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Topological groups, μ -types and their stabilizers

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Abstract. We consider an arbitrary topological group G definable in a structure \mathcal{M} , such that some basis for the topology of G consists of sets definable in \mathcal{M} . To each such group G we associate a compact G -space of partial types, $S_G^\mu(M) = \{p_\mu : p \in S_G(M)\}$, which is the quotient of the usual type space $S_G(M)$ by the relation of two types being “infinitesimally close to each other”. In the o-minimal setting, if p is a definable type then it has a corresponding definable subgroup $\text{Stab}^\mu(p)$, which is the stabilizer of p_μ . This group is nontrivial when p is unbounded; in fact, it is a torsion-free solvable group.

Along the way, we analyze the general construction of $S_G^\mu(M)$ and its connection to the Samuel compactification of topological groups.

Keywords. o-minimality, definable groups, compactification

1. Introduction

It was shown in [PS] that in a group G definable in an o-minimal structure, one can associate to any definable unbounded curve $\gamma \subseteq G$ a definable one-dimensional torsion-free group H_γ . In fact, the group H_γ can be viewed as associated to the (definable) type p of γ at “ $+\infty$ ”. Our initial goal in the current article was to extend that result to arbitrary definable types in G and associate to any such p a definable group H_p , which is nontrivial if and only if p is unbounded (here, a type p is called *unbounded* if no formula in p defines a definably compact set with respect to the G -topology).

While working on the above we discovered interesting connections to general topological groups, G -spaces and their universal compactifications. Namely, consider an arbitrary topological group G , definable in some structure \mathcal{M} , with a basis for its topology consisting of sets definable in \mathcal{M} . Under these assumptions we view the partial type μ of all definable open subsets of G containing the identity as an “infinitesimal subgroup” and use it to define an equivalence relation on complete types in $S_G(M)$: $p \sim_\mu q$ if $\mu \cdot p = \mu \cdot q$ as partial types. It turns out that this equivalence relation is well behaved and

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the quotient space $S_G^\mu(M)$ is a compact G -space. Moreover, this construction recovers the Samuel compactification [S] in the case of an arbitrary topological group, when all subsets of G are definable in \mathcal{M} . In the case when G is a discrete group our analysis is already subsumed by the work of Newelski [N] and others (see for example [GPP]).

Returning to our original problem, the group H_p described above is just the stabilizer of the associated partial type p_μ under the action of G on $S_G^\mu(M)$. For the result below, we say that a complete type $p \in S_G(M)$ is μ -reduced, if its \sim_μ -class does not contain any complete type of lower dimension. We first prove (Claim 3.4) that any definable type q which is unbounded has a \sim_μ -equivalent definable type p of positive dimension which is μ -reduced. Summarizing our main results we have (Theorems 3.10 and 3.26):

Theorem 1.1. *Let G be a definable group in an o-minimal expansion of a real closed field. Then to any definable μ -reduced type $p \in S_G(M)$ there is associated a definable, torsion-free group $H_p = \text{Stab}^\mu(p) \supseteq \text{Stab}(p)$ with $\dim H_p = \dim p$. In particular, $\dim H_p > 0$ if and only if p is unbounded.*

Regarding the general setting of a topological group G which is definable in a structure \mathcal{M} and has a basis of definable sets, we prove in Appendix A the following (Theorem A.15):

Theorem 1.2. *The quotient $S_G^\mu(M)$ is a compact Hausdorff space on which the group $G(M)$ acts continuously. The map $g \mapsto \text{tp}(g/M)_\mu$ embeds $G(M)$ as a dense subset of $S_G^\mu(M)$.*

Thus, $S_G^\mu(M)$ is a G -ambit, and we show that it is the greatest G -ambit within the so-called definably separable G -ambits.

While our main interest is in the group G itself, we found it useful to treat the more general case of G acting definably on a definable set X , and this is the setting throughout the first part of the paper.

A uniformity vs. a type-definable equivalence relation. As already pointed out, our construction of $S_G^\mu(M)$ recovers to a great extent the work of Samuel [S] on compactifications of uniform spaces. Samuel works in a set endowed with a *uniformity*, a collection of subsets of X^2 which satisfies certain conditions and gives rise to a topology on X . As we explain in Appendix A, the notion of a uniformity is the same as the model-theoretic notion of a type-definable equivalence relation on X . Thus, we carry out some of the work in this more general setting.

Outline of the paper. We begin in Section 2 by a general discussion of topological groups and μ -types. In Section 3 we move to the o-minimal setting and analyze the group $\text{Stab}^\mu(p)$ in detail. In Appendix A we carry out the above mentioned analysis of the general case and describe the space $S_G^\mu(M)$ in detail. In Appendix B we prove a technical result which is needed for the o-minimal case.

General conventions. We fix a complete first order theory T and a large saturated enough model \mathbb{U} of T . When D is a definable set and $\varphi(x)$ a formula, we say that $\varphi(x)$ is a *D -formula* if $\mathbb{U} \models \varphi(x) \rightarrow x \in D$. We extend this definition to types (possibly incomplete) by saying that a type $q(x)$ is a *D -type* if $q(x) \vdash x \in D$. We let $\mathcal{L}_D(M)$ be

the collection of all D -formulas with parameters in M and let $S_D(M)$ denote the set of all complete D -types over M .

We use the predicate D to denote both a definable set and a formula defining it. Thus, we write for example $g \in D$ or $p \vdash D$, where D is thought of as a definable set in the first case and a formula in the latter. When D is a definable set over a model \mathcal{M} and $\mathcal{N} \succ \mathcal{M}$ then we will write $D(M)$ or $D(N)$ to denote the specific realization of D in the structures \mathcal{M} or \mathcal{N} . Very often when we work in a fixed structure \mathcal{M} we just write D instead of $D(M)$.

2. Topological groups and μ -types

2.1. On definable groups and group actions

Let G be a group. Recall that a G -set is a set X together with an action of G on it. If X is a G -set then for subsets $P \subseteq G$ and $Y \subseteq X$, we will denote by $P \cdot Y$ the set $\{x \cdot y : x \in P, y \in Y\} \subseteq X$, and if P (or Y) is a singleton $\{a\}$ then we just write $a \cdot Y$ (resp. $P \cdot a$).

When G is a topological group, X is a topological space and the action is continuous, (i.e. the map $(g, x) \mapsto g \cdot x$ is a continuous function from $G \times X$ to X), then X is called a G -space. All topological groups and compact spaces are assumed to be Hausdorff.

Throughout Section 2 we fix a small arbitrary $\mathcal{M} \prec \mathbb{U}$ and an M -definable group G . We also fix an M -definable set X with G acting definably on X , namely, the map $(g, x) \mapsto g \cdot x$ is definable in \mathcal{M} . We call X a definable G -set.

Notation 2.1. • If $\varphi(v)$ is a G -formula over M and $\psi(x)$ is an X -formula then by $\varphi \cdot \psi$ we will denote the X -formula

$$(\varphi \cdot \psi)(x) = \exists v \exists u (\varphi(v) \ \& \ \psi(u) \ \& \ x = v \cdot u).$$

• If $p(v)$ is a G -type and $r(x)$ is an X -type (possibly incomplete) then by $p \cdot r$ we will denote the X -type

$$(p \cdot r)(x) = \{(\varphi \cdot \psi)(x) : p(v) \vdash \varphi(v), r(x) \vdash \psi(x)\}.$$

Remark 2.2. (1) For the action of G on itself by left multiplication, the above definition gives us a notion of product of types, but for $p, q \in S_G(M)$ the type $p \cdot q$ is usually incomplete.

(2) If we identify an element $g \in G(M)$ with the complete type $\text{tp}(g/M)$, the above definition agrees with the usual definition of the action of $G(M)$ on the G -types and the M -definable G -sets.

(3) It is easy to see that if $\varphi(v)$ is a G -formula over M and $\psi(x)$ is an X -formula over M then

$$\varphi(M) \cdot \psi(M) = (\varphi \cdot \psi)(M).$$

(4) For a G -type $p(v)$ over M and an X -type $r(x)$ over M we have

$$p(\mathbb{U}) \cdot r(\mathbb{U}) = (p \cdot r)(\mathbb{U}).$$

But if $\mathcal{N} \succ \mathcal{M}$ is not $|M|^+$ -saturated then in general we only have the inclusion

$$p(N) \cdot r(N) \subseteq (p \cdot r)(N).$$

(5) It follows from (4) that if $p(v), q(v)$ are G -types over M and $r(x)$ is an X -type over M then

$$((p \cdot q) \cdot r)(x) = (p \cdot (q \cdot r))(x).$$

2.2. Topological groups and their infinitesimal types

For the rest of Section 2 we assume in addition that G is a topological group and furthermore that G has a basis for its topology consisting of sets definable in \mathcal{M} . Note that this is a rather weak assumption and for example does not imply that $G(\mathbb{U})$ is still a topological group. When we develop the theory further, we make a stronger assumption, that a basis for the topology of G is *uniformly definable* in \mathcal{M} (which is what Pillay calls in [P2] “a first order topological group”). This will be sufficient to ensure that $G(\mathbb{U})$ is a topological group.

Definition 2.3. The *infinitesimal type* of G is the partial G -type over M , denoted by $\mu_G(v)$ (or just by $\mu(v)$ if G is fixed), consisting of all formulas over M defining an open neighborhood of e .

Notice that the type $\mu(v)$ is not complete unless the topology on G is discrete. The next claim follows from the continuity of the group operations.

- Claim 2.4.** (1) $\mu(v) = \mu^{-1}(v)$.
 (2) $g \cdot \mu = \mu \cdot g$ for every $g \in G(M)$.
 (3) $\mu \cdot \mu = \mu$.

Corollary 2.5. For any elementary extension \mathcal{N} of \mathcal{M} the set $\mu(N)$ is a subgroup of $G(N)$ and every element of $G(M)$ normalizes $\mu(N)$.

Claim 2.6. For a partial X -type $\Sigma(x)$ over M and $p \in S_X(M)$, the following are equivalent:

- (1) $(\mu \cdot p) \cup \Sigma$ is consistent.
 (2) $p \vdash \mu \cdot \Sigma$.

Proof. We work in \mathbb{U} .

(1) \Rightarrow (2): Assume that $\mu \cdot p \cup \Sigma$ is consistent and fix in $G(\mathbb{U})$ and $X(\mathbb{U})$ elements $\epsilon \models \mu$ and $b \models p$, respectively, such that $\epsilon \cdot b \models \Sigma$. Let $\beta = \epsilon \cdot b$. Since $b = \epsilon^{-1}\beta$ and $\mu^{-1} = \mu$, we have $b \models \mu \cdot \Sigma$. Since p is a complete type, this implies $p \vdash \mu \cdot \Sigma$.

(2) \Rightarrow (1): Assume that $p \vdash \mu \cdot \Sigma$ and choose $\epsilon \models \mu$ and $b \models p$ such that $\epsilon^{-1} \cdot b \models \Sigma$. Since $\mu^{-1} = \mu$, we have $\epsilon^{-1} \cdot b \models \mu \cdot p$, so the result follows. \square

Claim 2.7. For $p, q \in S_X(M)$, the following conditions are equivalent:

- (1) The type $(\mu \cdot p)(x) \cup (\mu \cdot q)(x)$ is consistent.
- (2) $p(x) \vdash (\mu \cdot q)(x)$.
- (3) $\mu \cdot p = \mu \cdot q$ (here and below we consider two partial types over M to be equal if they are logically equivalent).

Proof. We apply Claim 2.6, by taking $\Sigma = \mu \cdot q$, and using $\mu \cdot \mu = \mu$. □

Notation 2.8. • For $p, q \in S_X(M)$ write $p \sim_\mu q$ if $\mu \cdot p = \mu \cdot q$ as partial types.

- We let $S_X^\mu(M)$ be the quotient of $S_X(M)$ by the equivalence relation \sim_μ , and use p_μ to denote the \sim_μ -equivalence class of a type p . Namely, $\mu \cdot p$ is a partial type and p_μ is the associated equivalence class.

Claim 2.9. For any $g \in G(M)$ and $p \in S_X(M)$ we have

$$g \cdot (\mu \cdot p) = \mu \cdot (g \cdot p).$$

Proof. Follows from Remark 2.2(5) and Claim 2.4(2). □

Thus the action of $G(M)$ on $S_X(M)$ preserves \sim_μ , so it induces an action of $G(M)$ on $S_G^\mu(M)$ by $g \cdot p_\mu = (g \cdot p)_\mu$. We will consider $S_X^\mu(M)$ as a G -set. In Appendix A we discuss other properties of $S_X^\mu(M)$.

2.3. μ -stabilizers

We still assume that X is a definable G -set (and X is not assumed to carry any topology).

2.3.1. Stabilizers of partial types. As already pointed out, the group $G(M)$ acts on $\mathcal{L}_X(M)$.

Definition 2.10. Let $\Sigma(x)$ be a partial X -type over M . We first define

$$\text{Stab}(\Sigma) = \{g \in G(M) : \text{for all } \varphi \in \mathcal{L}_X(M), \Sigma \vdash \varphi \Leftrightarrow \Sigma \vdash g \cdot \varphi\}.$$

For $\varphi(x) \in \mathcal{L}_X(M)$, consider the set

$$\{h \in G(M) : \Sigma \vdash h \cdot \varphi(x)\} \tag{2.1}$$

and define $\text{Stab}_\varphi(\Sigma) \subseteq G(M)$ to be the stabilizer of the above set (so in particular a subgroup).

The following is easy to verify:

Claim 2.11. For every partial X -type Σ over M ,

$$\text{Stab}(\Sigma) = \bigcap_{\Sigma \vdash \varphi} \text{Stab}_\varphi(\Sigma) = \bigcap_{\varphi \in \mathcal{L}_X(M)} \text{Stab}_\varphi(\Sigma).$$

Definition 2.12. We say that a partial type $\Sigma(x) \subseteq \mathcal{L}(M)$ is *definable* over $A \subseteq M$ if for every formula $\phi(x, y)$ there exists a formula $\chi(y) \in \mathcal{L}(A)$ such that for every $a \in M$, $\Sigma \vdash \phi(x, a)$ if and only if $M \models \chi(a)$.

For $\Sigma(x)$ a partial type definable over M , and $\mathcal{N} \succ \mathcal{M}$, we denote by $\Sigma|N$ the extension of Σ by definitions to a partial type over N . Namely, for $a \in N$ and $\phi \in \mathcal{L}(M)$, $\phi(x, a) \in \Sigma|N$ iff $\mathcal{N} \models \chi(a)$, for $\chi(y)$ as above.

Here is our main use of definability of types:

Proposition 2.13. *Assume that Σ is a definable partial X -type over M . Then $\text{Stab}(\Sigma)$ can be written as an intersection of M -definable subgroups. If in addition G has the Descending Chain Condition on M -definable subgroups then $\text{Stab}(\Sigma)$ is a definable subgroup of G .*

Proof. The fact that each of the sets in (2.1) is definable is immediate from the definability of Σ . It follows that each $\text{Stab}_\varphi(\Sigma)$ is definable, and therefore $\text{Stab}(\Sigma)$ is an intersection of definable groups. If in addition G has DCC, then the intersection is finite, hence definable. \square

Strengthening the assumptions. *From now on we assume that G has a uniformly definable basis*

$$\{B_t : t \in T\}$$

of open neighborhoods of the identity. We call such a G a definably topological group. As pointed out earlier, for $\mathcal{N} \succ \mathcal{M}$ the group $G(N)$ is again a topological group and the definable family $\{B_t : t \in T(N)\}$ forms a basis for the open neighborhoods of e .

We may identify the type μ with the collection of formulas $\{B_t : t \in T\}$. Note that μ itself is a definable partial type, over the parameters defining T . Indeed, for $a \in M$, $\mu \vdash \phi(x, a)$ if and only if $\mathcal{M} \models \exists t \in T \forall x (x \in B_t \rightarrow \phi(x, a))$. If $\mathcal{N} \succ \mathcal{M}$ then $\mu|N$ is just the infinitesimal type of $G(N)$ in the structure \mathcal{N} .

2.3.2. Stabilizers of μ -types

Notation 2.14. For $p \in S_X(M)$, we define the *infinitesimal stabilizer* $\text{Stab}^\mu(p)$ as

$$\text{Stab}^\mu(p) = \text{Stab}(\mu \cdot p).$$

Note that $\text{Stab}^\mu(p)$ contains the usual stabilizer of p , denoted by $\text{Stab}(p)$. The claim below and its proof was proposed to us by A. Pillay.

Claim 2.15. *If p is a complete type in $S_X(M)$ definable over $A \subseteq M$ then $\mu \cdot p$ is a partial type definable over A .*

Proof. Let $\phi(x, y)$ be a formula. For $b \in M$ we have $(\mu \cdot p)(x) \vdash \phi(x, b)$ if and only if $(\mu \cdot p)(x) \cup \{\neg\phi(x, b)\}$ is inconsistent, which by Claim 2.6 is equivalent to $p(x) \not\vdash (\mu \cdot \neg\phi(x, b))(x)$. Since p is a complete type, the latter condition is equivalent to the existence of $t \in T(M)$ such that $p(x) \vdash \neg(B_t \cdot \neg\phi(x, b))(x)$.

Since the type $p(x)$ is definable over A , there is a formula $\chi(u, y) \in \mathcal{L}(A)$ such that $p(x) \vdash \neg(B_t \cdot \neg\phi(x, b))(x)$ if and only if $\mathcal{M} \models \chi(t, b)$.

Thus $(\mu \cdot p)(x) \vdash \phi(x, b)$ if and only if $\mathcal{M} \models \exists t \in T (\chi(t, b))$. \square

Remark 2.16. The definition of $\mu \cdot p$ is canonical in the following sense: If $p \in S_X(M)$ is definable over M and $\mathcal{N} \succ \mathcal{M}$ then $(\mu|N) \cdot (p|N) = (\mu \cdot p)|N$.

Proposition 2.17. Assume that G has the Descending Chain Condition for definable subgroups and that $p \in S_X(M)$ is a definable type over M . Then $\text{Stab}^\mu(p)$ is an M -definable subgroup of G . Moreover, if $\mathcal{N} \succ \mathcal{M}$ then $\text{Stab}^\mu(p|N)$ is defined by the same formula.

Proof. We take $\Sigma = \mu \cdot p$, which is definable by the last claim. By Claim 2.13, $\text{Stab}^\mu(p)$ is definable in \mathcal{M} . □

The following claim follows from generalities about group actions.

Claim 2.18. For $p, q \in S_X(M)$ and $g \in G(M)$, if $g \cdot p_\mu = q_\mu$ then

$$\text{Stab}^\mu(q) = g \text{Stab}^\mu(p)g^{-1}.$$

Remark 2.19. In the case $X = G$ we always consider the action of G on itself by left multiplication. In particular, both stabilizer groups $\text{Stab}(p)$ and $\text{Stab}^\mu(p)$ are taken with respect to left multiplication. Note that they could turn out to be different groups for the opposite action. The following example is taken from [P3].

Let $G = \text{SL}(2, \mathbb{R})$. Consider the curve

$$\gamma(t) = \begin{pmatrix} t & 1 \\ 0 & t^{-1} \end{pmatrix}, \quad t > 0.$$

We will denote by $S \subseteq G$ the image of γ , and let $p(x)$ be the type on S corresponding to $t > \mathbb{R}$.

It is not hard to see that the μ -stabilizer of p with respect to left multiplication is

$$\left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R} \right\}$$

and with respect to right multiplication is

$$\left\{ \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} : r \in \mathbb{R}^{>0} \right\}.$$

2.4. A partial “standard part” map

In the o-minimal case, when \mathcal{N} is a tame extension of \mathcal{M} , there is a standard part map $\text{st}: \mathcal{O}_M(N) \rightarrow M$ from the set of M -bounded elements of N into M (see Section 3.2 for more details). Here we introduce an analogue of a standard part map without assuming o-minimality or tameness.

Let \mathcal{N} be an elementary extension of \mathcal{M} . By Corollary 2.5, $\mu(N)$ is a subgroup of $G(N)$ normalized by $G(M)$, hence $\mu(N) \cdot G(M)$ is a subgroup of $G(N)$, still normalized by $G(M)$. Abusing notation, we denote this subgroup by $\mathcal{O}_G(N)$.

Because $\mu(N) \cap G(M) = \{e\}$, there exists a surjective group homomorphism $\text{st}^*: \mathcal{O}_G(N) \rightarrow G(M)$ which sends every h to the unique $g \in G(M)$ such that

$h \in \mu(N)g$. We think of the map st^* as a partial “standard part” map from $G(N)$ into $G(M)$, so for $Y \subseteq G(N)$ we will use $\text{st}^*(Y)$, again abusing notation, to denote the image under st^* of the set $Y \cap \mathcal{O}_G(N)$.

Note that if $H \leq G(N)$ is any subgroup then $\text{st}^*(H)$ is a subgroup of $G(M)$. As we shall see later, under various assumptions this group is definable in M .

Example 2.20. If $G = \langle \mathbb{R}, + \rangle$ and \mathcal{R} is a nonstandard elementary extension of the real field then $\mathcal{O}_G(\mathcal{R})$ equals the subgroup of elements of G of finite size. If $G = \langle \mathbb{R}^{>0}, \cdot \rangle$ then $\mathcal{O}_G(\mathcal{R})$ is the collection of all positive elements $a \in \mathcal{R}$ such that both a and $1/a$ are finite.

Claim 2.21. Let $p, q \in S_X(M)$, let $\mathcal{N} \succ \mathcal{M}$ be $|M|^+$ -saturated and let $\alpha \models p$ and $\beta \models q$ be in N . For $a \in X(N)$, let $G_a = \{h \in G(N) : h \cdot a = a\}$.

- (1) If $p \sim_\mu q$ then $\text{st}^*(G_\alpha) = \text{st}^*(G_\beta)$.
- (2) The group $\text{st}^*(G_\alpha)$ is normal in $\text{Stab}^\mu(p)$.

Proof. (1) Note that if $p = q$ then we can immediately conclude that $\text{st}^*(G_\alpha) = \text{st}^*(G_\beta)$. Our result shows that it is sufficient to assume that p and q are μ -equivalent types.

Because $p \sim_\mu q$ we may replace β by another realization of q if needed and assume $\beta = \epsilon \cdot \alpha$ for some $\epsilon \in \mu(N)$. It follows that $G_\beta = \epsilon G_\alpha \epsilon^{-1}$. Also, it is not hard to see that $G_\beta \cap \mathcal{O}_G(N) = \epsilon(G_\alpha \cap \mathcal{O}_G(N))\epsilon^{-1}$. Because st^* is a homomorphism and $\epsilon \in \ker(\text{st}^*)$, we see that $\text{st}^*(G_\beta) = \text{st}^*(G_\alpha)$.

(2) We first note that the group $\text{st}^*(G_\alpha)$ is contained in $\text{Stab}^\mu(p)$. Indeed, assume that $g = \text{st}^*(h)$ for some $h \in G_\alpha$, so $g = \epsilon h$ for $\epsilon \in \mu(N)$. Then

$$g \cdot \alpha = \epsilon h \cdot \alpha = \epsilon \cdot \alpha \models \mu \cdot p,$$

so $g \in \text{Stab}^\mu(p)$.

To see that $\text{st}^*(G_\alpha)$ is normal in $\text{Stab}^\mu(p)$, we take $g \in \text{Stab}^\mu(p)$ (in particular, $g \in G(M)$), and then

$$g(\text{st}^*(G_\alpha))g^{-1} = \text{st}^*(gG_\alpha g^{-1}) = \text{st}^*(G_{g \cdot \alpha}) = \text{st}^*(G_\alpha),$$

where the last equality follows by (1), since $\text{tp}(g \cdot \alpha/M) \sim_\mu p$. Thus we showed that $\text{st}^*(G_\alpha)$ is normal in $\text{Stab}^\mu(p)$. □

Finally, we have:

Claim 2.22. For $p \in S_G(M)$ and $|M|^+$ -saturated extension $\mathcal{N} \succ \mathcal{M}$ we have:

- (1) $\text{Stab}^\mu(p) = \text{st}^*(p(N)p(N)^{-1})$.
- (2) $\text{Stab}^\mu(p) = \text{st}^*(p(N)\alpha^{-1})$ for every $\alpha \models p$ in N .

Proof. (1) Let us see that $\text{st}^*(p(N)p(N)^{-1}) \subseteq \text{Stab}^\mu(p)$. Indeed, assume that $g = \epsilon \cdot a \cdot b^{-1} \in G(M)$ for $\epsilon \in \mu(N)$ and $a, b \in p(N)$. Then clearly $g \cdot p$ is consistent with $\mu \cdot p$, so, since $g \cdot p$ is a complete type, it follows that $g \cdot p \vdash \mu \cdot p$. By Claim 2.7, $g\mu \cdot p = \mu \cdot p$, so $g \in \text{Stab}^\mu(p)$.

For the opposite inclusion, assume that $g \in \text{Stab}^\mu(p)$ and take an arbitrary $b \models p$. Since $g \cdot p \vdash \mu \cdot p$, there exist $\varepsilon \in \mu(N)$ and $a \in p(N)$ such that $\varepsilon \cdot a = g \cdot b$ (here we have used the saturation of N). It follows that $g = \varepsilon ab^{-1}$, so $g \in \text{st}^*(p(N)p(N)^{-1})$.

(2) Use the fact that if $g = \text{st}^*(\beta'\alpha'^{-1})$ is in M and $\beta', \alpha' \models p$, then by saturation of \mathcal{N} using an automorphism over M we can replace α' with α and β' with some other $\beta \models p$. □

This ends our discussion under the assumption that G is a general definably topological group.

3. The case of an o-minimal G

We assume in this section that G is a definable group in an o-minimal structure \mathcal{M} expanding a real closed field R .

By Pillay’s work [P1] we know that for any k , G has the structure of a definable C^k -manifold with respect to R , making G into a C^k -group. In particular it is a definably topological group in \mathcal{M} . Because G has DCC [P1], it follows from Propositions 2.17 and 2.13 that for a definable type $p \in S_G(M)$, the groups $\text{Stab}^\mu(p)$ and $\text{Stab}(p)$ are definable. Our goal in this section is to realize $\text{Stab}^\mu(p)$ as the image under the standard part map of a definable set in a bigger model. This will allow us to prove for example that $\text{Stab}^\mu(p)$ is a torsion-free group which is nontrivial unless p is bounded. It will also help us determine the dimension of $\text{Stab}^\mu(p)$.

Most of the work in this section concerns the action of G on itself by left multiplication, but towards the end we also consider general definable G -sets in the o-minimal setting.

3.1. Embedding G as an affine group

The following claim is known, but since we could not find a precise reference we provide an outline of a proof.

Claim 3.1. *The group G , with its C^k -manifold structure, is definably C^k -diffeomorphic to a closed C^k -submanifold of M^n for some n .*

Proof. This is done in two steps. First, we apply a result of Fischer [F, 1.3] to find a C^k -diffeomorphism between G and a C^k -submanifold of M^n for some n . Since Fischer states his result only for structures over the reals, we clarify several points: The main tool in his article, Corollary 1.2, which states that every closed set is the zero set of a definable C^k -function, can be replaced by [vdDM, 4.22], whose proof goes through word-for-word for a general real closed field. The rest of the argument (we only need Step 1 in that proof) can remain as is. We now identify G with its image in M^n .

Next we use the argument from [Cos, 6.18]: as above we find a definable C^k -function $h: M^n \rightarrow M$ whose zero locus is the closed set $\text{Cl}(G) \setminus G$ and identify G with the closed set $\{(x, t) \in M^{n+1} : x \in G, t \cdot h(x) = 1\}$. □

We assume from now on that G is a closed C^1 -submanifold of M^n , with the group operations C^1 -maps.

Remark 3.2. Note that now p is an unbounded G -type if and only if $p(\mathbb{U})$ is not contained in any bounded subset of \mathbb{U}^n .

3.2. On the standard part map

Recall that an elementary extension $\mathcal{N} \succ \mathcal{M}$ is called *tame* if for every $n \in N^k$ the type $\text{tp}(n/M)$ is definable.

Let p be a type over M and α be a realization of p . By definability of Skolem functions [vdD1] the definable closure of $M \cup \{\alpha\}$ is an elementary extension of \mathcal{M} that we will denote by $\mathcal{M}(\alpha)$. Clearly, when p is a definable type, the extension $\mathcal{M}(\alpha) \succeq \mathcal{M}$ is tame.

Let $\mathcal{N} \succ \mathcal{M}$ be a tame extension and $\mathcal{O} = \mathcal{O}_M(N)$ the set of M -bounded elements of N , i.e.

$$\mathcal{O} = \{n \in N : -m < n < m \text{ for some } m \in M\}.$$

We denote by $\nu = \nu_M(N)$ the set of M -infinitesimal elements of N , i.e.

$$\nu = \{n \in N : -m < n < m \text{ for any } m > 0 \in M\}.$$

Note that ν^n is the intersection of all M -definable open neighborhoods of $0 \in M^n$.

Because \mathcal{N} is a tame extension of \mathcal{M} , every element in \mathcal{O} is infinitesimally close to a (unique) element in M [MS, Theorem 2.1], hence we have the standard part map $\text{st}: \mathcal{O} \rightarrow M$ defined as: $\text{st}(n)$ is the unique $m \in M$ such that $n \in m + \nu$. We extend this definition to $\text{st}: \mathcal{O}^n \rightarrow M^n$. Note that in this case the map st is the same as our previous st^* with respect to the group $G = \langle M, + \rangle$ or its cartesian powers.

The group G is embedded as a closed C^k -submanifold of M^n . As before, the infinitesimal subgroup μ of G is given by a definable basis of G -open neighborhoods of $e \in G$. As the group topology agrees with the restriction of the M^n -topology, we have $\mu(N) = (e + \nu^n) \cap G(N)$. Thus a partial type which defines μ can be taken to be $\{B_\epsilon : \epsilon > 0, \epsilon \in M\}$, where we write B_ϵ for the intersection of the ϵ -ball in M^n around e with the set G .

Using continuity of the group operations it is not difficult to show that for any $g \in G(M)$ we have

$$g \cdot \mu(N) = (g + \nu^n) \cap G(N).$$

Since G is a closed subset, it follows that for any $a \in \mathcal{O}(N)^n \cap G(N)$ we have $\text{st}(a) \in G(M)$, so that $\mathcal{O}_G(N) = \mathcal{O}(N)^n \cap G(N)$. In particular, the standard part maps with respect to G and $\langle M^n, + \rangle$ coincide for elements of $G(N)$.

3.3. Reduced types

Definition 3.3. We say that a type $p \in S_G(M)$ is μ -reduced if for every $q \in S_G(M)$ with $q_\mu = p_\mu$ we have $\dim p \leq \dim q$.

By finiteness of dimension, for every $p \in S_G(M)$ we can find a μ -reduced $q \in S_G(M)$ with $p_\mu = q_\mu$. Note however that one \sim_μ -class can contain more than one μ -reduced type. For example in $\langle M^2, + \rangle$ the type of $(\alpha, 0)$ where $\alpha > M$ and the type of $(\alpha, 1/\alpha)$ are both μ -reduced, one-dimensional and \sim_μ -equivalent.

Our first goal is to show that for *definable* $p \in S_G(M)$ we can find a *definable* μ -reduced q with $p_\mu = q_\mu$.

Claim 3.4. *Let $p \in S_G(M)$ be a definable type. If p is not μ -reduced then there is a definable type $q \in S_G(M)$ with $q_\mu = p_\mu$ and $\dim q < \dim p$.*

Proof. Assume p is a definable type that is not μ -reduced. Then we can find a type $s(x) \in S_G(M)$ such that $s_\mu = p_\mu$ and $\dim s < \dim p$. We can choose an M -definable set $V \subseteq G$ such that $\dim V = \dim s$ and $s \vdash V$.

We fix some positive $r \in M$.

Since $p_\mu = s_\mu$ we have

$$p(x) \vdash \exists y \exists z (x = y \cdot z \ \& \ y \in B_r \ \& \ z \in V). \tag{3.1}$$

We choose a realization α of p and let $\mathcal{N} = \mathcal{M}\langle\alpha\rangle$. Since the type p is definable, \mathcal{N} is a tame extension of \mathcal{M} .

Using (3.1) we find that there are $\beta \in V(N)$ and $g^* \in B_r(N)$ such that $\alpha = g^* \cdot \beta$.

Since $g^* \in B_r(N)$, we have $g^* \in \mathcal{O}^n$. Let $g = \text{st}(g^*) \in M$. Then $g^* = g \cdot \varepsilon$ for some $\varepsilon \in \mu(N)$. We have $\alpha = g \cdot (\varepsilon \cdot \beta)$.

Let $q'(x) = \text{tp}(\beta/M)$. It is a definable type with $\dim q' \leq \dim V < \dim p$.

By Claim 2.9 we also have $p_\mu = g \cdot q'_\mu$. Now the type $q(x) = g^{-1} \cdot q'$ is definable of the same dimension as q' , and $q_\mu = p_\mu$. □

Corollary 3.5. *For any definable type $p \in S_G(M)$ there is a μ -reduced definable type $q \in S_G(M)$ with $p_\mu = q_\mu$.*

Finally, we easily obtain:

Claim 3.6. *If $p \in S_G(M)$ is a μ -reduced type then for every $g \in G(M)$, the type $g \cdot p$ is μ -reduced.*

We end with the following observation:

Fact 3.7. *For a definable type $p \in S_G(M)$, $p(\mathbb{U})$ is bounded if and only if p is μ -equivalent to an algebraic type $\text{tp}(g/M)$ for some $g \in G(M)$.*

Proof. If $p \sim_\mu \text{tp}(g/M)$ then any realization of p is infinitesimally close to g . But then any M -definable open set containing g must be in p , so p has formulas defining bounded sets. For the converse, if p contains a formula over M defining a bounded set D then $D(N) \subseteq \mathcal{O}_G(N)$, where $\mathcal{N} = \mathcal{M}\langle\alpha\rangle$ for some $\alpha \models p$. It follows that the standard part map is defined on $D(N)$, and in particular there exists $g \in G(M)$ infinitesimally close to α . □

3.4. Redefining $\text{Stab}^\mu(p)$ using the standard part map

Our main goal in this section is to show that for a definable μ -reduced type p we can find an M -definable set S in p such that for any realization $\alpha \models p$ we have $\text{Stab}^\mu(p) = \text{st}(S\alpha^{-1})$. We first clarify the notation.

Since p is a definable type, the structure $\mathcal{N} = \mathcal{M}\langle\alpha\rangle$ is tame and we work in \mathcal{N} . As before, \mathcal{O} is the convex hull of M in \mathcal{N} and we have the standard part map $\text{st}: \mathcal{O}^n \rightarrow M^n$. By [vdD2, Corollary 1.3], for every set D definable in \mathcal{N} the image $\text{st}(D \cap \mathcal{O}^n)$ is an M -definable set. We are going to omit \mathcal{O}^n and just write $\text{st}(D)$ in this case. We still let B_r denote the intersection of the ball of radius r centered at e in M^n with G .

The first inclusion that we want is not difficult.

Claim 3.8. *Let $p \in S_G(M)$ be a definable type and S an M -definable set in p . Then for every realization $\alpha \models p$ we have*

$$\text{Stab}^\mu(p) \subseteq \text{st}(S\alpha^{-1}),$$

where st is taken in the structure $\mathcal{N} = \mathcal{M}\langle\alpha\rangle$.

Proof. Let α be a realization of p .

Assume $g \in \text{Stab}^\mu(p)$. Then $g \cdot p \vdash \mu \cdot p$.

As in Claim 3.4, for every positive $r \in M$ we have

$$g \cdot p(x) \vdash \exists y \exists z (x = y \cdot z \ \& \ y \in B_r \ \& \ z \in S).$$

Thus for every positive $r \in M$ we have

$$\mathcal{N} \models \exists y \exists z (g\alpha = y \cdot z \ \& \ y \in B_r \ \& \ z \in S).$$

In the structure \mathcal{N} we can now take the infimum of all $r > 0$ which satisfy the above. This infimum belongs to $v(N)$, hence we can find $\varepsilon \in v(N)$ such that

$$\mathcal{N} \models \exists y \exists z (g\alpha = y \cdot z \ \& \ y \in B_\varepsilon \ \& \ z \in S).$$

Hence there is $\beta \in S(N)$ (for z) and $g^* \in \mu(N)$ (for y) such that $g \cdot \alpha = g^* \cdot \beta$. Therefore $g = \text{st}(\beta \cdot \alpha^{-1})$. \square

Recall [vdD2, Proposition 1.10] that for every definable set V in an elementary tame extension \mathcal{N} of \mathcal{M} , we have $\dim \text{st}(V) \leq \dim V$. Since $\text{Stab}^\mu(p) \subseteq \text{st}(S\alpha^{-1})$ and we can choose S with $\dim p = \dim S$, we obtain:

Corollary 3.9. *For every definable type $p \in S_G(M)$, $\dim \text{Stab}^\mu(p) \leq \dim p$.*

We can now state our main theorem.

Theorem 3.10. *Let $p \in S_G(M)$ be a definable μ -reduced type. Then:*

- (1) $\dim \text{Stab}^\mu(p) = \dim p$.
- (2) *There exists an M -definable set S with $p \vdash S$ such that $\dim S = \dim p$, and for every $\alpha \models p$ and $\mathcal{N} = \mathcal{M}\langle\alpha\rangle$, we have:*

- (i) $\text{Stab}^\mu(p) = \text{st}(S\alpha^{-1})$.
- (ii) *The tangent space to $\text{Stab}^\mu(p)$ at e equals the standard part of the tangent space to $S\alpha^{-1}$ at e , i.e. $T(\text{Stab}^\mu(p))_e = \text{st}(T(S\alpha^{-1})_e)$.*

In particular, if p is not a bounded type in M^n then $\dim \text{Stab}^\mu(p) > 0$.

Remark 3.11. In the case when $\dim p = 1$, clauses (1) and (2)(i) of the above follow from [PS], and (2)(ii) is contained in [P3].

The proof of the above theorem will go through several steps and lemmas, and we divide it into several subsections. The main point of the proof is to find an appropriate set S .

3.4.1. The existence of S and the proof of clause (2)(i) in Theorem 3.10. During the rest of the proof we fix a definable μ -reduced type $p \in S_G(M)$, a realization $\alpha \models p$ and $\mathcal{N} = \mathcal{M}(\alpha)$. We work in the structure \mathcal{N} .

Notice that since p is μ -reduced, for every $g \in G(M)$, the type $g \cdot p$ is also μ -reduced. To simplify notation, from now on we write \mathcal{O} for $\mathcal{O}_G(N)$.

Claim 3.12. *For every M -definable set $Y \subseteq G(N)$, if $\dim Y < \dim p$ then $\mathcal{O} \cdot \alpha \cap Y = \emptyset$.*

Proof. Assume for contradiction that $\mathcal{O} \cdot \alpha \cap Y \neq \emptyset$ and let $\beta \in G(N)$ be a point in $Y \cap \mathcal{O} \cdot \alpha$. If we let $g = \text{st}(\beta\alpha^{-1})$ then $\text{tp}(\beta/M) \sim_\mu g \cdot p$ while $\dim \text{tp}(\beta/M) \leq \dim Y < \dim g \cdot p$. This contradicts the above observation that $g \cdot p$ is μ -reduced. \square

Claim 3.13. *There exists an M -definable set S in p such that every element of $S \cap (\mathcal{O} \cdot \alpha)$ realizes p .*

Proof. For every definable set $S \in p$, the set $S\alpha^{-1} \cap \mathcal{O}$ is a relatively definable subset of $\mathcal{O} \subseteq \mathcal{O}_M(N)^n$, hence, by Theorem B.2 in Appendix B (see also Example B.1), it has finitely many connected components. Namely, it can be written as a finite union of pairwise disjoint, relatively definable subsets of \mathcal{O} , each of which is clopen in $S\alpha^{-1} \cap \mathcal{O}$ and such that any other relatively definable clopen subset of $S\alpha^{-1} \cap \mathcal{O}$ contains one of those.

We choose an M -definable S in p such that $\dim S = \dim p$, S is a cell, and the number of connected components of $S\alpha^{-1} \cap \mathcal{O}$ is minimal. We claim that S has the desired property.

Indeed, assume not, namely there exists $\beta \in S \cap (\mathcal{O} \cdot \alpha)$ such that $\beta \models q \in S_G(M)$ and $q \neq p$. By Claim 3.12, we must have $\dim q = \dim p = \dim S$. Since $p \neq q$, there exists an M -definable set Y in q but not in p . We may assume that $Y \subseteq S$, and furthermore that Y is relatively open in S (because $\dim q = \dim S$), so $Y\alpha^{-1}$ is relatively open in $S\alpha^{-1}$.

We claim that $Y\alpha^{-1} \cap \mathcal{O}$ cannot be relatively closed in $S\alpha^{-1} \cap \mathcal{O}$. Indeed, if it were, then $Y\alpha^{-1} \cap \mathcal{O}$ would be clopen in $S\alpha^{-1} \cap \mathcal{O}$, and therefore would contain the whole connected component of β in $S\alpha^{-1} \cap \mathcal{O}$. This would imply that the number of components of $(S \setminus Y)\alpha^{-1} \cap \mathcal{O}$ is smaller than that of $S\alpha^{-1} \cap \mathcal{O}$, and furthermore $\alpha \in S \setminus Y$. Since $S \setminus Y$ is defined over M , we obtain a contradiction to the minimality of components in $S\alpha^{-1} \cap \mathcal{O}$.

Thus, $Y \cap \mathcal{O} \cdot \alpha$ is not closed in $S \cap \mathcal{O} \cdot \alpha$, so the set $\text{Fr}(Y) \cap \mathcal{O} \cdot \alpha$ is nonempty. However, $\text{Fr}(Y)$ is M -definable and its dimension is smaller than that of Y , so also smaller than $\dim p$. This contradicts Claim 3.12, so ends the proof of Claim 3.13. \square

In the rest of the proof we will show that any set S as in Claim 3.13 satisfies clause (2) of Theorem 3.10.

Claim 3.14. *Let S be as in Claim 3.13 and assume $\alpha \models p$. Then $\text{Stab}^\mu(p) = \text{st}(S\alpha^{-1})$. Moreover, for every M -definable set $S_1 \subseteq S$, if $p \vdash S_1$ then $\text{Stab}^\mu(p) = \text{st}(S_1\alpha^{-1})$.*

Proof. By Claim 3.8, we have $\text{Stab}^\mu(p) \subseteq \text{st}(S_1\alpha^{-1})$, so if we prove that $\text{st}(S\alpha^{-1}) \subseteq \text{Stab}^\mu(p)$ then we get equality.

Assume $g \in \text{st}(S\alpha^{-1})$. Then there exist $\epsilon \in \mu(N)$ and $\beta \in S$ such that $\epsilon g = \beta\alpha^{-1}$, so $\epsilon g\alpha = \beta$. By the choice of S we have $\beta \models p$, which implies that the types $g \cdot p(x)$ and $\mu \cdot p(x)$ are mutually consistent, so $g \in \text{Stab}^\mu(p)$. Thus, $\text{st}(S\alpha^{-1}) \subseteq \text{Stab}^\mu(p)$.

For the ‘‘moreover’’ part, it is easy to see that any such S_1 also satisfies Claim 3.13. \square

This ends the proof of clause (2)(i) in Theorem 3.10.

3.4.2. The dimension of $\text{Stab}^\mu(p)$ and the proof of clause (1) in Theorem 3.10. We start by proving, via a sequence of claims, a general proposition. Below we write $\dim_{\mathcal{M}}$ and $\dim_{\mathcal{N}}$ to emphasize that we compute the dimension of \mathcal{M} -definable and \mathcal{N} -definable sets, respectively. Recall that we are using $v(N)$ to denote the infinitesimals of \mathcal{M} in \mathcal{N} .

Proposition 3.15. *Let $U \subseteq N^k$ be an M -definable open set containing 0, and $Y \subseteq N^k$ be an N -definable, relatively closed C^1 -submanifold of U with $0 \in Y$. Assume that for every $h_1, h_2 \in v(N)^k \cap Y$ we have $\text{st}(T(Y)_{h_1}) = \text{st}(T(Y)_{h_2})$ (here $T(Y)_h$ is the tangent space of Y at h). Then $\dim_{\mathcal{M}} \text{st}(Y) = \dim_{\mathcal{N}} Y$.*

Proof. Assume that $\dim Y = \ell$. By [vdD2], $\dim_{\mathcal{M}} \text{st}(Y) \leq \dim_{\mathcal{N}} Y$ so it is sufficient to prove that $\dim_{\mathcal{M}} \text{st}(Y) \geq \ell$.

Let $H_0 = \text{st}(T(Y)_0)$. Then H_0 is a linear subspace of M^k of dimension ℓ . After a linear change of variables defined over M we may assume that $H_0 = M^\ell$, identified with the first ℓ coordinates of M^k . Let $\ell' = k - \ell$ and write $N^k = N^\ell \times N^{\ell'}$. We will denote by $\pi : N^k \rightarrow N^\ell$ the projection onto the first ℓ coordinates.

The following is a special case of the implicit function theorem.

Fact 3.16 (Implicit Function Theorem). *Let $Z \subseteq N^k$ be a definable C^1 -submanifold of dimension ℓ , $a \in Z$, $b = \pi(a)$, and $L = T(Z)_a$. Assume $\pi(L) = N^\ell$. Then locally, near a , Z is the graph of a function $F : N^\ell \rightarrow N^{\ell'}$, and L is the graph of the differential of F at b .*

Claim 3.17. *If $h \in Y \cap v(N)^k$ then, near h , Y is the graph of a function $F : N^\ell \rightarrow N^{\ell'}$. Moreover $\|dF_b\| \in v(N)$, where $b = \pi(h)$.*

Proof. This follows from our assumption on $T(Y)_h$ and Fact 3.16. \square

The following is easy to prove.

Claim 3.18. *Let $D \subseteq v(N)^\ell$ be an open ball centered at 0 and $F: D \rightarrow N^{\ell'}$ a definable smooth function. Assume that $F(0) = 0$ and $\|dF_b\| \in v(N)$ for all $b \in D$. Then $F(D) \subseteq v(N)^{\ell'}$, and therefore the graph of F is contained in $v(N)^k$.*

Combining Claims 3.17 and 3.18 with the fact that $Y \subseteq U$ is a closed submanifold, we obtain:

Claim 3.19. *Let $a > 0$ be in $v(N)$ and D_a be the open ball of radius a in N^ℓ centered at 0. Let Y_0^a be the connected component of $Y \cap (D_a \times N^{\ell'})$ containing 0. Then $Y_0^a \subseteq v(N)^k$ and it is the graph of a definable function $F: D_a \rightarrow N^{\ell'}$.*

By o-minimality we conclude:

Claim 3.20. *There is $a > v(N)$ in N such that Y contains the graph of a definable function $F: B_a \rightarrow N^{\ell'}$, where B_a is the open ball of radius a in N^k centered at 0. In particular, $\pi(Y)$ contains an open ball $B_b \subseteq N^\ell$ centered at 0 of radius b with $b \in M$.*

We can now complete the proof of Proposition 3.15. Because $\pi(Y)$ contains an M -definable ball around 0, the set $\text{st}(\pi(Y))$ contains an M -definable neighborhood of 0 in M^ℓ , so in particular its dimension is at least ℓ . But $\text{st}(\pi(Y)) = \pi(\text{st}(Y))$, hence $\dim_{\mathcal{M}} \text{st}(Y) \geq \ell$. □

We now return to the proof of clause (1) in Theorem 3.10. As before, we assume that $G \subseteq M^n$ is an embedded k -dimensional closed C^1 -submanifold.

Claim 3.21. *Let $Y \subseteq G$ be an M -definable set with $p \vdash Y$ and $\dim Y = \dim p = \ell$. Then for every positive $m \in M$, $B_m \cdot \alpha \cap Y$ is a closed ℓ -dimensional submanifold of $B_m \cdot \alpha$.*

Proof. By o-minimality, the set Z of all points in Y at which Y is **not** an ℓ -dimensional submanifold of G is an M -definable subset of smaller dimension. By Claim 3.12, we have $(B_m \cdot \alpha) \cap Z = \emptyset$. Similarly, the intersection of $\text{Fr}(Y)$ with $B_m \cdot \alpha$ is empty, so $B_m \cdot \alpha \cap Y$ is a closed submanifold. □

By working in an (M -definable) chart of G near e we identify G locally at e with an open neighborhood U of 0 in M^k . Since U is an open subset of M^k , for each $g \in U$ the tangent space $T(U)_g$ can be identified with M^k . Thus working in \mathcal{N} , for $g \in G(N) \cap U(N)$ we identify the tangent space $T(G)_g$ with N^k .

For $g \in G$, let $r_g: G \rightarrow G$ be right multiplication by g . The differential $d(r_g)_e$ of r_g at e is a linear isomorphism from $T(G)_e$ to $T(G)_g$, and for $g \in G(N) \cap U(N)$ we will view $d(r_g)_e$ as a linear isomorphism from N^k to N^k , i.e. an element of $\text{GL}_k(N)$. Thus we have a continuous, M -definable map $g \mapsto d(r_g)_e$ from $U(N)$ to $\text{GL}_k(N)$.

Let S be an M -definable set as in Claim 3.13. Then every $\beta \in \mathcal{O} \cdot \alpha \cap S$ realizes p and, by Claim 3.14, $\text{Stab}^\mu(p) = \text{st}(S\alpha^{-1})$. Replacing U with $B_1 \cap U$ if needed, and using Claim 3.21, we may assume that $S\alpha^{-1} \cap U(N)$ is a closed submanifold of $U(N)$.

Claim 3.22. *For every $h \in \mu(N) \cap S\alpha^{-1}$,*

$$\text{st}(T(S\alpha^{-1})_h) = \text{st}(T(S\alpha^{-1})_e).$$

Proof. If $h \in \mu(N) \cap S\alpha^{-1}$ then $h = \beta\alpha^{-1}$ for some $\beta \models p$. But then, since $\alpha \equiv_M \beta$, we have $\text{st}(T(S\beta^{-1})_e) = \text{st}(T(S\alpha^{-1})_e)$. The map r_h sends $S\beta^{-1}$ to $S\alpha^{-1}$ with e going to h , so its differential at e sends $T(S\beta^{-1})_e$ to $T(S\alpha^{-1})_h$. Since $h \in \mu(N)$ and the map $h \mapsto d(r_h)_e$ is continuous and M -definable, viewing $d(r_h)_e$ as an element of $\text{GL}_k(N)$ we can write it as $d(r_h)_e = I + \varepsilon$, where ε is a $k \times k$ matrix whose entries are in $\nu(N)$. Applying $d(r_h)_e$ to an orthonormal basis of $T(S\beta^{-1})_e$ (with respect to the standard dot product in N^k) we conclude that for each such basis vector v , $\text{st}(d(r_h)_e(v)) = \text{st}(v)$. It follows that $\text{st}(T(S\beta^{-1})_e) = \text{st}(T(S\alpha^{-1})_h)$, so we get the desired result. \square

Using Claim 3.22 and Proposition 3.15, we conclude that $\dim \text{Stab}^\mu(p) = \dim S = \dim p$, thus ending the proof of clause (1) in Theorem 3.10.

3.4.3. *Proof of clause (2)(ii) in Theorem 3.10.* As in the previous section, we identify G near e with an open M -definable subset $U \subseteq M^k$ containing 0. Again, shrinking U if needed we assume that U is contained in B_1 .

Note that in general it is not true that $T(\text{st}(Y))_a = \text{st}(T(Y)_{\text{st}(a)})$, even for a smooth definable manifold Y . However, we use the next fact, which easily follows from Mařková's result [M, Theorem 2.23]:

Fact 3.23. *Let $Y \subseteq \mathcal{O}(N)^k$ be an N -definable submanifold of dimension ℓ . Assume also that $\dim_{\mathcal{M}} \text{st}(Y) = \dim_N Y$. Then there exists $y \in Y$ such that $\text{st}(T(Y)_y) = T(\text{st}(Y))_{\text{st}(y)}$.*

Let S be an M -definable set as in Claim 3.13 and $Y = S\alpha^{-1} \cap U(N)$. We need to show that $T(\text{Stab}^\mu(p))_e = \text{st}(T(Y)_e)$.

We apply the above fact to Y , and fix $h \in Y$ with $\text{st}(T(Y)_h) = T(\text{st}(Y)_{\text{st}(h)})$.

As in the proof of Claim 3.22, $h = \beta\alpha^{-1}$ with $\beta \models p$, and

$$d(r_h)_e(T(S\beta^{-1})_e) = T(Y)_h.$$

Let $g = \text{st}(h)$. By continuity of the map $h \mapsto d(r_h)_e$ from $U(N)$ to $\text{GL}_k(N)$, we have

$$d(r_g)_e(\text{st}(T(S\beta^{-1})_e)) = \text{st}(T(Y)_h).$$

However, $\text{st}(T(S\beta^{-1})_e) = \text{st}(T(Y)_e)$ since $\beta \equiv_M \alpha$. We conclude that

$$d(r_g)_e(\text{st}(T(Y)_e)) = \text{st}(T(Y)_h).$$

By our assumption on h , we have $\text{st}(T(Y)_h) = T(\text{st}(Y))_g$, hence

$$d(r_g)_e(\text{st}(T(Y)_e)) = T(\text{st}(Y))_g.$$

By Claim 3.14 we have $\text{Stab}^\mu(p) \cap U = \text{st}(Y) \cap U$, hence $T(\text{Stab}^\mu(p))_g = T(\text{st}(Y))_g$, and

$$d(r_g)_e(\text{st}(T(Y)_e)) = T(\text{Stab}^\mu(p))_g.$$

Since $g \in \text{Stab}^\mu(p)$ and $\text{Stab}^\mu(p)$ is a subgroup of G , the map r_g is a diffeomorphism of $\text{Stab}^\mu(p)$ to itself, hence $T(\text{Stab}^\mu(p))_g = d(r_g)_e(T(\text{Stab}^\mu(p))_e)$, and $d(r_g)_e(\text{st}(T(Y)_e)) = d(r_g)_e(T(\text{Stab}^\mu(p))_e)$.

The linear map $d(r_g)_e$ is invertible, hence $\text{st}(T(Y)_e) = T(\text{Stab}^\mu(p))_e$, which is what we wanted to prove.

This ends the proof of Theorem 3.10.

3.5. The structure of $\text{Stab}^\mu(p)$

Below, \mathcal{M} is an o-minimal expansion of a real closed field and G a definable group.

Claim 3.24. *If $p \in S_G(M)$ is a definable type and H is an M -definable subgroup of G with $p \vdash H$ then $\text{Stab}^\mu(p) \subseteq H$.*

Proof. Let S, α and \mathcal{N} be as in Theorem 3.10. Replacing S by $S \cap H$ if needed, we may assume $S \subseteq H$. Then $S \cdot \alpha^{-1}$ is also contained in $H(N)$. Since H is closed in G , we get $\text{st}(S \cdot \alpha^{-1}) \subseteq H$. □

We will need the following fact about definable groups.

Fact 3.25. *Let G be a definable, definably connected group in \mathcal{M} . Then there exists an M -definable solvable, torsion-free subgroup $H_1 \subseteq G$ and a definably compact set $C \subseteq G$ such that $G = C \cdot H_1$. In particular, G/H_1 is a definably compact space.*

Proof. We first prove the existence of a torsion-free H_1 such that that G/H_1 is definably compact. This basically follows from the work of A. Conversano, but we give the details.

Use induction on $\dim G$. If G is not semisimple then it has an infinite M -definable normal abelian subgroup N . By induction the group G/N has an M -definable solvable torsion-free subgroup, which we may assume is of the form H/N , such that $(G/N)/(H/N)$ is a definably compact space. But then the group H is clearly solvable as well, and the quotient G/H is isomorphic to $(G/N)/(H/N)$, so definably compact. By [CP, Proposition 2.2], H has a maximal normal torsion-free definable subgroup $H_1 \trianglelefteq H$ with H/H_1 definably compact. It follows that the space G/H_1 is definably compact as well.

Assume now that G is semisimple. Then by [C, Theorem 1.2], G can be written as a product $K \cdot H_1$ of two subgroups with K definably compact and H_1 torsion-free (so necessarily solvable). This clearly implies that G/H_1 is definably compact.

Let us now prove the existence of a definably compact $C \subseteq G$ such that $G = C \cdot H_1$. We first note that G can be written as an increasing union $\bigcup_{r>0} B_r$ of open sets such that for each r , $\text{Cl}(B_r)$ is definably compact (here we use the fact that G is embedded in M^n). We also have $G/H_1 = \bigcup_r \pi(B_r)$, and since π is an open map, each $\pi(B_r)$ is open. Since G/H_1 is definably compact, there is r_0 such that $G/H_1 = \pi(B_{r_0})$, but then $G = B_{r_0} \cdot H_1$, so in particular $G = \text{Cl}(B_{r_0}) \cdot H_1$. □

Theorem 3.26. (1) *Let p be a definable type in $S_G(M)$. Then the group $H = \text{Stab}^\mu(p)$ is solvable, torsion-free, and $\dim H \leq \dim p$. In particular, $\text{Stab}(p)$ is torsion-free as well.*

(2) *In the opposite direction, if H is a torsion-free group definable over M then there exists a complete definable H -type p over M such that $\text{Stab}^\mu(p) = \text{Stab}(p) = H$.*

(3) *Any two maximal torsion-free definable subgroups of G are conjugate.*

Proof. (1) Let $H_1 \subseteq G$ be any M -definable torsion-free solvable group as in Fact 3.25 and let $C \subseteq G$ be a definably compact set, defined over M , such that $C \cdot H_1 = G$.

We take $\alpha \models p$ and work inside $\mathcal{N} = \mathcal{M}\langle\alpha\rangle$. There is some $g \in C(N)$ such that $g^{-1}\alpha = \beta \in H_1$. Because C is an M -definable, definably compact set, there exists some $g_0 \in C(M)$ such that $g_0g^{-1} \in \mu(N)$. It follows that $g_0^{-1}\alpha \in \mu(N) \cdot \beta$, and since $g_0 \in M$, we have $g_0^{-1}\alpha \models g_0^{-1} \cdot p$. If we let $q = \text{tp}(\beta/M)$ then this implies $(g_0^{-1} \cdot p)_\mu = q_\mu$.

Because $\beta \in H_1$, it follows from Claim 3.24 that $\text{Stab}^\mu(q)$ is a subgroup of H_1 , and hence $\text{Stab}^\mu(p)$ is conjugate, by an element of $G(M)$, to a definable subgroup of H_1 . Clearly, every such group is solvable and torsion-free.

(2) This is exactly [CP, Proposition 4.7].

(3) Let $H_1 \subseteq G$ be a definable torsion-free subgroup as in Fact 3.25. Let $H \subseteq G$ be a maximal definable torsion-free subgroup. By (2) there is a complete H -type p with $\text{Stab}^\mu(p) = \text{Stab}(p) = H$. By the proof of (1), $H = \text{Stab}^\mu(p)$ is conjugate to a subgroup $H_2 < H_1$. As H is maximal, we get $H_2 = H_1$. □

3.6. Stabilizers of types in definable G -sets

All that we have done so far in the o-minimal setting was to analyze $S_G^\mu(M)$. We present here several consequences for definable G -sets and leave the more substantial investigation for further research. We thus fix a definable group G and a definable G -set X in an o-minimal expansion \mathcal{M} of a real closed field.

Proposition 3.27. (1) *The group $\mu(\mathbb{U})$ is torsion-free.*

(2) *If \mathcal{N} is a tame extension of \mathcal{M} then for every N -definable subgroup H , the M -definable group $\text{st}(H)$ has the same dimension as H .*

Proof. (1) Assume for contradiction that $\mu(\mathbb{U})$ contains an n -torsion point of G . Thus the set $\{g \in G : g \neq e \text{ and } g^n = e\}$ is an M -definable set whose closure contains e .

We consider the Lie algebra \mathfrak{g} , associated to G (see [PPS]). It is not hard to see that for every $n \in \mathbb{N}$, the differential of the map $g \mapsto g^n$ at e , call it $d_n : \mathfrak{g} \rightarrow \mathfrak{g}$, is just the map $v \mapsto nv$. Since it is invertible, the map $g \mapsto g^n$ must be injective in a small neighborhood of e , contradicting the fact that every neighborhood of e contains an n -torsion point.

(2) Assume that H is an N -definable subgroup of G . As noted before, for every $h \in H$ we have $d(r_h)_e(T(H)_e) = T(H)_h$. Hence $\text{st}(T(H)_h) = \text{st}(T(H)_e)$ for all $h \in \mu(\mathbb{U}) \cap H$. But then, by Proposition 3.15, $\dim \text{st}(H) = \dim H$. □

Regarding the above proposition, note that while $\dim \text{st}(H) = \dim H$, these groups could be quite different. For example, let H be $\text{SO}(2, R)^A$, where

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix},$$

for α realizing the type of an infinitely large element. The group H is definably compact, but

$$\text{st}(H) = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R} \right\}$$

is torsion-free.

We can now state two results on $\text{Stab}^\mu(p)$ for definable G -sets.

Proposition 3.28. *Assume that X is a definable G -set, $p \in S_X(M)$ a definable type and $\alpha \models p$ is in \mathbb{U} .*

- (1) *Let $G_\alpha = \{g \in G(\mathbb{U}) : g \cdot \alpha = \alpha\}$. Then $\dim \text{Stab}^\mu(p) \geq \dim G_\alpha$.*
- (2) *Assume that G acts transitively on X , and endow X with the induced topology (either through $S_X^\mu(M)$, see Corollary A.11, or by identifying X with G/G_a for some $a \in X(M)$). If p is unbounded with respect to this topology then $\dim \text{Stab}^\mu(p) > 0$.*

Proof. (1) We work in $\mathcal{N} = \mathcal{M}\langle\alpha\rangle$, a tame extension of \mathcal{M} . By Claim 2.21, the group $\text{st}(G_\alpha)$ is a subgroup of $\text{Stab}^\mu(p)$ definable in \mathcal{M} , and by Proposition 3.27, its dimension equals that of G_α . Hence $\dim \text{Stab}^\mu(p) \geq \dim G_\alpha$.

(2) Fix $a \in X(M)$. By definable choice we can find an M -definable set $Y \subseteq G$ which is in definable bijection with X via the map $\pi(g) = g \cdot a$. While π is not a homeomorphism of $Y \subseteq G$ and X (with its quotient topology), if $D \subseteq Y$ is a definable set such that $\text{Cl}(D)$ is definably compact in G , then $\pi(\text{Cl}(D))$ is definably compact with respect to the topology of X .

Assume now that $p \in S_X(M)$ is a definable type which is not bounded, namely, does not contain any definably compact X -formula. Let $q \in S_Y(M)$ be the pull-back of p under π (so $q \cdot a = p$). Then q is a definable G -type which, by the above discussion, is unbounded in G . By Theorem 3.10 the μ -stabilizer of q has positive dimension and it is easy to see that $\text{Stab}^\mu(q) \subseteq \text{Stab}^\mu(p)$. We have thus shown that $\text{Stab}^\mu(p)$ is infinite. \square

4. Some examples

4.1. The group $G = \langle \mathbb{R}^2, + \rangle$

Let \mathcal{M} be an o-minimal expansion of the real field, and $G = \langle \mathbb{R}^2, + \rangle$. Our goal is to understand the space $S_G(M)$ and the μ -stabilizers of types. Note that every type in $S_G(M)$ is definable [MS]. In our discussion below, p is a complete type in $S_G(M)$.

Bounded types. As pointed earlier, if p contains any formula which defines a bounded subset of \mathbb{R}^2 then it is μ -equivalent to a (type of an) element $g \in \mathbb{R}^2$, and hence $\text{Stab}^\mu(p) = \{e\}$ (here $e = (0, 0)$).

Unbounded types of dimension 1. We may assume that there is a definable unbounded curve $\gamma(t) = (\gamma_1(t), \gamma_2(t)) : (0, \infty) \rightarrow \mathbb{R}^2$ such that p is the type of γ “at ∞ ”. Clearly, p is μ -reduced, for otherwise it will be infinitesimally close to a point $g \in \mathbb{R}^2$, contradicting the fact that it is unbounded.

If we let

$$\vec{v} = \lim_{t \rightarrow \infty} \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}$$

then $\text{Stab}^\mu(p) = \mathbb{R}\vec{v}$.

Unbounded types of dimension 2. We prove here a general claim:

Claim 4.1. *If G is any definable group in an o-minimal structure \mathcal{M} and $p \in S_G(M)$ is a definable μ -reduced type with $\dim p = \dim G$ then $\text{Stab}(p) = \text{Stab}^\mu(p) = G$.*

Proof. By Claim 3.13, there is an M -definable set S in p such that for every $\alpha \models p$ and $m \in M$, every element in $(B_m \cdot \alpha) \cap S$ realizes p . Since $\dim S = \dim p = \dim G$, by Claim 3.12, every element in $B_m \cdot \alpha$ realizes p . Because $G(M)$ is the union of these $B_m(M)$'s, it follows that every element in $G(M) \cdot \alpha$ realizes p , so $G(M) = \text{Stab}(p)$. \square

Going back to our example, we may conclude that $\text{Stab}(p) = \text{Stab}^\mu(p) = \mathbb{R}^2$.

4.2. The action of $\text{SL}(2, \mathbb{R})$ on \mathbb{H}

We now let $G = \text{SL}(2, \mathbb{R})$ and consider its usual action on the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}.$$

The action is transitive and the stabilizer of each point $z \in \mathbb{H}$, call it G_z , is a conjugate of $\text{SO}(2, \mathbb{R})$.

Our goal is to understand μ -types in \mathbb{H} . We work in an o-minimal expansion \mathcal{M} of the field \mathbb{R} .

We are using the fact that there is a definable compactification of \mathbb{H} , call it $\overline{\mathbb{H}}$, such that the action of G can be definably extended to that of $\overline{\mathbb{H}}$. Namely,

$$\overline{\mathbb{H}} = \{z \in \mathbb{C} : \text{Im}(z) \geq 0\} \cup \{\infty\},$$

with the action of G on the real line given by the same linear fraction, and with $A \cdot z = \infty$ when $cz + d = 0$ and $A \cdot \infty = a/c$. Clearly, the action is transitive on $\overline{\mathbb{H}} \setminus \mathbb{H}$. The topology on $\overline{\mathbb{H}}$ is as follows: the induced topology on $\overline{\mathbb{H}} \cap \mathbb{C}$ is the Euclidean one, and neighborhoods of ∞ are of the form $g \cdot U$, where $U \subseteq \overline{\mathbb{H}}$ is a neighborhood of 0 (by transitivity we could have chosen here any point other than 0). The space $\overline{\mathbb{H}}$ is compact and the action of G is continuous. Note, however, that because $\overline{\mathbb{H}} \setminus \mathbb{H}$ is a closed orbit of G , it is not true for $B \subseteq G$ open and $z \in \overline{\mathbb{H}} \setminus \mathbb{H}$ that the set $B \cdot z$ is open.

Given a complete type $p \in S_{\mathbb{H}}(M)$ we say that $z \in \overline{\mathbb{H}}$ is a *limit point* of p if for every M -definable open neighborhood $U \subseteq \overline{\mathbb{H}}$ of z , the formula defining $U \cap \mathbb{H}$ is in p . Since p is a complete type and $\overline{\mathbb{H}}$ is a Hausdorff space, each p has at most one limit point in $\overline{\mathbb{H}}$.

Consider now the intersection $\bigcap_{p \vdash Y} \text{Cl}(Y)$, where $\text{Cl}(Y)$ denotes the closure of Y in $\overline{\mathbb{H}}$. Since $\overline{\mathbb{H}}$ is compact and the collection of these Y 's is finitely satisfiable, it follows that the intersection is nonempty. It is not hard to see that any z in this intersection is a limit point of p . Hence every $p \in S_{\mathbb{H}}(M)$ has a unique limit point, call it $\lim p$, in $\overline{\mathbb{H}}$.

Claim 4.2. *The map $p \mapsto \lim p$ factors through $S_{\mathbb{H}}^\mu(M)$ and induces a topological homeomorphism of the G -spaces $S_{\mathbb{H}}^\mu(M)$ and $\overline{\mathbb{H}}$.*

Proof. In order to see that the map factors through $S_{\mathbb{H}}^\mu(M)$ we need to show that if $p, q \in S_{\mathbb{H}}(M)$ and $z_p = \lim p \neq \lim q = z_q$ then $p_\mu \neq q_\mu$.

Without loss of generality, $z_p, z_q \in \mathbb{C} \cap \overline{\mathbb{H}}$. If either of them is in \mathbb{H} then it is easy to see that $p_\mu \neq q_\mu$, so we may assume that $z_p, z_q \in \mathbb{R}$. Now, let $D_p, D_q \subseteq \mathbb{C}$ be open discs centered at z_p, z_q , respectively, such that $D_p \cap D_q = \emptyset$. By continuity of the action

on $\overline{\mathbb{H}}$, there exist open discs $U_p \subseteq D_p$ around z_p and $U_q \subseteq D_q$ around z_q and a definable open neighborhood W of e in G such that $W \cdot U_p \subseteq D_p$ and $W \cdot U_q \subseteq D_q$. By definition of $\lim p$ and $\lim q$ it follows that $p \vdash U_p$ and $q \vdash U_q$, hence $\mu \cdot p \subseteq D_p$ and $\mu \cdot q \vdash U_q$, so $p_\mu \neq q_\mu$.

A small variation of the above argument shows that the induced map $\pi : S_{\mathbb{H}}^\mu(M) \rightarrow \overline{\mathbb{H}}$ is continuous. Since \mathbb{H} is dense in $\overline{\mathbb{H}}$ and $S_{\mathbb{H}}^\mu(M)$ is compact, the map is surjective. Let us see that it is also injective.

Assume that $\lim p = \lim q$; we will show that $p_\mu = q_\mu$. Without loss of generality, we may assume that $\lim p = \lim q = 0$, and to simplify matters it will be sufficient to take $q = \text{tp}(\beta i/\mathbb{R})$, with $\beta > 0$ in an elementary extension of \mathcal{M} infinitesimally close to 0. It is clearly enough to show that $p_\mu = q_\mu$. Let $\alpha \models p$ in some elementary extension. We will show that there exists $h \in \mu(G)$ such that $h \cdot \alpha \models q$.

Indeed, write $\alpha = \alpha_1 + \alpha_2 i$ with α_1, α_2 infinitesimally close to 0 and take

$$h = \begin{pmatrix} 1 & -\alpha_1 \\ 0 & 1 \end{pmatrix}.$$

Clearly, $h \in \mu(G)$ and we have $h \cdot \alpha = \alpha_2 i \models q$. It follows using Claim 2.7 that $p_\mu = q_\mu$.

We have thus shown that $\pi : S_{\mathbb{H}}^\mu(M) \rightarrow \overline{\mathbb{H}}$ is a continuous bijection. Since both spaces are compact and Hausdorff, it is actually a homeomorphism. \square

As an immediate corollary we obtain:

Claim 4.3. *If $p_\mu \in S_{\mathbb{H}}^\mu(M)$ and $z_p = \lim p \in \overline{\mathbb{H}}$ then $\text{Stab}^\mu(p) = G_{z_p}$. Hence, if $\lim p \in \mathbb{H}$ then $\text{Stab}^\mu(p)$ is a conjugate of $\text{SO}(2, \mathbb{R})$, and if $\lim p \in \overline{\mathbb{H}} \setminus \mathbb{H}$ then $\text{Stab}^\mu(p)$ is a conjugate of the solvable group*

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} : a, b \in \mathbb{R} \right\}.$$

Appendix A. More on the space $S_X^\mu(M)$

In topological dynamics the *Samuel compactification* $S(G)$ of a topological group G is the compactification of G with respect to the uniformity induced via the action of G on itself by left multiplication. Up to a homeomorphism it is the unique compact space with the following properties:

- (i) There is an embedding $\chi : G \rightarrow S(G)$ such that the image $\chi(G)$ is dense in $S(G)$, and group multiplication on G extends to a continuous map from $G \times S(G)$ to $S(G)$. (In particular $S(G)$ is a G -space.)
- (ii) If C is a compact G -space then for any $p \in C$ the map $g \mapsto g \cdot p$ from G to C extends uniquely to a continuous map from $S(G)$ to C .

We refer to the original article [S] and a survey [U] for more details on the Samuel compactification.

In this section we investigate further properties of $S_G^\mu(M)$ for a topological group G definable in a first order structure \mathcal{M} . We show that it has a natural compact topology and under the additional assumption that every complete G -type is definable, property (ii) above holds if we restrict ourselves to definably separable actions of G on compact spaces.

In the special case, when every subset of $G(M)$ is definable in \mathcal{M} , the space $S_G^\mu(M)$ is exactly the Samuel compactification $S(G)$ (by properties (i) and (ii) above). A slightly different model-theoretic approach to the Samuel compactification is contained in [GPP, Section 2].

If the topology on G is discrete then $S_G^\mu(M) = S_G(M)$. This case has already been studied in several papers (e.g. [N]).

A.1. The topology of $S_X^\mu(M)$

Setting: We work in the same setting as in Section 2.1. We fix a first order structure \mathcal{M} and by definable we always mean M -definable. We also fix a group G definable in \mathcal{M} . We assume that G is a topological group and that it has a basis of open neighborhoods of e consisting of definable sets. As in Section 2.1, we will denote by $\mu(v)$ the infinitesimal type of G . Here we always write G for $G(M)$.

If X is a definable G -set then by Claim 2.9, the action of G on X induces an action of G on $S_X^\mu(M)$. Our first goal is to show that $S_X^\mu(M)$ has a natural Hausdorff topology. With respect to this topology, $S_G^\mu(M)$ is compact and the group action is continuous, in other words $S_X^\mu(M)$ is a compact G -space (compact G -spaces are also called G -flows).

Since the relation \sim_μ on $S_X(M)$ is induced by a type-definable equivalence relation (see below), we present the basics in a more general setting, still denoting the underlying set by X .

Let X be a definable set in \mathcal{M} and $E(x, y)$ an X^2 -type over M which defines an equivalence relation on $X(\mathbb{U})$ (we call it a *type-definable equivalence relation* on X). We denote by E^* the associated equivalence relation on $S_X(M)$:

$$p E^* q \Leftrightarrow p(x) \cup q(y) \cup E(x, y) \text{ is consistent.}$$

For $p \in S_X(M)$, let $[p]$ denote its E^* -equivalence class.

Definition A.1. We equip the set $S_X^*(M) = S_X(M)/E^*$ with the quotient topology, namely a subset $F \subseteq S_X^*(M)$ is closed if and only if there is a partial X -type Σ over M such that

$$F = \{[p] \in S_X^*(M) : p \vdash \Sigma\}. \quad (\text{A.1})$$

Claim A.2. *The relation E^* is closed in $S_X(M) \times S_X(M)$, hence $S_X^*(M)$ is compact and the projection map $\pi : S_X(M) \rightarrow S_X^*(M)$ is closed.*

Proof. Take $(p, q) \notin E^*$. Then $p(x) \cup q(y) \cup E(x, y)$ is inconsistent, and therefore by compactness there are formulas $\varphi(x) \in p$ and $\psi(y) \in q$ such that $\{\varphi(x), \psi(y)\} \cup E(x, y)$ is inconsistent. The formulas φ and ψ define an open neighborhood $U_\varphi \times U_\psi$ of (p, q) which is disjoint from E^* , so E^* is closed.

We now conclude that $S_X^*(M)$ is Hausdorff, so by continuity of π it is compact, and π is a closed map. \square

As a corollary of the above we see that for each $p \in S_X(M)$, the E^* -equivalence class $[p] \subseteq S_X(M)$ is a closed subset of $S_X(M)$, so given by a partial type, which we denote by $\Phi_p(x)$. The next claim describes the topology on $S_X^*(M)$.

Claim A.3. *Assume that X and E are as above.*

(1) *A nonempty subset $F \subseteq S_X^*(M)$ is closed if and only if there is an X -type $\Sigma(x)$ over M such that*

$$F = \{[p] \in S_X^*(M) : p(x) \cup E(x, y) \cup \Sigma(y) \text{ is consistent}\}.$$

(2) *A basis for the open sets in the topology of $S_X^*(M)$ is the collection, as φ varies in $\mathcal{L}_X(M)$, of sets of the form*

$$U_\varphi^* := \{[p] \in S_X^*(M) : \Phi_p(x) \vdash \varphi(x)\}.$$

Proof. (1) follows from the description of closed sets in (A.1).

(2) If F is any closed set as in (1) then its complement is

$$\{[p] \in S_X^*(M) : p(x) \cup E(x, y) \cup \Sigma(y) \text{ is inconsistent}\},$$

which is the same as

$$\{[p] \in S_X^*(M) : \Phi_p(x) \cup \Sigma(x) \text{ is inconsistent}\}.$$

By compactness, this is the union over all $\varphi \in \Sigma$ of the sets

$$\{[p] \in S_X^*(M) : \Phi_p \vdash \neg\varphi\}.$$

Every such set is an open set of the form $U_{\neg\varphi}^*$. \square

Remark A.4. Note that the notion of a type-definable equivalence relation E on X is the same as that of a *uniformity* on X [K, Chapter 6]. Namely, the set $\{\phi(M^2) : \phi(x, y) \in E\}$ is a base for a uniformity on X . Conversely, given a uniformity \mathcal{U} on a set X , if we endow X^2 with a predicate for every set in \mathcal{U} , then the set of these predicates is a type-definable equivalence relation on X .

The space $S_X^*(M)$ that we described above is basically the compactification of a uniform space (a set together with a uniformity) described by Samuel [S].

We now return to the case when X is a definable G -space. Obviously,

$$\Phi(x, y) = \{\exists z (\theta(z) \wedge z \cdot x = y) : \theta(v) \in \mu\}$$

defines the equivalence relation $\mu(\mathbb{U}) \cdot x = \mu(\mathbb{U}) \cdot y$ on $X(\mathbb{U})$, and we have

$$p \sim_\mu q \Leftrightarrow p(x) \cup q(y) \cup \Phi(x, y) \text{ is consistent,}$$

therefore \sim_μ is the equivalence relation on $S_X(M)$ associated to Φ , whose quotient space is $S_X^\mu(M)$.

By our analysis above, the space $S_X^\mu(M)$ is compact and a basis for its topology is given by open sets of the form

$$U_\varphi^\mu = \{p_\mu \in S_X^\mu(M) : \mu \cdot p \vdash \varphi\}$$

as φ varies over the X -formulas.

Although parts of what we do below can still be presented in a more general setting, we stick to the case of definable G -sets.

Claim A.5. *The action of G on $S_X^\mu(M)$ (see Claim 2.9) is continuous, hence $S_X^\mu(M)$ is a G -space.*

Proof. It is sufficient to show that for each X -formula φ over M the set $W_\varphi = \{(g, p_\mu) : g \cdot p_\mu \in U_\varphi^\mu\}$ is open in the product topology on $G \times S_X^\mu(M)$.

Assume $(g, p_\mu) \in W_\varphi$. Then there are $\theta(v) \in \mu$ and $\psi(x) \in p(x)$ such that $(g \cdot \theta \cdot \psi)(x) \vdash \varphi(x)$. Since $\mu \cdot \mu = \mu$, there is $\theta_1(v) \in \mu$ such that $(\theta_1 \cdot \theta_1)(v) \vdash \theta(v)$. Let $\Theta_1 = \{h \in G : \mathcal{M} \models \theta_1(h)\}$. It is an open subset of G . For every $h \in \Theta_1(M)$ we have

$$((g \cdot h) \cdot (\theta_1 \cdot \psi))(x) \vdash \varphi(x).$$

It is easy to see that $g \cdot \Theta_1 \times U_{\theta_1 \cdot \psi}^\mu$ is an open subset of W_φ containing (g, p_μ) . □

Recall that in general when a group G acts on sets X and Y then a map $f: X \rightarrow Y$ is called *equivariant* if it commutes with the action of G , i.e. $f(g \cdot a) = g \cdot f(a)$ for all $g \in G$ and $a \in X$.

Also recall that if $f: X \rightarrow Y$ is an M -definable map between M -definable sets (and hence also from $S_X(M)$ to $S_Y(M)$) then f has a canonical extension to a map f_* from $\mathcal{L}_X(M)$ to $\mathcal{L}_Y(M)$ such that for a saturated elementary extension \mathbb{U} of \mathcal{M} we have $f(\Sigma(\mathbb{U})) = f_*(\Sigma)(\mathbb{U})$ for every X -type $\Sigma(x)$, and if $p \in S_X(M)$ then $f_*(p) \in S_Y(M)$.

Claim A.6. *Let X, Y be definable G -sets and let $f: X \rightarrow Y$ be an M -definable equivariant map. Then the map $f_*: S_X(M) \rightarrow S_Y(M)$ respects \sim_μ , namely for $p \in S_X(M)$ we have $f_*(\mu \cdot p) = \mu \cdot f_*p$. The induced map from $S_X^\mu(M)$ to $S_Y^\mu(M)$, still denoted by f_* , is equivariant and continuous.*

Proof. Since $f: X \rightarrow Y$ is definable, the map from $X(\mathbb{U})$ to $Y(\mathbb{U})$ that it defines is equivariant with respect to the action of $G(\mathbb{U})$. Using Remark 2.2 and properties of f_* we obtain

$$\begin{aligned} f_*(\mu \cdot p)(\mathbb{U}) &= f((\mu \cdot p)(\mathbb{U})) = f(\mu(\mathbb{U}) \cdot p(\mathbb{U})) \\ &= \mu(\mathbb{U}) \cdot f(p(\mathbb{U})) = \mu(\mathbb{U}) \cdot f_*(p)(\mathbb{U}) = (\mu \cdot f_*(p))(\mathbb{U}). \end{aligned}$$

Hence $f_*(\mu \cdot p) = \mu \cdot f_*(p)$, and it is not hard to see that the induced map from $S_X^\mu(M)$ to $S_Y^\mu(M)$ is equivariant. It is continuous since $f_*: S_X(M) \rightarrow S_Y(M)$ is continuous in the logic topology. □

Notation A.7. For a definable G -set X we will denote by χ_X the map from X to $S_X^\mu(M)$ defined as $\chi_X: a \mapsto \text{tp}(a/M)_\mu$.

Claim A.8. Let X be a definable G -set.

- (1) The image $\chi_X(X)$ is dense in $S_X^\mu(M)$.
- (2) Let $a, b \in X$. Then $\chi_X(a) = \chi_X(b)$ if and only if $b \in \overline{G_a} \cdot a$, where $G_a = \{g \in G : g \cdot a = a\}$ is the stabilizer of a in G , and $\overline{G_a}$ is its topological closure in G . In particular the stabilizer of every $a \in X$ is closed in G if and only if χ_X is injective.

Proof. (1) Since the set $\{\text{tp}(a/M) : a \in X\}$ is dense in $S_X(M)$ in the logic topology, its image in $S_X^\mu(M)$ under the projection map is dense as well.

(2) Assume first that $b \in \overline{G_a} \cdot a$; we will show that $\text{tp}(b/M) \vdash \mu \cdot \text{tp}(a/M)$. Choose $g \in \overline{G_a}$ with $b = g \cdot a$. Since g is in the closure of G_a , for every $\theta(v) \in \mu(v)$ there is $h_\theta \in G_a$ with $g \in \theta(M) \cdot h_\theta$. Thus $b \in \theta(M) \cdot h_\theta \cdot a = \theta(M) \cdot a$. Hence we obtain $\text{tp}(b/M) \vdash \theta \cdot \text{tp}(a/M)$ for every $\theta(x) \in \mu(x)$, so $\text{tp}(b/M) \vdash \mu \cdot \text{tp}(a/M)$.

For the opposite direction assume $\mu \cdot \text{tp}(b/M) = \mu \cdot \text{tp}(a/M)$, or equivalently $\text{tp}(b/M) \vdash \mu \cdot \text{tp}(a/M)$. First we claim that in this case b is in the G -orbit of a . Indeed, take any $\varphi(v) \in \mu(v)$. We have $\text{tp}(b/M) \vdash \varphi \cdot \text{tp}(a/M)$, hence $b = g \cdot a$ for some $g \in \varphi(M)$. Now we fix some $g \in G$ such that $b = g \cdot a$. For every $\theta(v) \in \mu(v)$ there is $h_\theta \in \theta(M)$ such that $b = h_\theta \cdot a$, hence $h_\theta^{-1}g \in G_a$. Since $\mu = \mu^{-1}$, and the sets $\theta(M) \cdot g, \theta(v) \in \mu$ form a basis of open neighborhoods of g , we conclude that g is in the closure of G_a . □

Thus if the stabilizer of each point $a \in X$ is closed, then the map χ_X is injective and we can consider X as a subset of $S_X^\mu(M)$. Notice that this is indeed the case when X is a definable G -set in an o-minimal structure, since all definable subgroups of G are closed.

The next claim describes the induced topology on X .

Claim A.9. Let X be a definable G -space. Assume the map $\chi_X: X \rightarrow S_X^\mu(M)$ is injective. Then after identifying X with $\chi(X)$ the topology on X induced by $S_X^\mu(M)$ is exactly the topology whose basis is

$$\{V \cdot a : V \text{ open in } G, a \in X\}.$$

In particular:

- (1) Every G -orbit in X is open and closed.
- (2) For every $a \in X$ the orbit $G \cdot a$ is homeomorphic to the factor space G/G_a (with the quotient topology) under the natural map $gG_a \mapsto g \cdot a$.

Proof. Let us see that every such $V \cdot a \subseteq X$ is indeed open in the induced topology. Without loss of generality, V is a definable open set given by a G -formula $\varphi(v)$. We claim that $U_{\varphi \cdot a}^\mu \cap \chi_X(X) = \chi_X(V \cdot a)$. Indeed, if $\text{tp}(b/M)_\mu \in U_{\varphi(x) \cdot a}^\mu$ then in particular $b \in V \cdot a$. Conversely, if $b \in V \cdot a$ then there is $g \in V$ with $b = g \cdot a$. But then there is an M -definable open neighborhood V_1 of g contained in V , so in particular $(\mu \cdot g) \cdot a \vdash V \cdot a$, hence $\text{tp}(b/M)_\mu \in U_{\varphi \cdot a}^\mu$.

Let us see that every open subset of $\chi_X(X)$ is a union of sets of the form $\chi_X(V \cdot a)$. Consider $\text{tp}(a/M)_\mu \in U_\psi^\mu \cap \chi_X(X)$ for some X -formula $\psi(x)$ and $x \in X$. Since $\mu \cdot a \vdash \psi$, there exists $\theta \in \mu$ with $\theta \cdot a \vdash \psi$. If we take $V = \theta(G)$ then $\chi_X(V \cdot a) \subseteq U_\psi^\mu$, so we can write $U_\psi^\mu \cap \chi_X(X)$ as a union of sets of this form.

(1) and (2) easily follow. \square

Remark A.10. It is not hard to see that in the previous claim properties (1) and (2) define a unique topology on X , and it is the strongest topology on X making the action of G on X continuous.

Corollary A.11. *If H is a definable closed subgroup of G then the space $X = G/H$ with the quotient topology embeds into the compact space $S_X^\mu(M)$, and the action of G on $S_X^\mu(M)$ is the unique continuous extension of the action of G on G/H . In particular, for $H = \{e\}$ the map χ_G is a topological embedding of G into $S_G^\mu(M)$, and under this embedding the action of G on $S_G^\mu(M)$ is the unique continuous extension of the action of G on itself by left multiplication.*

A.2. Definably separable actions

In [GPP] a map f from a definable set D to a compact space C was called definable if for any disjoint closed $C_1, C_2 \subseteq C$ their preimages $f^{-1}(C_1)$ and $f^{-1}(C_2)$ can be separated by definable subsets of D . Since we prefer to reserve the term “definable” for actual definable maps, we will use the term “definably separable” instead.

As in [GPP], we say that an action of a definable group G on a compact space C is *definably separable* if for every $x_0 \in C$ the map from G to C given by $g \mapsto g \cdot x_0$ is definably separable.

Lemma A.12. *Assume G acts continuously on a compact space C . Let $c_0 \in C$ and assume the map $f : G \rightarrow C$ given by $g \mapsto g \cdot c_0$ is definably separable. Then f can be uniquely extended to a continuous G -equivariant map $f_* : S_G^\mu(M) \rightarrow C$ (we identify G with its image in $S_G^\mu(M)$ under the map χ_G).*

Proof. For $C_0 \subseteq C$ we will denote by $\overline{C_0}$ the topological closure of C_0 in C .

The definition of the map is classical and goes back to Stone–Čech: If $p(v) \in S_G(M)$ then using definable separation of f and compactness of C it follows that the set

$$f[p] := \bigcap_{p(v) \vdash \varphi(v)} \overline{f(\varphi(M))}$$

is a singleton. This gives a map from $S_G(M)$ to C . We claim that for $p, q \in S_G(M)$ with $p \sim_\mu q$ we have $f[p] = f[q]$.

Assume not, and $f[p] \neq f[q]$ for some $p \sim_\mu q$, hence $f[p] \cap f[q] = \emptyset$. By compactness of C it then follows that there are $\varphi(v) \in p(v)$ and $\psi(v) \in q(v)$ such that $\overline{f(\varphi(M))} \cap \overline{f(\psi(M))} = \emptyset$.

In general, by standard compactness arguments, when a group G acts continuously on a compact space C , for any two disjoint closed subsets C_0, C_1 of C there is an open subset

\mathcal{O} of G containing e such that $\mathcal{O} \cdot C_1 \cap C_2 = \emptyset$. Therefore there is a formula $\theta(v) \in \mu$ such that

$$\theta(M) \cdot \overline{f(\varphi(M))} \cap \overline{f(\psi(M))} = \emptyset,$$

and in particular $\theta(M) \cdot f(\varphi(M)) \cap f(\psi(M)) = \emptyset$.

Therefore $(\theta(M) \cdot \varphi(M)) \cdot c_0 \cap \psi(M) \cdot c_0 = \emptyset$, and hence $\theta(M) \cdot \varphi(M) \cap \psi(M) = \emptyset$. But this contradicts the consistency of $\mu \cdot p$ and q .

For $p \in S_G(M)$ we define $f_*(p_\mu)$ to be the unique element in $f[p]$. It is not hard to check that the map f_* is continuous. The uniqueness of f_* and its G -equivariance follow from the density of G in $S_G^\mu(M)$. \square

The proof of the following claim is similar to the proof of [GPP, Lemma 3.7]

Lemma A.13. *Let X be a definable G -set, and assume $p \in S_X(M)$ is a definable type. Then the map $f_p: G \rightarrow S_X^\mu(M)$ given by $g \mapsto g \cdot p_\mu$ is definably separable. In particular, if every type $p \in S_X(M)$ is definable then the action of G on $S_X^\mu(M)$ is definably separable.*

Proof. Let C_1, C_2 be disjoint closed subsets of $S_X^\mu(M)$. By Claim A.3(1) there are X -types $\Sigma_1(x), \Sigma_2(x)$ such that

$$C_i = \{s_\mu(x) \in S_X^\mu(M) : s \vdash \mu \cdot \Sigma_i\}, \quad i = 1, 2.$$

It follows that $\mu \cdot \Sigma_1 \cup \mu \cdot \Sigma_2$ is inconsistent, so there are $\theta(v) \in \mu$ and $\psi_i(x) \in \Sigma_i(x)$ such that $(\theta \cdot \psi_1 \wedge \theta \cdot \psi_2)(x)$ is inconsistent. Obviously, $q_\mu \in C_i \Rightarrow q(x) \vdash (\theta \cdot \psi_i)(x)$.

Since $p(x)$ is a definable type, the sets $U_i = \{g \in G : g \cdot p(x) \vdash (\theta \cdot \psi_i)(x)\}$, $i = 1, 2$, are definable. They are disjoint and, by the above, $g \cdot p_\mu \in C_i \Rightarrow g \in U_i$, hence $f_p^{-1}(C_i) \subseteq U_i$. \square

Remark A.14. It follows from the proof that instead of the assumption of definability of p it is sufficient to assume that for any formula $\varphi(x)$ the set $\{g \in G : g \cdot p \vdash \varphi(x)\}$ is definable.

In the next theorem we list the main properties of $S_G^\mu(M)$.

Theorem A.15. (i) *The space $S_G^\mu(M)$ is compact. There is a topological embedding $\chi: G \rightarrow S_G^\mu(M)$ with image dense in $S_G^\mu(M)$, and the group multiplication of G extends to a continuous map from $G \times S_G^\mu(M)$ to $S_G^\mu(M)$.*

(ii) *Assume that every type $p \in S_G(M)$ is definable. Then the action of G on $S_G^\mu(M)$ is definably separable.*

(iii) *For any compact G -space C and $p \in C$, if the map $g \mapsto g \cdot p$ is definably separable then it extends uniquely to a continuous G -equivariant map from $S_G^\mu(M)$ to C .*

Remark A.16. (a) It follows from the above theorem that in the case when every type $p \in S_G^\mu(M)$ is definable, the space $S_G^\mu(M)$ has the same universal property as the Samuel compactification if one considers only definably separable actions of G on compact spaces.

(b) In general, the spaces $S_G^\mu(M)$ and the Samuel compactification $S(G)$ have very different properties. For example, by a theorem of W. Veech [V] every locally compact group acts freely on its Samuel compactification, but Theorem 3.26 implies that if H is a torsion-free group definable in an o-minimal expansion \mathcal{M} of the real field then H fixes some type $p_\mu \in S_H^\mu(M)$.

(c) Theorem 3.26 and Fact 3.25 give a complete description of minimal compact G -invariant subsets of $S_G^\mu(M)$ in the case when G is a group definable in an o-minimal expansion of the real field. They are exactly orbits of μ -types q_μ for types q whose stabilizers are maximal torsion-free subgroups of G . Indeed, each such orbit is compact since $G/\text{Stab}^\mu(q)$ is compact. If $Y \subseteq S_G^\mu(M)$ is a minimal compact G -invariant set then there exists a G -map $f : S_G^\mu(M) \rightarrow Y$ which, by minimality, necessarily sends the orbit of q_μ above onto Y . It follows that $Y = G \cdot f(q_\mu)$ and $\text{Stab}(q_\mu) \subseteq \text{Stab}(f(q_\mu))$. Because $\text{Stab}^\mu(q)$ is a maximal torsion-free group, $\text{Stab}(q_\mu) = \text{Stab}(f(q_\mu))$.

(d) As for Samuel compactifications, it is easy to see, by properties (ii) and (iii) in Theorem A.15, that when every type $p \in S_G^\mu(M)$ is definable, the action of G on $S_G^\mu(M)$ extends to a semigroup operation on $S_G^\mu(M)$.

Appendix B. On connected components of relatively definable subsets

We work in an o-minimal structure \mathcal{M} , and by definable we mean definable in \mathcal{M} .

Recall that every definable nonempty subset $X \subseteq M^n$ is a disjoint union of finitely many definably connected components X_1, \dots, X_k . These components are unique, up to a permutation, and characterized by the following two properties:

- (I) Every X_i is a nonempty definable subset of X that is clopen in X .
- (II) If $Y \subseteq X$ is a definable clopen set and $Y \cap X_i \neq \emptyset$ then $X_i \subseteq Y$.

In this section we show that the same is true for relatively definable subsets of convexly definable open sets.

Let $I \subseteq M$ be an interval and $\{V_i : i \in I\}$ a uniformly definable family of open subsets of M^n . A subset $\mathcal{J} \subseteq I$ is *downward closed* if for all $x < y$ in I , if $y \in \mathcal{J}$ then $x \in \mathcal{J}$. For a downward closed subset $\mathcal{J} \subseteq I$ we denote by $V_{\mathcal{J}}$ the open set

$$V_{\mathcal{J}} = \bigcup_{r \in \mathcal{J}} V_r,$$

and call such an open set *convexly definable*.

A subset $\mathcal{X} \subseteq V_{\mathcal{J}}$ is called *relatively definable* if $\mathcal{X} = X \cap V_{\mathcal{J}}$ for some definable set X .

Example B.1. Assume \mathcal{M} is an o-minimal expansion of a group and $\mathcal{M}_0 \prec \mathcal{M}$ is an elementary substructure. Then the convex hull $\mathcal{O}_{\mathcal{M}_0}(M)$ and all its cartesian powers are convexly definable open sets, via $V_r = (-r, r)$, for $r \in M_{\geq 0}$.

The following is the main theorem of this section.

Theorem B.2. *Let $V_{\mathcal{J}} \subseteq M^n$ be a convexly definable open set and $\mathcal{X} \subseteq V_{\mathcal{J}}$ a relatively definable set. Then there are disjoint nonempty relatively definable subsets $\mathcal{X}_1, \dots, \mathcal{X}_k \subseteq V_{\mathcal{J}}$ such that $\mathcal{X} = \bigcup_{i=1}^k \mathcal{X}_i$ and:*

- (1) *Every \mathcal{X}_i is clopen in \mathcal{X} in the topology induced from M^n .*
- (2) *If $\mathcal{Y} \subseteq \mathcal{X}$ is a relatively definable clopen set and $\mathcal{Y} \cap \mathcal{X}_i \neq \emptyset$ then $\mathcal{X}_i \subseteq \mathcal{Y}$.*

We call the sets \mathcal{X}_i above *the connected components of \mathcal{X}* .

In the rest of this section we prove the above theorem.

Replacing V_r by $\bigcup_{s \leq r} V_s$ if needed, we assume that $s \leq r$ implies $V_s \subseteq V_r$. Let $X \subseteq M^n$ be a definable set such that $\mathcal{X} = X \cap V_{\mathcal{J}}$. For $r \in I$ we denote by X_r the set $X \cap V_r$, so $\mathcal{X} = \bigcup_{r \in \mathcal{J}} X_r$.

For $r \in I$ and $\alpha \in X_r$ we denote by $X_r(\alpha)$ the definable connected component of X_r containing α .

The following claim follows easily from o-minimality and properties of connected components.

- Claim B.3.** (1) *The family $\{X_r(\alpha) : \alpha \in X, r \in I\}$ is uniformly definable.*
 (2) *For any $\alpha \in M^n$ and $r_1 < r_2 \in I$ we have $X_{r_1}(\alpha) \subseteq X_{r_2}(\alpha)$.*
 (3) *Assume $X_r(\alpha) \cap X_r(\beta) \neq \emptyset$. Then $X_r(\alpha) = X_r(\beta)$ and for all $r < s \in I$ we have $X_s(\alpha) = X_s(\beta)$.*

For $\alpha \in \mathcal{X}$ we define

$$\mathcal{X}(\alpha) = \bigcup_{r \in \mathcal{J}} X_r(\alpha).$$

The following claim follows from Claim B.3(3).

Claim B.4. *For $\alpha, \beta \in \mathcal{X}$, either $\mathcal{X}(\alpha) = \mathcal{X}(\beta)$ or $\mathcal{X}(\alpha) \cap \mathcal{X}(\beta) = \emptyset$.*

For $\alpha \in \mathcal{X}$ let us call the set $\mathcal{X}(\alpha)$ a *component* of \mathcal{X} .

Claim B.5. *\mathcal{X} has finitely many components.*

Proof. By o-minimality there is an integer N such that for every $r \in I$ the set X_r has at most N connected components. We claim that \mathcal{X} has at most N components.

Assume not. Then there are $\alpha_0, \dots, \alpha_N \in \mathcal{X}$ such that the sets $\mathcal{X}(\alpha_i), i \leq N$, are disjoint. We can choose $r \in \mathcal{J}$ such that all α_i are in X_r . But then X_r has at least $N + 1$ connected components, a contradiction. \square

Claim B.6. *Each component \mathcal{X}_i is relatively definable.*

Proof. The claim is obvious if \mathcal{J} has a least upper bound (in $M \cup \{+\infty\}$), since then \mathcal{X} and all \mathcal{X}_i are definable. Assume \mathcal{J} does not have a least upper bound. Choose $\alpha_1, \dots, \alpha_k$ in \mathcal{X} such that $\mathcal{X}_i = \mathcal{X}(\alpha_i)$.

For every $i < j$ consider the set of all $r \in I$ such that $X_r(\alpha_i) \cap X_r(\alpha_j) = \emptyset$. By Claim B.3(1), it is a definable set containing \mathcal{J} , hence contains an element $r_{ij} \in I$ that is not in \mathcal{J} .

Let $r^* = \min\{r_{ij} : i < j\}$. Notice $r^* \notin \mathcal{J}$. Since the set $X_{r^*}(\alpha_i)$ is definable and $\mathcal{X}_i = X_{r^*}(\alpha_i) \cap V_{\mathcal{J}}$, the set \mathcal{X}_i is relatively definable. \square

Claim B.7. *For every $\alpha \in \mathcal{X}$, the component $\mathcal{X}(\alpha)$ is clopen in \mathcal{X} with respect to the M^n -induced topology.*

Proof. Since \mathcal{X} is a disjoint union of finitely many $\mathcal{X}(\alpha)$'s we only need to show that each $\mathcal{X}(\alpha)$ is open.

Since every V_r is open, for any $r \in \mathcal{J}$ the set $X_r = \mathcal{X} \cap V_r$ is open in \mathcal{X} . The connected component $X_r(\alpha)$ of X_r is open in X_r . Hence $X_r(\alpha)$ is open in \mathcal{X} , and $\mathcal{X}(\alpha)$ is open in \mathcal{X} as a union of open sets. \square

Claim B.8. *Let \mathcal{Y} be a relatively definable clopen subset of \mathcal{X} . If $\mathcal{Y} \cap \mathcal{X}_i \neq \emptyset$ then $\mathcal{X}_i \subseteq \mathcal{Y}$.*

Proof. Assume $\mathcal{Y} \cap \mathcal{X}_i \neq \emptyset$ and choose $\alpha \in \mathcal{Y} \cap \mathcal{X}_i$. Since \mathcal{Y} is relatively definable, for every $r \in \mathcal{J}$ the set $\mathcal{Y} \cap V_r$ is a definable subset of X_r that is clopen in X_r . Thus it contains the definably connected component $X_r(\alpha)$, hence \mathcal{Y} contains $\mathcal{X}_i = \mathcal{X}(\alpha)$. \square

This finishes the proof of Theorem B.2.

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