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Cocenters and representations of affine Hecke algebras

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Abstract. In this paper, we study the relationship between the cocenter and the representation theory of affine Hecke algebras. The approach is based on the interaction between the rigid cocenter, an important subspace of the cocenter, and the dual object in representation theory, the rigid quotient of the Grothendieck group of finite-dimensional representations.

Keywords. Affine Hecke algebra, cocenter, density theorem, trace Paley–Wiener theorem

1. Introduction

Affine Hecke algebras appear naturally in the representation theory of reductive p -adic groups as convolution algebras of compactly supported functions, such as Iwahori–Hecke algebras [Bo], [IM], and their generalizations in the theory of types [HM], [BK], or as endomorphism algebras of certain projective generators [Be], [Hei]. The representation theory of affine Hecke algebras with (equal and unequal) parameters that are not roots of unity was extensively studied [KL], [CG], [Lu1, Lu2], [BM1, BM2], [Xi1], [Op, OS1, OS2, So], [Ree], [Kat]. The representation theory of affine Hecke algebras with parameter equal to a root of unity plays an important role in the study of modular representations of p -adic groups [Vi], and the simple modular representations for affine Hecke algebras associated to general linear groups were classified in [AM]. The representations of affine Hecke algebras of classical types at roots of unity were subsequently studied in [VV, SVV] via the theory of canonical bases, and for G_2 in [Xi2].

In this paper, we consider finite-dimensional representations of the affine Hecke algebra \mathcal{H} from the perspective of its relation to the cocenter of the algebra. The cocenter $\overline{\mathcal{H}}$ is the quotient of the algebra by the subspace of commutators, and thus appears naturally in duality (given by the trace) with the (complexification of the) Grothendieck group $R(\mathcal{H})$ of finite-dimensional \mathcal{H} -modules.

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1.1. We explain our main results. In the body of the paper, we consider the more general case of an affine Hecke algebra with an automorphism δ of the root system and the δ -twisted cocenter, but, for simplicity, in the introduction, we present the results in the untwisted case only. We also consider arbitrary (nonzero) parameters.

The affine Hecke algebra \mathcal{H} is a deformation of the group algebra $\mathbb{C}[\tilde{W}]$ of the extended affine Weyl group \tilde{W} . Let $cl(\tilde{W})$ denote the set of conjugacy classes in \tilde{W} . It is easy to see that for any two elements w, w' in the same conjugacy class of \tilde{W} , the images of w and w' in the cocenter of $\mathbb{C}[\tilde{W}]$ are the same and the set $\{[\mathcal{O}]; \mathcal{O} \in cl(\tilde{W})\}$ is a basis of the cocenter of $\mathbb{C}[\tilde{W}]$. Here $[\mathcal{O}]$ is the image of w in the cocenter of $\mathbb{C}[\tilde{W}]$ for any $w \in \mathcal{O}$.

However, for w, w' in a conjugacy class \mathcal{O} of \tilde{W} , the images in $\tilde{\mathcal{H}}$ of the standard basis elements T_w and $T_{w'}$ of \mathcal{H} are not the same in general. It is showed in [HN1] that if w and w' are of minimal length in \mathcal{O} , then the images of T_w and $T_{w'}$ in $\tilde{\mathcal{H}}$ are the same. We denote this image by $T_{\mathcal{O}}$. Moreover, the whole cocenter $\tilde{\mathcal{H}}$ is spanned by $\{T_{\mathcal{O}}; \mathcal{O} \in cl(\tilde{W})\}$.

Note that $cl(\tilde{W})$ is a countably infinite set and the set of irreducible representations of \mathcal{H} is an uncountably infinite set. To compare the cocenter with representations, we would like to develop a reduction method from infinite sets to finite sets.

A familiar object in the literature is the elliptic quotient $\bar{R}_0(\mathcal{H})$ obtained by taking the quotient of $R(\mathcal{H})$ by the span of all proper parabolically induced modules [BDK]. The elliptic representation theory of reductive p -adic groups and associated Hecke algebras has been an area of active research [Ar], [BDK], [Be], [Kaz1], [SS], [Ree], [OS2]. The dual object to $\bar{R}_0(\mathcal{H})$ is the elliptic cocenter $\tilde{\mathcal{H}}^{\text{ell}}$, the subspace of $\tilde{\mathcal{H}}$ on which all proper parabolically induced modules vanish. However, as shown in [BDK], the elliptic cocenter is very complicated.

1.2. The solution we provide in this paper is another quotient of $R(\mathcal{H})$ and another subspace of $\tilde{\mathcal{H}}$, which we call the rigid quotient and rigid cocenter, respectively.

Let us first describe our motivation which leads to the definitions, and then describe the main result and its consequences.

For simplicity, we only consider affine Hecke algebras associated to semisimple root data in this introduction. In the beginning, we would like to get a nice finite subset of $cl(\tilde{W})$. A natural choice is $cl(\tilde{W})_{\text{ell}}$, the set of elliptic conjugacy classes of \tilde{W} . The problem is that there is no relation between the subspace spanned by $\{T_{\mathcal{O}}; \mathcal{O} \in cl(\tilde{W})_{\text{ell}}\}$ and the elliptic cocenter.

The finite subset of $cl(\tilde{W})$ we use here is $cl(\tilde{W})_0$, the conjugacy classes of \tilde{W} with zero Newton point. We have the inclusion $cl(\tilde{W})_{\text{ell}} \subset cl(\tilde{W})_0 \subset cl(\tilde{W})$. The definition of $cl(\tilde{W})_0$ and the idea to use it in the study of affine Hecke algebras are inspired by a classical result in arithmetic geometry: Kottwitz's classification of σ -isocrystals [Kot1, Kot2].

Here we provide more content. While this is not needed in our theory of rigid cocenter and rigid quotient, it serves as motivation for it.

Let G be a connected reductive group split over \mathbb{Q}_p and $L = W(\bar{\mathbb{F}}_p)[1/p]$ be the completion of the maximal unramified extension of \mathbb{Q}_p . Let σ be the (relative) Frobenius morphism on $G(L)$, and $B(G)$ the set of σ -conjugacy classes of $G(L)$. Kottwitz showed that a σ -conjugacy class is determined by two invariants: the image of the Kottwitz map

$B(G) \rightarrow \pi(G)$, and the Newton map from $B(G)$ to the rational coweight lattice. A σ -conjugacy class is called *basic* if its Newton point is zero. Moreover, Kottwitz shows that any σ -conjugacy class comes from a basic σ -conjugacy class of some Levi subgroup $M(L)$ of $G(L)$ via the inclusion map $M \rightarrow G$.

In [He], the second-named author studied the natural map $cl(\tilde{W}) \rightarrow B(G)$. This map is finite-to-one and it is compatible with both the Kottwitz map and the Newton map. Thus we have a Cartesian diagram

$$\begin{array}{ccc} cl(\tilde{W})_0 & \longrightarrow & B(G)_{\text{basic}} \\ \downarrow & & \downarrow \\ cl(\tilde{W}) & \longrightarrow & B(G) \end{array}$$

where $B(G)_{\text{basic}}$ is the set of basic σ -conjugacy classes.

As the basic σ -conjugacy classes capture the whole of $B(G)$ in an essential way, we expect that $cl(\tilde{W})_0$ plays an essential role in the study of $\tilde{\mathcal{H}}$. This leads to the following definition of the *rigid cocenter*:

$$\tilde{\mathcal{H}}^{\text{rigid}} = \text{span}\{T_{\mathcal{O}}; \mathcal{O} \in cl(\tilde{W})_0\}.$$

It turns out that for generic parameters, the quotient of $R(\mathcal{H})$ dual to $\tilde{\mathcal{H}}^{\text{rigid}}$ is also a natural object. Namely, $\bar{R}(\mathcal{H})_{\text{rigid}}$ is defined as the quotient of $R(\mathcal{H})$ by $R(\mathcal{H})_{\text{diff-ind}}$, the span of differences of central twists of parabolically induced modules (Definition 6.1).

The main result concerning the rigid cocenter is the following theorem.

- Theorem 1.1.** (1) *The set $\{T_{\mathcal{O}}; \mathcal{O} \in cl(\tilde{W})_0\}$ is a basis of $\tilde{\mathcal{H}}^{\text{rigid}}$ for an affine Hecke algebra \mathcal{H} with arbitrary parameters.*
 (2) *Suppose that the parameters of the affine Hecke algebra are admissible in the sense of Definition 6.6. Then the trace pairing $\text{tr} : \mathcal{H} \times R(\mathcal{H}) \rightarrow \mathbb{C}$ induces a perfect pairing*

$$\text{tr} : \tilde{\mathcal{H}}^{\text{rigid}} \times \bar{R}(\mathcal{H})_{\text{rigid}} \rightarrow \mathbb{C}.$$

In particular, the dimension of $\bar{R}(\mathcal{H})_{\text{rigid}}$ equals the number of classes in $cl(\tilde{W})_0$.

- (3) *For arbitrary parameters, the trace map $\text{tr} : \tilde{\mathcal{H}}^{\text{rigid}} \rightarrow R(\mathcal{H})_{\text{rigid}}^*$ is surjective.*

Theorem 1.1 in particular explains where the name “rigid” for $\tilde{\mathcal{H}}^{\text{rigid}}$ comes from:

- The traces of any given element in $\tilde{\mathcal{H}}^{\text{rigid}}$ on parabolically induced modules are constant if we deform the central characters, and the image under the trace map of $\tilde{\mathcal{H}}^{\text{rigid}}$ gives all such “rigid” linear functions on the induced modules.
- The rigid cocenter $\tilde{\mathcal{H}}^{\text{rigid}}$ has a basis which is independent of the parameters of the Hecke algebra, and thus $\tilde{\mathcal{H}}^{\text{rigid}}$ is “rigid” if we deform the parameters.

As to the relation between the rigid and elliptic cocenters/quotients, clearly, $\tilde{\mathcal{H}}^{\text{ell}} \subset \tilde{\mathcal{H}}^{\text{rigid}}$ and $\bar{R}(\mathcal{H})_{\text{rigid}} \twoheadrightarrow \bar{R}_0(\mathcal{H})$. But as we show in the paper, the rigid cocenter, in fact, combines together the elliptic cocenters of all the semisimple parts of parabolic subalgebras,

up to a certain equivalence (see Proposition 7.5). Dually, the rigid quotient admits a section formed by the elliptic quotients of the semisimple parts of parabolic subalgebras, up to equivalence (Corollary 6.9). Thus the rigid cocenter/quotient allows us to study the elliptic theory for all the parabolic subalgebras at once.

As a consequence, we obtain the basis theorem for $\tilde{\mathcal{H}}$.

Theorem 1.2. *The set $\{T_{\mathcal{O}}; \mathcal{O} \in cl(\tilde{W})\}$ is a basis of $\tilde{\mathcal{H}}$ for an affine Hecke algebra \mathcal{H} with arbitrary parameters.*

This theorem in particular shows that the cocenter of \mathcal{H} has a description independent of the parameters. For finite Hecke algebras, Tits' deformation theorem says that for generic parameters, the finite Hecke algebra is isomorphic to the group algebra. However, affine Hecke algebras of the same type but different parameters are almost never isomorphic. The basis theorems for $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{H}}^{\text{rigid}}$ provide a substitute for Tits' deformation theorem in the affine setting.

1.3. As consequences of our approach, we obtain direct, algebraic proofs of the analogues of classical results from p -adic groups: the density theorem and the trace Paley–Wiener theorem. For p -adic groups, proofs of these results are known from [BDK] and [Kaz2] (see also [Da] and [FI]). For affine Hecke algebras with positive parameters, the density theorem and trace Paley–Wiener theorem are studied in [So, Theorem 3.4] by different methods.

We summarize our results in the next theorem.

Theorem 1.3. (1) (Density Theorem) *Suppose that the parameters of the affine Hecke algebra are admissible. Then the trace map $\text{tr} : \tilde{\mathcal{H}} \rightarrow R(\mathcal{H})^*$ is injective.*
 (2) (Trace Paley–Wiener Theorem) *For arbitrary parameters, the image of the trace map $\text{tr} : \tilde{\mathcal{H}} \rightarrow R(\mathcal{H})^*$ is the space $R^*(\mathcal{H})_{\text{good}}$ of good forms (Definition 3.5).*

We would like to point out that the density theorem fails for affine Hecke algebras at some roots of unity but the trace Paley–Wiener theorem always holds. The trace Paley–Wiener theorem for affine Hecke algebras at roots of unity seems to be a new result and will play a role in the modular representations of p -adic groups.

1.4. As a different application, in Section 9, we show how Theorem 1.1 can be used to simplify the arguments of [BM1, BM2] for the preservation of unitarity under the Borel functor for the category of smooth representations with Iwahori fixed vectors of a semisimple p -adic group G . Let I be an Iwahori subgroup and if $H \subset G$ is a closed subgroup containing I , let $\mathcal{H}(H//I)$ denote the Iwahori–Hecke algebra of I -biinvariant compactly supported functions with support in H . The Borel–Casselman correspondence says that the functor $V \mapsto V^I$ is an equivalence between the subcategory of smooth complex G -representations generated by their I -fixed vectors and the category of $\mathcal{H}(G//I)$ -modules. The main result of [BM1, BM2] (see also [BC]) is the following theorem conjectured by Borel.

Theorem 1.4 (Barbasch–Moy preservation of unitarity). *Under the functor $V \mapsto V^I$, the G -representation V is unitary if and only if the $\mathcal{H}(G//I)$ -module V^I is unitary.*

The classical arguments of [BM1, BM2] use two main ingredients. The first is an analogue of Vogan’s signature character [Vo] which expresses the signature of an irreducible hermitian representation in terms of the K -character (K a maximal compact open subgroup) of tempered modules. The second ingredient is a linear independence of the $\mathcal{H}(K_0//I)$ -characters (K_0 hyperspecial) of irreducible tempered representations with “real infinitesimal character” combined with a subtle reduction to real infinitesimal character in [BM2] via endoscopic groups. The rigid quotient provides a simplification of the argument (without the need for the reduction to real infinitesimal character) by: firstly, considering the K -characters for all maximal parahoric subgroups, and secondly, using the fact that, by Theorem 1.1(2), the characters of a basis of the rigid quotient (the basis can be chosen to consist of tempered representations) are linearly independent when restricted to the union of $\mathcal{H}(K//I)$ ’s.

1.5. We give an outline of the paper. In Sections 2 and 3, we establish the notation and give the basic definitions for the affine Hecke algebra (extended by outer automorphisms), the cocenter and $R(\mathcal{H})$. In Section 4, we recall in our setting the main definitions and properties of the induction and restriction functors, as introduced for p -adic groups in [BDK] and studied further in [Da]. In Section 5, we record the main cocenter results from [HN1, HN2] that we will use. In Sections 6 and 7, we define the rigid and elliptic cocenters and the rigid and elliptic quotients of $R(\mathcal{H})$, and prove the main results about the duality between the rigid cocenter and rigid quotient, in particular, Theorem 1.1. Some of these results are proven under the assumption that (PDT), the density theorem for parabolic subalgebras, holds. In Section 8, we prove (by induction) the basis theorem, density theorem, and trace Paley–Wiener theorem, as enumerated in Theorems 1.2 and 1.3. In particular, we now see that the assumption (PDT) can be removed from the previous results. We also obtain sharp bounds for the dimension of the elliptic quotient for arbitrary parameters. Finally, in Section 9, we give our application to the preservation of unitarity argument.

Notation. If \mathcal{B} is a complex associative algebra, denote by $\mathcal{B}\text{-mod}$ the category of \mathcal{B} -modules, by $R(\mathcal{B})$ the complexification of the Grothendieck group of finite-dimensional modules, and by $\text{Irr } \mathcal{B}$ the set of isomorphism classes of simple \mathcal{B} -modules.

2. Preliminaries

2.1. Let $\Phi = (X, R, X^\vee, R^\vee, \Pi)$ be a reduced based root datum. In particular,

- (a) X, X^\vee are free abelian groups of finite rank with a perfect pairing $\langle \cdot, \cdot \rangle : X \times X^\vee \rightarrow \mathbb{Z}$;
- (b) $R \subset X, R^\vee \subset X^\vee$ are the roots and coroots, respectively, in bijection $\alpha \leftrightarrow \alpha^\vee$;
- (c) $\Pi \subset R$ is the set of simple roots.

Let R^+ denote the positive roots defined by Π , and $\Pi^- = -\Pi^+$; let $R^{\vee,+}, R^{\vee,-}$ denote the corresponding positive and negative coroots. For every $\alpha \in R$, let $s_\alpha \in GL(X)$ be the

reflection $s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$. Let $W \subset GL(X)$ be the finite Weyl group generated by $S = \{s_\alpha; \alpha \in \Pi\}$.

2.2. The set of *affine roots* is $R^a = R^\vee \times \mathbb{Z}$. Let \leq be the partial order of R^\vee defined by $\beta^\vee \leq \alpha^\vee$ if $\alpha^\vee - \beta^\vee$ is a nonnegative integer linear combination of $\{\alpha^\vee; \alpha \in \Pi\}$. Set $R_m = \{\gamma \in R; \gamma^\vee \text{ is minimal in } (R^\vee, \leq)\}$. The *simple affine roots* are

$$\Pi^a = \{(\alpha, 0); \alpha \in \Pi\} \cup \{(\gamma^\vee, 1); \gamma \in R_m\}. \quad (2.1)$$

Define the *extended affine Weyl group* by

$$\tilde{W} = X \rtimes W. \quad (2.2)$$

We write a typical element in \tilde{W} as $t_x w$, $x \in X$, $w \in W$. Multiplication in \tilde{W} is then $(t_x w) \cdot (t_{x'} w') = t_{x+w(x')} w w'$. The group \tilde{W} acts on X by $(t_x w) \cdot y = x + w(y)$.

Define also the *affine Weyl group* by

$$W^a = Q \rtimes W, \quad (2.3)$$

where $Q \subset X$ is the root lattice, i.e., the \mathbb{Z} -span of R . The group W^a is an infinite Coxeter group generated by $S^a = S \cup \{t_{-\gamma} s_\gamma; \gamma \in R_m\}$. Define positive and negative affine roots as follows:

$$\begin{aligned} R^{a,+} &= (R^\vee \times \mathbb{Z}_{>0}) \cup (R^{\vee,+} \times \{0\}), \\ R^{a,-} &= (R^\vee \times \mathbb{Z}_{<0}) \cup (R^{\vee,-} \times \{0\}), \end{aligned} \quad (2.4)$$

and the length function $\ell : \tilde{W} \rightarrow \mathbb{Z}_{\geq 0}$ by

$$\ell(w) = \#\{\alpha_a \in R^{a,+}; w\alpha_a \in R^{a,-}\}, \quad w \in \tilde{W}. \quad (2.5)$$

Set $\Omega = \{w \in \tilde{W}; \ell(w) = 0\}$. Then $\tilde{W} = W^a \rtimes \Omega$ and $\Omega \cong X/Q$.

2.3. Fix a set $\mathfrak{q} = \{\mathfrak{q}(s); s \in S^a\}$ of invertible, commuting indeterminates such that $\mathfrak{q}(ws w^{-1}) = \mathfrak{q}(s)$ for all $w \in \tilde{W}$ with $ws w^{-1} \in S^a$, and let $\Lambda = \mathbb{C}[\mathfrak{q}(s)^{\pm 1}; s \in S^a]$.

Definition 2.1 (Iwahori–Matsumoto presentation). The *affine Hecke algebra* $\mathcal{H} = \mathcal{H}(\Phi, \mathfrak{q})$ is the Λ -algebra generated by $\{T_w; w \in \tilde{W}\}$ subject to the relations:

- (1) $T_w \cdot T_{w'} = T_{ww'}$, if $\ell(ww') = \ell(w) + \ell(w')$;
- (2) $(T_s + 1)(T_s - \mathfrak{q}(s)^2) = 0$, $s \in S^a$.

The algebra $\mathcal{H}(\Phi, \mathfrak{q})$ admits a second presentation, due to Bernstein and Lusztig, which we recall next. If $w \in W^a$ has a reduced expression $w = s_1 \cdots s_k$, $s_i \in S^a$, set $\mathfrak{q}(w) = \prod \mathfrak{q}(s_i)$. Extend this further to \tilde{W} by setting $\mathfrak{q}(u) = 1$ for all $u \in \Omega$.

Define $X_+ = \{x \in X; \langle x, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Pi\}$. If $x \in X$, write $x = x_1 - x_2$ with $x_1, x_2 \in X_+$. Define

$$\theta_x = \mathfrak{q}(t_{x_1})^{-1} \mathfrak{q}(t_{x_2}) T_{t_{x_1}} T_{t_{x_2}}^{-1}. \quad (2.6)$$

Then $\{T_w \theta_x; w \in W, x \in X\}$ forms a Λ -basis of \mathcal{H} , and we have the relations [Lu1, 3.3(b), Lemma 3.4, Propositions 3.6, 3.7]:

$$\begin{aligned} \theta_x \cdot \theta_{x'} &= \theta_{x+x'} \quad \text{for all } x, x' \in X; \quad \theta_0 = 1; & (2.7) \\ \theta_x T_s - T_s \theta_{s(x)} &= \begin{cases} (q(s)^2 - 1) \frac{\theta_x - \theta_{s(x)}}{1 - \theta_{-\alpha}} & \text{if } \alpha^\vee \notin 2X^\vee, \\ ((q(s)^2 - 1) + \theta_{-\alpha}(q(s)q(\tilde{s}) - q(s)q(\tilde{s})^{-1})) \frac{\theta_x - \theta_{s(x)}}{1 - \theta_{-2\alpha}} & \text{if } \alpha^\vee \in 2X^\vee, \end{cases} & (2.8) \end{aligned}$$

for $s = s_\alpha \in S$ and $x \in X$. Here \tilde{s} is defined as follows. Let $S(\alpha) \subset S^a$ be the connected component of the Coxeter graph containing s . When $\alpha^\vee \in 2X^\vee$, $S(\alpha)$ must be of affine type $\tilde{C}_l, l \geq 1$. Let \tilde{s} be the image of s under the nontrivial graph automorphism of \tilde{C}_l .

Denote $\mathcal{A} = \Lambda[\theta_x; x \in X]$, an abelian subalgebra of \mathcal{H} . The center of \mathcal{H} is $\mathcal{Z} = \mathcal{A}^W$ [Lu1, Proposition 3.11], [OS1, §2.1.3]. In particular, \mathcal{H} is finite over its center and the simple \mathcal{H} -modules are finite-dimensional.

The central characters are identified with elements of $T/W \times \text{Spec } \Lambda$, where $T = \text{Hom}_{\mathbb{Z}}[X, \mathbb{C}^\times]$.

Remark 2.2. When q is specialized to a set $\{q_s; s \in S^a\}$ of nonzero complex numbers (such that $q_{ws w^{-1}} = q_s$ for all $w \in \tilde{W}$ with $ws w^{-1} \in S^a$), we say that \mathcal{H} has arbitrary parameters.

2.4. Let $J \subset \Pi$ be given, and set $J^\vee = \{\alpha^\vee; \alpha \in J\}$. Let $R_J = R \cap \mathbb{Q}J$ and $R_J^\vee = \{\alpha^\vee; \alpha \in R_J\}$. Let $W_J \subset W$ be the parabolic subgroup defined by the reflections in J . Denote $\tilde{W}_J = X \rtimes W_J$, and let q_J be the restriction of q to \tilde{W}_J . Consider the based root datum $\Phi_J = (X, R_J, X^\vee, R_J^\vee, J)$ and the affine Hecke algebra $\mathcal{H}_J = \mathcal{H}(\Phi_J, q_J)$. This algebra can be identified with the subalgebra of $\mathcal{H}(\Phi, q)$ generated by $T_w, w \in W_J$, and $\theta_x, x \in X$. One calls it a *parabolic subalgebra* of \mathcal{H} .

Define the induction functor

$$i_J = \mathcal{H} \otimes_{\mathcal{H}_J} - : \mathcal{H}_J\text{-mod} \rightarrow \mathcal{H}\text{-mod}. \tag{2.9}$$

To define the “semisimple part” of \mathcal{H} , one introduces

$$X_J = X/X \cap (J^\vee)^\perp \quad \text{and} \quad X_J^\vee = X^\vee \cap \mathbb{Q}J^\vee, \tag{2.10}$$

where $(J^\vee)^\perp = \{x \in X; \langle x, \alpha^\vee \rangle = 0 \text{ for all } \alpha \in J\}$. Consider the root datum $\Phi_J^{\text{ss}} = (X_J, R_J, X_J^\vee, R_J^\vee, J)$ and the affine Hecke algebra $\mathcal{H}_J^{\text{ss}} = \mathcal{H}(\Phi_J^{\text{ss}}, q_J)$. For every

$$t \in T^J = \text{Hom}_{\mathbb{Z}}(X/X \cap \mathbb{Q}J, \mathbb{C}^\times), \tag{2.11}$$

let $\chi_t : \mathcal{H}_J \rightarrow \mathcal{H}_J^{\text{ss}}$ be the algebra homomorphism [OS1]

$$\chi_t(\theta_x T_w) = t(x) \theta_{x_J} T_w, \quad x \in X, w \in W, \tag{2.12}$$

where x_J is the image of x in X_J . For every $\sigma \in \mathcal{H}_J^{\text{ss}}\text{-mod}$ and every $t \in T^J$, one can therefore construct the parabolically induced module

$$X(J, \sigma, t) = i_J(\sigma \circ \chi_t). \tag{2.13}$$

3. Clifford theory for $\mathcal{H} \rtimes \langle \delta \rangle$

In this section, we consider the affine Hecke algebra \mathcal{H} together with an automorphism δ and its Clifford theory.

Even if one is only interested in the representation theory of \mathcal{H} , automorphisms of its parabolic subalgebras appear in the study of induction and restriction functors (see also §4).

3.1. Suppose δ is an automorphism of Φ of finite order d . Then δ induces an automorphism of \tilde{W} with $\delta(S^a) = S^a$. If the indeterminates q satisfy $q(\delta(w)) = q(w)$ for all $w \in \tilde{W}$, we can define an extension of the affine Hecke algebra \mathcal{H} by $\Gamma = \langle \delta \rangle$:

$$\mathcal{H}' = \mathcal{H} \rtimes \Gamma. \tag{3.1}$$

Set $\tilde{W}' = \tilde{W} \rtimes \Gamma$ and $W' = W \rtimes \Gamma$. The center of \mathcal{H}' is $\mathcal{A}^{W'}$, so the central characters are parameterized by points in $T/W' \times \text{Spec } \Lambda$.

If $J \subset \Pi$, set $\Gamma_J = \{\delta^i \in \Gamma; \delta^i(R_J) = R_J\}$. The parabolic subalgebra is then $\mathcal{H}'_J = \mathcal{H}_J \rtimes \Gamma_J$. We denote the induction functor again by $i_J : \mathcal{H}'_J\text{-mod} \rightarrow \mathcal{H}'\text{-mod}$.

If $K \subset S^a$, set $\Gamma_K = \{\delta^i \in \Gamma; \delta^i(K) = K\}$. We denote by W_K the subgroup generated by the reflections in K , and set $W'_K = W_K \rtimes \Gamma_K$. If W_K is finite, then we call it a *parahoric subgroup*.

3.2. We define the δ -commutators and cocenters. This section is analogous to [CH, Section 3.1], where similar notions in the setting of the graded affine Hecke algebra were considered.

Definition 3.1. If $h, h' \in \mathcal{H}$ and $i \in \mathbb{Z}$, define the δ^i -commutator of h and h' by $[h, h']_{\delta^i} = hh' - h'\delta^i(h)$. Let $[\mathcal{H}, \mathcal{H}]_{\delta^i}$ be the submodule of \mathcal{H} generated by all δ^i -commutators.

Denote by $\tilde{\mathcal{H}}^{[i]}$ the quotient of $\mathcal{H}/[\mathcal{H}, \mathcal{H}]_{\delta^i}$ by the image of $1 - \delta$. The following result was proved in [CH] for extended graded Hecke algebras, but the proof applies to any associative algebra extended by a finite cyclic group.

Proposition 3.2 ([CH, Proposition 3.1.1]). *Set $\tilde{\mathcal{H}}' = \mathcal{H}'/([\mathcal{H}', \mathcal{H}']$. Then:*

- (1) $\tilde{\mathcal{H}}' = \bigoplus_{i=0}^{d-1} \mathcal{H}\delta^i / ([\mathcal{H}', \mathcal{H}'] \cap \mathcal{H}\delta^i)$.
- (2) *The map $h \mapsto h\delta^i$ induces a linear isomorphism $\tilde{\mathcal{H}}'^{[i]} \rightarrow \mathcal{H}\delta^i / ([\mathcal{H}', \mathcal{H}'] \cap \mathcal{H}\delta^i)$.*

3.3. We discuss Clifford theory for \mathcal{H}' . This is standard and analogous to the graded affine Hecke algebra case from [CH, Section 3.2], and the proofs are identical.

Let $\Gamma = \langle \delta \rangle$. If (π, M) is a finite-dimensional \mathcal{H} -module, let $({}^{\delta^i}\pi, {}^{\delta^i}M)$ denote the \mathcal{H} -module with the action ${}^{\delta^i}\pi(h)m = \pi(\delta^{-i}(h))m$ for all $m \in M$ and $h \in \mathcal{H}$. If M is a simple module, define the *inertia group* $\Gamma_M = \{\delta^i; M \cong {}^{\delta^i}M\}$.

Fix a family of isomorphisms $\phi_{\delta^i} : M \rightarrow {}^{\delta^{-i}}M$ of \mathcal{H} -modules for each $\delta^i \in \Gamma_M$ such that $\phi_{\delta^{ki}} = \phi_{\delta^i}^k$. This is possible since Γ_M is cyclic.

If U is an irreducible Γ_M -module, there is an action of $\mathcal{H} \rtimes \Gamma_M$ on $M \otimes U$:

$$(h\delta^i)(m \otimes u) = h\phi_{\delta^i}(m) \otimes \delta^i u. \tag{3.2}$$

One can form the induced \mathcal{H}' -module $M \rtimes U = \text{Ind}_{\mathcal{H} \rtimes \Gamma_M}^{\mathcal{H}' \rtimes \Gamma}(X \otimes U)$.

Theorem 3.3 (cf. [RR, Appendix A]).

- (1) If M is an irreducible \mathcal{H} -module and U an irreducible Γ_M -module, the induced \mathcal{H}' -module $M \rtimes U$ is irreducible.
- (2) Every irreducible \mathcal{H}' -module is isomorphic to an $M \rtimes U$.
- (3) If $M \rtimes U \cong M' \rtimes U'$, then M, M' are Γ -conjugate \mathcal{H} -modules, and $U \cong U'$ as Γ_M -modules.

For every $\delta' \in \Gamma$, set $\text{lrr}^{\delta'} \mathcal{H} = \{M \in \text{lrr} \mathcal{H}; \delta' \in \Gamma_M\}$ and let $R^{\delta'}(\mathcal{H})$ denote the \mathbb{Z} -linear span of $\text{lrr}^{\delta'} \mathcal{H}$. If $(\pi, M) \in \text{lrr}^{\delta'} \mathcal{H}$, let $\phi_{\delta'} \in \text{End}_{\mathbb{C}}(M)$ be the intertwiner as before. The twisted trace is

$$\text{tr}^{\delta'}(\pi) : \mathcal{H} \rightarrow \mathbb{C}, \quad \text{tr}^{\delta'}(\pi)(h) = \text{tr}(\pi(h) \circ \phi_{\delta'}).$$

Let also $\text{tr}(\cdot, \cdot) : \mathcal{H}' \times R(\mathcal{H}') \rightarrow \mathbb{C}$ be the trace pairing, i.e., $\text{tr}(h, \pi) = \text{tr} \pi(h)$ for $h \in \mathcal{H}'$ and $\pi \in R(\mathcal{H}')$.

Lemma 3.4 (cf. [CH, Lemma 3.2.1]). Let $M \rtimes U$ be an irreducible \mathcal{H}' -module as in Theorem 3.3. For $h \in \mathcal{H}$ and $\delta' \in \Gamma$,

$$\text{tr}(h\delta', M \rtimes U) = \begin{cases} \delta'(U) \sum_{\gamma \in \Gamma/\Gamma_M} \text{tr}^{\delta'}(M)(\gamma^{-1}(h)) & \text{if } \delta' \in \Gamma_M, \\ 0 & \text{if } \delta' \notin \Gamma_M, \end{cases} \tag{3.3}$$

where $\delta'(U)$ is the root of unity by which δ' acts on U .

Let O be a Γ -orbit on $\text{lrr} \mathcal{H}$. Set $\Gamma_O = \Gamma_M$ for any $M \in O$. This is well-defined since Γ is cyclic. Then for any irreducible Γ_O -module U and $M \in O$, $M \rtimes U = \bigoplus_{M' \in O} M' \otimes U$ is independent of the choice of M . We denote it by $O \rtimes U$. By Theorem 3.3, $\text{lrr} \mathcal{H}' = \{O \rtimes U\}$, where O runs over Γ -orbits on $\text{lrr} \mathcal{H}$, and U runs over isomorphism classes of irreducible representations of Γ_O .

Suppose that $\delta^i \in \Gamma_O$. Let $U_{O,i}$ be the virtual representation of Γ_O whose character is the characteristic function on δ^i . Then

$$\{O \rtimes U_{O,i}; O \in \text{lrr} \mathcal{H}/\Gamma, i \in \mathbb{Z}/d\mathbb{Z} \text{ with } \delta^i \in \Gamma_O\}$$

is a basis of $R(\mathcal{H}')$.

Let $R^{[i]}(\mathcal{H}')$ be the subspace of $R(\mathcal{H}')$ spanned by $O \rtimes U_{O,i}$, where O runs over Γ -orbits on $\text{lrr} \mathcal{H}$ with $\delta^i \in \Gamma_O$. Then

$$R(\mathcal{H}') = \bigoplus_{i=0}^{d-1} R^{[i]}(\mathcal{H}'). \tag{3.4}$$

By definition, $R^{[i]}(\mathcal{H}')$ is a vector space with basis $(\text{lrr}^{\delta^i} \mathcal{H})_{\Gamma}$. The map $\text{lrr}^{\delta^i}(\mathcal{H}) \ni M \mapsto M \rtimes U_{\Gamma_M, i}$ induces an isomorphism $R^{\delta^i}(\mathcal{H})_{\Gamma, \mathbb{C}} \rightarrow R^{[i]}(\mathcal{H}')$. Here $R^{\delta^i}(\mathcal{H})_{\Gamma, \mathbb{C}}$ is the Γ -coinvariants of $R^{\delta^i}(\mathcal{H})_{\mathbb{C}}$.

By Lemma 3.4, for $0 \leq i, j < d$ with $i \neq j$, $\text{tr}(\mathcal{H}\delta^i, O \rtimes U_{O,j}) = 0$.

3.4. Define the trace linear map

$$\mathrm{tr} : \bar{\mathcal{H}}' \rightarrow R(\mathcal{H}')^*, \quad h \mapsto (f_h : R(\mathcal{H}') \rightarrow \mathbb{C}, f_h(\pi) = \mathrm{tr} \pi(h)). \quad (3.5)$$

This map is compatible with the decompositions from Proposition 3.2 and (3.4) as follows. Let $R_\delta^*(\mathcal{H}) = \mathrm{Hom}_{\mathbb{C}}(R^\delta(\mathcal{H}), \mathbb{C})$. The twisted trace map

$$\mathrm{tr}_\delta : \mathcal{H} \rightarrow R_\delta^*(\mathcal{H}), \quad h \mapsto (f_h^\delta : R^\delta(\mathcal{H}) \rightarrow \mathbb{C}, f_h^\delta(\pi) = \mathrm{tr}^\delta(\pi)), \quad (3.6)$$

descends to a linear map

$$\mathrm{tr}^\delta : \bar{\mathcal{H}}_\delta = \mathcal{H}/[\mathcal{H}, \mathcal{H}]_\delta \rightarrow R_\delta^*(\mathcal{H}). \quad (3.7)$$

Definition 3.5. A form $f \in R_\delta^*(\mathcal{H})$ is called a *trace form* if $f \in \mathrm{im} \mathrm{tr}^\delta$. Denote the subspace of trace forms by $R_\delta^*(\mathcal{H})_{\mathrm{tr}}$.

A form $f \in R_\delta^*(\mathcal{H})$ is called *good* if for every $J \subset I$ such that $\delta(J) = J$, and every $\sigma \in \mathrm{lrr}^\delta(\mathcal{H}_J^{\mathrm{SS}})$, the function $t \mapsto f(X(J, \sigma, t))$ is a regular function on $(T^J)^\delta$. Denote the subspace of good forms by $R_\delta^*(\mathcal{H})_{\mathrm{good}}$.

It is clear that $R_\delta^*(\mathcal{H})_{\mathrm{tr}} \subset R_\delta^*(\mathcal{H})_{\mathrm{good}}$.

4. Induction and restriction

In this section, we collect some known facts about induction and restriction maps.

4.1. If $K \subset J$ are given subsets of Π , we denote by $i_K^J : R(\mathcal{H}_K) \rightarrow R(\mathcal{H}_J)$ and by $r_K^J : R(\mathcal{H}_J) \rightarrow R(\mathcal{H}_K)$ the maps on complexified Grothendieck groups induced by the induction and restriction functors respectively. When J and K are δ -invariant, we also have the corresponding maps, denoted again by i_K^J and r_K^J , between $R^\delta(\mathcal{H}_K)$ and $R^\delta(\mathcal{H}_J)$.

It is obvious that

Lemma 4.1. For $L \subset K \subset J \subset \Pi$, $i_L^J = i_K^J \circ i_L^K$ and $r_L^J = r_L^K \circ r_K^J$.

The following ‘‘Mackey decomposition’’ is the analogue of [BDK, Lemma 5.4] and [Fl, Lemma 2.1] (see also [Mi, Theorem 1]).

Lemma 4.2. Let δ be an automorphism of Φ of finite order. If $K = \delta(K)$ and $J = \delta(J)$, then

$$r_K \circ i_J = \sum_{w \in {}^K W^J \cap W^\delta} i_{K_w}^K \circ w \circ r_{J_w}^J \quad (4.1)$$

as maps from $R^\delta(\mathcal{H}_J)$ to $R^\delta(\mathcal{H}_K)$, where ${}^K W^J$ is the set of representatives of minimal length for $W_K \backslash W / W_J$, $K_w = K \cap w J w^{-1}$ and $J_w = J \cap w^{-1} K w$.

4.2. We recall certain elements defined in [Lu1, Section 5.1]. Let \mathcal{F} be the quotient field of \mathcal{Z} . For every $\alpha \in \Pi$, set [Lu1, Section 3.8]

$$\mathcal{G}(\alpha) = \begin{cases} \frac{q(s_\alpha)^2\theta_\alpha - 1}{\theta_\alpha - 1} & \text{if } \alpha^\vee \notin 2X^\vee, \\ \frac{(q(s)\tilde{q}(\tilde{s})\theta_\alpha - 1)(q(s)\tilde{q}(\tilde{s})^{-1})\theta_\alpha + 1}{\theta_{2\alpha} - 1} & \text{if } \alpha^\vee \in 2X^\vee. \end{cases} \tag{4.2}$$

Set $\mathcal{A}_{\mathcal{F}} = \mathcal{A} \otimes_{\mathcal{Z}} \mathcal{F}$. This is naturally isomorphic to the quotient field of \mathcal{A} [Lu1, §3.12(a)]. Therefore, $\mathcal{G}(\alpha) \in \mathcal{A}_{\mathcal{F}}$. Define the following elements in $\mathcal{H}_{\mathcal{F}} = \mathcal{H} \otimes_{\mathcal{Z}} \mathcal{F}$:

$$\tau^\alpha = (T_{s_\alpha} + 1)\mathcal{G}(\alpha)^{-1} - 1. \tag{4.3}$$

By [Lu1, Proposition 5.2], the assignment $s_\alpha \mapsto \tau^\alpha$ for $\alpha \in \Pi$ extends to a unique group homomorphism $W \rightarrow \mathcal{H}_{\mathcal{F}}^\times$. Denote by τ_w the image of $w \in W$. Moreover,

$$f \tau_w = \tau_w w^{-1}(f) \quad \text{for all } w \in W, f \in \mathcal{A}_{\mathcal{F}}. \tag{4.4}$$

The following lemma is also well known—see for example [BM3, Section 1.6], where a similar statement was verified in the context of graded Hecke algebras.

Lemma 4.3. *Suppose $K, J \subset \Pi$ are such that $K = w(J)$, where $w \in {}^K W^J$. Then $T_{s_\alpha} \tau_{w^{-1}} = \tau_{w^{-1}} T_{s_\beta}$ for every $\alpha \in J$, where $\beta = w(\alpha) \in K$.*

Proof. We calculate:

$$\begin{aligned} T_{s_\alpha} \tau_{w^{-1}} &= ((\tau^\alpha + 1)\mathcal{G}(\alpha) - 1)\tau_{w^{-1}} \\ &= (\tau^\alpha + 1)\tau_{w^{-1}}\mathcal{G}(\beta) - \tau_{w^{-1}} \quad \text{by (4.4)} \\ &= (\tau_{w^{-1}}\tau^\beta + \tau_{w^{-1}})\mathcal{G}(\beta) - \tau_{w^{-1}} \quad \text{since } \tau \text{ is a homomorphism} \\ &= \tau_{w^{-1}}T_{s_\beta}. \end{aligned} \quad \square$$

Lemma 4.4. *Suppose that $K, J \subset \Pi$ are δ -stable subsets. Let ${}^K W^J(\delta)$ denote the subset of δ -fixed elements in ${}^K W^J$. If $w \in {}^K W^J(\delta)$ and $K = w(J)$, then $i_K \circ w = i_J$ in $R^\delta(\mathcal{H})$.*

Proof. Let $(\sigma \circ \chi_t, V)$ be an irreducible module in \mathcal{H}_J -mod, let $t \in T^J$, and let $(\sigma^w \circ \chi_{w(t)}, V^w = V) \in H_K$ -mod be its twist by w . We need to prove that in $R^\delta(\mathcal{H})$, $i_K(\sigma^w \circ \chi_{w(t)}) = i_J(\sigma \circ \chi_t)$. Since the characters of both sides are regular functions in t , it is sufficient to prove this for t generic. Define

$$\phi_w(\sigma, t) : \mathcal{H} \otimes_{\mathcal{H}_J} V \rightarrow \mathcal{H} \otimes_{\mathcal{H}_K} V^w, \quad h \otimes v \mapsto v\tau_{w^{-1}} \otimes v. \tag{4.5}$$

This is a well-defined intertwining operator. To see this, notice first that since we are assuming t is generic, $\tau_{w^{-1}}$ evaluated at $\sigma \otimes \chi_t$ has no poles. Secondly, by (4.4) and Lemma 4.3, if $h' \in \mathcal{H}_J$, then $h'\tau_{w^{-1}} = \tau_{w^{-1}}w(h')$, and thus $\phi_w(\sigma, t)(hh' \otimes v) = \phi_w(\sigma, t)(h \otimes (\sigma(h') \circ \chi_t)v)$. Since the action of \mathcal{H} on the induced modules is by left multiplication, ϕ_w is indeed an intertwiner. Finally, for generic t , $\phi_w(\sigma, t)$ is invertible, and the inverse is $\phi_{w^{-1}}(\sigma^w, w(t))$. \square

4.3. For every $J \subset \Pi$, let $\tilde{i}_J : \mathcal{H}'_J \rightarrow \mathcal{H}'$ denote the inclusion. Define $\tilde{r}_J : \mathcal{H}' \rightarrow \mathcal{H}'_J$ as follows. Given $h \in \mathcal{H}'$, let $\psi_h : \mathcal{H}' \rightarrow \mathcal{H}'$ be the linear map given by left multiplication by h . This can be viewed as a right \mathcal{H}'_J -module morphism. Since \mathcal{H}' is free of finite rank as a right \mathcal{H}'_J -module, one can consider $\text{tr } \psi_h \in \mathcal{H}'_J$. Set $\tilde{r}_J(h) = \text{tr } \psi_h$.

As before, for every $K \subset J$, we may also define \tilde{i}_K^J and \tilde{r}_K^J .

Suppose $\delta(J) = J$. Define $\tilde{\mathcal{H}}_{J,\delta} = \mathcal{H}_J / [\mathcal{H}_J, \mathcal{H}_J]_\delta$. Then \tilde{i}_J gives rise to a well-defined map (not necessarily injective)

$$\tilde{i}_J : \tilde{\mathcal{H}}_{J,\delta} \rightarrow \tilde{\mathcal{H}}_\delta.$$

Since $\tilde{r}_J[\mathcal{H}, \mathcal{H}]_\delta \subset [\mathcal{H}_J, \mathcal{H}_J]_\delta$, \tilde{r}_J descends to a well-defined map

$$\bar{r}_J : \tilde{\mathcal{H}}_\delta \rightarrow \tilde{\mathcal{H}}_{J,\delta}$$

sending $h + [\mathcal{H}, \mathcal{H}]_\delta$ to the image of $\tilde{r}_J(h\delta)\delta^{-1}$ in $\mathcal{H}_{J,\delta}$. Define the similar notions \bar{i}_K^J and \bar{r}_K^J .

Lemma 4.5 (cf. [CH, Lemma 4.5.1]). *The maps \bar{i}_J and \bar{r}_J are $\text{tr}(\cdot, \cdot)$ -adjoint to r_J and i_J , respectively.*

4.4. We introduce the following hypothesis:

(PDT) The density theorem holds for every proper parabolic subalgebra \mathcal{H}_J , $\delta(J) = J$.

The following properties are dual to those of i_J and r_J .

Lemma 4.6. *Suppose (PDT) holds. Then:*

- (i) For $L \subset K \subset J$, $\bar{i}_L^J = \bar{i}_K^J \circ \bar{i}_L^K$ and $\bar{r}_L^J = \bar{r}_L^K \circ \bar{r}_K^J$.
- (ii) If $K = \delta(K)$ and $J = \delta(J)$, then

$$\bar{r}_J \circ \bar{i}_K = \sum_{w \in {}^K W^J(\delta)} \bar{i}_{J_w}^J \circ w^{-1} \circ \bar{r}_{K_w}^K \tag{4.6}$$

as maps from $\tilde{\mathcal{H}}_{K,\delta} \rightarrow \tilde{\mathcal{H}}_{J,\delta}$, where $K_w = K \cap wJw^{-1}$ and $J_w = J \cap w^{-1}Kw$.

- (iii) If $w \in {}^K W^J(\delta)$ and $K = w(J)$, then

$$w^{-1} \circ \bar{r}_K = \bar{r}_J. \tag{4.7}$$

Remark 4.7. (1) We will see in Section 8 that the (PDT) hypothesis is not necessary.

(2) In general, if J, K , and w are as in Lemma 4.6(iii), then $\bar{i}_K \circ w \neq \bar{i}_J$ in $\tilde{\mathcal{H}}_\delta$. For example, suppose $\delta = 1$ and $J = K = \emptyset$, and take $\theta_x \in \tilde{\mathcal{H}}_\emptyset$. Then, for $w \in W$, we have $\theta_x \neq \theta_{w(x)}$ in $\tilde{\mathcal{H}}$ in general. To see this, one can use the (one-dimensional) Steinberg module St of \mathcal{H} , and the fact that $\text{tr}(\theta_x, \text{St}) \neq \text{tr}(\theta_{w(x)}, \text{St})$ in general.

4.5. To end this section, we recall the A -operators. For any $J = \delta(J) \subset \Pi$, set $N_J = \{z \in {}^J W^J; z = \delta(z), J = z(J)\}$.

As in [BDK, Section 5.5] (see also [Da, Section 2]), fix an order of the subsets K of every given size, and define

$$A_\ell = \prod_{K=\delta(K), |K|=|\Pi|-\ell} (i_K \circ r_K - |N_K|), \quad A = A_{|\Pi|} \circ \dots \circ A_1 : R^\delta(\mathcal{H}) \rightarrow R^\delta(\mathcal{H}). \tag{4.8}$$

Let

$$\bar{A}^\ell = \prod_{\delta(K)=K, |K|=|\Pi|-\ell} (\bar{i}_K \circ \bar{r}_K - |N_K|), \quad \bar{A} = \bar{A}^1 \circ \dots \circ \bar{A}^{|\Pi|}, \tag{4.9}$$

be the adjoint operator.

The following proposition is proved in [BDK, Sections 5.4, 5.5] and [Da, Proposition 2.5(i)].

Proposition 4.8. *We have $A^2 = aA$ for some $a \neq 0$ and*

$$\ker A = \sum_{J=\delta(J) \subsetneq I} i_J(R^\delta(\mathcal{H}_J)).$$

The following proposition is dual to Proposition 4.8 and it follows by adjunction.

Proposition 4.9. *Suppose (PDT) holds.*

- (1) $\bar{A}^2 = a\bar{A}$ for some $a \neq 0$.
- (2) Set $\tilde{\mathcal{H}}_\delta^{\text{ell}} = \bigcap_{J=\delta(J) \subsetneq I} \ker \bar{r}_J$. Then $\tilde{\mathcal{H}}_\delta^{\text{ell}} = \text{im } \bar{A}$.

5. Spanning sets of the cocenter

In this section, we recall the explicit description of the cocenter of \mathcal{H} obtained in [HN1] and [HN2].

5.1. We first recall some properties of the minimal length elements of conjugacy classes of \tilde{W}' .

We follow [HN1]. For $w, w' \in \tilde{W}'$ and $s \in S^a$, we write $w \xrightarrow{s} w'$ if $w' = sws$ and $\ell(w') \leq \ell(w)$. We write $w \rightarrow w'$ if there is a sequence $w = w_0, w_1, \dots, w_n = w'$ of elements in \tilde{W} such that for any $k, w_{k-1} \xrightarrow{s} w_k$ for some $s \in S^a$.

We call $w, w' \in \tilde{W}'$ *elementarily strongly conjugate* if $\ell(w) = \ell(w')$ and there exists $x \in \tilde{W}'$ such that $w' = xwx^{-1}$ and $\ell(xw) = \ell(x) + \ell(w)$ or $\ell(wx^{-1}) = \ell(x) + \ell(w)$. We call w, w' *strongly conjugate* if there is a sequence $w = w_0, w_1, \dots, w_n = w'$ such that for each i, w_{i-1} is elementarily strongly conjugate to w_i , and we write $w \sim w'$ in this case.

The following result is proved in [HN1, Theorem A].

Theorem 5.1. *Let \mathcal{O} be a conjugacy class of \tilde{W}' and \mathcal{O}_{\min} be the set of minimal length elements in \mathcal{O} . Then:*

- (1) For any $w' \in \mathcal{O}$, there exists $w'' \in \mathcal{O}_{\min}$ such that $w' \rightarrow w''$.
- (2) If $w', w'' \in \mathcal{O}_{\min}$, then $w' \sim w''$.

5.2. By definition, if $w \sim w'$, then $T_w \equiv T_{w'} \pmod{[\mathcal{H}', \mathcal{H}']}$. Let \mathcal{O} be a conjugacy class of \tilde{W}' . Let $T_{\mathcal{O}}$ be the image of T_w in $\tilde{\mathcal{H}}'$, where w is a minimal length element in \mathcal{O} . By Theorem 5.1(2), $T_{\mathcal{O}}$ is independent of the choice of w .

Moreover, we have the following result.

Theorem 5.2. *The elements $\{T_{\mathcal{O}}\}$, where \mathcal{O} ranges over all the conjugacy classes of \tilde{W}' , span $\tilde{\mathcal{H}}'$ as an Λ -module.*

The equal parameter case was proved in [HN1, Theorem 5.3]. The general case can be proved in the same way and we omit the details. In [HN1, Theorem 6.7], it is proved in the equal parameter case that $\{T_{\mathcal{O}}\}$ is a basis of $\tilde{\mathcal{H}}'$. The proof there is to use the base ring to reduce the question to the group algebra of \tilde{W}' , and then to prove the density theorem for the group algebra.

In this paper, we will prove that $\{T_{\mathcal{O}}\}$ is a basis of $\tilde{\mathcal{H}}'$ for arbitrary parameters after establishing the trace Paley–Wiener theorem and the density theorem.

5.3. The expression for a minimal length element T_w in \mathcal{O} is usually very complicated. To study the induction and restriction maps on the cocenter, we also need the Bernstein–Lusztig presentation of $\tilde{\mathcal{H}}'$ established in [HN2].

5.4. Let $X_{\mathbb{C}} = X \otimes_{\mathbb{Z}} \mathbb{C}$ and $N = |W'|$. For any $w \in \tilde{W}'$, we have $w^n = t_{\lambda}$ for some $\lambda \in X$. Let $v_w = \lambda/n$ and \bar{v}_w be the unique dominant element in the W_0 -orbit of v_w . Then the map $\tilde{W} \rightarrow X_{\mathbb{C}}, w \mapsto \bar{v}_w$, is constant on the conjugacy classes of \tilde{W}' . For any conjugacy class \mathcal{O} , we set $v_{\mathcal{O}} = \bar{v}_w$ for any $w \in \mathcal{O}$ and call it the *Newton point* of \mathcal{O} . We set

$$J_{\mathcal{O}} = \{\alpha \in \Pi; s_{\alpha}(v_{\mathcal{O}}) = v_{\mathcal{O}}\}. \tag{5.1}$$

We say that the Newton point of \mathcal{O} is *central* if $J_{\mathcal{O}} = \Pi$.

5.5. Let $p : \tilde{W}' = X \rtimes W' \rightarrow W'$ be the projection map. We call an element $w \in W'$ *elliptic* if $(X_{\mathbb{C}})^w \subset (X_{\mathbb{C}})^{W'}$, and an element $w \in \tilde{W}$ *elliptic* if $p(w)$ is elliptic in W' . A conjugacy class \mathcal{O} in \tilde{W}' is called *elliptic* if w is elliptic for some (equivalently any) $w \in \mathcal{O}$. By definition, if \mathcal{O} is elliptic, then $v_{\mathcal{O}} \in X_{\mathbb{C}}^{W'}$. In other words, elliptic conjugacy classes have central Newton point.

5.6. Let \mathfrak{N} be the set of pairs (J, C) , where $J \subset S$, C is an elliptic conjugacy class of $\tilde{W}_J \rtimes \Gamma_J$ and v_w is dominant for some (equivalently any) $w \in C$. For any $(J, C), (J', C')$ in \mathfrak{N} , we write $(J, C) \sim (J', C')$ if $v_w = v_{w'}$ for $w \in C$ and $w' \in C'$ and there exists $x \in W_{J_{v_w}} \rtimes \Gamma_{J_{v_w}}$ such that $x C x^{-1} = C'$.

By [HN2, Lemma 5.8], the map from \mathfrak{N} to the set of conjugacy classes of \tilde{W}' sending (J, C) to the unique conjugacy class \mathcal{O} of \tilde{W} with $C \subset \mathcal{O}$ gives a bijection from \mathfrak{N}/\sim to the set of conjugacy classes of \tilde{W} . We denote this map by τ . The following is proved in [HN2, Theorem B].

Theorem 5.3. *Let $(J, C) \in \mathfrak{N}$ and $\mathcal{O} = \tau(J, C)$. Then*

$$T_{\mathcal{O}} = \bar{i}_J(T_C^J).$$

Here T_C^J is the image of T_y in $\tilde{\mathcal{H}}'_J$ for any minimal length element y of C (with respect to the length function on $\tilde{W}_J \rtimes \Gamma_J$). However, in general, y is not a minimal length element in \mathcal{O} (with respect to the length function on \tilde{W}^J). The expression of $\bar{i}_J(T_C^J)$ involves the Bernstein–Lusztig presentation of \mathcal{H}'_J and \mathcal{H}' .

5.7. As we will see later, the conjugacy classes with central Newton points play a crucial role in this paper. Here we discuss a variation of §5.6 and Theorem 5.3.

Let \mathcal{O} be a conjugacy class of \tilde{W}' and $J = J_{\mathcal{O}}$. We may assume that $\mathcal{O} = \tau(K, C)$. Then it is easy to see that $K \subset J$. Let \mathcal{O}' be the conjugacy class of $\tilde{W}'_J \rtimes \Gamma_J$ such that $\mathcal{O}' = \tau_J(K, C)$. Then $J_{\mathcal{O}'} = J$ and $T_{\mathcal{O}'}^J = \bar{i}_K^J(T_C^K)$. Hence

$$T_{\mathcal{O}} = \bar{i}_K(T_C^K) = \bar{i}_J(T_{\mathcal{O}'}^J).$$

6. Elliptic quotients and rigid quotients

In this section, we discuss two natural quotients for $R^\delta(\mathcal{H})$: the elliptic quotient introduced in [BDK] and the rigid quotient, which we introduce below.

Unless specified otherwise, \mathcal{H} is assumed to have arbitrary parameters.

6.1. The elliptic quotient of $R^\delta(\mathcal{H})$ is defined to be

$$\bar{R}_0^\delta(\mathcal{H}) = R^\delta(\mathcal{H}) / \sum_{J=\delta(J) \subsetneq I} i_J(R^\delta(\mathcal{H}_J)).$$

Now we introduce the rigid quotient.

Definition 6.1. Set

$$R^\delta(\mathcal{H})_{\text{diff-ind}} = \text{span}\{i_J(\sigma) - i_J(\sigma \circ \chi_t); J = \delta(J), \sigma \in R^\delta(\mathcal{H}_J^{\text{ss}}), t \in \text{Hom}_{\mathbb{Z}}(X \cap \mathbb{Q}R / X \cap \mathbb{Q}J, \mathbb{C}^*)^\delta\} \subset R^\delta(\mathcal{H}), \tag{6.1}$$

and define the *rigid quotient* of $R^\delta(\mathcal{H})$ to be

$$\bar{R}^\delta(\mathcal{H})_{\text{rigid}} = R^\delta(\mathcal{H}) / R^\delta(\mathcal{H})_{\text{diff-ind}}. \tag{6.2}$$

Notice the presence of $X \cap \mathbb{Q}R$ in the definition, rather than X . This is to account for the fact that the root datum may not be semisimple.

In the rest of this section, we assume that the root datum Φ for \mathcal{H} is semisimple. We estimate the dimension of the elliptic quotient and rigid quotient of $R^\delta(\mathcal{H})$. By Proposition 4.9, the subspace $\tilde{\mathcal{H}}_\delta^{\text{ell}}$ of $\tilde{\mathcal{H}}$ is the image of \bar{A} if (PDT) holds.

Proposition 6.2. *Suppose that (PDT) holds and the root datum for \mathcal{H} is semisimple. Then the rank of the restriction of the trace map $\text{tr} : \tilde{\mathcal{H}}_\delta^{\text{ell}} \times \bar{R}_0^\delta(\mathcal{H}) \rightarrow \mathbb{C}$ equals $\dim \bar{R}_0^\delta(\mathcal{H})$. In particular, $\dim \bar{R}_0^\delta(\mathcal{H}) \leq \dim \tilde{\mathcal{H}}_\delta^{\text{ell}}$.*

Remark 6.3. In fact, we will see in Section 8 that the (PDT) assumption can be dropped.

Proof of Proposition 6.2. Suppose $\{\pi_1, \dots, \pi_k\}$ is a set in $R^\delta(\mathcal{H})$ such that its image in $\bar{R}_0^\delta(\mathcal{H})$ is linearly independent. Applying the operator A , one obtains a linearly independent set $\{A(\pi_1), \dots, A(\pi_k)\}$ in $R^\delta(\mathcal{H})$. This is because $A(\pi) \equiv a\pi$ in $\bar{R}_0^\delta(\mathcal{H})$, and $a \neq 0$.

Since the characters of simple modules are linearly independent, so are the characters of any linearly independent set in $R^\delta(\mathcal{H})$. Thus there exist h_1, \dots, h_k in $\bar{\mathcal{H}}$ such that the matrix $(\text{tr}(h_i\delta, A(\pi_j)))_{i,j}$ is invertible. By Lemma 4.5, the matrix $(\text{tr}(\bar{A}(h_i)\delta, \pi_j))_{i,j}$ is also invertible, hence $\{\bar{A}(h_i)\}$ is a linearly independent set in $\bar{\mathcal{H}}_\delta$. Now the conclusion follows from Proposition 4.9. \square

6.2. Let $\mathcal{I}^\delta = \{J \subset \Pi; J = \delta(J)\}$. For $J, J' \in \mathcal{I}^\delta$, we write $J \sim_\delta J'$ if there exists $w \in W^\delta \cap J'W^J$ such that $w(J) = J'$. For each \sim_δ -equivalence class in \mathcal{I}^δ , we choose a representative. We denote by $\mathcal{I}_\bullet^\delta \subset \mathcal{I}^\delta$ the set of representatives. Recall $N_J = \{z \in J W^J; z = \delta(z), J = z(J)\}$. Then N_J acts on $R^\delta(\mathcal{H}_J^{\text{ss}})$ and on $\bar{R}_0^\delta(\mathcal{H}_J^{\text{ss}})$.

By Lemmas 4.1 and 4.2, we have

Lemma 6.4. *If $K = \delta(K) \subsetneq \Pi$, then*

$$i_K(R^\delta(\mathcal{H}_K)_{\text{diff-ind}}) \subseteq R^\delta(\mathcal{H})_{\text{diff-ind}} \quad \text{and} \quad r_K(R^\delta(\mathcal{H})_{\text{diff-ind}}) \subseteq R^\delta(\mathcal{H}_K)_{\text{diff-ind}}.$$

The first result is an equality for the dimension of $\bar{R}^\delta(\mathcal{H})_{\text{rigid}}$.

Proposition 6.5. *Suppose that the root datum for \mathcal{H} is semisimple. Then*

$$\dim \bar{R}^\delta(\mathcal{H})_{\text{rigid}} = \sum_{J \in \mathcal{I}_\bullet^\delta} \dim \bar{R}_0^\delta(\mathcal{H}_J^{\text{ss}})^{N_J}.$$

Proof. Let $m = |\Pi|$ and consider first the natural projection

$$p_m : \bar{R}^\delta(\mathcal{H})_{\text{rigid}} \rightarrow \bar{R}_0^\delta(\mathcal{H}).$$

By definition,

$$\ker p_m = \sum_{J=\delta(J) \subsetneq \Pi} [i_J(R^\delta(\mathcal{H}_J))],$$

where $[i_J(R^\delta(\mathcal{H}_J))] = i_J(R^\delta(\mathcal{H}_J))/i_J(R^\delta(\mathcal{H}_J) \cap R^\delta(\mathcal{H})_{\text{diff-ind}})$. By Lemma 4.4, we can replace the right hand side by

$$\ker p_m = \sum_{J \in \mathcal{I}_\bullet^\delta} [i_J(R^\delta(\mathcal{H}_J)^{N_J})].$$

Let $r_J^{\text{ss}} = \chi_{1 \circ r_J} : R^\delta(\mathcal{H}) \rightarrow R^\delta(\mathcal{H}_J) \rightarrow R^\delta(\mathcal{H}_J^{\text{ss}})$. Let J_1, \dots, J_k be the representatives in $\mathcal{I}_\bullet^\delta$ of the maximal δ -stable proper subsets of Π . Then consider

$$p_{m-1} : \ker p_m \rightarrow \bigoplus_{i=1}^k \bar{R}_0(\mathcal{H}_{J_i}^{\text{ss}})^{N_{J_i}}, \quad p_{m-1} = \sum_i r_{J_i}^{\text{ss}}.$$

By Lemma 4.2, this map is well-defined. It is also surjective, given the definition of $\ker p_N$.

Continue in this way and define

$$p_{m-2} : \ker p_{m-1} \rightarrow \bigoplus_{J \in \mathcal{I}_\bullet^\delta, |J|=m-2} \bar{R}_0(\mathcal{H}_J^{ss})^{N_J}, \quad p_{m-2} = \sum_{J \in \mathcal{I}_\bullet^\delta, |J|=m-2} r_J^{ss},$$

etc., until $p_0 : \ker p_1 \rightarrow \bar{R}_0(\mathcal{H}_\emptyset^{ss})^W = \{0\}$.

Therefore, $\dim \bar{R}^\delta(\mathcal{H})_{\text{rigid}} = \sum_{j=0}^N \dim \text{im } p_j = \sum_{J \in \mathcal{I}_\bullet^\delta} \dim \bar{R}_0^\delta(\mathcal{H}_J^{ss})^{N_J}$. □

Definition 6.6. A specialization of the parameter q of \mathcal{H} is called *admissible* if Lusztig’s reduction [Lu1, Theorems 8.6, 9.3] from affine Hecke algebras to graded affine Hecke algebras holds for all the parabolic subalgebras of \mathcal{H} (in particular, including \mathcal{H} itself). For example, this is the case if q is specialized to $q(\alpha) = q^{L(\alpha)}$, where $L(\alpha) \in \mathbb{Z}_{\geq 0}$ and $q \in \mathbb{C}^\times$ has infinite order, or more generally, q ’s order is not small (so that both the numerators and denominators in the first displayed equation in the proof of [Lu1, Lemma 9.5] are nonzero).

6.3. Now we give a precise formula for the dimensions of the elliptic quotient and rigid quotient of $R^\delta(\mathcal{H})$ in terms of δ -conjugacy classes of \tilde{W} .

Let $cl(\tilde{W}, \delta)$ be the set of δ -conjugacy classes of \tilde{W} . Let $cl(\tilde{W}, \delta)_{\text{ell}}$ be the set of elliptic δ -conjugacy classes of \tilde{W} , and let $cl(\tilde{W}, \delta)_0$ be the set of δ -conjugacy classes \mathcal{O} of \tilde{W} such that $v_{\mathcal{O}} = 0$. Then

$$cl(\tilde{W}, \delta)_{\text{ell}} \subset cl(\tilde{W}, \delta)_0 \subset cl(\tilde{W}, \delta).$$

Let $cl(W, \delta)_{\text{ell}} \subset cl(W, \delta)$ denote the set of elliptic δ -conjugacy classes, respectively all δ -conjugacy classes in W .

By [HN2, Lemma 5.8], we have

$$\sum_{J \in \mathcal{I}_\bullet^\delta} |cl(\tilde{W}_J, \delta)_{\text{ell}}/N_J| = |cl(\tilde{W}, \delta)_0|. \tag{6.3}$$

For $s \in T = \text{Hom}_{\mathbb{Z}}(X, \mathbb{C}^*)$, set $W_s = \{w \in W; w \cdot s = s\}$. We call s *isolated* if W_s contains a δ -elliptic element of W . By [OS2, Theorem 3.2], Clifford–Mackey induction gives a natural isomorphism

$$\bar{R}_0^\delta(\tilde{W}) \cong \bigoplus_{s \in T/W' \text{ isolated}} \bar{R}^\delta(W_s). \tag{6.4}$$

Now suppose that we are given a finite group Φ of automorphisms that commutes with δ . The action of N on Φ induces actions on \tilde{W} and on T . We denote by $Z_N(s)$ the stabilizer of $s \in T$. Then the previous isomorphism implies

$$\bar{R}_0^\delta(\tilde{W})^N \cong \bigoplus_{s \in T/W'N \text{ isolated}} \bar{R}^\delta(W_s)^{Z_N(s)}, \tag{6.5}$$

and in particular, considering the dimensions of the spaces involved, one finds

$$|cl(\tilde{W}, \delta)_{\text{ell}}/N| = \sum_s |cl(W_s, \delta)_{\text{ell}}/Z_N(s)|, \tag{6.6}$$

where s ranges over N -orbits of isolated representatives in T/W' .

We have the following result on the dimension of the elliptic quotient.

Proposition 6.7. *Suppose that \mathfrak{q} is specialized to an admissible parameter and the root datum for \mathcal{H} is semisimple. Let N be a group of automorphisms on \tilde{W} that preserves S^a and commutes with δ . Then:*

- (1) $\dim \bar{R}_0^\delta(\mathcal{H})^N = |cl(\tilde{W}, \delta)_{\text{ell}}/N|$.
- (2) $\dim \bar{R}^\delta(\mathcal{H})_{\text{rigid}}^N = |cl(\tilde{W}, \delta)_0/N|$.

Remark 6.8. In particular, the parameterization of irreducible representations of an affine Hecke algebra with (possibly unequal) admissible parameters is the same as the parameterization of irreducible representations of the extended affine Weyl group, once we identify the elliptic quotient for the parabolic subalgebras of affine Hecke algebra and the group algebra.

Proof of Proposition 6.7. Recall that the complex torus T has a polar decomposition $T = T_c T_u$, where $T_c = \text{Hom}_{\mathbb{Z}}(X, S^1)$ is the compact part of the torus T and $T_u = \text{Hom}_{\mathbb{Z}}(X, \mathbb{R}_{>0})$. For every $s \in T_c/W'$, let $R^\delta(\mathcal{H})_s$ denote the span of the irreducible δ -stable \mathcal{H} -modules whose central characters have compact part s . Clearly, we have the decompositions

$$R^\delta(\mathcal{H}) = \bigoplus_{s \in T_c/W'} R^\delta(\mathcal{H})_s, \quad R^\delta(\mathcal{H})^N = \bigoplus_{s \in T_c/W'N} R^\delta(\mathcal{H})_s^{Z_N(s)}.$$

Let $\bar{R}_0^\delta(\mathcal{H})_s$ denote the image of $R^\delta(\mathcal{H})$ in $\bar{R}_0^\delta(\mathcal{H})$. Then, since the action of N preserves $R_{\text{Ind}}^\delta(\mathcal{H})$, we have

$$\bar{R}_0^\delta(\mathcal{H})^N = \bigoplus_{s \in T_c/W'N} \bar{R}_0^\delta(\mathcal{H})_s^{Z_N(s)}.$$

If s is not isolated, then $\bar{R}_0^\delta(\mathcal{H})_s = 0$, so the decomposition becomes

$$\bar{R}_0^\delta(\mathcal{H})^N = \bigoplus_{s \in T_c/W'N \text{ isolated}} \bar{R}_0^\delta(\mathcal{H})_s^{Z_N(s)}. \tag{6.7}$$

Let \mathbb{H}_s be the graded affine Hecke algebra constructed from \mathcal{H} and the W -orbit of s in [Lu1, Section 8] (see also [BM2, Section 3]). When s is isolated, the root system corresponding to \mathbb{H}_s is semisimple. By [Lu1, Theorems 8.3 and 9.2] and the fact that parabolic induction commutes with the reduction to graded Hecke algebras (e.g., [BM2, Theorem 6.2]), it follows that

$$\bar{R}_0^\delta(\mathcal{H})_s \cong \bar{R}_0^\delta(\mathbb{H}_s) \quad \text{and} \quad \bar{R}_0^\delta(\mathcal{H})_s^{Z_N(s)} \cong \bar{R}_0^\delta(\mathbb{H}_s)^{Z_N(s)}. \tag{6.8}$$

By [CH, Theorems B and C] applied to the semisimple graded Hecke algebra \mathbb{H}_s , there exists a perfect pairing between the finite-dimensional spaces

$$\text{tr} : \bar{R}_0^\delta(\mathbb{H}_s) \times (\bar{\mathbb{H}}_s)_{\delta,0} \rightarrow \mathbb{C}. \tag{6.9}$$

This implies that the subspace $\bar{R}_0^\delta(\mathbb{H}_s)^{Z_N(s)}$ is in perfect duality with the space of $Z_N(s)$ -coinvariants $((\bar{\mathbb{H}}_s)_{\delta,0})_{Z_N(s)}$. By [CH, Theorem 7.2.1], $(\bar{\mathbb{H}}_s)_{\delta,0}$ has a basis $\{w_C; C \in cl(W_s, \delta)_{\text{ell}}\}$, where w_C is a representative of C . This means that

$$\dim \bar{R}_0^\delta(\mathbb{H}_s)^{Z_N(s)} = \dim ((\bar{\mathbb{H}}_s)_{\delta,0})_{Z_N(s)} = |cl(W_s, \delta)_{\text{ell}}/Z_N(s)|. \tag{6.10}$$

Then (6.8) gives $\dim \bar{R}_0^\delta(\mathcal{H})_s^{Z_N(s)} = |cl(W_s, \delta)_{\text{ell}}/Z_N(s)|$, and therefore part (1) of the proposition follows from (6.7) and (6.6).

Part (2) follows from (1) and Propositions 6.5 and 6.6. □

6.4. In fact, the proof of Proposition 6.5 gives a section of $\bar{R}^\delta(\mathcal{H})_{\text{rigid}} \rightarrow R^\delta(\mathcal{H})$. To define it, recall the [BDK]-map defined in (4.8),

$$A : R^\delta(\mathcal{H}) \rightarrow R^\delta(\mathcal{H}),$$

which by Proposition 4.8 has

$$\ker A = R_1^\delta(\mathcal{H}), \quad \text{and so} \quad \text{im } A \cong \bar{R}_0^\delta(\mathcal{H}). \tag{6.11}$$

Let $A_J : R^\delta(\mathcal{H}_J) \rightarrow R^\delta(\mathcal{H}_J)$ be the similar maps for the parabolic subalgebras \mathcal{H}_J . As a consequence of the proof of Proposition 6.5, we have

Corollary 6.9. *If the root datum for \mathcal{H} is semisimple, a section of $\bar{R}^\delta(\mathcal{H})_{\text{rigid}}$ is given by*

$$\bigoplus_{J \in \mathcal{I}_\bullet^\delta} i_J \text{im}(A_J : R(\mathcal{H}_J^{\text{ss}}) \rightarrow R(\mathcal{H}_J^{\text{ss}}))^{N_J}.$$

7. Duality between $\tilde{\mathcal{H}}$ and $R(\mathcal{H})$

Our strategy to prove the main results is to use traces of finite-dimensional \mathcal{H} -modules to separate the elements $\{T_\mathcal{O}\}$ that span the cocenter $\tilde{\mathcal{H}}$.

7.1. Recall that $R_\delta^*(\mathcal{H}) = \text{Hom}_{\mathbb{C}}(R^\delta(\mathcal{H}), \mathbb{C})$. Let $(\bar{R}^\delta(\mathcal{H})_{\text{rigid}})^*$ be the subspace of $R_\delta^*(\mathcal{H})$ consisting of linear functions on $R^\delta(\mathcal{H})$ that vanish on $R^\delta(\mathcal{H})_{\text{diff-ind}}$. Define

$$\tilde{\mathcal{H}}^{\text{rigid}} = \text{span}\{T_\mathcal{O}; J_\mathcal{O} = \Pi\}.$$

We have the following separation theorem.

Theorem 7.1. *Suppose that (PDT) holds. Then*

$$\tilde{\mathcal{H}}_\delta^{\text{rigid}} = \{h \in \tilde{\mathcal{H}}_\delta; \text{tr}_\delta(h)(R^\delta(\mathcal{H})_{\text{diff-ind}}) = 0\}.$$

i.e., the elements $\{T_\mathcal{O}\}$, where \mathcal{O} ranges over the conjugacy classes in \tilde{W}' with $J_\mathcal{O} = \Pi$, span $\tilde{\mathcal{H}}_\delta^{\text{rigid}}$ as an Λ -module.

Proof. We assume that the root datum for \mathcal{H} is semisimple. The reduction from the reductive root system to the semisimple root system will be discussed in a more general setting in Section 8.1.

Let \mathcal{O} be a conjugacy class with $J_\mathcal{O} = \Pi$. There exist $K \subsetneq \Pi^a$ such that $T_\mathcal{O} = T_w$ (in $\tilde{\mathcal{H}}_\delta$) for some elliptic element w in the parahoric subgroup W'_K . Since W'_K is a finite group,

$$i_J(\sigma) \cong i_J(\sigma \circ \chi_t) \quad \text{as } \mathcal{H}'_K\text{-representations,}$$

for all $J = \delta(J) \subset \Pi$ and $t \in \text{Hom}(X \cap \mathbb{Q}R/X \cap \mathbb{Q}J, \mathbb{C}^*)^\delta$. This means that

$$\text{tr}(T_w, \pi) = 0 \quad \text{for all } \pi \in R^\delta(\mathcal{H})_{\text{diff-ind}},$$

and hence $T_\mathcal{O} = T_w \in \tilde{\mathcal{H}}_\delta^{\text{rigid}}$.

For the converse inclusion, suppose $\text{tr}_\delta(\sum_{\mathcal{O}} a_{\mathcal{O}} T_{\mathcal{O}})(R^\delta(\mathcal{H})_{\text{diff-ind}}) = 0$. Let J be a minimal subset of Π such that $J = J_{\mathcal{O}}$ for some \mathcal{O} with $a_{\mathcal{O}} \neq 0$. If $J = \Pi$, we are done. Otherwise, suppose $J \subsetneq \Pi$. Apply $i_J(\sigma_{t-t'}) = i_J(\sigma \circ (\chi_t - \chi_{t'}))$, where $\sigma \in R^\delta(\mathcal{H}_J)$ will be chosen conveniently later, to the linear combination to obtain

$$\begin{aligned} p_J(t, t') &= \sum_{\mathcal{O}} a_{\mathcal{O}} \text{tr}(T_{\mathcal{O}}, i_J(\sigma_{t-t'})) \\ &= \sum_{J_{\mathcal{O}}=J} a_{\mathcal{O}} \text{tr}(T_{\mathcal{O}}, i_J(\sigma_{t-t'})) + \sum_{J_{\mathcal{O}} \neq J} a_{\mathcal{O}} \text{tr}(T_{\mathcal{O}}, i_J(\sigma_{t-t'})) \end{aligned}$$

as a polynomial function in $t, t' \in \text{Hom}(X \cap \mathbb{Q}R/X \cap \mathbb{Q}J, \mathbb{C}^*)^\delta$.

In $p_J(t, t')$, the part $\sum_{J_{\mathcal{O}}=J}$ is more regular in t, t' than $\sum_{J_{\mathcal{O}} \neq J}$. Since by assumption $p_J(t, t') = 0$ for all t, t' , it follows that

$$\sum_{J_{\mathcal{O}}=J} a_{\mathcal{O}} \text{tr}(T_{\mathcal{O}}, i_J(\sigma_{t-t'})) = 0.$$

By Theorem 5.3 and §5.7, for any \mathcal{O} with $J_{\mathcal{O}} = J$, we have $T_{\mathcal{O}} = \bar{i}_J(T_{\mathcal{O}'})$ for some conjugacy class \mathcal{O}' of $\tilde{W}_J \rtimes \Gamma_J$ with $J_{\mathcal{O}'} = J$ and $\mathcal{O}' \subset \mathcal{O}$.

Set $h = \sum_{J_{\mathcal{O}}=J} a_{\mathcal{O}} T_{\mathcal{O}}^J$. By induction, $h \in (\tilde{\mathcal{H}}_{J,\delta}^{\text{rigid}})_{N_J}$. Then

$$\sum_{J_{\mathcal{O}}=J} a_{\mathcal{O}} \text{tr}(T_{\mathcal{O}}, i_J(\sigma_{t-t'})) = \sum_{J_{\mathcal{O}}=J} a_{\mathcal{O}} \text{tr}(\bar{i}_J(T_{\mathcal{O}'}^J), i_J(\sigma_{t-t'})) = \text{tr}(h, r_J \circ i_J(\sigma_{t-t'})) = 0. \tag{7.1}$$

By Lemmas 4.2 and 4.4, we have

$$r_J \circ i_J = \sum_{w \in N_J} w + \sum_{w \in {}^J W^{J(\delta)} \setminus N_J} i_{J_w}^J \circ r_{J_w}^J, \quad J_w = J \cap w^{-1} J w.$$

Notice that the second part of the sum involves only J_w that are proper subsets of J . By Lemma 6.4, we find that $\text{tr}(h, \cdot)$ vanishes on this part, and so $\text{tr}(h, \sum_{w \in N_J} w \circ \sigma_{t-t'}) = 0$. Since t, t' are arbitrary, this implies that $\sum_{w \in N_J} \text{tr}(h, w \circ \sigma) = 0$ for all $\sigma \in R^\delta(\mathcal{H}_J)$. Now specialize $\sigma \in R^\delta(\mathcal{H}_J)^{N_J}$ and get

$$\text{tr}(h, \sigma) = 0 \quad \text{for all } \sigma \in R^\delta(\mathcal{H}_J)^{N_J}. \tag{7.2}$$

The (PDT) assumption says that $\text{tr} : \tilde{\mathcal{H}}_{J,\delta} \times R^\delta(\mathcal{H}_J) \rightarrow \mathbb{C}$ is nondegenerate on the left. Passing to N_J -coinvariants and N_J -invariants, respectively, it follows that $\text{tr} : (\tilde{\mathcal{H}}_{J,\delta})_{N_J} \times R^\delta(\mathcal{H}_J)^{N_J} \rightarrow \mathbb{C}$ is also nondegenerate on the left. Then (7.2) gives $h = 0$ in $(\tilde{\mathcal{H}}_{J,\delta}^{\text{rigid}})_{N_J}$. Therefore $\sum_{J_{\mathcal{O}}=J} a_{\mathcal{O}} T_{\mathcal{O}} = 0$, and this is a contradiction. \square

7.2. When the specialized parameters of the Hecke algebra are not roots of unity, the rigid cocenter $\tilde{\mathcal{H}}^{\text{rigid}}$ appears naturally in relation with $K_0(\mathcal{H})$, the Grothendieck group

of finitely generated projective \mathcal{H} -modules. Let

$$\text{Rk} : K_0(\mathcal{H}) \rightarrow \bar{\mathcal{H}} \tag{7.3}$$

be the Hattori–Stallings rank map (see for example [Da, §1.2]). If $P \in K_0(\mathcal{H})$ is a finitely generated projective module, then P is a direct summand of \mathcal{H}^n for some positive integer n . As such, there exists an idempotent $e_P \in M_{n \times n}(\mathcal{H})$ such that $P = \mathcal{H}^n e_P$ as left \mathcal{H} -modules. The rank map is defined to be the image in $\bar{\mathcal{H}}$ of the matrix trace of e_P , i.e.,

$$\text{Rk}(P) = \text{Tr}(e_P) \text{ mod } [\mathcal{H}, \mathcal{H}]. \tag{7.4}$$

Example 7.2. For every subset $K \subset S^a$ with W_K finite, let $\mathcal{H}_K \subset \mathcal{H}$ be the (finite) parahoric Hecke algebra generated by $T_{s_i}, i \in K$. If τ is a simple finite-dimensional \mathcal{H}_K -module, form the “compactly induced” module $P(K, \tau) = \mathcal{H} \otimes_{\mathcal{H}_K} \tau$. When the parameters of \mathcal{H} are such that \mathcal{H}_K is a semisimple algebra, these modules are all finitely generated projective.

Lemma 7.3. *The image of the rank map Rk lies in $\bar{\mathcal{H}}^{\text{rigid}}$.*

Proof. Let $P \in K_0(\mathcal{H})$ be a finitely generated projective module. By Theorem 7.1, for $J \subset \Pi, (\sigma, V_\sigma) \in \mathcal{H}_J^{\text{ss-mod}}$ and $t \in \text{Hom}(X \cap \mathbb{Q}R/X \cap \mathbb{Q}J, \mathbb{C}^*)$, we need to show that $\text{tr}(\text{Rk}(P), i_J(\sigma_t))$ is independent of t , where $\sigma_t = \sigma \circ \chi_t$ are acting on the same vector space V_σ . By the adjunction property, $\text{tr}(\text{Rk}(P), i_J(\sigma \circ \chi_t)) = \text{tr}(\bar{r}_J(\text{Rk}(P)), \sigma \circ \chi_t)$. Let $r_J : K_0(\mathcal{H}) \rightarrow K_0(\mathcal{H}_J)$ be the restriction map. Then $\bar{r}_J(\text{Rk}(P)) = \text{Rk}(r_J(P))$. Since $r_J(P)$ is a finitely generated projective \mathcal{H}_J -module, let $e_P^J \in M_{n \times n}(\mathcal{H}_J)$ be a corresponding idempotent. Thus, we have arrived at $\text{tr}(\text{Tr}(e_P^J), \sigma \circ \chi_t)$. This is equivalent to computing the matrix trace of the family of idempotents $\sigma_t(e_P^J) \in \text{End}_{\mathbb{C}}[V_\sigma]$. Now $\{\sigma_t(e_P^J)\}$ is a continuous family of idempotents, and by the rigidity of the trace of an idempotent, $\text{Tr}(\sigma_t(e_P^J))$ is independent of t . \square

Remark 7.4. One may regard this result as the affine Hecke algebra analogue of the “Selberg principle” of [BB] for reductive p -adic groups [Da, Theorem 1.6].

7.3. We need a decomposition of $\bar{\mathcal{H}}_\delta^{\text{rigid}}$ dual to the one for $\bar{R}^\delta(\mathcal{H})_{\text{rigid}}$ from Proposition 6.5 and Corollary 6.9.

Proposition 7.5. *Suppose that the root datum for \mathcal{H} is semisimple and (PDT) holds. Then*

$$\bar{\mathcal{H}}_\delta^{\text{rigid}} = \bigoplus_{J \in \mathcal{I}_\bullet^\delta} \bar{i}_J(\bar{A}_J((\bar{\mathcal{H}}_J^{\text{ss}})^{\text{rigid}})_{N_J}).$$

Proof. The proof follows the lines of the proof of Proposition 6.5.

By Theorem 7.1, $\bar{\mathcal{H}}_\delta^{\text{ell}} \subseteq \bar{\mathcal{H}}_\delta^{\text{rigid}}$. By Proposition 4.8, $\bar{A}(\bar{\mathcal{H}}_\delta^{\text{rigid}}) = \bar{\mathcal{H}}_\delta^{\text{ell}}$. Hence $\mathcal{H}_\delta^{\text{ell}} = \bar{A}(\mathcal{H}_\delta^{\text{ell}}) \subseteq \bar{A}(\bar{\mathcal{H}}_\delta^{\text{rigid}}) \subseteq \bar{A}(\mathcal{H}_\delta) = \mathcal{H}_\delta^{\text{ell}}$. Thus

$$\bar{A}(\bar{\mathcal{H}}_\delta^{\text{rigid}}) = \bar{\mathcal{H}}_\delta^{\text{ell}} \subset \bar{\mathcal{H}}_\delta^{\text{rigid}}.$$

Set $\bar{p}_0 = \bar{A}|_{\bar{\mathcal{H}}_\delta^{\text{rigid}}}$. Since $\bar{A}^2 = a\bar{A}$, $a \neq 0$, by Proposition 4.9, we have $\bar{\mathcal{H}}_\delta^{\text{rigid}} = \bar{A}(\bar{\mathcal{H}}_\delta^{\text{rigid}}) \oplus \ker \bar{p}_0$.

Since $\bar{A} = a + \sum_{J=\delta(J) \subsetneq \Pi} c'_J \bar{\Delta}_J$, we have $\ker \bar{p}_0 \subset \sum_{J \in \mathcal{I}_\bullet^\delta, J \neq \Pi} \bar{i}_J(\bar{\mathcal{H}}_{\delta,J})$. Recall that $\bar{\mathcal{H}}_\delta^{\text{rigid}}$ is spanned by $T_{\mathcal{O}}$, with $J_{\mathcal{O}} = \Pi$. From Theorem 5.3, we then see that

$$\ker \bar{p}_0 \subseteq \sum_{J \in \mathcal{I}_\bullet^\delta, J \neq \Pi} \bar{i}_J(\bar{\mathcal{H}}_{\delta,J}^{\text{ss,rigid}})_{N_J}.$$

Consider the projection

$$\bar{p}_1 : \ker \bar{p}_0 \rightarrow \bigoplus_{J \in \mathcal{I}_\bullet^\delta, |J|=|\Pi|-1} \bar{i}_J(\bar{A}_J(\bar{\mathcal{H}}_{\delta,J}^{\text{ss,rigid}})_{N_J}).$$

By Theorem 5.3, the map is well-defined, i.e., the range of \bar{p}_1 is indeed a direct sum (rather than a sum). Moreover, \bar{p}_1 is surjective because the range of \bar{p}_1 is in $\bar{\mathcal{H}}_\delta^{\text{rigid}}$, and it is orthogonal to $\bar{\mathcal{H}}_\delta^{\text{ell}}$. Write $\ker \bar{p}_0 = \ker \bar{p}_1 \oplus \text{im } \bar{p}_1$ and continue as in the proof of Proposition 6.5. □

Now we state our main theorem.

Theorem 7.6. *Suppose that the root datum for \mathcal{H} is semisimple, (PDT) holds,¹ and \mathfrak{q} is specialized to an admissible parameter. Then:*

- (1) $\text{tr} : \bar{\mathcal{H}}_\delta^{\text{rigid}} \times \bar{R}(\mathcal{H})_{\text{rigid}}^\delta \rightarrow \mathbb{C}$ is a perfect pairing.
- (2) $\text{tr} : \bar{\mathcal{H}}_\delta^{\text{ell}} \times \bar{R}_0^\delta(\mathcal{H}) \rightarrow \mathbb{C}$ is a perfect pairing.

Proof. We show that the pairing tr is block upper triangular with respect to the decomposition of $\bar{\mathcal{H}}_\delta^{\text{rigid}}$ from Proposition 7.5 and the decomposition of $\bar{R}^\delta(\mathcal{H})_{\text{rigid}}$ from Corollary 6.9. For every $J \in \mathcal{I}_\bullet^\delta$, denote

$$\bar{\mathcal{H}}_\delta^{\text{rigid}}[J] = \bar{i}_J(\bar{A}_J((\bar{\mathcal{H}}_J^{\text{ss}})^{\text{rigid}})_{N_J}) \quad \text{and} \quad \bar{R}^\delta(\mathcal{H})_{\text{rigid}}[J] = i_J(A_J(R(\mathcal{H}_J^{\text{ss}}))^{N_J}).$$

Let $J, K \in \mathcal{I}_\bullet^\delta$. Then, for $h \in (\bar{\mathcal{H}}_K^{\text{ss}})^{\text{rigid}}$ and $\pi \in R(\mathcal{H}_J^{\text{ss}})$,

$$\begin{aligned} \text{tr}(\bar{i}_K(\bar{A}_K(h)), i_J(A_J(\pi))) &= \text{tr}(h, A_K \circ (r_K \circ i_J)(A_J(\pi))) \quad \text{by Lemma 4.5,} \\ &= \sum_{w \in {}^K W^J(\delta)} \text{tr}(h, A_K \circ (i_{K_w}^K \circ w \circ r_{J_w}^J)(A_J(\pi))) \quad \text{by Lemma 4.2.} \end{aligned}$$

In particular, if $J \not\supseteq K$ (as elements of $\mathcal{I}_\bullet^\delta$), then all K_w are proper subsets of K , and therefore A_K kills the induced modules. This means that $\text{tr} : \bar{\mathcal{H}}_\delta^{\text{rigid}}[K] \times \bar{R}(\mathcal{H})_{\text{rigid}}^\delta[J] \rightarrow \mathbb{C}$ is identically zero if $J \not\supseteq K$.

¹ In Section 8, the Density Theorem 8.2 is proven, based on an inductive argument, and therefore this assumption can then be removed.

On the other hand, if $J = K$, the only nonzero contribution to the trace comes from $w \in {}^K W^K(\delta)$ on the right hand side, and we obtain

$$\text{tr}(\bar{i}_K(\bar{A}_K(h)), i_K(A_K(\pi))) = |N_K| \text{tr}(\bar{A}_K(h), A_K(\pi)).$$

This is because $A_K(\pi)$ is chosen to be N_K -invariant.

The rank of this pairing equals the rank of $\text{tr}_K^{\text{ss}} : (\bar{\mathcal{H}}_{K,\delta}^{\text{ss},0})_{N_K} \times \bar{R}_0^\delta(\mathcal{H}_K^{\text{ss}})^{N_K} \rightarrow \mathbb{C}$, which, by Proposition 6.2, equals $\dim \bar{R}_0^\delta(\mathcal{H}_K^{\text{ss}})^{N_K}$.

Therefore,

$$\begin{aligned} \text{rank tr}(\bar{\mathcal{H}}_\delta^{\text{rigid}} \times \bar{R}(\mathcal{H})_\delta^{\text{rigid}}) &= \sum_{J \in \mathcal{I}_\bullet^\delta} \text{rank tr}(\bar{\mathcal{H}}_\delta^{\text{rigid}}[J] \times \bar{R}(\mathcal{H})_\delta^{\text{rigid}}[J]) \\ &= \sum_{J \in \mathcal{I}_\bullet^\delta} \dim \bar{R}_0^\delta(\mathcal{H}_J^{\text{ss}})^{N_J} = \dim \bar{R}(\mathcal{H})_\delta^{\text{rigid}}. \end{aligned} \tag{7.5}$$

This implies that $\dim \bar{\mathcal{H}}_\delta^{\text{rigid}} \geq \dim \bar{R}(\mathcal{H})_\delta^{\text{rigid}}$. On the other hand, from Theorem 7.1, we know that $\dim \bar{\mathcal{H}}_\delta^{\text{rigid}} \leq |cl(\tilde{W}, \delta)_0|$. When \mathfrak{q} is admissible, $\dim \bar{R}(\mathcal{H})_\delta^{\text{rigid}} = |cl(\tilde{W}, \delta)_0|$, and therefore $\dim \bar{\mathcal{H}}_\delta^{\text{rigid}} = \dim \bar{R}(\mathcal{H})_\delta^{\text{rigid}} = |cl(\tilde{W}, \delta)_0|$ in this case. This proves (1).

In light of the inequality in Proposition 6.2 (applied to each J), (1) implies that the equality of dimensions must hold for each J , i.e., $\dim \bar{\mathcal{H}}_\delta^{\text{rigid}}[J] = \dim \bar{R}^\delta(\mathcal{H})_{\text{rigid}}[J]$. In particular, when $J = \Pi$, this is claim (2). \square

7.4. Examples. In this subsection, we illustrate the pairing between the rigid quotient and the rigid cocenter in some concrete examples.

Example 7.7 ($SL(2)$). Let \mathcal{H} be the affine Hecke algebra for the root datum of $SL(2)$. This is generated by T_0 and T_1 subject to

$$T_i^2 = (q - 1)T_i + q, \quad i = 0, 1.$$

There are three conjugacy classes of finite order in \tilde{W} : s_0 (the affine reflection), s_1 (the finite reflection), and 1. Accordingly, we have the three basis elements of $\bar{\mathcal{H}}^{\text{rigid}}$: T_0 , T_1 , and 1.

There are four one-dimensional modules corresponding to $T_i \in \{-1, q\}$. A basis of the elliptic space $\bar{R}_0(\mathcal{H})$ in the admissible case is given by the classes of $\text{St} = (T_0 = -1, T_1 = -1)$ and any one of the two modules $\pi^+ = (T_0 = -1, T_1 = q)$ or $\pi^- = (T_0 = q, T_1 = -1)$. Choose the class of π^+ .

The third basis element of $\bar{R}(\mathcal{H})_{\text{rigid}}$ is the (tempered) principal series $i_\emptyset(1)$. The resulting 3×3 table is in Table 1.

The determinant of this matrix is $-(q + 1)^2$, and therefore the matrix is invertible for all $q \neq -1$.

To get a block upper-triangular matrix, one needs to replace T_i by $\bar{A}(T_i)$, $i = 0, 1$, as in Theorem 7.6.

Table 1. The rigid cocenter/quotient table for $SL(2)$

$SL(2)$	St	π^+	$i_{\emptyset}(1)$
T_0	-1	-1	$q - 1$
T_1	-1	q	$q - 1$
1	1	1	2

Example 7.8 ($PGL(2)$). Let \mathcal{H} be the affine Hecke algebra for the root datum of $PGL(2)$. This is generated by T_0, T_1, τ subject to

$$T_i^2 = (q - 1)T_i + q, \quad i = 0, 1, \quad \tau^2 = 1, \quad \tau \cdot T_0 = T_1 \cdot \tau.$$

There are three conjugacy classes of finite order in \tilde{W} : s_1 (the finite reflection), τ , and 1. Accordingly, we have the three basis elements of $\tilde{\mathcal{H}}^{\text{rigid}}$: T_1, τ , and 1.

There are four one-dimensional modules corresponding to $\tau \in \{1, -1\}$ and $T_1 \in \{-1, q\}$. A basis of the elliptic space $\bar{R}_0(\mathcal{H})$ is given by the classes of $\text{St}^{\pm} = (T_1 = -1, \tau = \pm 1)$.

The third basis element of $\bar{R}(\mathcal{H})_{\text{rigid}}$ is the (tempered) principal series $i_{\emptyset}(1)$. The resulting 3×3 table is in Table 2.

Table 2. The rigid cocenter/quotient table for $PGL(2)$

$PGL(2)$	St ⁻	St ⁺	$i_{\emptyset}(1)$
T_1	-1	-1	$q - 1$
τ	1	-1	0
1	1	1	2

The determinant of this matrix is $2(q + 1)$, and therefore the matrix is invertible for all $q \neq -1$.

Example 7.9 (Affine C_2). Let \mathcal{H} be the affine Hecke algebra attached to the affine diagram of type C_2 with three parameters

$$q_0 \implies q_1 \longleftarrow q_2. \tag{7.6}$$

It is generated by $T_i, i = 0, 1, 2$, subject to $T_i^2 = (q_i - 1)T_i + q_i$ and the braid relations.

There are nine conjugacy classes of finite order in \tilde{W} , including five elliptic classes. Representatives for the five elliptic classes are: $s_1s_2, (s_1s_2)^2, s_0s_2, s_0s_1$, and $(s_0s_1)^2$. The other four classes correspond to: s_0, s_1, s_2 , and 1.

A basis of the elliptic space $\bar{R}_0(\mathcal{H})$ in the admissible case can be constructed by lifting the five simple modules for the finite Hecke algebra $\mathcal{H}_f(C_2, q_1, q_2)$ under the algebra homomorphism $T_0 \mapsto -1, T_1 \mapsto T_1, T_2 \mapsto T_2$. We label these five modules by the bipartitions which parameterized the corresponding representations of the finite Weyl group: $2 \times 0, 11 \times 0, 0 \times 2, 0 \times 11$, and 1×1 . For the character table of the finite Hecke algebra of type C_2 , see [GP].

The remaining four modules needed for a basis of the rigid quotient $\bar{R}(\mathcal{H})_{\text{rigid}}$ can be chosen as the induced tempered modules: $i_{\{1\}}(\text{St})$, $i_{\{2\}}(\text{St})$, $i_{\{2\}}(\pi^+)$, and $i_{\emptyset}(1)$. Here π^+ is the one dimensional elliptic module for the Hecke algebra of $SL(2) = Sp(2)$ as in Table 1.

For this calculation, we computed the restrictions of the nine representations to the maximal finite Hecke subalgebras, using the Mackey formula, and then used the character table for the finite Hecke algebras of type C_2 and A_1 . The resulting rigid character table is given, because of page limitations, in (7.7) and (7.8).

C_2^{aff}	2×0	11×0	0×2	0×11	1×1
$T_1 T_2$	$q_1 q_2$	$-q_2$	$-q_1$	1	0
$(T_1 T_2)^2$	$(q_1 q_2)^2$	q_2^2	q_1^2	1	$-2q_1 q_2$
$T_0 T_2$	$-q_2$	$-q_2$	1	1	$1 - q_2$
$T_0 T_1$	$-q_1$	1	$-q_1$	1	$1 - q_1$
$(T_0 T_1)^2$	q_1^2	1	q_1^2	1	$q_1^2 + 1$
T_0	-1	-1	-1	-1	-2
T_1	q_1	-1	q_1	-1	$q_1 - 1$
T_2	q_2	q_2	-1	-1	$q_2 - 1$
1	1	1	1	1	2

(7.7)

C_2^{aff}	$i_{\{1\}}(\text{St})$	$i_{\{2\}}(\text{St})$	$i_{\{2\}}(\pi^+)$	$i_{\emptyset}(1)$
$T_1 T_2$	$1 - q_2$	$1 - q_1$	$q_2(q_1 - 1)$	$(q_1 - 1)(q_2 - 1)$
$(T_1 T_2)^2$	$q_2^2 - 2q_1 q_2 + 1$	$q_1^2 - 2q_1 q_2 + 1$	$q_1^2 q_2^2 + q_2^2 - 2q_1 q_2$	$q_1^2 q_2^2 + q_2^2 + q_1^2 + 1 - 4q_1 q_2$
$T_0 T_2$	$(q_0 - 1)(q_2 - 1)$	$2 - q_0 - q_2$	$q_0 q_2 + 1 - 2q_2$	$2(q_0 - 1)(q_2 - 1)$
$T_0 T_1$	$1 - q_0$	$1 - q_1$	$1 - q_1$	$(q_0 - 1)(q_1 - 1)$
$(T_0 T_1)^2$	$q_0^2 - 2q_0 q_1 + 1$	$q_1^2 - 2q_0 q_1 + 1$	$q_1^2 - 2q_0 q_1 + 1$	$q_0^2 q_1^2 + q_1^2 + q_0^2 + 1 - 4q_0 q_1$
T_0	$2q_0 - 2$	$q_0 - 3$	$q_0 - 3$	$4q_0 - 4$
T_1	$q_1 - 3$	$2q_1 - 2$	$2q_1 - 2$	$4q_1 - 4$
T_2	$2q_2 - 2$	$q_2 - 3$	$3q_2 - 1$	$4q_2 - 4$
1	4	4	4	8

(7.8)

The determinant of the 9×9 rigid table equals

$$-(1 + q_0)^3(1 + q_1)^3(1 + q_2)^3(q_0 + q_1)(q_1 + q_2)(1 + q_0 q_1)(1 + q_1 q_2).$$

This implies that the determinant of the rigid table above is nonzero if and only if all the finite Hecke algebras are semisimple.²

² For the semisimplicity criterion for the finite Hecke algebra of type B_n , see [DJ, Theorem 5.5].

Example 7.10 (Extended affine C_2). The Hecke algebra attached to the extended affine diagram of type C_2

$$q_2 \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} q_1 \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} q_2 \tag{7.9}$$

is isomorphic to the Hecke algebra attached to the affine diagram

$$1 \rightleftarrows q_2 \longleftarrow q_1 \tag{7.10}$$

so this is a particular case of the previous example with the appropriate specialization of the parameters. More precisely, let T_0, T_1, T_2, τ be the generators of the Hecke algebra attached to the diagram (7.9) with relations:

$$\begin{aligned} T_0^2 &= (q_2 - 1)T_0 + q_2, & T_1^2 &= (q_1 - 1)T_1 + q_1, & T_2^2 &= (q_2 - 1)T_2 + q_2, \\ \tau^2 &= 1, & \tau T_0 &= T_2\tau, & \tau T_1 &= T_1\tau, \\ T_0T_1T_0T_1 &= T_1T_0T_1T_0, & T_1T_2T_1T_2 &= T_2T_1T_2T_1, & T_0T_2 &= T_2T_0, \end{aligned}$$

and let T'_0, T'_1, T'_2 be the generators of the Hecke algebra attached to the diagram (7.10) with relations:

$$\begin{aligned} (T'_0)^2 &= (q_1 - 1)T'_0 + q_1, & (T'_1)^2 &= (q_2 - 1)T'_1 + q_2, & (T'_2)^2 &= 1, \\ T'_0T'_1T'_0T'_1 &= T'_1T'_0T'_1T'_0, & T'_1T'_2T'_1T'_2 &= T'_2T'_1T'_2T'_1, & T'_0T'_2 &= T'_2T'_0. \end{aligned}$$

The isomorphism is realized by

$$\tau \leftrightarrow T'_0, \quad T_0 \leftrightarrow T'_0T'_1T'_0, \quad T_1 \leftrightarrow T'_2, \quad T_2 \leftrightarrow T'_1. \tag{7.11}$$

The determinant of the rigid table equals (up to a scalar independent of q_1, q_2)

$$(1 + q_1)^3(1 + q_2)^5(q_1 + q_2)(1 + q_1q_2).$$

8. Some consequences

In this section, we prove the basis theorem, density theorem and trace Paley–Wiener theorem. They will be proven inductively, using Theorem 7.6.

Theorem 8.1 (Basis Theorem for arbitrary parameters). $\tilde{\mathcal{H}}'$ is a free Λ -module with basis $\{T_{\mathcal{O}}\}$, where \mathcal{O} ranges over all the conjugacy classes of \tilde{W}' .

Theorem 8.2 (Density Theorem for admissible parameters). When q is specialized to an admissible parameter, the trace map $\text{tr} : \tilde{\mathcal{H}}' \rightarrow R^*(\mathcal{H}')_{\text{good}}$ is bijective.

The results in the previous subsections regarding $\bar{R}^\delta(\mathcal{H})_{\text{rigid}}$ and $\tilde{\mathcal{H}}_\delta^{\text{rigid}}$ were proved under the assumption that the root datum is semisimple and the parameters are admissible. In order to apply an inductive argument using parabolic subalgebras, we need to show that a proof in the semisimple case is sufficient to derive the general case. First we need some elements of Clifford theory.

8.1. Suppose that A is an associative algebra and Γ a group acting on A . Set $A' = A \rtimes \mathbb{C}[\Gamma]$. The map $a\gamma \mapsto \gamma$ for $a \in A$ and $\gamma \in \Gamma$ induces a surjective linear map

$$\tau : A' \rightarrow \overline{\mathbb{C}[\Gamma]} = \bigoplus_{[\gamma] \in \text{cl}(\Gamma)} \mathbb{C}\gamma. \tag{8.1}$$

The fiber $\tau^{-1}(\gamma)$ is by definition the image of $A\gamma$ in A' . Similar to Section 3.2, we have:

Lemma 8.3. *The map $a\gamma \mapsto a$ induces an isomorphism $\tau^{-1}(\gamma) \cong \bar{A}^{[\gamma]}$, where $\bar{A}^{[\gamma]} = (A/[A, A]_{\gamma})_{Z_{\Gamma}(\gamma)}$, the space of $Z_{\Gamma}(\gamma)$ -coinvariants.*

Proposition 8.4. *If the statement of Theorem 8.1 holds when \mathcal{H}' is semisimple with admissible parameters, then it holds in general.*

Proof. Define the linear map

$$\pi : \bigoplus_{\mathcal{O} \in \text{cl}(\tilde{W}')} \Lambda \rightarrow \tilde{\mathcal{H}}', \quad (a_{\mathcal{O}}) \mapsto \sum a_{\mathcal{O}}T_{\mathcal{O}}.$$

By Theorem 5.2, this map is surjective. We need to show that also $\ker \pi = 0$. For this, it is sufficient to show that $\ker \pi \otimes_{\Lambda} \mathbb{C}_q = 0$ for generic parameters q . Fix now an admissible q and specialize $\mathcal{H}'_q = \mathcal{H}' \otimes_{\Lambda} \mathbb{C}_q$. We will prove that the set $\{T_{\mathcal{O}}; \mathcal{O} \in \text{cl}(\tilde{W}')\}$ is \mathbb{C} -linearly independent in \mathcal{H}'_q .

Let $\pi^q : \bigoplus_{\mathcal{O} \in \text{cl}(\tilde{W}')} \mathbb{C} \rightarrow \tilde{\mathcal{H}}'_q$ denote the map induced by π after specialization.

Rewrite $\tilde{W}' = \tilde{W} \rtimes \Gamma$ as $\tilde{W}' = W^a \rtimes \tilde{\Gamma}$, where $\tilde{\Gamma} = \Omega \rtimes \Gamma$. For every $\mathcal{O} \in \text{cl}(\tilde{W}')$, there exists a unique $[\gamma] \in \text{cl}(\tilde{\Gamma})$ such that $\mathcal{O} \cap W^a\gamma \neq \emptyset$. By Lemma 8.3, it is sufficient to prove that for a fixed $[\gamma] \in \text{cl}(\tilde{\Gamma})$, the set

$$\{T_{\mathcal{O}}; \mathcal{O} \in \text{cl}(\tilde{W}'), \mathcal{O} \cap W^a\gamma \neq \emptyset\}$$

is linearly independent.

Denote $\tilde{\Gamma}$ the image of $Z_{\Gamma}(\gamma)$ in $\text{Aut}(W^a, S^a)$, and by $\underline{\gamma}$ the image of γ . Moreover, set $\tilde{W}' = W^a \rtimes \tilde{\Gamma}$ and $\tilde{\mathcal{H}}'_q = \mathcal{H}_a \rtimes \mathbb{C}[\tilde{\Gamma}]$. Here \mathcal{H}_a is the affine Hecke algebra corresponding to W^a . Notice that $\tilde{\mathcal{H}}'_q$ is an affine Hecke algebra attached to a semisimple root datum and extended by a group of automorphisms, therefore the hypothesis of the proposition applies to it.

Let

$$\underline{\pi}^q : \bigoplus_{\mathcal{O} \in \text{cl}(\tilde{W}')} \mathbb{C} \rightarrow \tilde{\mathcal{H}}'_q, \quad (a_{\mathcal{O}}) \mapsto \sum a_{\mathcal{O}}T_{\mathcal{O}},$$

be the cocenter map. By the assumption in the proposition, $\underline{\pi}^q$ is bijective. Consider its restriction to the $\underline{\gamma}$ -fiber:

$$\underline{\pi}^q_{[\underline{\gamma}]} : \bigoplus_{\mathcal{O} \in \text{cl}(\tilde{W}'), \mathcal{O} \cap W^a\underline{\gamma} \neq \emptyset} \mathbb{C} \rightarrow \text{Im}(\mathcal{H}_{a,q}\underline{\gamma} \rightarrow \tilde{\mathcal{H}}'_q) \cong \overline{\mathcal{H}}_{a,q}^{[\underline{\gamma}]}, \tag{8.2}$$

where the last isomorphism is by Lemma 8.3. Since $\underline{\pi}^q$ is injective, so is $\underline{\pi}^q_{[\underline{\gamma}]}$.

There is a natural bijection

$$\begin{aligned} \kappa_\gamma : \{ \mathcal{O}; \mathcal{O} \in cl(\tilde{W}'), \mathcal{O} \cap W^a \gamma \neq \emptyset \} &\rightarrow \{ \underline{\mathcal{O}}; \underline{\mathcal{O}} \in cl(\tilde{W}'), \underline{\mathcal{O}} \cap W^a \underline{\gamma} \neq \emptyset \}, \\ [w\gamma] &\mapsto [w\underline{\gamma}], \quad w \in W^a. \end{aligned} \tag{8.3}$$

Similar to (8.2), define

$$\pi_{[\gamma]}^q : \bigoplus_{\mathcal{O} \in cl(\tilde{W}'), \mathcal{O} \cap W^a \gamma \neq \emptyset} \mathbb{C} \rightarrow \text{Im}(\mathcal{H}_{a,q} \gamma \rightarrow \overline{\mathcal{H}}'_q) \cong \overline{\mathcal{H}}_{a,q}^{[\gamma]}. \tag{8.4}$$

We wish to show that $\pi_{[\gamma]}^q$ is also injective, which would complete the proof. We have the commutative diagram

$$\begin{array}{ccc} \bigoplus_{\mathcal{O} \in cl(\tilde{W}'), \mathcal{O} \cap W^a \gamma \neq \emptyset} \mathbb{C} & \xrightarrow{\kappa_\gamma} & \bigoplus_{\underline{\mathcal{O}} \in cl(\tilde{W}'), \underline{\mathcal{O}} \cap W^a \underline{\gamma} \neq \emptyset} \mathbb{C} \\ \pi_{[\gamma]}^q \downarrow & & \downarrow \pi_{[\underline{\gamma}]}^q \\ \overline{\mathcal{H}}_{a,q}^{[\gamma]} & \xrightarrow{\cong} & \overline{\mathcal{H}}_{a,q}^{[\underline{\gamma}]} \end{array} \tag{8.5}$$

Since $\pi_{[\underline{\gamma}]}^q$ is injective and κ_γ is bijective, it follows that $\pi_{[\gamma]}^q$ is injective. □

8.2. Proofs of Theorems 8.1 and 8.2. In light of Proposition 8.4, it is sufficient to prove the theorems under the assumption that the root datum is semisimple. Assume that the indeterminate q is specialized to an admissible parameter q . By induction, we may assume that both the basis and density theorems hold for all proper parabolic subalgebras.

Suppose $\sum_{\mathcal{O}} a_{\mathcal{O}} T_{\mathcal{O}} \in \ker \text{tr}$. We claim that by induction, $\sum_{\mathcal{O}} a_{\mathcal{O}} T_{\mathcal{O}} \in \tilde{\mathcal{H}}_\delta^{\text{rigid}}$. Indeed, by the proof of Theorem 7.1 and the (PDT) assumption, $\sum_{J_{\mathcal{O}}=J} a_{\mathcal{O}} T_{\mathcal{O}} = 0$ for all $J \subsetneq \Pi$. By the inductive assumption on proper parabolic subalgebras, $a_{\mathcal{O}} = 0$ for all \mathcal{O} with $J_{\mathcal{O}} \neq J$.

Thus $\sum_{\mathcal{O} \in cl(\tilde{W}, \delta)_0} a_{\mathcal{O}} T_{\mathcal{O}} \in \ker \text{tr}$. Note that $\tilde{\mathcal{H}}_\delta^{\text{rigid}}$ is spanned by $T_{\mathcal{O}}$ for $\mathcal{O} \in cl(\tilde{W}, \delta)_0$ and $\dim \bar{R}_0^\delta(\mathcal{H}) = |cl(\tilde{W}, \delta)_0|$. By Theorem 7.6, $T_{\mathcal{O}}$ for $\mathcal{O} \in cl(\tilde{W}, \delta)_0$ forms a basis of $\tilde{\mathcal{H}}_\delta^{\text{rigid}}$. Hence $a_{\mathcal{O}} = 0$ for $\mathcal{O} \in cl(\tilde{W}, \delta)$. This concludes the proof when q is admissible.

Finally, since Theorem 8.1 holds for generic parameters, it holds for the indeterminate q as well. □

8.3. Since $\tilde{\mathcal{H}}_\delta$ is a Λ -module with basis $T_{\mathcal{O}}$, it is easy to see that the induction and restriction maps \bar{i}_J and \bar{r}_J depend algebraically on the parameters, therefore Lemma 4.6 and hence Propositions 4.9 and 6.2 hold unconditionally. Moreover, $\tilde{\mathcal{H}}_\delta^{\text{ell}} \subset \tilde{\mathcal{H}}_\delta^{\text{rigid}}$ and Proposition 7.5 holds without the (PDT) assumption.

We can now prove the trace Paley–Wiener theorem for arbitrary parameters.

Theorem 8.5. *For arbitrary parameters, the image of the map $\text{tr} : \tilde{\mathcal{H}}' \rightarrow R(\mathcal{H}')$ is $R^*(\mathcal{H}')_{\text{good}}$. In other words, $R_\delta^*(\mathcal{H})_{\text{tr}} = R_\delta^*(\mathcal{H})_{\text{good}}$.*

Proof. We first show that

$$\text{tr} : \tilde{\mathcal{H}}_\delta^{\text{ell}} \rightarrow \bar{R}_0^\delta(\mathcal{H})^* \text{ is surjective.} \tag{8.6}$$

By Proposition 4.8, we may regard $\bar{R}_0^\delta(\mathcal{H})$ as $A(R^\delta(\mathcal{H})) \subset R^\delta(\mathcal{H})$. Under this identification, $\bar{R}_0^\delta(\mathcal{H}) = \{\sigma \circ \chi_t; \sigma \in \bar{R}_0^\delta(\mathcal{H}^{ss}), t \in T_\Pi^\delta\}$. On the other hand, the natural projection map $\bar{\mathcal{H}}_\delta \rightarrow \overline{(\mathcal{H}^{ss})}_\delta$ is surjective and each fiber is isomorphic to $\bigoplus_{\lambda \in (\Pi^\vee)^\perp / \langle \delta \rangle} \mathbb{C}\theta_\lambda h$ for some $h \in \bar{\mathcal{H}}_\delta$. Note that $\bigoplus_{\lambda \in (\Pi^\vee)^\perp / \langle \delta \rangle} \mathbb{C}\theta_\lambda$ is naturally isomorphic to the set of regular functions on T_Π^δ . Thus (8.6) follows from Proposition 6.2 and the remark above for \mathcal{H}^{ss} .

The rest of the argument is as in [BDK, §4]. Let $f \in R_\delta^*(\mathcal{H})_{\text{good}}$. By (8.6), there exists $h \in \bar{\mathcal{H}}_\delta^{\text{ell}}$ such that $f'_h = f$ on $\bar{R}_0^\delta(\mathcal{H})$, where $f'_h = \text{tr}(h) \in R_\delta^*(\mathcal{H})_{\text{tr}}$. Modifying f to $f - f'_h$, we may assume that $f(\bar{R}_0^\delta(\mathcal{H})) = 0$.

Consider the adjoint operator

$$A^* = a + \sum_{J=\delta(J) \subsetneq \Pi} c'_J i_J^* \circ r_J^* : R_\delta^*(\mathcal{H}) \rightarrow R_\delta^*(\mathcal{H}).$$

Then $A^*(f)(R^\delta(\mathcal{H})) = f(\bar{R}_0^\delta(\mathcal{H})) = 0$. So $f = -\sum_{J=\delta(J) \subsetneq \Pi} c'_J/a i_J^* \circ r_J^*(f)$. It is immediate that $i_J^*(R_\delta^*(\mathcal{H})_{\text{good}}) \subset R_\delta^*(\mathcal{H}_J)_{\text{good}}$ and $r_J^*(R_\delta^*(\mathcal{H}_J)_{\text{tr}}) \subset R_\delta^*(\mathcal{H})_{\text{tr}}$, thus the claim follows by induction on J . \square

8.4. We record one more finiteness result for arbitrary parameters.

Proposition 8.6. *Suppose that Φ is a semisimple based root datum. Then*

$$\dim \bar{R}_0^\delta(\mathcal{H}) \leq \text{rank } \bar{\mathcal{H}}_\delta^{\text{ell}} = |\text{cl}(\tilde{W}, \delta)_{\text{ell}}|.$$

Proof. By Proposition 6.2 and §8.3, $\dim \bar{R}_0^\delta(\mathcal{H}) \leq \dim \bar{\mathcal{H}}_\delta^{\text{ell}}$ for arbitrary parameters. Since (PDT) holds for admissible parameters, $\text{rank } \bar{\mathcal{H}}_\delta^{\text{ell}} = \text{rank } \bar{\mathcal{H}}_\delta / \ker \bar{A}$ is finite for admissible parameters. Since $\bar{\mathcal{H}}_\delta$ is a Λ -module with basis $T_\mathcal{O}$ and the map \bar{A} depends algebraically on the parameters, $\text{rank } \bar{\mathcal{H}}_\delta / \ker \bar{A}$ is semicontinuous, i.e., the rank at any parameter is less than or equal to the rank at generic parameters. By Theorem 7.6(2) and Proposition 6.7 (2), $\text{rank } \bar{\mathcal{H}}_\delta^{\text{ell}} \leq |\text{cl}(\tilde{W}, \delta)_{\text{ell}}|$.

On the other hand, by Proposition 7.5,

$$\text{rank } \bar{\mathcal{H}}_\delta^{\text{rigid}} = \sum_{J \in \mathcal{I}_\delta^\bullet} \text{rank } \overline{(\mathcal{H}_J^{ss})}_\delta^0 / N_J.$$

For each J , the N_J action depends algebraically on the parameters, hence

$$\text{rank } \overline{(\mathcal{H}_J^{ss})}_\delta^0 / N_J \leq |\text{cl}(\tilde{W}_J, \delta)_{\text{ell}} / N_J|.$$

By §6.3,

$$\begin{aligned} |\text{cl}(\tilde{W}, \delta)_0| &= \text{rank } \bar{\mathcal{H}}_\delta^{\text{rigid}} = \sum_{J \in \mathcal{I}_\delta^\bullet} \text{rank } \overline{(\mathcal{H}_J^{ss})}_\delta^0 / N_J \leq \sum_{J \in \mathcal{I}_\delta^\bullet} |\text{cl}(\tilde{W}_J, \delta)_{\text{ell}} / N_J| \\ &= |\text{cl}(\tilde{W}, \delta)_0|. \end{aligned}$$

Thus $\text{rank } \overline{(\mathcal{H}_J^{ss})}_\delta^0 / N_J = |\text{cl}(\tilde{W}_J, \delta)_{\text{ell}} / N_J|$ for all J . In particular, we conclude that $\text{rank } \bar{\mathcal{H}}_\delta^{\text{ell}} = |\text{cl}(\tilde{W}, \delta)_{\text{ell}}|$. \square

Remark 8.7. If the root datum Φ is not semisimple, apply the constructions from Section 2.4 with $J = \Pi$. Set $Z(\Phi) = X/X \cap \mathbb{Q}\Pi$ and $\widehat{Z}(\Phi) = T^\Pi = \text{Hom}_{\mathbb{Z}}(Z(\Phi), \mathbb{C}^\times)$. For every $t \in \widehat{Z}(\Phi)$, let $\chi_t : \mathcal{H} \rightarrow \mathcal{H}^{\text{ss}}$ be the algebra homomorphism (2.12). These homomorphisms define a right $\widehat{Z}(\Phi)$ -action on $\text{Irr}^\delta \mathcal{H}$ and on $\Theta(\mathcal{H})$, which preserves the elliptic parts $\text{Irr}_0^\delta \mathcal{H}$ and $\Theta(\mathcal{H})_0$. Then Proposition 8.6 implies that the sets $\Theta^\delta(\mathcal{H})_0$ and $\text{Irr}_0^\delta(\mathcal{H})$ are finite unions of $\widehat{Z}(\Phi)$ -orbits.

9. Applications to smooth representations of reductive p -adic groups

9.1. Let k be a p -adic field of characteristic 0 with ring of integers \mathfrak{o} and residue field of cardinality q . Let \mathcal{G} be a connected semisimple group over k , and set $G = \mathcal{G}(k)$. Let G^0 be the subgroup of G generated by all of the compact open subgroups. A (one-dimensional) smooth character of G is called *unramified* if $\chi|_{G^0} = 1$. Let $X_u(G)$ denote the group of unramified characters of G . Let $R(G)$ denote the Grothendieck group of smooth \mathbb{C} -representations of G . Let $\mathcal{B}(G)$ be the set of components of the Bernstein center of G , as in [BK, §1]. Then we have a decomposition $R(G) = \prod_{\mathfrak{s} \in \mathcal{B}(G)} R^\mathfrak{s}(G)$.

If P is a (k -rational) parabolic subgroup of G with Levi decomposition $P = MN$, let $i_M : R(M) \rightarrow R(G)$ and $r_M : R(G) \rightarrow R(M)$ denote the map of normalized parabolic induction and the normalized Jacquet map, respectively (see for example [Be, Chapter III] or [Ren, Chapter VI]).

Analogously to the Hecke algebras definition 6.1, we define the rigid quotient of $R(G)$.

Definition 9.1. Set

$$R(G)_{\text{diff-ind}} = \text{span}\{i_M(\sigma) - i_M(\sigma \circ \chi); P = MN \text{ a parabolic subgroup, } \sigma \in R(M), \chi \in X_u(M)\} \tag{9.1}$$

and define the *rigid quotient* of $R(G)$ to be

$$\bar{R}(G)_{\text{rigid}} := R(G)/R(G)_{\text{diff-ind}}. \tag{9.2}$$

If $\mathfrak{s} \in \mathcal{B}(G)$, let $\bar{R}^\mathfrak{s}(G)_{\text{rigid}}$ be the image of the Bernstein component $R^\mathfrak{s}(G)$ in $\bar{R}(G)_{\text{rigid}}$.

9.2. We explain the role of the rigid cocenter in the correspondence of unitarizable representations from the p -adic group to Iwahori–Hecke algebras. Let G satisfy the same assumptions as in [Bo, §3].

Let I be an Iwahori subgroup of G . Let $\mathcal{C}_I(G)$ be the category of smooth G -representations generated by their I -fixed vectors. By a classical result of Casselman,

$$\mathcal{C}_I(G) = R^0(G),$$

where $R^0(G)$ is the Bernstein component where the simple objects are subquotients of unramified minimal principal series.

Let $\mathcal{H} = C_c^\infty(I \backslash G / I)$ be the *Iwahori–Hecke algebra*, i.e., the complex associative unital algebra of compactly supported, smooth, I -biinvariant complex functions, under

the convolution with respect to a fixed Haar measure μ on G . Normalize the Haar measure so that $\mu(I) = 1$. Given a representation $(\pi, V) \in \mathcal{C}_I(G)$, the algebra \mathcal{H} acts on V^I via

$$\pi(f)v = \int_G f(x)\pi(x)v \, d\mu(x), \quad f \in \mathcal{H}, v \in V^I.$$

Borel [Bo] showed that the functor

$$F_I : \mathcal{C}_I(G) \rightarrow \mathcal{H}\text{-mod}, \quad V \mapsto V^I, \quad (9.3)$$

is an equivalence of categories.

A smooth admissible G -representation (π, V) is called *hermitian* if V has a nondegenerate hermitian form $\langle \cdot, \cdot \rangle_G$ which is G -invariant, i.e., $\langle \pi(g)v, w \rangle_G = \langle v, \pi(g^{-1})w \rangle_G$ for all $g \in G$ and $v, w \in V$. A hermitian representation is called (*pre*)*unitary* if the hermitian form is positive definite.

In parallel, we have the similar definitions for \mathcal{H} -modules. Let $*$: $\mathcal{H} \rightarrow \mathcal{H}$ denote the conjugate-linear anti-involution given by $f^*(g) = \overline{f(g^{-1})}$ for $f \in \mathcal{H}$ and $g \in G$. Then an \mathcal{H} -invariant hermitian form on an \mathcal{H} -module (π, V^I) has the defining property $\langle \pi(f)v, w \rangle_{\mathcal{H}} = \langle v, \pi(f^*)w \rangle_{\mathcal{H}}$.

By restriction to I -fixed vectors, it is clear that if $(\pi, V) \in \mathcal{C}_I(G)$ is hermitian (resp., unitary), then $(\pi, V^I) \in \mathcal{H}\text{-mod}$ is hermitian (resp., unitary). It is also easy to see that if (π, V^I) is hermitian, then (π, V) is hermitian [BM1, (2.10)]. The difficult implication is the following:

Theorem 9.2. *Let $(\pi, V) \in \mathcal{C}_I(G)$ be an irreducible hermitian G -representation. If V^I is a unitary \mathcal{H} -module, then V is a unitary G -representation.*

The above theorem was proven in [BM1] under the assumption that G is split of adjoint type and V has “real infinitesimal character”. The latter assumption was removed in [BM2] using a reduction to graded Hecke algebras and endoscopic groups. The assumption that G is split adjoint was removed in [BC] (where the theorem was generalized to other Bernstein components as well), but the argument still involved the reduction to real infinitesimal character. In the next subsection, we explain that the present result on the rigid cocenter, Theorem 7.6(1), allows for a direct extension of the [BM1] argument to arbitrary representations V with Iwahori fixed vectors for any semisimple connected k -group G in the sense of [Bo]. This allows one to bypass the reduction to real infinitesimal character from [BM2, BC].

9.3. Fix a basis $\{\bar{V}_i\}_{1 \leq i \leq n}$ of $\bar{R}(\mathcal{H})_{\text{rigid}}$ consisting of genuine finite-dimensional \mathcal{H} -modules. By invoking Langlands classification or Corollary 6.9, it is clear that we may choose \bar{V}_i so that they are all tempered \mathcal{H} -modules. Under the functor (9.3), there exist smooth admissible G -representations V_i such that $V_i^I = \bar{V}_i$. It is well-known that F_I induces a bijection between tempered representations in the two categories, thus V_i are tempered G -representations.³

³ In fact, F_I induces a homeomorphism between the supports of the Plancherel measures in the two categories [BHK, Theorem B], but we will not need this more precise statement.

Since the functor (9.3) commutes with parabolic induction [BM1, discussion around Theorem 6.1], we immediately see that $\{V_i\}$ is a basis of $\bar{R}^0(G)_{\text{rigid}}$.

Let $(\pi, V) \in \mathcal{C}_I(G)$ be an irreducible, hermitian representation with G -invariant hermitian form $\langle \cdot, \cdot \rangle_G$. Let $K \supset I$ be a maximal parahoric subgroup. Following [Vo], one defines the K -signature of V as follows. Let (σ, U_σ) be a smooth irreducible K -representation with an implicitly fixed positive definite K -invariant form. The K -character of V is the formal sum

$$\theta_K(V) = \sum_{\sigma \in \hat{K}} m_\sigma \sigma, \quad \text{where } m_\sigma = \dim \text{Hom}_K[U_\sigma, V] < \infty. \tag{9.4}$$

Then the form $\langle \cdot, \cdot \rangle_G$ induces a form on the isotypic component $\text{Hom}_K[U_\sigma, V]$ of σ in V (if this is nonzero), and this form has signature (p_σ, q_σ) . Of course, $p_\sigma + q_\sigma = m_\sigma$. The K -signature character of V is the formal combination

$$\Sigma_K(V) = \left(\sum_{\sigma \in \hat{K}} p_\sigma \sigma, \sum_{\sigma \in \hat{K}} q_\sigma \sigma \right). \tag{9.5}$$

Since $\{V_i\}$ is a basis of $\bar{R}^0(G)_{\text{rigid}}$, in particular we have the following sharpening of [BM1, Theorem 5.3].

Lemma 9.3. *For every $K \supset I$, the K -character of any irreducible representation in $\mathcal{C}_I(G)$ is a linear combination of $\theta_K(V_i)$, $i = 1, \dots, n$.*

The same Jantzen filtration arguments as in [BM1, Section 5] lead then to a sharpened version of [BM1, Theorem 5.3], which is the p -adic analogue of Vogan’s signature theorem [Vo, Theorem 1.5].

Theorem 9.4 (cf. [BM1, Theorem 5.3]). *Let $\{V_i\}_{1 \leq i \leq n}$ be the fixed basis of $\bar{R}^0(G)_{\text{rigid}}$ constructed above, consisting of tempered representations. Let (π, V) be an arbitrary irreducible Hermitian G -representation in $\mathcal{C}_I(G)$. There exist integers a_i, b_i such that, for every $K \supset I$, the K -signature character of V equals*

$$\Sigma_K(V) = \left(\sum_{i=1}^n a_i \theta_K(V_i), \sum_{i=1}^n b_i \theta_K(V_i) \right). \tag{9.6}$$

We emphasize that in Theorem 9.4, the integers a_i, b_i do not depend on the choice of K , but only on V .

9.4. Analogous definitions of K -character and K -signature exist for \mathcal{H} -modules. Let $\mathcal{H}_K = C^\infty(I \backslash K / I)$ be the subalgebra of \mathcal{H} consisting of functions whose support is in K . If (σ, U_σ) is a K -type, then (σ, U_σ^I) is an \mathcal{H}_K -module. If V^I is a simple \mathcal{H} -module, then the \mathcal{H}_K -character of V^I is

$$\theta_{\mathcal{H}_K}(V^I) = \sum_{\sigma \in \hat{K}, U_\sigma^I \neq 0} m_\sigma \sigma, \tag{9.7}$$

and the \mathcal{H}_K -signature character is

$$\Sigma_{\mathcal{H}_K}(V^I) = \left(\sum_{\sigma \in \hat{K}, U_\sigma^I \neq 0} p_\sigma \sigma, \sum_{\sigma \in \hat{K}, U_\sigma^I \neq 0} q_\sigma \sigma \right). \tag{9.8}$$

Theorem 9.4 implies that the \mathcal{H}_K -signature character of V^I is

$$\Sigma_{\mathcal{H}_K}(V^I) = \left(\sum_{i=1}^n a_i \theta_{\mathcal{H}_K}(V_i^I), \sum_{i=1}^n b_i \theta_{\mathcal{H}_K}(V_i^I) \right), \quad (9.9)$$

with the same integers a_i, b_i as in Theorem 9.4.

Now we can complete the argument. Suppose that V^I is unitary as an \mathcal{H} -module. This means that the negative part of the K -signature must be 0 for all K . From (9.9), this means that

$$\sum_{i=1}^n b_i \theta_{\mathcal{H}_K}(V_i^I) = 0 \quad (9.10)$$

for all K . If $T_{\mathcal{O}}$ is a basis element of $\bar{\mathcal{H}}^{\text{rigid}}$ as in Theorem 8.1, there exists $K \supset I$ such that $T_{\mathcal{O}}$ is the delta function supported at a double coset $I\tilde{w}I$, $\tilde{w} \in K$. Therefore

$$\sum_{i=1}^n b_i \text{tr}(T_{\mathcal{O}}, V_i^I) = 0$$

for all basis elements $T_{\mathcal{O}}$ of the rigid cocenter $\bar{\mathcal{H}}^{\text{rigid}}$. Recall that $\{V_i^I\}$ form a basis of $\bar{R}(\mathcal{H})_{\text{rigid}}$. But then Theorem 7.6 implies that they are linearly independent over $\bar{\mathcal{H}}^{\text{rigid}}$, and therefore $b_i = 0$ for all $i = 1, \dots, n$.

Finally, this means that the negative part of the signature $\Sigma_K(V)$ is 0 in Theorem 9.4, and so V is unitary as a G -representation. This proves the implication:

if V^I is a unitary \mathcal{H} -module, then V is a unitary G -representation.

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References

- [AM] Ariki, S., Mathas, A.: The number of simple modules of the Hecke algebras of type $G(r, 1, n)$. *Math. Z.* **233**, 601–623 (2000) [Zbl 0955.20003](#) [MR 1750939](#)
- [Ar] Arthur, J.: On elliptic tempered characters. *Acta Math.* **171**, 73–138 (1993) [Zbl 0822.22011](#) [MR 1237898](#)
- [BC] Barbasch, D., Ciubotaru, D.: Unitary equivalences for reductive p -adic groups. *Amer. J. Math.* **135**, 1633–1674 (2013) [Zbl 1283.22011](#) [MR 3145006](#)
- [BM1] Barbasch, D., Moy, A.: A unitarity criterion for p -adic groups. *Invent. Math.* **98**, 19–37 (1989) [Zbl 0676.22012](#)
- [BM2] Barbasch, D., Moy, A.: Reduction to real infinitesimal character in affine Hecke algebras. *J. Amer. Math. Soc.* **6**, 611–635 (1993) [Zbl 0835.22016](#) [MR 1186959](#)
- [BM3] Barbasch, D., Moy, A.: Unitary spherical spectrum for p -adic classical groups. *Acta Appl. Math.* **44**, 3–37 (1996) [Zbl 0862.22015](#) [MR 1407038](#)

- [Be] Bernstein, J.: Representations of reductive p -adic groups. Lecture notes by K. Rummelhart, Harvard Univ. (1992)
- [BDK] Bernstein, J., Deligne, P., Kazhdan, D.: Trace Paley–Wiener theorem for reductive p -adic groups. *J. Anal. Math.* **47**, 180–192 (1986) [Zbl 0634.22011](#) [MR 0874050](#)
- [Be] Bezrukavnikov, R.: Homology properties of representations of p -adic groups related to geometry of the group at infinity. Ph.D. thesis, Univ. of Tel Aviv (1998)
- [BB] Blanc, P., Brylinski, J.-L.: Cyclic homology and the Selberg principle. *J. Funct. Anal.* **109**, 289–330 (1992) [Zbl 0783.55004](#) [MR 1186324](#)
- [Bo] Borel, A.: Admissible representations of a semi-simple group over a local field with vectors fixed under an Iwahori subgroup. *Invent. Math.* **35**, 233–259 (1976) [Zbl 0334.22012](#) [MR 0444849](#)
- [BHK] Bushnell, C., Henniart, G., Kutzko, P.: Types and explicit Plancherel formulae for reductive p -adic groups. In: *On Certain L-Functions*, Clay Math. Proc. 13, Amer. Math. Soc., Providence, RI, 55–80 (2011) [Zbl 1222.22011](#) [MR 2767510](#)
- [BK] Bushnell, C., Kutzko, P.: Smooth representations of reductive p -adic groups: structure theory via types. *Proc. London Math. Soc. (3)* **77**, 582–634 (1998) [Zbl 0911.22014](#) [MR 1643417](#)
- [CG] Chriss, N., Ginzburg, V.: *Representation Theory and Complex Geometry*. Birkhäuser, Boston (1997) [Zbl 0879.22001](#) [MR 1433132](#)
- [CH] Ciubotaru, D., He, X.: The cocenter of graded affine Hecke algebras and the density theorem. *J. Pure Appl. Algebra* **220**, 382–410 (2016) [Zbl 06499137](#) [MR 3393467](#)
- [Da] Dat, J.-F.: On the K_0 of a p -adic group. *Invent. Math.* **140**, 171–226 (2000) [Zbl 0981.22004](#) [MR 1779801](#)
- [DJ] Dipper, R., James, G.: Representations of Hecke algebra of type B_n . *J. Algebra* **146**, 454–481 (1992) [Zbl 0808.20016](#) [MR 1152915](#)
- [Fl] Flicker, Y.: Bernstein’s isomorphism and good forms. In: *K-Theory and Algebraic Geometry: Connections with Quadratic Forms and Division Algebras* (Santa Barbara, CA, 1992), Proc. Sympos. Pure Math. 58, Vol. II, Amer. Math. Soc., Providence, RI, 171–196 (1995) [Zbl 0840.22031](#) [MR 1327297](#)
- [GP] Geck, M., Pfeiffer, G.: *Characters of Finite Coxeter Groups and Iwahori–Hecke Algebras*. London Math. Soc. Monogr. 21, Clarendon Press, New York (2000) [Zbl 0996.20004](#) [MR 1778802](#)
- [He] He, X.: Geometric and homological properties of affine Deligne–Lusztig varieties. *Ann. Math.* **179** (2014), 367–404. [Zbl 1321.14039](#) [MR 3126571](#)
- [HN1] He, X., Nie, S.: Minimal length elements of extended affine Weyl group. *Compos. Math.* **150**, 1903–1927 (2014) [Zbl 1335.20036](#)
- [HN2] He, X., Nie, S.: P -alcoves, parabolic subalgebras and cocenters of affine Hecke algebras. *Selecta Math. (N.S.)* **21**, 995–1019 (2015) [Zbl 1325.20002](#) [MR 3366924](#)
- [Hei] Heiermann, V.: Opérateurs d’entrelacement et algèbres de Hecke avec paramètres d’un groupe réductif p -adique: le cas des groupes classiques. *Selecta Math. (N.S.)* **17**, 713–756 (2011) [Zbl 1246.22021](#) [MR 2827179](#)
- [HM] Howe, R. (with the collaboration of A. Moy): *Harish-Chandra Homomorphisms for p -adic Groups*. CBMS Reg. Conf. Ser. Math. 59, Amer. Math. Soc., Providence, RI (1985) [Zbl 0593.22014](#) [MR 0821216](#)
- [IM] Iwahori, N., Matsumoto, H.: On some Bruhat decomposition and the structure of the Hecke rings of p -adic Chevalley groups. *Inst. Hautes Études Sci. Publ. Math.* **25**, 5–48 (1965) [Zbl 0228.20015](#) [MR 0185016](#)
- [Kat] Kato, S.: An exotic Deligne–Langlands correspondence for symplectic groups. *Duke Math. J.* **148**, 305–371 (2009) [Zbl 1183.20002](#) [MR 2524498](#)

- [Kaz1] Kazhdan, D.: Cuspidal geometry of p -adic groups. *J. Anal. Math.* **47**, 1–36 (1986) [Zbl 0634.22009](#) [MR 0874042](#)
- [Kaz2] Kazhdan, D.: Representations of groups over close local fields. *J. Anal. Math.* **47**, 175–179 (1986) [Zbl 0634.22010](#) [MR 0874049](#)
- [KL] Kazhdan, D., Lusztig, G.: Proof of the Deligne–Langlands conjecture for Hecke algebras. *Invent. Math.* **87**, 153–215 (1987) [Zbl 0613.22004](#) [MR 0862716](#)
- [Ko1] Kottwitz, R.: Isocrystals with additional structure. *Compos. Math.* **56** (1985), 201–220. [Zbl 0597.20038](#) [MR 0809866](#)
- [Ko2] Kottwitz, R.: Isocrystals with additional structure. II. *Compos. Math.* **109** (1997), 255–339. [Zbl 0966.20022](#) [MR 1485921](#)
- [Lu1] Lusztig, G.: Affine Hecke algebras and their graded version. *J. Amer. Math. Soc.* **2**, 599–635 (1989) [Zbl 0715.22020](#) [MR 0991016](#)
- [Lu2] Lusztig, G.: Cuspidal local systems and graded algebras II. In: *Representations of Groups* (Banff, AB, 1994), Amer. Math. Soc., Providence, RI, 217–275 (1995) [Zbl 0841.22013](#) [MR 1357201](#)
- [Mi] Miemietz, V.: On representations of affine Hecke algebras of type B . *Algebr. Represent. Theory* **11**, 369–405 (2008) [Zbl 1167.20005](#) [MR 2417511](#)
- [Op] Opdam, E.: On the spectral decomposition of affine Hecke algebras. *J. Inst. Math. Jussieu* **3**, 531–648 (2004) [Zbl 1102.22009](#) [MR 2094450](#)
- [OS1] Opdam, E., Solleveld, M.: Discrete series characters for affine Hecke algebras and their formal degrees. *Acta Math.* **205**, 105–187 (2010) [Zbl 1214.20008](#) [MR 2736154](#)
- [OS2] Opdam, E., Solleveld, M.: Homological algebra for affine Hecke algebras. *Adv. Math.* **220**, 1549–1601 (2009) [Zbl 1195.20004](#) [MR 2493620](#)
- [RR] Ram, A., Ramagge, J.: Affine Hecke algebras, cyclotomic Hecke algebras and Clifford theory. *Trends in Mathematics*, Birkhäuser, 428–466 (2003) [Zbl 1063.20004](#) [MR 2017596](#)
- [Ree] Reeder, M.: Euler–Poincaré pairings and elliptic representations of Weyl groups and p -adic groups. *Compos. Math.* **129**, 149–181 (2001) [Zbl 1037.22039](#) [MR 1863300](#)
- [Ren] Renard, D.: Représentations des groupes réductifs p -adiques. *Cours Spécialisés 17*, Soc. Math. France, Paris (2010) [Zbl 1186.22020](#) [MR 2567785](#)
- [SS] Schneider, P., Stuhler, U.: Representation theory and sheaves on the Bruhat–Tits building. *Inst. Hautes Études Sci. Publ. Math.* **85**, 97–191 (1997) [Zbl 0892.22012](#) [MR 1471867](#)
- [SVV] Shan, P., Varagnolo, M., Vasserot, E.: Canonical bases and affine Hecke algebras of type D . *Adv. Math.* **227**, 267–291 (2011) [Zbl 1235.20006](#) [MR 2782195](#)
- [So] Solleveld, M.: Hochschild homology of affine Hecke algebras. *J. Algebra* **384**, 1–35 (2013) [Zbl 1305.20002](#) [MR 3045149](#)
- [VV] Varagnolo, M., Vasserot, E.: Canonical bases and affine Hecke algebras of type B . *Invent. Math.* **185**, 593–693 (2011) [Zbl 1239.20007](#) [MR 2827096](#)
- [Vi] Vignéras, M.-V.: Modular representations of p -adic groups and of affine Hecke algebras. In: *Proc. Int. Congress Math. (Beijing, 2002)*, Vol. 2, Higher Education Press, Beijing, 667–677 (2002) [Zbl 1151.11325](#) [MR 1957074](#)
- [Vo] Vogan Jr., D. A.: Unitarizability of certain series of representations. *Ann. of Math. (2)* **120**, 141–187 (1984) [Zbl 0561.22010](#) [MR 0750719](#)
- [Xi1] Xi, N.: Representations of affine Hecke algebras and based rings of affine Weyl groups. *J. Amer. Math. Soc.* **20**, 211–217 (2007) [Zbl 1150.20005](#) [MR 2257401](#)
- [Xi2] Xi, N.: Representations of affine Hecke algebras of type G_2 . *Acta Math. Sci. Ser. B Engl. Ed.* **29**, 515–526 (3) (2009) [Zbl 1212.20003](#) [MR 2514357](#)