J. Eur. Math. Soc. 19, 3377–3390

DOI 10.4171/JEMS/741

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Conical structure for shrinking Ricci solitons

Received December 14, 2014

Abstract. It is shown that a shrinking gradient Ricci soliton must be smoothly asymptotic to a cone if its Ricci curvature goes to zero at infinity.

Keywords. Shrinking Ricci soliton, asymptotically conical

1. Introduction

The purpose of this paper is to show that a shrinking gradient Ricci soliton must be smoothly asymptotic to a cone if its Ricci curvature goes to zero at infinity. Recall that a *gradient shrinking Ricci soliton* is a Riemannian manifold (M^n, g) for which there exists a potential function f such that

$$\mathrm{Ric} + \mathrm{Hess}(f) = \frac{1}{2} g, \tag{1.1}$$

where Ric is the Ricci curvature of M and $\mathrm{Hess}(f)$ the Hessian of f. Aside from its own interest as generalization of Einstein manifolds, Ricci solitons are important in the study of Ricci flows. Indeed, one easily verifies (see [11]) that $g(t) = (1-t)\phi_t^* g$, $-\infty < t < 1$, is a solution to the Ricci flow

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}(g(t)), \qquad g(0) = g,$$

for a suitably chosen family of diffeomorphisms ϕ_t on M with $\phi_0 = \mathrm{Id}$. So shrinking Ricci solitons may be regarded as self-similar solutions to the Ricci flows. It has been shown in [14] and [9] that the blow-ups around a type-I singularity point always converge to nontrivial gradient shrinking Ricci solitons. Therefore, it would be very much desirable to understand and even classify shrinking Ricci solitons.

In the case of dimension $n = 2$, according to [16], the only examples are either the sphere $\mathbb{S}^2$ or the Gaussian soliton, Euclidean space $\mathbb{R}^2$ together with the potential function $f(x) = \frac{1}{4}|x|^2$. For dimension $n = 3$, improving upon the breakthrough of Perelman [26],

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Mathematics Subject Classification (2010): 53C44, 53C21

Naber [24], Ni and Wallach [25], and Cao, Chen and Zhu [5] have concluded that a three-dimensional shrinking gradient Ricci soliton must be a quotient of the sphere $\mathbb{S}^3$, or $\mathbb{R}^3$, $\mathbb{S}^2 \times \mathbb{R}$.

For high-dimensional shrinking Ricci solitons, examples other than the spheres, the Gaussian solitons, and their products were constructed in [2, 17, 29, 15, 12]. This certainly indicates that it would be more complicated, if at all possible, to obtain a complete classification. Under some auxiliary conditions on the full curvature tensor, partial classification results have been established. In [24], Naber showed that a four-dimensional noncompact shrinking Ricci soliton of bounded nonnegative curvature operator must be a quotient of $\mathbb{R}^k \times \mathbb{S}^{4-k}$ with $k = 1, 2$. More recently, we proved that any n-dimensional gradient shrinking Ricci soliton of positive sectional curvature must be compact [22]. In dimension three, this is a result of Perelman [26]. Combining this with a theorem of Böhm and Wilking [1], one sees that an n-dimensional shrinking gradient Ricci soliton with positive curvature operator must be a quotient of the round sphere $\mathbb{S}^n$.

Classification results of different flavor are also known. For example, shrinking gradient Ricci solitons of vanishing Weyl tensor have been classified and must be finite quotients of $\mathbb{S}^n$, $\mathbb{R}^n$, or $\mathbb{S}^{n-1} \times \mathbb{R}$ (see [30, 13, 27, 8, 20]). More generally, in a recent work [6], Cao and Chen have shown that a Bach-flat gradient shrinking Ricci soliton is either Einstein, a finite quotient of the Gaussian shrinking soliton $\mathbb{R}^n$, or a finite quotient of $N^{n-1} \times \mathbb{R}$, where N^{n-1} is an Einstein manifold of positive scalar curvature. We refer the readers to the surveys [3, 4] for more results and further information.

In another direction, Kotschwar and Wang [18] have recently shown that any two shrinking Ricci solitons C^2 asymptotic to the same cone must be isometric. Here, by a *cone*, we mean a manifold $[0, \infty) \times \Sigma$ endowed with Riemannian metric $g_c = dr^2 + r^2 g_\Sigma$, where (Σ, g_Σ) is a closed $(n-1)$-dimensional Riemannian manifold. Denote $E_R = (R, \infty) \times \Sigma$ for $R \geq 0$ and define the dilation by λ to be the map $\rho_\lambda : E_0 \to E_0$ given by $\rho_\lambda(r, \sigma) = (\lambda r, \sigma)$. A Riemannian manifold (M, g) is said to be C^k *asymptotic to the cone* (E_0, g_c) if, for some $R > 0$, there is a diffeomorphism $\Phi : E_R \to M \setminus \Omega$ such that $\lambda^{-2} \rho_\lambda^* \Phi^* g \to g_c$ as $\lambda \to \infty$ in $C^k_{\mathrm{loc}}(E_0, g_c)$, where Ω is a compact subset of M.

In view of the result of [18], it becomes of interest to determine when a shrinking Ricci soliton is asymptotically conical. In our recent work [21], we have shown that this is the case for four-dimensional shrinking gradient Ricci solitons with scalar curvature converging to 0 at infinity. This result depends on the fact that the full curvature tensor Rm of a four-dimensional soliton is controlled by its scalar curvature S alone: $|\mathrm{Rm}| \leq c|\mathrm{Ric}| \leq cS$. While it remains to be seen whether such an estimate is true for high-dimensional shrinking Ricci solitons, by imposing an assumption on the Ricci curvature instead, we manage to obtain a parallel result as well. Consequently, the classification problem for such solitons is reduced to the one for cones.

Theorem 1.1. *Let (M, g, f) be a gradient shrinking Ricci soliton of dimension n with Ricci curvature convergent to zero at infinity. Then (M, g) is C^k asymptotic to a cone for all k.*

The following corollary provides a partial generalization to arbitrary dimension of the aforementioned result for the four-dimensional case.

Corollary 1.2. *Let (M, g, f) be a shrinking Ricci soliton of dimension n with bounded curvature. Assume that the scalar curvature converges to zero at infinity. Then (M, g) is C^k asymptotic to a cone for all k.*

Essential to the proof of Theorem 1.1 is a quadratic decay estimate for the Riemann curvature Rm. Once this is available, together with Shi's [28] derivative estimates of Rm, it is straightforward to deduce that (M, g) is asymptotically conical [18]. As demonstrated in [21], such a decay estimate follows from a maximum principle argument provided that the Riemann curvature tensor Rm converges to zero at infinity. So the heart of the proof is to conclude from Ric converging to 0 that Rm goes to 0 as well. Here, we are very much inspired by the work of [23], where it is shown that for a shrinking gradient Ricci soliton, its Riemann curvature is at most of polynomial growth if its Ricci curvature is bounded. However, we emphasize that our present argument differs significantly from [23] in technical details.

Finally, we point out that our argument only requires the Ricci curvature to be sufficiently small outside a compact set. More precisely, Theorem 1.1 continues to hold if one assumes instead that $|\mathrm{Ric}| \leq \delta$ near the infinity of M for some positive constant δ depending only on the dimension n.

2. Curvature estimates

In this section, we prove Theorem 1.1. We continue to denote by (M, g, f) an n-dimensional gradient shrinking Ricci soliton with potential function f.

Let us recall the following important identities:

$$\nabla_k R_{jk} = R_{jk} f_k = \tfrac{1}{2}\nabla_j S, \qquad \nabla_l R_{ijkl} = R_{ijkl} f_l = \nabla_j R_{ki} - \nabla_i R_{kj}. \tag{2.1}$$

As observed in [16], this implies $S + |\nabla f|^2 = f$ by adding a suitable constant to f. Since $S \geq 0$ by [10], we have $|\nabla f|^2 \leq f$.

Also, denoting $\Delta_f = \Delta - \langle \nabla f, \nabla \rangle$, we have

$$\Delta_f R_{ij} = R_{ij} - 2R_{ikjl} R_{kl}, \qquad \Delta_f \mathrm{Rm} = \mathrm{Rm} + \mathrm{Rm} * \mathrm{Rm}, \tag{2.2}$$

where $\mathrm{Rm} * \mathrm{Rm}$ denotes a quadratic expression in the Riemann curvature tensor.

Let us denote

$$D(r) := \{x \in M : f(x) \leq r\}.$$

Note that $D(r)$ is always compact, as by [7] there exists constant c such that

$$\tfrac{1}{4} r^2(x) - c r(x) \leq f(x) \leq \tfrac{1}{4} r^2(x) + c r(x) \quad \text{for } r(x) \geq 1. \tag{2.3}$$

Here $r(x)$ is the distance from x to a fixed point $x_0 \in M$. Also, by [7], the volume $V(r)$ of $D(r)$ satisfies

$$V(r) \leq c r^{n/2}. \tag{2.4}$$

We define a cut-off function ϕ with support in $D(r)$ by

$$\phi(x) = \begin{cases} \frac{1}{r}(r - f(x)) & \text{if } x \in D(r), \\ 0 & \text{if } x \in M \setminus D(r). \end{cases}$$

Let us choose $r_0 > 0$ large enough so that $f \geq 1$ and

$$|\mathrm{Ric}| \leq 1/p^5 \quad \text{on } M \setminus D(r_0). \tag{2.5}$$

Here and in the following, $p \geq 8n$ is a large enough constant depending only on the dimension n. In particular, since

$$\mathrm{Ric} + \mathrm{Hess}(f) = \tfrac{1}{2} g,$$

we have

$$\mathrm{Hess}(f) \geq \tfrac{1}{3} g \quad \text{on } M \setminus D(r_0). \tag{2.6}$$

Finally, let $a > 0$ be a constant satisfying

$$a \leq \tfrac{1}{4} p. \tag{2.7}$$

Throughout the paper, unless otherwise indicated, we will use C to denote a constant that may depend on the geometry of $D(r_0)$ and on p. We will denote by c a constant depending only on the dimension n but independent of p, and by $c(p)$ a constant depending on p. These constants may change from line to line.

We first prove the following lemma.

Lemma 2.1. *Let (M, g, f) be a gradient shrinking Ricci soliton of dimension n with*

$$\lim_{x \to \infty} |\mathrm{Ric}|(x) = 0.$$

Then, for p and a satisfying (2.7), we have

$$\int_M |\mathrm{Rm}|^p f^a < \infty.$$

Proof. Integrating by parts and using $\Delta f \leq n/2$, we get

$$\begin{aligned} -\frac{n}{2} \int_M |\mathrm{Rm}|^p f^a \phi^{2p} &\leq -\int_M |\mathrm{Rm}|^p (\Delta f) f^a \phi^{2p} \\ &= \int_M \langle \nabla |\mathrm{Rm}|^p, \nabla f \rangle f^a \phi^{2p} + a \int_M |\mathrm{Rm}|^p |\nabla f|^2 f^{a-1} \phi^{2p} + \int_M |\mathrm{Rm}|^p f^a \langle \nabla f, \nabla \phi^{2p} \rangle \\ &\leq \int_M \langle \nabla |\mathrm{Rm}|^p, \nabla f \rangle f^a \phi^{2p} + a \int_M |\mathrm{Rm}|^p f^a \phi^{2p}, \end{aligned}$$

where in the last line we have used the inequalities $\langle \nabla f, \nabla \phi^q \rangle \leq 0$ and $|\nabla f|^2 \leq f$.

Therefore, by the Bianchi identities we obtain

$$
\begin{aligned}
-(a+n/2)\int_M |\mathrm{Rm}|^p f^a\phi^{2p} &\le \int_M \langle \nabla|\mathrm{Rm}|^p, \nabla f\rangle\, f^a\phi^{2p}\\
&= p\int_M f_h(\nabla_h R_{ijkl})R_{ijkl}|\mathrm{Rm}|^{p-2} f^a\phi^{2p} = 2p\int_M f_h(\nabla_l R_{ijkh})R_{ijkl}|\mathrm{Rm}|^{p-2} f^a\phi^{2p}.
\end{aligned}
$$

It follows through integration by parts that

$$
\begin{aligned}
&-(a+n/2)\int_M |\mathrm{Rm}|^p f^a\phi^{2p}\\
&\le -2p\int_M R_{ijkh}f_{hl}R_{ijkl}|\mathrm{Rm}|^{p-2} f^a\phi^{2p} - 2p\int_M R_{ijkh}f_h(\nabla_l R_{ijkl})|\mathrm{Rm}|^{p-2} f^a\phi^{2p}\\
&\quad - 2p\int_M R_{ijkh}f_h R_{ijkl}(\nabla_l |\mathrm{Rm}|^{p-2}) f^a\phi^{2p} - 2ap\int_M R_{ijkh}f_h R_{ijkl} f_l|\mathrm{Rm}|^{p-2} f^{a-1}\phi^{2p}\\
&\quad + \frac{4p^2}{r}\int_M R_{ijkh}f_h R_{ijkl} f_l|\mathrm{Rm}|^{p-2} f^a\phi^{2p-1}. \qquad (2.8)
\end{aligned}
$$

Note that on $M\setminus D(r_0)$, by (2.6),

$$
-R_{ijkh}f_{hl}R_{ijkl} \le -\tfrac{1}{3}|\mathrm{Rm}|^2.
$$

Hence, there exists $C>0$ such that

$$
-2p\int_M R_{ijkh}f_{hl}R_{ijkl}|\mathrm{Rm}|^{p-2} f^a\phi^{2p} \le -\frac{2p}{3}\int_M |\mathrm{Rm}|^p f^a\phi^{2p} + C.
$$

Together with (2.1) and (2.7), one concludes from (2.8) that

$$
\begin{aligned}
\frac{p}{3}\int_M |\mathrm{Rm}|^p f^a\phi^{2p} &\le -2p\int_M R_{ijkh}f_h R_{ijkl}(\nabla_l |\mathrm{Rm}|^{p-2}) f^a\phi^{2p}\\
&\quad + \frac{4p^2}{r}\int_M |R_{ijkh}f_h|^2|\mathrm{Rm}|^{p-2} f^a\phi^{2p-1} + C. \qquad (2.9)
\end{aligned}
$$

By (2.1) again, we have

$$
\begin{aligned}
-2p\int_M R_{ijkh}f_h R_{ijkl}(\nabla_l |\mathrm{Rm}|^{p-2}) f^a\phi^{2p} &\le cp^2\int_M |\nabla \mathrm{Ric}|\,|\nabla \mathrm{Rm}|\,|\mathrm{Rm}|^{p-2} f^a\phi^{2p}\\
&\le cp^4\int_M |\nabla \mathrm{Ric}|^2|\mathrm{Rm}|^{p-1} f^a\phi^{2p} + c\int_M |\nabla \mathrm{Rm}|^2|\mathrm{Rm}|^{p-3} f^a\phi^{2p}.
\end{aligned}
$$

Hence, from (2.9) we obtain

$$
\begin{aligned}
\int_M |\mathrm{Rm}|^p f^a\phi^{2p} &\le cp^3\int_M |\nabla \mathrm{Ric}|^2|\mathrm{Rm}|^{p-1} f^a\phi^{2p} + \frac{c}{p}\int_M |\nabla \mathrm{Rm}|^2|\mathrm{Rm}|^{p-3} f^a\phi^{2p}\\
&\quad + \frac{cp}{r}\int_M |R_{ijkh}f_h|^2|\mathrm{Rm}|^{p-2} f^a\phi^{2p-1} + C. \qquad (2.10)
\end{aligned}
$$

We now estimate the first term on the right side of (2.10). Note that (2.2) implies

$$
\Delta_f |\mathrm{Ric}|^2 \ge 2|\nabla\,\mathrm{Ric}\,|^2 - 4|\mathrm{Ric}|^2|\mathrm{Rm}|, \qquad \Delta_f |\mathrm{Rm}|^{p-1} \ge -cp|\mathrm{Rm}|^p.
$$

Therefore,

$$\begin{aligned}
&\Delta_f(|\mathrm{Ric}|^2|\mathrm{Rm}|^{p-1})\\
&\quad= (\Delta_f|\mathrm{Ric}|^2)|\mathrm{Rm}|^{p-1} + |\mathrm{Ric}|^2\Delta_f|\mathrm{Rm}|^{p-1} + 2\langle\nabla|\mathrm{Ric}|^2, \nabla|\mathrm{Rm}|^{p-1}\rangle\\
&\quad\geq 2|\nabla\mathrm{Ric}|^2|\mathrm{Rm}|^{p-1} - cp|\mathrm{Ric}|^2|\mathrm{Rm}|^p - cp|\nabla\mathrm{Ric}|\,|\nabla\mathrm{Rm}|\,|\mathrm{Ric}|\,|\mathrm{Rm}|^{p-2}.
\end{aligned}$$

Using (2.5), we estimate on $M\setminus D(r_0)$

$$\begin{aligned}
cp|\nabla\mathrm{Ric}|\,|\nabla\mathrm{Rm}|\,|\mathrm{Ric}|\,|\mathrm{Rm}|^{p-2} &\leq |\nabla\mathrm{Ric}|^2|\mathrm{Rm}|^{p-1} + cp^2|\nabla\mathrm{Rm}|^2|\mathrm{Ric}|^2|\mathrm{Rm}|^{p-3}\\
&\leq |\nabla\mathrm{Ric}|^2|\mathrm{Rm}|^{p-1} + \frac{c}{p^4}|\nabla\mathrm{Rm}|^2|\mathrm{Rm}|^{p-3}.
\end{aligned}$$

Similarly, (2.5) implies that

$$cp|\mathrm{Ric}|^2|\mathrm{Rm}|^p \leq \frac{c}{p^4}|\mathrm{Rm}|^p$$

on $M\setminus D(r_0)$. Consequently, on $M\setminus D(r_0)$ we get

$$\Delta_f(|\mathrm{Ric}|^2|\mathrm{Rm}|^{p-1}) \geq |\nabla\mathrm{Ric}|^2|\mathrm{Rm}|^{p-1} - \frac{c}{p^4}|\mathrm{Rm}|^p - \frac{c}{p^4}|\nabla\mathrm{Rm}|^2|\mathrm{Rm}|^{p-3}.$$

Hence,

$$\begin{aligned}
\int_M |\nabla\mathrm{Ric}|^2|\mathrm{Rm}|^{p-1}f^a\phi^{2p} &\leq \int_M \Delta_f(|\mathrm{Ric}|^2|\mathrm{Rm}|^{p-1})f^a\phi^{2p}\\
&\quad+\frac{c}{p^4}\int_M|\mathrm{Rm}|^pf^a\phi^{2p} + \frac{c}{p^4}\int_M|\nabla\mathrm{Rm}|^2|\mathrm{Rm}|^{p-3}f^a\phi^{2p} + C. \qquad (2.11)
\end{aligned}$$

We use integration by parts on the first term on the right hand side in (2.11) to get

$$\begin{aligned}
&\int_M \Delta_f(|\mathrm{Ric}|^2|\mathrm{Rm}|^{p-1})f^a\phi^{2p}\\
&\quad= \int_M \Delta(|\mathrm{Ric}|^2|\mathrm{Rm}|^{p-1})f^a\phi^{2p} - \int_M\langle\nabla f, \nabla(|\mathrm{Ric}|^2|\mathrm{Rm}|^{p-1})\rangle f^a\phi^{2p}\\
&\quad= \int_M |\mathrm{Ric}|^2|\mathrm{Rm}|^{p-1}\Delta(f^a\phi^{2p}) + \int_M|\mathrm{Ric}|^2|\mathrm{Rm}|^{p-1}(\Delta f + a|\nabla f|^2f^{-1})f^a\phi^{2p}\\
&\qquad-\frac{2p}{r}\int_M|\mathrm{Ric}|^2|\mathrm{Rm}|^{p-1}|\nabla f|^2f^a\phi^{2p-1}. \qquad (2.12)
\end{aligned}$$

Observe that

$$\begin{aligned}
\Delta f^a &= af^{a-1}\Delta f + a(a-1)f^{a-2}|\nabla f|^2 \leq cp^2f^{a-1},\\
\Delta\phi^{2p} &= 2p\phi^{2p-1}\Delta\phi + 2p(2p-1)\phi^{2p-2}|\nabla\phi|^2 \leq \frac{cp^2}{r}\phi^{2p-2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\Delta(f^a\phi^{2p}) &= (\Delta f^a)\phi^{2p} + f^a\Delta\phi^{2p} + 2\langle\nabla f^a, \nabla\phi^{2p}\rangle\\
&\leq (\Delta f^a)\phi^{2p} + f^a\Delta\phi^{2p} \leq cp^2f^{a-1}\phi^{2p-2}.
\end{aligned}$$

Consequently, by Jensen's inequality,

$$\begin{aligned}\int_M |\mathrm{Ric}|^2|\mathrm{Rm}|^{p-1}\Delta(f^a\phi^{2p}) &\le cp^2\int_M |\mathrm{Ric}|^2|\mathrm{Rm}|^{p-1}f^{a-1}\phi^{2p-2}\\ &\le \frac{c}{p^4}\int_M |\mathrm{Rm}|^p f^a\phi^{2p} + c(p)\int_M f^{a-p} \le \frac{c}{p^4}\int_M |\mathrm{Rm}|^p f^a\phi^q + C,\end{aligned}$$

where in the last line we have used (2.4) and (2.7) to infer that $\int_M f^{a-p}\le C$. Note that by (2.5),

$$\begin{aligned}\int_M |\mathrm{Ric}|^2|\mathrm{Rm}|^{p-1}(\Delta f + a|\nabla f|^2 f^{-1})f^a\phi^{2p} &\le cp\int_M |\mathrm{Ric}|\,|\mathrm{Rm}|^p f^a\phi^{2p}\\ &\le \frac{c}{p^4}\int_M |\mathrm{Rm}|^p f^a\phi^{2p}.\end{aligned}$$

In conclusion, (2.12) becomes

$$\int_M \Delta_f(|\mathrm{Ric}|^2|\mathrm{Rm}|^{p-1})f^a\phi^{2p} \le \frac{c}{p^4}\int_M |\mathrm{Rm}|^p f^a\phi^{2p} + C. \tag{2.13}$$

Plugging (2.13) into (2.11), we obtain

$$\begin{aligned}&cp^3\int_M |\nabla\mathrm{Ric}|^2|\mathrm{Rm}|^{p-1}f^a\phi^{2p}\\ &\qquad\le \frac{c}{p}\int_M |\nabla\mathrm{Rm}|^2|\mathrm{Rm}|^{p-3}f^a\phi^{2p} + \frac{c}{p}\int_M |\mathrm{Rm}|^p f^a\phi^{2p} + C.\end{aligned}$$

So (2.10) may be rewritten as

$$\begin{aligned}&\int_M |\mathrm{Rm}|^p f^a\phi^{2p}\\ &\le \frac{c}{p}\int_M |\nabla\mathrm{Rm}|^2|\mathrm{Rm}|^{p-3}f^a\phi^{2p} + \frac{cp}{r}\int_M |R_{ijkh}f_h|^2|\mathrm{Rm}|^{p-2}f^a\phi^{2p-1} + C.\end{aligned} \tag{2.14}$$

We now use (2.1) and integration by parts to get

$$\begin{aligned}&\frac{cp}{r}\int_M |R_{ijkh}f_h|^2|\mathrm{Rm}|^{p-2}f^a\phi^{2p-1} = \frac{cp}{r}\int_M \nabla_j R_{ik}(R_{ijkh}f_h)|\mathrm{Rm}|^{p-2}f^a\phi^{2p-1}\\ &= -\frac{cp}{r}\int_M R_{ik}f_h(\nabla_j R_{ijkh})|\mathrm{Rm}|^{p-2}f^a\phi^{2p-1}\\ &\quad - \frac{cp}{r}\int_M R_{ik}R_{ijkh}f_h(\nabla_j|\mathrm{Rm}|^{p-2})f^a\phi^{2p-1} - \frac{cp}{r}\int_M R_{ik}f_{hj}R_{ijkh}|\mathrm{Rm}|^{p-2}f^a\phi^{2p-1}\\ &\quad - \frac{cap}{r}\int_M R_{ik}R_{ijkh}f_hf_j|\mathrm{Rm}|^{p-2}f^{a-1}\phi^{2p-1}\\ &\quad + \frac{cp(2p-1)}{r^2}\int_M R_{ik}R_{ijkh}f_hf_j|\mathrm{Rm}|^{p-2}f^a\phi^{2p-2}.\end{aligned}$$

The last three terms above can be bounded by

$$
\begin{aligned}
&-\frac{cp}{r}\int_M R_{ik} f_{hj} R_{ijkh}|\mathrm{Rm}|^{p-2} f^a \phi^{2p-1} - \frac{cap}{r}\int_M R_{ik} R_{ijkh} f_h f_j |\mathrm{Rm}|^{p-2} f^{a-1}\phi^{2p-1} \\
&\qquad + \frac{cp(2p-1)}{r^2}\int_M R_{ik} R_{ijkh} f_h f_j |\mathrm{Rm}|^{p-2} f^a \phi^{2p-2} \\
&\quad \le cp^2 \int_M |\mathrm{Ric}|\,|\mathrm{Rm}|^{p-1} f^{a-1}\phi^{2p-2} \le \frac{c}{p}\int_M |\mathrm{Rm}|^p f^a \phi^{2p} + c(p)\int_M |\mathrm{Ric}|^p f^{a-p} \\
&\quad \le \frac{c}{p}\int_M |\mathrm{Rm}|^p f^a \phi^{2p} + C.
\end{aligned}
$$

Furthermore, note that

$$
\begin{aligned}
&-\frac{cp}{r}\int_M R_{ik} R_{ijkh} f_h (\nabla_j |\mathrm{Rm}|^{p-2}) f^a \phi^{2p-1} - \frac{cp}{r}\int_M R_{ik} f_h (\nabla_j R_{ijkh}) |\mathrm{Rm}|^{p-2} f^a \phi^{2p-1} \\
&\qquad \le \frac{cp^2}{\sqrt{r}}\int_M |\mathrm{Ric}|\,|\nabla \mathrm{Rm}|\,|\mathrm{Rm}|^{p-2} f^a \phi^{2p-1} \\
&\qquad \le \frac{1}{p}\int_M |\nabla \mathrm{Rm}|^2 |\mathrm{Rm}|^{p-3} f^a \phi^{2p} + \frac{c(p)}{r}\int_M |\mathrm{Ric}|^2 |\mathrm{Rm}|^{p-1} f^a \phi^{2p-2}.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\frac{c(p)}{r}\int_M |\mathrm{Ric}|^2 |\mathrm{Rm}|^{p-1} f^a \phi^{2p-2} \\
&\qquad \le \frac{1}{p}\int_M |\mathrm{Rm}|^p f^a \phi^{2p} + c(p)\int_M |\mathrm{Ric}|^{2p} f^{a-p} \le \frac{1}{p}\int_M |\mathrm{Rm}|^p f^a \phi^{2p} + C,
\end{aligned}
$$

we see that

$$
\begin{aligned}
&-\frac{cp}{r}\int_M R_{ik} R_{ijkh} f_h (\nabla_j |\mathrm{Rm}|^{p-2}) f^a \phi^{2p-1} - \frac{cp}{r}\int_M R_{ik} f_h (\nabla_j R_{ijkh}) |\mathrm{Rm}|^{p-2} f^a \phi^{2p-1} \\
&\qquad \le \frac{1}{p}\int_M |\nabla \mathrm{Rm}|^2 |\mathrm{Rm}|^{p-3} f^a \phi^{2p} + \frac{1}{p}\int_M |\mathrm{Rm}|^p f^a \phi^{2p} + C.
\end{aligned}
$$

Using all these estimates it follows that

$$
\begin{aligned}
&\frac{cp}{r}\int_M |R_{ijkh} f_h|^2 |\mathrm{Rm}|^{p-2} f^a \phi^{2p-1} \\
&\qquad \le \frac{c}{p}\int_M |\nabla \mathrm{Rm}|^2 |\mathrm{Rm}|^{p-3} f^a \phi^{2p} + \frac{c}{p}\int_M |\mathrm{Rm}|^p f^a \phi^{2p} + C.
\end{aligned}
$$

From (2.14) we now conclude that

$$
\int_M |\mathrm{Rm}|^p f^a \phi^{2p} \le \frac{c}{p}\int_M |\nabla \mathrm{Rm}|^2 |\mathrm{Rm}|^{p-3} f^a \phi^{2p} + C. \tag{2.15}
$$

The formula $\Delta_f \mathrm{Rm} = \mathrm{Rm} + \mathrm{Rm} * \mathrm{Rm}$ implies

$$
\Delta_f |\mathrm{Rm}|^2 \ge 2|\nabla \mathrm{Rm}|^2 + 2|\mathrm{Rm}|^2 - c|\mathrm{Rm}|^3. \tag{2.16}
$$

We use this to estimate

$$2\int_M |\nabla \mathrm{Rm}|^2|\mathrm{Rm}|^{p-3} f^a\phi^{2p} \leq \int_M (\Delta|\mathrm{Rm}|^2)|\mathrm{Rm}|^{p-3} f^a\phi^{2p} - \int_M \langle\nabla f, \nabla|\mathrm{Rm}|^2\rangle|\mathrm{Rm}|^{p-3} f^a\phi^{2p} - 2\int_M |\mathrm{Rm}|^{p-1} f^a\phi^{2p} + c\int_M |\mathrm{Rm}|^p f^a\phi^{2p}.$$

We have

$$\begin{aligned}
-\int_M \langle\nabla f, \nabla|\mathrm{Rm}|^2\rangle|\mathrm{Rm}|^{p-3} f^a\phi^{2p} &= -\frac{2}{p-1}\int_M \langle\nabla f, \nabla|\mathrm{Rm}|^{p-1}\rangle f^a\phi^{2p}\\
&= \frac{2}{p-1}\int_M ((\Delta f) f^a + a|\nabla f|^2 f^{a-1})|\mathrm{Rm}|^{p-1}\phi^{2p}\\
&\quad - \frac{4p}{p-1}\frac{1}{r}\int_M |\nabla f|^2 f^a|\mathrm{Rm}|^{p-1}\phi^{2p-1}\\
&\leq \frac{2a+n}{p-1}\int_M |\mathrm{Rm}|^{p-1} f^a\phi^{2p}.
\end{aligned}$$

Since $\frac{2a+n}{p-1} \leq 2$ by (2.7), we conclude that

$$2\int_M |\nabla \mathrm{Rm}|^2|\mathrm{Rm}|^{p-3} f^a\phi^{2p} \leq \int_M (\Delta|\mathrm{Rm}|^2)|\mathrm{Rm}|^{p-3} f^a\phi^{2p} + c\int_M |\mathrm{Rm}|^p f^a\phi^{2p}. \tag{2.17}$$

Integrating by parts, we have

$$\begin{aligned}
&\int_M (\Delta|\mathrm{Rm}|^2)|\mathrm{Rm}|^{p-3} f^a\phi^{2p}\\
&= -\int_M \langle\nabla|\mathrm{Rm}|^2, \nabla|\mathrm{Rm}|^{p-3}\rangle f^a\phi^{2p} - a\int_M \langle\nabla f, \nabla|\mathrm{Rm}|^2\rangle|\mathrm{Rm}|^{p-3} f^{a-1}\phi^{2p}\\
&\quad + \frac{2p}{r}\int_M \langle\nabla f, \nabla|\mathrm{Rm}|^2\rangle|\mathrm{Rm}|^{p-3} f^a\phi^{2p-1}\\
&\leq -a\int_M \langle\nabla f, \nabla|\mathrm{Rm}|^2\rangle|\mathrm{Rm}|^{p-3} f^{a-1}\phi^{2p} + \frac{2p}{r}\int_M \langle\nabla f, \nabla|\mathrm{Rm}|^2\rangle|\mathrm{Rm}|^{p-3} f^a\phi^{2p-1}.
\end{aligned}$$

However,

$$\begin{aligned}
-a\int_M \langle\nabla f, \nabla|\mathrm{Rm}|^2\rangle|\mathrm{Rm}|^{p-3} f^{a-1}\phi^{2p} &= -\frac{2a}{p-1}\int_M \langle\nabla f, \nabla|\mathrm{Rm}|^{p-1}\rangle f^{a-1}\phi^{2p}\\
&= \frac{2a}{p-1}\int_M \big((\Delta f) f^{a-1} + (a-1)|\nabla f|^2 f^{a-2}\big)|\mathrm{Rm}|^{p-1}\phi^{2p}\\
&\quad - \frac{4ap}{p-1}\frac{1}{r}\int_M |\nabla f|^2 f^{a-1}|\mathrm{Rm}|^{p-1}\phi^{2p-1}\\
&\leq cp\int_M |\mathrm{Rm}|^{p-1} f^{a-1}\phi^{2p} \leq c\int_M |\mathrm{Rm}|^p f^a\phi^{2p} + c(p)\int_M f^{a-p}\\
&\leq c\int_M |\mathrm{Rm}|^p f^a\phi^{2p} + C.
\end{aligned}$$

Finally, a similar argument implies that

$$\begin{aligned}\frac{2p}{r}\int_M \langle \nabla f, \nabla |\mathrm{Rm}|^2\rangle |\mathrm{Rm}|^{p-3} f^a \phi^{2p-1} &= \frac{4p}{(p-1)r}\int_M \langle \nabla f, \nabla |\mathrm{Rm}|^{p-1}\rangle f^a \phi^{2p-1}\\ &= \frac{4p}{(p-1)r}\int_M |\mathrm{Rm}|^{p-1}\big((-\Delta f) f^a - a|\nabla f|^2 f^{a-1}\big)\phi^{2p-1}\\ &\quad + \frac{4p(2p-1)}{(p-1)r^2}\int_M |\nabla f|^2 |\mathrm{Rm}|^{p-1} f^a \phi^{2p-2}\\ &\le cp\int_M |\mathrm{Rm}|^{p-1} f^{a-1}\phi^{2p-2} \le c\int_M |\mathrm{Rm}|^p f^a \phi^{2p} + c(p)\int_M f^{a-p}\\ &\le c\int_M |\mathrm{Rm}|^p f^a\phi^{2p} + C.\end{aligned}$$

Using these estimates in (2.17) implies that

$$\int_M |\nabla \mathrm{Rm}|^2 |\mathrm{Rm}|^{p-3} f^a \phi^{2p} \le c\int_M |\mathrm{Rm}|^p f^a \phi^{2p} + C.$$

Plugging this in (2.15) and choosing p so large that $c/p \le 1/2$, we arrive at

$$\int_M |\mathrm{Rm}|^p f^a \phi^{2p} \le C.$$

Note that $\phi \ge 1/2$ on $D(r/2)$. This implies that

$$\int_{D(r/2)} |\mathrm{Rm}|^p f^a \le C.$$

Since r is arbitrary, this proves the lemma. □

We are now ready to prove the main theorem of the paper.

Theorem 2.2. *Let (M, g, f) be a gradient shrinking Ricci soliton of dimension n with Ricci curvature convergent to zero at infinity. Then (M, g) is C^k asymptotic to a cone for all k.*

Proof. Applying Lemma 2.1 for $a = p/4$ and using (2.3), one sees that for any point $x \in M$,

$$\int_{B_x(1)} |\mathrm{Rm}|^p \le C(d(x_0, x) + 1)^{-p/2}. \tag{2.18}$$

We will deduce a pointwise estimate from this by using the De Giorgi–Nash–Moser iteration. First, we derive a differential inequality for $|\mathrm{Rm}|^2$. From (2.16) we get

$$\Delta_f |\mathrm{Rm}|^2 \ge 2|\nabla \mathrm{Rm}|^2 - c|\mathrm{Rm}|^3.$$

However, estimating

$$|\langle \nabla f, \nabla |\mathrm{Rm}|^2\rangle| \le |\nabla \mathrm{Rm}|^2 + |\nabla f|^2 |\mathrm{Rm}|^2,$$

we get

$$\Delta|\mathrm{Rm}|^2 \geq -u|\mathrm{Rm}|^2, \quad \text{where} \quad u := c(|\mathrm{Rm}| + f).$$

Note that the (Dirichlet) Sobolev constant of $B_x(1)$ depends only on the dimension n, the Ricci curvature bound and Perelman's invariant (see [23]). So the De Giorgi–Nash–Moser iteration (see e.g. [19, Chapter 19]) implies that

$$|\mathrm{Rm}|(x) \leq C\left(\int_{B_x(1)} u^n + 1\right)^{1/p}\left(\int_{B_x(1)} |\mathrm{Rm}|^p\right)^{1/p}. \tag{2.19}$$

By (2.18) and the Bishop volume comparison we get

$$\int_{B_x(1)} |\mathrm{Rm}|^n \leq \left(\int_{B_x(1)} |\mathrm{Rm}|^p\right)^{n/p} \mathrm{Vol}(B_x(1))^{(p-n)/p} \leq C(d(x_0, x) + 1)^{-n/2}.$$

Hence,

$$\int_{B_x(1)} u^n \leq C(d(x_0, x) + 1)^{2n}.$$

Together with (2.18) and (2.19) we get

$$|\mathrm{Rm}|(x) \leq C(d(x_0, x) + 1)^{-1/4}.$$

In particular, this shows that $\lim_{x\to\infty} |\mathrm{Rm}| = 0$.

Now from

$$\Delta_f|\mathrm{Rm}| \geq |\mathrm{Rm}| - c|\mathrm{Rm}|^2$$

and the fact that $|\mathrm{Rm}| \to 0$ at infinity, it follows as in [21] that Rm decays quadratically,

$$|\mathrm{Rm}|(x) \leq c(d(x_0, x) + 1)^{-2}. \tag{2.20}$$

For completeness, we include the details here. Denote $w := |\mathrm{Rm}|$. From (2.16) we get

$$\Delta_f w \geq w - c_0 w^2$$

for some $c_0 > 0$. Since w converges to zero at infinity, there exists r_0 sufficiently large such that $w < 1/(8c_0)$ on $M \setminus D(r_0)$. As $\Delta_f(f) = n/2 - f$, we get

$$\Delta_f(f^{-1}) = -\Delta_f(f)f^{-2} + 2|\nabla f|^2 f^{-3} \leq f^{-1}.$$

Similarly,

$$\Delta_f(f^{-2}) = 2(f - n/2)f^{-3} + 6|\nabla f|^2 f^{-4} \geq 2f^{-2} - nf^{-3}.$$

Hence, for r_0 sufficiently large, it follows that $\Delta_f(f^{-2}) \geq \frac{3}{2}f^{-2}$ on $M \setminus D(r_0)$. Define

$$v := w - \beta f^{-1} + 2c_0\beta^2 f^{-2},$$

where $\beta := r_0/(4c_0)$. From the above we get

$$\begin{aligned}\Delta_f v &\geq w - c_0 w^2 - \beta f^{-1} + 3c_0\beta^2 f^{-2} = v - c_0 w^2 + c_0\beta^2 f^{-2} \\ &= v - c_0(w - \beta f^{-1})(w + \beta f^{-1}) \geq v - c_0 v(w + \beta f^{-1}).\end{aligned} \tag{2.21}$$

Assume that there exists a point $x \in M \setminus D(r_0)$ such that $v(x) > 0$. Since

$$v < \frac{1}{8c_0} - \frac{\beta}{r_0} + \frac{2c_0\beta^2}{r_0^2} = 0 \quad \text{on } \partial D(r_0)$$

and $v \to 0$ at infinity, we conclude that v achieves a positive maximum in the interior of $M \setminus D(r_0)$. Applying the maximum principle to (2.21) we get

$$c_0(w + \beta f^{-1}) \geq 1 \tag{2.22}$$

at the maximum point of v. However, recall that $w < 1/(8c_0)$ and $\beta = r_0/(4c_0)$. This contradicts (2.22). In conclusion, $v \leq 0$ on $M \setminus D(r_0)$, which implies (2.20).

Now (2.20) and Shi's derivative estimates imply that the derivatives of the Riemann curvature tensor satisfy

$$|\nabla^k \mathrm{Rm}|(x) \leq c_k (d(x_0, x) + 1)^{-k-2}$$

for all $k \geq 1$. From this, it follows that (M, g) is C^k asymptotic to a cone for all k. We refer to [18] for more details. The theorem is proved. □

In dimension four, a stronger result proved in [21] says that if (M, g, f) has scalar curvature converging to zero at infinity, then (M, g) is asymptotically conical. While it is unclear to us whether this is true in arbitrary dimension, we do have the following partial result.

Corollary 2.3. *Let (M, g, f) be a shrinking Ricci soliton of dimension n with bounded curvature. Assume that the scalar curvature converges to zero at infinity. Then (M, g) is C^k asymptotic to a cone for all k.*

Proof. Recall that $(M, g(t))$ satisfies the Ricci flow equation, where $g(t) := (1-t)\phi_t^* g$ and $\frac{d\phi_t}{dt} = \frac{1}{1-t}\nabla f$ with $\phi_0 = \mathrm{Id}$. Since (M, g) has bounded curvature, it follows that $(M, g(t))$ is a noncollapsed type I ancient solution by [24]. So, for any $x_k \to \infty$, a subsequence of $(M, g(t), x_k)$ converges in the pointed Cheeger–Gromov sense to $(M_\infty, g_\infty(t), x_\infty)$. Moreover, $(M_\infty, g_\infty(t), x_\infty)$ is also an ancient solution to the Ricci flow. Clearly, the scalar curvature, of $(M_\infty, g_\infty(0))$ at x_∞ vanishes. So, according to [10], the scalar curvature of $(M_\infty, g_\infty(t))$ must be zero identically. By the evolution equation of the scalar curvature $\partial_t S = \Delta S + 2|\mathrm{Ric}|^2$, we see that this ancient solution is Ricci flat. This implies that for any sequence $x_k \to \infty$, there exists a subsequence x_{k_j} such that the Ricci curvature of (M, g) satisfies $|\mathrm{Ric}|(x_{k_j}) \to 0$. It is then easy to conclude that the Ricci curvature converges to zero at infinity. Now the corollary follows by appealing to Theorem 2.2. □

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