J. Eur. Math. Soc. 19, 3355–3376

DOI 10.4171/JEMS/740

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A new isoperimetric inequality for elasticae

Received December 10, 2014

Abstract. For a smooth curve γ, we define its elastic energy as $E(\gamma) = \frac{1}{2}\int_\gamma k^2(s)ds$ where $k(s)$ is the curvature. The main purpose of the paper is to prove that among all smooth, simply connected, bounded open sets of prescribed area in $\mathbb{R}^2$, the disc has the boundary with the least elastic energy. In other words, for any bounded simply connected domain Ω, the following isoperimetric inequality holds: $E^2(\partial\Omega)A(\Omega) \geq \pi^3$. The analysis relies on the minimization of the elastic energy of drops enclosing a prescribed area, for which we also give an analytic answer.

Keywords. Euler elasticae, minimization of elastic energy, isoperimetric inequality, curvature

1. Introduction

Let Ω be a smooth, bounded simply connected open set in the plane (the exact smoothness which is required will be made precise in Section 2) and let $\partial\Omega$ denote its boundary. Following L. Euler, we define its *elastic energy* as

$$E(\partial\Omega) = \frac{1}{2}\int_{\partial\Omega} k^2(s)\, ds \tag{1}$$

where s is the arc length parameter and k is the curvature. We will denote by $A(\Omega)$ the area of Ω and by $L(\Omega)$ its perimeter. The aim of this paper is to prove the following isoperimetric inequality.

Theorem 1.1. *For any bounded, smooth, simply connected, nonempty open set $\Omega \subseteq \mathbb{R}^2$,*

$$E^2(\partial\Omega)A(\Omega) \geq \pi^3, \tag{2}$$

where equality holds only for the disc.

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Mathematics Subject Classification (2010): Primary 49Q10; Secondary 53A04, 51M16

Since for any disc B_R we have $E^2(\partial B_R)A(B_R) = \pi^3$, we deduce that for every $A_0 > 0$, the disc is the unique solution for the minimization problem

$$\min\{E(\partial\Omega) : A(\Omega) = A_0,\ \Omega \text{ a bounded, smooth, simply connected open set in } \mathbb{R}^2\}.$$

More precisely, if we perform any scaling of ratio t, we have $E(t\partial\Omega) = t^{-1}E(\partial\Omega)$ and $A(t\partial\Omega) = t^2 A(\partial\Omega)$. Therefore, it is classical to prove that the following three minimization problems are equivalent (in the sense that any solution of one gives a solution of the others after a suitable scaling):

(i) $\min E^2(\partial\Omega)A(\Omega)$,
(ii) $\min\{E(\partial\Omega) : A(\Omega) \le A_0\}$,
(iii) $\min\{E(\partial\Omega) + A(\Omega)\}$.

Let us make some comments. For a detailed bibliography on closed elasticae, we refer to the classical [8] or the more recent [9]. Inequality (2) was already known for convex domains. Indeed, by a famous inequality due to M. Gage [6], for any bounded convex domain we have

$$\frac{E(\partial\Omega)A(\Omega)}{L(\Omega)} \ge \frac{\pi}{2}$$

with equality for the disc. Therefore,

$$E^2(\partial\Omega)A(\Omega) \ge E^2(\partial\Omega)A(\Omega)\frac{4\pi A(\Omega)}{L^2(\Omega)} \ge \frac{\pi^2}{4} \times 4\pi = \pi^3,$$

the first inequality being the classical isoperimetric inequality, and the second the Gage inequality. If the convexity hypothesis is dropped, then the Gage inequality is false (as shown by the counter-example of Figure 1).

The simple connectedness assumption is necessary. Indeed, if we take as a domain Ω the ring

$$\Omega_R = \{(x, y) : R < \sqrt{x^2 + y^2} < R + 1/R\},$$

we get

$$E(\partial\Omega_R) = \frac{\pi}{R} + \frac{\pi R}{R^2+1}, \quad \text{while} \quad A(\Omega_R) = \pi\left(R + \frac{1}{R}\right)^2 - \pi R^2 = 2\pi + \pi/R^2.$$

showing that $E^2(\partial\Omega_R)A(\Omega_R) \to 0$ as $R \to \infty$.

In the same way, the boundedness assumption is also necessary. Let us consider the following unbounded domain, subgraph of a Gaussian function, but with finite area and elastic energy:

$$\Omega_\alpha = \{(x, y) \in \mathbb{R}^2 : -\infty < x < \infty,\ 0 < y < e^{-\alpha x^2/2}\}.$$

We have

$$A(\Omega_\alpha) = \int_{-\infty}^{\infty} e^{-\alpha x^2/2}\,dx = \sqrt{2\pi/\alpha},$$

while

$$E(\partial\Omega_\alpha) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{(\alpha^2x^2-\alpha)^2 e^{-\alpha x^2}}{(1+\alpha^2x^2e^{-\alpha x^2})^{5/2}}\,dx = \frac{\alpha^{3/2}}{2}\int_{-\infty}^{\infty} \frac{(u^2-1)^2 e^{-u^2}}{(1+\alpha u^2 e^{-u^2})^{5/2}}\,du,$$

and we see that $E^2(\partial\Omega_\alpha)A(\Omega_\alpha) \to 0$ as $\alpha \to 0$.

This shows that the assumptions in Theorem 1.1 cannot be weakened.

Our strategy is to solve the following equivalent version of problem (2):

$$\min\{E(\partial\Omega) + A(\Omega) : \Omega \subseteq \mathbb{R}^2 \text{ open, smooth, bounded, simply connected}\}, \tag{3}$$

and to prove that the solution is a disc.

The proof follows the direct method of the calculus of variations (existence, regularity, analysis of the optimality conditions), but the existence part is by no means easy. In fact we need a control on the perimeter of a minimizing sequence (which is not a priori bounded) and have to handle the fact that the geometric limit of a minimizing sequence may not be smooth any more, in the sense that tangential self-intersections could occur.

Indeed, the boundedness constraints on $E(\partial\Omega)$ and $A(\Omega)$ do not ensure that the perimeter is uniformly bounded, as shown by a counter-example like a dumbell (see Figure 1). In order to deal with minimizing sequences having a diameter going to to infinity, our strategy follows the idea introduced by De Giorgi [4] for the analysis of the isoperimetric inequality. First, we introduce an artificial boundedness constraint: we shall assume that all our competing sets lie in a ball of radius R centered at the origin. In a second step, we prove that if R is large enough, the optimal set does not touch the boundary of the ball (up to a suitable translation), and so we will be able to write optimality conditions on the full boundary, and consequently deduce that the set is the disc.

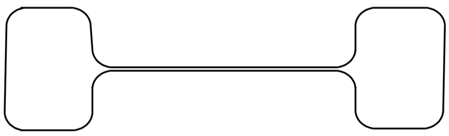

Fig. 1. A dumbbell with bounded area and elastic energy with a large perimeter.

In order to handle the self-intersection points, we analyse the minimization of the elastic energy of *drops* enclosing a fixed area, i.e. closed loops without self-intersection points, which are smooth except at one point, where the tangents are opposite. For this class of sets, we can easily eliminate the situations in which the limit of a minimizing sequence has self-intersections. Consequently, we give a complete characterization of the optimal drop, which turns out to be unique. We refer to Section 3 for a precise definition of drops.

Here is our plan.

- Let $R > 0$. We analyze the problem

$$\min\{E(\partial\Omega) + A(\Omega) : \Omega \subseteq B_R \text{ an open, smooth, simply connected } drop\}. \tag{4}$$

  There exists $R_0 > 0$ such that for $R \geq R_0$ the optimal drop does not touch the boundary of B_R. As a consequence of the optimality conditions, we give an analytic description of the optimal drop and deduce it is unique, independent of R.

- For every $R \geq R_0$ we consider

$$\min\{E(\partial\Omega) + A(\Omega) : \Omega \subseteq B_R \text{ open, smooth, simply connected}\}. \tag{5}$$

  We prove the existence of a solution which does not touch the boundary of B_R. The possible self-intersection points of a geometric limit of a minimizing sequence are eliminated by direct comparison with the disc, since their energy would be at least the double of the energy of the optimal drop. Consequently, we can write the optimality conditions on the full boundary and deduce that there are only four sets which satisfy the optimality conditions. By direct observation, the disc is the solution.
- We conclude that the solution of (3) is the disc, and so inequality (2) holds, with equality if and only if Ω is a disc.

2. Preliminaries

All curves $\gamma : [0, L] \to \mathbb{R}^2$ considered are parametrized by arc length. We denote by θ the angle of the tangent to γ to the axis Ox. The curvature of γ at $\gamma(s)$ will be denoted $k(s)$ and it is equal to $\theta'(s)$. Since we shall work with curves with finite elastic energy, the function θ belongs to the Sobolev space $H^1(0, L)$. By the embedding $H^1(0, L) \subseteq C^{0,\alpha}[0, L]$ for any $\alpha < 1/2$, the function θ is in particular continuous.

All curves we work with in this paper have finite *elastic energy*

$$E(\gamma) = \frac{1}{2} \int_{[0,L]} |\theta'(s)|^2 \, ds < \infty.$$

We start with a series of three technical lemmas.

Lemma 2.1. *Let $\gamma : [0, L] \to \mathbb{R}^2$ be a curve parametrized by arc length such that $E(\gamma) < \infty$. Then for $\delta = \pi^2/(32E(\gamma))$ the curve is locally the graph of a 1-Lipschitz function on each interval of size $\delta/\sqrt{2}$.*

Proof. Fix s_0 and assume that $\theta(s_0) = 0$. By the Cauchy–Schwarz inequality we get, for every $s \in (s_0, s_0 + \delta)$,

$$|\theta(s)| \leq \delta^{1/2}(2E(\gamma))^{1/2} \leq \pi/4,$$

which gives the conclusion. □

Lemma 2.2. *Let $\gamma : [0, L] \to \mathbb{R}^2$ be a curve parametrized by arc length such that $E(\gamma) < \infty$. If $\varepsilon > 0$ and $0 \leq s < t \leq L$ are such that*

$$|\theta(s) - \theta(t)| = \varepsilon,$$

then

$$\int_{[s,t]} |\theta'|^2 \geq \varepsilon^2/L.$$

Proof. As $\int_{[0,L]} |\theta'|^2\, ds < \infty$, we write

$$|\theta(s) - \theta(t)| = \left|\int_s^t \theta'(u)\, du\right| \leq |t-s|^{1/2}\left(\int_{[s,t]} |\theta'|^2\right)^{1/2},$$

which gives the result. □

Remark 2.3. The idea eneging from the lemma is that if there is an ε-variation of the angle, then the elastic energy on that section of the curve is at least a constant times ε^2, the constant depending on the global length of the curve.

Let B_R be a ball of radius R.

Lemma 2.4. *Let $\gamma : [0, L] \to \mathbb{R}^2$ be a smooth loop parametrized by arc length such that $E(\gamma) < \infty$ and $\gamma([0, L]) \subseteq B_R$. Then*

$$L \leq 2R^2 E(\gamma).$$

Proof. Denoting $\gamma(s) = (x(s), y(s))$, we have

$$L = \int_0^L [x'^2(s) + y'^2(s)]\, ds = -\int_0^L [x(s)x''(s) + y(s)y''(s)]\, ds.$$

But $|x(s)x''(s)+y(s)y''(s)| \leq (x^2(s)+y^2(s))^{1/2}(x''^2(s)+y''^2(s))^{1/2} \leq R|k(s)|$. Therefore, the conclusion of the lemma follows from the Cauchy–Schwarz inequality

$$L^2 \leq R^2 L \int_0^L k^2(s)\, ds.$$ □

Assume that a simply connected open set Ω is bounded by a loop γ such that $E(\gamma) < \infty$, and γ has no self-intersections on an interval $(s_0, s_0 + L)$. Assume moreover that for all perturbations of the form $\mathrm{Id} + tV$, $V \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$, such that $(\operatorname{supp} V) \cap \partial\Omega = (\operatorname{supp} V) \cap \gamma|_{(s_0,s_0+L)}$ the shape derivative of $E(\gamma) + A(\Omega)$ is vanishing at $t = 0$ (see [7, Chapter 5] for details on the shape derivative). We shall call such a piece of curve $\gamma|_{(s_0,s_0+L)}$ a *free branch* and denote it $\tilde{\gamma}$.

Theorem 2.5 (Optimality conditions). *Let $\tilde{\gamma}$ be any free branch of a minimizer Ω of the energy $E(\partial\Omega) + A(\Omega)$. Then $s \mapsto k(s)$ is C^∞ on $\tilde{\gamma}$ and satisfies:*

(B1) $k'' = -\frac{1}{2}k^3 + 1$,
(B2) $k'^2 = -\frac{1}{4}k^4 + 2k + 2C$ *for some constant C,*
(B3) *there exists $Q \in \mathbb{R}^2$ such that for all $M \in \tilde{\gamma}$, $|QM|^2 = 2k + 2C$ for some constant C,*
(B4) *there exists $Q \in \mathbb{R}^2$ such that for all $M \in \tilde{\gamma}$, $\overrightarrow{QM} \cdot \vec{\nu} = \frac{1}{2}k^2$ where $\vec{\nu}$ is the exterior normal vector to $\partial\Omega$.*

Remark 2.6. The point Q in (B3) and (B4) is the same (see the proof below). The constant C in (B2) and (B3) is also the same. To see that, take a point M_{M} on $\tilde{\gamma}$ where the curvature k is maximum. If such a point does not exist, just extend the curve with the same ODE. Then, according to (B3), $|QM_{\mathrm{M}}|$ is also maximum and the normal derivative of the boundary at this point is $\overrightarrow{QM}_{\mathrm{M}}/|QM_{\mathrm{M}}|$. Therefore (B4) yields $|QM_{\mathrm{M}}| = \frac{1}{2}k^2$, and plugging this into (B3) gives (B2), because $k' = 0$ at this point, with the same constant.

Proof of Theorem 2.5. Let us first prove (B3). For that purpose we use the expression of the elastic energy and the area parametrized with the angle θ. We have (see [2] for more details)

$$E(\tilde{\gamma}) = \frac{1}{2}\int_{\tilde{\gamma}} \theta'^2\, ds =: e(\theta), \qquad A(\Omega) = \iint_T \cos\theta(u)\sin\theta(s)\, du\, ds =: a(\theta)$$

where T is the triangle $\{(u,s) \in \mathbb{R}^2 : 0 \le u \le s \le L(\Omega)\}$. We write L for $L(\Omega)$. Thus we are led to minimize the sum $e(\theta) + a(\theta)$ with the following constraints (the starting point and the ending point of the branch $\tilde{\gamma}$ are fixed):

$$\int_0^L \cos(\theta(s))\, ds = x(L) - x(0), \qquad \int_0^L \sin(\theta(s))\, ds = y(L) - y(0). \tag{6}$$

The derivative of $e(\theta)$ is (for a compactly supported perturbation v)

$$\langle de(\theta), v\rangle = \int_0^L \theta' v'\, ds = -\int_0^L \theta'' v\, ds,$$

while the derivative of $a(\theta)$ is given by

$$\langle da(\theta), v\rangle = \iint_T [\cos\theta(u)\cos\theta(s)v(s) - \sin\theta(s)\sin\theta(u)v(u)]\, du\, ds.$$

Using (6) and Fubini, we can write

$$\iint_T \sin\theta(s)\sin\theta(u)v(u)\, du\, ds$$
$$= (y(L) - y(0))\int_0^L \sin\theta(s)v(s)\, ds - \iint_T \sin\theta(u)\sin\theta(s)v(s)\, du\, ds.$$

Therefore, the optimality condition for the constrained problem reads: there exist Lagrange multipliers λ_1, λ_2 such that, for any v,

$$-\int_0^L \theta'' v\, ds + \int_0^L \left(\cos\theta(s)\int_0^s \cos\theta(u)\, du + \sin\theta(s)\int_0^s \sin\theta(u)\, du\right) v(s)\, ds$$
$$= (y(L) - y(0))\int_0^L \sin\theta(s)v(s)\, ds - \lambda_1\int_0^L \sin\theta(s)v(s)\, ds + \lambda_2\int_0^L \cos\theta(s)v(s)\, ds, \tag{7}$$

which implies (thanks to $x'(s) = \cos\theta(s)$, $y'(s) = \sin\theta(s)$)

$$-\theta'' + x'(x - x(0)) + y'(y - y(0)) = (y(L) - y(0) - \lambda_1)y' + \lambda_2 x'. \tag{8}$$

By integration, we get (B3) on setting $Q = (x(0) + \lambda_2, y(L) - \lambda_1)$.

Now, the C^∞ regularity of $k(s)$ (and $\theta(s)$) comes from a bootstrap argument and equation (8). The first condition (B1) comes from the classical *shape derivative* of the elastic energy (under the small perturbation defined above). Following e.g. [2, Appendix], we see that it is given by

$$dE(\partial\Omega, V) = -\int_{\tilde{\gamma}} \left(\frac{1}{2}k(s)^3 + k''(s)\right)\langle V, \nu\rangle\, ds,$$

while the derivative of the area is classically

$$dA(\Omega, V) = \int_{\tilde{\gamma}} \langle V, \nu\rangle\, ds,$$

Condition (B1) follows since the derivative of $E + A$ must vanish for any admissible V. We obtain condition (B2) by multiplying (B1) by k' and integrating.

At last, differentiating twice (B3) we get $k' = \overrightarrow{QM} \cdot \vec{\tau}$ (where $\vec{\tau}$ is the tangent vector) and $k'' = 1 - k\overrightarrow{QM} \cdot \vec{\nu}$. Using (B1) we see that $\frac{1}{2}k^3 = k\overrightarrow{QM} \cdot \vec{\nu}$, so (B4) holds where $k \neq 0$. Since k is a solution of the ODE (B1), and therefore can be written with elliptic functions, it can only vanish at isolated points, and thus (B4) holds everywhere by continuity of both members. □

In the following lemma, we assume that the simply connected open set Ω is a minimizer of the energy $E(\partial\Omega) + A(\Omega)$.

Lemma 2.7. *Any free branch of a minimizer Ω has length L uniformly bounded by*

$$L \leq 146.$$

Proof. We work with a free branch of $\tilde{\gamma}$ on $s \in (s_0, s_0 + L)$ and use the optimality conditions above. We also know that the elastic energy of this branch is less than the total energy of the best disc B, so that

$$E(\tilde{\gamma}) \leq E(\partial B) + A(B) = 3\pi 2^{-2/3}. \tag{9}$$

We consider two cases. Assume first that $C \leq 1$ on this branch (C is defined above in (B2), (B3)). Then we know from (ODE3) in the Appendix that

$$k(s) \leq k_{\mathrm{M}}(C) \leq k_{\mathrm{M}}(1) \leq 7/3. \tag{10}$$

Then, from (B3),

$$|QM|^2 \leq 14/3 + 2 = 20/3,$$

hence the arc is contained in the disc centered at Q with radius $R_0 = \sqrt{20/3}$.

On the other hand, if we put the origin at Q, then

$$L(\tilde{\gamma}) = L = \int_0^L (x'^2 + y'^2)\, dx = (xx' + yy')|_0^L - \int_0^L (xx'' + yy'')\, ds.$$

But $|x(L)x'(L) + y(L)y'(L)| \leq R_0$ and $|x(0)x'(0) + y(0)y'(0)| \leq R_0$, while by Cauchy–Schwarz and (9),

$$\left| \int_0^L [xx'' + yy'']\, ds \right| \leq R_0 \int_0^L |k|\, ds \leq R_0 \sqrt{L 2E(\tilde{\gamma})} \leq R_0 \sqrt{L 3\pi 2^{1/3}}.$$

Therefore,

$$L \leq 2\sqrt{20/3} + \sqrt{(20/3) \times 3\pi \times 2^{1/3}}\, \sqrt{L}, \tag{11}$$

which implies (as soon as $C \leq 1$)

$$L \leq 90. \tag{12}$$

Second case: $C \geq 1$ for this branch. In this case, from (ODE3) in the Appendix we have

$$k_{\mathrm{M}}(C) \geq k_{\mathrm{M}}(1) \geq 9/4, \qquad k_{\mathrm{m}}(C) \leq k_{\mathrm{m}}(1) \leq -9/10.$$

We decompose the interval $I = (s_0, s_0 + L)$ into three parts (some could be empty), $I = I_- \cup I_0 \cup I_+$ where

$$I_- = \{s \in I : k(s) \leq 0\},$$
$$I_0 = \{s \in I : 0 < k(s) < 2^{1/3}\},$$
$$I_+ = \{s \in I : 2^{1/3} \leq k(s)\},$$

and we are going to prove that the length of each part is uniformly bounded, by a controlled constant. First of all, we have seen that the integral of k^2 on a period satisfies (see (ODE4) in the Appendix)

$$\frac{1}{2} \int_0^T k^2\, ds \geq \frac{\pi}{4} \sqrt{\frac{22}{3}}.$$

Following (9), this implies that we cannot have more than three periods on each free branch. We begin with I_+. Obviously

$$E(\tilde{\gamma}) \geq \frac{1}{2} \int_{I_+} k^2\, ds \geq \frac{1}{2} 2^{2/3} |I_+|,$$

therefore

$$|I_+| \leq 3\pi \times 2^{-2/3} \times 2^{1/3} \leq 8. \tag{13}$$

For I_0, we consider one of its connected components, say (α, β). Since $k_{\mathrm{M}}(C) \geq 9/4 > 2^{1/3}$ and $k_{\mathrm{m}}(C) \leq -9/10 < 0$, we cannot have any local minimum or local maximum of k in I_0 according to (ODE2) from the Appendix. Therefore, k is either increasing from $k(\alpha)$ to $k(\beta)$, or decreasing from $k(\alpha)$ to $k(\beta)$. Moreover, there are at most six such connected components because there are at most three periods of k. Let us consider the case of k increasing from $k(\alpha)$ to $k(\beta)$, the other one being similar. We have $0 \le k(\alpha) \le k(\beta) \le 2^{1/3}$. By (B1), $k'' \ge 0$ on (α, β), so that k is convex. Therefore

$$k(\alpha) + k'(\alpha)(s-\alpha) \le k(s), \tag{14}$$

which implies

$$k(\alpha) + k'(\alpha)(\beta-\alpha) \le k(\beta) \le 2^{1/3}.$$

Now $k(\alpha) \ge 0$ and $k'(\alpha) = \sqrt{2C + 2k(\alpha) - \frac{1}{4}k^4(\alpha)} \ge \sqrt{2C}$, thus $\sqrt{2}(\beta-\alpha) \le \sqrt{2C}(\beta-\alpha) \le 2^{1/3}$ or $\beta - \alpha \le 2^{-1/6}$. Since there are at most six such intervals, we have

$$|I_0| \le 6 \times 2^{-1/6} \le 6. \tag{15}$$

Finally, we consider the case of I_-. The set I_- is not empty only when $C > 0$ and $k_{\mathrm{m}} < 0$. The set I_- is composed of connected components $[\alpha, \beta]$ such that $k(\alpha) = k(\beta) = 0$ or is included in such connected components. Since we want to estimate the length of I_- from above, it suffices to look for the length of such connected components. There are at most three such (identical) components and $k\left(\frac{\alpha+\beta}{2}\right) = k_{\mathrm{m}}$ by symmetry.

Now, the elastic energy of such a component satisfies

$$E(\tilde{\gamma}_{\alpha_1,\beta_1}) = \frac{1}{2}\int_{\alpha_1}^{\beta_1} k^2\,ds = \int_{(\alpha_1+\beta_1)/2}^{\beta_1} k^2\,ds = \int_{\alpha_1}^{(\alpha_1+\beta_1)/2} k^2\,ds. \tag{16}$$

We denote by $L_- = \beta_1 - \alpha_1$ the length of this component. By convexity, on $(\alpha_1, (\alpha_1+\beta_1)/2)$ we have

$$k(s) \le \frac{2k_{\mathrm{m}}}{L_-}(s-\alpha_1) \le 0,$$

thus

$$E(\tilde{\gamma}_{\alpha_1,\beta_1}) \ge \int_{\alpha_1}^{\alpha_1+L_-/2} \frac{4k_{\mathrm{m}}^2}{L_-^2}(s-\alpha_1)^2\,ds = \frac{k_{\mathrm{m}}^2}{6}L_-.$$

Now, for $C \ge 1$ we have (see (ODE3) in the Appendix) $k_{\mathrm{m}}^2 \ge k_{\mathrm{m}}^2(1) \ge 81/100$ and $E(\tilde{\gamma}_{\alpha_1,\beta_1}) \le 3\pi 2^{-2/3}$. Therefore

$$L_- \le \tfrac{600}{81} \times 3\pi 2^{-2/3} \le 44,$$

and the total length of I_- satisfies

$$|I_-| \le 3L_- \le 132. \tag{17}$$

In conclusion, for $C \ge 1$ the total length of the branch satisfies (by gathering (13), (15), (17))

$$L \le 132 + 8 + 6 = 146.$$ □

3. The optimal drop

In this section we prove the existence of a best *drop* minimizing the sum of the elastic energy and the area enclosed. We introduce the class of admissible *Jordan drops* consisting of simply connected open sets Ω bounded by a Jordan curve γ of finite length, which satisfies

$$\theta(0) = \theta(L_\gamma) - \pi, \qquad E(\gamma) < \infty,$$

where L_γ is the length of γ. A drop will be denoted (Ω, γ), Ω being the open set enclosed by the Jordan curve γ (all Jordan curves are oriented in the positive sense).

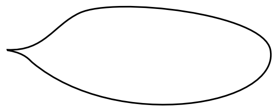

Fig. 2. A drop.

The class of Jordan drops is not closed under Haudsdorff convergence, since tangential contacts may occur in the limit of a sequence of Jordan drops. If this situation occurs for a minimizing sequence, we shall focus only on the loop which is the boundary of *a suitably chosen* connected component of the limit set, which turns out to be a Jordan drop. This selection is possible thanks to a priori geometric information on the minimizing sequence.

For some $R > 0$, we consider the problem

$$\inf\{E(\gamma) + A(\Omega) : (\Omega, \gamma) \text{ is a Jordan drop, } \Omega \subseteq B_R\}. \tag{18}$$

Note that by a similar argument to that for Lemma 2.4, the length of a Jordan drop γ cannot exceed $8R^2E(\gamma)$. Indeed, the same argument works for the drop: if the singularity lies at the origin, we have $x^2 + y^2 \le 4R^2$ since the diameter of the drop is less than $2R$.

Here is the main result.

Theorem 3.1. *Problem (18) has at least one solution.*

Remark 3.2. With no assumptions on the radius R, it could be possible that the optimal drop (Ω, γ) touches the boundary of the ball but, as we shall prove, it may not have self-intersections.

For simplicity of the notation, the ball B_R will be denoted B. We start with the following.

Lemma 3.3. *Let (Ω, γ) be a drop contained in B. If for some $\varepsilon > 0$ there exist $0 \le s < t \le L_\gamma$ with*

$$\theta(t) = \theta(s) - \pi - \varepsilon$$

then there exists a new drop $(\tilde\Omega, \tilde\gamma)$ in B such that

$$\int_{\tilde\gamma} |\tilde\theta'|^2 \le \int_\gamma |\theta'|^2 - \frac{\varepsilon^2}{2L_\gamma} \quad \textit{and} \quad A(\tilde\Omega) \le A(\Omega).$$

Proof. Assume s and t satisfy the hypotheses. Then, from continuity of θ, there exist $s < \bar{s} < \bar{t} < t$ such that

$$\theta(\bar{s}) = \theta(s) - \varepsilon/2 \quad \text{and} \quad \theta(\bar{t}) = \theta(t) + \varepsilon/2.$$

Moreover, there exist $\bar{s} \leq s' < t' \leq \bar{t}$ such that

$$\theta(t') = \theta(\bar{t}), \quad \theta(s') = \theta(\bar{s}) \quad \text{and} \quad \theta(u) \in (\theta(t'), \theta(s')) \quad \text{for every } u \in (s', t').$$

Indeed, we define

$$t' = \inf\{t > \bar{s} : \theta(t) = \theta(\bar{t})\}, \qquad s' = \sup\{s < t' : \theta(s) = \theta(\bar{s})\}.$$

Then the curve $\gamma|_{[s',t']}$ is a graph in the direction $\theta(s')$, by the choice of s' and t'. Setting the orientation of the curve in the trigonometric sense, we are in a configuration similar to Figure 4. Using the graph property, we can translate continuously the piece of the curve $\gamma|_{[s',t']}$ in a parallel way in the direction $\theta(s')$ until this piece touches again γ and add the two parallel segments described by the points $\gamma(s')$, $\gamma(t')$ in the newly created curve (Figure 4, right).

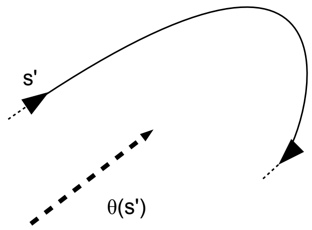

Fig. 3. The curve is a graph in the direction $\theta(s')$.

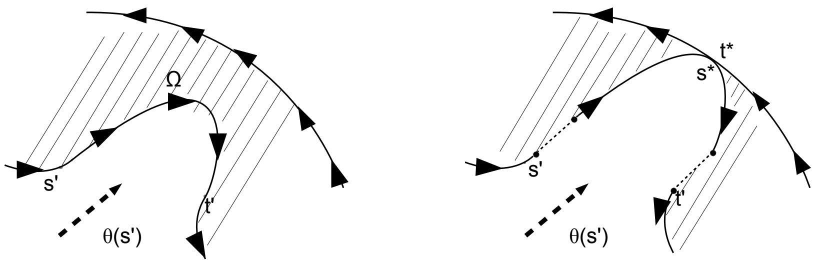

Fig. 4. Translation of $\gamma|_{[s',t']}$ in the direction $\theta(s')$.

We denote by $s_\alpha \in [s', t']$ and $t_\alpha \in [0, L] \setminus [s', t']$ the couples of touching points. We denote by s_1, respectively s_2, the minimal and maximal values of s_α. Then one of the curves starting at s_2 and ending at t_2, or starting at t_1 and ending at s_1, is a drop. Precisely, it is the one which does not contain the point $\gamma(0)$. Without losing generality we can assume it is the curve $s_2 \to t_2$; we rename the point $(s_2, t_2) = (s^*, t^*)$ and denote this curve $\tilde{\gamma}$. We notice that $\tilde{\gamma}$ cannot touch the piece $\gamma|_{[\bar{s},s']}$ any more. If there were a contact point, this contact is generated by the translation of $\gamma|_{[s',t']}$ and has to be precisely

(s^*, t^*). But in this case, t^* lies in the interval $[\bar{t}, s']$, so the curve starting at t^* and ending at s^* is a drop, which does not touch the piece $\gamma|_{[t',\bar{t}]}$.

In this way, we built a new drop $(\tilde{\Omega}, \tilde{\gamma})$, which encloses a domain contained in Ω, and in view of Lemma 2.2 has an elastic energy at least $\varepsilon^2/(4L_\gamma)$ smaller. □

Proof of Theorem 3.1. Let (Ω_n, γ_n) be a minimizing sequence of drops. We may assume that $E(\gamma_n)$, $A(\Omega_n)$ and L_{γ_n} are convergent. Assume that for every n we have $L_{\gamma_n} \le L^*$. In order to work on a fixed Sobolev space $H^1(0, L^*)$, we assume that θ_n is formally extended by the constant $\theta_n(L_{\gamma_n})$ on $(L_{\gamma_n}, L^*]$. Up to a subsequence, we can assume that θ_n converges uniformly on $[0, L^*]$ to some function θ. We define the limit curve γ in the following way: $L_\gamma = \lim_{n\to\infty} L_{\gamma_n}$ and $\gamma : [0, L_\gamma] \to \mathbb{R}^2$, $\gamma(s) = \int_0^s e^{i\theta(t)}\, dt + a$, where $a = \lim_{n\to\infty} \gamma_n(0)$.

Fix $\varepsilon > 0$. Then, from the previous lemma, for every $s < t$ and n large enough we have

$$\theta_n(t) \ge \theta_n(s) - \pi - \varepsilon.$$

Indeed, otherwise we could replace (Ω_n, γ_n) by $(\tilde{\Omega}_n, \tilde{g}_n)$ decreasing the energy by a fixed amount $\varepsilon^2/(4L^*)$, where L^* is a bound of the lengths. This contradicts the minimality of the sequence.

In particular, passing to the limit we find that for every $\varepsilon > 0$ and every $s < t$,

$$\theta(t) \ge \theta(s) - \pi - \varepsilon.$$

Since ε is arbitrary, we get

$$\theta(t) \ge \theta(s) - \pi. \tag{19}$$

From the compactness of the class of closed subsets of $\overline{B}_R$ endowed with the Hausdorff metric, and the embedding of $H^1(0, L^*)$ into $C^{0,\alpha}[0, L^*]$, we may assume that for some open set $\Omega \subseteq B_R$,

$$\Omega_n^c \xrightarrow{H} \Omega^c,$$

and the convergence of θ_n leads to

$$\gamma_n([0, L_{\gamma_n}]) \xrightarrow{H} \gamma([0, L_\gamma]).$$

We refer to [3] or [7] for precise properties of Hausdorff convergence. We know that in general $1_\Omega \le \liminf_{n\to\infty} 1_{\Omega_n}$, so that $A(\Omega) \le \lim_{n\to\infty} A(\Omega_n)$. Nevertheless, in our situation of the perimeters being uniformly bounded, we get $1_{\Omega_n} \to 1_\Omega$ in $L^1(B_R)$. Moreover, $\partial\Omega \subseteq \gamma([0, L_\gamma])$ and Ω is simply connected (i.e. any loop contained in Ω is homotopic to a point in Ω), but not necessarily connected. The curve γ is possibly self-intersecting, but not crossing, i.e. at every self-intersecting point, the tangent line is the same, and while looking locally around the point, the pieces of curve passing through it are (in view of Lemma 2.1) graphs of functions. From the simple connectedness hypothesis, these functions are necessarily ordered. From Lemma 2.2 and the fact that the elastic energy is finite, the number of pieces of curve passing through the touching point is uniformly finite. The situation displayed in Figure 5 may occur.

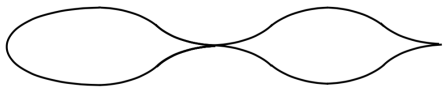

Fig. 5. Self-touching curve, disconnecting the limit.

We shall prove that γ cannot have self-intersection points, other than the type above, in which case we cut at the self-intersection point, keeping only the drop given by the left loop and decreasing in this way both the elastic energy and the area. The key ingredients are the local representation of the curve as a graph and inequality (19). We shall analyze the different contact types between two pieces of γ. Since the curves are graphs on an interval $[-l/\sqrt{2}, l/\sqrt{2}]$, and the representing functions are ordered, we shall look at the orientation of each piece.

Case 1: Opposite orientation, not disconnecting. Two branches of γ touching at some point $\gamma(s) = \gamma(t)$ are represented as graphs of functions g_s, g_t on $[-l/\sqrt{2}, l/\sqrt{2}]$. We assume that $g_s(0) = \gamma(s) = \gamma(t) = g_t(0)$ and choose the couple (s, t) such that for some $\varepsilon > 0$ we have

$$\forall u \in (0, \varepsilon) \quad g_s(u) > g_t(u),$$

otherwise we change the contact point. This inequality would imply the existence of points $s' > s$ and $t' < t$ such that $\theta(t') < \theta(s') - \pi$, in contradiction with (19), so that this situation cannot occur.

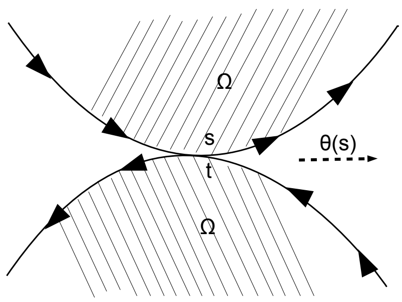

Fig. 6. Case 1: opposite orientation, not disconnecting.

Case 2: Contact of two branches of the same orientation. By simple connectedness, this situation implies that the touching point $\gamma(s)$ belongs to at least three branches, in particular between the graphs of g_s and g_t there is a graph corresponding to a piece with opposite orientation. There are two possibilities: either this new contact corresponds to a point $t' \in (t, L)$ or to $s' \in (0, s)$. The first situation is in fact Case 1 between the contact points s and t'. The second situation also leads to Case 1, but for the contact points s' and t', so we conclude that the second case cannot hold.

Case 3: Opposite orientation, disconnecting. This is the only remaining possibility for self-intersections. There may be several contact points, but every contact point is simple,

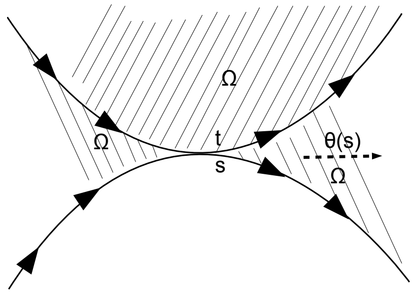

Fig. 7. Case 2: same orientation

otherwise we would fall in Case 2. So let $\{(s_\alpha, t_\alpha)\}_\alpha$ be the couple of parameters corresponding to the contact points. Because of simple connectedness and absence of contact points as in Cases 1 and 2, we know that if $s_\alpha < s_\beta$ then $t_\beta < t_\alpha$. Consequently, we can identify the contact point (s^*, t^*) such that between s^* and t^* there is no other contact, by setting $s^* = \sup_\alpha s_\alpha$ and $t^* = \inf_\alpha t_\alpha$. Of course, s^* and t^* cannot coincide. Indeed, in view of Lemma 2.2 applied to $\gamma|_{[s_\alpha,t_\alpha]}$, the elastic energy would then blow up. So $\gamma|_{[s^*,t^*]}$ is a Jordan curve for which all the area enclosed is part of Ω, since otherwise, by simple connectedness, a branch of the curve must pass through the contact point, bringing us to Case 2.

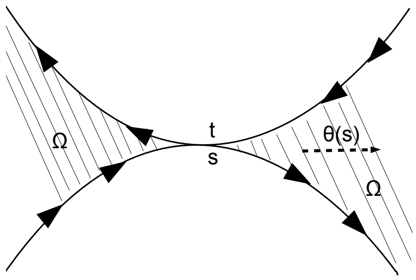

Fig. 8. Case 3: simple touch, disconnecting.

So $\gamma|_{[s^*,t^*]}$ is a drop, with elastic energy lower than γ and enclosing an area less than or equal to $A(\Omega)$. This means that $\gamma|_{[s^*,t^*]}$ is a solution for problem (18). □

Lemma 3.4. *There exists R_0 such that if the radius R of the ball B_R in Theorem 3.1 satisfies $R \geq R_0$, then there exists a translation of the optimal drop which does not touch the boundary of B_R.*

Proof. The proof relies on Lemma 2.7. Assume that (Ω^*, γ^*) is an optimal drop for problem (18) which touches the boundary, such that there is no translation moving the drop at positive distance from the boundary. This means that the touching points between γ^* and B_R are distributed in such a way that they do not fall in an arc of length less than πR.

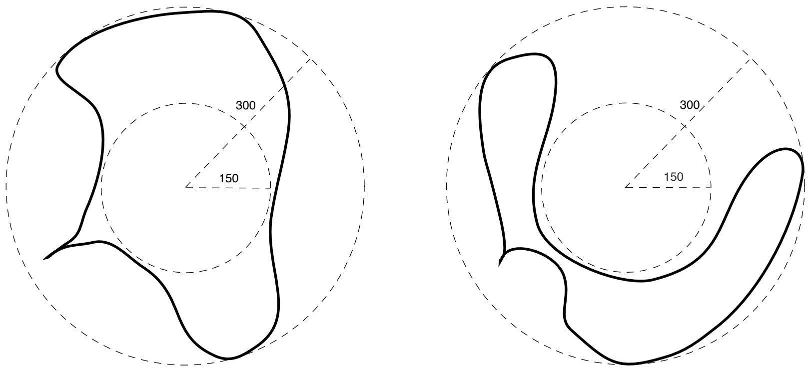

Fig. 9. An optimal drop touching the boundary.

Fix $R_0 = 300$. This means that the optimal drop which touches the boundary of the ball B_{300} centered at the origin of radius 300 cannot touch the boundary of the ball B_{150} of radius 150, otherwise at least one free branch would have length larger than 146, in contradiction with Lemma 2.7. Two situations may occur, as shown in Figure 9. Either the origin lies inside Ω^*, and so B_{150} has also to lie inside Ω^*, or the origin is not inside in Ω^* and so $\Omega^* \cap B_{150} = \emptyset$. The first situation is excluded since the energy of (Ω^*, γ^*) would be larger than the area $\pi \cdot 150^2$ of the disc of radius 150, in contradiction with its optimality. The second situation is excluded since there would be a free branch of length larger than 146. □

Theorem 3.5. *There exists a unique optimal drop* (Ω^*, γ^*) *which minimizes the energy* $E(\gamma) + A(\Omega)$ *among all Jordan drops in* $\mathbb{R}^2$. *It is fully characterized by the optimality conditions* (B1)–(B4) *with a unique constant C which can be determined. Moreover*

$$E(\gamma^*) + A(\Omega^*) > \pi > 3\pi 2^{-5/2} = \tfrac{1}{2}[E(\partial B_{2^{-1/3}}) + A(B_{2^{-1/3}})].$$

Figure 10 gives the representation of the optimal drop.

Proof. Existence follows from Theorem 3.1 and Lemma 3.4. The optimality conditions (B1)–(B4) can be written on the whole γ (except at the singularity) according to Theorem 2.5. We start for $s = 0$ at the origin, which is the singular point with horizontal tangent ($\theta(0) = 0$). By (B4) and star-shape property, the point Q is necessarily on the x-axis, the curvature $k(s)$ is negative for $s > 0$ small, and $k(s) \to 0$ as $s \to 0$. The function $k(s)$ is periodic but we will prove below (see the end of the proof) that we have only one period for the optimal drop and the curve is symmetric about the x-axis. Therefore to characterize the optimal drop, we can proceed in the following way: for any constant $C > 0$, we solve the ODE

$$\begin{cases} k'' = -\frac{1}{2}k^3 + 1, \\ k(0) = 0, \\ k'(0) = -\sqrt{2C}, \end{cases} \tag{20}$$

which has a unique solution. Let us denote by s_M the value where k is maximum with $k(s_M) = k_M$ (respectively s_m and $k_m = k(s_m)$ for the minimum). The point M_M of

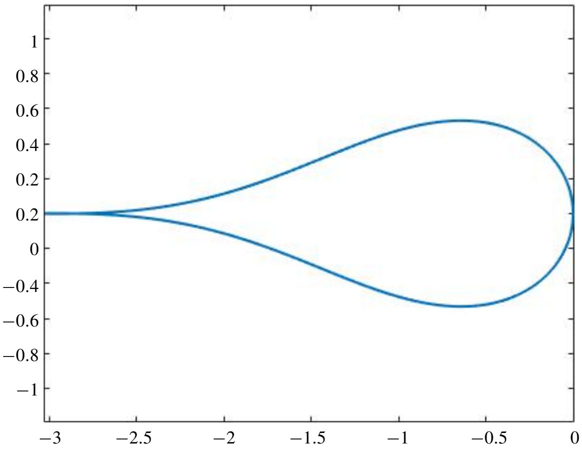

Fig. 10. The optimal drop.

abscissa s_M is necessarily on the x-axis and its tangent is vertical. Thus, we look for the value of C for which $\theta(s_M) = \int_0^{s_M} k(s)\,ds = \pi/2$.

We claim that conversely, if we find a value of C for which $\int_0^{s_M} k(s)\,ds = \pi/2$, then we have found the optimal drop. Indeed, since it satisfies the optimality conditions, it suffices to check that the curve we obtain by $x(s) = \int_0^s \cos\theta(t)\,dt$ and $y(s) = \int_0^s \cos\theta(t)\,dt$ with $\theta(s) = \int_0^s k(t)\,dt$ is an admissible drop. Since M_M is the point where the curvature is maximum, according to (B3), it is the point on γ which is the farthest to Q. But since the tangent is vertical at this point, it is necessarily on the x-axis: $y(s_M) = 0$, and the total length of the curve is $2s_M$. Now, since k is symmetric with respect to s_M (see (ODE1) in the Appendix), we have $k(s_M + t) = k(s_M - t)$, and after integration, $\theta(s_M + t) = \pi - \theta(s_M - t)$. This identity gives $\theta(2s_M) = \pi$ and

$$\begin{aligned} x(2s_M) &= \left(\int_0^{s_M} + \int_{s_M}^{2s_M}\right) \cos\theta(t)\,dt = \int_0^{s_M} [\cos\theta(t) + \cos(\pi - \theta(t))]\,dt = 0, \\ y(2s_M) &= \left(\int_0^{s_M} + \int_{s_M}^{2s_M}\right) \sin\theta(t)\,dt = \int_0^{s_M} [\sin\theta(t) + \sin(\pi - \theta(t))]\,dt \\ &= 2y(s_M) = 0, \end{aligned}$$

which shows that the curve γ is a drop.

Thus to prove uniqueness of the optimal drop, we need to prove that we can find only one $C > 0$ for which $I(C) := \int_0^{s_M} k(s)\,ds = \pi/2$. Let us write

$$\int_0^{s_M} k(s)\,ds = \left(\int_0^{2s_m} + \int_{2s_m}^{s_M}\right) k(s)\,ds = 2\int_0^{s_m} k(s)\,ds + \int_{2s_m}^{s_M} k(s)\,ds,$$

where we have used the symmetry of k with respect to s_m (see (ODE1)). This symmetry also shows that $k(2s_m) = 0$. We are going to prove uniqueness of C (and therefore of the optimal drop) by proving that the function $C \mapsto \int_0^{s_M} k(s)\,ds$ is strictly decreasing. Let us perform the change of variable $u = k(s)$ in each integral above. It follows, by using (B2) to express k', that

$$\begin{aligned}\int_{2s_m}^{s_M} k(s)\,ds &= \int_0^{k_M} \frac{u}{\sqrt{2C + 2u - u^4/4}}\,du,\\ \int_0^{s_m} k(s)\,ds &= -\int_0^{k_m} \frac{u}{\sqrt{2C + 2u - u^4/4}}\,du.\end{aligned} \tag{21}$$

Now to compute the derivative of the first integral $I_1(C)$ with respect to C, we make the change of variable $u = k_M x$. This yields

$$I_1(C) = \int_0^1 \frac{k_M^2 x}{\sqrt{2C + 2k_M x - k_M^4 x^4/4}}\,dx.$$

We compute the derivative of I_1 using $\frac{dk_M}{dC} = 2/(k_M^3 - 2)$ (see (ODE3) in the Appendix) and an easy computation gives

$$\frac{dI_1}{dC} = \int_0^1 \frac{6k_M^2 x(x-1)}{(k_M^3 - 2)(2C + 2k_M x - k_M^4 x^4/4)^{3/2}}\,dx,$$

which is clearly negative. In the same way, for the second integral $I_2(C) = \int_0^{s_m} k(s)\,ds$ we get

$$\frac{dI_2}{dC} = -\int_0^1 \frac{6k_m^2 x(x-1)}{(k_m^3 - 2)(2C + 2k_m x - k_m^4 x^4/4)^{3/2}}\,dx,$$

which is also negative, proving the uniqueness of a solution C for the equation $I_1(C) + 2I_2(C) = \pi/2$. Let us remark that a simple computation yields $I(0) = 2\pi/3$, while the limit of $I(C)$ when C goes to ∞ is $-\pi/2$, confirming that there exists a solution to our problem.

Let us estimate from below the energy of the optimal drop. Denote by $s_1 = 2s_m$ the first positive zero of k; we recall that s_m is the first minimum of k and $k_m = k(s_m)$, and s_M the first maximum of k and $k_M = k(s_M)$. From (B2), k_m and k_M are the real roots of the polynomial (which is concave)

$$P_C(X) = -\tfrac{1}{4}X^4 + 2X + 2C. \tag{22}$$

The maximum of P_C is at $X = 2^{1/3}$ and $P_C(0) = 2C$. We have

$$k_m < 0 \le 2^{1/3} \le k_M \tag{23}$$

(k_m must be negative, otherwise the set Ω^* would be convex).

Moreover, when C increases, then $k_M(C)$ is increasing while $k_m(C)$ is decreasing (with increasing absolute value $|k_m(C)|$), because we translate the curve $y = -\frac{1}{4}x^4 + 2x$ up).

If we denote by $S = k_M + k_m$ and $P = k_m k_M$ the sum and the product of those two roots, classical elimination and relation between roots provide

$$S^2 = P - 8C/P, \quad -8/S = P + 8C/P, \tag{24}$$

while the two complex roots $z_0, \overline{z}_0$ satisfy $z_0 + \overline{z}_0 = -S$, $z_0\overline{z}_0 = -8C/P$.

Since $P \le 0$ and $C > 0$, the last equation gives $S > 0$. Let us come back to the computation of the elastic energy of the optimal drop (Ω^*, γ^*),

$$E(\gamma^*) = \int_0^{s_M} k^2\, ds.$$

Now $\int_0^{s_M} k^2\, ds \ge \int_{s_m}^{s_M} k^2\, ds$ and k is increasing from s_m to s_M (since k' can only vanish at zeroes of $P_C(X)$, which only correspond to maxima k_M and minima k_m). We perform the change of variable $x = k(s)$ on this interval: $dx = k'(s)\, ds = \sqrt{2C + 2k - \frac{1}{4}k^4}\, ds$. Therefore

$$E(\gamma^*) \ge \int_{s_m}^{s_M} k^2\, ds = \int_{k_m}^{k_M} \frac{x^2}{\sqrt{2C + 2x - \frac{1}{4}x^4}}\, dx.$$

We want to find a lower bound of this integral. For this purpose, we write (following (22))

$$P_C(x) = \tfrac{1}{4}(k_M - x)(x - k_m)(x^2 + Sx - 8C/P).$$

Now, the parabola $y = \frac{1}{4}(x^2 + Sx - 8C/P)$ is symmetric with respect to $-S/2$, and since $(k_M + k_m)/2 = S/2 \ge -S/2$, the maximum of y on the interval $[k_m, k_M]$ is equal to

$$F^2 = \tfrac{1}{4}(k_M^2 + Sk_M - 8C/P) = \tfrac{1}{4}(2k_M^2 + k_m k_M - 8C/P) = \tfrac{1}{4}(3k_M^2 + 2k_m k_M + k_m^2), \tag{25}$$

where we have used (24) for the last equality. Thus

$$\int_{k_m}^{k_M} \frac{x^2}{\sqrt{2C + 2x - \frac{1}{4}x^4}}\, dx \ge \frac{1}{F}\int_{k_m}^{k_M} \frac{x^2}{\sqrt{(k_M - x)(x - k_m)}}\, dx.$$

This last integral can be computed explicitly and gives

$$E(\gamma^*) \ge \frac{1}{F}\, \frac{3k_M^2 + 2k_m k_M + 3k_m^2}{4}\, \frac{\pi}{2}. \tag{26}$$

We have $F \le \frac{1}{2}\sqrt{3k_M^2 + 2k_m k_M + 3k_m^2}$, and (26) gives

$$E(\gamma^*) \ge \frac{\pi}{4}\sqrt{3k_M^2 + 2k_m k_M + 3k_m^2}. \tag{27}$$

It remains to get a bound for the quantity $H = 3k_M^2 + 2k_m k_M + 3k_m^2$ which depends only on C. We discuss two cases.

Case A. If $C \geq 1$, then $H = k_M^2 + 2k_M(k_M + k_m) + 3k_m^2 \geq k_M^2 + 3k_m^2$. Both mappings $C \mapsto k_m^2$, $C \mapsto k_M^2$ are increasing, thus $C \geq k_M^2(1) + 3k_m^2(1)$. We study $P_1(X) = -\frac{1}{4}X^4 + 2X + 2$. Since

$$P_1\left(\tfrac{7}{3}\right) = -\tfrac{241}{324} \quad \text{and} \quad P_1\left(\tfrac{9}{4}\right) = \tfrac{95}{1024},$$

we get

$$\tfrac{9}{4} \leq k_M(1) \leq \tfrac{7}{3}. \tag{28}$$

From $P_1(-1) = -\frac{1}{4}$ and $P_1\left(-\frac{9}{10}\right) = \frac{1439}{40000}$, we get

$$-1 \leq k_m(1) \leq -\tfrac{9}{10}. \tag{29}$$

It follows that $H \geq \left(\frac{9}{4}\right)^2 + 3\left(\frac{9}{10}\right)^2 = \frac{2997}{400} \approx 7.4925$.

Case B. In the case $0 \leq C \leq 1$, we use $k_M^2(C) \geq k_M^2(0) = 4$, $k_m^2(C) \geq 0$ and $|k_M(C)k_m(C)| \leq |k_M(1)k_m(1)| \leq 7/3$ to get

$$H = 3k_M^2 + 2k_m k_M + 3k_m^2 \geq 12 - 14/3 = 22/3 = 7.333\ldots$$

So in any case, $H \geq 22/3$. It follows from (26) that

$$E(\gamma^*) \geq \tfrac{1}{4}\pi\sqrt{22/3}. \tag{30}$$

Now, integrating (B4) on the curve, we get $2A(\Omega^*) = \int_{\gamma^*} \overrightarrow{QM} \cdot \vec{\nu}\, ds = \frac{1}{2}\int_{\gamma^*} k^2\, ds = E(\gamma^*)$.

Therefore

$$E(\gamma^*) + A(\Omega^*) = \tfrac{3}{2}E(\gamma^*) \geq \tfrac{3\pi}{8}\sqrt{22/3} > \pi > 3\pi 2^{-5/3}. \tag{31}$$

Let us now conclude by proving that the optimal drop has only one period of the function $k(s)$. The estimate (30) we get is actually true on any possible period. Therefore, if we have a solution (γ_2^*, Ω_2^*) with at least two periods, we would have $E(\gamma_2^*) \geq \frac{1}{2}\pi\sqrt{22/3}$, therefore as in (31) its total energy would satisfy $E(\gamma_2^*) + A(\Omega_2^*) > 2\pi$. Now, proceeding in a similar way as we did for the estimate from below, we can get (details omitted) an estimate from above for an optimal drop with only one period which is

$$E(\gamma^*) + A(\Omega^*) \leq 2\pi$$

(the exact value is $E(\gamma^*) + A(\Omega^*) \simeq 4.6823$); therefore, any critical point with more than one period cannot be optimal. □

4. Proof of Theorem 1.1

With the notation of Sections 2 and 3 we return to problem (3), and write

$$\inf\{E(\gamma) + A(\Omega) : \Omega \text{ smooth, bounded, simply connected, } \partial\Omega = \gamma\}. \tag{32}$$

First of all we recall that among all circles, the optimal one has radius $r = 2^{-1/3}$. Let $R \geq 300$ and let us solve the problem

$$\inf\{E(\gamma) + A(\Omega) : \Omega \text{ smooth, bounded, simply connected, } \Omega \subseteq B_R, \partial\Omega = \gamma\}. \tag{33}$$

By the same arguments as in Section 3, a minimizing sequence will converge to a couple (Ω, γ). Two possibilities may occur. Assume first that there are self-intersections. In this case the limiting couple (Ω, γ) contains at least two drops, as in Case 3 of Theorem 3.1. Following Theorem 3.5, this configuration cannot be optimal since the energy of Ω is larger than the double of the optimal energy of a drop, so it is excluded.

The second situation is that (Ω, γ) does not have self-intersections. Since the radius is large enough, for a suitable translation the loop does not touch the boundary of the ball, as in Lemma 3.4. Moreover, in this case the optimality conditions $\overrightarrow{OM} \cdot \vec{\nu} = \frac{1}{2}k^2$ can be written on the full boundary.

Remark 4.1. This condition recalls the result of Ben Andrews [1, Theorem 1.5] which, under the hypothesis of positive curvature, would allow one directly to conclude that the curve is a circle of radius $2^{-1/3}$ (by direct computation). As the curvature is not known to be positive, we use again the optimality conditions. Actually, Andrews's result does not hold true for nonconvex curves. Indeed, Figure 11 (which have been obtained using the optimality conditions) shows a curve which satisfies $\overrightarrow{OM} \cdot \vec{\nu} = \frac{1}{2}k^2$ on the whole boundary.

If the curvature is not constant, we can assert that Ω is star-shaped and the structure of γ is a union of periods consisting of two branches γ_1 and γ_2, where $\gamma_1 : [0, l] \to \mathbb{R}^2$ is a branch of the curve with increasing curvature such that $\gamma(0) = k_{\mathrm{m}}, \gamma(l) = k_{\mathrm{M}}$ and $\gamma_2 : [l, 2l] \to \mathbb{R}^2$ is a congruent branch with decreasing curvature from k_{M} to k_{m}. Following (9) and (ODE4), γ consists of one, two or three periods (γ_1, γ_2) (as explained in the proof of Theorem 3.5). From the optimality conditions (B1)–(B4) one can eliminate any of those three configurations, since their energy is much larger than the one of the ball. Indeed, in the case of two or three periods, a couple (γ_1, γ_2) has a cap $\gamma_{(l-a,l+a)}$, where a is chosen such that $\nu_{\gamma(a)}$ is orthogonal to the segment $O\gamma(l)$. As in Figure 11 (left), we can cut and reflect along the line $\gamma(l-a), \gamma(l+a)$ to get a new domain with smaller

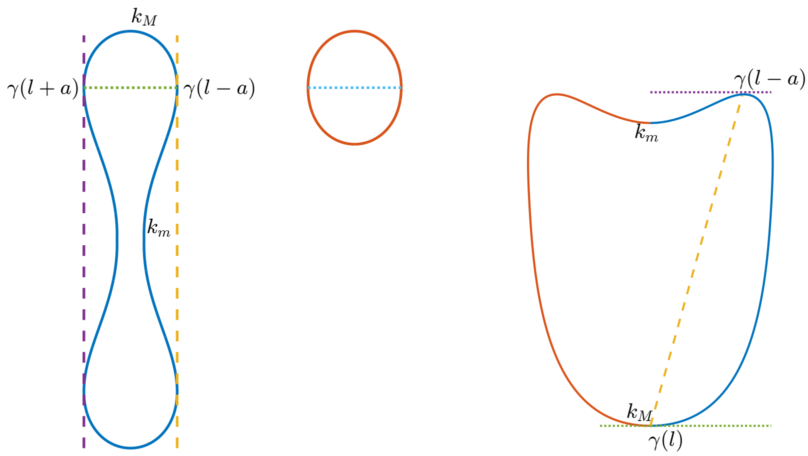

Fig. 11. The case of more than one period (left). The case of one period (right).

area and smaller elastic energy. If there is only one period, the argument is similar since we can center-symmetrize the branch from $\gamma(l-a)$ to $\gamma(l)$, where a is chosen such that the normal at $\gamma(l-a)$ is parallel to the segment $O\gamma(l)$ (see Figure 11, right). The symmetrization center is the middle of the segment joining $\gamma(l-a)$ to $\gamma(l)$.

Both previous constructions are admissible as a consequence of the optimality conditions (B1)–(B4).

5. Appendix: analysis of the ODE issued from optimality conditions

In this section, we give several properties of the following ODE in nonstandard form:

$$k'^2 = -\tfrac{1}{4}k^4 + 2k + 2C,$$

where $C \in \mathbb{R}$ is a constant. This ODE is issued from the optimality conditions on a free branch of a minimizer for our problem (see Theorem 2.5). We also refer the reader to reference [2] for related analysis.

Clearly, $C \geq -\frac{3}{4}2^{1/3} \approx -0.944$, otherwise the right hand side is negative. We denote by $k_{\mathrm{m}}(C) \leq k_{\mathrm{M}}(C)$ the two real roots of the polynomial $P_C(X) = -\frac{1}{4}X^4 + 2X + 2C$, or simply $k_{\mathrm{m}}, k_{\mathrm{M}}$ if there is no ambiguity.

Here we gather some immediate facts concerning this ODE.

(ODE1) The solution of the ODE is periodic (the period is denoted by T), symmetric with respect to its minimum or maximum.

(ODE2) The only local minima (maxima) are actually global minima (maxima, respectively) and correspond to $k = k_{\mathrm{m}}$ ($k = k_{\mathrm{M}}$, respectively), and k is monotone between these two values.

(ODE3) The mapping $C \mapsto k_{\mathrm{M}}(C)$ is increasing and its range is from $2^{1/3}$ to ∞, while the mapping $C \mapsto k_{\mathrm{m}}(C)$ is decreasing and its range is from $-\infty$ to $2^{1/3}$. Moreover, $k_{\mathrm{m}}(C) < 0$ when $C > 0$, $9/4 \leq k_{\mathrm{M}}(1) \leq 7/3$, $-1 \leq k_{\mathrm{m}}(1) \leq -9/10$, $-C \leq k_{\mathrm{m}}(C)$. Furthermore, $k_{\mathrm{M}}(C) \geq 2 + C$ for $-3/2 \times 2^{1/3} \leq C \leq 0$.

(ODE4) The integral $\frac{1}{2}\int_0^T k^2\,ds$ on one period is estimated from below

$$\frac{1}{2}\int_0^T k^2\,ds \geq \frac{\pi}{4}\sqrt{\frac{22}{3}}.$$

The proof of (ODE1) is classical, either working with the closed orbit, or using an explicit form of the solution thanks to elliptic functions.

The proof of (ODE2) is easy since k' can vanish only at the zeroes of P_C.

For the proof of (ODE3) we notice that

$$\frac{dk_{\mathrm{M}}}{dC} = \frac{2}{k_{\mathrm{M}}^3 - 2} > 0 \quad \text{and} \quad \frac{dk_{\mathrm{m}}}{dC} = \frac{2}{k_{\mathrm{m}}^3 - 2} < 0,$$

$k_{\mathrm{m}}(0) = 0$, $k_{\mathrm{M}}(0) = 2$, $P_C(-C) < 0 \Rightarrow k_{\mathrm{m}}(C) \geq -C$, $P_C(2+C) = -C\big[\frac{1}{4}C^3 + 2C^2 + 6C + 4\big] \geq 0$ and the bounds for $k_{\mathrm{m}}(1), k_{\mathrm{M}}(1)$ have been obtained in (28), (29).

The proof of (ODE4): we have already proved this inequality in Section 3 when $C \geq 0$. In the case $-\frac{3}{4}2^{1/3} \leq C < 0$, we have $k_{\mathrm{m}} \geq -C > 0$ and $k_{\mathrm{M}} \geq 2 + C > 0$, so $3k_{\mathrm{M}}^2 + 2k_{\mathrm{m}}k_{\mathrm{M}} + 3k_{\mathrm{m}}^2 \geq 4C^2 + 8C + 12 \geq 8 \geq 22/3$, and the result follows in the same way.

Acknowledgments and history. This work was initiated during a stay of the two authors at the Isaac Newton Institute, Cambridge in March 2014 during the programme "Free Boundary Problems and Related Topics". The authors are grateful to the Institute for the very good and stimulating atmosphere. The results of this paper were announced in several conferences (e.g. Petropolis in August 2014, Linz in October 2014). While completing this manuscript, we learnt from Bernd Kawohl that he, with Vincenzo Ferone and Carlo Nitsch, proved our Theorem 1.1 in a different way (see [5]). We also want to warmly thank the referees who allowed us to significantly improve the writing of this paper (in particular the proof of Lemma 2.1).

The authors were supported by the Isaac Newton Institute programme "Free Boundary Problems and Related Topics" 2014 and the ANR Optiform research programme, ANR-12-BS01-0007.

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