing filtration $F^r\mathcal{C}(\mathfrak{g}; k) := \text{Sym}^{\geq r}(\mathfrak{g}[1])$. In particular, we have $H_\bullet[\mathbf{B}_{\text{Lie}}(\mathfrak{g})^*] = H^{-\bullet}(\mathfrak{g}; k)$, where H^\bullet denotes the *continuous* Lie algebra cohomology. Thus, the isomorphism (6.8) explicitly relates the homological and cohomological Chevalley–Eilenberg complexes. In several interesting cases, the complex of continuous Lie cochains of \mathfrak{g} coincides with the complex of Lie cochains of \mathfrak{g} for degree reasons. One such case is when \mathfrak{g} is concentrated in degree 0. Another is when \mathfrak{g} is *cohomologically graded* and 2-connected (i.e., concentrated in cohomological degrees ≥ 2). In addition, Theorems 6.2 and 6.3 hold in the bigraded setting in which case $(-)^*$ means the bigraded dual. It is natural to regard $\mathbf{B}_{\text{Lie}}(\mathfrak{g})^*$ (rather than the full homologically graded k -linear dual of $\mathbf{B}_{\text{Lie}}(\mathfrak{g})$) as the complex of Lie cochains of \mathfrak{g} in the bigraded setting.

6.3. Derived representation schemes

6.3.1. *Convolution Lie algebras.* For a fixed $\mathfrak{G} \in \text{DGLC}_k$ and $A \in \text{DGCA}_k$, we define a Lie bracket on $\mathbf{Hom}(\mathfrak{G}, A)$ by

$$[f, g] := m \circ (f \otimes g) \circ]-[$$

where $m : A \otimes A \rightarrow A$ is the multiplication map on A , and $]-[: \mathfrak{G} \rightarrow \mathfrak{G} \otimes \mathfrak{G}$ is the Lie cobracket on \mathfrak{G} . For $\mathfrak{G} \in \text{DGLC}_k$ fixed, this gives a functor

$$\mathbf{Hom}(\mathfrak{G}, -) : \text{DGCA}_k \rightarrow \text{DGLA}_k, \tag{6.10}$$

which we call a *convolution functor*.

6.3.2. *The left adjoint functor.* For arbitrary elements ξ, η in a DG Lie algebra \mathfrak{a} and for $x \in \mathfrak{G}$, let $(]x[, \xi, \eta)$ denote the image of $x \otimes \xi \otimes \eta$ under the composite map

$$\mathfrak{G} \otimes \mathfrak{a} \otimes \mathfrak{a} \xrightarrow{]x[} \mathfrak{G} \otimes \mathfrak{G} \otimes \mathfrak{a} \otimes \mathfrak{a} \xrightarrow{\cong} (\mathfrak{G} \otimes \mathfrak{a})^{\otimes 2} \twoheadrightarrow \mathbf{Sym}^2(\mathfrak{G} \otimes \mathfrak{a}) \twoheadrightarrow \mathbf{Sym}_k(\mathfrak{G} \otimes \mathfrak{a}) \tag{6.11}$$

The following proposition describes the left adjoint for the Lie convolution algebra functor $\mathbf{Hom}(\mathfrak{G}, -)$.

Proposition 6.4. *The functor (6.10) has a left adjoint $(-)_\mathfrak{G} : \text{DGLA}_k \rightarrow \text{DGCA}_k$, which is given by*

$$\mathfrak{a} \mapsto \mathfrak{a}_\mathfrak{G} := \mathbf{Sym}_k(\mathfrak{G} \otimes \mathfrak{a}) / \langle x \otimes [\xi, \eta] - (]x[, \xi, \eta) \rangle,$$

where $(]x[, \xi, \eta)$ is defined by (6.11).

Proof. For $B \in \text{DGCA}_k$ consider natural maps

$$\begin{array}{ccc}
 \text{Hom}_{\text{DGLA}_k}(\mathfrak{a}, \mathbf{Hom}(\mathfrak{G}, B)) & \hookrightarrow & \text{Hom}_{\text{Com}_k}(\mathfrak{a}, \mathbf{Hom}(\mathfrak{G}, B)) \\
 \downarrow \simeq & & \downarrow = \\
 & & \text{Hom}_{\text{Com}_k}(\mathfrak{G} \otimes \mathfrak{a}, B) \\
 \downarrow \simeq & & \downarrow \simeq \\
 \text{Hom}_{\text{DGCA}_k}(\mathfrak{a}_{\mathfrak{G}}, B) & \hookrightarrow & \text{Hom}_{\text{DGCA}_k}(\mathbf{Sym}_k(\mathfrak{G} \otimes \mathfrak{a}), B)
 \end{array}$$

The map

$$\text{Hom}_{\text{DGA}_k}(\mathfrak{a}, \mathbf{Hom}(\mathfrak{G}, B)) \hookrightarrow \text{Hom}_{\text{DGCA}_k}(\mathbf{Sym}_k(\mathfrak{G} \otimes \mathfrak{a}), B) \tag{6.12}$$

obtained by following the upper right part of the above diagram is explicitly given by

$$(f : \mathfrak{a} \rightarrow \mathbf{Hom}(\mathfrak{G}, B)) \mapsto [\hat{f} : \mathbf{Sym}_k(\mathfrak{G} \otimes_k \mathfrak{a}) \rightarrow B, x \otimes \xi \mapsto (-1)^{|x||\xi|} f(\xi)(x)].$$

Further, by a straightforward calculation, f is a DG Lie algebra homomorphism iff $f([\xi, \eta]) = [f(\xi), f(\eta)]$ iff for all $\xi, \eta \in \mathfrak{a}$ and $x \in \mathfrak{G}$, $\hat{f}(x \otimes [\xi, \eta]) - ([x, \xi], \eta) = 0$. This shows that the image of the map (6.12) is precisely $\text{Hom}_{\text{DGCA}_k}(\mathfrak{a}_{\mathfrak{G}}, B)$. This proves the desired proposition. \square

As a consequence of Proposition 6.4, we obtain:

Theorem 6.5. *The pair of functors $(-)_{\mathfrak{G}} : \text{DGLA}_k \rightleftarrows \text{DGCA}_k : \mathbf{Hom}(\mathfrak{G}, -)$ is a Quillen pair. As a result, the functor $(-)_{\mathfrak{G}}$ has a (total) left derived functor*

$$\mathbb{L}(-)_{\mathfrak{G}} : \text{Ho}(\text{DGLA}_k) \rightarrow \text{Ho}(\text{DGCA}_k), \quad \mathfrak{a} \mapsto (Q\mathfrak{a})_{\mathfrak{G}},$$

where $Q\mathfrak{a} \xrightarrow{\sim} \mathfrak{a}$ is any cofibrant resolution in DGLA_k .

Proof. By Proposition 6.4 and [DS, Remark 9.8], it suffices to check that $\mathbf{Hom}(\mathfrak{G}, -)$ preserves degreewise surjections and quasi-isomorphisms. This is obvious. \square

The functor $\mathbf{Hom}(\mathfrak{G}, -)$ can be modified naturally to give a functor on *augmented* commutative DG algebras

$$\mathbf{Hom}(\mathfrak{G}, -) : \text{DGCA}_{k/k} \rightarrow \text{DGLA}_k, \quad A \mapsto \mathbf{Hom}(\mathfrak{G}, \bar{A}).$$

The left adjoint $(-)_{\mathfrak{G}}$ of $\mathbf{Hom}(\mathfrak{G}, -) : \text{DGCA}_{k/k} \rightarrow \text{DGLA}_k$ is the functor assigning to each $\mathfrak{a} \in \text{DGLA}_k$ the commutative DG algebra $\mathfrak{a}_{\mathfrak{G}}$ equipped with the canonical augmentation

$$\varepsilon : \mathfrak{a}_{\mathfrak{G}} \rightarrow k$$

corresponding to $0 \in \text{Hom}_{\text{DGLA}_k}(\mathfrak{a}, \mathbf{Hom}(\mathfrak{G}, k))$ under the adjunction (6.13). As in Theorem 6.5, it is easy to verify that the pair of functors $(-)_{\mathfrak{G}} : \text{DGLA}_k \rightleftarrows \text{DGCA}_{k/k} : \mathbf{Hom}(\mathfrak{G}, -)$ is Quillen. Hence, $(-)_{\mathfrak{G}}$ has a left derived functor

$$\mathbb{L}(-)_{\mathfrak{G}} : \text{Ho}(\text{DGLA}_k) \rightarrow \text{Ho}(\text{DGCA}_{k/k}),$$

which after applying the forgetful functor $\text{Ho}(\text{DGCA}_{k/k}) \rightarrow \text{Ho}(\text{DGCA}_k)$ coincides with $\mathbb{L}(-)_{\mathfrak{G}} : \text{Ho}(\text{DGLA}_k) \rightarrow \text{Ho}(\text{DGCA}_k)$.

6.3.3. *Derived representation schemes.* Let \mathfrak{g} be a finite-dimensional Lie algebra over k . Let $\mathfrak{G} := \mathfrak{g}^*$, the dual Lie coalgebra. Then

$$\mathbf{Hom}(\mathfrak{G}, A) = \mathbf{Hom}(\mathfrak{g}^*, A) \cong \mathfrak{g} \otimes A =: \mathfrak{g}(A).$$

Thus, the commutative DG algebra $\mathfrak{a}_{\mathfrak{g}} := \mathfrak{a}_{\mathfrak{G}}$ represents the functor

$$\mathrm{Rep}_{\mathfrak{g}}(\mathfrak{a}) : \mathrm{DGCA}_k \rightarrow \mathbf{Sets}, \quad A \mapsto \mathrm{Hom}_{\mathrm{DGLA}_k}(\mathfrak{a}, \mathfrak{g}(A)),$$

that is, there is a natural isomorphism of sets

$$\mathrm{Hom}_{\mathrm{DGCA}_k}(\mathfrak{a}_{\mathfrak{g}}, A) \cong \mathrm{Hom}_{\mathrm{DGLA}_k}(\mathfrak{a}, \mathfrak{g}(A)). \tag{6.13}$$

As in the associative case, we now define

$$\mathrm{DRep}_{\mathfrak{g}}(-) := \mathbb{L}(-)_{\mathfrak{g}} : \mathrm{Ho}(\mathrm{DGLA}_k) \rightarrow \mathrm{Ho}(\mathrm{DGCA}_k).$$

We call $\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})$ the *derived representation scheme parametrizing representations of \mathfrak{a} in \mathfrak{g}* . Further, if G is a Lie group whose Lie algebra is \mathfrak{g} , G acts (via the adjunction (6.13)) on $\mathfrak{a}_{\mathfrak{g}}$ by automorphisms for any $\mathfrak{a} \in \mathrm{DGLA}_k$. One can therefore form the subfunctor

$$(-)_{\mathfrak{g}}^G : \mathrm{DGLA}_k \rightarrow \mathrm{DGCA}_k, \quad \mathfrak{a} \mapsto \mathfrak{a}_{\mathfrak{g}}^G,$$

of $(-)_{\mathfrak{g}}$. An argument using Brown’s lemma similar to the proof of [BKR, Theorem 2.6] shows that the functor $(-)_{\mathfrak{g}}^G$ has a total left derived functor

$$\mathrm{DRep}_{\mathfrak{g}}(-)^G := \mathbb{L}(-)_{\mathfrak{g}}^G : \mathrm{Ho}(\mathrm{DGLA}_k) \rightarrow \mathrm{Ho}(\mathrm{DGCA}_k).$$

We define the representation homologies

$$H_{\bullet}(\mathfrak{a}, \mathfrak{g}) := H_{\bullet}(\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})), \quad H_{\bullet}(\mathfrak{a}, \mathfrak{g})^G := H_{\bullet}(\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})^G).$$

More generally, \mathfrak{g} acts (via the adjunction (6.13)) on $\mathfrak{a}_{\mathfrak{g}}$ by derivations. One can therefore form the functor

$$(-)_{\mathfrak{g}}^{\mathrm{ad} \mathfrak{g}} : \mathrm{DGLA}_k \rightarrow \mathrm{DGCA}_k, \quad \mathfrak{a} \mapsto \mathfrak{a}_{\mathfrak{g}}^{\mathrm{ad} \mathfrak{g}}.$$

Again, an argument paralleling the proof of [BKR, Theorem 2.6] shows that the functor $(-)_{\mathfrak{g}}^{\mathrm{ad} \mathfrak{g}}$ has a total left derived functor

$$\mathrm{DRep}_{\mathfrak{g}}(-)^{\mathrm{ad} \mathfrak{g}} := \mathbb{L}(-)_{\mathfrak{g}}^{\mathrm{ad} \mathfrak{g}} : \mathrm{Ho}(\mathrm{DGLA}_k) \rightarrow \mathrm{Ho}(\mathrm{DGCA}_k).$$

We define

$$H_{\bullet}(\mathfrak{a}, \mathfrak{g})^{\mathrm{ad} \mathfrak{g}} := H_{\bullet}[\mathrm{DRep}_{\mathfrak{g}}^{\mathrm{ad} \mathfrak{g}}(\mathfrak{a})].$$

Note that if \mathfrak{g} is the Lie algebra of a reductive Lie group G , the functors $(-)_{\mathfrak{g}}^G$ and $(-)_{\mathfrak{g}}^{\mathrm{ad} \mathfrak{g}}$ coincide. Hence, in this situation, their derived functors coincide as well.

The discussion in Section 6.3.2 points out that $\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})$ can be viewed as an object in $\mathrm{Ho}(\mathrm{DGCA}_{k/k})$ (rather than $\mathrm{Ho}(\mathrm{DGCA}_k)$). In the same way, $\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})^{\mathrm{ad} \mathfrak{g}}$ can be viewed as an object in $\mathrm{Ho}(\mathrm{DGCA}_{k/k})$. Similarly, if G is a Lie group whose Lie algebra is \mathfrak{g} , one can consider $\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})^G$ as an object in $\mathrm{Ho}(\mathrm{DGCA}_{k/k})$. In this case, $\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})^G \cong \mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})^{\mathrm{ad} \mathfrak{g}}$.

6.3.4. *Representation homology and Lie cohomology.* The following proposition leads to the main result of this section (Theorem 6.7).

Proposition 6.6. *For any $\mathfrak{G} \in \text{DGLC}_k$, the following diagram commutes (up to isomorphism of functors):*

$$\begin{array}{ccc}
 \text{DGCC}_{k/k} & \xrightleftharpoons[\mathbf{B}_{\text{Lie}}]{\Omega_{\text{Comm}}} & \text{DGLA}_k \\
 \mathfrak{G} \bar{\otimes} \downarrow & & \uparrow \mathbf{Hom}(\mathfrak{G}, -) \\
 \text{DGLC}_k & \xrightleftharpoons[\mathbf{B}_{\text{Comm}}]{\Omega_{\text{Lie}}} & \text{DGCA}_{k/k}
 \end{array}$$

where $\mathfrak{G} \bar{\otimes} C := \mathfrak{G} \otimes \bar{C}$ for any $C \in \text{DGCC}_{k/k}$. Thus, there is an isomorphism of functors

$$(-)_{\mathfrak{G}} \circ \Omega_{\text{Comm}} \cong \Omega_{\text{Lie}} \circ (\mathfrak{G} \bar{\otimes} -) = \mathcal{C}^c \circ (\mathfrak{G} \bar{\otimes} -). \tag{6.14}$$

Proof. For any $C \in \text{DGCC}_{k/k}$ and $\mathfrak{L} \in \text{DGLA}_k$, let $\text{Tw}(C, \mathfrak{L})$ denote the set of Maurer–Cartan elements (i.e., elements satisfying $d\alpha + \frac{1}{2}[\alpha, \alpha] = 0$) in the DG Lie algebra $\mathbf{Hom}(\bar{C}, \mathfrak{L})$. Similarly, for $\mathfrak{L}^c \in \text{DGLC}_k$ and $A \in \text{DGA}_{k/k}$, $\text{Tw}(\mathfrak{L}^c, A)$ will denote the set of Maurer–Cartan elements in the DG Lie algebra $\mathbf{Hom}(\mathfrak{L}^c, \bar{A})$. Now, for any $A \in \text{DGCA}_{k/k}$ and $C \in \text{DGCC}_{k/k}$, we have

$$\begin{aligned}
 \text{Hom}_{\text{DGCA}_{k/k}}(\Omega_{\text{Comm}}(C)_{\mathfrak{G}}, A) &\cong \text{Hom}_{\text{DGLA}_k}(\Omega_{\text{Comm}}(C), \mathbf{Hom}(\mathfrak{G}, \bar{A})) \\
 &\cong \text{Tw}(C, \mathbf{Hom}(\mathfrak{G}, \bar{A})) \cong \text{Tw}(\mathfrak{G} \bar{\otimes} C, A) \\
 &\cong \text{Hom}_{\text{DGCA}_{k/k}}(\mathcal{C}^c(\mathfrak{G} \bar{\otimes} C), A).
 \end{aligned}$$

The first isomorphism above is Proposition 6.4, the second is from [Hi], the third is because the DG Lie algebras $\mathbf{Hom}(\bar{C}, \mathbf{Hom}(\mathfrak{G}, A))$ and $\mathbf{Hom}(\mathfrak{G} \bar{\otimes} C, \bar{A})$ are isomorphic by the standard Hom-tensor duality, and the fourth is from arguments similar to those in [Hi] proving the second. (6.14) follows from this by Yoneda’s Lemma. The rest of the proposition follows from (6.14) by adjunction. \square

Theorem 6.7. (a) *Suppose that $\mathfrak{G} = \mathfrak{g}^*$ for some Lie algebra \mathfrak{g} . If $\Omega_{\text{Comm}}(C) \xrightarrow{\sim} \mathfrak{a}$ is a quasi-isomorphism for some $C \in \text{DGCC}_{k/k}$, then*

$$\text{DRep}_{\mathfrak{g}}(\mathfrak{a}) \cong \mathcal{C}^c(\mathfrak{g}^*(\bar{C}); k), \quad \text{DRep}_{\mathfrak{g}}(\mathfrak{a})^{\text{ad } \mathfrak{g}} \cong \mathcal{C}^c(\mathfrak{g}^*(C), \mathfrak{g}^*; k).$$

In particular,

$$\mathbf{H}_{\bullet}(\mathfrak{a}, \mathfrak{g}) \cong \mathbf{H}_{\bullet}(\mathfrak{g}^*(\bar{C}); k), \quad \mathbf{H}_{\bullet}(\mathfrak{a}, \mathfrak{g})^{\text{ad } \mathfrak{g}} \cong \mathbf{H}_{\bullet}(\mathfrak{g}^*(C), \mathfrak{g}^*; k).$$

(b) *Suppose, in addition, that \mathfrak{g} is finite-dimensional and C is either finite-dimensional in each degree or bigraded. Let $A = C^*$ (with bigraded duals being taken in the bigraded setting). Then*

$$\text{DRep}_{\mathfrak{g}}(\mathfrak{a}) \cong \mathcal{C}(\mathfrak{g}(\bar{A}); k)^*, \quad \text{DRep}_{\mathfrak{g}}(\mathfrak{a})^{\text{ad } \mathfrak{g}} \cong \mathcal{C}(\mathfrak{g}(A), \mathfrak{g}; k)^*.$$

Here, $(-)^*$ stands for continuous or bigraded dual as the case might be.¹⁵ In particular,

$$H_{\bullet}(\mathfrak{a}, \mathfrak{g}) \cong H^{-\bullet}(\mathfrak{g}(\bar{A}); k), \quad H_{\bullet}(\mathfrak{a}, \mathfrak{g})^{\text{ad } \mathfrak{g}} \cong H^{-\bullet}(\mathfrak{g}(A), \mathfrak{g}; k).$$

Here, “Lie cohomology” means “continuous Lie cohomology” or “Lie cohomology in the bigraded setting” as the case might be.

Proof. That $\text{DRep}_{\mathfrak{g}}(\mathfrak{a}) \cong \mathcal{C}^c(\mathfrak{g}^*(\bar{C}); k)$ is immediate from Proposition 6.6. The isomorphism $\text{DRep}_{\mathfrak{g}}(\mathfrak{a})^{\text{ad } \mathfrak{g}} \cong \mathcal{C}^c(\mathfrak{g}^*(C), \mathfrak{g}^*; k)$ then follows from the fact that $\mathcal{C}^c(\mathfrak{g}^*(C), \mathfrak{g}^*; k) = \mathcal{C}^c(\mathfrak{g}^*(\bar{C}); k)^{\text{ad } \mathfrak{g}}$. This proves (a). Part (b) follows from (a) and (6.8). \square

Remark. Recall that, by (6.4), $\Omega(C) \cong \mathcal{U}(\Omega_{\text{Comm}}(C))$ for any $C \in \text{DGCC}_{k/k}$.

Now, take $\mathfrak{g} = \mathfrak{gl}_n(k)$, $\mathfrak{G} = \mathfrak{g}^* \in \text{DGLC}_k$ and $M = \mathcal{M}_n(k)^* \in \text{DGC}_k$. The following proposition clarifies the relation between derived representation schemes of Lie algebras and their associative counterparts.

Proposition 6.8. *Let $C \in \text{DGCC}_{k/k}$. Then there is a natural isomorphism of commutative DG algebras*

$$\Omega(C)_n \cong \Omega_{\text{Comm}}(C)_{\mathfrak{gl}_n}. \tag{6.15}$$

Hence, for any DG Lie algebra \mathfrak{a} ,

$$\text{DRep}_n(\mathcal{U}(\mathfrak{a})) \cong \text{DRep}_{\mathfrak{gl}_n}(\mathfrak{a}). \tag{6.16}$$

Proof. For any $B \in \text{DGCA}_k$, we have

$$\begin{aligned} \text{Hom}_{\text{DGCA}_k}(\Omega(C)_n, B) &\cong \text{Hom}_{\text{DGA}_k}(\Omega(C), \mathcal{M}_n(B)) \\ &\cong \text{Hom}_{\text{DGA}_k}(\mathcal{U}(\Omega_{\text{Comm}}(C)), \mathcal{M}_n(B)) \\ &\cong \text{Hom}_{\text{DGLA}_k}(\Omega_{\text{Comm}}(C), \mathfrak{gl}_n(B)) \\ &\cong \text{Hom}_{\text{DGCA}_k}(\Omega_{\text{Comm}}(C)_{\mathfrak{gl}_n}, B). \end{aligned}$$

The isomorphism (6.15) now follows from Yoneda’s Lemma. To prove (6.16), note that for any $\mathfrak{a} \in \text{DGLA}_k$, one can find $C \in \text{DGCC}_{k/k}$ such that $\Omega_{\text{Comm}}(C) \rightarrow \mathfrak{a}$ is a cofibrant resolution in DGLA_k (for example, $C = \mathbf{B}_{\text{Lie}}(\mathfrak{a})$). Then $\Omega(C) \cong \mathcal{U}(\Omega_{\text{Comm}}(C)) \rightarrow \mathcal{U}(\mathfrak{a})$ is a cofibrant resolution of $\mathcal{U}(\mathfrak{a})$ in DGA_k . Hence,

$$\text{DRep}_n(\mathcal{U}(\mathfrak{a})) \cong \Omega(C)_n \cong \Omega_{\text{Comm}}(C)_{\mathfrak{gl}_n} \cong \text{DRep}_{\mathfrak{gl}_n}(\mathfrak{a}). \quad \square$$

7. Derived Harish-Chandra homomorphism and Drinfeld traces

The aim of this section is to construct the derived Harish-Chandra homomorphism and trace maps for representation schemes of Lie algebras. Our starting point is the observation of Section 4.2 interpreting the derived HC homomorphism for associative algebras in terms of Chevalley–Eilenberg complexes.

¹⁵ See remarks after Theorem 6.3.

Throughout this section, \mathfrak{g} will denote a finite-dimensional reductive Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ its Cartan subalgebra and W the Weyl group of \mathfrak{g} . Recall that in this case, if V is any k -vector space with an action of the reductive group G whose Lie algebra is \mathfrak{g} , then $V^G \cong V^{\text{ad } \mathfrak{g}}$.

7.1. *The derived Harish-Chandra homomorphism*

The natural inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ defines a homomorphism of Lie coalgebras $\mathfrak{g}^* \rightarrow \mathfrak{h}^*$ and hence, for any $C \in \text{DGCC}_{k/k}$, a morphism of commutative DG algebras

$$\Phi_{\mathfrak{g}}(C) : \mathcal{C}^c(\mathfrak{g}^*(C), \mathfrak{g}^*; k) \rightarrow \mathcal{C}^c(\mathfrak{h}^*(C), \mathfrak{h}^*; k). \tag{7.1}$$

The following proposition is a generalization of Lemma 4.1.

Proposition 7.1. *The image of $\Phi_{\mathfrak{g}}(C)$ lies in the DG subalgebra $\mathcal{C}^c(\mathfrak{h}^*(C), \mathfrak{h}^*; k)^W$ of chains that are invariant under the action of the Weyl group W of \mathfrak{g} .*

Proof. As \mathfrak{h} is an abelian Lie algebra, we have an isomorphism $\mathcal{C}^c(\mathfrak{h}^*(C), \mathfrak{h}^*(k); k) \cong \mathcal{C}^c(\mathfrak{h}^*(\bar{C}); k)$. The map $\Phi_{\mathfrak{g}}(C)$ is thus the restriction of the natural map $\mathcal{C}^c(\mathfrak{g}^*(C); k) \rightarrow \mathcal{C}^c(\mathfrak{h}^*(\bar{C}); k)$ to $\mathcal{C}^c(\mathfrak{g}^*(C), \mathfrak{g}^*; k) \cong \mathcal{C}^c(\mathfrak{g}^*(\bar{C}); k)^{\text{ad } \mathfrak{g}} \cong \mathcal{C}^c(\mathfrak{g}^*(\bar{C}); k)^G$, where G is the Lie group attached to \mathfrak{g} . Now, let N denote the normalizer of \mathfrak{h} in G , so that there is a surjective group homomorphism $N \rightarrow W$. Since W acts naturally on \mathfrak{h}^* , so does N . Thus, N acts on $\mathfrak{h}^*(\bar{C})$ as well, making $\mathfrak{h}^*(\bar{C})$ a DG-Lie coalgebra with N -action. This, in turn, induces an N -action on the commutative DG algebra $\mathcal{C}^c(\mathfrak{h}^*(\bar{C}); k)$. On the other hand, the adjoint action of G on \mathfrak{g} makes $\mathfrak{g}^*(\bar{C})$ a DG Lie coalgebra with G -action (and hence N -action). Thus, the commutative DG algebra $\mathcal{C}^c(\mathfrak{g}^*(\bar{C}); k)$ acquires a G -action (and hence an N -action). Since the map $\mathfrak{g}^* \rightarrow \mathfrak{h}^*$ is N -equivariant, the map $\mathfrak{g}^*(\bar{C}) \rightarrow \mathfrak{h}^*(\bar{C})$ is N -equivariant as well. Therefore, the map $\mathcal{C}^c(\mathfrak{g}^*(\bar{C}); k) \rightarrow \mathcal{C}^c(\mathfrak{h}^*(\bar{C}); k)$ is N -equivariant. Since any element of $\mathcal{C}^c(\mathfrak{g}^*(C), \mathfrak{g}^*(k); k)$ is G -invariant (and hence N -invariant), any element in the image of $\Phi_{\mathfrak{g}}(C)$ is N -invariant (and hence W -invariant). □

Thus, we have a morphism of commutative DG algebras

$$\Phi_{\mathfrak{g}}(C) : \mathcal{C}^c(\mathfrak{g}^*(C), \mathfrak{g}^*; k) \rightarrow \mathcal{C}^c(\mathfrak{h}^*(C), \mathfrak{h}^*; k)^W,$$

which we call the *derived Harish-Chandra homomorphism*. Suppose that there exists a quasi-isomorphism $\Omega_{\text{Comm}}(C) \xrightarrow{\sim} \mathfrak{a}$ for some Lie algebra \mathfrak{a} . By Theorem 6.7, the derived Harish-Chandra homomorphism can be viewed as a map in $\text{Ho}(\text{DGCA}_{k/k})$

$$\Phi_{\mathfrak{g}}(C) : \text{DRep}_{\mathfrak{g}}(\mathfrak{a})^G \rightarrow \mathcal{C}^c(\mathfrak{h}^*(C), \mathfrak{h}^*; k)^W.$$

It follows that the derived Harish-Chandra homomorphism induces the map

$$H_{\bullet}(\Phi_{\mathfrak{g}}) : H_{\bullet}(\mathfrak{a}, \mathfrak{g})^G \rightarrow H_{\bullet}(\mathfrak{h}^*(C), \mathfrak{h}^*; k)^W.$$

If C has zero differential, then $\mathcal{C}^c(\mathfrak{h}^*(C), \mathfrak{h}^*; k)$ also has zero differential, and hence in this case, $H_{\bullet}(\Phi_{\mathfrak{g}})$ maps $H_{\bullet}(\mathfrak{a}, \mathfrak{g})^G$ to $\text{Sym}_k(\mathfrak{h}^* \otimes \bar{C}[-1])^W$.

Example 7.2. Let $k = \mathbb{C}$ and let $\mathfrak{a} := \mathbb{C}\langle x \rangle$ be the one-dimensional Lie algebra with x having weight 1 and homological degree 0. The cocommutative coalgebra $C := \mathbf{Sym}^c(\mathfrak{a}[1]) = \wedge^c(sx)$ is Koszul dual to \mathfrak{a} . In this case, $\mathfrak{g}^*(C) \cong \mathfrak{g}^*(sx) \oplus \mathfrak{g}^*$ and $C^c(\mathfrak{g}^*(C), \mathfrak{g}^*; \mathbb{C}) = \mathbf{Sym}(\mathfrak{g}^*)^G$. In this case, the derived Harish-Chandra homomorphism becomes the Chevalley isomorphism

$$\mathbf{Sym}(\mathfrak{g}^*)^G \xrightarrow{\sim} \mathbf{Sym}(\mathfrak{h}^*)^W.$$

7.2. Drinfeld homology

We now proceed with constructing the analogues of the trace maps (2.18) in the case of Lie algebras. As the first step, we introduce an appropriate version of cyclic homology for Lie algebras that will relate to representation homology via the trace maps.

7.2.1. Drinfeld [Dr] introduced the functor

$$\lambda : \mathbf{DGLA}_k \rightarrow \mathbf{Com}_k, \quad \mathfrak{a} \mapsto \mathbf{Sym}^2(\mathfrak{a}) / \langle [x, y] \cdot z - x \cdot [y, z] : x, y, z \in \mathfrak{a} \rangle,$$

that assigns to a Lie algebra \mathfrak{a} (the target of) the universal invariant bilinear form on \mathfrak{a} . As shown in [GK], this functor plays a role of the cyclic functor (2.14) on the category of Lie algebras: its left derived $\mathfrak{L}\lambda$ exists and defines the analogue of cyclic homology for Lie algebras (cf. [GK, Theorem (5.3)]).

More generally, extending Drinfeld’s construction, for an integer $d \geq 1$, we define

$$\lambda^{(d)} : \mathbf{DGLA}_k \rightarrow \mathbf{Com}_k, \quad \mathfrak{a} \mapsto \mathbf{Sym}^d(\mathfrak{a}) / [\mathfrak{a}, \mathbf{Sym}^d(\mathfrak{a})].$$

This functor assigns to a Lie algebra \mathfrak{a} (the target of) the universal invariant multilinear form of degree d on \mathfrak{a} ; in particular, for $d = 2$, we have $\lambda^{(2)} = \lambda$.

Note that the symmetric invariant d -multilinear forms $\mathfrak{g} \times \cdots \times \mathfrak{g} \rightarrow k$ are in one-to-one correspondence with linear maps $\lambda^{(d)}(\mathfrak{g}) \rightarrow k$. To be precise, a nondegenerate pairing

$$\mathbf{Sym}^d(\mathfrak{g}) \times \mathbf{Sym}^d(\mathfrak{g}^*) \rightarrow k$$

induces a nondegenerate pairing

$$\lambda^{(d)}(\mathfrak{g}) \times \mathbf{Sym}^d(\mathfrak{g}^*)^{\mathrm{ad} \mathfrak{g}} \rightarrow k.$$

The next theorem generalizes the result of [GK, Theorem (5.3)] in the case of the Lie operad.

Theorem 7.3. For each $d \geq 1$, the functor $\lambda^{(d)}$ has a (total) left derived functor given by

$$\mathbb{L}\lambda^{(d)} : \mathbf{Ho}(\mathbf{DGLA}_k) \rightarrow \mathbf{Ho}(\mathbf{Com}_k), \quad \mathfrak{a} \mapsto \lambda^{(d)}(\mathfrak{L}),$$

where $\mathfrak{L} \xrightarrow{\sim} \mathfrak{a}$ is a cofibrant resolution of \mathfrak{a} in \mathbf{DGLA}_k .

Proof. Suppose that $\mathfrak{L} \in \mathbf{DGLA}_k$ is cofibrant and that $f, g : \mathfrak{L} \rightarrow \mathfrak{a}$ are homotopic. Then, by [BRZ, Lemma 2.1], there exists $h : \mathfrak{L} \rightarrow \mathfrak{a} \otimes k[t, dt]$ such that $h(0) = f$ and $h(1) = g$. Here, $\deg(t) = 0$ and $d(t) := dt$ and $h(a)$ denotes postcomposition of h with the map $\mathrm{Id}_{\mathfrak{a}} \otimes \mathrm{ev}_a$ where $\mathrm{ev}_a : k[t, dt] \rightarrow k$ is the map taking t to a and dt to 0 for any $a \in k$.

Note that for any $B \in \text{DGCA}_k$, one has natural maps in Com_k

$$\lambda^{(d)}(\mathfrak{a} \otimes B) \rightarrow \lambda^{(d)}(\mathfrak{a}) \otimes B. \tag{7.2}$$

One therefore has a map $H : \lambda^{(d)}(\mathfrak{L}) \rightarrow \lambda^{(d)}(\mathfrak{a}) \otimes k[t, dt]$ given by the composition

$$\lambda^{(d)}(\mathfrak{L}) \xrightarrow{\lambda^{(d)}(h)} \lambda^{(d)}(\mathfrak{a} \otimes k[t, dt]) \xrightarrow{(7.2)} \lambda^{(d)}(\mathfrak{a}) \otimes k[t, dt].$$

Clearly, $H(0) = \lambda^{(d)}(f)$ and $H(1) = \lambda^{(d)}(g)$. It follows that the maps $\lambda^{(d)}(f)$ and $\lambda^{(d)}(g)$ are homotopic in Com_k .

Now, if $f : \mathfrak{L} \rightarrow \mathfrak{L}'$ is a weak equivalence between cofibrant objects in DGLA_k , there exists a $g : \mathfrak{L}' \rightarrow \mathfrak{L}$ in DGLA_k such that fg and gf are homotopic to the respective identities. It follows that $\lambda^{(d)}(fg)$ and $\lambda^{(d)}(gf)$ are homotopic to the respective identities as well. Hence, $\lambda^{(d)}(f)$ is a quasi-isomorphism. In other words, the functors $\lambda^{(d)}$ take weak equivalences between cofibrant objects to weak equivalences. The desired theorem now follows from [DS, Proposition 9.3]. \square

We let $\text{HC}_\bullet^{(d)}(\text{Lie}, \mathfrak{a})$ denote the homology $H_\bullet[\mathbb{L}\lambda^{(d)}(\mathfrak{a})]$ and refer to it as *Drinfeld homology*. Note that $\lambda^{(1)}$ is just the abelianization functor: $\mathfrak{a} \mapsto \mathfrak{a}/[\mathfrak{a}, \mathfrak{a}]$, and hence $\text{HC}_\bullet^{(1)}(\text{Lie}, \mathfrak{a}) \cong H_{\bullet+1}(\mathfrak{a}; k)$ for any Lie algebra \mathfrak{a} (see, e.g., [BFR, Example 1, Sect. 2.6]). For $d = 2$, $\text{HC}_\bullet^{(2)}(\text{Lie}, \mathfrak{a})$ is precisely the Lie cyclic homology introduced by Getzler and Kapranov [GK] (and denoted $\text{HA}_\bullet(\text{Lie}, \mathfrak{a})$ in that paper). In general, the meaning of the homology groups $\text{HC}_\bullet^{(d)}(\text{Lie}, \mathfrak{a})$ is clarified by the following theorem which is one of the main results of this section.

Theorem 7.4. *Let $\mathfrak{a} \in \text{DGLA}_k$. The reduced cyclic homology of the universal enveloping algebra $\mathcal{U}(\mathfrak{a})$ of the Lie algebra \mathfrak{a} has a natural Hodge-type decomposition*

$$\overline{\text{HC}}_\bullet[\mathcal{U}(\mathfrak{a})] \cong \bigoplus_{d=1}^{\infty} \text{HC}_\bullet^{(d)}(\text{Lie}, \mathfrak{a}). \tag{7.3}$$

Proof. Let $C \in \text{DGCC}_{k/k}$ be a coalgebra Koszul dual to the Lie algebra \mathfrak{a} (for example, $C = \mathbf{B}_{\text{Lie}}(\mathfrak{a})$). Then we have a cofibrant resolution $\mathbf{\Omega}_{\text{Comm}}(C) \xrightarrow{\sim} \mathfrak{a}$ in DGLA_k . For a graded k -vector space V , there are natural isomorphisms of graded vector spaces

$$T_k(V)_{\natural} \cong T_k(V)/(k + [V, T_k(V)]) \cong T_k(V)/(k + [L_k(V), T_k(V)]),$$

where $L_k(V) \subset T_k(V)$ is the free (graded) Lie algebra generated by V . It follows that

$$\mathbf{\Omega}(C)_{\natural} \cong \mathbf{\Omega}(C)/[\mathbf{\Omega}_{\text{Comm}}(C), \mathbf{\Omega}(C)] \tag{7.4}$$

as complexes of k -vector spaces. By (6.4), $\mathbf{\Omega}(C) \cong \mathcal{U}(\mathbf{\Omega}_{\text{Comm}}(C))$. On the other hand, since $\mathbf{\Omega}_{\text{Comm}}(C)$ is a DG Lie algebra, we have an isomorphism of DG $\mathbf{\Omega}_{\text{Comm}}(C)$ -modules

$$\mathbf{Sym}_k[\mathbf{\Omega}_{\text{Comm}}(C)] \xrightarrow{\sim} \mathcal{U}_k[\mathbf{\Omega}_{\text{Comm}}(C)] \tag{7.5}$$

given by the symmetrization map. Therefore, writing

$$\text{Sym}^d[\mathbf{\Omega}_{\text{Comm}}(C)]_{\natural} := \text{Sym}^d(\mathbf{\Omega}_{\text{Comm}}(C))/[\mathbf{\Omega}_{\text{Comm}}(C), \text{Sym}^d(\mathbf{\Omega}_{\text{Comm}}(C))],$$

we get the decomposition

$$\Omega(C)_{\natural} \cong \bigoplus_{d=1}^{\infty} \text{Sym}^d[\Omega_{\text{Comm}}(C)]_{\natural}. \tag{7.6}$$

Note that $\Omega(C) \xrightarrow{\sim} \mathcal{U}(\mathfrak{a})$ is a cofibrant resolution in $\text{DGA}_{k/k}$. By [BKR, Proposition 3.1],

$$H_{\bullet}[\Omega(C)_{\natural}] \cong \overline{\text{HC}}_{\bullet}(\mathcal{U}(\mathfrak{a})).$$

On the other hand,

$$H_{\bullet}[\text{Sym}^d(\Omega_{\text{Comm}}(C))_{\natural}] \cong \text{HC}_{\bullet}^{(d)}(\text{Lie}, \mathfrak{a})$$

since C is Koszul dual to \mathfrak{a} . This proves the desired theorem. □

7.2.2. Lie–Hodge decomposition. Let $\mathfrak{a} \in \text{DGLA}_k$ and let $C \in \text{DGCC}_{k/k}$ be Koszul dual to \mathfrak{a} . By (6.4), $\Omega(C) \cong \mathcal{U}(\Omega_{\text{Comm}}(C))$. From this isomorphism, $\Omega(C)$ acquires the structure of a (primitively generated) cocommutative DG Hopf algebra whose DG Lie algebra of primitives is $\Omega_{\text{Comm}}(C)$. Let $m_p : \Omega(C)^{\otimes p} \rightarrow \Omega(C)$ denote the p -fold product and let $\Delta^p : \Omega(C) \rightarrow \Omega(C)^{\otimes p}$ denote the p -fold coproduct. For each $p \geq 2$, define the Adams operation

$$\psi^p := m_p \circ \Delta^p : \Omega(C) \rightarrow \Omega(C).$$

Note $\psi^p \circ \psi^q = \psi^{pq}$. The following proposition is dual to [FT2, Props. 5.3.4–5.3.6].

Proposition 7.5. *The Adams operations ψ^p , $p \geq 2$, descend to Adams operations*

$$\psi^p : \Omega(C)_{\natural} \rightarrow \Omega(C)_{\natural}, \quad p \geq 2.$$

It is verified without difficulty that on the image of $\text{Sym}^d(\Omega_{\text{Comm}}(C))$ in $\Omega(C)$ under the symmetrization map (7.5), ψ^p coincides with multiplication by p^d . Therefore,

Proposition 7.6. *ψ^p acts on the direct summand $\text{Sym}^d[\Omega_{\text{Comm}}(C)]_{\natural}$ of (7.6) by multiplication by p^d .*

Corollary 7.7. *There are Adams operations*

$$\psi^p : \overline{\text{HC}}_{\bullet}[\mathcal{U}(\mathfrak{a})] \rightarrow \overline{\text{HC}}_{\bullet}[\mathcal{U}(\mathfrak{a})], \quad p \geq 2.$$

Further, $\text{HC}_{\bullet}^{(d)}(\text{Lie}, \mathfrak{a})$ is precisely the (graded) eigenspace corresponding to the eigenvalue p^d of ψ^p for each $p \geq 2$.

Corollary 7.7 justifies referring to (7.3) as a Hodge decomposition. When C is the k -linear dual of a smooth commutative algebra A , the decomposition (7.3) can be related to the Hodge decomposition of the cyclic cohomology $\overline{\text{HC}}_{\bullet}^{\circ}(A)$ by the following proposition, which is an immediate consequence of [FT2, Corollary 6.5.1].

Proposition 7.8. *Let $\mathfrak{a} \in \text{DGLA}_k$, and let $C \in \text{DGCC}_{k/k}$ be a cocommutative coalgebra Koszul dual to \mathfrak{a} . Assume that $C \cong A^*$ for some smooth commutative k -algebra A . Then*

$$\text{HC}_{\bullet}^{(d)}(\text{Lie}, \mathfrak{a}) \cong \overline{\text{HC}}_{(d-1)}^{\circ}(A)[-1].$$

In particular, $\mathrm{HC}_\bullet^{(2)}(\mathrm{Lie}, \mathfrak{a})$ is isomorphic (up to a shift) to the Harrison cohomology of A . More generally, for $C \in \mathrm{DGCC}_{k/k}$, one can define a Hodge decomposition for $\overline{\mathrm{HC}}_\bullet(C)$ dual to the decomposition defined in [L, Theorem 4.6.7]. When $C = A^*$, this Hodge decomposition coincides with that on $\overline{\mathrm{HC}}_\bullet(A)$ (after the obvious identification is made). Proposition 7.8 therefore shows that the homology of the direct summand $\mathrm{Sym}^d(\Omega_{\mathrm{Comm}}(C))_{\mathfrak{g}}[1]$ of $\Omega(C)_{\mathfrak{g}}[1]$ should be denoted by $\overline{\mathrm{HC}}_\bullet^{(d-1)}(C)$. Thus,

$$\mathrm{HC}_\bullet^{(d)}(\mathrm{Lie}, \mathfrak{a}) \cong \overline{\mathrm{HC}}_\bullet^{(d-1)}(C)[-1]. \tag{7.7}$$

If \mathfrak{a} is a finite-dimensional abelian Lie algebra, one can compute $\mathrm{HC}_\bullet^{(d)}(\mathrm{Lie}, \mathfrak{a})$ explicitly, using the classical Hochschild–Kostant–Rosenberg Theorem. Precisely, we have a natural isomorphism

$$\mathrm{HC}_\bullet^{(d)}(\mathrm{Lie}, \mathfrak{a}) \cong \mathrm{Sym}^d(\mathfrak{a}) \otimes \wedge^\bullet(\mathfrak{a})/d[\mathrm{Sym}^{d+1}(\mathfrak{a}) \otimes \wedge^{\bullet-1}(\mathfrak{a})], \tag{7.8}$$

where d is the de Rham differential on the (algebraic) de Rham complex of the symmetric algebra $A = \mathrm{Sym}(\mathfrak{a})$. For the proof of (7.8) and further properties of the Lie–Hodge decomposition (7.3) we refer the reader to [BF⁺]. Here, we only remark that Proposition 7.8 also holds in the bigraded setting. In that case, $\overline{\mathrm{HC}}_\bullet(A)$ should be viewed as the cohomology of the bigraded dual of the reduced cyclic chain complex of A .

7.3. Drinfeld trace maps

Let \mathfrak{g} be a finite-dimensional reductive Lie algebra over k . The adjunction (6.13) gives a universal representation

$$\pi_{\mathfrak{g}} : \mathfrak{a} \rightarrow \mathfrak{g} \otimes \mathfrak{a}_{\mathfrak{g}}$$

for any $\mathfrak{a} \in \mathrm{DGLA}_k$. Let $\mathfrak{L} \xrightarrow{\sim} \mathfrak{a}$ be a cofibrant resolution in DGLA_k . For any $d \geq 1$, consider the composite map

$$\lambda^{(d)}(\mathfrak{L}) \xrightarrow{\lambda^{(d)}(\pi_{\mathfrak{g}})} \lambda^{(d)}(\mathfrak{g} \otimes \mathfrak{L}_{\mathfrak{g}}) \xrightarrow{(7.2)} \lambda^{(d)}(\mathfrak{g}) \otimes \mathfrak{L}_{\mathfrak{g}}. \tag{7.9}$$

Proposition 7.9. *The image of the composite map (7.9) lies in $\lambda^{(d)}(\mathfrak{g}) \otimes \mathfrak{L}_{\mathfrak{g}}^{\mathrm{ad} \mathfrak{g}}$.*

Proof. Equip \mathfrak{L} with the trivial \mathfrak{g} -action. Then $\pi_{\mathfrak{g}} : \mathfrak{L} \rightarrow \mathfrak{g} \otimes \mathfrak{L}_{\mathfrak{g}}$ is \mathfrak{g} -equivariant. It follows that $\lambda^{(d)}(\pi_{\mathfrak{g}})$ is \mathfrak{g} -equivariant as well. On the other hand, it is easy to verify that the map (7.2) from $\lambda^{(d)}(\mathfrak{g} \otimes \mathfrak{L}_{\mathfrak{g}})$ to $\lambda^{(d)}(\mathfrak{g}) \otimes \mathfrak{L}_{\mathfrak{g}}$ is also \mathfrak{g} -equivariant. Since \mathfrak{g} acts trivially on $\lambda^{(d)}(\mathfrak{g})$, the desired proposition follows. \square

One therefore obtains the maps

$$\mathrm{Tr}_{\mathfrak{g}} : \lambda^{(d)}(\mathfrak{L}) \rightarrow \lambda^{(d)}(\mathfrak{g}) \otimes \mathfrak{L}_{\mathfrak{g}}^{\mathrm{ad} \mathfrak{g}}.$$

For $d = 2$, the vector space $\lambda^{(d)}(\mathfrak{g})$ is one-dimensional: indeed, there is a unique (up to a scalar factor) invariant bilinear form on \mathfrak{g} (the Cartan–Killing form). Hence, at the level of homology, the map $\mathrm{Tr}_{\mathfrak{g}}$ induces a *canonical* trace map

$$\mathrm{Tr}_{\mathfrak{g}} : \mathrm{HC}_\bullet(\mathrm{Lie}, \mathfrak{a}) \rightarrow \mathrm{H}_\bullet(\mathfrak{a}, \mathfrak{g})^{\mathrm{ad} \mathfrak{g}}. \tag{7.10}$$

More generally, let $P \in \text{Sym}^d(\mathfrak{g}^*)^{\text{ad } \mathfrak{g}}$. Then one has the trace map

$$\text{Tr}_{\mathfrak{g}}^P : \lambda^{(d)}(\mathcal{L}) \xrightarrow{\text{Tr}_{\mathfrak{g}}} \lambda^{(d)}(\mathfrak{g}) \otimes \mathfrak{L}_{\mathfrak{g}}^{\text{ad } \mathfrak{g}} \xrightarrow{P(-) \otimes \text{Id}} \mathfrak{L}_{\mathfrak{g}}^{\text{ad } \mathfrak{g}}.$$

Recall that we have the Chevalley isomorphism

$$\text{Sym}(\mathfrak{g}^*)^{\text{ad } \mathfrak{g}} \cong \text{Sym}(\mathfrak{h}^*)^W \cong k[\bar{P}_1, \dots, \bar{P}_l]$$

where $\deg(\bar{P}_i) = d_i$ for $1 \leq i \leq l$ and the d_i are the fundamental degrees of \mathfrak{g} . Choosing $P_i \in \text{Sym}(\mathfrak{g}^*)^{\text{ad } \mathfrak{g}}$ corresponding to \bar{P}_i under Chevalley's isomorphism, we get a family of trace maps

$$\text{Tr}_{\mathfrak{g}}^{(d_i)} := \text{Tr}_{\mathfrak{g}}^{P_i} : \lambda^{(d_i)}(\mathcal{L}) \rightarrow \mathfrak{L}_{\mathfrak{g}}^{\text{ad } \mathfrak{g}},$$

which yields a homomorphism of commutative DG algebras

$$\text{Sym}_k[\text{Tr}_{\bullet}(\mathcal{L})] : \text{Sym}_k\left[\bigoplus_{i=1}^l \lambda^{(d_i)}(\mathcal{L})\right] \rightarrow \mathfrak{L}_{\mathfrak{g}}^{\text{ad } \mathfrak{g}}. \tag{7.11}$$

We refer to (7.11) as the *Drinfeld trace map*. We shall sometimes abuse this terminology and use it for closely related maps as well. At the level of homology, (7.11) gives

$$\text{Sym}_k[\text{Tr}_{\bullet}(\mathfrak{a})] : \text{Sym}_k\left[\bigoplus_{i=1}^l \text{HC}_{\bullet}^{(d_i)}(\text{Lie}, \mathfrak{a})\right] \rightarrow \text{H}_{\bullet}(\mathfrak{a}, \mathfrak{g})^{\text{ad } \mathfrak{g}}. \tag{7.12}$$

In particular, if $\mathcal{L} = \Omega_{\text{Comm}}(C)$, then by (7.7), the above map becomes

$$\text{Sym}_k[\text{Tr}_{\bullet}(\mathfrak{a})] : \text{Sym}_k\left[\bigoplus_{i=1}^l \overline{\text{HC}}_{\bullet}^{(d_i-1)}(C)[-1]\right] \rightarrow \text{H}_{\bullet}(\mathfrak{a}, \mathfrak{g})^{\text{ad } \mathfrak{g}}. \tag{7.13}$$

Remark. The Drinfeld trace map depends on the choice of the P_i , $1 \leq i \leq l$. This choice in turn depends precisely on the choice of an isomorphism $\text{Sym}(\mathfrak{h}^*)^W \cong k[\bar{P}_1, \dots, \bar{P}_l]$.

Example 7.10. Let $\mathfrak{a} := k.x$ be a one-dimensional Lie algebra over k with generator x having weight 1 and homological degree 0. Note that \mathfrak{a} is a free (and therefore cofibrant) DG Lie algebra. Since \mathfrak{a} is also abelian,

$$\mathbb{L}\lambda^{(d)}(\mathfrak{a}) \cong \text{Sym}^d(\mathfrak{a}) = k.x^d.$$

In this case, $\mathfrak{a}_{\mathfrak{g}}^{\text{ad } \mathfrak{g}} = \text{Sym}(\mathfrak{g}^*)^{\text{ad } \mathfrak{g}}$ and the map

$$\lambda^{(d)}(\mathfrak{a}) \rightarrow \mathfrak{a}_{\mathfrak{g}}^{\text{ad } \mathfrak{g}} \otimes \lambda^{(d)}(\mathfrak{g})$$

becomes the map dual to the nondegenerate pairing

$$\lambda^{(d)}(\mathfrak{g}) \otimes \text{Sym}^d(\mathfrak{g}^*)^{\text{ad } \mathfrak{g}} \rightarrow k.$$

It follows that for a fixed choice of isomorphism

$$\text{Sym}(\mathfrak{h}^*)^W \cong k[\bar{P}_1, \dots, \bar{P}_l],$$

the Drinfeld trace becomes the map

$$k[\bar{P}_1, \dots, \bar{P}_l] \rightarrow \text{Sym}(\mathfrak{g}^*)^{\text{ad } \mathfrak{g}}, \quad \bar{P}_i \mapsto P_i,$$

where the variable \bar{P}_i is identified with x^{d_i} . Since P_i corresponds to \bar{P}_i under the Chevalley restriction isomorphism, the Drinfeld trace is indeed a generalization of the map inverse to the Chevalley restriction isomorphism. Combining this observation with Example 7.2, we conclude that when \mathfrak{a} is a one-dimensional Lie algebra, the derived Harish-Chandra homomorphism and the Drinfeld trace are mutually inverse (quasi-)isomorphisms.

8. Derived commuting schemes

In this section, we turn to our main example: the derived commuting scheme associated to a finite-dimensional reductive Lie algebra \mathfrak{g} . Our goal is to state a general version of Conjecture 1 for \mathfrak{g} , deduce the corresponding constant term identity and present some evidence in favor of this conjecture. The next section will explain the relation to the famous Macdonald conjectures proved in [Ch] and [FGT].

8.1. Main conjecture

Let \mathfrak{a} be an abelian Lie algebra of dimension $r \geq 1$. In this case $\mathcal{U}\mathfrak{a} = \text{Sym}(\mathfrak{a})$ and the graded coalgebra $C := \mathbf{Sym}^c(\mathfrak{a}[1])$ is Koszul dual to \mathfrak{a} . Explicitly, $C = k \oplus \mathfrak{a} \oplus \wedge^2 \mathfrak{a} \oplus \dots \oplus \wedge^r \mathfrak{a}$, where $\wedge^i \mathfrak{a}$ has homological degree i . For a reductive Lie algebra \mathfrak{g} , the derived Harish-Chandra homomorphism (7.1) then becomes

$$\Phi_{\mathfrak{g}} : \text{DRep}_{\mathfrak{g}}(\mathfrak{a})^G \rightarrow \mathbf{Sym}[\mathfrak{h}^* \otimes (\mathfrak{a} \oplus \wedge^2 \mathfrak{a} \oplus \dots \oplus \wedge^r \mathfrak{a})]^W,$$

where $\wedge^i \mathfrak{a}$ has homological degree $i - 1$. If $\dim \mathfrak{a} = 2$, $\Phi_{\mathfrak{g}}$ induces on the 0-th homology the map

$$k[\text{Rep}_{\mathfrak{g}}(\mathfrak{a})]^G \rightarrow \text{Sym}[\mathfrak{h}^* \otimes \mathfrak{a}]^W,$$

which is known to be an isomorphism, at least when \mathfrak{g} is complex semisimple and $\text{Rep}_{\mathfrak{g}}(\mathfrak{a})//G$ is reduced (see, e.g., [Ha, Sect. 6.2]). It is therefore, reasonable to make the following conjecture extending Conjecture 4.6.

Conjecture 8.1. *Let $\dim(\mathfrak{a}) = 2$. Then, for any reductive Lie algebra \mathfrak{g} over k ,*

$$\Phi_{\mathfrak{g}} : \text{DRep}_{\mathfrak{g}}(\mathfrak{a})^G \rightarrow \mathbf{Sym}(\mathfrak{h}^* \oplus \mathfrak{h}^* \oplus \mathfrak{h}^*[1])^W$$

is a quasi-isomorphism (at least when the quotient commuting scheme $\text{Rep}_{\mathfrak{g}}(\mathfrak{a})//G$ is reduced).

Remark. The scheme $\text{Rep}_{\mathfrak{g}}(\mathfrak{a})$ in Conjecture 8.1 is precisely the classical commuting scheme of the Lie algebra \mathfrak{g} . It is known that the underlying variety of $\text{Rep}_{\mathfrak{g}}(\mathfrak{a})$ is irreducible for any semisimple complex Lie algebra \mathfrak{g} (see [R]). However, the question of whether $\text{Rep}_{\mathfrak{g}}(\mathfrak{a})$ (or even $\text{Rep}_{\mathfrak{g}}(\mathfrak{a})^G$) is a reduced scheme remains open in general.

Conjecture 8.1 can be restated in elementary terms, without using the language of derived representation schemes. To this end, define the DG algebra $(k[\mathfrak{g} \times \mathfrak{g}] \otimes \wedge \mathfrak{g}^*, d)$, with \mathfrak{g}^*

being in homological degree 1 and with differential $d : \mathfrak{g}^* \rightarrow k[\mathfrak{g} \times \mathfrak{g}]$ given by

$$d\varphi(\xi, \eta) := \varphi([\xi, \eta]), \quad \forall \xi, \eta \in \mathfrak{g}, \varphi \in \mathfrak{g}^*.$$

The adjoint action of G on \mathfrak{g} induces the diagonal G -action on $(k[\mathfrak{g} \times \mathfrak{g}] \otimes \wedge \mathfrak{g}^*, d)$, and this last action commutes with the differential. We may therefore consider the invariant DG algebra $(k[\mathfrak{g} \times \mathfrak{g}] \otimes \wedge \mathfrak{g}^*, d)^G$. Since the functions on \mathfrak{g} restrict to \mathfrak{h} , there is a natural DG algebra homomorphism

$$\Phi_{\mathfrak{g}} : (k[\mathfrak{g} \times \mathfrak{g}] \otimes \wedge \mathfrak{g}^*, d) \rightarrow k[\mathfrak{h} \times \mathfrak{h}] \otimes \wedge \mathfrak{h}^*,$$

where the right-hand side has zero differential. It is easy to see that the image of $(k[\mathfrak{g} \times \mathfrak{g}] \otimes \wedge \mathfrak{g}^*, d)^G$ under $\Phi_{\mathfrak{g}}$ lies in $(k[\mathfrak{h} \times \mathfrak{h}] \otimes \wedge \mathfrak{h}^*)^W$, and we have the following proposition.

Proposition 8.2. *Conjecture 8.1 is equivalent to the following statement: the DG algebra map*

$$\Phi_{\mathfrak{g}} : (k[\mathfrak{g} \times \mathfrak{g}] \otimes \wedge \mathfrak{g}^*, d)^G \rightarrow (k[\mathfrak{h} \times \mathfrak{h}] \otimes \wedge \mathfrak{h}^*)^W \tag{8.1}$$

is a quasi-isomorphism.

Proof. By Theorem 6.7, there is an isomorphism

$$\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})^G \cong \mathcal{C}^c(\mathfrak{g}^*(C), \mathfrak{g}^*; k)$$

where $C = \mathbf{Sym}^c(\mathfrak{a}[1])$. Choose any basis $\{x, y\}$ of \mathfrak{a} over k . Then, as graded algebras,

$$\mathcal{C}^c(\mathfrak{g}^*(C), \mathfrak{g}^*; k) \cong \mathbf{Sym}(\mathfrak{g}^*.x \oplus \mathfrak{g}^*.y \oplus \mathfrak{g}^*.\theta)$$

where $\mathfrak{g}^*.x := \mathfrak{g}^* \otimes x$, etc., and x, y have homological degree 0 and $\theta := s^{-1}(sx \wedge sy)$ has homological degree 1. Hence,

$$\mathcal{C}^c(\mathfrak{g}^*(C), \mathfrak{g}^*; k) \cong k[\mathfrak{g} \times \mathfrak{g}] \otimes \wedge \mathfrak{g}^*.$$

In particular, $\mathbf{Sym}(\mathfrak{g}^*.x \oplus \mathfrak{g}^*.y)$ can be identified with $k[\mathfrak{g} \times \mathfrak{g}]$. A direct computation using (3.1) then shows that for any $\varphi \in \mathfrak{g}^*$, the differential of the generator $\frac{1}{2}\varphi.\theta$ in $\mathcal{C}^c(\mathfrak{g}^*(C), \mathfrak{g}^*; k)$ is equal to the function $d\varphi \in k[\mathfrak{g} \times \mathfrak{g}]$ satisfying $d\varphi(\xi, \eta) = \varphi([\xi, \eta])$. This identifies $\mathcal{C}^c(\mathfrak{g}^*(C), \mathfrak{g}^*; k)$ with $(k[\mathfrak{g} \times \mathfrak{g}] \otimes \wedge \mathfrak{g}^*, d)^G$. The identification of $\mathcal{C}^c(\mathfrak{h}^*(C), \mathfrak{h}^*; k)$ with $k[\mathfrak{h} \times \mathfrak{h}] \otimes \wedge \mathfrak{h}^*$ is obvious. Since the derived Harish-Chandra homomorphism $\Phi_{\mathfrak{g}} : \mathcal{C}^c(\mathfrak{g}^*(C), \mathfrak{g}^*; k) \rightarrow \mathcal{C}^c(\mathfrak{h}^*(C), \mathfrak{h}^*; k)^W$ is indeed restriction of cocycles from \mathfrak{g} to \mathfrak{h} , it coincides with the map $\Phi_{\mathfrak{g}} : (k[\mathfrak{g} \times \mathfrak{g}] \otimes \wedge \mathfrak{g}^*, d)^G \rightarrow (k[\mathfrak{h} \times \mathfrak{h}] \otimes \wedge \mathfrak{h}^*)^W$ defined above. This proves the desired proposition. \square

8.2. A constant term identity

We now assume that $k = \mathbb{C}$ and compare the Euler characteristics of DG algebras in Conjecture 8.1. Write $\mathfrak{a} = \mathbb{C}x \oplus \mathbb{C}y$ with $[x, y] = 0$, and denote $B_{\mathfrak{g}} := \mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})$. Recall that as graded algebras,

$$B_{\mathfrak{g}} \cong \mathbf{Sym}_{\mathbb{C}}(\mathfrak{g}^*.x \oplus \mathfrak{g}^*.y \oplus \mathfrak{g}^*.\theta) \cong \mathbf{Sym}_{\mathbb{C}}(\mathfrak{g}.x \oplus \mathfrak{g}.y \oplus \mathfrak{g}.\theta), \tag{8.2}$$

where x, y have homological degree 0 and θ has homological degree 1. The DG algebra $B_{\mathfrak{g}}$ is equipped with an additional \mathbb{Z}^2 -weight grading, with the subspaces $\mathfrak{g}.x$ and $\mathfrak{g}.y$ having weights $(1, 0)$ and $(0, 1)$, and $\mathfrak{g}.\theta$ having weight $(1, 1)$.

Now, for $h \in \mathfrak{h}$, define $\chi(B_{\mathfrak{g}}, q, t, e^h)$ to be the character-valued Euler characteristic

$$\sum_{a,b \geq 0} \sum_{i \in \mathbb{Z}} (-1)^i \text{Tr}(e^h|_{(B_{\mathfrak{g}})_{i,a,b}}) q^a t^b,$$

where $V_{a,b}$ denotes the component of a \mathbb{Z}^2 -graded vector space V of weight (a, b) .

Lemma 8.3.

$$\chi(B_{\mathfrak{g}}, q, t, e^h) = \frac{(1-qt)^l}{(1-q)^l(1-t)^l} \prod_{\alpha \in R} \frac{(1-qt e^{\alpha(h)})}{(1-q e^{\alpha(h)})(1-t e^{\alpha(h)})},$$

where $l = \dim_{\mathbb{C}}(\mathfrak{h})$ is the rank of \mathfrak{g} and R is the root system associated to \mathfrak{g} .

Proof. Since $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_{\alpha}$, it follows from (8.2) that

$$B_{\mathfrak{g}} \cong B_{\mathfrak{h}} \otimes \bigotimes_{\alpha \in R} B_{\mathfrak{g}_{\alpha}},$$

where $B_{\mathfrak{h}} := \mathbf{Sym}(\mathfrak{h}.x \oplus \mathfrak{h}.y \oplus \mathfrak{h}.\theta)$ and $B_{\mathfrak{g}_{\alpha}} := \mathbf{Sym}(\mathfrak{g}_{\alpha}.x \oplus \mathfrak{g}_{\alpha}.y \oplus \mathfrak{g}_{\alpha}.\theta)$. Since e^h acts as the identity on \mathfrak{h} and by multiplication by $e^{\alpha(h)}$ on the root space \mathfrak{g}_{α} , we have

$$\chi(B_{\mathfrak{h}}, q, t, e^h) = \frac{(1-qt)^l}{(1-q)^l(1-t)^l}, \quad \chi(B_{\mathfrak{g}_{\alpha}}, q, t, e^h) = \frac{(1-qt e^{\alpha(h)})}{(1-q e^{\alpha(h)})(1-t e^{\alpha(h)})}. \tag{8.3}$$

The desired lemma now follows from the multiplicativity of the Euler characteristic. \square

Let $Q = Q(R)$ be the root lattice of \mathfrak{g} , and let $\mathbb{Z}[Q]$ denote the group ring of Q . For each $\alpha \in R \subset Q(R)$, write $e^{\alpha} \in \mathbb{Z}[Q]$ for the corresponding element in $\mathbb{Z}[Q]$, and denote by $\text{CT} : \mathbb{Z}[Q] \rightarrow \mathbb{Z}$ the map assigning to a polynomial in $\mathbb{Z}[Q]$ its constant term which does not involve any e^{α} . This constant term map naturally extends to the ring $\mathbb{Z}[Q][[q, t]]$ of formal power series over $\mathbb{Z}[Q]$: specifically, we define $\text{CT} : \mathbb{Z}[Q][[q, t]] \rightarrow \mathbb{Z}[[q, t]]$ by

$$\text{CT} \sum_{a,b \geq 0} P_{a,b}(e^{\alpha}) q^a t^b := \sum_{a,b \geq 0} \text{CT}[P_{a,b}(e^{\alpha})] q^a t^b. \tag{8.4}$$

As corollary of Lemma 8.3, we now get

Corollary 8.4. *The weighted Euler characteristic of $B_{\mathfrak{g}}^G$ is given by*

$$\chi(B_{\mathfrak{g}}^G, q, t) = \frac{1}{|W|} \frac{(1-qt)^l}{(1-q)^l(1-t)^l} \text{CT} \left\{ \prod_{\alpha \in R} \frac{(1-qt e^{\alpha})(1-e^{\alpha})}{(1-q e^{\alpha})(1-t e^{\alpha})} \right\}.$$

Proof. It suffices to verify the above formula for the compact real form of the group G , which we also denote by G . We have

$$\chi(B_{\mathfrak{g}}^G, q, t) = \int_G \chi(B_{\mathfrak{g}}, q, t, \text{Ad } g) dg,$$

where dg is the Haar measure on G normalized by $\int_G dg = 1$. Now, let T be the maximal torus of G corresponding to R . For $\alpha \in R$, regard e^α as a character of T . Then, by Weyl's integration formula,

$$\int_G \chi(B_{\mathfrak{g}}, q, t, \text{Ad } g) dg = \frac{1}{|W|} \int_T \chi(B_{\mathfrak{g}}, q, t, \text{Ad } \tau) \prod_{\alpha \in R} (1 - e^\alpha(\tau)) d\tau$$

where $d\tau$ is the normalized Haar measure on T . The result follows immediately from Lemma 8.3 and the fact that $\int_T e^\alpha(\tau) d\tau = 0$ for any nonzero root α . \square

Next, we look at the right-hand side of Conjecture 8.1. We write

$$A_{\mathfrak{g}} := \mathbf{Sym}_{\mathbb{C}}(\mathfrak{h}^* \oplus \mathfrak{h}^* \oplus \mathfrak{h}^*[1]) \cong \mathbf{Sym}_{\mathbb{C}}(\mathfrak{h}.x \oplus \mathfrak{h}.y \oplus \mathfrak{h}.\theta),$$

where x, y are of homological degree 0 and θ of homological degree 1. Again, the algebra $A_{\mathfrak{g}}$ has an additional \mathbb{Z}^2 -grading, with $\mathfrak{h}.x$ and $\mathfrak{h}.y$ having weights $(1, 0)$ and $(0, 1)$ and $\mathfrak{h}.\theta$ having weight $(1, 1)$. For an element $w \in W$, let $\{\lambda_1, \dots, \lambda_l\}$ be the eigenvalues w under the natural action of W on \mathfrak{h} . Then

$$\chi(A_{\mathfrak{g}}, q, t, w) = \prod_{i=1}^l \frac{(1 - qt\lambda_i)}{(1 - q\lambda_i)(1 - t\lambda_i)} = \frac{\det(1 - qtw)}{\det(1 - qw) \det(1 - tw)},$$

where 'det' is taken on $\text{End } \mathfrak{h}$. By the classical Molien formula, we get

$$\chi(A_{\mathfrak{g}}^W, q, t) = \frac{1}{|W|} \sum_{w \in W} \frac{\det(1 - qtw)}{\det(1 - qw) \det(1 - tw)}.$$

Conjecture 8.1 therefore implies the following constant term identity generalizing (5.2):

Conjecture 8.5. *The following identity holds:*

$$\frac{(1 - qt)^l}{(1 - q)^l (1 - t)^l} \text{CT} \left\{ \prod_{\alpha \in R} \frac{(1 - qte^\alpha)(1 - e^\alpha)}{(1 - qe^\alpha)(1 - te^\alpha)} \right\} = \sum_{w \in W} \frac{\det(1 - qtw)}{\det(1 - qw) \det(1 - tw)}.$$

Note that Conjecture 8.5 can be equivalently rewritten as

$$\int_G \frac{\det(1 - qt \text{Ad } g)}{\det(1 - q \text{Ad } g) \det(1 - t \text{Ad } g)} dg = \frac{1}{|W|} \sum_{w \in W} \frac{\det(1 - qtw)}{\det(1 - qw) \det(1 - tw)}. \tag{8.5}$$

For $t = 0$, this simplifies to the well-known identity

$$\int_G \frac{dg}{\det(1 - q \text{Ad } g)} = \frac{1}{|W|} \sum_{w \in W} \frac{1}{\det(1 - qw)} = \prod_{i=1}^l \frac{1}{1 - q^{d_i}}, \tag{8.6}$$

which is obtained by equating the Poincaré series of both sides of the Chevalley isomorphism

$$\mathbf{Sym}(\mathfrak{g}^*)^G \cong \mathbf{Sym}(\mathfrak{h}^*)^W.$$

In (8.6), the numbers d_i are the fundamental degrees of W , i.e. the degrees of algebraically independent elements generating $\text{Sym}(\mathfrak{h}^*)^W$. If we expand both sides of (8.5) as power series in t and compare the corresponding Taylor coefficients under t^k , we get a sequence of identities extending (8.6) for $k \geq 1$. Thus, Conjecture 8.5 answers a question of I. Macdonald posed in [M, Remark 2, p. 997].

We now provide some evidence for Conjectures 8.1 and 8.5. We recall that we have already proved Conjecture 8.1 for \mathfrak{gl}_2 (Theorem 4.7) and Conjecture 8.5 for \mathfrak{gl}_n for an arbitrary n (Theorem 5.5).

8.3. Lower order terms

First, we show that the Taylor expansion in q, t of both sides of Conjecture 8.5 agree up to degree 2. In fact, in their normalized form (8.5) the quadratic terms are independent of the root system.

Proposition 8.6. *The first terms of both sides of (8.5) (viewed as power series in q, t) are equal to*

$$1 + q^2 + qt + t^2 + \dots,$$

where the dots stand for terms of degree at least 3 in q, t .

Proposition 8.6 follows from Lemma 8.7 below that evaluates both sides of (8.5) as integrals of the same ratio of determinants over a compact Lie group for a real irreducible nontrivial representation (we formally think of W as a compact Lie group of dimension 0).

Lemma 8.7. *Let $\rho: G \rightarrow \text{GL}(V)$ be a nontrivial irreducible representation of a compact Lie group G over \mathbb{R} with invariant measure dg and volume $|G|$. Then*

$$\frac{1}{|G|} \int_G \frac{\det(1 - qt\rho(g))}{\det(1 - q\rho(g)) \det(1 - t\rho(g))} dg = 1 + q^2 + qt + t^2 + \dots.$$

Proof. For any endomorphism a of a finite-dimensional vector space,

$$\frac{\det(1 - qta)}{\det(1 - qa) \det(1 - ta)} = 1 + (q+t-qt) \text{tr } a + \frac{1}{2}(q+t)^2 (\text{tr } a)^2 + \frac{1}{2}(q^2+t^2) \text{tr}(a^2) + \dots.$$

If ρ is irreducible and nontrivial then its character has norm 1 and is orthogonal to the trivial character. Thus

$$\frac{1}{|G|} \int_G \text{tr } \rho(g) dg = 0, \quad \frac{1}{|G|} \int_G |\text{tr } \rho(g)|^2 dg = 1.$$

Now real representations have real-valued characters and the Frobenius–Schur theorem holds:

$$\frac{1}{|G|} \int_G \text{tr } \rho(g^2) dg = 1.$$

The average over the group is then

$$1 + \frac{1}{2}(q+t)^2 + \frac{1}{2}(q^2+t^2) + \dots = 1 + q^2 + qt + t^2 + \dots.$$

This finishes the proof of the lemma and of Proposition 8.6. □

8.4. The case of \mathfrak{sl}_n

Our next result is

Theorem 8.8. *For any $n \geq 2$, Conjecture 8.1 for \mathfrak{sl}_n is equivalent to Conjecture 8.1 for \mathfrak{gl}_n .*

Proof. Let \mathfrak{t}_n denote the Cartan subalgebra of \mathfrak{sl}_n comprising the diagonal matrices with trace 0. The short exact sequence of Lie algebras

$$0 \rightarrow \mathfrak{sl}_n \rightarrow \mathfrak{gl}_n \xrightarrow{\text{Tr}} k \rightarrow 0$$

has a canonical splitting, giving an isomorphism of Lie algebras

$$\mathfrak{gl}_n \xrightarrow{\sim} \mathfrak{sl}_n \oplus k, \quad M \mapsto (M - n^{-1} \text{Tr}(M)\text{Id}_n, \text{Tr}(M)). \tag{8.7}$$

This restricts to an isomorphism of diagonal Cartan subalgebras $\mathfrak{h}_n \cong \mathfrak{t}_n \oplus k$, which is equivariant under the action of the Weyl group S_n (with S_n acting trivially on k).

Let $C := \mathbf{Sym}^c(\mathfrak{a}[1])$ where $\mathfrak{a} := k.x \oplus k.y$ is the two-dimensional abelian Lie algebra. The isomorphism (8.7) induces an isomorphism $\mathfrak{gl}_n^*(C) \cong \mathfrak{sl}_n^*(C) \oplus C$. Hence,

$$\mathcal{C}^c(\mathfrak{gl}_n^*(C), \mathfrak{gl}_n^*(k); k) \cong \mathcal{C}^c(\mathfrak{sl}_n^*(C), \mathfrak{sl}_n^*(k); k) \otimes \mathcal{C}^c(\bar{C}; k). \tag{8.8}$$

Similarly, the S_n -equivariant isomorphism $\mathfrak{h}_n \cong \mathfrak{t}_n \oplus k$ yields

$$\mathcal{C}^c(\mathfrak{h}_n^*(C), \mathfrak{h}_n^*(k); k)^{S_n} \cong \mathcal{C}^c(\mathfrak{t}_n^*(C), \mathfrak{t}_n^*(k); k)^{S_n} \otimes \mathcal{C}^c(\bar{C}; k). \tag{8.9}$$

Let $\Phi_{\mathfrak{gl}_n}$ (resp., $\Phi_{\mathfrak{sl}_n}$) denote the derived Harish-Chandra homomorphisms for \mathfrak{gl}_n (resp., \mathfrak{sl}_n). Since the isomorphism $\mathfrak{h}_n \cong \mathfrak{t}_n \oplus k$ is the restriction of $\mathfrak{gl}_n \cong \mathfrak{sl}_n \oplus k$ to \mathfrak{h}_n , one has the following commutative diagram:

$$\begin{array}{ccc} \mathcal{C}^c(\mathfrak{gl}_n^*(C), \mathfrak{gl}_n^*(k); k) & \xrightarrow{\cong} & \mathcal{C}^c(\mathfrak{sl}_n^*(C), \mathfrak{sl}_n^*(k); k) \otimes \mathcal{C}^c(\bar{C}; k) \\ \Phi_{\mathfrak{gl}_n} \downarrow & & \Phi_{\mathfrak{sl}_n} \otimes \text{Id} \downarrow \\ \mathcal{C}^c(\mathfrak{h}_n^*(C), \mathfrak{h}_n^*(k); k)^{S_n} & \xrightarrow{\cong} & \mathcal{C}^c(\mathfrak{t}_n^*(C), \mathfrak{t}_n^*(k); k)^{S_n} \otimes \mathcal{C}^c(\bar{C}; k) \end{array}$$

Hence, $\Phi_{\mathfrak{gl}_n}$ is a quasi-isomorphism if $\Phi_{\mathfrak{sl}_n}$ is. Conversely, if $\Phi_{\mathfrak{sl}_n}$ is not a quasi-isomorphism, choose k to be the minimum homological degree such that $H_k(\Phi_{\mathfrak{sl}_n})$ is not an isomorphism. Since $\mathcal{C}^c(\bar{C}; k) \cong k[x, y, \theta]$,

$$H_k(\Phi_{\mathfrak{gl}_n}) \cong H_k(\Phi_{\mathfrak{sl}_n}) \otimes \text{Id}_{k[x,y]} \oplus H_{k-1}(\Phi_{\mathfrak{sl}_n}) \otimes \text{Id}_{k[x,y,\theta]},$$

and so $H_k(\Phi_{\mathfrak{gl}_n})$ is not an isomorphism either. This proves the desired theorem. □

As a consequence of (the proof of) Theorem 8.8, we have

Corollary 8.9. *Conjecture 8.5 holds for \mathfrak{sl}_n for all $n \geq 2$.*

Proof. We need to show the equality of Euler characteristics:

$$\chi(\text{DRep}_{\mathfrak{sl}_n}(\mathfrak{a})^{\text{SL}_n}, q, t) = \chi(\mathcal{C}^c(\mathfrak{t}_n^*(C), \mathfrak{t}_n^*(k); k)^{S_n}, q, t).$$

In view of isomorphisms (8.8) and (8.9), we have

$$\begin{aligned} \chi(\text{DRep}_{\mathfrak{sl}_n}(\mathfrak{a})^{\text{SL}_n}, q, t) \cdot \chi(k[x, y, \theta], q, t) &= \chi(\mathcal{C}^c(\mathfrak{g}_n^*(C), \mathfrak{g}_n^*(k); k), q, t), \\ \chi(\mathcal{C}^c(\mathfrak{t}_n^*(C), \mathfrak{t}_n^*(k); k)^{S_n}, q, t) \cdot \chi(k[x, y, \theta], q, t) &= \chi(\mathcal{C}^c(\mathfrak{h}_n^*(C), \mathfrak{h}_n^*(k); k)^{S_n}, q, t). \end{aligned}$$

Note that $\chi(k[x, y, \theta], q, t) = \frac{1-qt}{(1-q)(1-t)}$. Hence, $\chi(k[x, y, \theta], q, t)$ is an invertible power series in q, t . The result follows now from Theorem 5.5, which says that the Euler characteristics of $\mathcal{C}^c(\mathfrak{g}_n^*(C), \mathfrak{g}_n^*(k); k)$ and $\mathcal{C}^c(\mathfrak{h}_n^*(C), \mathfrak{h}_n^*(k); k)^{S_n}$ are equal. \square

8.5. Orthogonal and symplectic Lie algebras

We now verify that Conjecture 8.1 holds in the limit $n \rightarrow \infty$ for orthogonal and symplectic Lie algebras. We begin with definitions of these classical Lie algebras.

8.5.1. *The orthogonal Lie algebras \mathfrak{so}_{2n+1} .* Fix a basis $\{e_1, \dots, e_{2n+1}\}$ of k^{2n+1} . Let \mathcal{M}_{2n+1} denote the matrix of the nondegenerate symmetric bilinear form Q satisfying

$$\begin{aligned} Q(e_{2i-1}, e_{2i}) &= Q(e_{2i}, e_{2i-1}) = 1, & 1 \leq i \leq n, \\ Q(e_{2n+1}, e_{2n+1}) &= 1, \\ Q(e_i, e_j) &= 0 & \text{otherwise.} \end{aligned}$$

Define $\text{SO}(2n + 1)$ to be the group of invertible matrices of determinant 1 preserving the bilinear form Q . In other words, $\text{SO}(2n + 1)$ is the group of invertible matrices P of size $2n + 1$ and determinant 1 satisfying

$$\mathcal{M}_{2n+1} = P^t \mathcal{M}_{2n+1} P.$$

The Lie algebra \mathfrak{so}_{2n+1} of $\text{SO}(2n+1)$ is therefore the Lie subalgebra of \mathfrak{gl}_{2n+1} comprising those matrices X satisfying

$$X^t \mathcal{M}_{2n+1} + \mathcal{M}_{2n+1} X = 0.$$

Further, let \mathcal{M}_{2n} denote the matrix of the nondegenerate symmetric bilinear form Q satisfying

$$\begin{aligned} Q(e_{2i-1}, e_{2i}) &= Q(e_{2i}, e_{2i-1}) = 1, & 1 \leq i \leq n, \\ Q(e_i, e_j) &= 0 & \text{otherwise.} \end{aligned}$$

The Lie algebra \mathfrak{so}_{2n} can similarly be defined as the Lie subalgebra of \mathfrak{gl}_{2n} comprising those matrices X that satisfy

$$X^t \mathcal{M}_{2n} + \mathcal{M}_{2n} X = 0.$$

Padding on the right and bottom with 0's gives an embedding $\mathfrak{so}_n \hookrightarrow \mathfrak{so}_{n+1}$ for all n . We may therefore form the direct limits

$$\mathfrak{so}_{2\infty+1} := \varinjlim \mathfrak{so}_{2n+1}, \quad \mathfrak{so}_{2\infty} := \varinjlim \mathfrak{so}_{2n}, \quad \mathfrak{so}_\infty := \varinjlim \mathfrak{so}_n.$$

It is easy to see that one has isomorphisms of Lie algebras $\mathfrak{so}_{2\infty+1} \cong \mathfrak{so}_\infty \cong \mathfrak{so}_{2\infty}$.

8.5.2. *Lie algebras with (anti-)involution.* An *(anti-)involution* on a DG Lie algebra \mathfrak{g} is a map of complexes $\sigma : \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying $\sigma^2 = \text{Id}$ and

$$\sigma([X, Y]) = (-1)^{|X||Y|}[\sigma(Y), \sigma(X)] \quad \forall X, Y \in \mathfrak{g}.$$

Note that multiplication by -1 in an anti-*involution* on every Lie algebra \mathfrak{g} . For terminological brevity, we refer to an anti-*involution* on a Lie algebra as an *involution*. Further, for any involution σ on a DG Lie algebra \mathfrak{g} , the (-1) -eigenspace of σ is a DG Lie subalgebra of \mathfrak{g} . For example, if \mathcal{M}_{2n+1} is as in Section 8.5.1, then

$$X \mapsto \mathcal{M}_{2n+1}^t X^t \mathcal{M}_{2n+1} = \mathcal{M}_{2n+1} X^t \mathcal{M}_{2n+1}$$

is an involution on \mathfrak{gl}_{2n+1} . The (-1) -eigenspace of this involution is precisely \mathfrak{so}_{2n+1} .

Note that any involution on a DG Lie algebra \mathfrak{L} extends to an involution on its universal enveloping algebra $\mathcal{U}\mathfrak{L}$. Equip $\mathcal{M}_n(k)$ with an involution. Recall from [BR2] that for an involutive DG algebra A , one has the commutative DG algebra $\text{Rep}_n^*(A)$ of functions on the DG scheme parametrizing involution preserving representations of A to $\mathcal{M}_n(k)$. Let \mathfrak{g}_n^- denote the (-1) -eigenspace of this involution. The following proposition is proven in [BR2].

Proposition 8.10. *For any DG Lie algebra \mathfrak{L} , we have an isomorphism of commutative DG algebras*

$$k[\text{Rep}_{\mathfrak{g}_n^-}(\mathfrak{L})] \cong k[\text{Rep}_n^*(\mathcal{U}\mathfrak{L})]$$

where the involution on $\mathcal{U}\mathfrak{L}$ is the extension of the involution on \mathfrak{L} given by multiplication by -1 .

Suppose that $\mathfrak{L} \xrightarrow{\sim} \mathfrak{g}$ is a cofibrant resolution in DGLA_k . Then $\mathcal{U}\mathfrak{L} \xrightarrow{\sim} \mathcal{U}\mathfrak{g}$ is a cofibrant resolution in $\text{DGA}_{k/k}$. Let σ denote the involution on $\mathcal{U}\mathfrak{L}$ (resp., $\mathcal{U}\mathfrak{g}$) extending multiplication by -1 on \mathfrak{L} (resp., \mathfrak{g}). Suppose that the involution chosen on $\mathcal{M}_n(k)$ is trace preserving. Let G be a Lie subgroup of $\text{GL}_n(k)$ whose Lie algebra is \mathfrak{g}_n^- . For A an involutive DG algebra, denote

$$A_{\mathfrak{g}_n^-, \sigma} := A/(k + [A, A] + \text{Im}(1 - \sigma)).$$

Then there is a natural morphism of complexes

$$\text{Tr}_n : \mathcal{U}(\mathfrak{L})_{\mathfrak{g}_n^-, \sigma} \rightarrow k[\text{Rep}_n^*(\mathcal{U}\mathfrak{L})]^G, \tag{8.10}$$

induced by the trace map $\text{Tr}_n[\mathcal{U}(\mathfrak{L})] : \mathcal{U}(\mathfrak{L})_{\mathfrak{g}_n^-, \sigma} \rightarrow (\mathcal{U}\mathfrak{L})_n^{\text{GL}}$ (see [BR2]). This gives the commutative diagram at the level of homology groups:

$$\begin{array}{ccc} \overline{\text{HC}}_{\bullet}(\mathcal{U}\mathfrak{g}) & \xrightarrow{\text{Tr}_n} & \text{H}_{\bullet}(\mathcal{U}\mathfrak{g}, n)^{\text{GL}} \\ \downarrow & & \downarrow \\ \overline{\text{HD}}_{\bullet}(\mathcal{U}\mathfrak{g}) & \xrightarrow{\text{Tr}_n} & \text{H}_{\bullet}^*(\mathcal{U}\mathfrak{g}, n)^G \end{array} \tag{8.11}$$

Here, $\overline{\text{HD}}_{\bullet}(\mathcal{U}\mathfrak{g})$ denotes the reduced dihedral homology of $\mathcal{U}\mathfrak{g}$ with respect to the involution on $\mathcal{U}\mathfrak{g}$ that extends the involution on \mathfrak{g} given by multiplication by -1 .

Let $\mathfrak{a} = k.x \oplus k.y$. Let $A := \mathcal{U}\mathfrak{a} \cong k[x, y]$ denote the universal enveloping algebra of \mathfrak{a} equipped with the involution that takes x (resp., y) to $-x$ (resp., $-y$). Let $C := \mathbf{Sym}^c(\mathfrak{a}[1])$. Then $\mathfrak{L} := \mathbf{\Omega}_{\text{Comm}}(C)$ is a cofibrant resolution of the abelian Lie algebra \mathfrak{a} . Explicitly, \mathfrak{L} is the free Lie algebra generated by x, y, θ with x, y in degree 0 and θ in degree 1 satisfying $d\theta = [x, y]$. Its universal enveloping algebra is the free algebra $R := k\langle x, y, \theta : d\theta = [x, y] \rangle$ resolving $k[x, y]$. By Proposition 8.10, we have

$$\text{DRep}_{\mathfrak{so}_{2n+1}}(\mathfrak{a}) \cong k[\text{Rep}_{\mathfrak{so}_{2n+1}}(\mathfrak{L})] \cong k[\text{Rep}_{2n+1}^*(R)].$$

Hence, the trace maps (8.10) become

$$\text{Tr}_{2n+1} : R_{\natural, \sigma} \rightarrow k[\text{Rep}_{\mathfrak{so}_{2n+1}}(\mathfrak{L})]^{\text{SO}_{2n+1}},$$

which give the following maps at the level of homology groups:

$$\text{Tr}_{2n+1} : \overline{\text{HD}}_{\bullet}(A) \rightarrow \mathbf{H}_{\bullet}(\mathfrak{a}, \mathfrak{so}_{2n+1})^{\text{SO}_{2n+1}}.$$

The inclusion of Lie algebras $\mathfrak{so}_{2n+1} \hookrightarrow \mathfrak{so}_{2n+3}$ induces a (degreewise surjective) homomorphism of commutative DG algebras

$$\mu_{2n+1} : k[\text{Rep}_{\mathfrak{so}_{2n+3}}(\mathfrak{L})] \rightarrow k[\text{Rep}_{\mathfrak{so}_{2n+1}}(\mathfrak{L})].$$

It is easy to verify that μ_{2n+1} maps $k[\text{Rep}_{\mathfrak{so}_{2n+3}}(\mathfrak{L})]^{\text{SO}_{2n+3}}$ to $k[\text{Rep}_{\mathfrak{so}_{2n+1}}(\mathfrak{L})]^{\text{SO}_{2n+1}}$ and that

$$\mu_{2n+1} \circ \text{Tr}_{2n+3} = \text{Tr}_{2n+1}.$$

Hence, one obtains a map of complexes

$$\text{Tr}_{\infty} : R_{\natural, \sigma} \rightarrow k[\text{Rep}_{\mathfrak{so}_{2\infty+1}}(\mathfrak{L})]^{\text{SO}_{2\infty+1}} := \varprojlim_n k[\text{Rep}_{\mathfrak{so}_{2n+1}}(\mathfrak{L})]^{\text{SO}_{2n+1}}.$$

Here, the projective limit is taken in the category of *bigraded commutative* DG algebras. By multiplicativity, one obtains a morphism of commutative DG algebras

$$\mathbf{Sym}(\text{Tr}_{\infty}) : \mathbf{Sym}(R_{\natural, \sigma}) \rightarrow k[\text{Rep}_{\mathfrak{so}_{2\infty+1}}(\mathfrak{L})]^{\text{SO}_{2\infty+1}}.$$

The following result is a consequence of the stabilization theorem [BR1, Theorem 4.4] for involutive DG algebras. We however sketch a different proof of this statement.

Proposition 8.11. $\mathbf{Sym}(\text{Tr}_{\infty}) : \mathbf{Sym}(R_{\natural, \sigma}) \rightarrow k[\text{Rep}_{\mathfrak{so}_{2\infty+1}}(\mathfrak{L})]^{\text{SO}_{2\infty+1}}$ is an isomorphism of bigraded commutative DG algebras.

Proof. Let $E := C^*$ denote the bigraded dual of C . Then the complex $R_{\natural, \sigma}$, whose homology is $\overline{\text{HD}}_{\bullet}(k[x, y])$, coincides on the nose with the complex $D(E)^*[-1]$, where $*$ denotes bigraded dual and $D(E)$ is the complex computing the relative skew dihedral homology of E with respect to the trivial involution. On the other hand, by Theorem 6.7,

$$k[\text{Rep}_{\mathfrak{so}_{2n+1}}(\mathfrak{L})]^{\text{SO}_{2n+1}} \cong \mathcal{C}(\mathfrak{so}_{2n+1}(E), \mathfrak{so}_{2n+1}(k); k)^*.$$

Hence,

$$k[\text{Rep}_{\mathfrak{so}_{2\infty+1}}(\mathfrak{L})]^{\text{SO}_{2\infty+1}} \cong \mathcal{C}(\mathfrak{so}_{2\infty+1}(E), \mathfrak{so}_{2\infty+1}(k); k)^*.$$

Note that $\mathfrak{so}_{2\infty+1} \cong \mathfrak{so}_\infty$. The map $\mathbf{Sym}(\mathrm{Tr}_\infty)$ can then be verified to be the bigraded dual of the map

$$\mathcal{C}(\mathfrak{so}_\infty(E), \mathfrak{so}_\infty(k); k) \rightarrow \mathbf{Sym}(D(E)[1])$$

induced by the map of complexes denoted in [L, Section 10.2.3] by $\mathrm{tr}_* \circ \theta_*$. By the relative version of [LP, Theorem 5.5] (see also [L, Section 10.5.7]), this last map is an isomorphism of DG coalgebras. This proves the desired proposition. \square

On homologies, Tr_∞ gives a map

$$\mathrm{Tr}_\infty : \overline{\mathrm{HD}}_\bullet(A) \rightarrow \mathbf{H}_\bullet(\mathfrak{a}, \mathfrak{so}_{2\infty+1})^{\mathrm{SO}_{2\infty+1}} := \mathbf{H}_\bullet(k[\mathrm{Rep}_{\mathfrak{so}_{2\infty+1}}(\mathfrak{L})]^{\mathrm{SO}_{2\infty+1}}).$$

By Proposition 8.11, $\mathbf{Sym}(\mathrm{Tr}_\infty)$ gives an isomorphism on homologies

$$\mathbf{Sym}(\mathrm{Tr}_\infty) : \mathbf{Sym}(\overline{\mathrm{HD}}_\bullet(A)) \xrightarrow{\sim} \mathbf{H}_\bullet(\mathfrak{a}, \mathfrak{so}_{2\infty+1})^{\mathrm{SO}_{2\infty+1}}.$$

We therefore need to compute the (relative) dihedral homology $\overline{\mathrm{HD}}_\bullet(A)$. Note that it is the space of covariants (and hence invariants) of the \mathbb{Z}_2 -action on $\overline{\mathrm{HC}}_\bullet(A)$ induced by the chosen involution on A . On the other hand, $\overline{\mathrm{HC}}_0(A) \cong A/k$ and $\overline{\mathrm{HC}}_1(A) \cong \Omega^1(A)/dA \cong A \cdot ydx$, and the other reduced cyclic homologies of A vanish. Let A^{odd} and A^{even} denote the subspaces of $A = k[x, y]$ spanned by all monomials of odd and even weight respectively (for this definition, x and y are both taken to have weight 1).

Lemma 8.12. *There are isomorphisms of vector spaces*

$$\overline{\mathrm{HD}}_0(A) \cong A^{\mathrm{ev}}/k, \quad \overline{\mathrm{HD}}_1(A) \cong A^{\mathrm{odd}} \cdot ydx.$$

Proof. Recall from [BKR, Example 4.1] that the monomial $x^k y^l$ in A/k is identified with the cyclic chain $x^k y^l$ in $R_{\mathfrak{q}}$. The involution on R maps $x^k y^l$ to $(-1)^{k+l} y^l x^k$, which coincides with $(-1)^{k+l} x^k y^l$ in $R_{\mathfrak{q}}$. Hence, $x^k y^l$ is a nontrivial basis element of $R/(k + [R, R] + \mathrm{Im}(1 - \sigma))$ iff $k + l$ is even.

Similarly, [BKR, Example 4.1] tells us that the basis element $x^k y^l dx$ ($l \geq 1$) of $\overline{\mathrm{HC}}_1(A)$ is identified with the cycle $\sum_{i=0}^{l-1} y^{l-1-i} x^k y^i \theta$ in $R_{\mathfrak{q}}$. This cycle is mapped by the involution on R to $(-1)^{k+l} \sum_{i=0}^{l-1} \theta y^i x^k y^{l-1-i}$, which coincides with $(-1)^{k+l} \sum_{i=0}^{l-1} y^{l-1-i} x^k y^i \theta$ in $R_{\mathfrak{q}}$. It follows that $x^k y^l dx$ is a nontrivial basis element in $\overline{\mathrm{HD}}_1(A)$ iff $k + l$ is even and $l \geq 1$. This proves the desired lemma. \square

8.5.3. The derived Harish-Chandra homomorphism for the odd orthogonal case. Note that for \mathfrak{so}_{2n+1} , one can choose the Cartan subalgebra \mathfrak{h}_{2n+1} to be the abelian Lie subalgebra of \mathfrak{so}_{2n+1} spanned by the basis elements $h_i := e_{2i-1, 2i-1} - e_{2i, 2i}$, $1 \leq i \leq n$. The Weyl group of \mathfrak{so}_{2n+1} is $(\mathbb{Z}_2)^n \rtimes S_n$, where elements of S_n permute the h_i 's and the element $\gamma_i := (1, \dots, \gamma, \dots, 1)$ of $(\mathbb{Z}_2)^n$ with the generator γ of \mathbb{Z}_2 in the i -th coordinate transforms h_i to $-h_i$ and leaves other h_j 's unchanged.

The derived Harish-Chandra homomorphism therefore becomes a morphism of commutative DG algebras

$$\Phi : k[\mathrm{Rep}_{\mathfrak{so}_{2n+1}}(\mathfrak{L})]^{\mathrm{SO}_{2n+1}} \rightarrow k[x_1, \dots, x_n, y_1, \dots, y_n, \theta_1, \dots, \theta_n]^{(\mathbb{Z}_2)^n \rtimes S_n}$$

where elements of S_n simultaneously permute the x_i 's, y_i 's and θ_i 's, and γ_i multiplies the generators x_i , y_i and θ_i by -1 and leaves the other generators unchanged. In the limit, the derived Harish-Chandra homomorphism becomes the map

$$\Phi : k[\text{Rep}_{\mathfrak{so}_{2\infty+1}}(\mathcal{L})]^{\text{SO}_{2\infty+1}} \rightarrow k[x_1, \dots; y_1, \dots; \theta_1, \dots]^{(\mathbb{Z}_2)^\infty \rtimes S_\infty}$$

where

$$k[x_1, \dots; y_1, \dots; \theta_1, \dots]^{(\mathbb{Z}_2)^\infty \rtimes S_\infty} := \varprojlim_n k[x_1, \dots, x_n, y_1, \dots, y_n, \theta_1, \dots, \theta_n]^{(\mathbb{Z}_2)^n \rtimes S_n}.$$

Here, the limit is taken in the category of *bigraded* commutative algebras.

For a word $w \in R$, let $|w|_x$ (resp., $|w|_y$, $|w|_\theta$) denote the number of occurrences of x (resp, y , θ) in w . Note that the homological degree of w is $|w|_\theta$. Let $|w| := |w|_x + |w|_y + |w|_\theta$ be the length of w and keep the notation Tr_{2n+1} for the composite map $R \rightarrow R_{\mathfrak{h}, \sigma} \rightarrow k[\text{Rep}_{\mathfrak{so}_{2n+1}}(\mathcal{L})]^{\text{SO}_{2n+1}}$.

Lemma 8.13. $(\Phi \circ \text{Tr}_{2n+1})(w) = (1 + (-1)^{|w|}) \sum_{i=1}^n x_i^{|w|_x} y_i^{|w|_y} \theta_i^{|w|_\theta}$ for any $w \in R$.

In particular, if $|w|$ is of odd length, or of homological degree ≥ 2 , then $(\Phi \circ \text{Tr}_{2n+1})(w)$ vanishes. Lemmas 8.12 and 8.13 enable us to explicitly compute the map

$$H_\bullet(\Phi) \circ \text{Tr}_{2n+1} : \overline{\text{HD}}_\bullet(A) \rightarrow k[x_1, \dots, x_n, y_1, \dots, y_n, \theta_1, \dots, \theta_n]^{(\mathbb{Z}_2)^n \rtimes S_n}.$$

Lemma 8.14. We have

$$(H_\bullet(\Phi) \circ \text{Tr}_{2n+1})(x^k y^l) = 2 \sum_{i=1}^n x_i^k y_i^l \quad \text{for } k + l \text{ even,}$$

$$(H_\bullet(\Phi) \circ \text{Tr}_{2n+1})(x^k y^l dx) = 2l \sum_{i=1}^n x_i^k y_i^{l-1} \theta_i \quad \text{for } k + l \text{ even.}$$

Theorem 8.15. (i) $\Phi : k[\text{Rep}_{\mathfrak{so}_{2\infty+1}}(\mathcal{L})]^{\text{SO}_{2\infty+1}} \rightarrow k[x_1, \dots; y_1, \dots; \theta_1, \dots]^{(\mathbb{Z}_2)^\infty \rtimes S_\infty}$ is a quasi-isomorphism.

(ii) $H_\bullet(\Phi) : H_\bullet(\mathfrak{a}, \mathfrak{so}_{2n+1})^{\text{SO}_{2n+1}} \rightarrow k[x_1, \dots, x_n, y_1, \dots, y_n, \theta_1, \dots, \theta_n]^{(\mathbb{Z}_2)^n \rtimes S_n}$ is a surjection.

Proof. We first prove (i). By Proposition 8.11, it suffices to prove that the composite map

$$\text{Sym}(R_{\mathfrak{h}, \sigma}) \xrightarrow{\cong} k[\text{Rep}_{\mathfrak{so}_{2\infty+1}}(\mathcal{L})]^{\text{SO}_{2\infty+1}} \xrightarrow{\Phi} k[x_1, \dots; y_1, \dots; \theta_1, \dots]^{(\mathbb{Z}_2)^\infty \rtimes S_\infty}$$

is a quasi-isomorphism. This is equivalent to verifying that the map

$$\text{Sym}_k(\overline{\text{HD}}_\bullet(A)) \xrightarrow{\text{Sym}(H_\bullet(\Phi) \circ \text{Tr}_\infty)} k[x_1, \dots; y_1, \dots; \theta_1, \dots]^{(\mathbb{Z}_2)^\infty \rtimes S_\infty}$$

is an isomorphism. For $\mathfrak{a} := (s, p, l) \in \{0, 1\} \times \mathbb{Z}_{\geq 0}^2$, let $P_\mathfrak{a}$ denote the power sum $\sum_{i \geq 1} x_i^p y_i^l \theta_i^s$. Call an element $\mathfrak{a} := (s, p, l)$ of $\{0, 1\} \times \mathbb{Z}_{\geq 0}^2$ even if $s + p + l$ is even. By Lemma 8.12, $\overline{\text{HD}}_\bullet(A)$ can be identified with the bigraded vector space V spanned

by basis vectors $q_{\mathbf{a}}$ for $\mathbf{a} \in \{0, 1\} \times \mathbb{Z}_{\geq 0}^2$ even. The basis element $q_{(1,p,l)}$ corresponds to the form $\frac{1}{2(l+1)}x^p y^{l+1} dx$ in $\overline{\text{HD}}_1(A)$, and the basis element $q_{(0,p,l)}$ corresponds to the form $\frac{1}{2}x^p y^l$ in $\overline{\text{HD}}_0(A)$. Lemma 8.14 implies that $H_{\bullet}(\Phi) \circ \text{Tr}_{\infty}$ maps $q_{\mathbf{a}}$ to the power sum $P_{\mathbf{a}}$ for each even \mathbf{a} in $\{0, 1\} \times \mathbb{Z}_{\geq 0}^2$. (i) therefore follows once we verify that $k[x_1, \dots, y_1, \dots; \theta_1, \dots]^{(\mathbb{Z}_2)^{\infty} \times S_{\infty}}$ is isomorphic to the graded symmetric algebra generated by the power sums $P_{\mathbf{a}}$ for \mathbf{a} even.

To see this, note that for any $n \geq 2$, the orbit sum

$$O(x_1^{\alpha_1} \dots x_n^{\alpha_n} y_1^{\beta_1} \dots y_n^{\beta_n} \theta_1^{\gamma_1} \dots \theta_n^{\gamma_n}) := \sum_{\sigma \in (\mathbb{Z}_2)^n \rtimes S_n} \sigma(x_1^{\alpha_1} \dots x_n^{\alpha_n} y_1^{\beta_1} \dots y_n^{\beta_n} \theta_1^{\gamma_1} \dots \theta_n^{\gamma_n}) \tag{8.12}$$

is nonzero iff the triples $(\gamma_i, \alpha_i, \beta_i)$ are even for $1 \leq i \leq n$ and $(\gamma_i, \alpha_i, \beta_i) \neq (\gamma_j, \alpha_j, \beta_j)$ whenever $1 \leq i < j \leq n$ and $\gamma_i = \gamma_j = 1$. Starting with this observation and proceeding as in the proof of Proposition 4.8, one can show the following analog of Proposition 4.8 without difficulty.

Proposition 8.16. (i) *The homomorphism of bigraded commutative algebras*

$$\text{Sym}_k(V) \rightarrow k[x_1, \dots, x_n, y_1, \dots, y_n, \theta_1, \dots, \theta_n]^{(\mathbb{Z}_2)^n \times S_n}, \quad q_{\mathbf{a}} \mapsto P_{\mathbf{a}} \text{ for } \mathbf{a} \text{ even,}$$

induces an isomorphism of bigraded vector spaces

$$\text{Sym}_k^{\leq n}(V) \cong k[x_1, \dots, x_n, y_1, \dots, y_n, \theta_1, \dots, \theta_n]^{(\mathbb{Z}_2)^n \times S_n}.$$

(ii) *The homomorphism*

$$\text{Sym}_k(V) \rightarrow k[x_1, \dots, y_1, \dots; \theta_1, \dots]^{(\mathbb{Z}_2)^{\infty} \times S_{\infty}}, \quad q_{\mathbf{a}} \mapsto P_{\mathbf{a}},$$

is an isomorphism of bigraded commutative algebras.

Proposition 8.16(ii) is precisely what we needed to verify to complete the proof of (i). Proposition 8.16(i) implies that the natural map

$$k[x_1, \dots, y_1, \dots; \theta_1, \dots]^{(\mathbb{Z}_2)^{\infty} \times S_{\infty}} \rightarrow k[x_1, \dots, x_n, y_1, \dots, y_n, \theta_1, \dots, \theta_n]^{(\mathbb{Z}_2)^n \times S_n}$$

is surjective. (ii) is now immediate from (i). □

8.5.4. The even orthogonal and symplectic cases. The DG algebras on both sides of the Harish-Chandra homomorphism in the even orthogonal case coincide in the limit with their odd counterparts. Hence, the analog of Theorem 8.15(i) holds in the even orthogonal case as well. The Weyl group for \mathfrak{so}_{2n} is however $(\mathbb{Z}_2)^{n-1} \times S_n$ rather than $(\mathbb{Z}_2)^n \times S_n$. Here, $(\mathbb{Z}_2)^{n-1}$ is the subgroup of $(\mathbb{Z}_2)^n$ comprising those elements that flip the signs of an even number of the Cartan basis elements h_i . In this case, the map

$$k[x_1, \dots, y_1, \dots; \theta_1, \dots]^{(\mathbb{Z}_2)^{\infty} \times S_{\infty}} \rightarrow k[x_1, \dots, x_n, y_1, \dots, y_n, \theta_1, \dots, \theta_n]^{(\mathbb{Z}_2)^{n-1} \times S_n} \tag{8.13}$$

is not surjective for $n \geq 2$. For example, the element $x_1 \dots x_n$ of $k[x_1, \dots, x_n, y_1, \dots, y_n, \theta_1, \dots, \theta_n]^{(\mathbb{Z}_2)^{n-1} \rtimes S_n}$ is not in the image of (8.13). The argument we used to deduce part (ii) of Theorem 8.15 from part (i) in the odd orthogonal case does not therefore work in the even case.

Remark. Note, however, that the elements $x_1 \dots x_n$, and similarly the elements of the form

$$O(x_1 \dots x_k y_{k+1} \dots y_n) \quad \text{and} \quad O(x_1 \dots x_k y_{k+1} \dots y_{n-1} \theta_n),$$

where $O(\dots)$ denotes the orbit sum as in (8.12), do not lead to trivial counterexamples to Conjecture 8.1. In fact, these elements are the images of the following ‘Pfaffian’ cocycles in $k[\text{Rep}_{\mathfrak{so}_{2n}}(\mathcal{L})]^{SO_{2n}}$:

$$\sum_{\sigma} (-1)^{\sigma} x_{\sigma(1)\sigma(2)} \dots x_{\sigma(2k-1)\sigma(2k)} y_{\sigma(2k+1)\sigma(2k)} \dots y_{\sigma(2n-1)\sigma(2n)}$$

and

$$\sum_{\sigma} (-1)^{\sigma} x_{\sigma(1)\sigma(2)} \dots x_{\sigma(2k-1)\sigma(2k)} y_{\sigma(2k+1)\sigma(2k)} \dots y_{\sigma(2n-3)\sigma(2n-2)} \theta_{\sigma(2n-1)\sigma(2n)}.$$

A small computation shows that the latter element is indeed a cocycle, and it cannot be exact, since its expression contains no repeated indices, while repeated indices are necessarily introduced by the differential (recall that the differential of θ_{ij} is given by $\sum_k (x_{ik} y_{kj} - y_{ik} x_{kj})$).

The case of \mathfrak{sp}_{2n} is analogous to that of \mathfrak{so}_{2n+1} . The only modification here is a different involution on $\mathcal{M}_{2n}(k)$: it is given by $X \mapsto \mathcal{M}_{2n}^t X^t \mathcal{M}_{2n}$, where \mathcal{M}_{2n} is the matrix of the nondegenerate skew-symmetric bilinear form Q on k^{2n} satisfying

$$\begin{aligned} Q(e_{2i-1}, e_{2i}) &= -Q(e_{2i}, e_{2i-1}) = 1, & 1 \leq i \leq n, \\ Q(e_i, e_j) &= 0 & \text{otherwise.} \end{aligned}$$

In particular, the obvious analogues of both parts of Theorem 8.15 hold in the symplectic case.

9. Macdonald conjectures

In this section, we explain how our Conjecture 8.1 is related to the strong Macdonald conjecture proved in [FGT]. The key point is to consider the G -equivariant derived commuting scheme $\text{DRep}_{\mathfrak{g}}(\mathfrak{a})^G$ of the two-dimensional abelian Lie algebra \mathfrak{a} with ‘shifted’ homological degrees. Following the conventions of [FGT], we will assume in this section that $k = \mathbb{C}$. As in the previous section, \mathfrak{g} will denote an arbitrary finite-dimensional reductive Lie algebra with associated complex algebraic group G .

9.1. Graded commuting schemes

Let $\mathfrak{a} := \mathbb{C}.u \oplus \mathbb{C}.v$ be a homologically graded Lie algebra with trivial bracket, where u has degree -1 and v has degree -2 . Let $z \in (\mathfrak{a}[1])^*$ and $w \in (\mathfrak{a}[1])^*$ denote basis vectors dual to $su \in \mathfrak{a}[1]$ and $sv \in \mathfrak{a}[1]$ respectively. Thus z and w have homological degrees 0

and 1. We equip \mathfrak{a} with a \mathbb{Z}^2 -weight grading by setting the weight of u to be $(1, 0)$ and that of v to be $(0, 1)$. By convention, the weight of the dual of a finite-dimensional weight-homogeneous vector space W will coincide with that of W . Also, in this section, $(-)^*$ will mean bigraded dual.

Lemma 9.1. $\text{DRep}_{\mathfrak{g}}(\mathfrak{a})^G \cong \mathcal{C}(\mathfrak{g}[z, w], \mathfrak{g}; \mathbb{C})^*$.

Proof. Since \mathfrak{a} is abelian, a cofibrant resolution of \mathfrak{a} in DGL_k is given by $\Omega_{\text{Com}}(\text{Sym}^c(\mathfrak{a}[1]))$. Note that the bigraded dual of $\text{Sym}^c(\mathfrak{a}[1])$ is exactly $\mathbb{C}[z, w]$. By Theorem 6.7,

$$\text{DRep}_{\mathfrak{g}}(\mathfrak{a})^G \cong \mathcal{C}^c(\mathfrak{g}^*(\text{Sym}^c(\mathfrak{a}[1])), \mathfrak{g}^*; \mathbb{C}).$$

The desired lemma now follows from the isomorphism (6.8). □

We have the following restatement of the (now proven) strong Macdonald conjecture (see [FGT, Theorem 1.5]) in terms of derived representation schemes.

Theorem 9.2. *The map*

$$\text{Sym}_k[\text{Tr}_{\mathfrak{g}}(\mathfrak{a})] := \text{Sym}_k\left[\bigoplus_{i=1}^l \text{Tr}_{\mathfrak{g}}^{(d_i)}\right] : \text{Sym}_k\left[\bigoplus_{i=1}^l \mathbb{L}\lambda^{(d_i)}(\mathfrak{a})\right] \rightarrow \text{DRep}_{\mathfrak{g}}(\mathfrak{a})^G$$

is a quasi-isomorphism. In particular, the induced map

$$\text{Sym}_k[\text{Tr}_{\mathfrak{g}}(\mathfrak{a})] : \text{Sym}_k\left[\bigoplus_{i=1}^l \text{HC}_{\bullet}^{(d_i)}(\text{Lie}, \mathfrak{a})\right] \rightarrow \mathbf{H}_{\bullet}(\mathfrak{a}, \mathfrak{g})^G \tag{9.1}$$

is an isomorphism of graded algebras.

Proof. Since the bigraded dual of $\text{Sym}^c(\mathfrak{a}[1])$, namely $\mathbb{C}[z, w]$, is a smooth graded commutative algebra, Proposition 7.8 applies. Thus, the map (9.1) becomes

$$\text{Sym}_k[\text{Tr}_{\mathfrak{g}}(\mathfrak{a})] : \text{Sym}_k\left[\bigoplus_{i=1}^l \overline{\text{HC}}_{\bullet-1}^{(m_i)}(\mathbb{C}[z, w])^*\right] \rightarrow \mathcal{C}(\mathfrak{g}[z, w], \mathfrak{g}; \mathbb{C})^*, \tag{9.2}$$

where $m_i, 1 \leq i \leq r$, are the exponents of \mathfrak{g} . It turns out that (9.2) is precisely the map in [Te, (3.1)]; [FGT, Theorem 1.5] states that this map is a quasi-isomorphism. □

Corollary 9.3. $\mathbf{H}_{\bullet}(\mathfrak{a}, \mathfrak{g})^G$ is isomorphic to the symmetric algebra with one generator in homological degree $-(2m + 1)$ and weight $(n + 1, m)$ and one generator of homological degree $-(2m + 2)$ and weight $(n, m + 1)$ for each exponent m of \mathfrak{g} and each $n \geq 0$.

Proof. Indeed, [FGT, (1.7)] shows that $\overline{\text{HC}}_{2m+1}^{(m)}(\mathbb{C}[z, w])$ is identified with the (weight graded) vector space $\mathbb{C}[z].w(dw)^m$ for each exponent m of \mathfrak{g} . The duals of the basis elements $z^n w(dw)^m$ of $\overline{\text{HC}}_{2m+1}^{(m)}(\mathbb{C}[z, w])$ give generators of $\text{Sym}_k[\bigoplus_{i=1}^l \overline{\text{HC}}_{\bullet-1}^{(m_i)}(\mathbb{C}[z, w])^*]$ having homological degree $-2m - 2$ and weight $(n, m + 1)$ for each $n \geq 0$ and for each exponent m of \mathfrak{g} . Similarly, [FGT, (1.7)] shows that for each exponent m of \mathfrak{g} , $\overline{\text{HC}}_{2m}^{(m)}(\mathbb{C}[z, w])$ is identified with the (weight graded) vector space $\mathbb{C}[z].dzw(dw)^{m-1}$. The duals of $z^n dzw(dw)^{m-1}$ give generators of $\text{Sym}_k[\bigoplus_{i=1}^l \overline{\text{HC}}_{\bullet-1}^{(m_i)}(\mathbb{C}[z, w])^*]$ having homological degree $-2m - 1$ and weight $(n + 1, m)$. Theorem 9.2 (more precisely, its proof) then implies the desired corollary. □

9.2. Lie algebra (co)homology and parity

For a fixed pair (p, r) of integers, let $\mathfrak{a}_{p,r} := \mathbb{C}\langle u \rangle \oplus \mathbb{C}\langle v \rangle$ denote the homologically graded abelian Lie algebra, with u and v having degrees p and r respectively. We now demonstrate that much of the behavior of $\text{DRep}_{\mathfrak{g}}(\mathfrak{a}_{p,r})^G$ depends only on the parity of p and r .

9.2.1. *Functors on complexes.* Let Γ be an abelian group. Let $\Gamma\text{-Com}$ denote the category of Γ -weight graded complexes of \mathbb{C} -vector spaces. In other words, any $V \in \Gamma\text{-Com}$ is a direct sum of subcomplexes,

$$V := \bigoplus_{\gamma \in \Gamma} V_{\gamma}.$$

Here, V_{γ} is the subcomplex of V of weight γ . Let $\Gamma\text{-Com}_2$ denote the category of Γ -weight graded \mathbb{Z}_2 -homologically graded complexes. One has a functor

$$\Pi : \Gamma\text{-Com} \rightarrow \Gamma\text{-Com}_2, \quad V \mapsto \left(\bigoplus_{n \in \mathbb{Z}} V_{2n} \right) \rightleftharpoons \left(\bigoplus_{n \in \mathbb{Z}} V_{2n+1} \right).$$

The functor Π (for ‘parity’) remembers only the parity of the homological grading while retaining the Γ -weight grading. Note that the functor Π is faithful but not full. For most of this section, $\Gamma = \mathbb{Z}^2$. The following lemma is obvious.

Lemma 9.4. *A morphism $\phi \in \Gamma\text{-Com}$ is a quasi-isomorphism iff $\Pi(\phi)$ is a quasi-isomorphism in $\Gamma\text{-Com}_2$.*

For any $k \geq 1$, there is a functor

$$F_k : \mathbb{Z}^2\text{-Com} \rightarrow \mathbb{Z}\text{-Com}$$

that assigns to a \mathbb{Z}^2 -weight graded complex $V = \bigoplus_{(a,b) \in \mathbb{Z}^2} V_{(a,b)}$ the \mathbb{Z} -weight graded complex $F_k(V)$ with component of weight r being $F_k(V)_r = \bigoplus_{a+kb=r} V_{(a,b)}$. Note that F_k only changes weights without changing the homological grading or differential.

9.2.2. As before, we equip u and v with \mathbb{Z}^2 -weights $(1, 0)$ and $(0, 1)$ respectively. The proof of the following lemma is essentially the same as that of Lemma 9.1.

Lemma 9.5. *There is an isomorphism of DG algebras*

$$\text{DRep}_{\mathfrak{g}}(\mathfrak{a}_{p,r})^G \cong \mathcal{C}(\mathfrak{g}[z, w], \mathfrak{g}; \mathbb{C})^*,$$

where z has degree $-(p + 1)$ and weight $(1, 0)$ and w has degree $-(r + 1)$ and weight $(0, 1)$.

Hence, we get

Proposition 9.6. (i) *The weighted Euler characteristic of $\text{DRep}_{\mathfrak{g}}(\mathfrak{a}_{p,r})^G$ depends only on the parities of p and r .*
 (ii) *If either the derived Harish-Chandra map*

$$\Phi_{\mathfrak{g}} : \text{DRep}_{\mathfrak{g}}(\mathfrak{a})^G \rightarrow \mathbf{Sym}_{\mathbb{C}}(\mathfrak{h}^* \otimes \overline{\mathbf{Sym}^c(\mathfrak{a}[1])})^W$$

or the Drinfeld trace map

$$\mathbf{Sym}_k[\mathrm{Tr}_{\mathfrak{g}}(\mathfrak{a})] : \mathbf{Sym}_k\left[\bigoplus_{i=1}^l \mathbb{L}\lambda^{(d_i)}(\mathfrak{a})\right] \rightarrow \mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})^G$$

is a quasi-isomorphism for $\mathfrak{a}_{p,r}$, then it is a quasi-isomorphism for $\mathfrak{a}_{p',r'}$ whenever p, p' and r, r' have the same parity.

Proof. Note that if p, p' and r, r' have the same parity, then

$$\Pi[\Phi_{\mathfrak{g}}(\mathfrak{a}_{p,r})] = \Pi[\Phi_{\mathfrak{g}}(\mathfrak{a}_{p',r'})], \quad \Pi(\mathbf{Sym}_k[\mathrm{Tr}_{\mathfrak{g}}(\mathfrak{a}_{p,r})]) = \Pi(\mathbf{Sym}_k[\mathrm{Tr}_{\mathfrak{g}}(\mathfrak{a}_{p',r'})]).$$

The desired proposition therefore follows from Lemma 9.4. □

Thus, when p, r are both *even*, we expect that the derived Harish-Chandra homomorphism gives the quasi-isomorphism

$$\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a}_{p,r})^G \xrightarrow{\sim} \mathbf{Sym}(\mathfrak{h}^*[p] \oplus \mathfrak{h}^*[r] \oplus \mathfrak{h}^*[p+r+1])^W,$$

and we get Conjectures 8.1 and 8.5.

On the other hand, when p, r are of *opposite* parity, by Theorem 9.2, the Drinfeld trace map must be a quasi-isomorphism, so that the homology of $\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a}_{p,r})^G$ is a free graded commutative algebra, and we get the classical (q, t) -Macdonald identity (see Section 9.3 below).

9.3. Euler characteristics

Let $B_{\mathfrak{g}} := \mathcal{C}[\mathfrak{g}[z, w]/\mathfrak{g}; \mathbb{C}]$. It follows from Lemma 9.1 that the weighted Euler characteristic of $\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})^G$ coincides with that of its bigraded dual, namely, $B_{\mathfrak{g}}^{\mathrm{ad}\mathfrak{g}} = \mathcal{C}(\mathfrak{g}[z, w], \mathfrak{g}; \mathbb{C})$. We compute this Euler characteristic by computing the character-valued Euler characteristic

$$\chi(B_{\mathfrak{g}}, q, t, e^h) := \sum_{a,b \geq 0} \sum_{i \in \mathbb{Z}} (-1)^i \mathrm{Tr}(e^h|_{(B_{\mathfrak{g}})_{a,b}}) q^a t^b.$$

Lemma 9.7.

$$\chi(B_{\mathfrak{g}}, q, t, e^h) = \prod_{n \geq 0} \left[\left(\frac{1 - q^{n+1}}{1 - q^n t} \right)^l \prod_{\alpha \in R} \frac{1 - q^{n+1} e^\alpha}{1 - q^n t e^\alpha} \right],$$

where $l := \dim_{\mathbb{C}}(\mathfrak{h})$ is the rank of \mathfrak{g} .

Proof. The proof is identical to that of Lemma 8.3. □

Note that $B_{\mathfrak{g}}^G = B_{\mathfrak{g}}^{\mathrm{ad}\mathfrak{g}}$, where G is the complex reductive Lie group whose Lie algebra is \mathfrak{g} . Hence, we get

Corollary 9.8.

$$\chi(B_{\mathfrak{g}}^{\mathrm{ad}\mathfrak{g}}, q, t) = \frac{1}{|W|} \prod_{n \geq 0} \left(\frac{1 - q^{n+1}}{1 - q^n t} \right)^l \mathrm{CT} \left\{ \prod_{n \geq 0} \prod_{\alpha \in R} \frac{1 - q^n e^\alpha}{1 - q^n t e^\alpha} \right\}. \tag{9.3}$$

Proof. The proof is similar to the proof of Corollary 8.4 from Lemma 8.3. □

As a consequence, we obtain the following proposition:

Proposition 9.9. *The following identity holds:*

$$\frac{1}{|W|} \text{CT} \left\{ \prod_{n \geq 0} \prod_{\alpha \in R} \frac{1 - q^n e^\alpha}{1 - q^n t e^\alpha} \right\} = \prod_{n \geq 0} \prod_{i=1}^l \frac{(1 - q^n t)(1 - q^{n+1} t^{m_i})}{(1 - q^{n+1})(1 - q^n t^{m_i+1})}, \tag{9.4}$$

where $m_i := d_i - 1$ are the exponents of the Lie algebra \mathfrak{g} .

Proof. Let l be the rank of \mathfrak{g} . The Euler characteristic of $\text{DRep}_{\mathfrak{g}}(\mathfrak{a})^G$ equals that of its homology. By Corollary 9.3, the latter Euler characteristic is equal to

$$\prod_{n \geq 0} \prod_{i=1}^l \frac{1 - q^{n+1} t^{m_i}}{1 - q^n t^{m_i+1}}. \tag{9.5}$$

On the other hand, by Lemma 9.1, the Euler characteristic of $\text{DRep}_{\mathfrak{g}}(\mathfrak{a})^G$ is equal to that of $B_{\mathfrak{g}}^{\text{ad } \mathfrak{g}}$. By Corollary 9.8, the Euler characteristic of $B_{\mathfrak{g}}^{\text{ad } \mathfrak{g}}$ also equals the right-hand side of (9.3). Equating the right-hand side of (9.3) with the expression (9.5) and multiplying both sides by $\prod_{n \geq 0} \left(\frac{1 - q^{n+1}}{1 - q^n t}\right)^l$, we obtain the desired identity. □

Remark. The left-hand side of (9.4) is exactly the expression denoted by $\frac{1}{|W|} [\Delta_{q^{1/2}, t^{1/2}}]_0$ in [Kir]. Thus (9.4) is equivalent to the standard (q, t) -version of Macdonald’s constant term identity [Kir, (2.7)], which is the special case of the inner product identity for $P_\lambda = 1$ [Kir, Theorem 2.4].

9.3.1. The q -Macdonald identity. Let $V \in \mathbb{Z}^2\text{-Com}$ be such that the subcomplexes $V_{a,b}$ of weight (a, b) are finite-dimensional and nonzero only for $a, b \geq 0$. Recall the definition of the functor $F_k : \mathbb{Z}^2\text{-Com} \rightarrow \mathbb{Z}\text{-Com}$ above. There is an equality of Euler characteristics

$$\chi(V, q, q^k) = \chi(F_k(V), q).$$

As a result, we get

Lemma 9.10.

$$\chi(\mathcal{C}(\mathfrak{g}[z]/z^k, \mathfrak{g}; \mathbb{C}), q) = \chi(B_{\mathfrak{g}}^{\text{ad } \mathfrak{g}}, q, q^k).$$

Proof. Consider the DG Lie algebra $\mathfrak{g}_\partial[z, w] := (\mathfrak{g}[z, w], \partial w = z^k)$, which is quasi-isomorphic to $\mathfrak{g}[z]/z^k$. Hence, $\mathcal{C}(\mathfrak{g}_\partial[z, w], \mathfrak{g}; \mathbb{C})$ is quasi-isomorphic to $\mathcal{C}(\mathfrak{g}[z]/z^k, \mathfrak{g}; \mathbb{C})$. Note that the differential on $\mathcal{C}(\mathfrak{g}_\partial[z, w], \mathfrak{g}; \mathbb{C})$ is obtained by twisting that on $B_{\mathfrak{g}}^{\text{ad } \mathfrak{g}} = \mathcal{C}(\mathfrak{g}[z, w], \mathfrak{g}; k)$ with the differential induced by ∂ . The latter differential also preserves weights provided we apply the functor F_k to $B_{\mathfrak{g}}^{\text{ad } \mathfrak{g}}$. Hence,

$$\chi(\mathcal{C}(\mathfrak{g}[z]/z^k, \mathfrak{g}; \mathbb{C}), q) = \chi(\mathcal{C}(\mathfrak{g}_\partial[z, w], \mathfrak{g}; \mathbb{C}), q) = \chi(F_k(B_{\mathfrak{g}}^{\text{ad } \mathfrak{g}}), q) = \chi(B_{\mathfrak{g}}^{\text{ad } \mathfrak{g}}, q, q^k). \tag{9.6}$$

On the other hand, the identity (9.4) implies

Lemma 9.11. *The following identity holds:*

$$\frac{1}{|W|} \text{CT} \left\{ \prod_{j=0}^{k-1} \prod_{\alpha \in R} (1 - q^j e^\alpha) \right\} = \prod_{i=1}^l \prod_{j=1}^{k-1} \frac{1 - q^{km_i+j}}{1 - q^j} = \prod_{i=1}^l \binom{kd_i - 1}{k - 1}_q \tag{9.6}$$

Proof. Setting $t = q^k$ in (9.4), we obtain (9.6). Equivalently, we can multiply both sides of (9.6) by $\prod_{j=1}^{k-1} (1 - q^j)^l$ to obtain

$$\frac{1}{|W|} \prod_{j=1}^{k-1} (1 - q^j)^l \text{CT} \left\{ \prod_{j=0}^{k-1} \prod_{\alpha \in R} (1 - q^j e^\alpha) \right\} = \prod_{i=1}^l \prod_{j=1}^{k-1} (1 - q^{km_i+j}) \tag{9.7}$$

By Corollary 9.8 and Lemma 9.10, the left-hand side of the above identity is the Euler characteristic of $\mathcal{C}(\mathfrak{g}[z]/z^k, \mathfrak{g}; \mathbb{C})$. That this Euler characteristic equals the right-hand side of the above identity follows from [FGT, Theorem A], which explicitly computes $H_\bullet(\mathfrak{g}[z]/z^k, \mathfrak{g}; \mathbb{C})$ as a free bigraded commutative algebra and specifies the weights and homological degrees of a set of homogeneous generators. We remind the reader that [FGT, Theorem A] is proven from [FGT, Theorem 1.5] by replacing $\mathfrak{g}[z]/z^k$ by the quasi-isomorphic DG Lie algebra $\mathfrak{g}_\partial[z, w]$ and computing $H_\bullet(\mathfrak{g}_\partial[z, w], \mathfrak{g}; \mathbb{C})$ by appealing to [FGT, Theorem 1.5] (which computes $H_\bullet(\mathfrak{g}[z, w], \mathfrak{g}; \mathbb{C})$) and using a simple spectral sequence argument. \square

Remark. Being the Euler characteristic of $\mathcal{C}(\mathfrak{g}[z]/z^k, \mathfrak{g}; \mathbb{C})$, the left-hand side of (9.7) can be rewritten as

$$\int_G \prod_{j=1}^{k-1} \det(1 - q^j \text{Ad } g) dg.$$

The identity (9.6) is thus equivalent to

$$\int_G \prod_{j=1}^{k-1} \det(1 - q^j \text{Ad } g) dg = \prod_{i=1}^l \prod_{j=1}^{k-1} (1 - q^{km_i+j}), \tag{9.8}$$

which is a version of the original Macdonald identity (see [M, Conjecture 3.1’]).

Appendix. Derived representation schemes of algebras over an operad

In this Appendix, we construct (derived) representation schemes for (DG) algebras over an arbitrary binary quadratic operad. Our generalization covers the representation schemes of associative algebras studied in [BKR, BR1], the representation algebras defined in [Tu] as well as the representation schemes of Lie algebras introduced in Section 6. We also construct canonical trace maps from operadic cyclic homology to representation homology, generalizing the derived character maps of [BKR]. For cyclic operads our notion of operadic cyclic homology agrees with that of [GK]. The main result of this section, Theorem A.4, unifies Theorems 3.2 and 6.7.

Throughout this section, \mathcal{P} will denote a finitely generated binary quadratic operad and $\mathcal{Q} = \mathcal{P}^\dagger$ will stand for its (quadratic) Koszul dual. Unlike Getzler and Kapranov [GK] who work with cyclic operads, which are not necessarily binary quadratic, we will work

with binary quadratic operads, which are not necessarily cyclic. This is due to the following fact that we need for Theorem A.4: if A is a DG \mathcal{P} -algebra and if B is a DG \mathcal{Q} -algebra then $A \otimes B$ has a natural DG Lie algebra structure (see [LV, Prop. 7.6.5]).

A.1. *Internal Hom-functor and the representation functor*

Let DGPA (resp., DGQA) denote the category of DG \mathcal{P} -algebras (resp., DG \mathcal{Q} -algebras) over k . If \mathfrak{C} is a DG \mathcal{P} -coalgebra, the complex $\mathbf{Hom}(\mathfrak{C}, \bar{B})$ naturally acquires the structure of a DG \mathcal{P} -algebra for any $B \in \text{DGCA}_{k/k}$. We therefore have a functor

$$\mathbf{Hom}_*(\mathfrak{C}, -) : \text{DGCA}_{k/k} \rightarrow \text{DGPA}, \quad B \mapsto \mathbf{Hom}(\mathfrak{C}, \bar{B}),$$

where $*$ indicates that we form the convolution algebra with the augmentation ideal \bar{B} . Let $A \in \text{DGPA}$. Each generating operation $m \in \mathcal{P}(2)$ gives k -linear maps

$$m_A : A \otimes A \rightarrow A, \quad m_{\mathfrak{C}} : \mathfrak{C} \rightarrow \mathfrak{C} \otimes \mathfrak{C}.$$

For $a, b \in A$ and $x \in \mathfrak{C}$, define $(m_{\mathfrak{C}}(x), a, b) \in \mathbf{Sym}_k(\mathfrak{C} \otimes A)$ to be the image of the element $x \otimes a \otimes b \in \mathfrak{C} \otimes A \otimes A$ under the composite map

$$\begin{aligned} \mathfrak{C} \otimes A \otimes A &\xrightarrow{m_{\mathfrak{C}} \otimes \text{Id} \otimes \text{Id}} \mathfrak{C} \otimes \mathfrak{C} \otimes A \otimes A \xrightarrow{\text{Id} \otimes \tau_{23} \otimes \text{Id}} (\mathfrak{C} \otimes A) \otimes (\mathfrak{C} \otimes A) \\ &\xrightarrow{\text{can}} \mathbf{Sym}^2(\mathfrak{C} \otimes A) \hookrightarrow \mathbf{Sym}_k(\mathfrak{C} \otimes A). \end{aligned}$$

The proof of the following proposition is a straightforward generalization of that of Proposition 6.4. We therefore leave it to the interested reader.

Proposition A.1. *The functor $\mathbf{Hom}_*(\mathfrak{C}, -) : \text{DGCA}_{k/k} \rightarrow \text{DGPA}$ has a left adjoint*

$$\mathfrak{C} \times - : \text{DGPA}_k \rightarrow \text{DGCA}_{k/k}, \quad A \mapsto \mathfrak{C} \times A := \mathbf{Sym}_k(\mathfrak{C} \otimes A) / \mathcal{I}_{A, \mathfrak{C}},$$

where $\mathcal{I}_{A, \mathfrak{C}}$ is the ideal generated by the elements

$$x \otimes m_A(a, b) - (m_{\mathfrak{C}}(x), a, b), \quad m \in \mathcal{P}(2), \quad x \in \mathfrak{C}, a, b \in A.$$

In particular, there is a natural isomorphism

$$\text{Hom}_{\text{DGPA}}(A, \mathbf{Hom}_*(\mathfrak{C}, B)) \cong \text{Hom}_{\text{DGCA}_{k/k}}(\mathfrak{C} \times A, B). \tag{A.1}$$

The category DGPA has a natural model structure where fibrations are degreewise surjections and weak equivalences are quasi-isomorphisms (see [H]). It is easy to verify that the functor $\mathbf{Hom}_*(\mathfrak{C}, -)$ preserves fibrations (i.e. degreewise surjections) and acyclic fibrations. Hence, we have

Theorem A.2. (a) *The following functors form a Quillen pair:*

$$\mathfrak{C} \times - : \text{DGPA} \rightleftarrows \text{DGCA}_{k/k} : \mathbf{Hom}_*(\mathfrak{C}, -).$$

(b) *The functor $\mathfrak{C} \times - : \text{DGPA} \rightleftarrows \text{DGCA}_{k/k}$ has a (total) left derived functor*

$$\mathfrak{C} \times^{\mathbb{L}} - : \text{Ho}(\text{DGPA}) \rightarrow \text{Ho}(\text{DGCA}_{k/k}), \quad A \mapsto \mathfrak{C} \times QA,$$

where $QA \xrightarrow{\sim} A$ is any cofibrant resolution of A in DGPA.

(c) For any $A \in \text{DGPA}$ and $B \in \text{DGCA}_{k/k}$, there is a canonical isomorphism

$$\text{Hom}_{\text{Ho}(\text{DGPA})}(A, \mathbf{Hom}(\mathcal{C}, \bar{B})) \cong \text{Hom}_{\text{Ho}(\text{DGCA}_{k/k})}(\mathcal{C} \times A, B).$$

We define the *derived representation scheme* of A over the coalgebra \mathcal{C} by

$$\text{DRep}_{\mathcal{C}}(A) := \mathcal{C} \overset{\mathbb{L}}{\times} A \in \text{Ho}(\text{DGCA}_{k/k})$$

and the *representation homology* of A over \mathcal{C} by

$$\mathbf{H}_{\bullet}(A, \mathcal{C}) := \mathbf{H}_{\bullet}[\text{DRep}_{\mathcal{C}}(A)].$$

A.2. Representation homology vs Lie homology: the operadic setting

Recall that \mathcal{Q} denotes the Koszul dual of the operad \mathcal{P} . The main result of this section, Theorem A.4, identifies the representation homology $\mathbf{H}_{\bullet}(A, \mathcal{C})$ with the homology of the Lie coalgebra $\mathcal{C} \otimes C$ constructed over a \mathcal{Q} -coalgebra C Koszul dual to A ; this unifies Theorems 3.2 and 6.7 in the main body of the paper. We begin by recalling some facts on twisting chains and Koszul duality in the operadic setting. The main reference for this material is [LV].

A.2.1. Twisting chains and Koszul duality. Let DGQC denote the category of conilpotent DG \mathcal{Q} -coalgebras. Let $A \in \text{DGPA}$ and $C \in \text{DGQC}$. The following proposition is a special case of [LV, Proposition 11.1.1].

- Lemma A.3.** (a) *The complex $\mathbf{Hom}(C, A)$ has the natural structure of a DG Lie algebra.*
 (b) *If \mathcal{C} is a DG \mathcal{P} -coalgebra, the complex $\mathcal{C} \otimes C$ has the natural structure of a (conilpotent) DG Lie coalgebra.*

Proof. We note that $\mathbf{Hom}(C, A)$ has the natural structure of an algebra over the Hadamard product $\mathcal{P} \otimes_{\mathbb{H}} \mathcal{Q}$ of the operads \mathcal{P} and \mathcal{Q} (see [LV, Section 5.1.12]). The Lie algebra structure on $\mathbf{Hom}(C, A)$ comes from the morphism of operads $\text{Lie} \rightarrow \mathcal{P} \otimes_{\mathbb{H}} \mathcal{Q}$ in [LV, Prop. 7.6.5]. This proves (a).

On the other hand, $\mathcal{C} \otimes C$ is naturally a DG coalgebra over the operad $\mathcal{P} \otimes_{\mathbb{H}} \mathcal{Q}$. The DG Lie coalgebra structure on $\mathcal{C} \otimes C$ then comes from the morphism of operads $\text{Lie} \rightarrow \mathcal{P} \otimes_{\mathbb{H}} \mathcal{Q}$ in [LV, Prop. 7.6.5]. The conilpotency of $\mathcal{C} \otimes C$ follows from the conilpotency of C . This proves (b). \square

A *twisting chain* from C to A is a degree -1 element $\tau \in \mathbf{Hom}(C, A)$ satisfying the Maurer–Cartan equation

$$d\tau + \frac{1}{2}[\tau, \tau] = 0.$$

Let $\text{Tw}(C, A)$ denote the set of all twisting chains from C to A . It can be shown (see [LV, Prop. 11.3.1]) that, for a fixed \mathcal{P} -algebra A , the functor

$$\text{Tw}(-, A) : \text{DGQC} \rightarrow \text{Sets}, \quad C \mapsto \text{Tw}(C, A),$$

is representable; the corresponding coalgebra $\mathbf{B}_{\mathcal{P}}(A) \in \text{DGQC}$ is called the *bar construction* of A (see [LV, Sect. 11.2.1]). Dually, for a fixed coalgebra C , the functor

$$\text{Tw}(C, -) : \text{DGPA} \rightarrow \text{Sets}, \quad A \mapsto \text{Tw}(C, A),$$

is corepresentable; the corresponding algebra $\mathbf{\Omega}_{\mathcal{Q}}(C) \in \text{DGPA}$ is called the *cobar construction* of C (see [LV, Sect. 11.2.5]). Thus, we have canonical isomorphisms

$$\text{Hom}_{\text{DGPA}}(\mathbf{\Omega}_{\mathcal{Q}}(C), A) = \text{Tw}(C, A) = \text{Hom}_{\text{DGQC}}(C, \mathbf{B}_{\mathcal{P}}(A)) \tag{A.2}$$

showing that $\mathbf{\Omega}_{\mathcal{Q}} : \text{DGQC} \rightleftarrows \text{DGQC} : \mathbf{B}_{\mathcal{P}}$ are adjoint functors. We say that $C \in \text{DGQC}$ is *Koszul dual* to $A \in \text{DGPA}$ if there is a quasi-isomorphism $\mathbf{\Omega}_{\mathcal{Q}}(C) \xrightarrow{\sim} A$ in DGPA. Note that since $\mathbf{\Omega}_{\mathcal{Q}}(C)$ is free as a graded \mathcal{P} -algebra (see [LV, Sect. 11.2.5]), $\mathbf{\Omega}_{\mathcal{Q}}(C) \xrightarrow{\sim} A$ is a cofibrant resolution of A in DGPA when C is Koszul dual to A . Further, if \mathcal{P} is a Koszul operad, there is at least one $C \in \text{DGQC}$ Koszul dual to A , namely, $\mathbf{B}_{\mathcal{P}}(A)$ (see [LV, Sect. 11.3.3]).

A.2.2. . For $C \in \text{DGQC}$ and for any DG \mathcal{P} -coalgebra \mathfrak{C} , let $\mathcal{L}ie^c(\mathfrak{C} \otimes C)$ denote $\mathfrak{C} \otimes C$ equipped with the Lie coalgebra structure from Lemma A.3.

Theorem A.4. *There is a natural isomorphism of functors from DGQC to $\text{DGCA}_{k/k}$:*

$$(\mathfrak{C} \times -) \circ \mathbf{\Omega}_{\mathcal{Q}}(-) \cong \mathbf{\Omega}_{\text{Lie}}[\mathcal{L}ie^c(\mathfrak{C} \otimes -)].$$

As a result, if $C \in \text{DGQC}$ is Koszul dual to $A \in \text{DGPA}$, there are isomorphisms in $\text{Ho}(\text{DGCA}_{k/k})$:

$$\text{DRep}_{\mathfrak{C}}(A) \cong \mathcal{C}^c(\mathcal{L}ie^c(\mathfrak{C} \otimes C); k).$$

Consequently,

$$\mathbf{H}_{\bullet}(A, \mathfrak{C}) \cong \mathbf{H}_{\bullet}(\mathcal{L}ie^c(\mathfrak{C} \otimes C); k).$$

Proof. Let $C \in \text{DGQC}$. For any $B \in \text{DGCA}_{k/k}$, we have natural isomorphisms

$$\begin{aligned} \text{Hom}_{\text{DGCA}_{k/k}}(\mathfrak{C} \times \mathbf{\Omega}_{\mathcal{Q}}(C), B) &\cong \text{Hom}_{\text{DGPA}}(\mathbf{\Omega}_{\mathcal{Q}}(C), \mathbf{Hom}(\mathfrak{C}, \bar{B})) \\ &\cong \text{Tw}(C, \mathbf{Hom}(\mathfrak{C}, \bar{B})) \cong \text{Tw}(\mathcal{L}ie^c(\mathfrak{C} \otimes C), \bar{B}) \\ &\cong \text{Hom}_{\text{DGCA}_{k/k}}(\mathbf{\Omega}_{\text{Lie}}[\mathcal{L}ie^c(\mathfrak{C} \otimes C)], B). \end{aligned}$$

The DG Lie algebra structure on $\mathbf{Hom}(C, \mathbf{Hom}(\mathfrak{C}, \bar{B}))$ comes from Lemma A.3 and the fact that $\mathbf{Hom}(\mathfrak{C}, \bar{B})$ acquires the structure of a \mathcal{P} -algebra. The third isomorphism above is from the isomorphism of DG Lie algebras $\mathbf{Hom}(C, \mathbf{Hom}(\mathfrak{C}, \bar{B})) \cong \mathbf{Hom}(\mathcal{L}ie^c(\mathfrak{C} \otimes C), \bar{B})$ coming from hom-tensor duality (and Lemma A.3). The natural isomorphism

$$\mathfrak{C} \times \mathbf{\Omega}_{\mathcal{Q}}(C) \cong \mathbf{\Omega}_{\text{Lie}}[\mathcal{L}ie^c(\mathfrak{C} \otimes C)] = \mathcal{C}^c(\mathcal{L}ie^c(\mathfrak{C} \otimes C); k)$$

follows from the natural isomorphism

$$\text{Hom}_{\text{DGCA}_{k/k}}(\mathfrak{C} \times \mathbf{\Omega}_{\mathcal{Q}}(C), B) \cong \text{Hom}_{\text{DGCA}_{k/k}}(\mathbf{\Omega}_{\text{Lie}}[\mathcal{L}ie^c(\mathfrak{C} \otimes C)], B)$$

by Yoneda’s Lemma. This proves the first statement of the theorem.

If $C \in \text{DGQC}$ is Koszul dual to $A \in \text{DGPA}$, then $\Omega_{\mathcal{Q}}(C) \xrightarrow{\sim} A$ is a cofibrant resolution of A . Hence, $\text{DRep}_{\mathcal{C}}(A) \cong \mathcal{C} \times \Omega_{\mathcal{Q}}(C) \cong \mathcal{C}^c(\text{Lie}^c(\mathcal{C} \otimes C); k)$. This proves the second assertion in the above theorem. The final statement of the theorem now follows immediately. \square

A.3. Examples

A.3.1. Derived rerepresentation schemes for associative algebras. When $\mathcal{P} = \mathcal{Q} = \text{Ass}$, the operad governing associative algebras, DGPA becomes DGA, the category of *nonunital* DG k -algebras. Recall that there is an equivalence of categories (see Section 2.2.1):

$$\text{DGA} \rightarrow \text{DGA}_{k/k}, \quad A \mapsto A' := k \oplus A.$$

The inverse is the functor that assigns to each augmented DG algebra A its augmentation ideal \bar{A} . In this case, Proposition A.1 is a generalization of [Tu, Lemma 2.1] to the DG setting. Let \mathcal{C} be a counital DG coalgebra. Denote the composite functor

$$\text{DGA}_{k/k} \rightarrow \text{DGA} \xrightarrow{\mathcal{C} \times -} \text{DGCA}_{k/k}$$

by $\mathcal{C} \times -$. For $A \in \text{DGA}_{k/k}$, one has a natural isomorphism

$$\mathcal{C} \times A \cong (\mathcal{C} \triangleright A)_{\text{ab}},$$

where

$$\mathcal{C} \triangleright A := T_k(\mathcal{C} \otimes A) / \langle m \otimes ab - (m^{(1)} \otimes a) \otimes (m^{(2)} \otimes b), m \otimes 1 - u(m).1 \rangle. \quad (\text{A.3})$$

Here, $u(m) \in k$ and $\Delta(m) = \sum m^{(1)} \otimes m^{(2)}$ are the counit and the coproduct of \mathcal{C} evaluated at an element $m \in \mathcal{C}$. This operation is introduced in [AJ, Section 3.4], where it is called the *Sweedler product*.

Note that the algebra homomorphisms $\mathcal{C} \triangleright A \rightarrow A$ correspond precisely to the left actions $\mathcal{C} \otimes A \rightarrow A$ commuting with the multiplication $A \otimes A \rightarrow A$ and the unit map $k \rightarrow A$ on A . This clarifies the meaning of the defining relations and the notation for the algebra $\mathcal{C} \triangleright A$ (cf. [Maj]).

Now, if $\mathcal{C} = \mathcal{M}_n^*(k)$ is the finite-dimensional coalgebra dual to the matrix algebra $\mathcal{M}_n(k)$, the functor $\mathbf{Hom}(\mathcal{C}, -)$ is canonically isomorphic to $\mathcal{M}_n(-)$. Hence, by adjunction (see Proposition 2.3), we have a natural isomorphism

$$A_n \cong \mathcal{M}_n^*(k) \times A.$$

This allows one to regard the representation functor (2.7) as a special case of the general construction of this section. In particular, all results of Section 2.4 (along with Theorem 3.2) follow from results proved in this section (e.g., Theorem 2.6 is a special case of Theorem A.2, Theorem 3.2 is a special case of Theorem A.4, etc.)

Remark. The existence of the noncommutative ‘lifting’ $\mathcal{C} \triangleright A$ of $\mathcal{C} \times A$ appears to be a special phenomenon that occurs only in the case $\mathcal{P} = \mathcal{Q} = \text{Ass}$. In this case, the natural morphism of operads $\text{Lie} \rightarrow \text{Ass} \otimes_{\mathbb{H}} \text{Ass}$ factors through the associative operad. Hence, from the point of view of a coalgebra-algebra pair over a Koszul dual pair of operads, it is the abelianization of the Sweedler product (rather than the Sweedler product itself) that is the natural construction.

A.3.2. *Derived representation schemes of Lie algebras.* When $\mathcal{P} = \text{Lie}$, the operad governing Lie algebras, then $\mathcal{Q} = \text{Comm}$, the operad governing commutative algebras. In this case, the constructions of this section specialize to those of Section 6. For example, Proposition A.1 specializes to Proposition 6.4. Theorem A.4 specializes to Proposition 6.6 and Theorem 6.7, etc.

A.4. *\mathcal{P} -cyclic homology*

We now construct derived trace maps relating (operadic) cyclic homology to (operadic) representation homology. Our construction generalizes the construction of traces in [BKR] as well as Drinfeld traces of Section 7.3.

A.4.1. *Invariant bilinear forms.* We say that a symmetric bilinear form B on a DG \mathcal{P} -algebra is *invariant* if for all $m \in \mathcal{P}(2)$,

$$B(m(a, b), c) = (-1)^{|a||m|} B(a, m(b, c)) \quad \text{for all } a, b, c \in A.$$

Let A_{\natural} denote the target of the universal invariant bilinear form on A ; this is equal to the quotient of $A \otimes A$ by the subcomplex spanned by the images of the maps

$$(m \otimes \text{Id} - \text{Id} \otimes m) : A \otimes A \otimes A \rightarrow A \otimes A,$$

where m runs over all (homogeneous) elements of $\mathcal{P}(2)$. If \mathcal{P} is a *cyclic* binary quadratic operad, our notion of an invariant bilinear form agrees with that of [GK] (cf. [GK, Prop. 4.3]). In this case, A_{\natural} is denoted in [GK] by $\lambda(A)$.

Dually, if \mathcal{C} is a DG \mathcal{P} -coalgebra, a (homogeneous) 2-tensor α is called *invariant* if it is a (homogeneous) element of $\mathcal{C} \otimes \mathcal{C}$ such that for all $m \in \mathcal{P}(2)$, the composite map of complexes

$$k[|\alpha|] \xrightarrow{\alpha} \mathcal{C} \otimes \mathcal{C} \xrightarrow{m \otimes \text{Id} - \text{Id} \otimes m} \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C}$$

vanishes. We denote by \mathcal{C}^{\natural} the subcomplex of invariant tensors on \mathcal{C} .

The following theorem is a generalization of [GK, Theorem 5.3] to arbitrary binary quadratic operads. Since its proof is similar to that of Theorem 7.3, we omit it in order to avoid being repetitive.

Theorem A.5. *The functor*

$$(-)_{\natural} : \text{DGPA} \rightarrow \text{Com}_k, \quad A \mapsto A_{\natural},$$

has a (total) left derived functor

$$\mathbb{L}(-)_{\natural} : \text{Ho}(\text{DGPA}) \rightarrow \text{Ho}(\text{Com}_k), \quad A \mapsto R_{\natural},$$

for any cofibrant resolution $R \xrightarrow{\sim} A$ in DGPA.

We denote the homology $H_{\bullet}(\mathbb{L}A_{\natural})$ by $\text{HC}_{\bullet}(\mathcal{P}, A)$ and call it the *\mathcal{P} -cyclic homology* of A . When $\mathcal{P} = \text{Lie}$ and $A = \mathfrak{a}$, this is exactly the Lie cyclic homology $\text{HC}_{\bullet}(\text{Lie}, \mathfrak{a})$ used in Section 7 (which [GK] denotes by $\text{HA}_{\bullet}(\text{Lie}, A)$).

A.5. *Traces*

Suppose that the DG \mathcal{P} -coalgebra \mathcal{C} is equipped with a degree 0 invariant 2-tensor

$$\text{coTr} : k \rightarrow \mathcal{C}^\natural.$$

Then, for any $A \in \text{DGPA}$, there is a natural morphism of DG \mathcal{P} -algebras

$$\pi_A : A \rightarrow \mathbf{Hom}_*(\mathcal{C}, \mathcal{C} \times A) \hookrightarrow \mathbf{Hom}(\mathcal{C}, \mathcal{C} \times A),$$

where the first arrow corresponds under the adjunction (A.1) to the identity morphism on $\mathcal{C} \times A$. Denote by $\pi_{A \otimes A}$ the composite map

$$\begin{aligned} A \otimes A &\xrightarrow{\pi_A \otimes \pi_A} \mathbf{Hom}(\mathcal{C}, \mathcal{C} \times A) \otimes \mathbf{Hom}(\mathcal{C}, \mathcal{C} \times A) \\ &\rightarrow \mathbf{Hom}(\mathcal{C}^{\otimes 2}, (\mathcal{C} \times A)^{\otimes 2}) \rightarrow \mathbf{Hom}(\mathcal{C} \otimes \mathcal{C}, \mathcal{C} \times A), \end{aligned}$$

where the last arrow is induced by the product on $\mathcal{C} \times A$. The map of complexes

$$A \otimes A \xrightarrow{\pi_{A \otimes A}} \mathbf{Hom}(\mathcal{C} \otimes \mathcal{C}, \mathcal{C} \times A) \rightarrow \mathbf{Hom}(\mathcal{C}^\natural, \mathcal{C} \times A)$$

induced by the inclusion $\mathcal{C}^\natural \hookrightarrow \mathcal{C} \otimes \mathcal{C}$ clearly factors through A_{\natural} , giving

$$A_{\natural} \rightarrow \mathbf{Hom}(\mathcal{C}^\natural, \mathcal{C} \times A).$$

Composing this last map with the map $\mathbf{Hom}(\mathcal{C}^\natural, \mathcal{C} \times A) \rightarrow \mathbf{Hom}(k, \mathcal{C} \times A) = \mathcal{C} \times A$ induced by coTr , we get

$$\text{Tr}_{\mathcal{C}} : A_{\natural} \rightarrow \mathcal{C} \times A. \tag{A.4}$$

We remark that, in general, $\text{Tr}_{\mathcal{C}}$ depends on the choice of the invariant 2-tensor coTr on \mathcal{C} ; however, instead of making that choice, we could work with the universal invariant 2-tensor $\mathcal{C}^\natural \hookrightarrow \mathcal{C} \otimes \mathcal{C}$.

By construction, $\text{Tr}_{\mathcal{C}}$ gives a natural transformation of functors

$$(-)_{\natural} \rightarrow \mathcal{C} \times - : \text{DGPA} \rightarrow \text{Com}_k. \tag{A.5}$$

It therefore gives a natural transformation of *derived* functors

$$\text{Tr}_{\mathcal{C}} : \mathbb{L}(-)_{\natural} \rightarrow \mathcal{C} \overset{\mathbb{L}}{\times} - : \text{Ho}(\text{DGPA}) \rightarrow \text{Ho}(\text{Com}_k). \tag{A.6}$$

Applying this to $A \in \text{DGPA}$ and taking homologies, one obtains maps of graded vector spaces

$$\text{Tr}_{\mathcal{C}} : \text{HC}_{\bullet}(\mathcal{P}, A) \rightarrow \text{H}_{\bullet}(A, \mathcal{C}). \tag{A.7}$$

We now give a few examples of the trace (A.7).

A.5.1. Example. Let \mathcal{P} be a cyclic binary quadratic operad. Suppose that \mathfrak{C} is a finite-dimensional \mathcal{P} -coalgebra in homological degree 0. Let $\mathcal{S} := \mathfrak{C}^*$ and let $\text{tr} : \mathcal{S}_{\natural} \rightarrow k$ denote the dual of the map coTr . Let $G := \text{Aut}(\mathcal{S}, \text{tr})$ denote the algebraic group of trace preserving automorphisms of the \mathcal{P} -algebra \mathcal{S} (see [G, Section 6]). Suppose A is a \mathcal{P} -algebra concentrated in homological degree 0. In this case, $\mathfrak{C} \times A = k[\text{Rep}_{\mathcal{P}}(A, \mathcal{S})]$. The trace (A.7) induces a trace map

$$\text{Tr}_{\mathcal{S}} : A_{\natural} \rightarrow k[\text{Rep}_{\mathcal{P}}(A, \mathcal{S})]$$

on 0-th homologies. It is easy to verify that the image of $\text{Tr}_{\mathcal{S}}$ actually lies in $k[\text{Rep}_{\mathcal{P}}(A, \mathcal{S})]^G$. The map

$$\text{Tr}_{\mathcal{S}} : A_{\natural} \rightarrow k[\text{Rep}_{\mathcal{P}}(A, \mathcal{S})]^G$$

appears in [G, Section 6].

A.5.2. Example. Let $\mathcal{P} = \text{Ass}$ and let $\mathfrak{C} = \mathcal{M}_n^*(k)$. Let A be a unital DG k -algebra. Then (see [GK, Section 4]) $A_{\natural} \cong A/[A, A]$. Dually, since $\mathcal{M}_n^*(k)$ is counital, $[\mathcal{M}_n^*(k)]^{\natural}$ can be identified with the cocommutator subspace of $\mathcal{M}_n^*(k)$. This subspace is a one-dimensional subspace of $\mathcal{M}_n^*(k)$ generated by the dual of the usual trace map. Choosing that dual as our cotrace and applying the construction in this section, one sees that the trace map (A.7) becomes the trace map

$$\text{Tr}_n(A)_{\bullet} : \text{HC}_{\bullet}(A) \rightarrow H_{\bullet}(A, n)$$

constructed in [BKR, Section 4].

A.5.3. Example. Let $\mathcal{P} = \text{Lie}$ and let $\mathfrak{C} = \mathfrak{g}^*$ where \mathfrak{g} is a semisimple Lie algebra. Then \mathfrak{C}^{\natural} is a one-dimensional k -vector space generated by the Killing form on \mathfrak{g} . If we take the Killing form as our invariant 2-tensor, the trace (A.7) specializes to the canonical trace (7.10).

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References

- [AJ] Anel, M., Joyal, A.: Sweedler theory for (co)algebras and the bar-cobar constructions. [arXiv:1309.6952](https://arxiv.org/abs/1309.6952) (2013)
- [BF⁺] Berest, Yu., Felder, G., Patotski, A., Ramadoss, A. C., Willwacher, T.: Chern–Simons forms and higher character maps of Lie representations. *Int. Math. Res. Notices* **2017**, 158–212 [MR 3632101](https://doi.org/10.1093/imrn/rnx001)

- [BFR] Berest, Yu., Felder, G., Ramadoss, A.: Derived representation schemes and noncommutative geometry. In: Expository Lectures on Representation Theory, *Contemp. Math.* 607, Amer. Math. Soc., 113–162 (2014) [Zbl 1333.16014](#) [MR 3204869](#)
- [BKR] Berest, Yu., Khachatryan, G., Ramadoss, A.: Derived representation schemes and cyclic homology. *Adv. Math.* **245**, 625–689 (2013) [Zbl 1291.14006](#) [MR 3084440](#)
- [BR1] Berest, Yu., Ramadoss, A.: Stable representation homology and Koszul duality. *J. Reine Angew. Math.* **715**, 143–187 (2016) [Zbl 06591393](#) [MR 3507922](#)
- [BR2] Berest, Yu., Ramadoss, A.: Derived character varieties and skein relations. In preparation
- [BRZ] Berest, Yu., Ramadoss, A. C., Zhang, Y.: Dual Hodge decompositions and derived Poisson brackets. *Selecta Math.* **23**, 2029–2070 (2017) [MR 3663602](#)
- [BV] Burghelca, D., Vigué-Poirrier, M.: Cyclic homology of commutative algebras. In: Algebraic Topology—Rational Homotopy (Louvain-La-Neuve, 1986), *Lecture Notes in Math.* 1318, Springer, Berlin, 51–72 (1987) [Zbl 0666.13007](#) [MR 0952571](#)
- [Ch] Cherednik, I.: Double affine Hecke algebras and Macdonald’s conjectures. *Ann. of Math.* (2) **141**, 191–216 (1995) [Zbl 0822.33008](#) [MR 131403](#)
- [C] Chevalley, C.: The Betti numbers of the exceptional simple Lie groups. In: Proc. Int. Congress of Mathematicians (Cambridge, MA, 1950), Vol. II, Amer. Math. Soc., 21–24 (1952) [Zbl 0049.15704](#) [MR 00445316](#)
- [CG] Chriss, N., Ginzburg, V.: Representation Theory and Complex Geometry. Birkhäuser, Boston (1997) [Zbl 0879.22001](#) [MR 1433132](#)
- [CK] Ciocan-Fontanine, I., Kapranov, M.: Derived Quot schemes. *Ann. Sci. École Norm. Sup.* **34**, 403–440 (2001) [Zbl 1050.14042](#) [MR 1839580](#)
- [Do] Domokos, M.: Vector invariants of a class of pseudo-reflection groups and multisymmetric syzygies. *J. Lie Theory* **19**, 507–525 (2009) [Zbl 1221.13007](#) [MR 2583916](#)
- [Dr] Drinfeld, V. G.: On quasitriangular quasi-Hopf algebras and on a group that is closely connected with $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. *Leningrad Math. J.* **2**, 829–860 (1991) [Zbl 0728.16021](#) [MR 1080203](#)
- [DS] Dwyer, W. G., Spaliński, J.: Homotopy theories and model categories. In: Handbook of Algebraic Topology, Elsevier, 73–126 (1995) [Zbl 0869.55018](#) [MR 1361887](#)
- [EM] Eilenberg, S., MacLane, S.: On the groups $H(\Pi, n)$, I. *Ann. of Math.* **58**, 55–106 (1953) [Zbl 0050.39304](#)
- [Fa] Farinati, M. A.: On the derived invariance of cohomology theories for coalgebras. *Algebras Represent. Theory* **6**, 303–331 (2003) [Zbl 1033.16016](#) [MR 2000964](#)
- [F] Feigin, B.: On the cohomology of the Lie algebra of vector fields and of the current algebra. *Selecta Math. Soviet.* **7**, 49–62 (1988) [Zbl 0657.17009](#) [MR 0967082](#)
- [FT1] Feigin, B., Tsygan, B.: Additive K-theory and crystalline cohomology. *Funct. Anal. Appl.* **19**, 124–132 (1985) [Zbl 0585.18007](#) [MR 0800920](#)
- [FT2] Feigin, B., Tsygan, B.: Additive K-theory. In: K-theory, Arithmetic and Geometry (Moscow, 1984–1986), *Lecture Notes in Math.* 1289, Springer, Berlin, 67–209 (1987) [Zbl 0635.18008](#) [MR 0923136](#)
- [FGT] Fishel, S., Grojnowski, I., Teleman, C.: The strong Macdonald conjectures and Hodge theory on the loop Grassmannian. *Ann. of Math.* **168**, 175–220 (2008) [Zbl 1186.17010](#) [MR 2415401](#)
- [GK] Getzler, E., Kapranov, M.: Cyclic operads and cyclic homology. In: Geometry, Topology, & Physics, Conf. Proc. *Lecture Notes Geom. Topology IV*, Int. Press, Cambridge, MA, 167–201 (1995) [Zbl 0883.18013](#) [MR 1358617](#)
- [G] Ginzburg, V.: Noncommutative symplectic geometry, quiver varieties and operads. *Math. Res. Lett.* **8**, 377–400 (2001) [Zbl 1113.17306](#) [MR 1839485](#)

- [Ha] Haiman, M. D.: Conjectures on the quotient ring by diagonal invariants. *J. Algebraic Combin.* **3**, 17–76 (1994) [Zbl 0803.13010](#) [MR 1256101](#)
- [H1] Hanlon, P.: Cyclic homology and the Macdonald conjectures. *Invent. Math.* **86**, 131–159 (1986) [Zbl 0604.17007](#) [MR 0853448](#)
- [H2] Hanlon, P.: Some conjectures and results concerning the homology of nilpotent Lie algebras. *Adv. Math.* **84**, 91–134 (1990) [Zbl 0716.17018](#) [MR 1075234](#)
- [HC] Harish-Chandra: Differential operators on a semisimple Lie algebra. *Amer. J. Math.* **79**, 87–120 (1957) [Zbl 0072.01901](#) [MR 0084104](#)
- [H] Hinich, V.: Homological algebra of homotopy algebras. *Comm. Algebra* **25**, 3291–3323 (1997) [Zbl 0894.18008](#) [MR 1465117](#)
- [Hi] Hinich, V.: DG coalgebras as formal stacks. *J. Pure Appl. Algebra* **162**, 209–250 (2001) [Zbl 1020.18007](#) [MR 1843805](#)
- [J] Joseph, A.: On a Harish-Chandra homomorphism. *C. R. Math. Acad. Sci. Paris* **324**, 759–764 (1997) [Zbl 1002.17007](#) [MR 1446576](#)
- [Ka] Kaygun, A.: Loday–Quillen–Tsygan theorem for coalgebras. [arXiv:math/0411661](#) (2004)
- [Ke] Keller, B.: A-infinity algebras, modules and functor categories. In: *Trends in Representation Theory of Algebras and Related Topics*, *Contemp. Math.* 406, Amer. Math. Soc., 67–93 (2006) [Zbl 1121.18008](#) [MR 2258042](#)
- [Kh] Khoroshkin, A.: Highest weight categories and Macdonald polynomials. [arXiv:1312.7053](#) (2013)
- [Kir] Kirillov, A. A.: Lectures on affine Hecke algebras and Macdonald’s conjectures. *Bull. Amer. Math. Soc.* **34**, 251–292 (1997) [Zbl 0884.05100](#) [MR 1441642](#)
- [K] Kontsevich, M.: Formal (non)commutative symplectic geometry. In: *The Gelfand Mathematical Seminars, 1990–1992*, Birkhäuser Boston, Boston, MA, 173–187 (1993) [Zbl 0821.58018](#) [MR 1247289](#)
- [L] Loday, J.-L.: *Cyclic Homology*. 2nd ed., Grundlehren Math. Wiss. 301, Springer, Berlin (1998) [Zbl 0885.18007](#) [MR 1600246](#)
- [LP] Loday, J.-L., Procési, C.: Homology of symplectic and orthogonal algebras. *Adv. Math.* **69**, 93–108 (1988) [Zbl 0716.17019](#) [MR 0937318](#)
- [LQ] Loday, J.-L., Quillen, D.: Cyclic homology and the Lie algebra homology of matrices. *Comment. Math. Helv.* **59**, 565–591 (1984) [Zbl 0565.17006](#) [MR 0780077](#)
- [LV] Loday, J.-L., Vallette, B.: *Algebraic Operads*. Grundlehren Math. Wiss. 346, Springer, Heidelberg (2012) [Zbl 1260.18001](#) [MR 2954392](#)
- [M] Macdonald, I. G.: Some conjectures for root systems. *SIAM J. Math. Anal.* **13**, 988–1007 (1982) [Zbl 0498.17006](#) [MR 0674768](#)
- [Maj] Majid, S.: *Foundations of Quantum Group Theory*. Cambridge Univ. Press, Cambridge (1995) [Zbl 0857.17009](#) [MR 1381692](#)
- [Me] Meinrenken, E.: *Clifford Algebras and Lie Theory*. Springer, Heidelberg (2013) [Zbl 1267.15021](#) [MR 3052646](#)
- [Mo] Moore, J. C.: Differential homological algebra. In: *Actes du Congrès International des Mathématiciens (Nice, 1970)*, Tome 1, Gauthier-Villars, Paris, 335–339 (1971) [Zbl 0337.18009](#) [MR 0436178](#)
- [N] Nekrasov, N.: Seiberg–Witten prepotential from instanton counting. *Adv. Theor. Math. Phys.* **7**, 831–864 (2003) [Zbl 1056.81068](#) [MR 2045303](#)
- [P] Procési, C.: The invariant theory of $n \times n$ matrices. *Adv. Math.* **19**, 306–381 (1976) [Zbl 0331.15021](#) [MR 0419491](#)
- [Q1] Quillen, D.: *Homotopical Algebra*. Lecture Notes in Math. 43, Springer, Berlin (1967) [Zbl 0168.20903](#) [MR 0223432](#)

- [Q2] Quillen, D.: Rational homotopy theory. *Ann. of Math.* **90**, 205–295 (1969) [Zbl 0191.53702](#) [MR 0258031](#)
- [Q3] Quillen, D.: Algebra cochains and cyclic cohomology. *Inst. Hautes Études Sci. Publ. Math.* **68**, 139–174 (1989) [Zbl 0689.57026](#) [MR 1001452](#)
- [R] Richardson, R. W.: Commuting varieties of semisimple Lie algebras and algebraic groups. *Compos. Math.* **38**, 311–327 (1979) [Zbl 0409.17006](#) [MR 0535074](#)
- [SW] Sinha, D., Walters, B.: Lie coalgebras and rational homotopy theory, I: graph coalgebras. *Homology Homotopy Appl.* **13**, 263–292 (2011) [Zbl 1231.55009](#) [MR 2861231](#)
- [St] Stasheff, J.: The intrinsic bracket on the deformation complex of an associative algebra. *J. Pure Appl. Algebra* **89**, 231–235 (1993) [Zbl 0786.57017](#) [MR 1239562](#)
- [Te] Teleman, C.: Some Hodge theory from Lie algebras. In: *Motives, Polylogarithms and Hodge Theory. Part II*, Int. Press, Somerville, MA, 731–744 (2002) [Zbl 1051.14009](#) [MR 1978717](#)
- [T] Tsygan, B.: Homology of matrix Lie algebras over rings and the Hochschild homology. *Uspekhi Mat. Nauk* **38**, no. 2, 217–218 (1983) (in Russian) [Zbl 0518.17002](#) [MR 0695483](#)
- [Tu] Turaev, V.: Poisson–Gerstenhaber brackets in representation algebras. *J. Algebra* **402**, 435–478 (2014) [Zbl 1308.17027](#) [MR 3160431](#)
- [Va1] Vaccarino, F.: Linear representations, symmetric products and the commuting scheme. *J. Algebra* **317**, 634–641 (2007) [Zbl 1155.13007](#) [MR 2362934](#)
- [Va2] Vaccarino, F.: Moduli of linear representations, symmetric products and the non-commutative Hilbert scheme. In: *Geometric Methods in Representation Theory. II*, Sémin. Congr. 24-II, Soc. Math. France, 437–458 (2012) [Zbl 1314.14004](#) [MR 3203038](#)
- [VdB] Van den Bergh, M.: Non-commutative quasi-Hamiltonian spaces. In: *Poisson Geometry in Mathematics and Physics*, *Contemp. Math.* 450, Amer. Math. Soc., 273–299 (2008) [Zbl 1236.58019](#) [MR 2397630](#)
- [Wa] Wambst, M.: Complexes de Koszul quantiques et homologie cyclique. *C. R. Acad. Sci. Paris Sér. I Math.* **314**, 977–982 (1992) [Zbl 0760.16004](#) [MR 1168520](#)
- [W] Weyl, H.: *The Classical Groups. Their Invariants and Representations*. Princeton Univ. Press, Princeton, NJ (1939) [Zbl 0020.20601](#) [MR 0000255](#)