J. Eur. Math. Soc. 19, 2911–2946
DOI 10.4171/JEMS/731

José F. Alves · Carla L. Dias · Stefano Luzzatto · Vilton Pinheiro

SRB measures for partially hyperbolic systems whose central direction is weakly expanding

Received October 15, 2014

Abstract. We consider partially hyperbolic C^{1+} diffeomorphisms of compact Riemannian manifolds of arbitrary dimension which admit a partially hyperbolic tangent bundle decomposition $E^s \oplus E^{cu}$. Assuming the existence of a set of positive Lebesgue measure on which f satisfies a weak nonuniform expansivity assumption in the centre unstable direction, we prove that there exist at most a finite number of transitive attractors each of which supports an SRB measure. As part of our argument, we prove that each attractor admits a Gibbs–Markov–Young geometric structure with integrable return times. We also characterize in this setting SRB measures which are liftable to Gibbs–Markov–Young structures.

Keywords. SRB measures, Lyapunov exponents, nonuniform expansion, GMY structures

1. Introduction

1.1. Physical measures

Let M be a compact Riemannian manifold with a normalized Riemannian volume Leb which we will refer to as Lebesgue measure, and let $f : M \to M$ be a C^{1+} diffeomorphism of M (meaning that f is C^1 with Hölder continuous derivative). For a Borel probability measure μ on M we define the *basin* of μ by

$$\mathcal{B}_\mu := \left\{ x \in M : \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) \to \int \varphi\, d\mu \text{ for all continuous } \varphi : M \to \mathbb{R} \right\}.$$

Then $\mathcal{B}_\mu$ is the set of points whose orbits are asymptotically uniformly distributed with respect to μ. A priori there is no reason for the basin to be nonempty but under certain

J. F. Alves: Departamento de Matemática, Faculdade de Ciências, Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal; e-mail: jfalves@fc.up.pt, http://www.fc.up.pt/cmup/jfalves

C. L. Dias: Instituto Politécnico de Portalegre, Lugar da Abadessa, Apartado 148, 7301-901 Portalegre, Portugal; e-mail: carlald.dias@gmail.com

S. Luzzatto: Abdus Salam International Centre for Theoretical Physics, Strada Costiera 11, Trieste, Italy; e-mail: luzzatto@ictp.it, http://www.ictp.it/~luzzatto

V. Pinheiro: Departamento de Matemática, Universidade Federal da Bahia, Av. Ademar de Barros s/n, 40170-110 Salvador, Brazil; e-mail: viltonj@ufba.br

Mathematics Subject Classification (2010): 37A05, 37C40, 37D25, 37D30

conditions it is possible to prove that it is and that indeed it is "large" in certain respects. The classical Ergodic Theorem of Birkhoff implies, for example, that if μ is ergodic and invariant then $\mu(\mathcal{B}_\mu) = 1$. If μ is singular with respect to Lebesgue measure, which it usually is, this does not guarantee that the basin has positive Lebesgue measure which is, in some sense, the reference measure with respect to which we want to describe the dynamics. This motivates the following definition. We say that μ is a *physical measure* if

$$\mathrm{Leb}(\mathcal{B}_\mu) > 0.$$

There are examples of systems without physical measures (e.g. the identity map) as well as examples of systems with an infinite number of physical measures. These examples, however, are somewhat "pathological" and the *Palis conjecture* [36] says that typical dynamical systems admit at least one and at most a finite number of physical measures and that their basins have full Lebesgue measure in M.

A particular type of physical measures are the so-called *Sinai–Ruelle–Bowen*, or *SRB*, measures which have the property of having nonzero Lyapunov exponents μ-almost everywhere and admitting a system of conditional measures such that the conditional measures on unstable manifolds are absolutely continuous with respect to the Lebesgue measures Leb_γ on these manifolds induced by the restriction of the Riemannian structure [14, 37, 39, 50]. The main result of this paper is to prove the existence and finiteness of SRB physical measures in a natural and relatively general class of partially hyperbolic diffeomorphisms. We mention that some of the specific constants in the conditions in the following results depend on the choice of Riemannian metric, thus it is implicit in our assumptions that it is sufficient for the conditions to be satisfied for some choice of Riemannian metric.

Theorem A. *Let $f : M \to M$ be a C^{1+} diffeomorphism, $K \subseteq M$ a forward invariant compact set on which f admits a Df-invariant continuous tangent bundle splitting $T_K M = E^s \oplus E^{cu}$, and suppose there exists $\lambda \in (0, 1)$ such that for all $x \in K$,*

$$\|Df|_{E^s_x}\| < \lambda \quad \text{and} \quad \|Df|_{E^s_x}\| \cdot \|Df^{-1}|_{E^{cu}_{f(x)}}\| < \lambda. \tag{1}$$

Suppose moreover that there exists $H \subseteq K$ with $\mathrm{Leb}(H) > 0$ and an $\epsilon > 0$ such that for all $x \in H$,

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|Df^{-1}|_{E^{cu}_{f^j(x)}}\| < -\epsilon. \tag{2}$$

Then

(a) *there exist closed invariant transitive sets $\Omega_1, \dots, \Omega_\ell$ such that for Lebesgue almost every $x \in H$ we have $\omega(x) = \Omega_j$ for some $1 \le j \le \ell$;*
(b) *there exist physical SRB measures $\mu_1, \dots, \mu_\ell$ supported on the sets $\Omega_1, \dots, \Omega_\ell$, whose basins have nonempty interior, such that for Lebesgue almost every $x \in H$ we have $x \in \mathcal{B}_{\mu_j}$ for some $1 \le j \le \ell$.*

We refer to the second part of (1) as *domination property.*

This result generalizes the well-known and often cited result of [4] in which the same conclusions are obtained under the stronger assumption that condition (2) holds with a lim sup instead of a lim inf. While this may seem like a technical detail, we emphasize that the lim inf condition (2) implies that the growth only needs to be verified on a subsequence of iterates, in contrast to the same conditions with lim sup, which needs to be verified for *all* sufficiently large times. The difference between these two assumptions leads to the failure of a key technical property which means that the techniques and methods that we use below to deal with the weaker assumption are completely different from those used in [4]—see further discussion in Section 2 and Remark 4.3 below.

It is, in some sense, impossible to find an example of a diffeomorphism that satisfies our condition (2) and not the a priori stronger condition in [4], because the conclusions of Theorem A imply that the limit in (2) exists, and therefore, *a fortiori*, the two conditions are equivalent. The spirit of our result is therefore mainly of a theoretical nature, but of course it may very well be possible to encounter a situation in which only our weaker condition is verifiable in practice. While it is not immediately obvious how to construct such an example, it may be instructive to remark that a robust (C^1 open) class of examples of diffeomorphisms is constructed in [4] which satisfies the conditions of that paper (and therefore also the a priori weaker conditions of our Theorem A) which rely on the property stated in [4, Lemma A.1] showing that typical points spend a uniformly positive frequency of times in certain good regions B of the manifold, i.e. there exists some $\epsilon_0 > 0$ such that $\liminf n^{-1}\,\mathrm{Leb}\{0 \le i < n : f^i(x) \in B\} \ge \epsilon_0$. In principle, a situation in which this property cannot be verified directly but can be replaced by a weaker statement of the form $\limsup n^{-1}\,\mathrm{Leb}\{0 \le i < n : f^i(x) \in B\} \ge \epsilon_0$, i.e. a situation in which there is an a priori weaker control of the recurrence in B, might lead to the verification of condition (2) and therefore to the desired conclusions on the existence and finiteness of SRB measures.

1.2. Transitivity and uniqueness of physical measures

We remark that our argument also works if we let $\epsilon = 0$ in (2), in which case we get a countable number of transitive sets and corresponding SRB measures supported on them. On the other hand, as the ergodic basins $\mathcal{B}_{\mu_j}$ have nonempty interior, in the special case when f is transitive, partially hyperbolic and weakly nonuniformly expanding along E^{cu} on the whole manifold M, we get the following consequence.

Corollary B. *Under the assumptions of Theorem A, suppose f is transitive and* $\mathrm{Leb}(H) = 1$. *Then $\omega(x) = M$ for Lebesgue a.e. x, and f has a unique SRB measure with* $\mathrm{Leb}(\mathcal{B}_\mu) = 1$.

We emphasize that the statement of Corollary B is nontrivial and makes full use of the specific formulations of the results in Theorem A which derive from our construction and proof, and in particular is not just an immediate consequence of the finiteness of ergodic components under the additional transitivity assumption. Indeed, notice that there are no uniqueness statements in [4] where a different strategy is used for the proof of finiteness. The conclusions of Corollary B are obtained in [13] in the quite distinct setting of a partially hyperbolic diffeomorphism with an invariant tangent bundle decomposition $E^{cs} \oplus E^u$ (i.e. with uniform expansion and nonuniform contraction, see Section 2 for an in-depth historical literature review of the various kinds of partial hyperbolicity and corresponding results) with a particularly strong form of transitivity, namely that all the leaves of the unstable foliation are dense in M. This does not seem like a natural assumption in our setting since we do not have an unstable foliation, but if such a condition could be verified in a specific example, it seems likely that the conclusions would follow here also.

We note that just adding the transitivity condition to the assumptions of Theorem A, i.e. relaxing the assumption that $\text{Leb}(H) = 1$ to $\text{Leb}(H) > 0$, is probably not sufficient to obtain the conclusions of Corollary B. Nevertheless, if required in specific applications, it should be possible to find some additional conditions from which the result follows without the a priori assumption that H has full measure, e.g. those used in [13] and mentioned in the previous paragraph. Alternatively, assuming transitivity, $\text{Leb}(H) > 0$ and that H contains an open set, would then make it natural to prove (or assume) that H is open and dense, since condition (2) is invariant along orbits and it is thus natural to assume that H is both forward and backward invariant (and, by transitivity, a forward and backward invariant open set is also dense). In this case it should then be possible, using uniform contraction in the direction of E^s and the existence of a stable foliation, to deduce that $\text{Leb}(H) = 1$ and thus obtain the conclusions.

1.3. Gibbs–Markov–Young structures

The main part of the proof of Theorem A consists in the construction of a finite collection of nontrivial, and not at all a priori expected, geometric structures which we call *Gibbs–Markov–Young*, or *GMY*, structures for the map f. The precise definition is rather long and technical, and so, in order not to interrupt the flow of the presentation, we postpone it to Section 1.5. For the formal statement of our results we just mention that such a structure consists of an *induced map* $F = f^R : \Lambda \to \Lambda$, defined on some set $\Lambda \subseteq M$ by an inducing time function $R : \Lambda \to \mathbb{N}$. Various combinatorial and geometrical conditions need to be satisfied, as well as some control over the return time function which is generally unbounded; we give all the precise formulations in Section 1.5. For the moment we just remark that the required control on the return time function is referred to as "integrability of the return times". The assumptions of the following theorem are exactly those of Theorem A, but we restate them here in order to keep the statement self-contained.

Theorem C. *Let $f : M \to M$ be a C^{1+} diffeomorphism, $K \subseteq M$ a forward invariant compact set on which f admits a Df-invariant continuous tangent bundle splitting $T_K M = E^s \oplus E^{cu}$, and suppose there exists $\lambda \in (0, 1)$ such that for all $x \in K$,*

$$\|Df|_{E^s_x}\| < \lambda \quad \text{and} \quad \|Df|_{E^s_x}\| \cdot \|Df^{-1}|_{E^{cu}_{f(x)}}\| < \lambda.$$

Suppose moreover that there exists $H \subseteq K$ with $\text{Leb}(H) > 0$ and an $\epsilon > 0$ such that for all $x \in H$,

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|Df^{-1}|_{E^{cu}_{f^j(x)}}\| < -\epsilon. \tag{3}$$

If there exists a closed invariant transitive set Ω such that $\omega(x) = \Omega$ for every $x \in H$, then there exists a GMY structure $\Lambda \subseteq \Omega$ with integrable return times.

The statement in item (a) of Theorem A, i.e. the existence of a finite number of closed invariant transitive sets Ω, will be proved in Sections 3 and 4. Then Theorem C provides a GMY structure for each of these sets. Classical results imply that the map F which defines the GMY structure admits a unique SRB measure ν with respect to which the return time function R is integrable, and therefore we can define a probability measure

$$\mu = \frac{1}{\int R \, d\nu} \sum_{j=0}^{\infty} f_*^j (\nu|_{\{R>j\}}), \tag{4}$$

which is one of the finite number of required SRB measures for f [49, Section 2]. Therefore item (b) of Theorem A follows from (a) and Theorem C.

1.4. Positive Lyapunov exponents and liftability of measures

SRB measures associated to GMY structures through the formula (4) are, a priori, a special kind of SRB measure. The additional structure may be useful in obtaining information on various other properties of the dynamics and of the measure, such as for example statistical properties of the dynamics with respect to μ, like decay of correlations, large deviations, limit theorems, etc.; see for instance [49, 28, 34, 35, 42]. SRB measures which can be written in the form (4) are said to be *liftable* (to a GMY structure). We have the following very natural question.

Question. Is every SRB measure μ liftable?

In the setting of partially hyperbolic systems as above, we can give a full characterization of liftable measures.

Theorem D. *Let $f : M \to M$ be a C^{1+} diffeomorphism, $K \subseteq M$ a forward invariant compact set on which f admits a Df-invariant continuous tangent bundle splitting $T_K M = E^s \oplus E^{cu}$, and suppose there exists $\lambda \in (0, 1)$ such that for all $x \in K$,*

$$\|Df|_{E^s_x}\| < \lambda \quad \text{and} \quad \|Df|_{E^s_x}\| \cdot \|Df^{-1}|_{E^{cu}_{f(x)}}\| < \lambda.$$

Then for any invariant probability μ supported on K the following conditions are equivalent:

(a) *μ is an SRB measure and there exists an $\epsilon > 0$ such that for μ-a.e. x and every unit vector $v \in E^{cu}_x$,*

$$\limsup_{n\to\infty} \frac{1}{n} \log \|Df^n(x)v\| > \epsilon; \tag{5}$$

(b) *μ is liftable to a GMY structure on K.*

We remark that the implication (b)⇒(a) follows essentially from the definition of GMY structure, so the nontrivial part is (a)⇒(b). If we replace condition (5), which can be described as saying that μ has *positive Lyapunov exponents* in the E^{cu} direction, by condition (3), then this implication is essentially already contained in Theorem C, but we emphasize that *condition* (5) *is strictly weaker than* (3): see for instance the example in [2, Section 4]. We will use in an essential way the assumption of the existence of an invariant measure μ to show that (5) implies that *some iterate* of f satisfies (3), after which we can apply the techniques in the proof of Theorem C to conclude the proof of Theorem D.

In view of these remarks a natural and interesting question is whether a version of Theorem C, and therefore also of Theorem A, could be proved under an assumption of positive Lyapunov exponents such as in (5), which is in some sense more natural, instead of condition (3). This turns out to be a challenging question and, without the a priori assumption of the existence of an invariant measure, we do not know how to derive (3) from (5), even for some higher iterate of f, nor do we know how to carry out the construction of the GMY structure directly from the assumption (5). It is interesting to observe, however, that condition (5), unlike (3), does not depend on the choice of metric. One possible approach to the problem may be via the following

Conjecture 1. Assume that f is partially hyperbolic and satisfies (5) along E^{cu} on a set H. Then there exists a Riemannian metric on M such that f satisfies (3) on H.

If this conjecture is true, we immediately obtain the conclusions of Theorem C and therefore of Theorem A and of Corollary B under the a priori weaker condition of positive Lyapunov exponents. Pushing this theme even further we mention a long-standing conjecture of Viana along these lines.

Conjecture (Viana conjecture, [47]). If a smooth map has only nonzero Lyapunov exponents at Lebesgue almost every point, then it admits some SRB measure.

So far, apart from the one-dimensional setting, it has been extremely difficult to work directly with Lyapunov exponents and it has been necessary to introduce stronger versions of nonuniform contraction and expansion as in [13, 4] and the present paper. Viana's conjecture forms an important part of the motivation for introducing increasingly weaker conditions such as that of Theorem A above and questions such as that in Conjecture 1 above. In this direction we also mention the remarkable recent results in [22] where quite new techniques are introduced to construct SRB measures for some diffeomorphisms which are nonuniformly hyperbolic but do not have a continuous dominated splitting.

1.5. Precise definition of Gibbs–Markov–Young structure

We now give the precise formal definition of the GMY structures. These geometric structures were introduced in [49] and have been applied to study the existence and properties of physical measures in certain classes of dynamical systems.

An embedded disk $\gamma \subset M$ is called an *unstable manifold* if $d(f^{-n}(x), f^{-n}(y)) \to 0$ exponentially fast as $n \to \infty$ for all $x, y \in \gamma$; similarly $\gamma \subset M$ is called a *stable manifold* if $d(f^n(x), f^n(y)) \to 0$ exponentially fast as $n \to \infty$. We say that $\Gamma^u = \{\gamma^u\}$ is a *continuous family of C^1 unstable manifolds* if there is a compact set K^s, a unit disk D^u in some $\mathbb{R}^n$, and a map $\Phi^u \colon K^s \times D^u \to M$ such that

- $\gamma^u = \Phi^u(\{x\} \times D^u)$ is an unstable manifold;
- Φ^u maps $K^s \times D^u$ homeomorphically onto its image;
- $x \mapsto \Phi^u|_{\{x\}\times D^u}$ defines a continuous map from K^s into $\mathrm{Emb}^1(D^u, M)$.

Here $\mathrm{Emb}^1(D^u, M)$ denotes the space of C^1 embeddings of D^u into M. Continuous families of C^1 stable manifolds are defined similarly.

We say that a set $\Lambda \subset M$ has a *hyperbolic product structure* if there exist a continuous family $\Gamma^u = \{\gamma^u\}$ of local unstable manifolds and a continuous family $\Gamma^s = \{\gamma^s\}$ of local stable manifolds such that

- $\Lambda = (\bigcup \gamma^u) \cap (\bigcup \gamma^s)$;
- $\dim \gamma^u + \dim \gamma^s = \dim M$;
- each γ^s meets each γ^u in exactly one point;
- stable and unstable manifolds are transversal with angles bounded away from 0.

If $\Lambda \subset M$ has a product structure, we say that $\Lambda_0 \subset \Lambda$ is an *s-subset* if Λ_0 also has a product structure and its defining families Γ_0^s and Γ_0^u can be chosen with $\Gamma_0^s \subset \Gamma^s$ and $\Gamma_0^u = \Gamma^u$; *u-subsets* are defined analogously. For convenience we shall use the following notation: given $x \in \Lambda$, let $\gamma^*(x)$ denote the element of Γ^* containing x, for $* = s, u$. Also, for each $n \geq 1$ let $(f^n)^u$ denote the restriction of the map f^n to γ^u-disks and let $\det D(f^n)^u$ be the Jacobian of $D(f^n)^u$.

We say that f admits a *Gibbs–Markov–Young* (*GMY*) *structure* if there exist a set Λ with hyperbolic product structure and constants $C > 0$ and $0 < \beta < 1$, depending on f and Λ, satisfying the following additional properties:

(P_0) *Detectable*: $\mathrm{Leb}_\gamma(\Lambda) > 0$ for each $\gamma \in \Gamma^u$.

(P_1) *Markov*: there are pairwise disjoint s-subsets $\Lambda_1, \Lambda_2, \ldots \subset \Lambda$ such that

(a) $\mathrm{Leb}_\gamma((\Lambda \setminus \bigcup \Lambda_i) \cap \gamma) = 0$ for each $\gamma \in \Gamma^u$;

(b) for each $i \in \mathbb{N}$ there is $R_i \in \mathbb{N}$ such that $f^{R_i}(\Lambda_i)$ is a u-subset, and for all $x \in \Lambda_i$,

$$f^{R_i}(\gamma^s(x)) \subset \gamma^s(f^{R_i}(x)) \quad \text{and} \quad f^{R_i}(\gamma^u(x)) \supset \gamma^u(f^{R_i}(x)).$$

(P_2) *Contraction on stable leaves*: for all $\gamma^s \in \Gamma^s$, $x, y \in \gamma^s$ and $n \geq 1$,

$$\mathrm{dist}(f^n(y), f^n(x)) \leq C\beta^n.$$

(P_3) *Backward contraction on unstable leaves*: for all $\gamma^u \in \Gamma^u$, $x, y \in \Lambda_i \cap \gamma^u$ and $0 \leq n < R_i$,

$$\mathrm{dist}(f^n(y), f^n(x)) \leq C\beta^{R_i - n}\, \mathrm{dist}(f^{R_i}(x), f^{R_i}(y)).$$

(P_4) *Bounded distortion*: for all $\gamma^u \in \Gamma^u$ and $x, y \in \Lambda_i \cap \gamma^u$,

$$\log \frac{\det D(f^{R_i})^u(x)}{\det D(f^{R_i})^u(y)} \leq C\, \mathrm{dist}(f^{R_i}(x), f^{R_i}(y)).$$

(P_5) *Regularity of the foliations*:

(a) for all $\gamma^s \in \Gamma^s$, $x, y \in \gamma^s$ and $n \geq 1$,

$$\log \prod_{i=n}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(y))} \leq C\beta^n;$$

(b) given $\gamma, \gamma' \in \Gamma^u$, we define $\Theta\colon \gamma \cap \Lambda \to \gamma' \cap \Lambda$ by taking $\Theta(x)$ equal to $\gamma^s(x) \cap \gamma'$. Then Θ is absolutely continuous and

$$\frac{d(\Theta_* \operatorname{Leb}_\gamma)}{d \operatorname{Leb}_{\gamma'}}(x) = \prod_{i=0}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(\Theta^{-1}(x)))}.$$

We define a return time function $R : \Lambda \to \mathbb{N}$ by $R|_{\Lambda_i} = R_i$ and we say that the GMY structure has *integrable return times* if for some (and hence all) $\gamma \in \Gamma^u$, we have

$$\int_{\gamma \cap \Lambda} R \, d \operatorname{Leb}_\gamma < \infty. \tag{6}$$

1.6. Outline of the paper

In Section 2 we give a relatively detailed discussion of the previously existing related results in the literature, in order in particular to clarify the position and significance of the results presented here. We also discuss some aspects of our strategy and how it is developed in response to a failure of previous approaches to work with our weaker assumptions. In Section 3 we give an abstract criterion for verifying that at most a finite number of transitive topological attractors exist for a given set. In Section 4 we show that this criterion is satisfied by the set H in Theorem A, thus proving item (a) in the theorem. It is then sufficient to restrict our attention to one of these attractors, as in the setting of Theorem C. In Section 5 we give the full combinatorial inductive "recipe" for the construction of the GMY structure. In Section 6 we prove that this recipe does indeed account for the dynamics of almost every point, and in Section 7 that all the formal conditions in the definition of GMY structure are satisfied, except for the integrability of the return times which is proved in Section 8, thus completing the proof of Theorem C. Combining the conclusion of Theorem C with the comments at the beginning of Section 1.4 we get item (b) of Theorem A and the "if" direction of Theorem D; the other implication is proved in Section 9.

2. Historical background and context

2.1. Uniform hyperbolicity

The problem of the existence and finiteness of physical measures was first formulated, and solved, by Anosov, Smale, Ruelle and Bowen [14, 15, 43, 44] in the 1970's, in the setting of *uniformly hyperbolic* diffeomorphisms $f : M \to M$, i.e. systems which admit a continuous invariant tangent bundle decomposition $TM = E^s \oplus E^u$ such that the differential map is uniformly contracting on E^s and uniformly expanding on E^u, on compact manifolds. Their basic strategy was to choose some essentially arbitrary local unstable manifold γ^u, consider the normalized volume, or Lebesgue measure, on γ, and define the sequence of probability measures

$$\mu_n := \frac{1}{n} \sum_{i=0}^{n-1} f_*^i \operatorname{Leb}_\gamma . \tag{7}$$

By compactness of M the space of probability measures is compact in the weak-star topology, and thus there exists some probability measure μ and some subsequence $n_j \to \infty$ such that

$$\mu_{n_j} \to \mu$$

in the weak-star topology as $j \to \infty$. It is then a relatively straightforward argument, using the uniform expansivity and the uniform size of local unstable manifolds at every point, to show that μ is f-invariant and satisfies one of the key properties of SRB measures which is, as mentioned above, that it admits a local "disintegration" into conditional measures on local unstable manifolds which are absolutely continuous with respect to Lebesgue measure on such manifolds. Together with the second key property of *absolute continuity* of the stable foliation, this implies that μ is indeed an SRB measure and in particular a physical measure.

2.2. *Nonuniform and partial hyperbolicity*

Since the 1970's, ongoing research has achieved extensions of this result to increasingly general classes of dynamical systems, but things get significantly harder as soon as the uniform hyperbolicity conditions are relaxed in any way; progress has been slow and has required the development of increasingly sophisticated new arguments and techniques. There are two main natural directions in which the uniform hyperbolicity assumptions can be relaxed: one is *nonuniform hyperbolicity* where the decomposition $TM = E^s \oplus E^u$ is only measurable and the contraction and expansion estimates are only asymptotic [11, 37]; the other is *partial hyperbolicity* where the tangent bundle decomposition takes the form $TM = E^s \oplus E^c \oplus E^u$, which is still assumed to be continuous, and to admit uniform contraction and expansion estimates in E^s and E^u respectively, but also includes a "central" direction on which, in principle, very little is assumed [16, 38]. Notice that neither of these conditions implies the other.

A vast literature exists concerning the properties of systems satisfying such weak hyperbolicity conditions and several papers address specifically the existence and finiteness of SRB measures, but it turns out that both of these classes of systems, in full generality, are extremely difficult to study. On a very heuristic level, the nonuniformly hyperbolic setting, while maintaining the important property of absolute continuity of the stable foliation, means that we lose the uniformity of several estimates including in particular the size of the local stable manifolds, making it difficult to control the sequence (7) and to show that the limit point μ has absolutely continuous conditional measures; on the other hand, the partially hyperbolic setting maintains certain uniform estimates, including the sizes of local stable and unstable manifolds, but we lose in general the absolute continuity of the central foliation (or even the existence of that foliation).

These difficulties are clearly reflected in the way the subject has developed over the years. A first pioneering paper is [39] in which *Gibbs* measures are constructed for partially hyperbolic systems. Gibbs measures are in some sense analogous to SRB measures in their intrinsic structure relative to the geometry of the invariant manifolds of the diffeomorphisms (in the sense that they have absolutely continuous conditional measures on unstable manifolds, but are not necessarily physical measures because of the possible lack of absolute continuity or even existence of a foliation tangent to the central subbundle). To overcome this problem, most results require the dynamics to be a combination of partially hyperbolic *and* nonuniformly hyperbolic in the central direction. Moreover, most of the time the dynamics is assumed to be *either nonuniformly contracting* in the central direction, which for simplicity we will refer to by saying that the splitting has the form $E^{cs} \oplus E^u$, *or nonuniformly expanding* in the central direction, which for simplicity we will refer to by saying that the splitting has the form $E^s \oplus E^{cu}$; a mixture of both nonuniform contraction and nonuniform expansion in the central direction would essentially present the same difficulties as a fully nonuniformly hyperbolic system. We remark that the notation E^{cs} and E^{cu} does not imply the existence of a further splitting $E^{cs} = E^s \oplus E^s$ or $E^{cu} = E^u \oplus E^u$ which may or may not exist; for certain results, for example concerning some geometrical properties of the systems, the existence or not of this further splitting can be relevant, but for most of the results on the existence and finiteness of SRB measures this turns out not to be a significant issue.

2.3. The "mostly contracting" case

In the light of the observations made above, perhaps the "easiest" of the two cases mentioned above is when the splitting is of the form $E^{sc} \oplus E^u$. Indeed, in this case (modulo some not completely trivial technical difficulties), the uniform expansivity of E^u implies uniform estimates on the growth and size of local unstable manifolds and thus makes it possible to show that the limit measures μ obtained from a sequence as in (7) above admit absolutely continuous conditional measures by similar arguments to the uniformly hyperbolic case. Moreover, the nonuniform contraction of E^{cs} implies, by standard theory of nonuniform hyperbolicity, that the centre-stable foliation is absolutely continuous and we conclude that μ is an SRB, and thus physical, measure. This strategy was implemented to show the existence and finiteness of SRB measures in this setting in [13]. Additional results include conditions which imply uniqueness of the SRB measure for a diffeomorphism and its perturbations [18, 19, 48], mixing properties of this measure [20, 21, 25], and even differentiability with respect to perturbations [26].

2.4. The "mostly expanding" case

In the case of a decomposition of the form $E^s \oplus E^{cu}$, the absolute continuity of the stable foliation follows immediately from the uniformity of the contraction in E^s, but the properties of the sequence of measures μ_n as in (7) are much more difficult to control due to the highly irregular pattern of growth of the derivative and consequently of the length of pieces of unstable manifolds. This means that not all points on the initial local unstable manifold γ have neighbourhoods which grow to some sufficiently large fixed size with bounded distortion at every iterate, which is the case when the expansion is uniform. A key technical tool to overcome this problem is the notion of a *hyperbolic time*, which was introduced in [1] and, for a given point x, is exactly the time when x has a neighbourhood which grows to large scale with bounded distortion (see Definition 4.1

and Lemma 4.4 below). It is also the key tool in [4] where the existence and finiteness of SRB measures was first proved under the assumption that there exists a set $H \subseteq K$ with $m(H) > 0$ and an $\epsilon > 0$ such that for all $x \in H$,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|Df^{-1}|_{E^{cu}_{f^j(x)}}\| < -\epsilon. \tag{8}$$

Notice that (8) is a stronger version of (2) with lim sup replacing lim inf. The key technical consequence of the difference between the two conditions is contained in Lemma 4.2 which says that under condition (2) almost every point x has an infinite sequence of hyperbolic times, while under (8), almost every point has an infinite sequence of hyperbolic times and this sequence has *positive density at infinity*, i.e. the statement of Lemma 4.2 holds with lim inf instead of lim sup [1, 4].

The difference between these two statements is the reason why completely different techniques are required for the proof in the present paper as compared to [4]. Indeed, the positive density of hyperbolic times means that there is a positive density of times at which the dynamics is essentially as in the uniformly hyperbolic case. This means that it is actually possible to adapt, albeit in a nontrivial way, the argument mentioned above used in the uniformly hyperbolic and mostly contracting cases.

Indeed, recalling the definition of the pushforward of a measure we have $f^i_* \mathrm{Leb}_\gamma(A) = \mathrm{Leb}_\gamma(f^{-i}(A)) = \mathrm{Leb}_\gamma(\{x : f^i(x) \in A\})$, and so it is clear that each of the measures $f^i_* \mathrm{Leb}_\gamma$ coming into the definition of the measures μ_n in (7) is supported on the image $f^i(\gamma)$ of the starting chosen piece of local unstable manifold. In the case when f is uniformly expanding along E^u we can basically divide up all of γ into pieces, each of which grows to large scale with bounded distortion at time i, and thus $f^i_*(\gamma)$ is supported on some collection of uniformly large unstable disks. Therefore the same is true for μ_n, and this is the crucial property used to show that any limiting measure μ has absolutely continuous conditional measures.

In the nonuniformly expanding case it is not true that for every i we can divide γ into pieces each of which grows to large scale with bounded distortion at time i. Instead this will be true just for *some* points in γ, precisely those for which i is a hyperbolic time and the images at time i of other parts of γ may be very small and/or very distorted. In particular it is no longer the case that $f^i_*(\gamma)$ is supported on a collection of uniformly large unstable disks. Nevertheless some points do eventually have hyperbolic times, and therefore *some* parts of the measures $f^i_*(\gamma)$, and hence μ_n, are supported on some such collection of uniformly large unstable disks. Thus it is possible to write the measures μ_n as

$$\mu_n = \mu'_n + \mu''_n$$

where μ'_n is the "good" part of the measure supported on a collection of uniformly large unstable disks and μ''_n is the "bad" part on which we have little control. Using condition (8) and the fact that it implies that almost every point has a positive density of hyperbolic times, it is shown in [4] that the good part of the measure μ'_n forms a proportion of the overall measure μ_n that is uniformly bounded below in n and that it is therefore possible to essentially recover a version of the original argument of Sinai, Ruelle, Bowen and show that there exists a limit measure μ' which has absolutely continuous conditional measures and is therefore an SRB measure.

In this paper, the weaker condition (2) does not imply a positive density of hyperbolic times, and therefore is not sufficient to imply that the mass of μ'_n is uniformly bounded below. Thus any hope of adapting further the classical argument breaks down in a fundamental and essentially unrecoverable way. For this reason we have used a completely different strategy via the construction of the geometric GMY structure as stated in Theorem C. In certain respects there are of course still some similarities with the classical approach in the sense that the GMY structure also relies on constructing some region where large unstable disks accumulate, and indeed the problem of the possibly low asymptotic frequency of hyperbolic times does not disappear in this approach but is rather "translated" into the problem of integrability of the return times. It turns out that it is possible to resolve the problem in this framework with a remarkably simple argument, given in Section 8 below, which nevertheless relies heavily on the specific and careful set up of the construction of the induced map.

We close this section by mentioning some other related papers in similar settings: [3, 8, 23, 27, 29, 46]. We also remark that we have restricted our discussion to diffeomorphisms, but the same kind of questions, and the Palis conjecture, also apply to endomorphisms—see for example [9, 10, 17, 24, 32, 31] for the setting of one-dimensional maps, [45] in two dimensions, and [5, 6, 40, 41] for the construction of SRB measures in certain higher-dimensional settings.

3. Ergodic components

Let X be a compact metric space and μ a Borel probability measure on X. Let $f : X \to X$ be a measurable map, not necessarily preserving the measure μ. Given $x \in X$, the *stable set* of x is

$$W^s(x) = \{y \in X : \mathrm{dist}(f^j(x), f^j(y)) \to 0 \text{ as } j \to \infty\}.$$

Notice that "$x \sim y$ if and only if $y \in W^s(x)$" defines an equivalence relation on X. In particular, we will use the transitivity of this relation. If $U \subset X$, let

$$W^s(U) = \bigcup_{x \in U} W^s(x).$$

We recall that a set $U \subseteq X$ is *invariant* if $f^{-1}(U) = U$. We now introduce a notion which is key to our argument. We say that $Y \subseteq X$ is *μ-unshrinkable* if it is an invariant set with $\mu(Y) > 0$ and there exists a $\delta > 0$ such that for every invariant set $U \subseteq Y$ we have

$$\mu(U) > 0 \ \Rightarrow \ \mu(W^s(U)) > \delta.$$

Proposition 3.1. *Suppose $Y \subseteq X$ is μ-unshrinkable. Then there exist a finite number of closed invariant subsets $\Omega_1, \dots, \Omega_\ell$ of X such that for μ-almost every $x \in Y$ we have $\omega(x) = \Omega_j$ for some $1 \le j \le \ell$.*

We will split the proof of Proposition 3.1 into two lemmas. To do this we need to introduce some additional concepts. We say that a set S is *s-saturated* if $W^s(S) = S$. We say that S is a *u-ergodic component* if it is invariant, s-saturated, and any subset $S' \subset S$ which is also invariant and s-saturated satisfies $\mu(S) = \mu(S')$ or $\mu(S') = 0$.

Lemma 3.2. *Suppose $Y \subseteq X$ is μ-unshrinkable. Then Y is contained (μ mod 0) in the union of a finite number of u-ergodic components.*

Proof. Let $Y_1 = Y$ and let

$$\mathcal{F}(Y_1) := \{W^s(U) : U \subseteq Y_1,\ f^{-1}(U) = U,\ \text{and } \mu(W^s(U)) > 0\}.$$

Note that $\mathcal{F}(Y_1)$ is nonempty because $W^s(Y_1) \in \mathcal{F}(Y_1)$. Moreover, we claim that

$$[W, W' \in \mathcal{F}(Y_1) \text{ and } \mu(W \setminus W') > 0] \Rightarrow W \setminus W' \in \mathcal{F}(Y_1). \tag{9}$$

To see this, let $U, U' \subseteq Y_1$ be invariant sets such that $W = W^s(U)$ and $W' = W^s(U')$. We claim that

$$W \setminus W' = W^s(U \setminus W^s(U')). \tag{10}$$

Notice that $U \setminus W^s(U') \subseteq Y_1$, and also $U \setminus W^s(U')$ is invariant because both $U, W^s(U')$ are invariant. Therefore (10) implies (9).

To prove (10), we prove first of all that $W \setminus W' \subseteq W^s(U \setminus W^s(U'))$. Suppose $x \in W \setminus W'$, i.e. $x \in W^s(U)$ and $x \notin W^s(U')$. This means that there exists $u \in U$ such that $x \in W^s(u)$ and also that $x \notin W^s(u')$ for any $u' \in U'$, which implies that $x \notin W^s(z)$ for any $z \in W^s(U')$ by the transitivity of $\sim$ mentioned above. This proves the $\subseteq$ inclusion. To prove $\supseteq$, let $x \in W^s(U \setminus W^s(U'))$. Then clearly $x \in W$ and $x \in W^s(y)$ for some $y \in U \setminus W^s(U')$. It just remains to show that $x \notin W'$. For contradiction, suppose that $x \in W' = W^s(U')$; then $x \in W^s(u')$ for some $u' \in U'$, and so, as $x \sim y$, we have $y \in W^s(u')$, which contradicts the fact that $y \in U \setminus W^s(U')$. This completes the proof of (10) and hence of (9).

Now consider the partial order on $\mathcal{F}(Y_1)$ defined by *strict inclusion*, meaning that $W \succ W'$ if $W \supset W'$ and $\mu(W \setminus W') > 0$. We claim that for this partial order, every totally ordered subset of $\mathcal{F}(Y_1)$ is finite, and in particular it has a lower bound. Indeed, for contradiction suppose that there is an infinite sequence $W_1 \succ W_2 \succ \cdots$ in $\mathcal{F}(Y_1)$, i.e. $W_1 \supset W_2 \supset \cdots$ with $\mu(W_k \setminus W_{k+1}) > 0$ for all $k \geq 1$. Then

$$\sum_{k\geq 1} \mu(W_k \setminus W_{k+1}) = \mu(W_1) < \infty,$$

and therefore $\mu(W_k \setminus W_{k+1}) \to 0$ as $k \to \infty$. Since $W_k \setminus W_{k+1} \in \mathcal{F}(Y_1)$ by (9), this contradicts our assumptions that $Y_1 = Y$ is μ-unshrinkable. This shows that every totally ordered subset of $\mathcal{F}(Y_1)$ has a lower bound. Thus by Zorn's Lemma there exists at least one minimal element $W^s(U_1) \in \mathcal{F}(Y_1)$, which must therefore necessarily be a u-ergodic component.

We now let $Y_2 := Y_1 \setminus W^s(U_1)$, which is again invariant. If $\mu(Y_2) = 0$ then $Y = Y_1$ is essentially contained in $W^s(U_1)$, which is a u-ergodic component, and thus we are done. On the other hand, if $\mu(Y_2) > 0$ we can repeat the entire argument above to obtain a set $U_2 \subseteq Y_2$ and a u-ergodic component $W^s(U_2)$. Inductively, we construct a collection of disjoint u-ergodic components $W^s(U_1), \dots, W^s(U_r)$ and continue as long as $\mu(Y \setminus W^s(U_1) \cup \dots \cup W^s(U_r)) > 0$. But, as $\mu(W^s(U_j)) \geq \delta$ for all $1 \leq j \leq r$ by the assumption that Y is μ-unshrinkable, this process will stop and we will get the conclusion. □

Lemma 3.3. *Suppose $S \subseteq X$ is a u-ergodic component. Then there exists a closed invariant set $\Omega \subseteq X$ such that $\omega(x) = \Omega$ for μ-almost every $x \in S$.*

Proof. Given any open set $B \subset X$, let

$$B_\omega := \{x \in S : \omega(x) \cap B \neq \emptyset\}.$$

Then B_ω is invariant and s-saturated, and therefore, by the assumption that S is u-ergodic, $\mu(B_\omega) = 0$ or $\mu(B_\omega) = \mu(S)$. Now, let $Z_1 = X$ and $\mathcal{C}_1$ be any finite covering of X by open balls of radius 1. By the previous considerations, for every $B \in \mathcal{C}_1$ we have $\mu(B_\omega) = 0$ or $\mu(B_\omega) = \mu(S)$, and therefore, since we only have a finite number of elements in $\mathcal{C}_1$, there exists at least one $B_\omega \in \mathcal{C}_1$ such that $\mu(B_\omega) = \mu(S)$. Let

$$\mathcal{C}_1' = \{B \in \mathcal{C}_1 : \mu(B_\omega) = 0\} \quad \text{and} \quad Z_2 = Z_1 \setminus \bigcup_{B \in \mathcal{C}_1'} B.$$

Then Z_2 is a nonempty compact set and $\omega(x) \subseteq Z_2$ for μ-almost every $x \in S$. We can therefore repeat the procedure with a finite cover $\mathcal{C}_2$ of Z_2 by open balls of radius $1/2$, and, by induction, construct sequences $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_1', \mathcal{C}_2', \dots$ and $Z_1, Z_2, \dots$ such that $Z_1 \supset Z_2 \supset \cdots$ is a sequence of nonempty compact sets and $\omega(x) \subset Z_j$ for almost every $x \in S$. In particular,

$$\omega(x) \subseteq \Omega := \bigcap_{n \geq 1} Z_n.$$

It just remains to show that $\Omega \subseteq \omega(x)$ for μ-almost every $x \in S$. Indeed, given $y \in \Omega$ we have $y \in Z_n$ for every $n \geq 1$, and therefore there is some $B^{(n)} \in \mathcal{C}_n \setminus \mathcal{C}_n'$ such that $y \in B^{(n)}$. Since $\operatorname{diam}(B^{(n)}) \to 0$ as $n \to \infty$, this implies that $\bigcap_n B^{(n)} = \{y\}$. Moreover, as $B^{(n)} \in \mathcal{C}_n \setminus \mathcal{C}_n'$, we have $\mu(B_\omega^{(n)}) = \mu(S)$, and therefore $\omega(x) \cap B^{(n)} \neq \emptyset$ for μ-almost all $x \in S$. This implies that $y \in \omega(x)$ for μ-almost all $x \in S$, and as $\omega(x)$ is closed and invariant, the statement follows. □

4. Transitive attractors

Here we prove the topological part of Theorem A. Throughout this section we assume that the assumptions of the theorem hold. Let $f : M \to M$ be a C^{1+} diffeomorphism, $K \subset M$ a forward invariant compact set on which f is partially hyperbolic, and $H \subseteq K$ a set with $\operatorname{Leb}(H) > 0$ on which f is weakly nonuniformly expanding along E^{cu}. The main result of this section is the following proposition.

Proposition 4.1. *There exist closed invariant sets* $\Omega_1, \dots, \Omega_\ell \subseteq K$ *such that for Lebesgue almost every* $x \in H$ *we have* $\omega(x) = \Omega_j$ *for some* $1 \le j \le \ell$*. Moreover, each* Ω_j *is transitive and contains a cu-disk* Δ_j *of radius* $\delta_1/4$ *on which* f *is weakly nonuniformly expanding along* E^{cu} *for* Leb_{Δ_j}*-almost every point in* Δ_j*.*

We first prove some preliminary lemmas. We remark that K is not assumed to contain any open sets. We therefore fix continuous extensions of the two subbundles E^s and E^{cu} to some compact neighbourhood V of K, which we still denote E^s and E^{cu}. We do not require these extensions to be Df-invariant. Given $0 < a < 1$, we define the *centre-unstable cone field* $C_a^{cu} = (C_a^{cu}(x))_{x \in V}$ *of width* a by

$$C_a^{cu}(x) = \{v_1 + v_2 \in E_x^s \oplus E_x^{cu} : \|v_1\| \le a\|v_2\|\}. \tag{11}$$

We define the *stable cone field* $C_a^s = (C_a^s(x))_{x \in V}$ *of width* a in a similar way, just reversing the roles of the subbundles in (11). We fix $a > 0$ and V small enough so that the domination condition in (1) remains valid in the two cone fields:

$$\|Df(x)v^s\| \cdot \|Df^{-1}(f(x))v^{cu}\| < \lambda \|v^s\| \cdot \|v^{cu}\|$$

for every $v^s \in C_a^s(x)$, $v^{cu} \in C_a^{cu}(f(x))$ and any point $x \in V \cap f^{-1}(V)$. Note that the centre-unstable cone field is forward invariant: $Df(x)C_a^{cu}(x) \subset C_a^{cu}(f(x))$ whenever $x, f(x) \in V$. Actually, the domination property together with the invariance of $E^{cu}|_K$ implies that $Df(x)C_a^{cu}(x) \subset C_{\lambda a}^{cu}(f(x)) \subset C_a^{cu}(f(x))$ for every $x \in K$, and this extends to any $x \in V \cap f^{-1}(V)$ just by continuity.

Definition 4.1. Given $0 < \sigma < 1$, we say that n is a σ*-hyperbolic time* for $x \in K$ if

$$\prod_{j=n-k+1}^{n} \|Df^{-1}|_{E^{cu}_{f^j(x)}}\| \le \sigma^k \quad \text{for all } 1 \le k \le n.$$

The next result gives the existence of (infinitely many) σ-hyperbolic times for points satisfying the weak nonuniform expansion condition (2). For a proof see [7, Corollary 5.3].

Lemma 4.2. *There are* $\sigma, \theta > 0$ *such that if* (2) *holds for* $x \in K$*, then*

$$\limsup_{n\to\infty} \frac{1}{n} \#\{1 \le j \le n : j \text{ is a } \sigma\text{-hyperbolic time for } x\} \ge \theta.$$

Remark 4.3. We remark that the lim sup in the conclusions of Lemma 4.2 is directly related to the lim inf in condition (2). Replacing that lim inf with a lim sup, as in [4], would allow us to replace the lim sup in the statement of Lemma 4.2 with a lim inf and thus obtain positive frequency at infinity of hyperbolic times, which plays a crucial role in the argument used in [4, Corollary 3.2] to prove the existence of SRB measures. The fact that we do not have such a positive frequence is exactly the reason why we cannot use those arguments here.

Hyperbolic times are defined pointwise but, as we shall see below, some important properties can be derived for a neighbourhood of the reference point at a hyperbolic time. From now on we fix σ and θ as in Lemma 4.2. Now observe that, by continuity of the derivative, we can choose $a > 0$ and $\delta_1 > 0$ sufficiently small so that the δ_1-neighbourhood of K is contained in V and

$$\|Df^{-1}(f(y))v\| \leq \sigma^{-1/4}\|Df^{-1}|_{E^{cu}_{f(x)}}\| \, \|v\| \tag{12}$$

for all $x \in K$, $y \in V$ with $\mathrm{dist}(x, y) \leq \delta_1$, and $v \in C_a^{cu}(y)$. From now on we fix these values of a, δ_1 so that (12) holds.

We say that an embedded C^1 submanifold $D \subset V$ is a *cu-disk* if the tangent subspace to D at each point $x \in D$ is contained in the corresponding cone $C_a^{cu}(x)$. Then $f(D)$ is also a cu-disk, if it is contained in V, by the domination property. Given any disk $D \subset M$, we use $\mathrm{dist}_D(x, y)$ to denote the distance between $x, y \in D$, measured along D.

Lemma 4.4. *Let D be a cu-disk. There exists $C_1 > 1$ such that if n is a σ-hyperbolic time for $x \in K \cap D$, then there exists a neighbourhood $V_n^+(x)$ of x in D such that f^n maps $V_n^+(x)$ diffeomorphically onto a cu-disk $B^u_{2\delta_1}(f^n(x))$ of radius $2\delta_1$ around $f^n(x)$. Moreover, for every $1 \leq k \leq n$ and $y, z \in V_n^+(x)$ we have*

(1) $\mathrm{dist}_{f^{n-k}(V_n^+(x))}(f^{n-k}(y), f^{n-k}(z)) \leq \sigma^{3k/4}\, \mathrm{dist}_{f^n(V_n^+(x))}(f^n(y), f^n(z))$;

(2) $\log \dfrac{|\det Df^n|_{T_yD}|}{|\det Df^n|_{T_zD}|} \leq C_1\, \mathrm{dist}_{f^n(D)}(f^n(y), f^n(z))$;

(3) *for any Borel sets $X, Y \subset V_n^+(x)$,*

$$\frac{\mathrm{Leb}_{f^n(V_n^+(x))}(f^n(X))}{\mathrm{Leb}_{f^n(V_n^+(x))}(f^n(Y))} \leq C_1 \frac{\mathrm{Leb}_{V_n^+(x)}(X)}{\mathrm{Leb}_{V_n^+(x)}(Y)}.$$

The first two items are proved in [4, Lemma 2.7 & Proposition 2.8], and the third is a standard consequence of the second. Notice that the factor $\sigma^{3/4}$ in the first item differs from the factor $\sigma^{1/2}$ in [4, Lemma 2.7] simply because we have chosen $\delta_1 > 0$ sufficiently small so that (12) holds, in contrast to [7, (6)] where $\delta_1 > 0$ is chosen so that a similar conclusion holds with $\sigma^{1/2}$ in place of $\sigma^{1/4}$.

Remark 4.5. Notice that if we replace the assumption that n is a σ-hyperbolic time in Lemma 4.4 with the assumption that n is a σ^α-hyperbolic time for some $\alpha > 1/4$ then the conclusions of the lemma continue to hold with $\sigma^{\alpha-1/4}$ instead of $\sigma^{3/4}$ in item (1), where the term $1/4$ comes from (12).

Now we define $V_n(x) \subseteq V_n^+(x)$, where $f^n(V_n(x)) = B^u_{\delta_1}(f^n(x))$ is the cu-disk of radius δ_1 around $f^n(x)$ contained in $B^u_{2\delta_1}(f^n(x))$ as in Lemma 4.4. The sets $V_n(x)$ are called *hyperbolic pre-disks* and their images $f^n(V_n(x))$ *hyperbolic disks*. The following result is proved in [7, Proposition 5.5].

Lemma 4.6. *Let D be a cu-disk and $U \subseteq H$ with* $\mathrm{Leb}_D(U) > 0$*. Then there exists a sequence of sets $\cdots \subseteq W_2 \subseteq W_1 \subseteq D$ and a sequence of integers $n_1 < n_2 < \cdots$ such that*

(1) *W_k is contained in some hyperbolic pre-disk with hyperbolic time n_k;*
(2) *$D_k := f^{n_k}(W_k)$ is a cu-disk of radius $\delta_1/4$;*
(3) $\displaystyle\lim_{k\to\infty} \frac{\mathrm{Leb}_{D_k}(f^{n_k}(U \cap D))}{\mathrm{Leb}_{D_k}(D_k)} = 1.$

Now we are in a position to prove Proposition 4.1. We define

$$\widetilde{H} := \bigcup_{n\in\mathbb{Z}} f^n(H).$$

Then $\widetilde{H}$ is clearly invariant and $\mathrm{Leb}(\widetilde{H}) > 0$.

Lemma 4.7. *$\widetilde{H}$ is* Leb*-unshrinkable.*

Proof. It is sufficient to show that there exists $\delta > 0$ such that for every f-invariant set $U \subseteq \widetilde{H}$ with $\mathrm{Leb}(U) > 0$ we have $\mathrm{Leb}(W^s(U)) > \delta$. We remark that in the proof of this assertion, to be given in the following paragraphs, we will only use the assumption that U is forward invariant. This allows us to assume without loss of generality that $U \subseteq K$. Indeed, if U is invariant of positive Lebesgue measure, then it must intersect K in a set of positive Lebesgue measure, and as K is forward invariant, also $U \cap K$ is forward invariant. Clearly, if $\mathrm{Leb}(W^s(U \cap K)) > \delta$ then also $\mathrm{Leb}(W^s(U)) > \delta$. In particular, as $U \subseteq K$ it admits a partially hyperbolic structure, and as also $U \subseteq \widetilde{H}$, it is weakly nonuniformly expanding along E^{cu}.

Now we show that there exists a cu-disk $D \subseteq V$ such that $\mathrm{Leb}_D(U) > 0$. Recall that V is the neighbourhood of K introduced at the beginning of this section. To see this, consider a Lebesgue density point p of U. Notice that T_pM has a partially hyperbolic splitting $E^s_p \oplus E^{cu}_p$ and we can consider a neighbourhood of the origin foliated by disks parallel to the E^{cu} subspace whose images under the exponential map $\exp_p$ are cu-disks in the manifold. Since $\exp_p$ is a local diffeomorphism, the preimage of U under the exponential map has positive volume in T_pM and full density at the origin. By Fubini, at least one of the disks above must intersect this set in positive relative volume, and the same must hold for its image under the exponential map.

Now let $D \subseteq V$ be a cu-disk satisfying $\mathrm{Leb}_D(U) > 0$, as in the previous paragraph. Consider the sequences $\cdots \subseteq W_2 \subseteq W_1 \subseteq D$ and $n_1 < n_2 < \cdots$ given by Lemma 4.6. By Lemma 4.6(3) the relative measure of $f^{n_k}(U \cap D)$ in D_k converges to 1. Since U is forward invariant, we conclude that the relative measure of U in D_k converges to 1, and therefore $\mathrm{Leb}_{D_k}(U) \to \delta_1/4$ as $k \to \infty$. Since $U \subseteq K$ and all points of U have local stable manifolds of uniform size, and since the foliation defined by these local stable manifolds is absolutely continuous, it follows that $\widetilde{H}$ is Leb-unshrinkable. □

The previous result, together with Proposition 3.1, implies that there exist closed invariant sets $\Omega_1, \ldots, \Omega_\ell$ such that for Lebesgue almost every $x \in H$ we have $\omega(x) = \Omega_j$ for some $1 \le j \le \ell$. This gives the first assertion of Proposition 4.1. We divide the proof of the remaining part of Proposition 4.1 into the next two lemmas.

Lemma 4.8. *Each $\Omega = \Omega_j$ contains a cu-disk Δ of radius $\delta_1/4$ on which f is weakly nonuniformly expanding along E^{cu} for Leb_Δ-almost every point in Δ.*

Proof. Let

$$A^{(n)} = \{x \in H : \mathrm{dist}(f^k(x), \Omega) \leq 1/n \text{ for every } k \geq 0\}.$$

Since the set of points $x \in H$ with $\omega(x) = \Omega$ has positive Lebesgue measure, we clearly have $\mathrm{Leb}(A^{(n)}) > 0$ for every $n \geq 1$. Then, by the same arguments used in the proof of Lemma 4.7, with $A^{(n)}$ playing the role of U, there exists a cu-disk $D^{(n)} \subseteq V$ such that $\mathrm{Leb}_{D^{(n)}}(A^{(n)}) > 0$, and corresponding sequences $\cdots \subseteq W_2^{(n)} \subseteq W_1^{(n)} \subseteq D^{(n)}$, $n_1 < n_2 < \cdots$ (also depending on n, but we omit the superscript here for obvious reasons) and cu-disks $D_k^{(n)} = f^{n_k}(W_k^{(n)})$ such that

$$\mathrm{Leb}_{D_k^{(n)}}(A^{(n)}) \to \delta_1/4 \quad \text{as } k \to \infty. \tag{13}$$

Let $p_k^{(n)}$ denote the centre of the disk $D_k^{(n)}$. Up to taking a subsequence, we may assume that the sequence $\{p_k^{(n)}\}$ converges to a point $p^{(n)} \in K$, and up to taking a further subsequence, and using Ascoli–Arzelà and the fact that the disks $D_k^{(n)}$ have tangent directions contained in the cu-cones, we may assume that the sequence $\{D_k^{(n)}\}$ converges uniformly, as $k \to \infty$, to some cu-disk $\Delta^{(n)}$ of radius $\delta_1/4$. Notice that each $\Delta^{(n)}$ is necessarily contained in a neighbourhood of Ω of radius $1/n$.

We claim tht f is weakly nonuniformly expanding along E^{cu} for $\mathrm{Leb}_{\Delta^{(n)}}$-almost every point in $\Delta^{(n)}$. To see this, recall first of all that the property of weak nonuniform expansion is an asymptotic property, and therefore if it is satisfied by a point x then it is satisfied by every point $y \in W^s(x)$. Moreover, every point of $\Delta^{(n)}$ has a local stable manifold of uniform size, and the foliation by those local stable manifolds is absolutely continuous. Since the sequence $\{D_k^{(n)}\}$ converges uniformly to $\Delta^{(n)}$, for large k, the disks $D_k^{(n)}$ will intersect the stable foliation through points of $\Delta^{(n)}$, and therefore, by (13) and the fact that $A^{(n)} \subseteq H$, it follows that f is weakly nonuniformly expanding along E^{cu} for $\mathrm{Leb}_{\Delta^{(n)}}$-almost every point in $\Delta^{(n)}$.

Now, arguing as above, we can consider a subsequence of $\Delta^{(n)}$'s converging uniformly to some cu-disk Δ of radius $\delta_1/4$ such that f is weakly nonuniformly expanding along E^{cu} for Leb_Δ-almost every point in Δ. As each $\Delta^{(n)}$ is contained in a neighbourhood of Ω of radius $1/n$ and Ω is closed, it follows that $\Delta \subseteq \Omega$. □

Lemma 4.9. *$f|_\Omega$ is transitive.*

Proof. Recall that by construction there exists some point (in fact a positive Lebesgue measure set of points) in H whose ω-limit set coincides with Ω. The orbit of any such point must eventually hit the stable manifold of some point in $\Delta \subseteq \Omega$. As points in the same stable manifold have the same ω-limit sets, we conclude that there exists a point of Ω whose orbit is dense in Ω. □

5. Construction on a reference leaf

In this section we describe an algorithm for the construction of a partition of some subdisk of Δ which is the basis of the construction of the GMY structure. We first fix some $1 \le j \le \ell$ and for the rest of the paper we let $\Omega = \Omega_j$ and $\Delta = \Delta_j$ be as in Proposition 4.1. We also fix a constant $\delta_s > 0$ so that the local stable manifolds $W^s_{\delta_s}(x)$ are defined for all points $x \in K$. For any subdisk $\Delta' \subset \Delta$ we define

$$\mathcal{C}(\Delta') = \bigcup_{x \in \Delta'} W^s_{\delta_s}(x).$$

Let π denote the projection from $\mathcal{C}(\Delta')$ onto Δ' along local stable leaves. We say that a centre-unstable disk $\gamma^u \subset M$ *u-crosses* $\mathcal{C}(\Delta')$ if $\pi(\gamma) = \Delta'$ for some connected component γ of $\gamma^u \cap \mathcal{C}(\Delta')$.

Remark 5.1. We will often be considering *cu*-disks which *u*-cross $\mathcal{C}(\Delta')$. By continuity of the stable foliation, if we choose δ_s sufficiently small, then the diameter and Lebesgue measure of such disks intersected with $\mathcal{C}(\Delta')$ are very close to those of Δ', respectively. To simplify the notation and the calculations below we will ignore this difference as it has no significant effect on the estimates.

Lemma 5.2. *Given $N \in \mathbb{N}$, there exists $\delta_2 = \delta_2(N, \delta_1) > 0$ such that if $\gamma^u \subset \Omega$ is a cu-disk of radius $\delta_1/2$ centred at z, then $f^m(\gamma^u)$ contains a cu-disk of radius δ_2 centred at $f^m(z)$, for each $1 \le m \le N$.*

Proof. We first prove the result for $j = 1$. Let z be the centre of γ^u. Let $f(y)$ be a point in $\partial f(\gamma^u)$ minimizing the distance from $f(z)$ to $\partial f(\gamma^u)$, and let η_1 be a curve of minimal length in $f(\gamma^u)$ connecting $f(z)$ to $f(y)$. Letting $\eta_0 = f^{-1}(\eta_1)$ and $\dot\eta_1(x)$ be the tangent vector to the curve η_1 at the point x, we have

$$\|Df^{-1}(w)\dot\eta_1(x)\| \le C\|\dot\eta_1(x)\|, \quad \text{where} \quad C = \max_{x \in M} \|Df^{-1}(x)\| \ge 1.$$

Hence,

$$\text{length}(\eta_0) \le C\,\text{length}(\eta_1).$$

Since η_0 is a curve connecting z to $y \in \partial\gamma^u$, we have length$(\eta_0) \ge \delta_1/2$, and so

$$\text{length}(\eta_1) \ge C^{-1}\,\text{length}(\eta_0) \ge C^{-1}\delta_1/2.$$

Thus $f(\gamma^u)$ contains the *cu*-disk γ^u_1 of radius $C^{-1}\delta_1/2$ around $f(z)$.

If we now make γ^u_1 play the role of γ^u and $f^2(z)$ play the role of $f(z)$, the argument above proves that $f(\gamma^u_1)$ contains a *cu*-disk of radius $C^{-2}\delta_1/2^2$ centred at $f^2(z)$. Inductively, we prove that $f^m(\gamma^u)$ contains a *cu*-disk of radius $C^{-m}\delta_1/2^m \ge C^{-N}\delta_1/2^N$ around $f^m(z)$, for each $1 \le m \le N$. We take $\delta_2 = C^{-N}\delta_1/2^N$. □

Lemma 5.3. *There are $p \in \Delta$ and $N_0 \ge 1$ such that for all $\delta_0 > 0$ sufficiently small and each hyperbolic pre-disk $V_n(x) \subseteq \Delta$ there is $0 \le m \le N_0$ such that $f^{n+m}(V_n(x))$ intersects $W^s_{\delta_s/2}(p)$ and u-crosses $\mathcal{C}(B^u_{\delta_0}(p))$, where $B^u_{\delta_0}(p)$ is the ball in Δ of radius δ_0 centred at p.*

Proof. First of all we observe that, as the subbundles in the dominated splitting have angles uniformly bounded away from zero, given any $\rho > 0$ there is $\alpha = \alpha(\rho) > 0$, with $\alpha \to 0$ as $\rho \to 0$, for which the following holds: if $x, y \in \Omega$ satisfy $\operatorname{dist}(x, y) < \rho$ and $\operatorname{dist}_{\gamma^u}(y, \partial\gamma^u) > \delta_1$ for some *cu*-disk $\gamma^u \subset \Omega$, then $W^s_{\delta_s}(x)$ intersects γ^u in a point z with

$$\operatorname{dist}_{W^s_{\delta_s}(x)}(z, x) < \alpha \quad \text{and} \quad \operatorname{dist}_{\gamma^u}(z, y) < \delta_1/2.$$

Take $\rho > 0$ small enough so that $4\alpha < \delta_s$. Since $f|_\Omega$ is transitive, we may choose $q \in \Omega$ and $N_0 \in \mathbb{N}$ such that both:

- $W^s_{\delta_s/4}(q)$ intersects Δ in a point p with $\operatorname{dist}_\Delta(p, \partial\Delta) > 0$; and
- $\{f^{-N_0}(q), \dots, f^{-1}(q), q\}$ is ρ-dense in Ω.

Given a hyperbolic pre-disk $V_n(x) \subseteq \Delta$ we know by definition that $f^n(V_n(x))$ is a *cu*-disk of radius δ_1 centred at $y = f^n(x)$ inside Ω. Consider $0 \le m \le N_0$ such that $\operatorname{dist}(f^{-m}(q), y) < \rho$. Then, by the choice of ρ and α, the set $W^s_{\delta_s}(f^{-j}(q))$ intersects $f^n(V_n(x))$ in a point z with $\operatorname{dist}_{W^s_{\delta_s}(f^{-j}(q))}(z, f^{-j}(q)) < \alpha < \delta_s/4$ and $\operatorname{dist}_{f^n(V_n(x))}(z, y) < \delta_1/2$. In particular, $f^n(V_n(x))$ contains a *cu*-disk γ^u of radius $\delta_1/2$ centred at z. It follows from Lemma 5.2 that $f^m(\gamma^u)$ contains a *cu*-disk of radius $\delta_2 = \delta_2(N_0, \delta_1) > 0$ centred at $f^m(z) \in W^s(p)$. Moreover, as distances are not expanded under iterations of points in the same stable manifold, we have

$$\operatorname{dist}_{W^s(p)}(f^m(z), p) \le \operatorname{dist}_{W^s(p)}(f^m(z), q) + \operatorname{dist}_{W^s(p)}(q, p) \le \delta_s/4 + \delta_s/4,$$

which means that $f^{n+m}(V_n(x))$ intersects $W^s_{\delta_s/2}(p)$. Also, choosing $\delta_0 > 0$ sufficiently small (depending only on δ_2) we see that $f^{n+m}(V_n(x))$ *u*-crosses $\mathcal{C}(B^u_{\delta_0}(p))$. □

We now fix $p \in \Delta$, $N_0 \ge 1$ and $\delta_0 > 0$ sufficiently small so that the conclusions of Lemma 5.3 hold. Considering the constant

$$K_0 = \max_{x \in M}\{\|Df^{-1}(x)\|, \|Df(x)\|\}, \tag{14}$$

we choose in particular $\delta_0 > 0$ so small that

$$2\delta_0 K_0^{N_0}\sigma^{-N_0} < \delta_1 K_0^{-N_0}. \tag{15}$$

Now we define

$$\Delta_0 = B^u_{\delta_0}(p) \quad \text{and} \quad \mathcal{C}_0 = \mathcal{C}(\Delta_0). \tag{16}$$

We also choose $\delta_0 > 0$ so small that any *cu*-disk intersecting $W^s_{3\delta_s/4}$ cannot reach the top or bottom parts of $\mathcal{C}_0$, i.e. the boundary points of the local stable manifolds $W^s_{\delta_s}(x)$ through points $x \in \Delta_0$. For every $n \ge 1$ we define

$$H_n = \{x \in \Delta \cap H : n \text{ is a hyperbolic time for } x\}.$$

It follows from Lemma 4.4 that for each $x \in H_n \cap \Delta_0$ there exists a hyperbolic pre-disk $V_n(x) \subset \Delta$. Then by Lemma 5.3 there are $0 \le m \le N_0$ and a centre-unstable disk $\omega^x_n \subseteq \Delta$ such that

$$\pi(f^{n+m}(\omega^x_n)) = \Delta_0. \tag{17}$$

We remark that condition (17) may in principle hold for several values of m. For definiteness, we shall always assume that m takes the smallest possible value. Notice that ω_n^x is associated to x by construction, but does not necessarily contain x.

Now we describe an inductive partitioning algorithm which gives rise to a (Leb mod 0) partition $\mathcal{P}$ of the cu-disk Δ_0.

Base step. We observe that since $\|Df\|$ is uniformly bounded, for any $n \geq 1$, all hyperbolic pre-disks $V_n(x)$ contain a ball of some radius $\tau_n > 0$ which depends only on n. In particular, by compactness, the set H_n is covered by a finite number of hyperbolic pre-disks $V_n(x)$.

We fix some large $n_0 \in \mathbb{N}$ and ignore any dynamics occurring up to time n_0. Then there exist ℓ_{n_0} and points $z_1, \ldots, z_{\ell_{n_0}} \in H_{n_0}$ such that

$$H_{n_0} \cap \Delta_0 \subset S_{n_0} := V_{n_0}(z_1) \cup \cdots \cup V_{n_0}(z_{\ell_{n_0}}).$$

We now choose a maximal subset of points $x_1, \ldots, x_{j_{n_0}} \in \{z_1, \ldots, z_{\ell_{n_0}}\}$ such that the corresponding sets $\omega_{n_0}^{x_i}$ of type (17) are pairwise disjoint and contained in Δ_0, and let

$$\mathcal{P}_{n_0} = \{\omega_{n_0}^{x_1}, \ldots, \omega_{n_0}^{x_{j_{n_0}}}\}.$$

These are the elements of the partition $\mathcal{P}$ constructed in the n_0-th step of the algorithm. Let

$$\Delta_{n_0} = \Delta \setminus \bigcup_{\omega \in \mathcal{P}_{n_0}} \omega.$$

For each $0 \leq i \leq j_{n_0}$, we define the inducing time

$$R|_{\omega_{n_0}^{x_i}} = n_0 + m_i$$

where $0 \leq m_i \leq N$ is the integer associated to $\omega_{n_0}^{x_i}$ as in (17).

Inductive step. We now assume inductively that the construction has been carried out up to time $n-1$ for some $n > n_0$. More precisely, for each $n_0 \leq k \leq n-1$ we have a collection of pairwise disjoint sets $\mathcal{P}_k = \{\omega_k^{x_1}, \ldots, \omega_k^{x_{j_k}}\}$ which "return" at time $k+m$ with $0 \leq m \leq N$, and such that for any $k \neq k'$, any two sets $\omega \in \mathcal{P}_k$ and $\omega' \in \mathcal{P}_{k'}$ are disjoint. We also have a set Δ_k which is the set of points which do not yet have an associated return time. To construct all relevant objects at time n, we first observe, as before, that there are $z_1, \ldots, z_{\ell_n} \in H_n$ such that

$$H_n \cap \Delta_{n-1} \subset S_n := V_n(z_1) \cup \cdots \cup V_n(z_{\ell_n}),$$

and we choose a maximal subset of points $x_1, \ldots, x_{j_n} \in \{z_1, \ldots, z_{\ell_n}\}$ such that the corresponding sets of type (17) are pairwise disjoint and contained in Δ_{n-1}. Then we let

$$\mathcal{P}_n = \{\omega_n^{x_1}, \ldots, \omega_n^{x_{j_n}}\}.$$

These are the elements of the partition $\mathcal{P}$ constructed in the n-th step of the algorithm. We also define the set of points of Δ_0 which do not belong to partition elements constructed up to this point:

$$\Delta_n = \Delta_0 \setminus \bigcup_{\omega \in \mathcal{P}_{n_0} \cup \cdots \cup \mathcal{P}_n} \omega.$$

For each $0 \le i \le j_n$ we set

$$R|_{\omega_n^{x_i}} = n + m_i$$

where $0 \le m_i \le N$ is the integer associated to $\omega_{n_0}^{x_i}$ as in (17). Note that for each $n \ge n_0$ one has

$$H_n \cap \Delta_0 \subset S_n \cup \bigcup_{\omega \in \mathcal{P}_{n_0} \cup \cdots \cup \mathcal{P}_n} \omega. \tag{18}$$

More specifically, we have $H_n \cap \Delta_{n-1} \subset S_n$, i.e. all points in Δ_{n-1} which have a hyperbolic time at time n are "covered" by S_n, while the points which have a hyperbolic time at time n but which are already contained in previously constructed partition elements, are trivially "covered" by the union of these partition elements. The inclusion (18) will be crucial in the proof of Proposition 6.1 and in Section 8 to prove the integrability of the return times. This inductive construction allows us to define the family

$$\mathcal{P} = \bigcup_{n \ge n_0} \mathcal{P}_n$$

of pairwise disjoint subsets of Δ_0. In the next section we prove

Proposition 5.4. *$\mathcal{P}$ forms a* Leb *mod* 0 *partition of* Δ_0.

The statement in Proposition 5.4 seems intuitively obvious. After all, the elements of $\mathcal{P}$ are automatically disjoint by construction and almost every point x has a basis of arbitrarily small neighbourhoods which in time grow to large scale and return to the base within a finite number of iterates, and each of these returns is a candidate for the creation of an element of $\mathcal{P}$ containing x. The potential problem is that each time such an opportunity arises, the region ω^x which returns either may not contain x or cannot be chosen because it overlaps a previously constructed element of $\mathcal{P}$. Thus it is theoretically conceivable a priori that $\mathcal{P}$ may not have full measure in Δ_0.

6. Partition on the reference leaf

In this section we prove Proposition 5.4. The key step is the following result.

Proposition 6.1. $\sum_{n=n_0}^{\infty} \mathrm{Leb}_\Delta(S_n) < \infty$.

Proof of Proposition 5.4 assuming Proposition 6.1. Recall that $\Delta_0 \supset \Delta_{n_0} \supset \Delta_{n_0+1} \supset \cdots$, where Δ_n is the set of points which do not belong to any element of the collection $\mathcal{P}$ constructed up to time n. It is enough to show that

$$\mathrm{Leb}_\Delta\Big(\bigcap_n \Delta_n\Big) = 0. \tag{19}$$

To prove this, notice that by Proposition 6.1, the sum of the Leb_Δ-measures of the sets S_n is finite. It follows from the Borel–Cantelli Lemma that Leb_Δ-almost every $x \in \Delta_0$ belongs only to finitely many S_n's, and therefore one can find n such that $x \notin S_j$ for $j \geq n$. Since Leb_Δ-almost every $x \in \Delta_0$ has infinitely many hyperbolic times, it follows from (18) that $x \in \omega$ for some $\omega \in \mathcal{P}_{n_0} \cup \cdots \cup \mathcal{P}_n$, and therefore (19) holds. □

To prove Proposition 6.1 it will be useful to decompose S_n, which is simply the union of all hyperbolic pre-balls $V_n(z_i)$ associated to points $z_1, \ldots, z_{\ell_n} \in H_n \cap \Delta_{n-1}$, into several pieces as follows. Recall that we have points $x_1, \ldots, x_{j_n} \in \{z_1, \ldots, z_{\ell_n}\}$ which give rise to elements of $\mathcal{P}_n$. We collect the corresponding hyperbolic pre-balls in the set

$$V_n = \bigcup_{i=1}^{j_n} V(x_i).$$

The remaining pre-balls are associated to the points

$$Z_n = \{z_1, \ldots, z_{\ell_n}\} \setminus \{x_1, \ldots, x_{j_n}\}$$

which do not give rise to elements of $\mathcal{P}_n$ because the corresponding domains ω^x intersect some previously (or currently) constructed partition element, i.e. some $\omega \in \mathcal{P}_{n_0} \cup \cdots \cup \mathcal{P}_n \cup \{\Delta_0^c\}$. We need to further distinguish them according to precisely which of these regions they intersect. Thus, for any $\omega \in \mathcal{P}_{n_0} \cup \cdots \cup \mathcal{P}_n \cup \{\Delta_0^c\}$ define

$$Z_n^\omega = \{z \in Z_n : \omega_n^z \cap \omega \neq \emptyset\}$$

and its *n-satellite*

$$S_n^\omega = \bigcup_{z \in Z_n^\omega} V_n(z).$$

Clearly,

$$S_n = \bigcup_{\omega \in \mathcal{P}_{n_0} \cup \cdots \cup \mathcal{P}_n \cup \{\Delta_0^c\}} S_n^\omega \cup V_n.$$

We can now prove two technical lemmas which will allow us to prove Proposition 6.1.

Lemma 6.2. *There exists $C_2 > 0$ such that for any $n \geq k \geq n_0$ and any $\omega \in \mathcal{P}_k$ we have*

$$\mathrm{Leb}_\Delta(S_n^\omega) \leq C_2 \,\mathrm{Leb}_\Delta\Big(\bigcup_{z \in Z_n^\omega} \omega_n^z\Big).$$

Proof. We note first of all that, from the construction above, two distinct points z_1, z_2 with the same hyperbolic time n can give rise to the same associated disks $\omega_n^{z_1} = \omega_n^{z_2}$. We prove here that the measure of the union of the hyperbolic pre-disks $V_n(z)$ associated to points $z \in Z_n^\omega$ which give rise to the same disk ω_n^z is comparable to the measure of ω_n^z. More precisely, we will show that for every $n \geq 1$ and $z_1, \ldots, z_N \in H_n$ with $\omega_n^{z_i} = \omega_n^{z_1}$ for $1 \leq i \leq N$ we have

$$\mathrm{Leb}_\Delta\Big(\bigcup_{i=1}^{N} V_n(z_i)\Big) \leq C_2 \,\mathrm{Leb}_\Delta(\omega_n^{z_1}). \tag{20}$$

Notice that (20) implies the statement in the lemma. Indeed, consider a subdivision of the set S_n^ω of all hyperbolic pre-disks associated to the points of Z_n^ω into a finite number of classes such that all hyperbolic pre-disks in each class have the same associated set ω_n^z. Then apply (20) to each. This gives the statement in the lemma.

Thus we just need to prove (20). For simplicity of notation, for $1 \le i \le N$, we write $U_i = V_n(z_i)$ and $B_i = f^n(V_i)$. We define

$$X_1 = U_1 \quad \text{and} \quad X_i = U_i \setminus \bigcup_{j=1}^{i-1} U_j \quad \text{for } 2 \le i \le N.$$

Similarly

$$Y_1 = B_1 \quad \text{and} \quad Y_i = B_i \setminus \bigcup_{j=1}^{i-1} B_j \quad \text{for } 2 \le i \le N.$$

Observe that the X_i's are pairwise disjoint sets whose union coincides with the union of the U_i's, and similarly for the Y_i's and B_i's. Recalling that $\omega_n^{z_i} = \omega_n^{z_1}$ for $1 \le i \le N$, by Lemma 4.4(3) we have

$$\frac{\mathrm{Leb}_\Delta(X_i)}{\mathrm{Leb}_\Delta(\omega_n^{z_1})} \le C_1 \frac{\mathrm{Leb}_{f^n(\Delta)}(Y_i)}{\mathrm{Leb}_{f^n(\Delta)}(f^n(\omega_n^{z_1}))}.$$

Hence

$$\begin{aligned}\frac{\mathrm{Leb}_\Delta(U_1 \cup \cdots \cup U_N)}{\mathrm{Leb}_\Delta(\omega_n^{z_1})} &= \frac{\sum_{i=1}^N \mathrm{Leb}_\Delta(X_i)}{\mathrm{Leb}_\Delta(\omega_n^{z_1})} \\ &\le C_1 \frac{\sum_{i=1}^N \mathrm{Leb}_{f^n(\Delta)}(Y_i)}{\mathrm{Leb}_{f^n(\Delta)}(f^n(\omega_n^{z_1}))} = C_1 \frac{\mathrm{Leb}_{f^n(\Delta)}(B_1 \cup \cdots \cup B_N)}{\mathrm{Leb}_{f^n(\Delta)}(f^n(\omega_n^{z_1}))}.\end{aligned}$$

We just need to show that the right hand side is bounded above, and for this it is sufficient to show that the denominator $\mathrm{Leb}_{f^n(\Delta)}(f^n(\omega_n^{z_1}))$ is bounded below. This is clearly true, because by definition of $\omega_n^{z_1}$ there exists $m \le N_0$ such that $f^{n+m}(\omega_n^{z_1})$ is a *cu*-disk of radius δ_0. □

Remark 6.3. The argument used to prove (20) gives in particular that for each $1 \le i \le j_n$ we have $\mathrm{Leb}_\Delta(V(x_i)) \le C_2 \,\mathrm{Leb}_\Delta(\omega_n^{x_i})$.

The next lemma shows that, for each n and m fixed, the Lebesgue measure on the disk H of the union of candidates ω_{n+m}^z which intersect an element of a partition is proportional to the Lebesgue measure of this element. The proportionality constant can actually be made uniformly summable in n.

Lemma 6.4. *There exists $C_3 > 0$ such that for all $n \ge k \ge n_0$ and $\omega \in \mathcal{P}_k$ we have*

$$\mathrm{Leb}_\Delta\Big(\bigcup_{z \in Z_n^\omega} \omega_n^z\Big) \le C_3 \sigma^{n-k}\, \mathrm{Leb}_\Delta(\omega).$$

Proof. By construction, given $\omega \in \mathcal{P}_k$, there is some hyperbolic pre-disk $V_k(y)$ such that

$$\omega \subset V_k(y) \subset V_k^+(y)$$

and the images $f^k(V_k(y))$ and $f^k(V_k^+(y))$ are respectively cu-disks $B^u_{\delta_1} \subset B^u_{2\delta_1}$ centred at $f^k(y)$. Moreover, there exists some integer $0 \leq \ell \leq N_0$ such that $f^{k+\ell}(V_k(y))$ u-crosses $\mathcal{C}_0$ and $f^{k+\ell}(\omega)$ is the part of $f^{k+\ell}(V_k(y))$ which projects onto Δ_0. Moreover, $f^k(V_k^+(y))$ is a δ_1-neighbourhood of $f^k(V_k(y))$, and so $f^{k+\ell}(V_k^+(y))$ contains a $\delta_1 K_0^{-N_0}$-neighbourhood of $f^{k+\ell}(V_k(y))$, where K_0 is defined in (14). In particular,

$$f^{k+\ell}(V_k^+(y)) \text{ contains a } \delta_1 K_0^{-N_0}\text{-neighbourhood of } \partial f^{k+\ell}(\omega). \tag{21}$$

For any $n \geq k$ we let

$$A^0_{n,k} = \{z \in f^{k+\ell}(V_k^+(y)) : \mathrm{dist}_{f^{k+\ell}(V_k^+(y))}(z, \partial f^{k+\ell}(\omega)) \leq 2\delta_0 K_0^{N_0} \sigma^{n-(k+N_0)}\},$$
$$A^1_{n,k} = \{z \in f^{k+\ell}(\omega) : \mathrm{dist}_{f^{k+\ell}(V_k^+(y))}(z, \partial f^{k+\ell}(\omega)) \leq 2\delta_1 K_0^{N_0} \sigma^{n-(k+N_0)}\}.$$

Observe that $A^0_{n,k}$ and $A^1_{n,k}$ are both annuli surrounding the boundary of $f^{k+\ell}(\omega)$ in $f^{k+\ell}(V_k^+(y))$, with $A^0_{n,k}$ containing this boundary in its interior, whereas $A^1_{n,k}$ shares this boundary with $f^{k+\ell}(\omega)$ and is fully contained in $f^{k+\ell}(\omega)$.

A straightforward calculation shows that there is a constant $C > 0$, independent of k and n, such that

$$\mathrm{Leb}_{f^{k+\ell}(V_k^+(y))}(A^i_{n,k}) \leq C\sigma^{n-k}, \quad i = 0, 1. \tag{22}$$

Now we see that for $z \in Z^\omega_n$ we have $f^{k+\ell}(\omega^z_n)$ contained in $A^0_{n,k}$ or $A^1_{n,k}$, depending on the following two possible cases:

(1) $\omega^z_n \subseteq \omega$. By Lemma 4.4(1) (see also Remark 5.1), for each ω^z_n with $z \in Z^\omega_n$ we have

$$\mathrm{diam}_{f^{k+\ell}(\omega^z_n)}(f^{k+\ell}(\omega^z_n)) \leq \mathrm{diam}_{f^{k+\ell}(\omega^z_n)}(f^{k+\ell}(V_n(z))) \leq 2\delta_1 K_0^{N_0} \sigma^{n-(k+N_0)}. \tag{23}$$

Noting that as $z \notin \omega$ and $\omega^z_n \subseteq \omega$, $V_n(z)$ necessarily intersects the boundary of ω, and so $f^{k+\ell}(V_n(z))$ intersects $\partial f^{k+\ell}(\omega)$. It follows from (23) that

$$f^{k+\ell}(\omega^z_n) \subseteq A^1_{n,k}. \tag{24}$$

(2) $\omega^z_n \nsubseteq \omega$. In this case, ω^z_n necessarily intersects the boundary of ω because $z \in Z^\omega_n$. Once more by Lemma 4.4(1) (see also Remark 5.1), we have

$$\mathrm{diam}_{f^{k+\ell}(\omega^z_n)}(f^{k+\ell}(\omega^z_n)) \leq 2\delta_0 K_0^{N_0} \sigma^{n-(k+N_0)}, \tag{25}$$

where we have used the fact that ω^z_n is contained in some hyperbolic pre-disk $V_n(z)$ and the term $K_0^{N_0}$ comes from the fact that ω^z_n may require up to N_0 iterates to go from $f^n(V_n(z))$ to the cylinder $\mathcal{C}_0$, u-crossing it. Since ω^z_n intersects the boundary of ω, $f^{k+\ell}(\omega^z_n)$ intersects the boundary of $f^{k+\ell}(\omega)$. Recalling that $\sigma < 1$ and (15), it follows from (21) and (25) that

$$f^{k+\ell}(\omega^z_n) \subseteq A^0_{n,k}. \tag{26}$$

Therefore

$$\frac{\mathrm{Leb}_{V_k^+(y)}(\bigcup_{z\in Z_n^\omega}\omega_n^z)}{\mathrm{Leb}_{V_k^+(y)}(\omega)} \le \widetilde{C}\,\frac{\mathrm{Leb}_{f^{k+\ell}(V_k^+(y))}(f^{k+\ell}(\bigcup_{z\in Z_n^\omega}\omega_n^z))}{\mathrm{Leb}_{f^{k+\ell}(V_k^+(y))}(f^{k+\ell}(\omega))}$$
$$\le \widetilde{C}\,\frac{\mathrm{Leb}_{f^{k+\ell}(V_k^+(y))}(A^0_{n,k}) + \mathrm{Leb}_{f^{k+\ell}(V_k^+(y))}(A^1_{n,k})}{\mathrm{Leb}_{f^{k+\ell}(V_k^+(y))}(f^{k+\ell}(\omega))},$$

where $\widetilde{C} > 0$ is a uniform constant that incorporates the distortion at the hyperbolic time k given by Lemma 4.4 and the distortion of f^ℓ with $\ell \le N_0$. If we now recall that $f^{k+\ell}(\omega)$ u-crosses $\mathcal{C}_0$, the result follows by (22), (24) and (26). □

Proof of Proposition 6.1. Observe that

$$\sum_{n=n_0}^{\infty}\mathrm{Leb}_\Delta(S_n) \le \sum_{n=n_0}^{\infty}\mathrm{Leb}_\Delta(S_n^{\Delta_0^c}) + \sum_{k=n_0}^{\infty}\sum_{\omega\in\mathcal{P}_k}\sum_{n=k}^{\infty}\mathrm{Leb}_\Delta(S_n^\omega) + \sum_{n=n_0}^{\infty}\mathrm{Leb}_\Delta(V_n). \quad (27)$$

We start by estimating the sum with respect to the satellites of Δ_0^c. Notice that from Lemma 4.4 it follows that all hyperbolic pre-disks $V_n(x)$ have diameter $\le 2\delta_1\sigma^n$. Therefore

$$S_n^{\Delta_0^c} \subset \{x \in \Delta_0 : \operatorname{dist}(x, \partial\Delta_0) < 2\delta_1\sigma^n\},$$

and so we can find $\zeta > 0$ such that

$$\mathrm{Leb}_\Delta(S_n^{\Delta_0^c}) \le \zeta\sigma^n.$$

This obviously implies that the part of the sum related to Δ_0^c in (27) is finite.

Consider now $n \ge k \ge n_0$. By Lemmas 6.2 and 6.4, for any $\omega \in \mathcal{P}_k$ we have

$$\mathrm{Leb}_\Delta(S_n^\omega) \le C_2C_3\sigma^{n-k}\,\mathrm{Leb}_\Delta(\omega).$$

It follows that

$$\sum_{k=n_0}^{\infty}\sum_{\omega\in\mathcal{P}_k}\sum_{n=k}^{\infty}\mathrm{Leb}_\Delta(S_n^\omega) \le C_2C_3\sum_{k=n_0}^{\infty}\sum_{\omega\in\mathcal{P}_k}\sum_{j=0}^{\infty}\sigma^j\,\mathrm{Leb}_\Delta(\omega)$$
$$= C_2C_3\frac{1}{1-\sigma}\sum_{k=n_0}^{\infty}\sum_{\omega\in\mathcal{P}_k}\mathrm{Leb}_\Delta(\omega) \le C_2C_3\frac{1}{1-\sigma}\,\mathrm{Leb}_\Delta(\Delta).$$

Finally, by Remark 6.3 we have

$$\sum_{n=n_0}^{\infty}\mathrm{Leb}_\Delta(V_n) \le C_2\sum_{n=n_0}^{\infty}\sum_{\omega\in\mathcal{P}_n}\mathrm{Leb}_\Delta(\omega) \le C_2\,\mathrm{Leb}_\Delta(\Delta),$$

and this gives the conclusion. □

7. The GMY structure

We are now ready to define the GMY structure on Ω as at the beginning of Section 5. Consider the centre-unstable disk $\Delta_0 \subset \Delta$ as in (16) and the Leb_Δ mod 0 partition $\mathcal{P}$ of Δ_0 defined in Section 5. We define

$$\Gamma^s = \{W^s_{\delta_s}(x) : x \in \Delta_0\}.$$

Moreover, we define Γ^u as the set of all local unstable manifolds contained in $\mathcal{C}_0$ which u-cross $\mathcal{C}_0$. Clearly, Γ^u is nonempty because $\Delta_0 \in \Gamma^u$. We need to see that the union of the leaves in Γ^u is compact. This follows ideas that we have already used to prove Proposition 4.1. By the domination property and the Ascoli–Arzelà Theorem, any limit leaf γ_∞ of leaves in Γ^u is still a cu-disk u-crossing $\mathcal{C}_0$. Thus, by definition of Γ^u, we have $\gamma_\infty \in \Gamma^u$. We thus define our set Λ with hyperbolic product structure as the intersection of these families of stable and unstable leaves. The cylinders $\{\mathcal{C}(\omega)\}_{\omega\in\mathcal{P}}$ then clearly form a countable collection of s-subsets of Λ that play the role of the sets $\Lambda_1, \Lambda_2, \dots$ in (P_1) with the corresponding return times $R(\omega)$. It just remains to check that conditions (P_1)–(P_5) hold.

7.1. *Markov property and contraction on stable leaves*

Condition (P_1) is essentially an immediate consequence of the construction. We just need to check that $f^{R(\omega)}(\mathcal{C}(\omega))$ is a u-subset for any $\omega \in \mathcal{P}$. Indeed, choosing the integer n_0 in the base step of the inductive algorithm sufficiently large, and using the fact that the local stable manifolds are uniformly contracted by forward iterations under f, we can easily see that the "height" of $f^{R(\omega)}(\mathcal{C}(\omega))$ is at most $\delta_s/4$. Hence, by the choice of δ_0 we have $f^{R(\omega)}(\mathcal{C}(\omega))$ made up of cu-unstable disks contained in $\mathcal{C}_0$. Moreover, as $f^{R(\omega)}(\omega)$ u-crosses $\mathcal{C}_0$, the same occurs with the local unstable leaves that form $\mathcal{C}(\omega)$, and so (P_1) holds. (P_2) is clearly satisfied under our assumptions.

7.2. *Backward contraction and bounded distortion*

The backward contraction on unstable leaves and bounded distortion, respectively properties (P_3) and (P_4), follow from Lemma 4.4. Indeed, by construction, for each $\omega \in \mathcal{P}$ there is a hyperbolic pre-ball $V_{n(\omega)}(x)$ containing ω associated to some point $x \in D$ with σ-hyperbolic time $n(\omega)$ satisfying $R(\omega) - N_0 \le n(\omega) \le R(\omega)$. It is sufficient to prove (P_3) and (P_4) at time $n = n(\omega)$ instead of $R(\omega)$ since the two differ by a finite and uniformly bounded number of iterations whose contribution to the estimates is also uniformly controlled.

An immediate consequence of (12) is that if $y \in K$ satisfies $\mathrm{dist}(f^j(x), f^j(y)) \le \delta_1$ for $0 \le j \le n-1$, then n is a $\sigma^{3/4}$-hyperbolic time for y, i.e.

$$\prod_{j=n-k+1}^{n} \|Df^{-1}|_{E^{cu}_{f^j(y)}}\| \le \sigma^{3k/4} \quad \text{for all } 1 \le k \le n.$$

Therefore, taking $\delta_s, \delta_0 < \delta_1/2$, for any $\gamma \in \Gamma^u$ we find that n is a $\sigma^{3/4}$-hyperbolic time for every point in $\mathcal{C}_\omega \cap \gamma$. The backward contraction on unstable leaves and bounded distortion are then consequences of Lemma 4.4 (recall Remark 4.5).

7.3. *Regularity of the foliations*

Property (P_5) is standard for uniformly hyperbolic attractors. In the rest of this section we shall adapt classical ideas to our setting.

We begin with the statement of a useful lemma on vector bundles whose proof can be found in [30, Theorem 6.1]. Let us recall that a metric d on E is *admissible* if there is a complementary bundle E' over X, and an isomorphism $h\colon E \oplus E' \to X \times B$ to a product bundle, where B is a Banach space, such that d is induced from the product metric on $X \times B$.

Lemma 7.1. *Let $p\colon E \to X$ be a vector bundle over a metric space X endowed with an admissible metric. Let $D \subset E$ be the unit ball bundle, and $F\colon D \to D$ a map covering a Lipschitz homeomorphism $f\colon X \to X$. Assume that there is $0 \le \kappa < 1$ such that for each $x \in X$ the restriction $F_x\colon D_x \to D_x$ satisfies $\mathrm{Lip}(F_x) \le \kappa$. Then*

(1) *there is a unique section $\sigma_0\colon X \to D$ whose image is invariant under F;*
(2) *if $\kappa\,\mathrm{Lip}(f)^\alpha < 1$ for some $0 < \alpha \le 1$, then σ_0 is Hölder continuous with exponent α.*

Proposition 7.2. *Let $f : M \to M$ be a C^1 diffeomorphism and $\Omega \subset M$ a compact invariant set with a dominated splitting $T_\Omega M = E^{cs} \oplus E^{cu}$. Then the fiber bundles E^{cs} and E^{cu} are Hölder continuous on Ω.*

Proof. We consider only the centre-unstable bundle as the other one is similar. For each $x \in \Omega$ let L_x be the space of bounded linear maps from E^{cu}_x to E^{cs}_x and let L^1_x denote the unit ball around $0 \in L_x$. We define $\Gamma_x : L^1_x \to L^1_{f(x)}$ as the graph transform induced by $Df(x)$:

$$\Gamma_x(\mu_x) = (Df|_{E^{cs}_x}) \cdot \mu_x \cdot (Df^{-1}|_{E^{cu}_{f(x)}}).$$

Consider the vector bundle L over Ω whose fiber over each $x \in \Omega$ is L_x, and let L^1 be its unit ball bundle. Then $\Gamma : L^1 \to L^1$ is a bundle map covering $f|_\Omega$ with

$$\mathrm{Lip}(\Gamma_x) \le \|Df|_{E^{cs}_x}\| \cdot \|Df^{-1}|_{E^{cu}_{f(x)}}\| \le \lambda < 1.$$

Let c be a Lipschitz constant for $f|_\Omega$, and choose $0 < \alpha \le 1$ so small that $\lambda c^\alpha < 1$. By Lemma 7.1 there exists a unique section $\sigma_0\colon M \to L^1$ whose image is invariant under Γ and which satisfies the Hölder condition with exponent α. This unique section is necessarily the null section. □

The next result gives precisely (P_5)(a).

Corollary 7.3. *There are $C > 0$ and $0 < \beta < 1$ such that for all $y \in \gamma^s(x)$ and $n \ge 0$,*

$$\log \prod_{i=n}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(y))} \le C\beta^n.$$

Proof. As we are assuming that Df is Hölder continuous, it follows from Proposition 7.2 that $\log|\det Df^u|$ is Hölder continuous. The conclusion is then an immediate consequence of the uniform contraction on stable leaves. □

To prove (P_5)(b) we introduce some useful notions. We say that $\phi : N \to P$, where N and P are submanifolds of M, is *absolutely continuous* if it is an injective map for which there exists $J : N \to \mathbb{R}$ such that

$$\mathrm{Leb}_P(\phi(A)) = \int_A J \, d\,\mathrm{Leb}_N.$$

J is called the *Jacobian* of ϕ. Property (P_5)(b) can be restated in the following terms:

Proposition 7.4. *Given $\gamma, \gamma' \in \Gamma^u$, define $\phi\colon \gamma' \to \gamma$ by $\phi(x) = \gamma^s(x) \cap \gamma$. Then ϕ is absolutely continuous and the Jacobian of ϕ is given by*

$$J(x) = \prod_{i=0}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(\phi(x)))}.$$

One can easily deduce from Corollary 7.3 that this infinite product converges uniformly. The remainder of this section is devoted to the proof of Proposition 7.4. We start with a general result about the convergence of Jacobians whose proof is given in [33, Theorem 3.3].

Lemma 7.5. *Let N and P be manifolds, P with finite volume, and for each $n \geq 1$, let $\phi_n : N \to P$ be an absolutely continuous map with Jacobian J_n. Assume that*

(1) *ϕ_n converges uniformly to an injective continuous map $\phi : N \to P$;*
(2) *J_n converges uniformly to an integrable function $J : N \to \mathbb{R}$.*

Then ϕ is absolutely continuous with Jacobian J.

For the sake of completeness, we observe that there is a slight difference in our definition of absolute continuity. In contrast to [33], and for reasons that will become clear below, we do not impose the continuity of the maps ϕ_n. However, the proof of [33, Theorem 3.3] uses only the continuity of the limit function ϕ, and so it still works in our case.

Consider now $\gamma, \gamma' \in \Gamma^u$ and $\phi\colon \gamma' \to \gamma$ as in Proposition 7.4. The proof of the next lemma is given in [33, Lemma 3.4] for uniformly hyperbolic diffeomorphisms. Nevertheless, one can easily see that it is obtained as a consequence of [33, Lemma 3.8] whose proof uses only the existence of a dominated splitting.

Lemma 7.6. *For each $n \geq 1$, there is an absolutely continuous $\pi_n : f^n(\gamma) \to f^n(\gamma')$ with Jacobian G_n satisfying*

(1) $\lim_{n\to\infty} \sup_{x\in\gamma} \mathrm{dist}_{f^n(\gamma')}\big(\pi_n(f^n(x)), f^n(\phi(x))\big) = 0$;
(2) $\lim_{n\to\infty} \sup_{x\in f^n(\gamma)} |1 - G_n(x)| = 0.$

We consider the sequence of consecutive return times for points in Λ,

$$r_1 = R \quad \text{and} \quad r_{n+1} = r_n + R \circ f^{r_n} \quad \text{for } n \geq 1.$$

Notice that these return time functions are defined Leb_γ-almost everywhere on each $\gamma \in \Gamma^u$ and are piecewise constant.

Remark 7.7. Using the sequence of return times one can easily construct a sequence $(\mathcal{Q}_n)_n$ of Leb_γ mod 0 partitions by s-subsets of Λ with r_n constant on each element of $\mathcal{Q}_n$, for which (P_1)–(P_5) hold when we take r_n playing the role of R and the elements of $\mathcal{Q}_n$ playing the role of the s-subsets. Moreover, the constants $C > 0$ and $0 < \beta < 1$ can be chosen not depending on n.

We define, for each $n \geq 1$, the map $\phi_n : \gamma \to \gamma'$ as

$$\phi_n = f^{-r_n} \pi_{r_n} f^{r_n}. \tag{28}$$

It is straightforward to check that ϕ_n is absolutely continuous with Jacobian

$$J_n(x) = \frac{|\det (Df^{r_n})^u(x)|}{|\det (Df^{r_n})^u(\phi_n(x))|} \cdot G_{r_n}(f^{r_n}(x)). \tag{29}$$

Observe that these functions are defined Leb_γ-almost everywhere. So, we may find a Borel set $A \subset \gamma$ with full Leb_γ-measure on which they are all defined. We extend ϕ_n to γ simply by considering $\phi_n(x) = \phi(x)$ and $J_n(x) = J(x)$ for all $n \geq 1$ and $x \in \gamma \setminus A$. Since A has zero Leb_γ-measure, J_n is still the Jacobian of ϕ_n.

Proposition 7.4 is now a consequence of Lemma 7.5 together with the next one.

Lemma 7.8. *$(\phi_n)_n$ converges uniformly to ϕ, and $(J_n)_n$ converges uniformly to J.*

Proof. It is sufficient to prove the convergence of each sequence restricted to A described above. In particular, the expressions of ϕ_n and J_n are given by (28) and (29) respectively.

Let us first prove the case of $(\phi_n)_n$. Using the backward contraction on unstable leaves given by (P_3) and recalling Remark 7.7, we may write, for each $x \in \gamma$,

$$\begin{aligned}\mathrm{dist}_{\gamma'}(\phi_n(x), \phi(x)) &= \mathrm{dist}_{\gamma'}\big(f^{-r_n}\pi_{r_n} f^{r_n}(x), f^{-r_n} f^{r_n}\phi(x)\big)\\ &\leq C\beta^{r_n}\, \mathrm{dist}_{f^{r_n}(\gamma')}\big(\pi_{r_n} f^{r_n}(x), f^{r_n}\phi(x)\big).\end{aligned}$$

Since $r_n \to \infty$ as $n \to \infty$ and $\mathrm{dist}_{f^{r_n}(\gamma')}(\pi_{r_n} f^{r_n}(x), f^{r_n}\phi(x))$ is bounded, by Lemma 7.6, we have the uniform convergence of ϕ_n to ϕ.

Let us now deal with the case of the Jacobians $(J_n)_n$. By (29), we have

$$J_n(x) = \frac{|\det (Df^{r_n})^u(x)|}{|\det (Df^{r_n})^u(\phi(x))|} \cdot \frac{|\det (Df^{r_n})^u(\phi(x))|}{|\det (Df^{r_n})^u(\phi_n(x))|} \cdot G_{r_n}(f^{r_n}(x)).$$

Using the chain rule and Corollary 7.3, it easily follows that the first term in the product above converges uniformly to $J(x)$. Moreover, by Lemma 7.6, the third term converges uniformly to 1. It remains to see that the middle term also converges uniformly to 1. Recalling Remark 7.7, by bounded distortion we have

$$\frac{|\det (Df^{r_n})^u(\phi(x))|}{|\det (Df^{r_n})^u(\phi_n(x))|} \le \exp\big(C\,\mathrm{dist}_{f^{r_n}(\gamma')}(f^{r_n}(\phi(x)),\, f^{r_n}(\phi_n(x)))^\eta\big)$$
$$= \exp\big(C\,\mathrm{dist}_{f^{r_n}(\gamma')}(f^{r_n}(\phi(x)), \pi_{r_n}(f^{r_n}(x)))^\eta\big).$$

Similarly we obtain

$$\frac{|\det (Df^{r_n})^u(\phi(x))|}{|\det (Df^{r_n})^u(\phi_n(x))|} \ge \exp\big(-C\,\mathrm{dist}_{f^{r_n}(\gamma')}(f^{r_n}(\phi(x)), \pi_{r_n}(f^{r_n}(x)))^\eta\big).$$

The conclusion then follows from Lemma 7.6. □

8. Integrability of the return time

In the previous sections we have constructed a GMY structure on Ω. To complete the proof of Theorem C it just remains to show that this GMY structure has integrable return times as in (6). Recall first that the existence of a GMY structure implies the existence of an induced map $F : \Lambda \to \Lambda$ with an invariant probability measure ν (see remarks following Theorem C). This measure can be disintegrated into a family of conditional measures on the unstable leaves $\{\gamma^u\}$ with conditional measures which are equivalent to Lebesgue measure with densities bounded by uniform constants above and below [49, Lemma 2]. We fix one such unstable leaf $\gamma \in \Gamma^u$ and let $\bar\nu$ denote the conditional measure associated to ν and equivalent to Lebesgue measure. The integrability of the return times with respect to Lebesgue measure as in (6) therefore follows immediately from the next result.

Proposition 8.1. *The inducing time function R is $\bar\nu$-integrable.*

Proof. We first introduce some notation. For $x \in \Delta$ we consider the orbit $x, f(x), \dots, f^{n-1}(x)$ of the point x under iteration by f for some large value of n. In particular x may undergo several full returns to Δ before time n. Then we define the following quantities:

$$H^{(n)}(x) := \text{number of hyperbolic times for } x \text{ before time } n,$$
$$S^{(n)}(x) := \text{number of times } x \text{ belongs to a satellite before time } n,$$
$$R^{(n)}(x) := \text{number of returns of } x \text{ before time } n.$$

Each time x has a hyperbolic time, it either has a return within some finite and uniformly bounded number of iterations, or by definition it belongs to a satellite. Therefore there exists some constant $\kappa > 0$ independent of x and n such that

$$R^{(n)}(x) + S^{(n)}(x) \ge \kappa H^{(n)}(x).$$

Notice that x may belong to a satellite or have a return without it having a hyperbolic time itself, since it may belong to a hyperbolic pre-disk of some other point y which has a hyperbolic time. Dividing the above inequality through by n we get

$$\frac{R^{(n)}(x)}{n} + \frac{S^{(n)}(x)}{n} \ge \frac{\kappa H^{(n)}(x)}{n}.$$

Recalling that the hyperbolic times have uniformly positive asymptotic frequency, we see that there exists a constant $\theta > 0$ such that $H^{(n)}(x)/n \geq \theta$ for all n sufficiently large, and therefore rearranging the left hand side above gives

$$\frac{R^{(n)}(x)}{n}\left(1+\frac{S^{(n)}(x)}{R^{(n)}(x)}\right) \geq \kappa\theta > 0.$$

Moreover $S^{(n)}(x)/R^{(n)}(x)$ converges by Birkhoff's Ergodic Theorem to precisely the average number of times, $\int S\,d\nu$, that typical points belong to satellites before they return, and from Proposition 6.1 it follows that $\int S\,d\nu < \infty$. Therefore, we have

$$\frac{R^{(n)}(x)}{n} \geq \kappa' > 0 \tag{30}$$

for all sufficiently large n where κ' can be chosen arbitrarily close to $\kappa\theta/(1+\int S\,d\bar{\nu})$, which is independent of x and n. To conclude the proof notice that $n/R^{(n)}(x)$ is the average return time over the first n iterations, and thus converges to $\int \bar{R}\;d\bar{\nu}$ by Birkhoff's Ergodic Theorem. This holds even if we do not assume a priori that $\bar{R}$ is integrable, since it is a positive function and thus $\int \bar{R}\,d\bar{\nu}$ is always well defined and lack of integrability necessarily implies $\int \bar{R}\,d\bar{\nu} = \infty$. Thus, assuming towards a contradiction that $\int \bar{R}\,d\bar{\nu} = \infty$ gives $n/R^{(n)}(x) \to \int \bar{R}\,d\bar{\nu} = \infty$, and therefore $R^{(n)}(x)/n \to 0$. This contradicts (30), and therefore we have $\int \bar{R}\,d\bar{\nu} < \infty$ as required. □

9. Liftability

In this section we complete the proof of Theorem D. The "if" part of this result is well known and we refer to it in the comments preceding Theorem D. We therefore just need to show that every SRB measure with positive Lyapunov exponents in the E^{cu} direction is liftable. To achieve this, first of all let Ω denote the support of the given SRB measure μ. Then Ω is invariant under f, and thus under any positive iterate of f. We will show in the following proposition that there exists some $N \geq 1$ such that f^N on Ω is nonuniformly expanding, and thus weakly nonuniformly expanding, along E^{cu}. We can then apply the conclusions of Theorem C to obtain a GMY structure for f^N with integrable return time function R. This easily gives a corresponding GMY structure for f with return time function NR, which is therefore still integrable and, as explained above, gives rise to an SRB measure. By uniqueness of SRB measures it follows that this measure coincides with μ, thus proving that μ is liftable.

Proposition 9.1. *There exists $N \geq 1$ such that f^N is nonuniformly expanding along E^{cu} on a set of positive Lebesgue measure.*

Proof. We prove first of all that there exists $N \geq 1$ such that

$$\int \log \|(Df^N|_{E_x^{cu}})^{-1}\|\,d\mu < 0. \tag{31}$$

Indeed, by assumption all Lyapunov exponents of f along E^{cu} are positive and therefore all Lyapunov exponents of f^{-1} along E^{cu} are negative. Thus, considering the cocycle

$(x, v) \mapsto (f^{-1}(x), Df^{-1}(x)v)$, Oseledets' Theorem implies that there exists λ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|Df^{-1}|_{E^{cu}_{f^{-n+1}(x)}} \cdots Df^{-1}|_{E^{cu}_x}\| = \lambda < 0 \tag{32}$$

where λ is the largest Lyapunov exponent of f^{-1} [12, Addendum 4]. By the chain rule and the inverse function theorem, we have

$$Df^{-1}|_{E^{cu}_{f^{-n+1}(x)}} \cdots Df^{-1}|_{E^{cu}_x} = (Df^n|_{E^{cu}_{f^{-n}(x)}})^{-1}. \tag{33}$$

Since the sequence

$$\phi_n = \log \|(Df^n|_{E^{cu}_{f^{-n}(x)}})^{-1}\|$$

satisfies $\phi_{n+m} \le \phi_n + \phi_m \circ f^{-n}$, using the invariance of μ with respect to f^{-1} and Kingmann's Subadditive Ergodic Theorem we have, for μ-almost every x,

$$\lim_{n\to\infty} \frac{1}{n} \log \|(Df^n|_{E^{cu}_{f^{-n}(x)}})^{-1}\| = \inf_{n\ge 1} \frac{1}{n} \int \log \|(Df^n|_{E^{cu}_{f^{-n}(x)}})^{-1}\| \, d\mu,$$

which, together with (32) and (33), gives (31).

Notice that μ may not be ergodic for f^N, but it can have at most N ergodic components. Indeed, notice first of all that any subset C which is f^N-invariant and has positive measure, satisfies $\mu(C) \ge 1/N$: Assume for contradiction that $\mu(C) < 1/N$ and consider the set $\bigcup_{j=0}^{N-1} f^{-j}(C)$. We have

$$0 < \mu\Big(\bigcup_{j=0}^{N-1} f^{-j}(C)\Big) \le \sum_{j=0}^{N-1} \mu(f^{-j}(C)) < 1.$$

This gives a contradiction, because the set is f-invariant and μ is ergodic. Now, if (f^N, μ) is not ergodic, then we decompose M into a union of two f^N-invariant disjoint sets of positive measure. If the restriction of μ to one of these sets is not ergodic, then we iterate this process. Note that this must stop after a finite number of steps with at most N disjoint subsets, since f^N-invariant sets of positive measure have their measure bounded from below by $1/N$.

Thus, (f^N, μ) has at most N ergodic components. By (31), at least one of these components, whose support we denote by Σ, satisfies $\int_\Sigma \log \|(Df^N|_{E^{cu}_x})^{-1}\| \, d\mu < 0$. Hence, by Birkhoff's Ergodic Theorem, for μ-almost every $x \in \Sigma$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|(Df^N|_{E^{cu}_{f^{Nj}(x)}})^{-1}\| = \int_\Sigma \log \|(Df^N|_{E^{cu}_x})^{-1}\| \, d\mu < 0.$$

This proves that f^N is nonuniformly expanding along E^{cu} for μ-almost every point in Σ. Since μ is an SRB measure, conditional measures of μ on local unstable manifolds are absolutely continuous with respect to Lebesgue measure. In particular there is some local unstable manifold γ^u on which we have nonuniform expansion for a set of points of positive Leb_{γ^u}-measure. Considering the union of the local stable manifolds through these points and the absolute continuity of the stable foliation, we get the result. □

Acknowledgments. JFA was partially supported by Fundação Calouste Gulbenkian, CMUP, the European Regional Development Fund through the Programme COMPETE, and FCT under the projects PTDC/MAT/099493/2008, PTDC/MAT/120346/2010 and PEst-C/MAT/UI0144/2011. CLD was supported by FCT. The research described in this paper was also supported by BREUDS.

References

[1] Alves, J. F.: SRB measures for non-hyperbolic systems with multidimensional expansion, Ann. Sci. École Norm. Sup. **228**, 1203–1236 (2000) Zbl 0955.37012 MR 1743717

[2] Alves, J. F., Araújo, V., Saussol, B.: On the uniform hyperbolicity of some nonuniformly hyperbolic systems. Proc. Amer. Math. Soc. **131**, 1303–1309 (2003) Zbl 1014.37018 MR 1948124

[3] Alves, J. F., Araújo, V., Vásquez, C.: Stochastic stability of non-uniformly hyperbolic diffeomorphisms. Stoch. Dynam. **7**, 299–333 (2007) Zbl 1210.37022 MR 2351040

[4] Alves, J. F., Bonatti, C., Viana, M.: SRB measures for partially hyperbolic systems whose central direction is mostly expanding. Invent. Math. **140**, 351–398 (2000) Zbl 0962.37012 MR 1749677

[5] Alves, J. F., Dias, C. L., Luzzatto, S.: Geometry of expanding absolutely continuous invariant measures and the liftability problem. Ann. Inst. H. Poincaré Anal. Non Linéaire **30**, 101–120 (2013) Zbl 06154084 MR 3011293

[6] Alves, J. F., Luzzatto, S., Pinheiro, V.: Markov structures and decay of correlations for non-uniformly expanding dynamical systems. Ann. Inst. H. Poincaré Anal. Non Linéaire **22**, 817–839 (2005) Zbl 1134.37326 MR 2172861

[7] Alves, J. F., Pinheiro, V.: Topological structure of (partially) hyperbolic sets with positive volume. Trans. Amer. Math. Soc. **360**, 5551–5569 (2008) Zbl 1149.37018 MR 2415085

[8] Alves, J. F., Pinheiro, V.: Gibbs–Markov structures and limit laws for partially hyperbolic attractors with mostly expanding central direction. Adv. Math. **223**, 1706–1730 (2010) Zbl 1185.37011 MR 2592507

[9] Araújo, V., Luzzatto, S., Viana, M.: Invariant measures for interval maps with critical points and singularities. Adv. Math. **221**, 1428–1444 (2009) Zbl 1184.37032 MR 2522425

[10] Avila, A., Lyubich, M., de Melo, W.: Regular or stochastic dynamics in real analytic families of unimodal maps. Invent. Math. **154**, 451–550 (2003) Zbl 1050.37018 MR 2018784

[11] Barreira, L., Pesin, Ya.: Nonuniform Hyperbolicity. Encyclopedia Math. Appl. 115, Cambridge Univ. Press (2007) Zbl 1144.37002 MR 2348606

[12] Bochi, J.: The multiplicative ergodic theorem of Oseledets. http://www.mat.puc-rio.br/~jairo/docs/oseledets.pdf

[13] Bonatti, C., Viana, M.: SRB measures for partially hyperbolic systems whose central direction is mostly contracting. Israel J. Math. **115**, 157–193 (2000) Zbl 0996.37033 MR 1749677

[14] Bowen, R.: Equilibrium States and Ergodic Theory of Anosov Diffeomorphisms. Lecture Notes in Math. 470, Springer (1975) Zbl 1172.37001 MR 2423393

[15] Bowen, R., Ruelle, D.: The ergodic theory of Axiom A flows. Invent. Math. **29**, 181–202 (1975) Zbl 0311.58010 MR 0380889

[16] Brin, M. I., Pesin, Ya. B.: Partially hyperbolic dynamical systems. Uspekhi Mat. Nauk **28**, no. 3, 169–170 (1973) (in Russian) Zbl 0285.58010 MR 0391178

[17] Bruin, H., Luzzatto, S., van Strien, S.: Decay of correlations in one-dimensional dynamics. Ann. Sci. École Norm. Sup. **36**, 621–646 (2003) Zbl 1039.37021 MR 2013929

[18] Burns, K., Dolgopyat, D., Pesin, Ya.: Partial hyperbolicity, Lyapunov exponents and stable ergodicity. J. Statist. Phys. **108**, 927–942 (2002) Zbl 1124.37308 MR 1933439

[19] Burns, K., Dolgopyat, D., Pesin, Ya., Pollicott, M.: Stable ergodicity for partially hyperbolic attractors with negative central exponents. J. Modern Dynam. **2**, 63–81 (2008) Zbl 1145.37021 MR 2366230

[20] Castro, A.: Backward inducing and exponential decay of correlations for partially hyperbolic attractors. Israel J. Math. **130**, 29–75 (2002) Zbl 1186.37041 MR 1919371

[21] Castro, A.: Fast mixing for attractors with a mostly contracting central direction. Ergodic Theory Dynam. Systems **24**, 17–44 (2004) Zbl 1115.37023 MR 2041259

[22] Climenhaga, V., Dolgopyat, D., Pesin, Ya.: Non-stationary non-uniform hyperbolicity: SRB measures for dissipative maps. Preprint http://front.math.ucdavis.edu/1405.6194

[23] Cowieson, W., Young, L.-S.: SRB measures as zero-noise limits. Ergodic Theory Dynam. Systems **25**, 1115–1138 (2005) Zbl 1098.37020 MR 2158399

[24] Díaz-Ordaz, K., Holland, M., Luzzatto, S.: Statistical properties of one-dimensional maps with critical points and singularities. Stoch. Dynam. **6**, 423–458 (2006) Zbl 1130.37362 MR 2285510

[25] Dolgopyat, D.: On dynamics of mostly contracting diffeomorphisms. Comm. Math. Phys. **213**, 181–201 (2000) Zbl 0964.37020 MR 1782146

[26] Dolgopyat, D.: On differentiability of SRB states for partially hyperbolic systems. Invent. Math. **155**, 389–449 (2004) Zbl 1059.37021 MR 2031432

[27] Field, M., Nicol, M.: Ergodic theory of equivariant diffeomorphisms: Markov partitions and stable ergodicity. Mem. Amer. Math. Soc. **169**, no. 803 (2004) Zbl 1073.37030 MR 2045641

[28] Gouëzel, S.: Berry–Esseen theorem and local limit theorem for non-uniformly expanding maps. Ann. Inst. H. Poincaré Probab. Statist. **41**, 997–1024 (2005) Zbl 1134.60323 MR 2172207

[29] Hatomoto, J.: Diffeomorphisms admitting SRB measures and their regularity. Kodai Math. J. **29**, 211–226 (2006) Zbl 108.37019 MR 2247431

[30] Hirsch, M. W., Pugh, C. C.: Stable manifolds and hyperbolic sets. In: Global Analysis, Proc. Sympos. Pure Math. 14 (Berkeley, CA, 1968), Amer. Math. Soc., Providence, RI, 133–163 (1970) Zbl 0215.53001 MR 0271991

[31] Kozlovski, O., Shen, W., van Strien, S.: Density of hyperbolicity in dimension one. Ann. of Math. (2) **166**, 145–182 (2007) Zbl 1138.37013 MR 2342693

[32] Lyubich, M.: Almost every real quadratic map is either regular or stochastic. Ann. of Math. (2) **156**, 1–78 (2002) Zbl 1160.37356 MR 1935840

[33] Mañé, R.: Ergodic Theory and Differentiable Dynamics. Springer, Berlin (1987) Zbl 0616.28007 MR 0889254

[34] Melbourne, I., Nicol, M.: Almost sure invariance principle for non-uniformly hyperbolic systems. Comm. Math. Phys. **260**, 131–1456 (2005) Zbl 1084.37024 MR 2175992

[35] Melbourne, I., Nicol, M.: Large deviations for non-uniformly hyperbolic systems. Trans. Amer. Math. Soc. **360**, 6661–6676 (2008) Zbl 1151.37031 MR 2434305

[36] Palis, J.: A global perspective for non-conservative dynamics. Ann. Inst. H. Poincaré Anal. Non Linéaire **22**, 485–507 (2005) Zbl 1143.37016 MR 2145722

[37] Pesin, Ya. B.: Characteristic Lyapunov exponents, and smooth ergodic theory. Uspekhi Mat. Nauk **32**, no. 4, 55–112 (1977) (in Russian) Zbl 0383.58011 MR 0466791

[38] Pesin, Ya. B.: Lectures on Partial Hyperbolicity and Stable Ergodicity. Zurich Lectures Adv. Math., Eur. Math. Soc., Zürich (2004) Zbl 1098.37024 MR 2068774

[39] Pesin, Ya., Sinai, Ya.: Gibbs measures for partially hyperbolic attractors. Ergodic Theory Dynam. Systems **2**, 417–438 (1982) Zbl 0519.58035 MR 0721733

[40] Pinheiro, V.: Sinai–Ruelle–Bowen measures for weakly expanding maps. Nonlinearity **19**, 1185–1200 (2006) Zbl 1100.37014 MR 2222364

[41] Pinheiro, V.: Expanding measures. Ann. Inst. H. Poincaré Anal. Non Linéaire **28**, 889–939 (2011) Zbl 1254.37026 MR 2859932

[42] Rey-Bellet, L., Young, L.-S.: Large deviations in non-uniformly hyperbolic dynamical systems. Ergodic Theory Dynam. Systems **28**, 587–612 (2008) Zbl 1154.37331 MR 3366637

[43] Ruelle, D.: A measure associated with Axiom A attractors. Amer. J. Math. **98**, 619–654 (1976) Zbl 0355.58010 MR 0415683

[44] Sinai, Ya. G.: Gibbs measure in ergodic theory. Russian Math. Surveys **27**, 21–69 (1972) Zbl 0255.28016 MR 0399421

[45] Tsujii, M.: Physical measures for partially hyperbolic surface endomorphisms. Acta Math. **194**, 37–132 (2005) Zbl 1105.37022 MR 2231338

[46] Vásquez, C. H.: Statistical stability for diffeomorphisms with dominated splitting. Ergodic Theory Dynam. Systems **27**, 253–283 (2007) Zbl 147.37019

[47] Viana, M.: Dynamics: a probabilistic and geometric perspective. In: Proc. Int. Congress of Mathematicians (Berlin, 1998), Vol. I, Doc. Math. **1998**, Extra Vol. I, 557–578 Zbl 0911.58013 MR 1648047

[48] Viana, M., Yang, J.: Physical measures and absolute continuity for one-dimensional center direction. Ann. Inst. H. Poincaré Anal. Non Linéaire **30**, 845–877 (2013) Zbl 06295444 MR 3103173

[49] Young, L.-S.: Statistical properties of dynamical systems with some hyperbolicity. Ann. of Math. **147**, 585–650 (1998) Zbl 0945.37009 MR 1637655

[50] Young, L.-S.: What are SRB measures, and which dynamical systems have them? J. Statist. Phys. **108**, 733–754 (2002) Zbl 1124.37307 MR 1933431