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The dynamical Manin–Mumford problem for plane polynomial automorphisms

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Abstract. Let f be a polynomial automorphism of the affine plane. In this paper we consider the possibility for it to possess infinitely many periodic points on an algebraic curve C. We conjecture that this happens if and only if f admits a time-reversal symmetry; in particular the Jacobian $\mathrm{Jac}(f)$ must be a root of unity.

As a step towards this conjecture, we prove that the Jacobian and all its Galois conjugates lie on the unit circle in the complex plane. Under mild additional assumptions we are able to conclude that indeed $\mathrm{Jac}(f)$ is a root of unity.

We use these results to show in various cases that any two automorphisms sharing an infinite set of periodic points must have a common iterate, in the spirit of recent results by Baker–DeMarco and Yuan–Zhang.

Keywords. Dynamical Manin–Mumford problem, polynomial automorphisms of the plane, dynamical heights, arithmetic equidistribution, non-Archimedean dynamics, non-uniform hyperbolicity

Contents

Introduction

In this paper we discuss the following problem in the case of polynomial automorphisms of the affine plane.

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Dynamical Manin-Mumford Problem. *Let X be a quasi-projective variety and $f : X \to X$ a dominant endomorphism. Describe all positive-dimensional irreducible subvarieties $C \subset X$ such that the Zariski closure of the set of preperiodic*[1] *points of f contained in C is Zariski dense in C.*

In case (X, f) is the dynamical system induced on an abelian variety (defined over a number field) by multiplication by an integer ≥ 2, it is a deep theorem originally due to M. Raynaud (and formerly known as the Manin–Mumford conjecture) that any such C is a translate of an abelian subvariety by a torsion point. Several generalizations of this theorem have appeared since then, concerning abelian and semiabelian varieties over fields of arbitrary characteristic. We refer to [PR02, Roe08] for an account on the different approaches to these results.

S.-W. Zhang [Zh95] conjectured that a similar result should hold in the more general setting of polarized[2] endomorphisms. More precisely, he asked whether any subvariety containing a Zariski dense set of periodic points is itself preperiodic. This conjecture was recently disproved by D. Ghioca, T. Tucker and S.-W. Zhang [GTZ11], who proposed a modified statement (see also [Pa13]). Some positive results on Zhang's conjecture are also available—see for instance [MS14].

Our goal is to explore this problem when f is a polynomial automorphism of the affine plane $\mathbb{A}^2$, defined over a field of characteristic zero.

Let us first collect a few facts on the dynamics of these maps. A dynamical classification of polynomial automorphisms was given by S. Friedland and J. Milnor [FM89], based on a famous theorem of H. W. E. Jung. They proved that any polynomial automorphism is conjugate to one of the following forms:

- an affine map,
- an elementary automorphism, that is, a map of the form $(x, y) \mapsto (ax + b, y + P(x))$ with $a \neq 0$, b a constant and P is a polynomial,
- a polynomial automorphism f satisfying $\deg(f^n) = \deg(f)^n \geq 2$ for every integer $n \geq 1$.

In the last case the integer $\deg(f) \geq 1$ denotes the maximum of the degrees of the components of f in any set of affine coordinates. An automorphism falling into this category will be referred to as *of Hénon type*. Observe that the set of periodic points of an affine or an elementary map is algebraic, hence the dynamical Manin–Mumford problem is uninteresting in these cases. We shall therefore restrict our attention to Hénon-type automorphisms.

Suppose that f is an automorphism of Hénon type that is conjugate to its inverse by an involution σ possessing a curve C of fixed points. Such a map is usually called *reversible* [GM03a, GM03b]. Then any point $p \in C \cap f^{-n}(C)$ is periodic of period $2n$, and we verify in §7 that $\#(C \cap f^{-n}(C))$ indeed grows to infinity. Thus the pair (f, C) falls into the framework of the Manin–Mumford problem. On the other hand, it is a theorem by

[1] That is, satisfying $f^n(p) = f^m(p)$ for some $n > m \geq 0$.

[2] This means that X is projective, and $f^*L \simeq L^{\otimes q}$ for some ample line bundle $L \to X$ and an integer $q \geq 2$.

E. Bedford and J. Smillie [BS91] that there exists no f-invariant algebraic curve. These examples motivate the following conjecture.

Conjecture 1 (Dynamical Manin–Mumford conjecture for complex polynomial automorphisms of the affine plane). *Let f be a complex polynomial automorphism of Hénon type of the affine plane. Assume that there exists an irreducible algebraic curve C containing infinitely many periodic points of f. Then there exists an involution σ of the affine plane whose set of fixed points is C and an integer $n \geq 1$ such that $\sigma f^n \sigma = f^{-n}$.*

Recall that the Jacobian $\mathrm{Jac}(f)$ of a polynomial automorphism is a non-zero constant. If f is reversible then $\mathrm{Jac}(f) = \pm 1$. In particular, if f^n is reversible for some n then $\mathrm{Jac}(f)$ must be a root of unity.

Our first main result can thus be seen as a step towards Conjecture 1 (see Remark 4.4 below for comments about the symmetry asserted in the conjecture).

Theorem A. *Let f be a polynomial automorphism of Hénon type of the affine plane, defined over a field of characteristic zero. Assume that there exists an algebraic curve containing infinitely many periodic points of f. Then $\mathrm{Jac}(f)$ is algebraic over $\mathbb{Q}$ and all its Galois conjugates have complex modulus 1.*

In particular, if $\mathrm{Jac}(f)$ is an algebraic integer then it is a root of unity.

Using a specialization argument, one can reduce the proof to the case where f and C are both defined over a number field $\mathbb{L}$ (see §5). Fix an algebraic closure $\mathbb{L}^{\mathrm{alg}}$ of $\mathbb{L}$. Modifying a construction of S. Kawaguchi [Ka06], C.-G. Lee [Le13] built a dynamical height function $h_f : \mathbb{A}^2(\mathbb{L}^{\mathrm{alg}}) \to \mathbb{R}_+$. This height is associated to a continuous semi-positive adelic metrization of the ample line bundle $\mathcal{O}_{\mathbb{P}^2}(1)$ (in the sense of Zhang [Zh95]) and $h_f(p) = 0$ if and only if p is periodic. We refer the reader to the survey [CL11] for a detailed account on these concepts.

When f and C are defined over a number field, Theorem A is now a consequence of the following effective statement.

Theorem A′. *Let f be a polynomial automorphism of Hénon type of the affine plane, defined over a number field $\mathbb{L}$. Assume that there exists an Archimedean place v such that $|\mathrm{Jac}(f)|_v \neq 1$. Then for any algebraic curve C defined over $\mathbb{L}$ there exists a positive constant $\varepsilon = \varepsilon(C) > 0$ such that the set $\{p \in C(\mathbb{L}^{\mathrm{alg}}) : h_f(p) \leq \varepsilon\}$ is finite.*

Let us briefly explain the strategy of the proof. We follow the approach of L. Szpiro, E. Ullmo and S.-W. Zhang [SUZ97, Ul98, Zh98] to the Bogomolov conjecture whose statement is the analog of Theorem A′ in case f is the doubling map on an abelian variety.

The first step is to describe the asymptotic distribution of the periodic points lying on C. Pick any place v on $\mathbb{L}$ and denote by $\mathbb{C}_v$ the completion of the algebraic closure of the completion of $\mathbb{L}$ relative to the norm v. Write $\|(x, y)\|_v = \max\{|x|, |y|\}$ and $\log^+ = \max\{\log, 0\}$. Then it can be shown that the sequence of functions $d^{-n} \log^+ \|f^n(x, y)\|_v$ converges uniformly on bounded sets in $(\mathbb{C}_v)^2$ to a continuous "Green" function $G_v^+ : (\mathbb{C}_v)^2 \to \mathbb{R}_+$ satisfying the invariance property $G^+ \circ f = dG^+$, where $d = \deg(f)$.

Its zero locus $\{G^+ = 0\}$ coincides with the set of points in $(\mathbb{C}_v)^2$ with bounded forward orbits.

Replacing f by its inverse, we define a function G^- in a similar way and we set $G = \max(G^+, G^-)$. These Green functions were first introduced and studied in the context of complex polynomial automorphisms by J. H. Hubbard [Hu86], E. Bedford and J. Smillie [BS91] and J. E. Fornæss and N. Sibony [FS92].

The key observation is that the asymptotic distribution of periodic points on C can be understood by applying suitable equidistribution results for points of small height on curves. These results were developed by various authors in successively greater generality and the version we use here is due to P. Autissier [Au01] and A. Thuillier [Th05]. More precisely, we prove that the collection of functions $\{G_v^+\}_v$ (resp. $\{G_v^-\}_v$) induces a continuous semipositive metrization of $\mathcal{O}_C(1)$. Then the Autissier–Thuillier theorem implies that the probability measures equidistributed over Galois conjugates of periodic points in C converge to a multiple of $\Delta G_v^+|_C$ (resp. $\Delta G_v^-|_C$) at any place[3] when the period tends to infinity.

From this one deduces that for each v the functions G_v^+ and G_v^- are proportional on C (up to a harmonic function).

The second step is to use this information on the Green functions to infer that f is conservative at Archimedean places. The argument relies on Pesin's theory and is quite technical so that let us first explain how the mechanism works under a more restrictive assumption.

Suppose indeed that there exists a hyperbolic periodic point p in the regular locus $\mathrm{reg}(C)$ of C, with multipliers u, s satisfying $|u| > 1 > |s|$, and assume moreover that the local unstable manifold $W^u_{\mathrm{loc}}(p)$, the local stable manifold $W^s_{\mathrm{loc}}(p)$, and the curve C are pairwise transverse. Using the invariance property of G^+, we show that the local Hölder exponent ϑ_+ of G^+ at p along $W^u_{\mathrm{loc}}(p)$ satisfies $|u|^{\vartheta_+} = d$. Using a rescaling argument reminiscent of that used by X. Buff and A. Epstein [BE09], we then show that this Hölder exponent is actually equal to that of $G^+|_C$. Applying the same argument to f^{-1}, we find that the local Hölder exponent ϑ_- of G^- along the stable manifold satisfies $|s|^{-\vartheta_-} = d$. But since $G^+|_C$ and $G^-|_C$ are proportional, ϑ_- and ϑ_+ must be equal. This proves that $|\mathrm{Jac}(f)| = |us| = 1$.

Unfortunately we cannot ensure the existence of such a saddle point at an archimedean place. It turns out that working at all places (Archimedean or not) resolves this difficulty. Adapting the above argument then leads to our next main result.

Theorem B. *Let f be a polynomial automorphism of Hénon type, defined over a field of characteristic zero. Assume that there exists an irreducible curve C containing infinitely many periodic points of f. Suppose in addition that the following transversality statement is true:*

(T) *There exists a periodic point $p \in \mathrm{reg}(C)$ such that T_pC is not periodic under the induced action of f.*

Then $\mathrm{Jac}(f)$ *is a root of unity.*

[3] At a non-Archimedean place, Δ stands for the Laplacian operator as defined by Thuillier.

Observe that this result is very much in the spirit of [GTZ11, Conjecture 2.4]. Let us also note that if C contains a saddle point at an Archimedean place, then the transversality assumption (T) is superfluous (see Theorem 4.3).

Returning to the proof of Theorem A we get around the issue of the existence of a hyperbolic periodic point on C and that of the transversality of its invariant manifolds with C by applying Pesin's theory of non-uniform hyperbolicity, in combination with the theory of laminar currents, in the spirit of the work of E. Bedford, M. Lyubich and J. Smillie [BLS93a]. This allows us to estimate the Hölder exponent of G^+ at generic points and relate it to the positive Lyapunov exponent of the so-called *equilibrium measure* $\mu_f := (dd^c)^2 \max(G^+, G^-)$. This is an ergodic invariant measure which has remarkable properties; in particular it describes the asymptotic distribution of periodic orbits [BLS93b].

The proportionality of G^+ and G^- on C finally implies that the positive and negative exponents are opposite, thereby showing that $|\mathrm{Jac}(f)| = 1$.

The key input of Pesin's theory into our argument is to guarantee the *transversality* of stable and unstable manifolds at a μ_f-generic point with the curve C.

A dual way to state Theorem A is to say that the intersection of the set of periodic points with any curve is finite when $|\mathrm{Jac}(f)| \neq 1$. We expect that the following stronger uniform statement holds.

Conjecture 2. *Let f be a complex polynomial automorphism of Hénon type such that $|\mathrm{Jac}(f)| \neq 1$. Then for any algebraic curve C, the cardinality of the set of periodic points of f lying on C is bounded from above by a constant depending only on the degree of C, the degree of f and the Jacobian* $\mathrm{Jac}(f)$.

We indicate in §3.3 how to adapt the arguments of Theorem A′ to confirm a weaker form of this conjecture.

The automatic uniformity statement obtained by T. Scanlon [Sc04], based on ideas of E. Hrushovski, implies that such a bound would follow from (a restricted version of) the dynamical Manin–Mumford problem for product maps of the form $(f, \dots, f)$ acting on $(\mathbb{A}^2)^n$. Even though this problem seems very delicate, we are able to address some cases of the dynamical Manin–Mumford problem for special product maps of Hénon type.

In the second part of the paper we prove the following two theorems.

Theorem C. *Let f and g be polynomial automorphisms of Hénon type of the affine plane defined over a number field. If f and g share a set of periodic points that is Zariski dense, then there exist non-zero $n, m \in \mathbb{Z}$ such that $f^n = g^m$.*

Theorem D. *Let f and g be polynomial automorphisms of Hénon type of the affine plane with complex coefficients such that $|\mathrm{Jac}(f)| \neq 1$. If f and g share an infinite set of periodic points, then there exist non-zero $n, m \in \mathbb{Z}$ such that $f^n = g^m$.*

Notice that these two statements concern product maps (f, g) such that the diagonal in $\mathbb{A}^2 \times \mathbb{A}^2$ admits a Zariski dense set of periodic points.[4]

[4] Observe that in Theorem D, this Zariski density follows from Theorem A.

We also show in §6.4 that Theorem C holds for a pair of automorphisms sharing infinitely many periodic *cycles*.

Observe that Theorems C and D are analogs in our setting of recent results due to M. Baker and L. DeMarco [BdM11] and X. Yuan and S.-W. Zhang [YZ13a, YZ13b].

We believe that these results hold under the following weaker assumption.

Conjecture 3. *Suppose f and g are complex polynomial automorphisms of Hénon type sharing infinitely many periodic points. Then $f^n = g^m$ for some non-zero integers n and m.*

The proof of Theorems C and D goes as follows. The hypothesis implies that the equidistribution theorem for points of small height (X. Yuan [Yu08], C.-G. Lee [Le13]) can be applied. Therefore f and g have the same equilibrium measure. If it happens that f and g are simultaneously conjugate to automorphisms that extend to birational maps on $\mathbb{P}^2$ contracting the line at infinity to a point that is not indeterminate, then it is not difficult to see that the Green functions of f and g coincide. We can then invoke a theorem of S. Lamy [La01] to conclude that f and g have a common iterate.

Otherwise we use the equality of equilibrium measures at all places to infer that f and g, as well as any automorphism belonging to the group generated by f and g, have the same sets of periodic points. Then we use Lamy's geometric group-theoretic description of $\mathrm{Aut}[\mathbb{A}^2]$ to reduce the situation to the previous one.

Theorem D is obtained from Theorem C by a specialization argument. Theorem A is used in the course of the proof, which explains the need of an extra hypothesis on the Jacobian of one of the maps.

The plan of the paper is as follows. In §1 we gather a number of facts on the dynamics of polynomial automorphisms over arbitrary metrized fields, including equidistribution theorems for points of small height. Then in §2 we show how these equidistribution results apply in our situation. Theorems A and B are respectively established in §3 and 4, assuming that f and C are defined over a number field. The extension to general ground fields is achieved in §5. The proofs of Theorems C and D are given in §6. Finally, §7 is devoted to a discussion of reversible mappings.

1. Polynomial automorphisms over a metrized field

In the first four sections, we fix an arbitrary complete (non-trivially) metrized field $(L, |\cdot|)$ of characteristic zero that is algebraically closed. In §1.6 and §1.7, we work over a number field $\mathbb{L}$.

1.1. Potential theory over a non-Archimedean curve

In this section we suppose that the norm on L is non-Archimedean and give a brief account of Thuillier's potential theory on curves [Th05].

We pick any algebraic curve C defined over L (possibly singular). We shall work with the analytification C^{an} of C in the sense of Berkovich [Be90, §3.4]. If $U \subset C$

is an affine Zariski open subset of C, then its analytification U^{an} is defined as the set of multiplicative seminorms on the ring $L[U]$ of regular functions whose restriction to L equals $|\cdot|$, endowed with the topology of pointwise convergence. Any closed point $p \in C$ defines a point in U^{an} given by $L[U] \ni \phi \mapsto |\phi(p)| \in \mathbb{R}_+$.

The space C^{an} is then constructed by patching together the sets U^{an} where U ranges over any affine cover of C. In this way, one obtains a locally compact and connected space. There is a distinguished set of compact subsets of C^{an} that forms a basis for its topology; its elements are referred to as *strictly L-affinoid subdomains*. We refer to [Be90, §3] for a formal definition. For us it will be sufficient to say that each affinoid subdomain A has a finite boundary, and admits a canonical retraction to its skeleton $\text{Sk}(A) \subset A$ which is the geometric realization of a finite graph. We write $r_A : A \to \text{Sk}(A)$ for this retraction.

Any of these skeletons comes equipped with a canonical integral affine structure, hence with a metric. One can thus make sense of the notion of a harmonic function on $\text{Sk}(A)$. By definition this is a continuous function that is piecewise affine and such that the sum of the directional derivatives at any point (including the endpoints) is zero.

A *harmonic* function $h : U \to \mathbb{R}$ defined on a (Berkovich) open subset U of the regular part $\text{reg}(C)^{\text{an}}$ of C is a continuous function such that for all subdomains A the map $h|_{\text{Sk}(A)}$ is harmonic.

For any invertible function $\phi \in L[U]$ defined on an affine open subset $U \subset C$, the function $\log|\phi|$ is harmonic on U^{an} [Th05, Proposition 2.3.20]. However it is not true that any harmonic function can be locally expressed as the logarithm of an invertible function [Th05, Lemme 2.3.22]. This discrepancy with the complex case will not, however, affect our arguments.

Proposition 1.1. *Pick any open subset U of $\text{reg}(C)^{\text{an}}$, and suppose that h_n is a sequence of harmonic functions defined on U that converges uniformly. Then its limit $\lim_n h_n$ is harmonic.*

Proof. This is a consequence of the following fact: Suppose we are given a sequence of convex functions on a real interval that converges locally uniformly. Then the limit is convex and the directional derivatives also converge at any point. □

Proposition 1.2 ([Th05, Proposition 2.3.13]). *Suppose u is a non-negative harmonic function such that $h(p) = 0$ for some point $p \in U$. Then h is constant in a neighborhood of p.*

Pick any connected open subset $U \subset \text{reg}(C)^{\text{an}}$. An upper semicontinuous function $u : U \to \mathbb{R} \cup \{-\infty\}$ is said to be *subharmonic* if it is not identically $-\infty$ and satisfies the condition that for any strictly L-affinoid subdomain A and any harmonic function h on A the inequality $u|_{\partial A} \leq h|_{\partial A}$ implies $u \leq h$ on A.

One can check that the set of subharmonic functions is a positive convex cone that is stable under taking maxima, contains all functions of the form $\log|\phi|$ for any regular function ϕ, and is stable under decreasing sequences[5] [Th05, Proposition 3.1.9].

[5] We shall be concerned *only* with subharmonic functions that are uniform limits of positive linear combinations of maxima of functions of the form $\log|\phi|$.

To any subharmonic function u defined on an open set $U \subset \mathrm{reg}(C)^{\mathrm{an}}$ is associated a unique positive Radon measure Δu supported on U that satisfies the following properties:

- $\Delta(au + v) = a\Delta u + \Delta v$ for any two subharmonic functions u, v and any positive constant $a > 0$;
- for any regular function ϕ, the Poincaré–Lelong formula holds:

$$\Delta \log |\phi| = \sum_{\phi(p)=0} \mathrm{ord}_p(\phi)\delta_p;$$

- for any decreasing sequence $u_n \to u$, Δu_n converges to Δu in the weak sense of measures.

We shall use the following properties of this Laplacian operator.

Proposition 1.3. *Let $u : U \to \mathbb{R} \cup \{-\infty\}$ be any subharmonic function. Then u is harmonic if and only if $\Delta u = 0$.*

Proof. If u is harmonic then $\pm u$ are subharmonic as in [Th05, Définition 3.1.5], hence $\pm\Delta u$ is a positive measure by [Th05, Théorème 3.4.8], and $\Delta u = 0$.

Conversely, if $\Delta u = 0$ then u is harmonic by [Th05, Corollaire 3.4.9]. □

Proposition 1.4. *Suppose U, V are open subsets of the Berkovich analytifications of two smooth algebraic curves, and $f : U \to V$ is an isomorphism. Let u be any subharmonic function on V. Then $u \circ f$ is subharmonic on U, and $\Delta(u \circ f) = f^*\Delta u$.*

Proof. The first statement follows from [Th05, Proposition 3.1.13], and the second from [Th05, Proposition 3.2.13]. □

Remark 1.5. The above discussion was restricted to harmonic and subharmonic functions on an open subset contained in the *regular part* of an algebraic curve. Throughout the paper we will be led to consider such functions on curves which are not a priori smooth. In every such case it is understood that the notions of harmonicity or subharmonicity are considered only on the regular part of the curve.

Remark 1.6. Suppose u is a *bounded* function that is subharmonic on $\mathrm{reg}(C)$, and let $\pi : \widetilde{C} \to C$ be the normalization of C. Then $u \circ \pi$ extends to a subharmonic function on $\widetilde{C}$ and $\Delta(u|_{\mathrm{reg}(C)})$ coincides with $\pi_*\Delta(u \circ \pi)$. We shall simply write Δu in this case.

1.2. Dynamics of regular automorphisms

Following Sibony [Si99] we say that a polynomial automorphism of the affine plane $f : \mathbb{A}^2 \to \mathbb{A}^2$ is *regular* if its extension as a rational map to the projective plane $F : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ contracts the line at infinity H_∞ to a point p_+ that is *not* indeterminate for F. It follows that p_+ is a superattracting fixed point, and that its inverse map contracts H_∞ to the (single) indeterminacy point p_- of F.

The degree of a polynomial map is the maximum of the degrees of its (two) components. By [FM89], up to a linear change of coordinates, any regular polynomial automorphism of degree ≥ 2 is the composition of finitely many maps of the form

$$(x, y) \mapsto (ay, x + P(y))$$

where $a \in L^*$ and P is a polynomial of degree ≥ 2.

A polynomial automorphism of $\mathbb{A}^2$ will be said to be *of Hénon type* if it is conjugate (in the group of automorphisms) to a regular automorphism of degree ≥ 2. A complex polynomial automorphism has positive topological entropy if and only if it is of Hénon type.

Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be any regular polynomial automorphism of degree $d \geq 2$ and such that $p_+ = [0 : 1 : 0]$ and $p_- = [1 : 0 : 0]$ in homogeneous coordinates on $\mathbb{P}^2$.

Fix a constant $C > 0$, and define

$$\begin{aligned} V &= \{p = (x, y) \in L^2 : \|p\| = \max\{|x|, |y|\} \leq C\}, \\ V^+ &= \{(x, y) \in L^2 : |y| \geq \max\{|x|, C\}\}, \\ V^- &= \{(x, y) \in L^2 : |x| \geq \max\{|y|, C\}\}. \end{aligned}$$

It can be shown that if C was chosen sufficiently large, then $f(V^+) \subset V^+$, and more precisely

$$\frac{1}{d} \log|y \circ f| \geq \log|y| - \text{const} \tag{1.1}$$

for any point in V^+. The same kind of inequality holds when f is replaced by its inverse, so we similarly obtain $f^{-1}(V^-) \subset V^-$. This implies $f(V) \subset V \cup V^+$.

Set

$$K = \left\{p \in L^2 : \sup_{n\in\mathbb{Z}} \|f^n(p)\| < \infty\right\}, \qquad K^\pm = \left\{p \in L^2 : \sup_{n\geq 0} \|f^{\pm n}(p)\| < \infty\right\}.$$

The next result easily follows from the above properties.

Lemma 1.7. *We have*

- $K = K^+ \cap K^- \subset V$;
- $L^2 \setminus K^+ = \bigcup_{n\geq 0} f^{-n}(V^+)$ *and* $K^+ \subset V \cup V^-$;
- $L^2 \setminus K^- = \bigcup_{n\geq 0} f^{-n}(V^-)$ *and* $K^- \subset V \cup V^+$.

Proposition 1.8. *The sequences $d^{-n} \log^+ \|f^{\pm n}(p)\|$ converge pointwise as $n \to \infty$ to respective functions $G^\pm$ which are continuous and non-negative on L^2. These functions moreover satisfy:*

(1) $G^\pm \circ f^{\pm 1} = dG^\pm$;
(2) $G^\pm(p) - \log^+ \|p\|$ *extends to a continuous functions on $\mathbb{P}^2(L) \setminus \{p_\mp\}$ that is bounded from above;*
(3) $\{G^\pm = 0\} = K^\pm$.

This result follows from [BS91] over the complex numbers, and [Ka13, Theorem A] or [In14, §2] over a non-Archimedean field.

Sketch of proof. Since f is regular, we know that $d^{-1}\log^+\|f(p)\| \leq \log^+\|p\| + \text{const}$ on L^2. The converse inequality holds on V^+ by (1.1). From this we infer that the sequence $d^{-n}\log^+\|f^n(p)\|$ converges uniformly on the f-invariant set $V^+ \cup V$, so that G^+ is well-defined, continuous and satisfies (1) there. Since any point eventually lands in $V^+ \cup V$ under positive iteration, using (1) we can extend G^+ to a globally defined continuous function. Property (2) is a consequence of the estimate $d^{-1}\log^+\|f(p)\| \leq \log^+\|p\| + \text{const}$. To get (3), it is enough to show that $G^+(p) > 0$ when $p \in V^+$. Again this is a consequence of (1.1). □

The following fact will be crucial in our work. It follows from [BS91, Proposition 4.2] whose proof works over any field.

Proposition 1.9. *A polynomial automorphism of Hénon type admits no invariant algebraic curve.*

1.3. Invariant measures

We keep the notation of the previous subsection. Our purpose is to construct an invariant measure from the functions G^+ and G^-.

Proposition 1.10. *The function $G = \max\{G^+, G^-\}$ is a continuous non-negative function on L^2 that satisfies:*

(1) *$G(p) - \log^+\|p\|$ extends to a continuous function to $\mathbb{P}^2(L)$;*
(2) *$\{G = 0\} = K$;*
(3) *G is the uniform limit of the sequence of continuous functions*

$$\max\{d^{-n}\log^+\|f^n\|, d^{-n}\log^+\|f^{-n}\|\}.$$

Sketch of proof. Properties (1) and (2) are direct consequences of items (2) and (3) of Proposition 1.8.

To prove the convergence in (3), we argue as follows (see [Ka13, Theorem 2.3] for related arguments): for large B and small ε consider the domain $V^+_{B,\varepsilon}$ defined by

$$V^+_{B,\varepsilon} := \{(x, y) \in L^2 : |y| \geq B,\ |x| \leq \varepsilon|y|\}.$$

Define similarly $V^-_{B,\varepsilon}$ by exchanging the roles of x and y, and let $W = L^2 \setminus (V^+_{B,\varepsilon} \cup V^-_{B,\varepsilon})$. It follows from the proof of Proposition 1.8 that $d^{-n}\log\|f^n(p)\|$ converges uniformly to G^+ on $V^+_{B,\varepsilon} \cup W$. Likewise, $d^{-n}\log\|f^{-n}(p)\|$ converges uniformly to G^- on $V^-_{B,\varepsilon} \cup W$. In particular, $d^{-n}\max\{\log^+\|f^n\|, \log^+\|f^{-n}\|\}$ converges uniformly to $G = \max\{G^+, G^-\}$ on W.

We know from Proposition 1.8 that $G^+(p) \geq \log^+\|p\| + \text{const}$ on V^+, hence on $V^+_{B,\varepsilon}$. We claim that for every $A > 0$, we can choose (ε, B) so that $G^-(p) \leq \log^+\|p\| - A$ on $V^+_{B,\varepsilon}$. In particular $G^+ > G^-$ there. The same argument shows that $\log^+\|f^n\| > \log^+\|f^{-n}\|$ on $V^+_{B,\varepsilon}$, so we conclude that

$$\frac{1}{d^n}\max\{\log^+\|f^n\|, \log^+\|f^{-n}\|\} = \frac{1}{d^n}\log^+\|f^n\| \to G^+ = \max\{G^+, G^-\}$$

on that set. Of course reversing the roles of f and f^{-1} we get the same result on $V^-_{B,\varepsilon}$, and property (3) follows.

To prove the desired estimate, observe that by our assumptions on f, we have $f^{-1}(x, y) = (x^d, 0) + \text{l.o.t.}$ In particular, if $|x| \le \varepsilon|y|$, then for $B \ge B(\varepsilon)$ we get $\|f^{-1}(p)\| \lesssim \varepsilon^d|y|^d \le \varepsilon^d\|p\|^d$. Since $G^-(p) \le \log^+\|p\| + \text{const}$ on L^2, using the invariance relation for G^- we find that for $p \in V^+_{B,\varepsilon}$,

$$G^-(p) = \frac{1}{d}G^-(f^{-1}(p)) \le \frac{1}{d}\log^+\|f^{-1}(p)\| + \text{const} \le \frac{1}{d}\log(\varepsilon^d\|p\|^d) + \text{const},$$

hence the result. □

Now assume that $(L, |\cdot|)$ is Archimedean, i.e. $L = \mathbb{C}$ endowed with its standard hermitian norm. In this case, G^+ and G^- are continuous plurisubharmonic functions on $\mathbb{C}^2$, and so is G. Using Bedford–Taylor's theory it is possible to make sense of the Monge–Ampère operator of G and define the positive measure $\mu_f := (dd^c)^2G$. It is an f-invariant probability measure whose support is included in K. We refer to [BLS93a] for details on its ergodic properties.

Pick any irreducible algebraic curve C and denote by $\text{reg}(C)$ its set of regular points. Then $\mu_{f,C}$ is by definition the Laplacian of the function G restricted to $\text{reg}(C)$. Since G is continuous and $G(p) - \log^+\|p\|$ is bounded, this measure carries no mass on singletons and its total mass equals $\deg(C)$.

When $(L, |\cdot|)$ is non-Archimedean, the analogues of the measures μ_f and $\mu_{f,C}$ have been constructed by Chambert-Loir [CL06, CL11].

Indeed, the function G induces a metrization $|\cdot|_G$ on the line bundle $\mathcal{O}(1)_{\mathbb{P}^2}$ by setting $|\sigma|_G := \exp(-G)$, where σ is the section corresponding to the constant function 1 on $\mathbb{A}^2$. Proposition 1.10 implies that the metrization $|\cdot|_G$ is a continuous semipositive metric in the sense of [CL11, §3.1].

The measure μ_f is defined as a probability measure on the Berkovich analytic space $\mathbb{A}^{2,\text{an}}_L$.

When the affine plane is replaced by an irreducible curve C, the measure $\mu_{f,C}$ is a positive measure on the analytification C^{an} of C in the sense of Berkovich. It can be defined using Thuillier's theory recalled in §1.1 as $\mu_{f,C} := \Delta G|_{\text{reg}(C)}$. Its mass is again equal to $\deg(C)$.

Remark 1.11. It is a priori not clear from the definition that μ_f is f-invariant. Over the complex numbers this invariance is obtained from the identity $\mu_f = dd^cG^+ \wedge dd^cG^-$ and the equations $f^*dd^cG^+ = (\deg(f))dd^cG^+$ and $f^*dd^cG^- = (\deg(f))^{-1}dd^cG^-$. Over a non-Archimedean field, an intersection theory of positive closed $(1, 1)$-currents has been proposed in [CLD12, Gub13] and it is likely that this theory provides the right tool to extend the above complex arguments.

When f is defined over a number field, one can proceed in a different (although less satisfactory) way. A theorem of C. G. Lee (see Theorem 1.18 below) asserts that μ_f describes the distribution of periodic points which implies it to be invariant under the dynamics. This setting covers our needs in this paper.

Since Green functions vary continuously with parameters, it is then possible to extend this result to the case where f is defined over the completion of the algebraic closure of any p-adic field.

The study of the ergodic properties of μ_f remains to be addressed.

1.4. Saddle fixed points

In this subsection we let f be any analytic germ fixing the point $0 \in \mathbb{A}^2_L$. The fixed point 0 is said to be a *saddle* when the eigenvalues u, s of $Df(0)$ satisfy $|u| > 1 > |s|$.

Given a small bidisk B around 0, we let $W^s_{\mathrm{loc}}(0)$ (resp. $W^u_{\mathrm{loc}}(0)$) be the set of points $p \in B$ such that for every $n \geq 0$, $f^n(p) \in B$ (resp. $f^{-n}(p) \in B$). It follows that if $p \in W^s_{\mathrm{loc}}(0)$ (resp. $p \in W^u_{\mathrm{loc}}(0)$) then $\lim_{n\to\infty} f^n(p) = 0$ (resp. $\lim_{n\to\infty} f^{-n}(p) = 0$). It is known that W^s_{loc} and W^u_{loc} are graphs of analytic functions in a neighborhood of 0 tangent to the respective eigendirections of df, hence intersect transversely. We refer to [HY83, Theorem A.1] for a proof that works over any metrized field. We refer to $W^s_{\mathrm{loc}}(0)$ (resp. $W^u_{\mathrm{loc}}(0)$) as the *local stable* (resp. *unstable*) *manifold* (or curve) of 0.

It is easy to see that one may always make a change of coordinates, such that $W^u_{\mathrm{loc}}(0) = \{y = 0\}$ and $W^s_{\mathrm{loc}}(0) = \{x = 0\}$. We will need the following more precise normal form.

Lemma 1.12. *There exist coordinates* (x, y) *near* 0 *in which* f *assumes the form*

$$f(x, y) = \big(ux(1 + xyg_1(x, y)), sy(1 + xyg_2(x, y))\big),$$

where g_1, g_2 *are analytic functions.*

Note that in this set of coordinates, f is linear along the stable and unstable manifolds.

Proof. By straightening the local stable and unstable manifolds, f can be written in the form

$$f(x, y) = \big(ux(1 + \text{h.o.t.}), sy(1 + \text{h.o.t.})\big) \quad \text{with} \quad |u| > 1 \text{ and } |s| < 1,$$

and we want to make this expression more precise. First, by the Schröder linearization theorem (which holds for arbitrary L [HY83]) we can make a local change of coordinates depending only on x in which $f|_{W^u_{\mathrm{loc}}(0)}$ becomes linear. Doing the same for y, we reach the form

$$f(x, y) = \big(ux(1 + yg^{(0)}(x, y)), sy(1 + xh^{(0)}(x, y)\big). \tag{1.2}$$

Let us now focus on the first coordinate in (1.2). We want to get rid of monomials of the form xy^j for $j > 0$. Reorder the expression of f so that it reads

$$f(x, y) = \big(uxa_1(y) + x^2a_2(y) + \cdots, syb_1(x) + y^2b_2(x) + \cdots\big),$$

where the a_j and b_j are analytic and $a_1(0) = b_1(0) = 1$. We want to find local coordinates in which $a_1(y) \equiv 1$. For this, set $(x', y') = (\varphi(y)x, y)$ with $\varphi(0) = 1$. Notice that in the coordinates (x', y'), f preserves the coordinate axes and remains linear along them, so f is still of the form (1.2). In the new coordinates, f can be expressed as

$$(x', y') \mapsto \left(ua_1(y') \frac{\varphi(sy')}{\varphi(y')} x' + O((x')^2), sy' + O(x') \right),$$

so to achieve $a_1(y') \frac{\varphi(sy')}{\varphi(y')} = 1$ it is enough to choose $\varphi(y') = \prod_{n=0}^{\infty} a_1(s^n y')$, which is well-defined for sufficiently small y', since $|s| < 1$ and $a_1(0) = 1$. Doing the same in the second variable, and renaming the coordinates as (x, y), we obtain

$$f(x, y) = \big(ux + x^2 a_2(y) + \cdots, sy + y^2 b_2(x) + \cdots\big).$$

Going back to the form (1.2), we get the desired result. □

1.5. Stable manifolds of polynomial automorphisms

In the case of polynomial automorphisms local stable manifolds can be globalized and have the following structure.

Proposition 1.13. *Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be any polynomial automorphism and assume that 0 is a saddle fixed point for f. Denote by s the eigenvalue of $Df(0)$ lying in the unit disk. Then the global stable manifold $W^s(0) := \{p \in L^2 : f^n(p) \to 0\}$ is an immersed affine line. More precisely, there exists an analytic injective immersion $\phi_s : \mathbb{A}^1 \to \mathbb{A}^2$ with image $W^s(0)$ such that $f \circ \phi_s(\zeta) = \phi_s(s\zeta)$ for every $\zeta \in \mathbb{A}^1$.*

Proof. We saw in §1.4 that for a sufficiently small neighborhood N of 0 the intersection $W^s_{\mathrm{loc}}(0) = W^s(0) \cap N$ is parameterized by an analytic immersion $\phi_s : B(0, 1) \to N$ which satisfies

$$f \circ \phi_s(\zeta) = \phi_s(s\zeta). \tag{1.3}$$

Since f is an automorphism, it follows that $W^s(0) = \bigcup_{n \geq 0} f^{-n}(W^s_{\mathrm{loc}}(0))$, and using the functional equation we may extend the analytic immersion ϕ_s by setting

$$\phi_s(\zeta) = f^{-n} \phi_s(s^n \zeta)$$

for every $\zeta \in L$ and sufficiently large n. □

Proposition 1.14. *Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be a polynomial automorphism of Hénon type of $\mathbb{A}^2$ and assume that 0 is a saddle fixed point for f. Then the restriction of G^- to $W^s(0)$ is not identically 0. In particular $G^-|_{W^s(0)}$ cannot vanish identically in a neighborhood of the origin. The same results hold for $G^+|_{W^u(0)}$.*

Proof. Suppose for contradiction that $G^- \equiv 0$ on $W^s(0)$. By Proposition 1.8(3), we have $W^s(0) \subset K^-$. Since the successive images by f of any point in $W^s(0)$ converge to 0, it follows that $W^s(0) \subset K^+$, hence $W^s(0) \subset K$.

We conclude that the image of the analytic map $\phi_s : \mathbb{A}^1 \to \mathbb{A}^2$ lies in a bounded domain, hence it is a constant by Liouville's theorem (see [Rob00] for the non-Archimedean case). This contradicts the fact that ϕ_s is an immersion.

The second statement follows from the invariance relation (1.3). □

1.6. Heights associated to adelic metrics on the affine space

We now assume that $\mathbb{L}$ is a number field, and we fix an algebraic closure $\mathbb{L}^{\mathrm{alg}}$ of $\mathbb{L}$. Let $\mathcal{M}_{\mathbb{L}}$ be the set of *places* of $\mathbb{L}$, that is, non-trivial multiplicative norms on $\mathbb{L}$ modulo equivalence. For each place $v \in \mathcal{M}_{\mathbb{L}}$ we denote by $|\cdot|_v$ the unique representative that is normalized in such a way that its restriction to $\mathbb{Q}$ is either the standard Archimedean norm or the p-adic norm satisfying $|p| = p^{-1}$ for some prime $p > 1$.

Then for any $x \in \mathbb{L}$, the *product formula* $\prod_{v\in\mathcal{M}_{\mathbb{L}}} |x|_v^{n_v} = 1$ holds. Here the integer n_v is the degree of the field extension of the completion of $\mathbb{L}$ over the completion of $\mathbb{Q}$ relative to $|\cdot|_v$.

Fix an integer $d \geq 1$. We let $\mathbb{C}_v$ be the completion of the algebraic closure of the completion of $\mathbb{L}$ relative to the absolute value $|\cdot|_v$. We continue to denote by $|\cdot|_v$ the unique extension to $\mathbb{C}_v$ of the absolute value on $\mathbb{L}$. For any $p = (x_1, \ldots, x_d) \in (\mathbb{C}_v)^d$ we shall write $\|p\|_v = \max\{|x_1|_v, \ldots, |x_d|_v\}$ (or simply $\|p\|$ when there is no risk of confusion).

Recall that the standard height of a point $p \in \mathbb{A}^d(\mathbb{L}^{\mathrm{alg}})$ is defined by the formula

$$h(p) = \frac{1}{\deg(p)} \sum_{v\in\mathcal{M}_{\mathbb{L}}} \sum_{q\in O(p)} n_v \log^+ \|q\|_v$$

where $O(p)$ denotes the orbit of p under the absolute Galois group of $\mathbb{L}$, and $\deg(p)$ is the cardinality of $O(p)$.

As in [CL11] we use heights that are associated to semipositive adelic metrics on ample line bundles. We present here this notion in a form that is tailored to our needs.

Suppose $X \subset \mathbb{A}^d_{\mathbb{L}}$ is an irreducible affine variety, and fix an embedding $\mathbb{A}^d_{\mathbb{L}} \subset \mathbb{P}^d_{\mathbb{L}}$. For us, X will always be either a curve or $\mathbb{A}^2_{\mathbb{L}}$.

A *semipositive adelic metric* on the (ample) line bundle $\mathcal{O}_X(1)$ is a collection $\{G_v\}_{v\in\mathcal{M}_{\mathbb{L}}}$ of functions $G_v : X(\mathbb{L}^{\mathrm{alg}}) \to \mathbb{R}$ such that

(M1) the function $G_v(p) - \log^+ \|p\|_v$ extends continuously to the closure of X in $\mathbb{P}^d$ for each place v;
(M2) $G_v(p) = \log^+ \|p\|_v$ for all but finitely many v;
(M3) G_v is plurisubharmonic for each Archimedean v;
(M4) for each non-Archimedean, the function G_v is a uniform limit of positive multiples of functions of the form $\log\max\{|P_1|_v, \ldots, |P_r|_v\}$ with $P_i \in \mathbb{L}[x_1, \ldots, x_d]$.

To any such semipositive adelic metric $\{G_v\}_{v\in\mathcal{M}_{\mathbb{L}}}$ is associated a height defined on $X(\mathbb{L}^{\mathrm{alg}})$ by

$$h_G(p) = \frac{1}{\deg(p)} \sum_{v\in\mathcal{M}_{\mathbb{L}}} \sum_{q\in O(p)} n_v G_v(q),$$

and such that $\sup_{X(\mathbb{L}^{\mathrm{alg}})} |h_G - h| < \infty$.

For any place v, one can also associate to the metrization G_v a positive measure $\mathrm{MA}(G_v)$ which is defined as in §1.3.

When v is Archimedean, $\mathrm{MA}(G_v)$ is the Monge–Ampère measure mass of $\deg(X)$ defined using Bedford–Taylor's theory.

When v is non-Archimedean, the measure $\mathrm{MA}(G_v)$ is defined by Chambert-Loir [CL06] as a positive measure of mass $\deg(X)$ on the analytification of X over $\mathbb{C}_v$ in the sense of Berkovich. Its definition relies in an essential way on condition (M4) above.

When X is a curve, $\mathrm{MA}(G_v)$ is alternatively defined as the Laplacian of $G_v|_{\mathrm{reg}(C)}$ (in the sense of Thuillier when v is a non-Archimedean place).

1.7. Metrizations associated to polynomial automorphisms of Hénon type and equidistribution

Assume that f is a regular polynomial automorphism of degree ≥ 2 defined over a number field $\mathbb{L}$.

The following two results are direct consequences of the definitions and Propositions 1.8 and 1.10.

Proposition 1.15. *For any regular polynomial automorphism f of degree ≥ 2 the collection $\{G_{v,f}\}$ defines a semipositive adelic metric on $\mathcal{O}(1)_{\mathbb{P}^2}$.*

In what follows, we denote by h_f the height associated to this collection of metrics.

Proposition 1.16. *Let f be a regular polynomial automorphism of degree $d \geq 2$. As above, denote by p_+ the fixed point at infinity of the rational extension of f onto $\mathbb{P}^2$. Then for any irreducible algebraic curve C whose Zariski closure $\bar{C}$ in $\mathbb{P}^2$ intersects the line at infinity at the single point p_+, the collection $\{G^+_{v,f}|_C\}$ defines a semipositive adelic metric on $\mathcal{O}(1)_{\bar{C}}$.*

We will need two versions of the equidistribution theorem for points of small height: one for curves, and one for the affine plane $\mathbb{A}^2$. The first statement follows from a statement due to P. Autissier [Au01, Proposition 4.7.1] at an Archimedean place and to A. Thuillier [Th05, Théorème 4.3.6] when the curve is smooth, and to X. Yuan [Yu08, Theorem 3.1] in full generality.

Theorem 1.17 (Equidistribution for points of small height on a curve). *Let f be a polynomial automorphism of Hénon type defined over $\mathbb{L}$. Suppose C is an irreducible curve of the affine space $\mathbb{A}^2$ that is defined over $\mathbb{L}$ whose Zariski closure in $\mathbb{P}^2$ intersects the line at infinity only at p_+. Suppose that we are given an infinite sequence of distinct points $p_m \in C(\mathbb{L}^{\mathrm{alg}})$ such that $h_{G^+}(p_m) \to 0$. Then, for any place $v \in \mathcal{M}_{\mathbb{L}}$, the convergence*

$$\frac{1}{\deg(p_m)} \sum_{q \in O(p_m)} \delta_q \to \frac{1}{\deg(C)} \Delta(G^+_v|_{\mathrm{reg}(C_v)}) \tag{1.4}$$

holds in the weak topology of measures, where $O(p_m)$ is the orbit of x_m under the action of the absolute Galois group of $\mathbb{L}$.

For simplicity, we write C_v for the analytification of C over the field $\mathbb{C}_v$.

Proof of Theorem 1.17. To keep the argument as short as possible we directly apply Yuan's result [Yu08, Theorem 3.1]. To do so one needs to check that the height of C induced by the metrization given by $\{G_v^+\}_v$ on $\mathcal{O}(1)_C$ is equal to zero. We refer to [CL11] for the definition of this quantity. Now for a curve it follows from e.g. [Zh95, Theorem 1.10] that this height is equal to

$$e = \operatorname*{ess\,inf}_C h_{G^+} := \sup_{\#F<\infty} \inf_{p\in C\setminus F} h_{G^+}(p).$$

Our assumption implies that $\inf_{p\in C\setminus F} h_{G^+}(p) \le \liminf_n h_{G^+}(p_n) = 0$. On the other hand, we have $G_v^+ \ge 0$ at all v, hence $h_{G^+}(p) \ge 0$ for every $p \in C$. Therefore $e = 0$ as required. □

The next result, still based on Yuan's theorem [Yu08], is due to C.-G. Lee [Le13, Theorem A].

Theorem 1.18 (Equidistribution theorem for periodic points of Hénon maps). *Let f be an automorphism of Hénon type defined over $\mathbb{L}$. Let $(p_m)_{m\ge0}$ be any sequence of distinct periodic points such that the set $\{p_m\} \cap C$ is finite for any irreducible curve $C \subset \mathbb{A}^2_{\mathbb{L}}$. Then, for any place $v \in \mathcal{M}_{\mathbb{L}}$, the convergence*

$$\frac{1}{\deg(p_m)} \sum_{q\in O(p_m)} \delta_q \to \mathrm{MA}(G_v) \tag{1.5}$$

holds in the weak topology of measures, where $O(x_m)$ is the orbit of x_m under the action of the absolute Galois group of $\mathbb{L}$.

Write $\mu_{f,v} := \mathrm{MA}(G_v)$ and $K_{f,v} := \{x \in \mathbb{A}^{2,\mathrm{an}}_{\mathbb{L}_v^{\mathrm{alg}}} : \|f^n(x)\| = O(1)\}$. The previous result immediately implies

Corollary 1.19. *Let f be an automorphism of Hénon type defined over a number field $\mathbb{L}$, and let v be any place on $\mathbb{L}$. Then the probability measure $\mu_{f,v}$ is f-invariant and is supported on $K_{f,v}$.*

We expect this result to be valid without any assumption on the field of definition (see Remark 1.11).

2. Applying the equidistribution theorem

A key step of the proofs of Theorems A and B is the use of equidistribution theorems for points of small height. In doing so we follow the approach to the Manin–Mumford conjecture initiated by Szpiro–Ullmo–Zhang [SUZ97]. In our setting this results in the following proposition.

Proposition 2.1. *Let f be a regular polynomial automorphism, and C an irreducible algebraic curve in the affine plane, both defined over a number field $\mathbb{L}$. Suppose there exists a sequence of distinct points $p_n \in C(\mathbb{L}^{\mathrm{alg}})$ such that $h_f(p_n) \to 0$. Then for any place $v \in \mathcal{M}_{\mathbb{L}}$ there exist a positive rational number $\alpha := \alpha(C, f)$ and a continuous function $H_v : C_v \to \mathbb{R}$ such that*

$$G^+_{v,f} = \alpha\, G^-_{v,f} + H_v \quad \text{on } C_v$$

and $H_v|_{\mathrm{reg}(C_v)}$ is harmonic.

Let us stress that the constant α does not depend on the chosen place v, but only on the curve C and the automorphism f. When v is non-Archimedean, the harmonicity of H_v is understood in the sense of Thuillier (see §1.1).

Remark 2.2. When C has a single place at infinity,[6] it may be shown that H_v must be a constant, hence necessarily zero since it vanishes at any periodic point of f. Taking $H_v \equiv 0$ somewhat simplifies the proof of Theorem A. However, it seems delicate to prove the vanishing of H_v in the general case of a curve with several places at infinity.

Proof of Proposition 2.1. Let p_n be a sequence of distinct points in $C(\mathbb{L}^{\mathrm{alg}})$ with $h_f(p_n) \to 0$, and fix any place $v \in \mathcal{M}_{\mathbb{L}}$. To simplify notation we denote by $[F_n]$ the normalized equidistributed integration measure on the Galois orbit of p_n. It is a probability measure supported on the analytification C_v of C over $\mathbb{C}_v$ in the sense of Berkovich.

By [BS91, Proposition 4.2] together with Proposition 1.9, there exists an integer $k \geq 0$ such that $f^k(C)$ intersects the line at infinity in $\mathbb{P}^2_{\mathbb{L}}$ only at the superattracting point p_+.

By Proposition 1.16, the metrization given by $\{G^+_{v,f}\}_v$ is semipositive adelic. Let h_{G^+} be the associated height. Since

$$0 \leq G^+_{f,v} \leq G_{f,v} \leq d^k G_{f,v} \circ f^{-k}$$

at all places, it follows that

$$h_{G^+}(f^k(p_n)) \leq h_f(f^k(p_n)) \leq d^k h_f(p_n) \to 0,$$

whence $h_{G^+}(p_n) \to 0$. Theorem 1.17 thus applies and we see that the sequence of probability measures $f^k_*[F_n]$ converges to the unique probability measure μ_k that is proportional to $\Delta(G^+_v|_{f^k(C_v)})$, that is,

$$\mu_k = \frac{1}{\deg(f^k(C))}\Delta(G^+_{f,v}|_{f^k(\mathrm{reg}(C_v))}).$$

Pulling back the convergence $f^k_*[F_n] \to \mu_k$ by the automorphism f^k, we get

$$\begin{aligned}\lim_n [F_n] &= \frac{1}{\deg(f^k(C))}(f^k)^*\Delta(G^+_{f,v}|_{f^k(\mathrm{reg}(C_v))})\\ &= \frac{1}{\deg(f^k(C))}\Delta(G^+_{f,v} \circ f^k|_{\mathrm{reg}(C_v)}) = \frac{d^k}{\deg(f^k(C))}\Delta(G^+_{f,v}|_{\mathrm{reg}(C_v)}).\end{aligned}$$

[6] That is, C intersects the line at infinity in $\mathbb{P}^2$ in a single point and is analytically irreducible there.

Proceeding in the same way for f^{-1} we deduce that there exist non-negative integers k, k' such that

$$\frac{d^k}{\deg(f^k(C))}\Delta(G^+_{f,v}|_{\mathrm{reg}(C_v)}) = \frac{d^{k'}}{\deg(f^{-k'}(C))}\Delta(G^-_{f,v}|_{\mathrm{reg}(C_v)}).$$

Therefore, there exists a positive rational number α_C depending only on C and f such that the restriction of $G^+_{f,v} - \alpha_C G^-_{f,v}$ to $\mathrm{reg}(C_v)$ is harmonic. □

For completeness, let us mention the following partial converse to Proposition 2.1.

Proposition 2.3. *Suppose that there exists a positive constant $\alpha > 0$, such that $G^+_{v,f} = \alpha G^-_{v,f}$ on C for each place v. Then there exists a sequence of points $p_n \in C(\mathbb{L}^{\mathrm{alg}})$ such that $h_f(p_n) \to 0$.*

Remark 2.4. If for every place v there exists a constant α_v such that $G^+_{v,f}|_C = \alpha_v G^-_{v,f}|_C$ then α_v does not depend on v. This follows from the fact that the mass of $\Delta G^\pm_v|_{\mathrm{reg}(C_v)}$ only depends on the geometry of the branches of C at infinity.

Proof of Proposition 2.3. Replacing C by $f^n(C)$ for n large enough, one may suppose that $\alpha > 1$ and that the completion of C intersects the line at infinity only at p_+. We claim that the height of C is zero. Indeed,

$$h_f(C) = \sum_v \int_{\mathrm{reg}(C_v)} G_v \Delta G_v|_{\mathrm{reg}(C_v)} + h_f(p_+) = \sum_v \int_{\mathrm{reg}(C_v)} G^+_v \Delta G^+_v|_{\mathrm{reg}(C_v)} = 0,$$

since $G^+_v \equiv 0$ on $\mathrm{supp}(\Delta G^+_v|_{\mathrm{reg}(C_v)})$. The fact that $h_f(p_+) = 0$ can be obtained from [Le13, Theorem 6.5] which asserts that

$$h_f(p) = \lim_n \frac{1}{\deg(f)^n} h_{\mathrm{naive}}(\phi_n(p))$$

where $\phi_n : \mathbb{P}^2 \to \mathbb{P}^4$ is the regular map whose restriction to $\mathbb{A}^2$ is defined by $\phi_n(p) = (f^n(p), f^{-n}(p))$ and h_{naive} is the naive height on $\mathbb{P}^4$ [HS00, §B.2]. An easy computation shows that $\phi_n(p_+)$ is independent of n, so the result follows.

We then conclude by applying the arithmetic Hilbert–Samuel theorem [Zh95, Theorem 1.10]. □

3. The DMM statement in the Archimedean dissipative case

Throughout this section we assume that f is a regular polynomial automorphism of $\mathbb{A}^2$ defined over a number field $\mathbb{L}$. We use the notation and results from Section 1. Our purpose is to establish Theorem A′. This in turn clearly implies Theorem A in the case where f and C are defined over a number field (the general case will be treated in §5).

The proof will be based on Pesin's theory of non-uniformly hyperbolic dynamical systems. We refer to [BLS93a] for a presentation adapted to our situation.

3.1. *Proof of Theorem A′*

Recall that h_f denotes the height associated to the semipositive adelic metric $\{G_{f,v}\}$. We suppose that there exists an irreducible curve C defined over $\mathbb{L}$ and a sequence of points $p_n \in C(\mathbb{L}^{\mathrm{alg}})$ with $h_f(p_n) \to 0$.

We want to prove that $|\mathrm{Jac}(f)|_v = 1$. To simplify notation we assume that $\mathbb{L} \subset \mathbb{C}$, drop the reference to v and work directly over $\mathbb{C}$ endowed with its standard absolute value.

Under our assumptions, we know from Proposition 2.1 that there exists a positive constant α such that $(G^+ - \alpha G^-)|_{\mathrm{reg}(C)}$ is harmonic, which implies that the positive measures $\mu_C^\pm := dd^c(G^\pm|_C)$ are proportional. Recall that since $G^\pm$ are continuous, these measures can simply be defined by taking the restriction to $\mathrm{reg}(C)$ and extending by zero at the singular points.

Denote by χ^u and χ^s the Lyapunov exponents of f relative to the measure $\mu = (dd^c)^2 G_f$. Recall that they are defined by

$$\chi^u = \lim_{n\to\infty} \int \log\|Df_p^n\|\, d\mu(p) \quad \text{and} \quad \chi^s = \lim_{n\to\infty} \int \log\|Df_p^{-n}\|\, d\mu(p).$$

We also write $\lambda^u = \exp(\chi^u)$ and $\lambda^s = \exp(-\chi^s)$. It is known that $\lambda^{u/s} \geq d > 1$, and $\lambda^u\lambda^s = |\mathrm{Jac}(f)|$ [BS92]. The main step of the proof of Theorem A′ is the following proposition, which computes the lower Hölder exponent of continuity of G^+ at a μ_C^+-generic point (observe that for such a point one has $G^+(p) = 0$).

Proposition 3.1. *For μ_C^+-almost every point p in C, one has*

$$\liminf_{r\to 0} \frac{1}{\log r} \log\Big[\sup_{d(p,q)\leq r,\, q\in C} G^+(q)\Big] = \vartheta_+,$$

where ϑ_+ is the unique positive real number satisfying $(\lambda^u)^{\vartheta_+} = d$.

Replacing f by f^{-1}, we see that a similar result holds for G^- at μ_C^--a.e. point, with λ^u replaced by $(\lambda^s)^{-1}$ and ϑ_+ by ϑ_- such that $(\lambda^s)^{-\vartheta_-} = d$. Observe that $\lambda^{u/s} \geq d$ implies that $\vartheta_\pm \in (0, 1]$.

If $\vartheta_+ = \vartheta_- = 1$ then $|\mathrm{Jac}(f)| = \lambda^u\lambda^s = d \cdot d^{-1} = 1$ and we are done. Thus we may assume that $\vartheta_- < 1$. Recall that $G^+ = \alpha G^- + H$ with $\alpha > 0$ and H a harmonic function on C. For a μ_C^+-generic point p, the preceding proposition yields

$$\begin{aligned}
\vartheta_+ &= \liminf_{r\to 0} \frac{1}{\log r} \log\Big[\sup_{d(p,q)\leq r,\, q\in C} G^+(q)\Big] \\
&= \liminf_{r\to 0} \frac{1}{\log r} \log\Big[\sup_{d(p,q)\leq r,\, q\in C} (\alpha G^- + H)(q)\Big].
\end{aligned}$$

Now observe that for any sequence $r_n \to 0$ and any sequence of points p_n at distance r_n from p we have

$$\liminf_n \frac{\log G^+(p_n)}{\log r_n} \geq \liminf_n \frac{1}{\log r_n} \log\Big[\sup_{d(p,q)\leq r_n,\, q\in C} G^+(q)\Big] \geq \vartheta_+.$$

Now pick p_n with $d(p_n, p) = r_n \to 0$ such that $\log G^-(p_n)/\log r_n \to \vartheta_- < 1$. This is possible because since μ_C^+ is proportional to μ_C^-, the corresponding notions of generic points coincide. Since H is smooth, if ε is so small that $(1+\varepsilon)\vartheta_- < 1$, we infer that

$$\alpha G^-(p_n) + H(p_n) \geq \alpha r_n^{(1+\varepsilon)\vartheta_-} + O(r_n) \geq \text{const} \cdot r_n^{(1+\varepsilon)\vartheta_-}$$

for large n, so that $1 > (1+\varepsilon)\vartheta_- \geq \vartheta_+$. Since ε was arbitrary, we conclude that $\vartheta_- \geq \vartheta_+$.

Applying the same argument with the roles of ϑ_+ and ϑ_- reversed, we conclude that $\vartheta_+ = \vartheta_-$, which implies that $\lambda^u = (\lambda^s)^{-1}$ and finally $|\mathrm{Jac}(f)| = 1$, as desired. □

Remark 3.2. It would perhaps be dynamically more significant to prove that the Hausdorff dimension of the measure $\Delta(G^+|_C)$ at a generic point is equal to the Hausdorff dimension of the measures induced by T^+ along the unstable lamination. The value of this dimension is precisely equal to the above constant ϑ_+, in virtue of Lai-Sang Young's formula [Yo82].

3.2. *Proof of Proposition 3.1*

The proposition relies on the interplay between Pesin's theory and the laminarity properties of the currents $T^\pm$. This is very close in spirit to the main results of [BLS93a].

Since its Lyapunov exponents are both non-zero, the measure μ_f is hyperbolic. Pesin's theory then asserts the existence of a family of Lyapunov charts, in which f expands (resp. contracts) in the horizontal (resp. vertical) direction. The precise statement is as follows (see [BP06, Theorem 8.14]). Let $\mathbb{B}(r) = \{(x, y) : \max\{|x|, |y|\} < r\}$ be the polydisk of radius r in $\mathbb{C}^2$. Then for any given $\varepsilon > 0$, there exists an f-invariant set E (the set of *regular points*) of full μ_f-measure, a measurable function $\rho : E \to (0, 1)$ and a family of charts $\varphi_p : \mathbb{B}(\rho(p)) \to \mathbb{C}^2$ defined for $p \in E$ and satisfying

(i) $\varphi_p(0) = p$ and $e^{-\varepsilon} < \rho(f(p))/\rho(p) < e^{\varepsilon}$;

(ii) if $f_p := \varphi_{f(p)}^{-1} \circ f \circ \varphi_p$, then

$$f_p(x, y) = \big(a^u(p)x + xh_1(x, y), a^s(p)y + yh_2(x, y)\big) \tag{3.1}$$

where $\lambda^u - \varepsilon \leq |a^u(p)| \leq \lambda^u + \varepsilon$, $\lambda^s - \varepsilon \leq |a^s(p)| \leq \lambda^s + \varepsilon$, and $\sup\{\|h_1\|, \|h_2\|\} < \varepsilon$;

(iii) there exist a constant $B > 0$ and a measurable function $A : E \to (0, \infty)$ such that

$$B^{-1}\|\varphi_p(q) - \varphi_p(q')\| \leq \|q - q'\| \leq A(p)\|\varphi_p(q) - \varphi_p(q')\| \tag{3.2}$$

with $e^{-\varepsilon} < A(f(p))/A(p) < e^{\varepsilon}$.

We denote by $W^u_{\mathrm{loc}}(p)$ (resp. $W^s_{\mathrm{loc}}(p)$) the image by φ_p of $\{x = 0\}$ (resp. $\{y = 0\}$). These will be referred to as the *local unstable* (resp. *stable*) *manifold at* p. Notice that (3.1) is slightly different from the corresponding statement in [BP06] as we have straightened the local stable and unstable manifolds. Observe also that on removing a set of measure 0 we may assume that $f^{-1}(W^u_{\mathrm{loc}}(p)) \subset W^u_{\mathrm{loc}}(f^{-1}(p))$ and $f(W^s_{\mathrm{loc}}(p)) \subset W^s_{\mathrm{loc}}(f(p))$.

For every point $p \in E$, we define the *global stable* and *unstable manifolds* by $W^s(p) = \bigcup_{n\geq 0} f^{-n} W^s_{\mathrm{loc}}(f^n(p))$ and $W^u(p) = \bigcup_{n\geq 0} f^n W^u_{\mathrm{loc}}(f^{-n}(p))$. These are embedded images of $\mathbb{C}$ respectively lying in K^+ and K^- like the stable and unstable manifolds of saddle points, as we saw in §1.4.

The next result follows from [BLS93a, Lemma 8.6], and will be proved afterwards. Recall that $\mu_C^+ := T^+ \wedge [C]$.

Lemma 3.3. *Let E denote as above the set of Pesin regular points for μ_f. Then for every subset $A \subset E$ of full μ_f-measure there exists $\bar{A} \subset C$ of full μ_C^+-measure such that if $\bar{p} \in \bar{A}$ then there exists $p \in A$ such that*

- *$\bar{p} \in W^s(p)$,*
- *$W^s(p)$ intersects C transversely at $\bar{p}$.*

With notation as in the lemma, pick any $\bar{p} \in \bar{E}$ and introduce the function

$$\theta_{\bar{p}}(r) = \sup\{G^+(q) : q \in C,\ d(\bar{p}, q) \leq r\}.$$

To prove the proposition we need to show that μ_C^+-a.s.,

$$\liminf_{r\to 0} \frac{\log(\theta_{\bar{p}}(r))}{\log r} = \vartheta_+ = \frac{\log d}{\log \lambda^u}.$$

Using Lemma 3.3, let $p \in E$ be such that $W^s(p)$ intersects C transversely at $\bar{p}$. Then there exists an integer N such that $f^N(\bar{p})$ lies in $W^s_{\mathrm{loc}}(f^N(p))$. By the invariance relation for G^+ and the differentiability of f, replacing C by $f^N(C)$ if needed, it is no loss of generality to assume that $N = 0$.

Choose an integer n, and pick a point $q \in B(\rho(p))$ such that $f^k(q) \in \varphi_{f^k(p)} B(\rho(f^k(p)))$ for all $0 \leq k \leq n$. Write $\varphi_p(q) = (x, y)$ so that $|x|, |y| \leq \rho(p)$, and let $(x_k, y_k) := \varphi^{-1}_{f^k(p)}(f^k(q))$. It then follows from (3.1) that $|y_n| \leq (\lambda^s + 2\varepsilon)^n |y_0|$ and

$$(\lambda^u - 2\varepsilon)^n |x_0| \leq |x_n| \leq (\lambda^u + 2\varepsilon)^n |x_0|. \tag{3.3}$$

Conversely, it follows a posteriori from these estimates that any point $q = \varphi_p(x, y)$ such that

$$|x| \leq \rho(p)(\lambda^u + 2\varepsilon)^{-n} e^{-\varepsilon n}$$

satisfies $|x_k| \leq |x|(\lambda^u + 2\varepsilon)^k \leq \rho(f^k(p))$, hence $f^k(x) \in \varphi_{f^k(p)} B(\rho(f^k(p)))$ for every $0 \leq k \leq n$.

We now estimate $\theta_{\bar{p}}(r)$ for a given small enough r. To estimate it from below, choose $q \in C$ such that $d(p, q) \leq r$ and $\theta_{\bar{p}}(r) = G^+(q)$. By (3.2) writing $\varphi_p(q) = (x, y)$, we infer that $|x| \leq Br$. To ease notation assume without loss of generality that $B = 1$. Choose n such that

$$\rho(p)(e^\varepsilon(\lambda^u + 2\varepsilon))^{-n} \leq r < \rho(p)(e^\varepsilon(\lambda^u + 2\varepsilon))^{-(n-1)}. \tag{3.4}$$

From (3.4) and the invariance relation for G^+ we get the upper bound

$$\log \theta_{\bar{p}}(r) = -\log(d^n) + \log G^+(f^n(q)) \leq -\log(d^n) + \max_{\varphi_p(B(\rho(p)))} G^+$$
$$\leq -n \log d + \text{const} \leq \frac{\log r - \log \rho(p)}{\varepsilon + \log(\lambda^u + 2\varepsilon)} \log d + \text{const},$$

where the first inequality on the second line follows from the fact that due to (3.2), $\bigcup_E \varphi_p(B(\rho(p))$ is bounded in $\mathbb{C}^2$. Letting $r \to 0$, we infer that

$$\liminf_{r\to 0} \frac{\log \theta_{\bar{p}}(r)}{\log r} \geq \frac{\log d}{\varepsilon + \log(\lambda^u + 2\varepsilon)}.$$

Since this holds for every ε, we conclude that

$$\liminf_{r\to 0} \frac{\log \theta_{\bar{p}}(r)}{\log r} \geq \frac{\log d}{\log \lambda^u}. \tag{3.5}$$

To prove the opposite inequality we proceed as follows. Let us introduce the auxiliary function $\psi : p \mapsto \sup_{W^u_{\text{loc}}(p)} G^+$. This is a measurable function that is uniformly bounded from above since $\bigcup_E \varphi_p(B(\rho(p))$ is bounded in $\mathbb{C}^2$. Likewise $\psi(p) > 0$ for every $p \in E$ since $W^u_{\text{loc}}(p)$ cannot be contained in K^+ [BLS93a, Lemma 2.8] (the argument is identical to that of Proposition 1.14).

Now fix a constant $g_0 > 0$ such that the set $\{\psi > g_0\}$ has positive μ_f-measure. By the Poincaré recurrence theorem, there exists a measurable set $A \subset E$ of full μ_f-measure such that for every $p \in A$, $\psi(f^{n_j}(p)) > g_0$ for infinitely many n_j's.

Let $\bar{A}$ be as in Lemma 3.3, and let $\bar{p} \in \bar{A}$ and p be as above. Stable manifold theory shows that there exists n_0 such that for $n \geq n_0$ the connected component of $\varphi^{-1}_{f^n(p)}(f^n(C))$ in $\mathbb{B}(\rho(f^n(p)))$ containing $\varphi^{-1}_{f^n(p)}(\bar{p})$ is a graph over the first coordinate which converges exponentially fast in the C^0 (hence C^1) topology to $\{y = 0\}$ [BP06, §8.2 and Theorem 8.13]; denote it by $C_{n, f^n(\bar{p})}$.

Since p belongs to A, we have $\psi(f^{n_j}(p)) > g_0$ for infinitely many n_j's. To ease notation we simply write n for n_j. Since G^+ is Hölder continuous, for such an iterate n we infer that

$$\sup\{G^+(w) : w \in C_{n, f^n(\bar{p})}\} \geq g_0 - \delta_n,$$

where δ_n is exponentially small. Let $w_n \in C_{n, f^n(\bar{p})}$ be a point at which $G^+(w_n) \geq g_0/2$. Consider now $f^{-n}(w_n)$ and denote $\varphi_p^{-1}(f^{-n}(w_n)) = (x_n, y_n)$. By (3.1), we have

$$|x_n| \leq \frac{\rho(f^n(p))}{(\lambda^u - 2\varepsilon)^n} \leq \frac{\text{const}}{(\lambda^u - 2\varepsilon)^n},$$

hence, since C is transverse to $W^s_{\text{loc}}(p)$, from (3.2) we get

$$d(f^{-n}(w_n), \bar{p}) \leq \frac{C_0}{(\lambda^u - 2\varepsilon)^n},$$

where C_0 does not depend on n. Therefore, setting $r_n = C_0/(\lambda^u - 2\varepsilon)^n$ and using the invariance relation for G^+ and the definition of w_n, we infer that

$$\theta(\bar{p}, r_n) \geq \frac{g_0}{2d^n} = \frac{g_0}{2}\left(\frac{r_n}{C_0}\right)^{\frac{\log d}{\log(\lambda^u - 2\varepsilon)}}.$$

Finally,

$$\limsup_{n\to\infty} \frac{\log\theta(\bar{p}, r_n)}{\log r_n} \leq \frac{\log d}{\log(\lambda^u - 2\varepsilon)},$$

thus

$$\liminf_{r\to 0} \frac{\log\theta(\bar{p}, r)}{\log r} \leq \frac{\log d}{\log \lambda^u}$$

which, along with (3.5), finishes the proof.

Proof of Lemma 3.3. The proof relies on the theory of laminar currents [BLS93a]. It is shown in [BLS93a, Theorem 7.4] that the positive closed (1, 1)-current $T^+ := dd^cG^+$ is laminar.

Recall that this means the following. First, we say that a current S in $\Omega \subset \mathbb{C}^2$ is *locally uniformly laminar* if every point in supp(S) admits a neighborhood B biholomorphic to a bidisk, such that in adapted coordinates, S can be locally written as $\int [\Delta_a]\, d\alpha(a)$, where the Δ_a are disjoint graphs over the first coordinate in B, and α is a positive measure on the space of such graphs. These disks will be said to be *subordinate* to S. Notice that a locally uniformly laminar current is always closed.

A current is *laminar* if for any $\varepsilon > 0$ there exists a finite family of disjoint open sets Ω^i, and for each i a locally uniformly laminar current $T^i \leq T$ such that the mass of $T - \sum_i T^i$ is smaller than ε. If R is any positive closed current in $\mathbb{C}^2$ such that the wedge product $T \wedge R$ is admissible, then slightly abusing notation we define the wedge product $(\sum_i T^i) \wedge R$ by $\sum_i (T^i \wedge R)|_{\Omega^i}$.

The geometric intersection product of a current of integration over a curve $[M]$ with a uniformly laminar current $T = \int [\Delta_a]\, d\alpha(a)$ is defined by

$$T \dot{\wedge} [M] = \int [\Delta_a \cap M]\, d\alpha(a)$$

where $[\Delta_a \cap M]$ is the atomic positive measure putting mass 1 at any intersection point of Δ_a and M. If T has continuous potential then $T \dot{\wedge} [M] = T \wedge [M]$ [BLS93a, Lemma 6.4].

Pick any open subset $\Omega \subset \mathbb{C}^2$, and let $0 < S^+ \leq T^+$ be any locally uniformly laminar current in Ω. Denote by $\mathbf{M}(\mu)$ the total mass of a given positive measure μ. We claim that

$$\lim_{n\to\infty} \mathbf{M}\left(\frac{1}{d^n}(f^n)^* S^+ \dot{\wedge} [C]\right) = \deg(C) \cdot \mathbf{M}(S^+ \wedge T^-). \tag{3.6}$$

To see this, we observe that the current S^+ has continuous potential by [BLS93a, Lemma 8.2] so that

$$\frac{1}{d^n}(f^n)^* S^+ \dot{\wedge} [C] = \frac{1}{d^n}(f^n)^* S^+ \wedge [C]$$

as positive measures in $f^{-n}(\Omega)$. Now $f^n : f^{-n}(\Omega) \to \Omega$ is an automorphism so that

$$\mathbf{M}\left(\frac{1}{d^n}(f^n)^* S^+ \dot{\wedge} [C]\right) = \mathbf{M}(S^+ \wedge d^{-n}(f^n)_*[C]).$$

Replacing C by some iterate we may assume that it intersects the line at infinity at p_+ only, hence $d^{-n}(f^n)_*[C] \to (\deg(C))T^-$ by [BS91, FS92]. Again since S^+ has continuous potential in Ω, the measures $S^+ \wedge d^{-n}(f^n)_*[C]$ converge to $\deg(C) S^+ \wedge T^-$. We conclude that $\mathbf{M}(d^{-n}(f^n)^* S^+ \dot{\wedge} [C])$ converges to $\deg(C)\mathbf{M}(S^+ \wedge T^-)$ as $n \to \infty$ as required.

Another result in [BLS93a] is that T^+ and T^- intersect geometrically (see also [Duj04]). This implies that for every $\varepsilon > 0$ there exists a current $T^+_\varepsilon \leq T^+$ which is a finite sum of uniformly laminar currents in disjoint open sets Ω^i as above, and such that $\mathbf{M}(T^+_\varepsilon \wedge T^-) \geq 1 - \varepsilon$. Then from (3.6) we deduce that for large n, the positive measures

$$\frac{1}{d^n}(f^n)^* T^+_\varepsilon \wedge [C] := \sum_i \frac{1}{d^n}(f^n)^* T^+_\varepsilon \wedge [C]|_{f^{-n}\Omega_i}$$

are dominated by μ^+_C, and

$$\mathbf{M}\left(\frac{1}{d^n}(f^n)^* T^+_\varepsilon \wedge [C]\right) \geq (1-\varepsilon)\deg(C) \sum_i \mathbf{M}(T_\varepsilon|_{\Omega_i} \wedge T^-)$$
$$\geq (1-\varepsilon)^2 \deg(C)\mathbf{M}(T^+ \wedge T^-) \geq (1-2\varepsilon)\deg(C).$$

Now recall that $T^+_\varepsilon|_{\Omega_i}$ has continuous potential for each i, hence does not charge any curve. By [BLS93a, Lemma 6.4] it follows that only transverse intersections between disks subordinate to $d^{-n}(f^n)^*(T^+_\varepsilon|_{\Omega_i})$ and $[C]$ need to be taken into account in the computation of the geometric intersection $d^{-n}(f^n)^*(T^+_\varepsilon|_{\Omega_i}) \dot{\wedge} [C]$. Further by [BLS93a, Corollary 8.8], almost every disk subordinate to $T^+_\varepsilon|_{\Omega_i}$ is an open subset of some stable curve $W^s(p)$ for some $p \in A$.

In particular, there exists a set B_n of total mass for the positive measure $d^{-n}(f^n)^* T^+_\varepsilon \wedge [C]$ such that for all points $q \in B_n \subset C$ there exists a point $p \in A$ such that $W^s(p)$ intersects C transversely at q. Since $\mu^+_C(B_n) \geq (1-2\varepsilon)\deg(C)$, the proof is complete. □

3.3. A uniform Theorem A′

In this section we indicate how our arguments can be modified so as to get the following statement.

Theorem A″. *Let f be a polynomial automorphism of Hénon type of the affine plane, defined over a number field $\mathbb{L}$. Assume that there exists an Archimedean place v such that $|\mathrm{Jac}(f)|_v \neq 1$. For any integer d, there exists a positive constant $\varepsilon(d) > 0$ and an integer $N(d) \geq 1$ such that for any algebraic curve C defined over $\mathbb{L}$ of degree at most d, the set $\{p \in C(\mathbb{L}^{\mathrm{alg}}) : h_f(p) \leq \varepsilon(d)\}$ contains at most $N(d)$ points.*

Proof. Suppose that there exists a sequence of curves C_m defined over $\mathbb{L}$ of degree d and finite sets $F_m \subset C_m$ invariant under the absolute Galois group of $\mathbb{L}$ such that $\#F_m \geq m$ and $h_f(F_m) \leq 1/m$. Let us show that $|\mathrm{Jac}(f)|_v = 1$.

Let μ_m be the probability measure equidistributed over F_m.

Lemma 3.4. *Any weak limit of the sequence (μ_m) is supported on a curve of degree d.*

Suppose first that any curve defined over $\mathbb{L}$ intersects only finitely many F_m's. Then Yuan's result [Yu08, Theorem 3.1] implies the equidistribution $\mu_n \to \mu_{f,v}$ (see [Le13, Theorem B]). However, $\mu_{f,v}$ gives no mass to curves so the previous lemma gives a contradiction.

We may thus suppose that there exists a curve D defined over $\mathbb{L}$ that contains infinitely many F_m's. Theorem A then applies to show that $|\mathrm{Jac}(f)|_v = 1$ as required. □

Proof of Lemma 3.4. For each m pick an equation $P_m = \sum_{i+j\leq d} a_{ij}^m x^i y^j$ of C_m such that $\max\{|a_{ij}^m|\} = 1$. Replacing F_m by a suitable subsequence we may assume that each coefficient a_{ij}^m converges to some a_{ij} in the completion of $\mathbb{L}$ with respect to the norm induced by v and we set $P = \sum_{i+j\leq d} a_{ij} x^i y^j$. Since the height of F_m is bounded from above, $\bigcup_m F_m$ is included in a fixed bounded set K in $(\mathbb{C}_v)^2$. We have $\sup_K |P_m - P| \to 0$, and this implies $\int |P|\, d\mu = \lim_m \int |P_m|\, d\mu_m = 0$. Therefore μ is supported on the curve $\{P = 0\}$. □

4. The DMM statement under a transversality assumption

This section is devoted to the proof of Theorem B in the number field case. Let f be a regular polynomial automorphism of $\mathbb{A}^2$, and C be an irreducible algebraic curve containing infinitely many periodic points, both defined over a number field $\mathbb{L}$. By the transversality assumption (T), replacing f by f^N if needed we assume that one of these periodic points $p \in \mathrm{Reg}(C)$ is fixed and satisfies $Df_p(T_pC) \neq T_pC$.

We want to show that $\mathrm{Jac}(f)$ is a root of unity. To do so it will be enough to prove that $|\mathrm{Jac}(f)|_v = 1$ for each place v.

If the place v is Archimedean, the equality $|\mathrm{Jac}(f)|_v = 1$ follows from Theorem A, so we will work at non-Archimedean places only.

Lemma 4.1. *Let f and C be as in Theorem B. Let p be any fixed point lying on C and denote by λ_1, λ_2 the two (possibly equal) eigenvalues of $Df(p)$. At any non-Archimedean place v, either $|\lambda_1|_v = |\lambda_2|_v = 1$ or p is a saddle.*

It is also not difficult to see that for all but finitely many places v, *all* periodic points are indifferent in the sense that their multipliers have norm 1.

Proof of Lemma 4.1. To reach a contradiction, without loss of generality we may assume that $|\lambda_1|_v \leq |\lambda_2|_v$, $|\lambda_1|_v < 1$ and $|\lambda_2|_v \leq 1$.

Since $|\lambda_1|_v \leq |\lambda_2|_v \leq 1$, it is classical that there exists a (bounded) neighborhood U of p that is forward invariant, in particular $U \subset K^+$. Indeed, performing a linear change of coordinates we can write $f(x, y) = (\lambda_1 x + \text{h.o.t.}, \lambda_2 y + \text{h.o.t.})$, and since v is non-Archimedean, it follows that if x and y are small enough, then $|\lambda_1 x + \text{h.o.t.}| = |\lambda_1|\,|x|$ and $|\lambda_2 y + \text{h.o.t.}| = |\lambda_2|\,|y|$.

Consequently, $G^+|_U \equiv 0$, hence from Proposition 2.1 we deduce that $G^-|_{U\cap C}$ is harmonic, and since $G^-(p) = 0$ and $G^- \geq 0$, by Proposition 1.2 we conclude that $G^-|_{U\cap C} \equiv 0$ as well. This implies that $f^{-n}(U \cap C) \subset K$ for all n, and since K is bounded, the Cauchy inequality implies that the norms $\|Df^{-n}(p)\|$ stay uniformly bounded in n along $U \cap C$.

If p is a sink, that is, $|\lambda_1|_v \leq |\lambda_2|_v < 1$, then $\|Df^{-n}(p)\|$ must grow exponentially and we readily get a contradiction. The semiattracting case $|\lambda_1|_v < |\lambda_2|_v = 1$ requires a few more arguments.

Assume first that λ_2 is not a root of unity. Then a theorem by Herman and Yoccoz [HY83] asserts that in this case λ_2 satisfies a Diophantine condition, hence so does the pair (λ_1, λ_2), and the fixed point p is analytically linearizable. Therefore there exist adapted coordinates (x, y) near p in which p is sent to the origin and f takes the form $f(x, y) = (\lambda_1 x, \lambda_2 y)$. Since $\|Df^{-n}(p)\|$ is uniformly bounded in n along $U \cap C$, in these coordinates C must be tangent to the y-axis, which contradicts our standing assumption (T).

If λ_2 is a root of unity, the argument is similar. Replace f by some iterate so that $\lambda_2 = 1$. To ease notation we work with f^{-1} instead of f. A theorem by Jenkins and Spallone [JS12, §4] asserts that there are coordinates (x, y) as above in which f^{-1} can be expressed as

$$f^{-1}(x, y) = (\lambda_1^{-1} x(1 + g(y)), h(y)) \quad \text{with} \quad g(0) = 0 \quad \text{and} \quad h(y) = y + \text{h.o.t.}$$

(recall that $|\lambda_1^{-1}| > 1$). From this we deduce that

$$f^{-n}(x, y) = (\lambda_1^{-n} x(1 + g_n(y)), h^n(y)) = \Big(\lambda_1^{-n} x \prod_{j=0}^{n-1} (1 + g(h^j(y))), h^n(y)\Big).$$

For ε small enough, if $|y|_v < \varepsilon$, using the ultrametric inequality we find that for every $0 \leq j \leq n$, $|h^j(y)|_v < \varepsilon$. It follows that $|1 + g(h^j(y))|_v = 1$.

Now as in the previous case, in the new coordinates the curve C must be tangent to the y-axis. Parameterize it as $t \mapsto (\psi(t), t)$, so that the curve $f^{-n}C$ is parameterized by

$$t \mapsto \Big(\lambda_1^{-n} \psi(t) \prod_{j=0}^{n-1} (1 + g(h^j(t))), h^n(t)\Big).$$

We see that the only possibility for it to be locally bounded as $n \to \infty$ is that $\psi \equiv 0$. As before we deduce that C is equal to the y-axis, which is invariant, and again we get a contradiction. □

Let us resume the proof of Theorem B. Pick any non-Archimedean place v, and recall that we wish to prove that $|\text{Jac}(f)|_v = 1$. To simplify notation we drop all indices referring to the place v. Let p be the fixed point satisfying (T), and denote by λ_i its eigenvalues.

Then $\mathrm{Jac}(f) = \lambda_1\lambda_2$. By the previous proposition either $|\lambda_1| = |\lambda_2| = 1$ and we are done, or p is a saddle. For notational consistency we denote by u (resp. s) the unstable (resp. stable) eigenvalue (which can be λ_1 or λ_2 depending on the place). By the transversality assumption (T), $W^u_{\mathrm{loc}}(p)$ and $W^s_{\mathrm{loc}}(p)$ are not tangent to C at p. Indeed since the tangent directions to $W^u_{\mathrm{loc}}(p)$ and $W^s_{\mathrm{loc}}(p)$ are given by the eigenvectors of $Df(p)$, this transversality does not depend on the place.

By Lemma 1.12 there are adapted coordinates (x, y) near p in which f takes the form

$$f(x, y) = \big(ux(1 + xyg_1(x, y)), sy(1 + xyg_2(x, y))\big). \tag{4.1}$$

By scaling the coordinates if necessary, we may assume that we work in the unit bidisk $\mathbb{B}$.

The following key renormalization lemma will be proven afterwards.

Lemma 4.2. *If $|x|, |y|$ are small enough, then for every $1 \le j \le n$, $f^j(x/u^n, y) \in \mathbb{B}$ and*

$$f^n(x/u^n, y) = (x, 0) + O(n\rho^n),$$

uniformly in (x, y), with $\rho := \max\{|u|^{-1}, |s|\} < 1$.

In the coordinates (x, y), we write C as a graph $y = \psi(x) = bx + \text{h.o.t.}$ over the first coordinate. Replacing y by by we can assume $b = 1$. Using Proposition 2.1 we set $\tilde{G}(x) = G^+(x, \psi(x)) = \alpha G^-(x, \psi(x)) + H(x, \psi(x))$. By Lemma 4.2, the continuity of G^+ and the invariance relation for G^+ we deduce that for small enough x,

$$d^n \tilde{G}(x/u^n) = G^+ \circ f^n(x/u^n, \psi(x/u^n)) \to G^+(x, 0). \tag{4.2}$$

Applying Lemma 4.2 to f^{-1}, for small y we have the following uniform convergence in a small disk:

$$f^{-n}(\psi^{-1}(s^n y), s^n y) \xrightarrow[n\to\infty]{} (0, y). \tag{4.3}$$

Now we claim that $|us| = 1$. Indeed, assume for contradiction that $|us| > 1$. Then setting $y_n = s^{-n}\psi(x/u^n)$, we get $y_n \to 0$ as $n \to \infty$. We write

$$d^n \tilde{G}(x/u^n) = \alpha G^- \circ f^{-n}(x/u^n, \psi(x/u^n)) + d^n H(x/u^n, \psi(x/u^n)), \tag{4.4}$$

and applying (4.3) we see that

$$G^- \circ f^{-n}(x/u^n, \psi(x/u^n)) = G^- \circ f^{-n}(\psi^{-1}(s^n y_n), s^n y_n) \xrightarrow[n\to\infty]{} G^-(0, 0) = 0.$$

Thus from (4.2) and (4.4) we infer that

$$d^n H(x/u^n, \psi(x/u^n)) \xrightarrow[n\to\infty]{} G^+(x, 0),$$

locally uniformly in the neighborhood of the origin. Since a limit of harmonic functions is harmonic (see Proposition 1.1), we conclude that G^+ is harmonic, hence identically zero on $W^u_{\mathrm{loc}}(p)$, thereby contradicting Proposition 1.14. This contradiction shows that at the place v we have $|us|_v = |\mathrm{Jac}(f)|_v = 1$, and completes the proof of Theorem B. □

Proof of Lemma 4.2. Recall that we work in the unit bidisk $\mathbb{B}$. Scaling the coordinates further, we may assume that the functions g_1, g_2 appearing in (4.1) are as small as we wish, say $\sup_{\mathbb{B}}\{|g_1|, |g_2|\} \leq \varepsilon$, where ε is a small positive constant whose value will be determined shortly.

Assume $(x_0, y_0) \in \mathbb{B}$ and denote by $(x_1, y_1) = f(x_0, y_0), \ldots, (x_k, y_k) = f^k(x_0, y_0)$ its sucessive iterates (whenever defined). Using (4.1) recursively, we obtain

$$x_k = u^k x_0 \prod_{j=0}^{k-1}(1 + x_j y_j g_1(x_j, y_j)) \quad \text{and} \quad y_k = s^k y_0 \prod_{j=0}^{k-1}(1 + x_j y_j g_2(x_j, y_j)).$$

We claim that if $|x_0| \leq B|u|^{-n}$ for a suitable constant B and $|y_0| \leq 1$ then the first n iterates of (x_0, y_0) are well-defined.

Indeed, assume by induction that the first $k-1$ iterates of (x_0, y_0) stay in $\mathbb{B}$ for some $k \leq n-1$. Then

$$|y_k| \leq |s|^k \prod_{j=0}^{k-1}(1 + |y_j|\varepsilon).$$

This will in turn be bounded by $A|s|^k$ if A is any constant satisfying $A \geq \prod_{j\geq 0}(1+A|s|^j\varepsilon)$. We leave it to the reader to check that if $\varepsilon < (1-|s|)/10$, then $A = 3$ will do. In what follows we work under this assumption.

Now assume that $|x_0| \leq \frac{1}{4}|u|^{-n}$ and let us show by induction that for small enough ε, $|x_j| \leq |u|^{j-n}$ for $0 \leq j \leq n$ (so that in particular $(x_j, y_j) \in \mathbb{B}$). Indeed, if this estimate holds for $0 \leq j \leq k-1$, then using the formula for x_k, we get

$$\begin{aligned}|x_k| &\leq |u|^k|x_0| \prod_{j=0}^{k-1}(1 + |x_j|3|s|^j\varepsilon) \leq \frac{1}{4}|u|^{k-n} \prod_{j=0}^{k-1}\left(1 + 3\varepsilon\frac{|us|^j}{|u|^n}\right) \\ &\leq \frac{1}{4}|u|^{k-n}\left(1 + \exp\left(\frac{3\varepsilon}{|u|^n}\sum_{j=0}^{n}|us|^j\right)\right) \\ &\leq \frac{1}{4}|u|^{k-n}\left(1 + \exp\left(3\varepsilon \sup_{n\geq 0}\max\left(\frac{|s|^n}{|us|-1}, \frac{n}{|u|^n}\right)\right)\right).\end{aligned}$$

Hence, on choosing ε sufficiently small (depending only on u and s), the exponential term is smaller than 2, and we are done.

To get the conclusion of the lemma, we simply reconsider the previous computation for $k = n$, and use the inequality

$$\left|\prod(1 + z_j) - 1\right| \leq \exp\left(\sum |z_j|\right) - 1$$

to obtain

$$\left|\frac{x_n}{u^n x_0} - 1\right| = \left|\prod_{j=0}^{n-1}\big(1 + x_j y_j g_1(x_j, y_j)\big) - 1\right| = O(\max(|s|^n, n/|u|^n)),$$

and we are done. ∎

In the next theorem, we give a direct argument for Theorem B under a more restrictive assumption which is reasonable from the dynamical point of view. We feel it is interesting to include it as it gives in this case a purely Archimedean proof of our main result. Observe that no transversality assumption is required.

Theorem 4.3. *Let f be a polynomial automorphism of the affine plane of Hénon type that is defined over a number field $\mathbb{L}$. Assume that there exists an algebraic curve C defined over $\mathbb{L}$ and containing infinitely many periodic points. Suppose there exists a periodic point $p \in C$ that is a saddle at some Archimedean place. Then* $\mathrm{Jac}(f)$ *is a root of unity.*

It follows from [BLS93b] that at the Archimedean place most periodic orbits of f are saddles, which makes the assumptions of the proposition natural. Still, there exist examples of polynomial automorphisms of $\mathbb{C}^2$ with infinitely many non-saddle periodic orbits, even in a conservative setting [Dua08].

Proof of Theorem 4.3. We do an analysis similar to that of the proof of Theorem B, starting from equation (4.2), and keeping the same notation. For simplicity we write $\mathbb{L}_v = \mathbb{C}$, and drop the v. By assumption there is a saddle point $p \in C$, with multipliers u and s. From Theorem A we know that $|us| = 1$. We assume that p is fixed and work in the local adapted coordinates (x, y) given by Lemma 1.12. Since we make no smoothness or transversality assumption here, we pick any local irreducible component of C at p, and parameterize it by $\Psi : t \mapsto (t^k, \psi(t))$ with $\psi(t) = t^l + \text{h.o.t.}$ By Proposition 2.1, for small $t \in \mathbb{C}$ we have

$$\tilde{G}(t) := G^+ \circ \Psi(t) = G^+(t^k, \psi(t)) = \alpha G^-(t^k, \psi(t)) + H(t^k, \psi(t)).$$

Swapping the stable and unstable directions if needed we may assume that $k \leq l$. Pick a kth root of u, denoted by $u^{1/k}$.

Applying the same reasoning as in (4.2) we get

$$d^n \tilde{G}(t/u^{n/k}) = G^+ \circ f^n(t^k/u^n, \psi(t/u^{n/k})) \to G^+(t^k, 0). \tag{4.5}$$

Since $k \leq l$ and $|us| = 1$, we see that $|s^k u^l| \geq 1$, from which we infer that $\psi(t/u^{n/k}) = O(s^n)$. Therefore we can do the same with f^{-n} to deduce that

$$\begin{aligned} d^n \tilde{G}(t/u^{n/k}) &= \alpha G^- \circ f^{-n}(t^k/u^n, \psi(t/u^{n/k})) + d^n H(t^k/u^n, \psi(t/u^{n/k})) \\ &= \alpha G^-(0, s^{-n}\psi(t/u^{n/k})) + o(1) + d^n H(t^k/u^n, \psi(t/u^{n/k})) \\ &= \alpha G^-(0, t^l/(s^k u^l)^{n/k}) + o(1) + d^n H(t^k/u^n, \psi(t/u^{n/k})). \end{aligned} \tag{4.6}$$

Arguing exactly as in the proof of Theorem B, we see that this is contradictory unless $|s^k u^l| = 1$, that is, $k = l$ (in particular, if C is smooth at p it must be transverse to $W^u_{\mathrm{loc}}(p)$ and $W^s_{\mathrm{loc}}(p)$).

Now since we work in the Archimedean setting, we can push the analysis further and proceed to prove that $us = \mathrm{Jac}(f)$ is a root of unity. Assume it is not. Choose any θ in the unit circle, and pick a subsequence (n_j) such that $(u^k s^k)^{n_j} \to \theta$. Observe that $s^{-n}\psi(x/u^{n/k}) \to x^k/\theta$. Then by (4.2) and (4.4) in the smooth case ($k = 1$), and (4.5) and (4.6) in the singular case, we infer that for small t,

$$\begin{aligned} G^+(t^k,0) &= \lim_{n_j\to\infty} G^+\circ f^{n_j}(\Psi(t/u^{n_j/k})) \\ &= \lim_{n_j\to\infty}[\alpha G^-\circ f^{-n_j}(\Psi(t/u^{n_j/k})) + d^{n_j} H\circ\Psi(t/u^{n_j/k})] \\ &= \alpha G^-(0,t^k/\theta) + \lim_{n_j\to\infty} d^{n_j} H\circ\Psi(t/u^{n_j/k}). \end{aligned}$$

Since θ was arbitrary and since a uniform limit of harmonic functions is harmonic, we see that the Laplacian of the function $t\mapsto G^-(0,t^k)$ is rotation-invariant in a neighborhood of the origin. Observe that this Laplacian can be written as $\kappa^*\Delta(G^-(0,t))$, where $\kappa: t\mapsto t^k$. Recall also that the support of $\Delta(G^-(0,t))$ equals $\partial(K^-\cap W^s_{\mathrm{loc}}(p))$, where the boundary is relative to the intrinsic topology on $W^s_{\mathrm{loc}}(p)$. Thus we conclude that relative the linearizing coordinate on $W^s(p)$, $\kappa^{-1}(\partial(K^-\cap W^s_{\mathrm{loc}}(p)))$ is rotation invariant, so $\partial(K^-\cap W^s_{\mathrm{loc}}(p))$ is rotation invariant as well. But since $dd^c(G^-|_{W^s_{\mathrm{loc}}(p)})$ gives no mass to points, p must be an accumulation point of $\partial(K^-\cap W^s_{\mathrm{loc}}(p))$. By rotation invariance, $K^-\cap W^s_{\mathrm{loc}}(p)$ will then contain small circles around the origin. By the maximum principle this implies that $G^-|_{W^s_{\mathrm{loc}}(p)}$ vanishes in a neighborhood of p, which contradicts Proposition 1.14. The proof is complete. □

Remark 4.4. It is a well-known idea in the dynamical study of plane polynomial automorphisms that the slices of $T^\pm$ by stable and unstable manifolds (or more generally by any curve) contain a great deal of information about f. For instance, as we saw in §3, the Lyapunov exponents of the maximal entropy measure can be read off from this data. The same holds for multipliers of all saddle periodic orbits. See also [BS98b] for a striking application of this circle of ideas.

The proof of Theorem 4.3 (with $k=1$, say) implies that in adapted coordinates a relation of the form $G^+(x,0) = G^-(0,x) + \tilde{H}$ holds, where $\tilde{H}$ is a harmonic function. So we deduce that an unstable slice of K^+ is *holomorphically equivalent* to a stable slice of K^-.

This rigidity suggests a strong form of symmetry between f and f^{-1}, which gives additional credibility to Conjecture 1.

5. Conclusion of the proof of Theorems A and B

Recall that when f and C are defined over a number field, Theorems A and B were established in §3 and §4 respectively. In this section we explain how a specialization argument allows us to extend these results to an arbitrary field K of characteristic zero. Our approach treats Theorems A and B simultaneously.

Notice that it is not clear how to use the Lefschetz Principle here because the statement that f has infinitely many periodic points on a curve does not belong to first order logic.

Since K has characteristic zero, replacing it by an algebraic extension if needed, we may assume that it contains the algebraic closure of its prime field. We fix an isomorphism of this algebraically closed field with $\mathbb{Q}^{\mathrm{alg}}$.

We first make a conjugacy so that f becomes a regular polynomial automorphism of degree $d \geq 2$. Pick a finitely generated $\mathbb{Q}^{\mathrm{alg}}$-algebra $R \subset K$ containing all the coefficients of f, f^{-1}, and of an equation defining C. We may assume that $\mathrm{Frac}(R) = K$, and we set $S = \mathrm{Spec}(R)$.

Let us start with a loose explanation of the proof. The field K is a function field over $\mathbb{Q}^{\mathrm{alg}}$, which we view as the function field of the variety S. For every $s \in S$, we substitute the corresponding value s into the coefficients of f, obtaining a map f_s. For generic s, we obtain a polynomial automorphism (Lemma 5.1), which satisfies the assumptions of Theorem A or B (Lemmas 5.2 and 5.3). Thus for every s, $\mathrm{Jac}(f_s)$ is of modulus 1, and we can conclude that the same holds for $\mathrm{Jac}(f)$. The details however require some algebro-geometric technology.

Let $\pi : \mathbb{A}^2_S \to S$ be the natural projection map. Observe that for every (scheme-theoretic) point $s \in S$, the fiber $\pi^{-1}(s)$ is canonically isomorphic to $\mathbb{A}^2_{\kappa(s)}$, where $\kappa(s)$ is the residue field of s. For such an s, we let $C_s := \pi^{-1}(s) \cap C$ be the *specialization* of C, and likewise we denote by f_s and f_s^{-1} the maps respectively induced by f and f^{-1} on $\mathbb{A}^2_{\kappa(s)}$.

Lemma 5.1. *There exists a non-empty open subset $S' \subset S$ such that for any $s \in S'$ the map f_s is a regular polynomial automorphism of degree d.*

Proof. Observe first that $f \circ f^{-1} = \mathrm{id}$, hence $f_s \circ f_s^{-1} = \mathrm{id}$ on $\mathbb{A}^2_{\kappa(s)}$, so f is an automorphism for all $s \in S$.

The condition of being regular of degree d can be stated as follows: Expand f as $f = f^{(0)} + f^{(1)} + \cdots + f^{(d)}$, where $f^{(i)} : \mathbb{A}^2 \to \mathbb{A}^2$ contains only homogeneous terms of degree i. Then f is regular of degree d if and only if the composition $f^{(d)} \circ f^{(d)}$ is not identically 0. Now the set of $s \in S$ where $f_s^{(d)} \circ f_s^{(d)} \not\equiv 0$ is open and non-empty, and the result follows. □

Lemma 5.2. *Suppose that $p \in C(R)$ is a* (*closed*) *periodic point for f lying in C and defined over R such that the transversality condition* (T) *holds for f. Then there exists an open subset $S' \subset S$ such that for any $s \in S'$ the condition* (T) *is also satisfied for f_s.*

Proof. The condition that p_s belongs to the regular locus of C_s is open since it is given by an equation of the form $d\phi_s(p_s) \neq 0$ where $\phi \in R[x, y]$ is an equation of C.

If p has exact period k, the condition that the period of p_s is also k is given by the open condition $f(p), \ldots, f^{k-1}(p) \neq p$.

The condition that T_pC is not invariant by an iterate of f is equivalent to saying that the vector $\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right)$ is not an eigenvector for df^{2k}, which is open. □

The key point is the following lemma.

Lemma 5.3. *Suppose that C contains infinitely many periodic points of f. Then for any $s \in S$ the curve C_s contains infinitely many periodic points of f_s.*

Before proving this lemma, let us show how to conclude the proof of Theorems A and B. Since $\mathrm{Jac}(f)$ lies in R, it may be viewed as a regular function from S to $\mathbb{A}^1_{\mathbb{Q}^{\mathrm{alg}}}$.

Fix any embedding $\mathbb{Q}^{\mathrm{alg}} \subset \mathbb{C}$. By Theorems A (resp. B) over $\mathbb{Q}^{\mathrm{alg}}$, we know that for any closed point $s \in S$, the algebraic number $\mathrm{Jac}(f)(s)$ has all its conjugates of modulus 1 (resp. it is a root of unity).

If $\mathrm{Jac}(f)$ were not a constant function then its image would contain an open affine subset of $\mathbb{A}^1_{\mathbb{Q}^{\mathrm{alg}}}$, and in particular at least one algebraic number of modulus different from 1. Therefore $\mathrm{Jac}(f)$ is a constant lying in $\mathbb{Q}^{\mathrm{alg}}$, and we are done. □

Proof of Lemma 5.3. For any $n \geq 1$, write $f^n = (f_1^n, f_2^n)$ in affine coordinates (x, y), and pick a defining equation $C = \{\phi = 0\}$ with $\phi \in R[x, y]$.

Given an integer $l \geq 1$, denote by $\mathcal{I}_{n,l}$ the coherent ideal sheaf generated by the polynomials $f_1^n - x$, $f_2^n - y$ and ϕ^l. Let $\mathcal{F}_{n,l}$ be the quotient sheaf $\mathcal{O}_{\mathbb{A}^2}/\mathcal{I}_{n,l}$, and denote by $X_{n,l}$ the S-subscheme of $\mathbb{A}^2_S$ defined by $\mathcal{F}_{n,l}$.

We claim that the map of schemes $\pi : X_{n,l} \to S$ is proper (hence finite since $X_{n,l}$ is a subscheme of $\mathbb{A}^2_S$). Taking this fact for granted we proceed with the proof of the lemma.

Since $\pi : X_{n,l} \to S$ is proper, the sheaf $\mathcal{G} := \pi_* \mathcal{F}_{n,l}$ is coherent on S. It follows from Nakayama's lemma [Ha77, Exercice II.5.8] that the function $s \mapsto \dim_{\kappa(s)} \mathcal{G}_s/\mathfrak{m}_s\mathcal{G}_s$ is upper semicontinuous. Now observe that

$$\mathcal{G}_s/\mathfrak{m}_s\mathcal{G}_s = \kappa(s)[x, y]/((f_1^n)_s - x, (f_2^n)_s - y, \phi_s^l).$$

Pick any closed point $s \in S$. By the Nullstellensatz, for l large enough and for any given point $p \in \pi^{-1}(s)$ the stalk of the coherent sheaf $\mathcal{F}_{n,l}/\mathfrak{m}_s\mathcal{F}_{n,l}$ at p coincides with the stalk of $\kappa(s)[x, y]/((f_1^n)_s - x, (f_2^n)_s - y)$. To simplify notation, denote by $\mu(p, f_s^n)$ the multiplicity of p as a fixed point for f_s^n, that is, the dimension of the finite-dimensional $\kappa(s)$-vector space $\mathcal{O}_{\mathbb{A}^2_{\kappa(s),p}}/((f_1^n)_s - x, (f_2^n)_s - y)$.

Lemma 5.4. *The sequence* $\{\mu(p, f_s^n)\}_n$ *is bounded.*

Proof. Since the residue field $\kappa(s)$ has characteristic zero and is finitely generated over $\mathbb{Q}^{\mathrm{alg}}$, we may embed it into $\mathbb{C}$ and assume f_s is a complex polynomial map. The result then follows from [SS74]. □

Then we have

$$\begin{aligned}\sum_{p \in C \cap \mathrm{Fix}(f_s^n)} \mu(p, f_s^n) &= \sum_{p \in C \cap \pi^{-1}(s)} \dim_{\kappa(s)} \mathcal{O}_{\mathbb{A}^2_{\kappa(s),p}}/((f_1^n)_s - x, (f_2^n)_s - y)\\ &= \sum_{p \in \mathbb{A}^2 \cap \pi^{-1}(s)} \dim_{\kappa(s)} \mathcal{O}_{\mathbb{A}^2_{\kappa(s),p}}/((f_1^n)_s - x, (f_2^n)_s - y, \phi_s^l)\\ &= \dim_{\kappa(s)} \mathcal{G}_s/\mathfrak{m}_s\mathcal{G}_s \geq \dim_K \mathcal{G}_\eta = \sum_{p \in C_K} \mu(p, f_K^n),\end{aligned}$$

where η denotes the generic point of S, and f_s (resp. f_K) is the map induced by f on $\mathbb{A}^2_{\kappa(s)}$ (resp. on $\mathbb{A}^2_K$).

By assumption we know that the quantity

$$\sum_{p \in C_K} \mu(p, f_K^n) \geq \mathrm{Card}(C_K \cap \mathrm{Per}(f_K^n))$$

tends to infinity as $n \to \infty$. It follows that $\sum_{p \in C_{\kappa(s)} \cap \mathrm{Fix}(f_s^n)} \mu(p, f_s^n) \to \infty$. By Lemma 5.4, we conclude that $\mathrm{Card}(C_{\kappa(s)} \cap \mathrm{Fix}(f_s^n))$ tends to infinity, as was to be shown.

It remains to prove that the projection map $\pi : X_{n,l} \to S$ is proper. Let X_n be the S-scheme defined by the equations $f_1^n - x$, $f_2^n - y$. Since $X_{n,l}$ is a subscheme of X_n, it is sufficient to show that $\pi : X_n \to S$ is proper.

To simplify notation we shall only treat the case $n = 1$. Consider the intersection of the diagonal Δ and the graph Γ of f in $\mathbb{P}^2_S \times \mathbb{P}^2_S$, and denote by Y its projection to the first factor. The projection map $Y \to S$ is projective, hence proper. Observe that for each $s \in S$, Y_s is the union of the support of $(X_1)_s$, which is finite by Proposition 1.9, and two points at infinity $p_-(s)$ and $p_+(s)$ corresponding to the (unique) indeterminacy point of f_s and its superattracting fixed point.

Let Y_+ and Y_- be the irreducible components of Y such that $(Y_\pm)_s = p_\pm(s)$ for all s. Since $p_+(s)$ is superattracting, the differential of f_s at $p_+(s)$ has no eigenvalue equal to 1, and the intersection of Δ and Γ is transverse at $p_+(s)$. It follows from the next lemma (which was indicated to us by A. Ducros) applied to the section $s \mapsto p_+(s)$ that Y_+ is a connected component of Y. Replacing f by f^{-1} we find that Y_- is also a connected component of Y.

We conclude that π is a projective map from $X_n = Y \setminus (Y_+ \cup Y_-)$ to S, hence it is proper. ◻

Lemma 5.5. *Let $f : Y \to S$ be a finite morphism of finite presentation and let $\sigma : S \to Y$ be a section of f. If $\mathcal{O}_{f^{-1}(s),\sigma(s)} = \kappa(s)$ for all $s \in S$ then $\sigma(S)$ is open in Y.*

Proof. Pick any $s \in S$ and let T be the spectrum of the henselization of $\mathcal{O}_{S,s}$. Denote by t the closed point of T. Since T is henselian, the finite T-scheme $X \times_S T$ is a disjoint union $\coprod T_i$ of spectra of local rings. Pick i_0 such that $\sigma_T(s) \in T_{i_0}$ (here σ_T is the section obtained from σ by base change).

The ring $\mathcal{O}(T_{i_0})$ is a module of finite type over $\mathcal{O}(T)$. Its rank over a closed point is equal to 1 by assumption, hence is at most 1 at any point of T. Since there is a section σ_T, this rank is actually equal to 1 everywhere, whence $\sigma_T(T) = T_{i_0}$.

It follows that there exists an étale morphism $U \to S$ whose image contains s, and there is a decomposition $Y \times_U S = \coprod U_i$ where each U_i is finite over U where $\sigma_U(U) = U_{i_0}$. The image of U_{i_0} in Y is an open subset Y' of Y. By construction, $Y' \subset \sigma(S)$ and Y' contains $\sigma(s)$. We conclude that $\sigma(S)$ is open. ◻

6. Automorphisms sharing periodic points

The main purpose of this section is to prove Theorems C and D.

6.1. The Bass–Serre tree of $\mathrm{Aut}[\mathbb{A}^2]$

Let us recall briefly how the group of polynomial automorphisms of the affine plane naturally acts on a tree. We refer to [La01] for details and to [Se77] for basics on (groups acting on) trees.

Denote by A (resp. E) the subgroup of affine (resp. elementary) automorphisms. The intersection $A \cap E$ consists of those automorphisms of the form $(x, y) \mapsto (ax + b, cy + dx + e)$ with $ac \neq 0$.

Jung's theorem states that $\mathrm{Aut}[\mathbb{A}^2]$ is the free amalgamated product of A and E over their intersection. This means that any automorphism $f \in \mathrm{Aut}[\mathbb{A}^2]$ can be written as a product

$$f = e_1 \circ a_1 \circ \cdots \circ e_s \circ a_s$$

with $e_i \in E$ and $a_i \in A$, and such a decomposition is unique up to replacing a product $e \circ a$ by $(e \circ h^{-1}) \circ (h \circ a)$ with $h \in A \cap E$.

The *Bass–Serre tree* $\mathcal{T}$ of $\mathrm{Aut}[\mathbb{A}^2]$ is the simplicial tree whose vertices are left cosets modulo A or E. In other words we choose a set S_A (resp. S_E) of representatives of the quotient of $\mathrm{Aut}[\mathbb{A}^2]$ under the right action of A (resp. of E). Then the vertices of $\mathcal{T}$ are in bijection with $\{hA\}_{h\in S_A} \cup \{hE\}_{h\in S_E}$. There is an edge joining hA to $h'E$ if $h' = h \circ a$ for some $a \in A$ or $h = h' \circ e$ for some $e \in E$.

We endow $\mathcal{T}$ with the unique tree metric giving length 1 to all edges. The left action of an automorphism f on cosets induces an action on $\mathcal{T}$ by isometries. It sends any vertex of the form hA (resp. hE) to fhA (resp. to fhE). Abusing notation we will simply denote this action by f.

If the action of f on $\mathcal{T}$ stabilizes a point of type hA (resp. hE) then it is conjugate to an affine (resp. elementary) map by h.

Otherwise f has no fixed point on $\mathcal{T}$ and it is of Hénon type. Then by definition its *axis* is the set of vertices minimizing the distance $d(t, ft)$. It is a unique geodesic (that is, a bi-infinite path in $\mathcal{T}$), which we denote by $\mathrm{Geo}(f)$. It is f-invariant and f acts on $\mathrm{Geo}(f)$ as a translation, whose length is a non-zero even integer.

For further reference let us isolate two statements from [La01].

Proposition 6.1. *Let f and g be automorphisms of Hénon type that do not share a non-trivial iterate. Then* $\mathrm{Geo}(f) \cap \mathrm{Geo}(g)$ *is bounded (possibly empty).*

Proof. By [La01, Corollaire 4.2], either $\mathrm{Geo}(f) = \mathrm{Geo}(g)$ or $\mathrm{Geo}(f) \cap \mathrm{Geo}(g)$ is bounded (possibly empty). By [La01, Théorème 5.4], if the first alternative holds, then f and g have a common iterate. □

Proposition 6.2. *If γ is a segment in the Bass–Serre tree $\mathcal{T}$, then there exists a polynomial automorphism φ with the property that if f is any polynomial automorphism whose invariant geodesic* $\mathrm{Geo}(f)$ *contains γ, then the map $\varphi^{-1} \circ f \circ \varphi$ is regular.*

Proof. It was observed in [La01, Remarque 2.3] that if f is a polynomial automorphism of Hénon type whose associated geodesic contains the edge joining $\mathrm{id}\,A$ to $\mathrm{id}\,E$, then f is cyclically reduced. Hence by [FM89, Theorem 2.6] it is conjugate by an affine map to a composition of generalized Hénon maps, so that in particular it is regular.

In the general case, since $\mathrm{Geo}(\varphi^{-1} f \varphi) = \varphi^{-1}\,\mathrm{Geo}(f)$, it is enough to pick φ such that $\varphi^{-1}(\gamma) = [\mathrm{id}\,A, \mathrm{id}\,E]$ and conclude by the above argument. □

6.2. *Proof of Theorem C*

In this section we assume that both automorphisms f and g are of Hénon type and defined over a number field $\mathbb{L}$. We suppose that they share a Zariski dense subset of periodic points, and wish to prove that they admit a common iterate. The proof is divided into three steps.

Step 1: *f and g have the same equilibrium measure at any place.* Assume that f and g share a set $\{p_m\}$ of periodic points which is Zariski dense in $\mathbb{A}^2$. We use a diagonal argument to extract a subset $\{p'_k\}$ satisfying the requirements of Theorem 1.18. For this, enumerate as $(C_q)_{q\in\mathbb{N}}$ all irreducible curves in $\mathbb{A}^2_{\mathbb{L}}$. We construct an auxiliary subsequence of (p_m) as follows. Let m_1 be the minimal integer such that $p_{m_1} \notin C_1$, and set $p'_1 = p_{m_1}$. Then define $m_2 > m_1$ to be the minimal integer such that $p_{m_2} \notin C_1 \cup C_2$, and set $p'_2 = p_{m_2}$. These integers exist since the set $\{p_m\}$ is Zariski dense. Continuing in this way one defines recursively a sequence (p'_k) of periodic points with the desired properties. In particular we conclude from Theorem 1.18 that $\mu_{f,v} = \mu_{g,v}$ for every place v.

Step 2: *f and g have the same set of periodic points.* The difficulty is that we do not assume that f and g are conjugate by the *same* automorphism to a regular map. If this happens, the conclusion follows rather directly from the work of Lamy [La01], as we will see in Step 3.

To overcome this problem, we proceed as follows. Fix a place v, and define $K_v(f) = \{p \in \mathbb{A}^{2,\mathrm{an}}_{\mathbb{C}_v} : \sup_{n\in\mathbb{Z}} |H^n(p)| < \infty\}$.

Lemma 6.3. *For any place v, the set $K_v(f)$ is the largest compact set in $\mathbb{A}^{2,\mathrm{an}}_{\mathbb{C}_v}$ such that*

$$\sup_{K_v(f)} |P| = \sup_{\mathrm{supp}(\mu_{f,v})} |P| \quad \textit{for all } P \in \mathbb{C}_v[\mathbb{A}^2].$$

Proof. Since $\mathrm{supp}(\mu_{f,v}) \subset K_v(f)$, it is sufficient to prove that the supremum of $|P|$ over $K_v(f)$ is attained at a point lying in $\mathrm{supp}(\mu_{f,v})$.

Suppose that P is a polynomial function, and pick any constant $C_0 > 0$ such that $\log(|P|/C_0) \le 0$ on $\mathrm{supp}(\mu_{f,v})$. Then the function

$$\tilde{G} := \max\{G, \log(|P|/(C_0 + \varepsilon))\}$$

is a continuous non-negative function on $\mathbb{A}^2_{\mathbb{C}_v}$ that induces a continuous semipositive metric on $\mathcal{O}(1)$. Since $\tilde{G} = G$ near $\mathrm{supp}(\mu_{f,v})$, from Corollary A.2 we deduce the equality of measures $\mathrm{MA}(\tilde{G}) = \mathrm{MA}(G)$, and it follows from Yuan–Zhang's theorem [YZ13a] that $\tilde{G} - G$ is a constant, hence $\tilde{G} = G$. It follows that $\log(|P|/(C_0 + \varepsilon)) \le 0$ on $K_v(f)$. By letting $\varepsilon \to 0$, we conclude that $\log(|P|/C_0) \le 0$ on $K_v(f)$. □

By analogy with the complex case, one can summarize the previous result by saying that the polynomially convex hull of $\mathrm{supp}(\mu_{f,v})$ is the set $K_{f,v}$. Since $\mu_{f,v} = \mu_{g,v}$ for all v, we conclude that $K_{f,v} = K_{g,v}$.

Now pick any periodic point p of f. At the place v, it belongs to $K_{f,v}$, hence to $K_{g,v}$. Since Lee's height can be computed by summing the local quantities $G_{g,v} :=$

$\max\{G^+_{g,v}, G^-_{g,v}\}$, and since $\{G_{g,v} = 0\} = K_{g,v}$, we conclude that the canonical g-height of p is zero, hence p is g-periodic.

In what follows, we actually need a stronger information.

Lemma 6.4. *Suppose f and g are polynomial automorphisms of the affine plane of Hénon type defined over a number field $\mathbb{L}$, satisfying the assumptions of Theorem C. Then for all places v over $\mathbb{L}$, and for any Hénon-type automorphism h belonging to the subgroup generated by f and g, one has $K_{h,v} = K_{g,v} = K_{f,v}$.*

Proof. We already know that $K_v := K_{g,v} = K_{f,v}$. Since this compact set is invariant by both f and g, it follows that h also preserves K_v, and this implies $K_v \subset K_{h,v}$ for all v. Now let F_n denote the set of points of period n for f. For all v, we have $F_n \subset K_{h,v}$, hence the canonical h-height of F_n is equal to 0. Extracting a subsequence if necessary, we may always assume that F_n is generic since the set of periodic points of a hyperbolic automorphism is Zariski dense. By Yuan's theorem, F_n is equidistributed with respect to the equilibrium measure of both K_v and $K_{h,v}$, and we conclude that $K_{h,v} = K_v$. □

Step 3: *f and g admit a common iterate.* We use Lamy's structure theory of subgroups of the group of polynomial automorphisms of the plane [La01]. Assume for contradiction that f and g admit no common iterate.

Lemma 6.5. *Under the above hypotheses, there exist two Hénon-type elements h_1, h_2 in the subgroup generated by f and g and a polynomial automorphism φ such that*

- *the subgroup H generated by h_1 and h_2 is a free non-abelian group;*
- *any element in H that is not the identity is of Hénon type;*
- *for any $h \in H$, the automorphism $\varphi^{-1} \circ h \circ \varphi$ is a regular automorphism of $\mathbb{A}^2_L$.*

The rest of the argument is now contained in [La01, Théorème 5.4]. We include it for the convenience of the reader. We may assume that h_1, h_2 are regular polynomial automorphisms of $\mathbb{A}^2_L$. By Lemma 6.4 we have $K_v := K_{h_1,v} = K_{h_2,v}$ at all places. Pick any Archimedean place v. Then $\mu := \mu_{h_1,v} = \mu_{h_2,v}$. Now since both h_1 and h_2 are regular, it follows that $G_1 := \max\{G^+_{h_1}, G^-_{h_1}\}$ and G_2 are both equal to the Siciak–Green function of K_v by [BS91, Proposition 3.9], and are hence equal.

Replacing h_1 by its inverse if necessary, we find that $G^+_{h_1} = G^+_{h_2}$ on a non-empty open set where the two functions are positive. Since these functions are pluriharmonic where they are non-zero, and since for a Hénon-type automorphism h, $\mathbb{C}^2 \setminus K^+$ is connected (indeed $\mathbb{C}^2 \setminus K^+ = \bigcup_{n\geq 0} h^{-n}(V^+)$), we deduce that they coincide everywhere. We conclude that the positive closed $(1, 1)$-currents $T := dd^c G^+_{h_1}$ and $dd^c G^+_{h_2}$ are equal. Now consider the commutator $h_3 = h_1 h_2 h_1^{-1} h_2^{-1}$. Observe that $h_3^* T = T$, and h_3 is regular by the previous lemma. Since the support of T has a unique point on the line at infinity, replacing h_3 by h_3^{-1} if needed we may suppose that this point is not an indeterminacy point of h_3. It then follows that the mass of $h_3^* T$ equals the degree of h_3 times the mass of T, which is contradictory. This completes the proof of Theorem C.

Proof of Lemma 6.5. By Proposition 6.1, the invariant geodesics $\mathrm{Geo}(f)$ and $\mathrm{Geo}(g)$ have empty or bounded intersection. Assume first that $\mathrm{Geo}(f) \cap \mathrm{Geo}(g)$ contains a segment. Pick an edge γ in this intersection. By Proposition 6.2 by conjugating we may assume f and g are regular automorphisms.

Set $h_1 = f^N$ and $h_2 = g^N$ where N is greater than the diameter of $\mathrm{Geo}(f) \cap \mathrm{Geo}(g)$. The invariant geodesics of these two automorphisms are equal to $\mathrm{Geo}(f)$ and $\mathrm{Geo}(g)$ respectively. Now pick any non-trivial word

$$h = h_1^{n_p} \circ h_2^{m_p} \circ \cdots \circ h_1^{n_1} \circ h_2^{m_1}$$

with $p \geq 1$ and all $n_i, m_i \in \mathbb{Z} \setminus \{0\}$. Then a ping-pong argument [La01, Proposition 4.3] shows that γ lies in the interior of the segment $[h(\gamma), h^{-1}(\gamma)]$, which implies that h is of Hénon type and $\mathrm{Geo}(h) \supset \gamma$. By Proposition 6.2 we conclude that h is also regular. This concludes the proof in this case.

Assume now that $\mathrm{Geo}(f)$ and $\mathrm{Geo}(g)$ are either disjoint, or their intersection is reduced to a singleton. Then there exists a unique segment $I = [\gamma_1, \gamma_2]$ in the tree with $I \cap \mathrm{Geo}(f) = \{\gamma_1\}$ and $I \cap \mathrm{Geo}(g) = \{\gamma_2\}$. Pick any element $\gamma \in I$, and N large enough such that the translation lengths of both f^N and g^N are larger than twice the diameter of I. Then both automorphisms $h_1 = f^N g^N$ and $h_2 = f^{2N} g^{2N}$ satisfy $\gamma \in [h_i(\gamma), h_i^{-1}(\gamma)]$, so that we can apply the same argument as in the previous case. The proof is complete. □

6.3. Proof of Theorem D

Here we assume that f and g are automorphisms of Hénon type with *complex* coefficients such that $|\mathrm{Jac}(f)| \neq 1$, and f and g share an infinite set $\mathcal{P}$ of periodic points.

As in Section 5, we pick a finitely generated $\mathbb{Q}^{\mathrm{alg}}$-algebra R such that f and g are defined over $K := \mathrm{Frac}(R)$ that is an integral domain, and we set $S = \mathrm{Spec}(R)$. As before we write f_s for the specialization of f at $s \in S$.

Towards a contradiction, assume that f and g have no common iterate. By Lemma 6.5 we can fix two elements h_1, h_2 in the subgroup generated by f and g such that any non-trivial element in $H := \langle h_1, h_2 \rangle$ is of Hénon type and regular.

Lemma 6.6. *There exists a non-empty open subset $S' \subset S$ such that for any $s \in S'$ the maps $h_{1,s}$ and $h_{2,s}$ as well as their commutators $(h_1 h_2 h_1^{-1} h_2^{-1})_s$, $(h_1 h_2^{-1} h_1^{-1} h_2)_s$ are regular polynomial automorphisms of Hénon type.*

Proof. The condition for an automorphism h to be regular is open since it amounts to saying that the indeterminacy loci of h and h^{-1} are disjoint. A theorem of J.-P. Furter [Fu99] asserts that an automorphism h is of Hénon type if and only if $\deg(h^2) > \deg(h)$. Since there exists an open set where $\deg(h_s) = \deg(h)$ and $\deg(h_s^2) = \deg(h^2)$, the result follows. □

Thus, replacing S by a Zariski dense open subset, we may assume that for all $s \in S$, the maps $h_{1,s}$, $h_{2,s}$, $(h_1 h_2 h_1^{-1} h_2^{-1})_s$ and $(h_1 h_2^{-1} h_1^{-1} h_2)_s$ are regular polynomial automorphisms of Hénon type.

Let us now pick $s \in S(\mathbb{Q}^{\mathrm{alg}})$ such that $\mathrm{Jac}(f_s)$ has at least one complex conjugate of norm $\neq 1$. It is always possible to find such a parameter since otherwise $\mathrm{Jac}(f)$ would be an algebraic number whose complex conjugates all lie on the unit circle, contradicting our assumption.

Our next claim is that $\mathcal{P}_s$ is infinite. Indeed, assume that $\mathcal{P}_s$ is finite, so it is included in the set of fixed points of f^{n_0}. For each $n \geq n_0$ denote by X_n the subscheme of $\mathbb{A}^2_S$ whose underlying space is

$$\mathcal{P} \cap \{(x, y) : f^n(x, y) = (x, y)\},$$

endowed with the scheme structure induced by the quotient sheaf

$$\mathcal{O}_{\mathbb{A}^2_S}/(f_1^n - x, f_2^n - y), \quad \text{where} \quad f^n = (f_1^n, f_2^n).$$

For any $p \in \mathcal{P}_s$, the ordinary multiplicity $e(p, X_{n,s})$ of p as a point in $X_{n,s}$ is equal to the multiplicity $\mu(p, f_s^n)$ as a fixed point for f_s^n. By the Shub–Sullivan Theorem [SS74], the sequence $e(p, X_{n,s})$ is bounded, hence $\sum_{p \in \mathcal{P}_s} e(p, X_{n,s})$ is bounded.

Arguing as in the proof of Lemma 5.3, we see that the map $X_n \to S$ is proper and finite, and by Nakayama's lemma we get

$$\#[\mathcal{P} \cap \{f^n = \mathrm{id}\}] \leq \sum_{p \in \mathcal{P}} e(p, X_n) \leq \sum_{p \in \mathcal{P}_s} e(p, X_{n,s}) < \infty.$$

Now observe that $\mathcal{P} \cap \{f^{n!} = \mathrm{id}\}$ contains all the periodic points in $\mathcal{P}$ of period $\leq n$, so the cardinality of this set tends to infinity as $n \to \infty$. This is contradictory, thereby showing that $\mathcal{P}_s$ is infinite.

To conclude the proof of the theorem, fix an Archimedean place v at which $|\mathrm{Jac}(f_s)|_v \neq 1$. Since $\mathcal{P}_s$ is infinite, Theorem A implies $\mathcal{P}_s$ is Zariski dense. By Lemma 6.4, h_1 and h_2 have the same K_v, and by arguing as in Step 3 on p. 3456, we conclude that the two pairs of functions $\{G^+_{h_{1,s}}, G^-_{h_{1,s}}\}$ and $\{G^+_{h_{2,s}}, G^-_{h_{2,s}}\}$ are identical. It follows that one of the commutators $(h_1 h_2 h_1^{-1} h_2^{-1})_s$ or $(h_1 h_2^{-1} h_1^{-1} h_2)_s$, denoted by h_3, leaves $G^+_s := G^+_{h_{1,s}}$ invariant, that is, $G^+_s \circ h_3 = G^+_s$. As we saw in Theorem C, since h_3 is a regular automorphism, this cannot be true. This contradiction finishes the proof. □

Remark 6.7. The argument uses in an essential way the fact that the place v is Archimedean. Indeed, we ultimately rely on the fact that for any two regular maps of Hénon type the equality $G_{h_1} = G_{h_2}$ forces that of $\{G^+_{h_1}, G^-_{h_1}\}$ and $\{G^+_{h_2}, G^-_{h_2}\}$. The corresponding statement over a non-Archimedean field k is not true.

Indeed, as above it can be shown that $\{G^+_{h_1}, G^-_{h_1}\} = \{G^+_{h_2}, G^-_{h_2}\}$ if and only if h_1 and h_2 admit a common iterate. On the other hand, if h is any regular automorphism such that h and h^{-1} have their coefficients in the ring of integers of k, then $G_h(x, y) = \log \max\{1, |x|, |y|\}$.

6.4. *Sharing cycles*

In this section we observe that a strengthening of Theorem C can be obtained if one assumes that two automorphisms of Hénon type share infinitely many periodic cycles (and not just periodic points).

Let us start with the following observation.

Proposition 6.8. *Let f be an automorphism of Hénon type defined over a number field $\mathbb{L}$. Let (F_m) be any sequence of disjoint periodic* cycles. *Then the sequence (μ_m) of probability measures equidistributed over the Galois conjugates of F_m converges weakly to $\mu_{f,v}$ for all places v.*

As a consequence we have

Corollary 6.9. *Let f and g be polynomial automorphisms of Hénon type of the affine plane, defined over a number field. If they share an infinite set of periodic cycles, then there exist non-zero $n, m \in \mathbb{Z}$ such that $f^n = g^m$.*

Indeed, to prove the corollary it suffices to repeat the proof of Theorem C starting from Step 2.

Proof of Proposition 6.8. We may assume that all F_m are Galois invariant. The result does not quite follow from Lee's argument of [Le13, Theorem A] since we do not assume that the set $\bigcup_m (F_m \cap C)$ is finite for every curve C. We claim however that for any algebraic curve C,

$$\#(C \cap F_m) = o(\#F_m) \quad \text{as } n \to \infty. \tag{6.1}$$

One then argues exactly as in [FG14, proof of Theorem 1] to conclude that μ_m converges to μ_f.

Let us justify (6.1). Suppose for contradiction that there exists $\varepsilon > 0$ such that

$$\#(C \cap F_m) \geq \varepsilon \, \#F_m.$$

First observe that the minimal period of all points in F_m tends to infinity since f^n admits only finitely many fixed points for any $n > 0$. Pick any integer $N > 1/\varepsilon$ and m large enough such that the periods of all points in F_m are larger than N. We claim that

$$\#\{p \in F_m \cap C : f^k(p) \in C \text{ for some } 0 < k \leq N\} \to \infty.$$

But this implies that $C \cap f^{-k}(C)$ is infinite for some $0 < k \leq N$, whence $f^k(C) = C$, a contradiction.

To prove the claim, let B denote the set of points in $F_m \cap C$ such that $f^k(p) \notin C$ for all $1 \leq k \leq N$, and let G be its complement in $F_m \cap C$. We want to estimate $\#G$. For this, we see that $\#(B \times N) \leq \#F_m$, hence

$$\#G \geq \#(F_m \cap C) - \#B \geq \varepsilon \#F_m - \frac{1}{N} \#F_m \to \infty$$

as required. □

7. Reversible polynomial automorphisms

A polynomial automorphism of $\mathbb{A}^2$ is said to be *reversible* if there exists a polynomial automorphism σ, which may or may not be an involution, such that $\sigma^{-1} f \sigma = f^{-1}$. Any such σ is then called a *reversor*.

Since invariance under time-reversal appears frequently in physical models, such mappings have attracted a lot of attention in the mathematical physics literature. In the context of plane polynomial automorphisms, reversible mappings were classified by Gómez and Meiss [GM03a, GM03b]. In particular they prove that the reversor σ is either affine or elementary and of finite (even) order. Moreover they show that when σ admits a curve of fixed points, then σ must be an involution conjugate to the affine involution $t : (x, y) \mapsto (y, x)$.

Our aim is to prove the following:

Proposition 7.1. *Suppose that f is a reversible polynomial automorphism of Hénon type and σ is an involution conjugating f to f^{-1}. Then any curve of fixed points of σ contains infinitely many periodic points of f.*

Specific examples include all Hénon transformations of Jacobian 1 that are of the form $(x, y) \mapsto (p(x) - y, x)$, for which the reversor is the affine involution t. So is the Hénon mapping $(x, y) \mapsto (-y, p(y^2) - x)$, of Jacobian -1. More generally, a mapping of the form $tH^{-1}tH$ is reversible with reversor t, where H denotes any polynomial automorphism.

Let us also observe that taking iterates is really necessary in Conjecture 1. Indeed, pick for some $n \geq 2$ a primitive n-th root of unity ζ, and let p be any polynomial such that $p(\zeta x) = \zeta p(x)$. Then the automorphisms $(p(x) - y, x)$, $(\zeta x, \zeta y)$ commute and the automorphism defined by $H := (p(x) - y, x) \circ (\zeta x, \zeta y)$ is not reversible but its n-th iterate is. Observe also that the Jacobian of H equals ζ^2.

Algebraic proof of Proposition 7.1. As observed above, we may assume that $\sigma = t$ so that the curve of fixed points is actually the diagonal $\Delta = \{x = y\}$.

Observe now that any point $p \in \Delta \cap f^n(\Delta)$ satisfies $f^{-n}(p) = \sigma f^n \sigma(p) = \sigma f^n(p) = f^n(p)$, and is thus periodic of period $2n$.

To conclude, it remains to prove that $\#(\Delta \cap f^n(\Delta)) \to \infty$. For any $p \in \Delta \cap f^n(\Delta)$, we denote by $\mu_n(p)$ the multiplicity of intersection of Δ and $f^n(\Delta)$ at p. We rely on the following result of Arnol'd [Ar93] (see also [SY14]).

Lemma 7.2. *For any $p \in \Delta$, the sequence $\mu_n(p)$ is bounded.*

Now since f is a polynomial automorphism of Hénon type, we may choose affine coordinates such that f extends to $\mathbb{P}^2$ as a regular map. Recall that f admits a superattracting point p_+ and an indeterminacy point p_- on the line at infinity, and $p_+ \neq p_-$. Write $\overline{\Delta}$ for the closure of the diagonal in $\mathbb{P}^2$.

By [BS91], there exists an integer $n_0 \geq 1$ such that $f^n(\overline{\Delta}) \ni p_+$ for all $n \geq n_0$ and $f^n(\overline{\Delta}) \ni p_-$ for all $n \leq -n_0$. It follows that for all $n \geq n_0$, the intersection $f^{-n_0}(\overline{\Delta}) \cap f^n(\overline{\Delta})$ is included in $\mathbb{A}^2$. By Bézout's Theorem we infer that

$$\sum_{p \in \Delta} \mu_n(p) = f^{-n_0}(\overline{\Delta}) \cdot f^{n-n_0}(\overline{\Delta}) = \deg(f^{-n_0}(\overline{\Delta})) \times \deg(f^{n-n_0}(\overline{\Delta})) \to \infty.$$

We conclude that there are infinitely many fixed points of f on Δ, for otherwise their multiplicities would have to grow to infinity, contradicting Lemma 7.2. □

Analytic proof of Proposition 7.1. Let us sketch an alternative argument for $\#(\Delta \cap f^n(\Delta)) \to \infty$, based on intersection theory of laminar currents.

For notational ease, assume that $n = 2k$ is even. Then $\#(\Delta \cap f^n(\Delta)) = \#(f^{-k}(\Delta) \cap f^k(\Delta))$. We know from [BS91] that the sequence of positive closed (1, 1)-currents $d^{-k}[f^k(\Delta)]$ converges to T^-, and likewise for T^+. It follows from [BS98a] that this convergence holds in a geometric sense. Informally this means that we can discard a part of $d^{-k}[f^k(\Delta)]$ of arbitrarily small mass, uniformly in k, so that the remaining part is made of disks of uniformly bounded geometry, which geometrically converge to the disks making up the laminar structure of T^-.

To state things more precisely, we follow the presentation of [Duj04]. Fix $\varepsilon > 0$. Given a generic subdivision $\mathcal{Q}$ of $\mathbb{C}^2$ by affine cubes of size $r > 0$, there exist uniformly laminar currents $T^-_{\mathcal{Q},k} \leq d^{-k}[f^k(\Delta)]$ and $T^+_{\mathcal{Q},k} \leq d^{-k}[f^{-k}(\Delta)]$ made of graphs in these cubes and such that the mass of $d^{-k}[f^{\pm k}(\Delta)] - T^{\pm}_{\mathcal{Q},k}$ is bounded by Cr^2 for some constant C [Duj04, Proposition 4.4]. Therefore, up to extracting a subsequence, the currents $T^{\pm}_{\mathcal{Q},k}$ converge to currents $T^{\pm}_{\mathcal{Q}} \leq T^{\pm}$ such that $\mathbf{M}(T^{\pm} - T^{\pm}_{\mathcal{Q}}) \leq Cr^2$. Then we infer from [Duj04, Theorem 4.2] that if r is smaller than some $r(\varepsilon)$, then $\mathbf{M}(T^+ \wedge T^- - T^+_{\mathcal{Q}} \wedge T^-_{\mathcal{Q}}) \leq \varepsilon/2$. Furthermore, only transverse intersections account for the wedge product $T^+_{\mathcal{Q}} \wedge T^-_{\mathcal{Q}}$. If we denote by $\dot{\wedge}$ the geometric intersection product for curves, which consists in putting a Dirac mass at any proper intersection, without counting multiplicities, we have the weak convergence

$$T^+_{\mathcal{Q},k} \,\dot{\wedge}\, T^-_{\mathcal{Q},k} \to T^+_{\mathcal{Q}} \,\dot{\wedge}\, T^-_{\mathcal{Q}} = T^+_{\mathcal{Q}} \wedge T^-_{\mathcal{Q}},$$

where the last equality follows from [Duj04, Theorem 3.1]. Hence the mass of $T^+_{\mathcal{Q},k} \dot{\wedge} T^-_{\mathcal{Q},k}$ is larger than $1 - \varepsilon$ for k large enough, which was the result to be proved. □

Remark 7.3. Observe that the analytic argument implies that Δ intersects $f^{-k}(\Delta)$ transversely at $\sim d^k$ points. We claim that this implies

$$\#(\mathrm{Per}(f^{2k}) \cap \Delta) = d^k(1 + o(1)),$$

that is, most of $\Delta \cap f^{-k}(\Delta)$ is made of points of exact period $2k$. Indeed, assume that this is not the case. Then there exists $\varepsilon > 0$ and a sequence $k_j \to \infty$ such that εd^{k_j} of these points have a period which is a proper divisor N of d^{2k_j}, in particular $N \leq d^{k_j}$. Let F_j be this set of points and $\nu_j = d^{-k_j} \sum_{p \in F_j} \delta_p$. By [BLS93b], the measure equidistributed on $\mathrm{Fix}(f^{k_j})$ (which has cardinality d^{k_j}) converges to μ_f. Thus any cluster limit ν of (ν_j) has mass at least ε and satisfies $\nu \leq \mu_f$. It follows that μ_f gives a mass of at least ε to Δ, which is contradictory.

Appendix. A complement on the non-Archimedean Monge–Ampère operator

Let K be any non-trivially valued complete non-Archimedean field. We prove

Theorem A.1. *Suppose $L \to X$ is an ample line bundle over a smooth K-variety of dimension d. Pick any semipositive continuous metrics $|\cdot|_1, |\cdot|_2$ in the sense of Zhang. Then*

$$\mathbf{1}_{\{|\cdot|_1 < |\cdot|_2\}} c_1(L, \min\{|\cdot|_1, |\cdot|_2\})^d = \mathbf{1}_{\{|\cdot|_1 < |\cdot|_2\}} c_1(L, |\cdot|_1)^d. \qquad \text{(A.1)}$$

As in [BFJ15, §5] this result implies

Corollary A.2. *Suppose $L \to X$ is an ample line bundle over a smooth K-variety of dimension d. Pick any two semipositive continuous metrics $|\cdot|_1, |\cdot|_2$ in the sense of Zhang and suppose that they coincide on an open set Ω in the analytification of X in the sense of Berkovich. Then the positive measures $c_1(L, |\cdot|_1)^d$ and $c_1(L, |\cdot|_2)^d$ coincide in Ω.*

Recall that in the main body of the text, we deal with metrics $|\cdot|$ on $\mathcal{O}(1) \to \mathbb{P}^d$, and the evaluation of the constant section 1 on the analytification of the affine space $\mathbb{A}^d \subset \mathbb{P}^d$ defines a function $G := \log|1|$. With this identification, one has $\mathrm{MA}(G) = c_1(\mathcal{O}(1), |\cdot|)^d$.

Remark A.3. This corollary also follows from the approach to pluripotential theory on Berkovich spaces developed by Chambert-Loir and Ducros [CLD12] since their definition of the curvature of a metric is purely local.

Proof of Theorem A.1. Assume first that the metrics $|\cdot|_1, |\cdot|_2$ are model metrics. This means that we can find a model $\mathfrak{X}$ of X over $\mathrm{Spec}(\mathcal{O}_K)$ and nef line bundles $\mathfrak{L}_1, \mathfrak{L}_2$ over $\mathfrak{X}$ whose restriction to the generic fiber of $\mathfrak{X}$ is L.

Observe that by [Gub98, Lemma 7.8], $\min\{|\cdot|_1, |\cdot|_2\}$ is also a model metric (maybe in some other model of X).

In that case $c_1(L, \min\{|\cdot|_1, |\cdot|_2\})^d$ and $c_1(L, |\cdot|_1)^d$ are both atomic measures, supported on divisorial points corresponding to irreducible components of the special fiber.

If E is such a component for which $(|\cdot|_1/|\cdot|_2)(x_E) < 1$ then $(|\cdot|_1/|\cdot|_2)(x_F) \leq 1$ for all irreducible components F of the special fiber intersecting E. It follows that $\mathfrak{L}_1|_E = \mathfrak{L}_2|_E$ as numerical classes on E, and hence $c_1(L, \min\{|\cdot|_1, |\cdot|_2\})^d\{x_E\}$ and $c_1(L, |\cdot|_1)^d\{x_E\}$ by the definition of Monge–Ampère measures of model functions.

In the general case, we may assume that we have sequences of model metrics $|\cdot|_{i,n}$ on L such that $|\cdot|_{i,n} \to |\cdot|_i$ uniformly on X^{an}. Observe that $\Omega := \{|\cdot|_1 < |\cdot|_2\}$ is open since both metrics are continuous. It suffices to prove that

$$\int h\, c_1(L, \min\{|\cdot|_1, |\cdot|_2\})^d = \int h\, c_1(L, |\cdot|_1)^d$$

for all continuous functions h whose support is contained in Ω and such that $0 \leq h \leq 1$.

Pick $\varepsilon > 0$ small and rational and write $\Omega_n := \{|\cdot|_{1,n}e^{-\varepsilon} < |\cdot|_{2,n}|\}$. For $n \gg 0$, we have $\Omega \subseteq \Omega_n$. Since $|\cdot|_{1,n}e^{-\varepsilon}$ and $|\cdot|_{2,n}|$ are both model metrics, we know that $c_1(L, \min\{|\cdot|_{1,n}e^{-\varepsilon}, |\cdot|_{2,n}\})^d = c_1(L, |\cdot|_{1,n})^d$ on Ω_n by the previous step. Since h is supported in $\Omega \subset \Omega_n$, we get

$$\int hc_1(L, \min\{|\cdot|_{1,n}e^{-\varepsilon}, |\cdot|_{2,n}\})^d = \int hc_1(L, |\cdot|_{1,n}e^{-\varepsilon})^d = \int hc_1(L, |\cdot|_{1,n})^d$$

for all n, and we conclude by letting $n \to \infty$ and $\varepsilon \to 0$ and using [CL06, Proposition 2.7]. □

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