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Tame class field theory for singular varieties over finite fields

Abstract. Schmidt and Spieß described the abelian tame fundamental group of a smooth variety over a finite field by using Suslin homology. In this paper we show that their result generalizes to singular varieties if one uses Weil–Suslin homology instead.

Keywords. Class field theory, Suslin homology, Weil-etale cohomology

1. Introduction

In the 1980's, Kato and Saito (based on ideas of Bloch) generalized the class field theory for smooth, projective curves over finite fields to smooth, projective varieties of arbitrary dimension [KaSa]: The map from the free abelian group generated by the closed points which sends a generator $x \in X$ to the image of the Frobenius of $k(x)$ under $\pi_1^{\mathrm{ab}}(k(x)) \to \pi_1^{\mathrm{ab}}(X)$ factors through the Chow group of zero cycles, and induces an isomorphism

$$r_X : \mathrm{CH}_0(X) \xrightarrow{\sim} \pi_1^{\mathrm{ab}}(X)_{\mathrm{W}}.$$

Here $\pi_1^{\mathrm{ab}}(X)_{\mathrm{W}}$ is the subgroup of elements in $\pi_1^{\mathrm{ab}}(X)$ whose images in the absolute Galois group of the finite base field are integral Frobenius powers.

If X is not necessarily projective but still smooth, then Schmidt and Spieß [SS, Sc] showed that the same result still holds if one replaces the Chow group by Suslin homology and the fundamental group by its tame version: The reciprocity map induces an isomorphism of finitely generated abelian groups

$$r_X : H_0^S(X, \mathbb{Z}) \xrightarrow{\sim} \pi_1^{\mathrm{t,ab}}(X)_{\mathrm{W}}.$$

This result does not extend to nonsmooth schemes: the example of the node shows that r_X is neither injective nor surjective in general. If X is normal, then r_X is surjective but an example of Matsumi–Sato–Asakura [MAS] shows that r_X may have a nontrivial kernel.

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In this paper, we show that the result of Schmidt and Spieß can be generalized to singular varieties if one uses a refined version of Suslin homology on the one hand, and replaces the fundamental group by the enlarged fundamental group of [SGA3, X, §6] on the other hand. We denote the abelian enlarged tame fundamental group by $\Pi_1^{t,ab}$ in order to distinguish it from the usual abelian tame fundamental group $\pi_1^{t,ab}$, which is its profinite completion. The groups coincide if X geometrically unibranch (e.g., normal). Our first result is:

Theorem 1.1. *For any connected scheme X, separated and of finite type over a finite field, the pro-group $\Pi_1^{t,ab}(X)_W$ is isomorphic to a (constant) finitely generated abelian group.*

On the other hand, recall from [Ge3] the definition of Weil–Suslin homology: Let $\mathbb{F}$ be the finite base field, $F \in \mathrm{Gal}_{\mathbb{F}}$ the Frobenius automorphism and $G \cong \mathbb{Z}$ the subgroup of $\mathrm{Gal}_{\mathbb{F}}$ generated by F. For an abelian group A, the groups $H_i^{WS}(X, A)$ are defined as the homology of the cone of $1 - F^*$ on the Suslin complex tensored by A of the base change $\bar{X}$ of X to the algebraic closure $\bar{\mathbb{F}}$ of $\mathbb{F}$. By definition, there are short exact sequences

$$0 \to H_i^S(\bar{X}, A)_G \to H_i^{WS}(X, A) \to H_{i-1}^S(\bar{X}, A)^G \to 0.$$

Furthermore, for all i there are natural maps

$$H_i^S(X, A) \to H_{i+1}^{WS}(X, A).$$

If X is smooth and A is finite, then it follows from the proof of Kato's conjecture by Jannsen, Kerz and Saito [KeSa] (and under resolution of singularities) that these maps are isomorphisms (for $i = 0$ this follows from the theorem of Schmidt–Spieß).

We define a refined reciprocity homomorphism

$$\mathrm{rec}_X : H_1^{WS}(X, \mathbb{Z}) \to \Pi_1^{t,ab}(X)_W$$

such that the composite with the natural map $H_0^S(X, \mathbb{Z}) \to H_1^{WS}(X, \mathbb{Z})$ is the reciprocity map r_X described above. Our main result, conjectured in [Ge3], is the following

Theorem 1.2. *For any connected scheme X, separated and of finite type over a finite field $\mathbb{F}$, the homomorphism*

$$\mathrm{rec}_X : H_1^{WS}(X, \mathbb{Z}) \to \Pi_1^{t,ab}(X)_W$$

is surjective. The kernel of rec_X contains the maximal divisible subgroup of $H_1^{WS}(X, \mathbb{Z})$, and is equal to it if resolution of singularities for schemes of dimension $\leq \dim X + 1$ holds over $\mathbb{F}$.

As a corollary, we obtain (under resolution of singularities) an isomorphism of profinite completions

$$\mathrm{rec}_X^{\wedge} : H_1^{WS}(X, \mathbb{Z})^{\wedge} \xrightarrow{\sim} \pi_1^{t,ab}(X).$$

Under Parshin's conjecture [Ge3], $H_1^{WS}(X, \mathbb{Z})$ is finitely generated, hence rec_X would be an isomorphism.

2. The fundamental group and tame coverings

2.1. Etale and Weil-etale cohomology

Let X be a scheme, separated and of finite type over a finite field $\mathbb{F}$. The absolute Galois group $\mathrm{Gal}_\mathbb{F}$ acts on $\bar{X} = X \times_\mathbb{F} \bar{\mathbb{F}}$ via its action on $\bar{\mathbb{F}}$. Pulling an etale sheaf $\mathcal{F}$ on X back to $\bar{\mathcal{F}}$ on $\bar{X}$, we obtain a sheaf with a $\mathrm{Gal}_\mathbb{F}$-action [SGA7, XIII, §1.1]. By using this action, the etale cohomology $H^*_{\mathrm{et}}(X, \mathcal{F})$ can be calculated as the cohomology of $R\Gamma_{\mathrm{Gal}_\mathbb{F}} R\Gamma_{\mathrm{et}}(\bar{X}, \bar{\mathcal{F}})$. The *Weil-etale cohomology* of $\mathcal{F}$ is by definition the cohomology of $R\Gamma_G R\Gamma_{\mathrm{et}}(\bar{X}, \bar{\mathcal{F}})$, where $G \cong \mathbb{Z}$ is the subgroup of $\mathrm{Gal}_\mathbb{F}$ generated by the Frobenius. Since $R\Gamma_{\mathrm{Gal}_\mathbb{F}}$ and $R\Gamma_G$ coincide on discrete torsion modules, etale and Weil-etale cohomology coincide on torsion sheaves (cf. [Ge1, §2] for a more detailed account).

We can calculate the Weil-etale cohomology of a sheaf $\mathcal{F}$ as follows: Choose an injective resolution $\mathcal{F} \to \mathrm{I}^\bullet$. Then

$$H^i_{\mathrm{et}}(X, \mathcal{F}) = H^i(\mathrm{I}^\bullet(\bar{X})^G) \quad \text{and} \quad H^i_{\mathrm{W}}(X, A) = H^i\big(\mathrm{I}^\bullet(\bar{X}) \xrightarrow{1-F^*} \mathrm{I}^\bullet(\bar{X})\big),$$

where F^* is the pull-back along the Frobenius, and the left complex is considered to be in homological degree 0. We form the double complex by using the negative of the differential in the second complex, i.e., the differential of the total complex has the form

$$\mathrm{I}^i(\bar{X}) \oplus \mathrm{I}^{i-1}(\bar{X}) \to \mathrm{I}^{i+1}(\bar{X}) \oplus \mathrm{I}^i(\bar{X}), \quad (\alpha, \beta) \mapsto (d\alpha, \alpha - F^*\alpha - d\beta). \quad (1)$$

From the definition, we obtain short exact sequences

$$0 \to H^{i-1}_{\mathrm{et}}(\bar{X}, \mathcal{F})_G \to H^i_{\mathrm{W}}(X, \mathcal{F}) \to H^i_{\mathrm{et}}(\bar{X}, \mathcal{F})^G \to 0, \quad (2)$$

as well as a homomorphism

$$H^i_{\mathrm{et}}(X, \mathcal{F}) \to H^i_{\mathrm{W}}(X, \mathcal{F}) \quad (3)$$

for each $i \geq 0$, which is an isomorphism for $i = 0$ and injective for $i = 1$.

By [SV1, Thm. 10.2], etale and qfh-cohomology of a constant sheaf A coincide. Hence, in order to calculate Weil-etale cohomology of A, we can also work with an injective resolution $A \to \mathrm{I}^\bullet$ in the big qfh-site over $\mathbb{F}$. If moreover A is a $\mathbb{Z}/m$-module for some $m \geq 1$, then we can also work with a resolution of A by injective h-sheaves of $\mathbb{Z}/m$-modules [SV1, Thm. 10.7].

For a regular connected curve C over a field k we consider the subgroup $H^1_{\mathrm{t}}(C, A) \subseteq H^1_{\mathrm{et}}(C, A)$ of *tame* cohomology classes (corresponding to those continuous homomorphisms $\pi^{\mathrm{et}}_1(C) \to A$ which factor through the tame fundamental group $\pi^t_1(C', C' - C)$, where C' is the regular compactification of C).

For a general k-scheme X, we call a cohomology class in $a \in H^1_{\mathrm{et}}(X, A)$ *tame* if for any morphism $f : C \to X$ with C a regular curve, we have $f^*(a) \in H^1_{\mathrm{t}}(C, A)$. The tame cohomology classes form a subgroup

$$H^1_{\mathrm{t}}(X, A) \subseteq H^1_{\mathrm{et}}(X, A).$$

The groups coincide if X is proper, if $p = 0$, or if $p > 0$ and A is p-torsion free, where p is the characteristic of the base field k. If X is smooth with smooth compactification X', then $H^1_t(X, \mathbb{Z}/p^r) \cong H^1_{\text{et}}(X', \mathbb{Z}/p^r)$ for any $r \geq 1$ by [GS, Prop. 2.10].

For X separated and of finite type over the finite field $\mathbb{F}$, we define the *tame Weil-etale cohomology* to be the subgroup

$$H^1_{W,t}(X, A) \subseteq H^1_W(X, A)$$

of those elements whose image in $H^1_{\text{et}}(\bar{X}, A)$ in (2) is tame.

Recall that an abstract blow-up square is a cartesian diagram

$$\begin{array}{ccc} Z' & \xrightarrow{i'} & X' \\ \pi' \downarrow & & \downarrow \pi \\ Z & \xrightarrow{i} & X \end{array} \tag{4}$$

such that i is a closed embedding, π is proper, and π induces an isomorphism $(X' - Z')_{\text{red}} \xrightarrow{\sim} (X - Z)_{\text{red}}$.

Proposition 2.1. *If in the abstract blow-up square* (4), *π is finite, or if the abelian group A is torsion, then there is an exact sequence*

$$0 \to H^0_W(X, A) \to H^0_W(X', A) \oplus H^0_W(Z, A) \to H^0_W(Z', A)$$
$$\to H^1_{W,t}(X, A) \to H^1_{W,t}(X', A) \oplus H^1_{W,t}(Z, A) \to H^1_{W,t}(Z', A).$$

Proof. Let Sch/$\mathbb{F}$ be the category of separated schemes of finite type over $\mathbb{F}$. For $S \in$ Sch/$\mathbb{F}$ we denote by $\mathbb{Z}_h(S)$ the h-sheaf of abelian groups associated with the presheaf defined by $U \mapsto \mathbb{Z}[\text{Mor}_{\mathbb{F}}(U, S)]$.

For any finite field extension $\mathbb{F}'/\mathbb{F}$, the base changes of X, Z, Z', X' to $\mathbb{F}'$ form an abstract blow-up square in a natural way. By the same argument as in the proof of [Ge2, Prop. 3.2] (for the eh-topology) or [SV2, Lem. 12.1] (for the cdh-topology), we have an exact sequence of h-sheaves on Sch/$\mathbb{F}$,

$$0 \to \mathbb{Z}_h(Z'_{\mathbb{F}'}) \to \mathbb{Z}_h(Z_{\mathbb{F}'}) \oplus \mathbb{Z}_h(X'_{\mathbb{F}'}) \to \mathbb{Z}_h(X_{\mathbb{F}'}) \to 0. \tag{5}$$

If A is torsion, then etale and h-cohomology with values in A agree by [SV1, Thm. 10.2]. Applying the functor $R\,\text{Hom}_h(-, A)$ and passing to the limit over all $\mathbb{F}'/\mathbb{F}$, we obtain the exact triangle

$$R\Gamma(\bar{X}_{\text{et}}, A) \to R\Gamma(\bar{X}'_{\text{et}}, A) \oplus R\Gamma(\bar{Z}_{\text{et}}, A) \to R\Gamma(\bar{Z}'_{\text{et}}, A) \to R\Gamma(\bar{X}_{\text{et}}, A)[1] \tag{6}$$

and the long exact sequence

$$\to H^i_{\text{et}}(\bar{X}, A) \to H^i_{\text{et}}(\bar{X}', A) \oplus H^i_{\text{et}}(\bar{Z}, A) \to H^i_{\text{et}}(\bar{Z}', A) \to H^{i+1}_{\text{et}}(\bar{X}, A) \to . \tag{7}$$

If π is finite, we have the qfh-version of the exact sequence (5) and obtain (6) and (7) for arbitrary A, since etale and qfh-cohomology with values in any abelian group agree by

[SV1, Thm. 10.7]. Applying $R\Gamma_G$ to (6) and taking cohomology, we obtain the long exact sequence

$$H_{\mathrm{W}}^i(X, A) \to H_{\mathrm{W}}^i(X', A) \oplus H_{\mathrm{W}}^i(Z, A) \to H_{\mathrm{W}}^i(Z', A) \to H_{\mathrm{W}}^{i+1}(X, A) \to \cdots. \quad (8)$$

By [GS, Prop. 5.1], (7) induces an exact sequence

$$0 \to H_{\mathrm{et}}^0(\bar{X}, A) \to H_{\mathrm{et}}^0(\bar{X}', A) \oplus H_{\mathrm{et}}^0(\bar{Z}, A) \to H_{\mathrm{et}}^0(\bar{Z}', A)$$
$$\xrightarrow{\delta} H_{\mathrm{t}}^1(\bar{X}, A) \to H_{\mathrm{t}}^1(\bar{X}', A) \oplus H_{\mathrm{t}}^1(\bar{Z}, A) \to H_{\mathrm{t}}^1(\bar{Z}', A). \quad (9)$$

Comparing (9) with the sequences (7) and (8), we obtain the statement of the proposition by a diagram chase. $\qquad\square$

2.2. The enlarged fundamental group

We recall the definition of the enlarged fundamental group of [SGA3, X, §6]: Let X be a connected, locally noetherian scheme. For a group G (considered as a constant group scheme over X), a *G-torsor* P over X is a nonempty etale X-scheme P (i.e., $\pi : P \to X$ is unramified, flat and locally of finite type) with a G-action $P \times_X G \to P$ such that $P \times_X G \to P \times_X P$, $(x, g) \mapsto (x, xg)$, is an isomorphism. By [Mi, Prop. 2.7] (see also Ex. 2.6 there), for any etale sheaf F on X we have a Hochschild–Serre spectral sequence

$$E_2^{rs} = H^r(G, H_{\mathrm{et}}^s(P, \pi^*F)) \Longrightarrow H_{\mathrm{et}}^{r+s}(X, F).$$

For a geometric point ξ of X, one defines $\Pi^1(X, \xi, G)$ to be the set of isomorphism classes of G-torsors over X pointed over ξ. The trivial G-torsor gives a distinguished element in $\Pi^1(X, \xi, G)$. For a G-torsor P on X and a group homomorphism $f : G \to H$, consider

$$f_*(P) := (P \times H)/G,$$

where G acts by $(t, h) \cdot g = (tg, f(g^{-1})h)$. Then $f_*(P)$ is an H-torsor over X and we obtain a functor

$$G \mapsto \Pi^1(X, \xi, G)$$

from groups to pointed sets. By [SGA3, X, §6], this functor is pro-represented by the enlarged fundamental pro-group $\Pi_1(X, \xi)$, i.e.,

$$\Pi^1(X, \xi, G) \cong \mathrm{Hom}_{\mathrm{pro\text{-}grps}}(\Pi_1(X, \xi), G).$$

Explicitly, there is a pro-system of groups $\Pi_1(X, \xi) = (G_i)_{i \in I}$ with I filtering, and a G_i-torsor P_i corresponding to the projection map $\Pi_1(X, \xi) \to G_i$ for all i such that for any transition map $\alpha_{ij} : G_i \to G_j$ in the system we have $P_j = (\alpha_{ij})_*(P_i)$, and for any morphism $\Pi_1(X, \xi) \to H$ represented by $f : G_j \to H$ in the filtered colimit, the corresponding H-torsor is $f_*(P_j)$.

Next we define the enlarged tame fundamental group by extending the notion of curve-tameness from [KeSc] to the enlarged context. A G-torsor P over a regular connected curve C over a field k is called *tame* if the projection $P \to C$ extends to an at most

tamely ramified covering of the regular compactification C' of C. If X is any scheme, separated and of finite type over k, and P is a G-torsor on X, then we call P *tame* if its pull-back to the normalization of any curve on X is a tame torsor. If $G = A$ is abelian, then an A-torsor is tame if and only if its associated class in $H^1_{et}(X, A)$ lies in $H^1_t(X, A)$.

The functor

$$G \mapsto \Pi^{1,t}(X, \xi, G)$$

sending G to the set of isomorphism classes of pointed tame G-torsors on X is pro-represented by the enlarged tame fundamental group $\Pi^t_1(X, \xi)$, a quotient of $\Pi_1(X, \xi)$ in the category of pro-groups.

The abelianizations $\Pi^{ab}_1(X)$ and $\Pi^{t,ab}_1(X)$ of $\Pi_1(X, \xi)$ and $\Pi^t_1(X, \xi)$ represent the restrictions of the respective functors to the category of abelian groups and are independent of the chosen base point ξ.

Lemma 2.2. *For any abelian group A we have*

$$\mathrm{Hom}_{\text{pro-grps}}(\Pi^{ab}_1(X), A) \cong H^1_{et}(X, A),$$

and similarly for the tame version.

Proof. Write $\Pi^{ab}_1(X) = (A_i)$ and P_i for the A_i-torsor corresponding to the projection $\Pi^{ab}_1(X) \to A_i$. We obtain a filtered direct system of Hochschild–Serre spectral sequences

$$H^r(A_i, H^s_{et}(P_i, A)) \to H^{r+s}_{et}(X, A)$$

inducing a system of short exact sequences

$$0 \to H^1(A_i, A) \to H^1_{et}(X, A) \to H^0(A_i, H^1_{et}(P_i, A)).$$

The right map is the zero map in the colimit over all i, because if the A-torsor P arises from a map $f : A_i \to A$, i.e., $P = f_*(P_i)$, then P trivializes over P_i. Finally,

$$\mathrm{colim}\, H^1(A_i, A) \cong \mathrm{colim}\, \mathrm{Hom}(A_i, A) = \mathrm{Hom}_{\text{pro-grps}}(\Pi^{ab}_1(X), A). \qquad \square$$

From now on let X be a connected scheme, separated and of finite type over the finite field $\mathbb{F}$. As above, we denote by G the subgroup of $\mathrm{Gal}_\mathbb{F}$ consisting of all integral powers of the Frobenius automorphism.

Definition 2.3. The *enlarged Weil-tame fundamental group* $\Pi^t_1(X, \xi)_W$ is defined by the cartesian diagram of pro-groups

$$
\begin{array}{ccc}
\Pi^t_1(X, \xi)_W & \longrightarrow & \Pi^t_1(X, \xi) \\
\downarrow & & \downarrow \\
G & \longrightarrow & \Pi_1(\mathbb{F})
\end{array}
$$

The abelianization $\Pi_1^{\mathrm{t,ab}}(X)_{\mathrm{W}}$ of $\Pi_1^{\mathrm{t}}(X, \xi)_{\mathrm{W}}$ fits into the analogous cartesian diagram

$$
\begin{array}{ccc}
\Pi_1^{\mathrm{t,ab}}(X)_{\mathrm{W}} & \longrightarrow & \Pi_1^{\mathrm{t,ab}}(X) \\
\downarrow & & \downarrow \\
G & \longrightarrow & \Pi_1(\mathbb{F})
\end{array}
$$

The profinite completion of $\Pi_1^{\mathrm{t,ab}}(X)_{\mathrm{W}}$ is the usual abelian (curve-)tame fundamental group $\pi_1^{\mathrm{t,ab}}(X)$ of [KeSc].

Proposition 2.4. *For any abelian group A, there is a functorial isomorphism*

$$
\mathrm{Hom}_{\text{pro-grps}}(\Pi_1^{\mathrm{ab}}(X)_{\mathrm{W}}, A) \cong H_{\mathrm{W}}^1(X, A)
$$

compatible with the isomorphism of Lemma 2.2, and there is a similar isomorphism for the tame version.

Proof. Replacing $\mathbb{F}$ by its maximal algebraic extension in $\Gamma(X, \mathcal{O}_X)$ changes G to a subgroup of finite index, but does not change the groups on both sides of the statement. Hence we may assume that X is geometrically connected [DG, I, §4, 6.7]. Setting $\bar{X} = X \times_{\mathbb{F}} \bar{\mathbb{F}}$, we have the exact sequence of pro-groups

$$
1 \to \Pi_1(\bar{X}, \xi) \to \Pi_1(X, \xi)_{\mathrm{W}} \to G \to 1.
$$

If we write $\Pi_1(X, \xi)_{\mathrm{W}} = (G_i)$, then $\Pi_1(\bar{X}, \xi) = (\bar{G}_i)$ with $\bar{G}_i = \ker(G_i \to G)$. We denote by $\bar{P}_i$ the $\bar{G}_i$-torsor over $\bar{X}$ associated with the projection map $\Pi_1(\bar{X}, \xi) \to \bar{G}_i$.

Consider the functor $\mathcal{F} \mapsto \Gamma(\bar{P}_i, \mathcal{F})$ from the category of etale sheaves on X to G_i-modules. The inclusion $A \to R\Gamma(\bar{P}_i, A)$ induces a map $R\Gamma_{G_i}(A) \to R\Gamma_{G_i} R\Gamma(\bar{P}_i, A)$ in the derived category of abelian groups. Since $R\Gamma_{G_i} = R\Gamma_G R\Gamma_{\bar{G}_i}$, we can write this map in the form

$$
R\Gamma_{G_i}(A) \to R\Gamma_G R\Gamma_{\bar{G}_i} R\Gamma(\bar{P}_i, A). \tag{10}
$$

Since taking global sections over $\bar{P}_i$ has an exact left adjoint, it sends injectives sheaves on X to injective $\mathbb{Z}\bar{G}_i$-modules, and we obtain $R\Gamma_{\bar{G}_i} R\Gamma(\bar{P}_i, A) = R\Gamma(\bar{X}, A)$. We can thus write (10) in the form

$$
R\Gamma_{G_i}(A) \to R\Gamma_G R\Gamma(\bar{X}, A). \tag{11}
$$

Taking cohomology, and passing to the colimit over i, we obtain maps

$$
H^n(\Pi_1^{\mathrm{ab}}(X)_{\mathrm{W}}, A) \to H_{\mathrm{W}}^n(X, A), \quad n \geq 0.
$$

For $n = 1$ this is the map of the proposition and we have to show that it is an isomorphism. For this we rewrite (11) in the form

$$
R\Gamma_G R\Gamma_{\bar{G}_i}(A) \to R\Gamma_G R\Gamma(\bar{X}, A) \tag{12}
$$

and consider the map of the associated spectral sequences, which degenerate to short exact sequences. In degree 1, we obtain the commutative diagram with exact lines

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A_G & \longrightarrow & H^1(\Pi_1(X)_{\mathrm{W}}, A) & \to & H^1(\Pi_1(\bar{X}), A)^G & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^0_{\mathrm{et}}(\bar{X}, A)_G & \longrightarrow & H^1_{\mathrm{W}}(X, A) & \longrightarrow & H^1_{\mathrm{et}}(\bar{X}, A)^G & \longrightarrow & 0
\end{array}
$$

Since X is geometrically connected, the left vertical map is an isomorphism. The right vertical map is an isomorphism by Lemma 2.2. Hence the middle map is an isomorphism.

To show the statement for the tame variant, we note that there is a similar diagram to the one above for the tame groups. Indeed, a torsor on X is tame if and only if its base change to $\bar{X}$ is tame, so we obtain the exact sequence

$$
1 \to \Pi_1^{\mathrm{t}}(\bar{X}, \xi) \to \Pi_1^{\mathrm{t}}(X, \xi)_{\mathrm{W}} \to G \to 1.
$$

On the other hand, the bottom sequence of the above diagram induces a similar sequence for the tame Weil-etale cohomology by definition. This time, the right vertical map is an isomorphism by the tame version of Lemma 2.2. □

For a finite disjoint union $X = \coprod X_i$ of connected schemes we write by abuse of notation

$$
\Pi_1^{\mathrm{t,ab}}(X)_{\mathrm{W}} = \prod_i \Pi_1^{\mathrm{t,ab}}(X_i)_{\mathrm{W}}.
$$

Theorem 2.5. *For any X, separated and of finite type over a finite field $\mathbb{F}$, $\Pi_1^{\mathrm{t,ab}}(X)_{\mathrm{W}}$ is isomorphic to a finitely generated abelian group.*

Proof. First assume that X is normal and connected. We claim that the kernel

$$
\Pi_1^{\mathrm{t,ab}}(X)^{\mathrm{geo}} := \ker\big(\Pi_1^{\mathrm{t,ab}}(X)_{\mathrm{W}} \to \Pi_1(\mathbb{F})_{\mathrm{W}} \cong \mathbb{Z}\big)
$$

is a finite abelian group. If X is smooth, this follows from the main theorem of Schmidt–Spieß [SS, Sc]. For a general normal X, choose a dense open smooth subscheme $U \subset X$. Then, by [SGA1, V, Prop. 8], $\Pi_1^{\mathrm{t,ab}}(U)^{\mathrm{geo}}$ surjects onto $\Pi_1^{\mathrm{t,ab}}(X)^{\mathrm{geo}}$, hence the latter group is finite.

Now let X be arbitrary. We can assume that X is connected and reduced. Let $X' \to X$ be the normalization. The cokernel $\Pi_1^{\mathrm{ab}}(X'/X)$ of $\Pi_1^{\mathrm{ab}}(X') \to \Pi_1^{\mathrm{ab}}(X)$ represents the functor $\Pi_{\mathrm{ab}}^1(X'/X)$ which sends an abelian group A to the set of isomorphism classes of A-torsors on X which trivialize over X'. We denote the tame version of this group by $\Pi_1^{\mathrm{t,ab}}(X'/X)$ and the cokernel of $\Pi_1^{\mathrm{t,ab}}(X')_{\mathrm{W}} \to \Pi_1^{\mathrm{t,ab}}(X)_{\mathrm{W}}$ by C. Consider the diagram

$$
\begin{array}{ccccccc}
\Pi_1^{\mathrm{ab}}(X') & \longrightarrow & \Pi_1^{\mathrm{ab}}(X) & \longrightarrow & \Pi_1^{\mathrm{ab}}(X'/X) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \\
\Pi_1^{\mathrm{t,ab}}(X') & \longrightarrow & \Pi_1^{\mathrm{t,ab}}(X) & \longrightarrow & \Pi_1^{\mathrm{t,ab}}(X'/X) & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow{\scriptstyle \beta} & & \\
\Pi_1^{\mathrm{t,ab}}(X')_{\mathrm{W}} & \longrightarrow & \Pi_1^{\mathrm{t,ab}}(X)_{\mathrm{W}} & \longrightarrow & C & \longrightarrow & 0
\end{array}
$$

Since $X' \to X$ is proper, a torsor on X which trivializes over X' is tame. Hence α is an isomorphism, and so is β. By [SGA3, X §6, p. 109], $\Pi_1^{ab}(X'/X)$ is a finitely generated abelian group, hence so is C. We have proved above that the geometric part of $\Pi_1^{t,ab}(X')_W$ (defined componentwise if X' is not connected) is finite. This implies that $\Pi_1^{t,ab}(X)_W$ is constant and finitely generated. $\qquad\square$

3. Weil–Suslin homology

Let k be a perfect field and X a scheme, separated and of finite type over k. We recall that, for a smooth k-scheme T, the group $\mathrm{Cor}(T, X)$ of finite correspondences is the free abelian group generated by closed integral $Z \subseteq T \times X$ which are finite and surjective over a component of T. The *Suslin complex* of X is the complex $C_\bullet(X) = \mathrm{Cor}(\Delta^\bullet, X)$, where $\Delta^i = \mathrm{Spec}\, k[T_0, \ldots, T_i]/(\sum T_i - 1)$. Set $\partial := \sum_{j=0}^{i}(-1)^j \delta_i^j \in \mathrm{Cor}(\Delta^{i-1}, \Delta^i)$, where $\delta_i^j : \Delta^{i-1} \to \Delta^i$, $j = 0, \ldots, i$, are the face maps. Then the differential $\mathrm{Cor}(\Delta^i, X) \to \mathrm{Cor}(\Delta^{i-1}, X)$ is the composition of the correspondences $x \mapsto x \circ \partial$. The following lemma is easy to check from the definitions:

Lemma 3.1. *Let $f : X \to Y$ be a morphism of schemes.*

(a) *If X and T are smooth and $c \in \mathrm{Cor}(T, X)$, then $(\mathrm{id}_T \times f)_* c = f \circ c$. Here the left hand side is a push-forward of cycles, and the right hand side is a composition of correspondences.*

(b) *If X and Y are smooth and $d \in \mathrm{Cor}(Y, Z)$, then $(f \times \mathrm{id}_Z)^* d = d \circ f$. Here the left hand side is a pull-back of cycles, and the right hand side is a composition of correspondences.*

(c) *If f is an automorphism of the smooth scheme X, then $f^* c = f_*^{-1} c$ for any cycle c.*

Let T and X be separated schemes of finite type over k and $\sigma \in \mathrm{Gal}(\bar{k}/k)$. Then σ acts on $\bar{X} = X \times_k \bar{k}$ via its action on $\bar{k}$, and on algebraic cycles by pull-back.

Lemma 3.2. *The action of σ on $\mathrm{Cor}_{\bar{k}}(\bar{T}, \bar{X})$ induced by pull-back of algebraic cycles sends a correspondence α to the composition $\sigma_X^{-1} \alpha \sigma_T$, where σ_X and σ_T are the automorphisms of $\bar{X}$ and $\bar{T}$ induced by σ. In other words, the following diagram of correspondences commutes:*

$$
\begin{array}{ccc}
\bar{T} & \xrightarrow{\sigma_T} & \bar{T} \\
{\scriptstyle\sigma^*\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\
\bar{X} & \xrightarrow{\sigma_X} & \bar{X}
\end{array}
\tag{13}
$$

Proof. From Lemma 3.1, we have

$$
\begin{aligned}
\sigma_X \circ \sigma^* \alpha &= (\mathrm{id}_T \times \sigma_X)_* (\sigma_T \times \sigma_X)^* \alpha \\
&= (\mathrm{id}_T \times \sigma_X^{-1})^* (\mathrm{id}_T \times \sigma_X)^* (\sigma_T \times \mathrm{id}_X)^* \alpha = \alpha \circ \sigma_T. \qquad\square
\end{aligned}
$$

Now we assume that $k = \mathbb{F}$ is a finite field. Let $\bar{X}$ be the extension to the algebraic closure $\bar{\mathbb{F}}$, and let F be the Frobenius automorphism of $\bar{\mathbb{F}}/\mathbb{F}$. Let $G \cong \mathbb{Z}$ be the Weil group of $\mathbb{F}$, generated by the Frobenius F.

Definition 3.3. *Weil–Suslin homology* $H_i^{\mathrm{WS}}(X, A)$ with coefficients in the abelian group A is defined as the homology of the cone of

$$C_\bullet(\bar{X}) \otimes A \xrightarrow{1-F^*} C_\bullet(\bar{X}) \otimes A,$$

i.e., the total complex of the double complex

$$
\begin{array}{ccccccc}
\cdots \longrightarrow & C_2(\bar{X}) \otimes A & \xrightarrow{-\partial} & C_1(\bar{X}) \otimes A & \xrightarrow{-\partial} & C_0(\bar{X}) \otimes A \\
 & \downarrow{\scriptstyle 1-F^*} & & \downarrow{\scriptstyle 1-F^*} & & \downarrow{\scriptstyle 1-F^*} \\
\cdots \longrightarrow & C_2(\bar{X}) \otimes A & \xrightarrow{\partial} & C_1(\bar{X}) \otimes A & \xrightarrow{\partial} & C_0(\bar{X}) \otimes A
\end{array}
$$

In degree i, the total complex consists of all elements

$$(x_i, x_{i-1}) \in \mathrm{Cor}(\bar{\Delta}^i, \bar{X}) \otimes A \oplus \mathrm{Cor}(\bar{\Delta}^{i-1}, \bar{X}) \otimes A$$

with differential

$$(x, y) \mapsto (x\partial + y - F^{-1}yF, -y\partial). \tag{14}$$

The spectral sequence for double complexes gives short exact sequences

$$0 \to H_i^{\mathrm{S}}(\bar{X}, A)_G \to H_i^{\mathrm{WS}}(X, A) \to H_{i-1}^{\mathrm{S}}(\bar{X}, A)^G \to 0 \tag{15}$$

where the left hand side and right hand side are the coinvariants and invariants with respect to G, respectively. The map $C_\bullet(X) \to C_\bullet(\bar{X})$, sending a generator $Z \subseteq X \times \Delta^i$ to its pull-back to the algebraic closure, has image in the kernel of $1 - F^*$. Therefore, we obtain natural maps, for $i \geq 0$,

$$H_i^{\mathrm{S}}(X, A) \to H_{i+1}^{\mathrm{WS}}(X, A). \tag{16}$$

Remark 3.4. For a torsion $\mathrm{Gal}_\mathbb{F}$-module M, we have $R\Gamma(\mathbb{F}, M) \cong [M \xrightarrow{1-F^*} M]$, where the last complex is concentrated in (cohomological) degrees zero and one. Hence, if A is a torsion group, then

$$H_i^{\mathrm{WS}}(X, A) = H^{1-i}\big(R\Gamma(\mathbb{F}, C_\bullet(\bar{X}) \otimes A)\big).$$

Remark 3.5. The definition of Weil–Suslin homology depends on the finite base field $\mathbb{F}$ (via $F \in \mathrm{Gal}_\mathbb{F}$). However, if $X \to \mathbb{F}$ factors through $\mathbb{F}'/\mathbb{F}$, then the Weil–Suslin homology of X does not depend on whether we consider X as a scheme over $\mathbb{F}'$ or over $\mathbb{F}$.

Proposition 3.6. *An abstract blow-up diagram* (4) *induces a long exact sequence of Weil–Suslin homology groups*

$$
\begin{aligned}
H_1^{\mathrm{WS}}(Z', A) &\to H_1^{\mathrm{WS}}(X', A) \oplus H_1^{\mathrm{WS}}(Z, A) \to H_1^{\mathrm{WS}}(X, A) \\
&\to H_0^{\mathrm{WS}}(Z', A) \to H_0^{\mathrm{WS}}(X', A) \oplus H_0^{\mathrm{WS}}(Z, A) \to H_0^{\mathrm{WS}}(X, A) \to 0.
\end{aligned}
$$

Proof. By definition of Weil–Suslin homology, only the terms $C_i(\bar{?})$ for $i \leq 2$ are involved in the definition of the terms in the sequence, hence a diagram chase shows that it suffices to show that in the short exact sequence of complexes

$$0 \to C_\bullet(\bar{Z}') \to C_\bullet(\bar{X}') \oplus C_\bullet(\bar{Z}) \to C_\bullet(\bar{X}) \to K_\bullet \to 0$$

one has $H_i(K_\bullet) = 0$ for $i \leq 2$. This was shown in [GS, proof of Prop. 5.2]. □

Since $H_0^S(\bar{\mathbb{F}}, \mathbb{Z}) = \mathbb{Z}$ and $H_i^S(\bar{\mathbb{F}}, \mathbb{Z}) = 0$ for $i \geq 1$, we can calculate the Weil–Suslin homology of the point by using (15) as follows.

Example 3.7.

$$H_i^{\mathrm{WS}}(\mathbb{F}, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & i = 0, 1, \\ 0, & i \geq 2. \end{cases}$$

Let C be a smooth, proper, geometrically connected curve over $\mathbb{F}$. By [Li], we have

$$H_i^S(\bar{C}, \mathbb{Z}) \cong \begin{cases} \mathrm{Pic}(\bar{C}), & i = 0, \\ \bar{\mathbb{F}}^\times, & i = 1, \\ 0, & i \geq 2. \end{cases}$$

Recalling $\bar{\mathbb{F}}_G^\times = 0$ and $\mathrm{Pic}(\bar{C})_G \cong \mathbb{Z}$, the exact sequence (15) yields the following:

Example 3.8. Let C be a smooth, proper, geometrically connected curve over $\mathbb{F}$. Then

$$H_i^{\mathrm{WS}}(C, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & i = 0, \\ \mathrm{Pic}(C), & i = 1, \\ \mathbb{F}^\times, & i = 2, \\ 0, & i \geq 3. \end{cases}$$

The following improves [Ge3, Prop. 7.8]:

Proposition 3.9. *Let X be a connected, separated scheme of finite type over $\mathbb{F}$. Then the structure map induces an isomorphism*

$$\deg : H_0^{\mathrm{WS}}(X, \mathbb{Z}) \xrightarrow{\sim} H_0^{\mathrm{WS}}(\mathbb{F}, \mathbb{Z}) \cong \mathbb{Z}.$$

Proof. We have $H_0^{\mathrm{WS}}(X, \mathbb{Z}) \cong H_0^S(\bar{X}, \mathbb{Z})_G$, hence deg is surjective, and it remains to show that its kernel is trivial. Since elements in $H_0^S(\bar{X}, \mathbb{Z})$ are represented by zero-cycles, any element of $H_0^{\mathrm{WS}}(X, \mathbb{Z})$ comes by push-forward from $H_0^{\mathrm{WS}}(C, \mathbb{Z})$ for some connected curve $C \subset X$ (use, e.g., [Mu, II, §6, Lemma]). We can therefore assume that $X = C$ is a connected curve.

If $C' \to C$ is finite and surjective, then $H_0^{\mathrm{WS}}(C', \mathbb{Z}) \to H_0^{\mathrm{WS}}(C, \mathbb{Z})$ is surjective. Moreover, any element of degree zero in $H_0^{\mathrm{WS}}(\bar{C}, \mathbb{Z})$ can be lifted to an element in the kernel of the multi-degree map

$$H_0^{\mathrm{WS}}(C', \mathbb{Z}) \to \mathbb{Z}^{\pi_0(C')}.$$

We can therefore assume that C is a normal, connected curve. Moreover, we can assume that C has an $\mathbb{F}$-rational point (use Remark 3.5). Let $\mathcal{A}$ be the semi-abelian Albanese variety of C. Then, by [SV1, Thm. 3.1] (and [Li] if C is proper), the degree zero part of $H_0^S(\bar{C}, \mathbb{Z})$ is isomorphic to $\mathcal{A}(\bar{\mathbb{F}})$. The G-coinvariants of this group are both finite and divisible, hence trivial. $\square$

Corollary 3.10. *The canonical injection*

$$H_1^{WS}(X, \mathbb{Z})/m \to H_1^{WS}(X, \mathbb{Z}/m)$$

is an isomorphism for any $m \geq 1$.

Proof. The cokernel is isomorphic to $\ker(H_0^{WS}(X, \mathbb{Z}) \xrightarrow{\cdot m} H_0^{WS}(X, \mathbb{Z})) = 0$. $\square$

4. Duality

We say that *resolution of singularities holds for schemes of dimension $\leq d$ over a perfect field k* if the following two conditions are satisfied.

(1) For any integral separated scheme of finite type X of dimension $\leq d$ over k there exists a projective birational morphism $Y \to X$ with Y smooth over k which is an isomorphism over the regular locus of X.
(2) For any integral smooth scheme X of dimension $\leq d$ over k and any birational proper morphism $Y \to X$ there exists a tower of morphisms $X_n \to X_{n-1} \to \cdots \to X_0 = X$ such that $X_n \to X_{n-1}$ is a blow-up with a smooth center for $i = 1, \ldots, n$, and the composite morphism $X_n \to X$ factors through $Y \to X$.

In this paper, we do not use the full duality statement below, but only the equality of the orders of the respective groups.

Theorem 4.1. *Let X be separated and of finite type over the finite field $\mathbb{F}$ of characteristic p. If m is prime to p, then there is a perfect pairing of finite groups*

$$H_i^{WS}(X, \mathbb{Z}/m) \times H_{et}^i(X, \mathbb{Z}/m) \to \mathbb{Z}/m.$$

If X is smooth and resolution of singularities for schemes of dimension $\leq \dim X + 1$ holds over $\mathbb{F}$, then there is a perfect pairing of finite groups for any $r \geq 1$,

$$H_1^{WS}(X, \mathbb{Z}/p^r) \times H_t^1(X, \mathbb{Z}/p^r) \to \mathbb{Z}/p^r.$$

Proof. By [Ge3, Thms. 5.4 and 5.5], we have a perfect pairing of finite groups

$$H_{i-1}^{GS}(X, \mathbb{Z}/m) \times H_{et}^i(X, \mathbb{Z}/m) \to \mathbb{Z}/m,$$

where H_{i-1}^{GS} is Galois–Suslin homology. By [Ge3, §7.1], we have $H_{i-1}^{GS}(X, \mathbb{Z}/m) \cong H_i^{WS}(X, \mathbb{Z}/m)$, showing the first statement (note that H_i^{WS} is denoted by H_i^{ar} in [Ge3]).

For the second statement, let X' be a smooth, proper variety containing X as a dense open subscheme. Then, by [GS, Prop. 6.2], we have $H_i^S(\bar{X}, \mathbb{Z}/p^r) \cong H_i^S(\bar{X}', \mathbb{Z}/p^r)$ for $i = 0, 1$, and the exact sequence (15) above implies that

$$H_1^{\mathrm{WS}}(X, \mathbb{Z}/p^r) \cong H_1^{\mathrm{WS}}(X', \mathbb{Z}/p^r).$$

Furthermore, $H_t^1(X, \mathbb{Z}/p^r) \cong H_{\mathrm{et}}^1(X', \mathbb{Z}/p^r)$ by [GS, Prop. 2.10]. Hence we may assume that $X = X'$ is smooth and proper.

Let $\mathbb{Z}_X^c$ be the complex of etale sheaves on X which associates to $U \to X$ the Bloch complex $z_0(U, \bullet)$. Then, for smooth proper X, $H_{\mathrm{et}}^i(X, \mathbb{Z}/p^r)$ is dual to $\mathrm{Ext}_X^{2-i}(\mathbb{Z}/p^r, \mathbb{Z}_X^c)$ by [Ge4, Thm. 5.1]. Furthermore,

$$
\begin{aligned}
\mathrm{Ext}_X^{2-i}(\mathbb{Z}/p^r, \mathbb{Z}_X^c) &\cong \mathrm{Ext}_{X,\mathbb{Z}/p^r}^{1-i}(\mathbb{Z}/p^r, \mathbb{Z}_X^c/p^r) \quad \text{(by [Ge4, Lem. 2.4])} \\
&\cong H_{\mathrm{et}}^{1-i}(X, \mathbb{Z}_X^c/p^r) \\
&\cong H^{1-i}\big(R\Gamma(\mathbb{F}, R\Gamma(\bar{X}_{\mathrm{et}}, \mathbb{Z}_{\bar{X}}^c/p^r))\big) \\
&\cong H^{1-i}\big(R\Gamma(\mathbb{F}, \mathbb{Z}_{\bar{X}}^c/p^r(\bar{X}))\big) \quad \text{(by [Ge4, Thm. 3.1])}.
\end{aligned}
$$

The natural map

$$C_\bullet(\bar{X}) \otimes \mathbb{Z}/p^r \to \mathbb{Z}_{\bar{X}}^c/p^r(\bar{X})$$

induces an isomorphism on H^j for $j = -1, 0$ by [SS, Thm. 2.7] (the assumption $\dim X \leq 2$ is unnecessary and not used in the proof). Hence for $i = 1$ we obtain

$$
\begin{aligned}
H^0\big(R\Gamma(\mathbb{F}, \mathbb{Z}_{\bar{X}}^c/p^r(\bar{X}))\big) &\cong H^0\big(R\Gamma(\mathbb{F}, C_\bullet(\bar{X}) \otimes \mathbb{Z}/p^r)\big) \\
&\cong H_1^{\mathrm{WS}}(X, \mathbb{Z}/p^r) \quad \text{(by Remark 3.4)}.
\end{aligned}
$$

This concludes the proof. $\qquad\square$

5. The reciprocity map

For any X and any abelian group A, we construct a functorial pairing

$$H_1^{\mathrm{WS}}(X, \mathbb{Z}) \times H_{\mathrm{W,t}}^1(X, A) \xrightarrow{\langle\,,\,\rangle} A, \tag{17}$$

which induces natural maps

$$\mathrm{Hom}(\Pi_1^{\mathrm{t,ab}}(X)_{\mathrm{W}}, A) \cong H_{\mathrm{W,t}}^1(X, A) \to \mathrm{Hom}(H_1^{\mathrm{WS}}(X, \mathbb{Z}), A)$$

for any abelian (pro-)group A, hence the Yoneda lemma induces

$$\mathrm{rec}_X : H_1^{\mathrm{WS}}(X, \mathbb{Z}) \to \Pi_1^{\mathrm{t,ab}}(X)_{\mathrm{W}}. \tag{18}$$

The pairing (17) should satisfy two conditions. First, the composite of rec_X with the natural map $H_0^S(X, \mathbb{Z}) \to H_1^{\mathrm{WS}}(X, \mathbb{Z})$ should be the map

$$r_X : H_0^S(X, \mathbb{Z}) \to \Pi_1^{\mathrm{t,ab}}(X)_{\mathrm{W}}$$

which sends the class $[x]$ of a closed point $x \in X$ to its Frobenius automorphism in $\Pi_1^{t,ab}(X)_W$.

Secondly, we want (17) to be compatible with the pairing over $\bar{\mathbb{F}}$ considered in [GS]: there we considered (over any algebraically closed field) a natural pairing

$$H_1^S(\bar{X}, \mathbb{Z}) \times H_t^1(\bar{X}, A) \to A, \tag{19}$$

which is defined by pulling back torsors along finite correspondences. For $A = \mathbb{Z}/m$ with $(m, \mathrm{char}(\bar{\mathbb{F}})) = 1$, the induced homomorphism

$$\mathrm{Hom}(H_1^S(\bar{X}, \mathbb{Z}), \mathbb{Z}/m) \to H_{et}^1(\bar{X}, \mathbb{Z}/m)$$

coincides with the composite of $\mathrm{Hom}(H_1^S(\bar{X}, \mathbb{Z}), \mathbb{Z}/m) \overset{can}{\hookrightarrow} H_S^1(\bar{X}, \mathbb{Z}/m)$ with the Suslin–Voevodsky comparison isomorphism [SV1, Cor. 7.8]

$$H_S^1(\bar{X}, \mathbb{Z}/m) \overset{SV}{\underset{\sim}{\to}} H_{et}^1(\bar{X}, \mathbb{Z}/m).$$

We want to construct the pairing (17) in such a way that the diagram

$$
\begin{array}{ccccccc}
H_1^S(\bar{X}, \mathbb{Z}) & \times & H_t^1(\bar{X}, A) & \overset{(19)}{\longrightarrow} & A \\
\downarrow & & \uparrow & & \| \\
H_1^{WS}(X, \mathbb{Z}) & \times & H_t^1(X, A) & \overset{(17)}{\longrightarrow} & A
\end{array}
$$

commutes. In [GS, §4] an explicit interpretation of (19) in terms of qfh-sheaves is given. This motivates the following construction of the pairing (17).

Let A be an abelian group and $A \to I^\bullet$ an injective resolution of the constant sheaf A in the category of qfh-sheaves. An element of $H_{W,t}^1(X, A)$ is represented by a pair

$$(\alpha, \beta) \in I^1(\bar{X}) \oplus I^0(\bar{X}) \quad \text{with} \quad d\alpha = 0, \ [\alpha] \in H_t^1(\bar{X}, A) \text{ and } d\beta = \alpha - F^*\alpha.$$

An element in $H_1^{WS}(X, \mathbb{Z})$ is represented by a pair

$$(x, y) \in \mathrm{Cor}(\bar{\Delta}^1, \bar{X}) \oplus \mathrm{Cor}(\bar{\Delta}^0, \bar{X}) \quad \text{with} \quad x\partial = F^{-1}yF - y.$$

Since $H_t^1(\bar{\Delta}^1, A) = 0 = H_{et}^1(\bar{\Delta}^0, A)$, we can find $s \in I^0(\bar{\Delta}^1)$ with $ds = x^*F^*\alpha \in I^1(\bar{\Delta}^1)$ and $t \in I^0(\bar{\Delta}^0)$ with $dt = y^*\alpha \in I^1(\bar{\Delta}^0)$. Then

$$\langle (x, y), (\alpha, \beta) \rangle := F^*t - t - \partial^*s + y^*\beta \tag{20}$$

lies in

$$A = H_{et}^0(\bar{\Delta}^0, A) = \ker(I^0(\bar{\Delta}^0) \overset{d}{\to} I^1(\bar{\Delta}^0)).$$

Indeed, we have

$$d(F^*t - t - \partial^*s + y^*\beta) = F^*y^*\alpha - y^*\alpha - \partial^*x^*F^*\alpha + y^*(\alpha - F^*\alpha)$$

$$= F^*y^*\alpha - (F^*y^*F^{*-1} - y^*)(F^*\alpha) - y^*F^*\alpha = 0.$$

One checks without difficulty that $\langle (x, y), (\alpha, \beta) \rangle$ does not depend on the choices of s and t.

Lemma 5.1. $\langle (x, y), (\alpha, \beta) \rangle \in A$ *only depends on the class of* (x, y) *in* $H_1^{\mathrm{WS}}(X, \mathbb{Z})$ *and on the class of* (α, β) *in* $H_{\mathrm{W,t}}^1(X, A)$.

Proof. For $\gamma \in \mathrm{I}^0(\bar{X})$, $s = x^* F^* \gamma$ and $t = y^* \gamma$ satisfy the condition and we have

$$\langle (x, y), (d\gamma, \gamma - F^*\gamma) \rangle = F^* t - t - \partial^* s + y^*(\gamma - F^*\gamma) = F^* y^* \gamma - y^* \gamma - (x\delta)^* F^* \gamma$$
$$= F^* y^* \gamma - y^* \gamma - F^* y^* \gamma + y^* F^* \gamma = 0.$$

Let $(u, v) \in \mathrm{Cor}(\bar{\Delta}^2, \bar{X}) \oplus \mathrm{Cor}(\bar{\Delta}^1, \bar{X})$. Since $H_{\mathrm{t}}^1(\bar{\Delta}^2, A) = 0 = H_{\mathrm{t}}^1(\bar{\Delta}^1, A)$, we can find $\sigma \in \mathrm{I}^0(\bar{\Delta}^2)$ with $d\sigma = u^* F^* \alpha$ and $\tau \in \mathrm{I}^0(\bar{\Delta}^1)$ with $d\tau = -v^*\alpha$. Then $s = \partial^* \sigma + F^* \tau - \tau - v^* \beta$ and $t = \partial^* \tau$ satisfy

$$ds = (u\partial + v - F^{-1} v F) F^* \alpha, \quad dt = (-v\partial)^* \alpha$$

and

$$\langle (u\partial + v - F^{-1} v F, -v\partial), (\alpha, \beta) \rangle = F^* t - t - \partial^* s - (v\partial)^* \beta$$
$$= F^* \partial^* \tau - \partial^* \tau - \partial^* (\partial + \sigma^* F^* \tau - \tau - v^* \beta) - \partial^* v^* \beta = 0. \qquad \square$$

By Lemma 5.1, we obtain the pairing (17) and the reciprocity map (18).

Lemma 5.2. *If* $f : X \to Y$ *is an* $\mathbb{F}$-*morphism, then the diagram*

$$
\begin{array}{ccccc}
H_1^{\mathrm{WS}}(X, \mathbb{Z}) & \times & H_{\mathrm{W,t}}^1(X, A) & \xrightarrow{\langle \, , \, \rangle_X} & A \\
f_* \downarrow & & f^* \uparrow & & \parallel \\
H_1^{\mathrm{WS}}(Y, \mathbb{Z}) & \times & H_{\mathrm{W,t}}^1(Y, A) & \xrightarrow{\langle \, , \, \rangle_Y} & A
\end{array}
$$

commutes, hence $\mathrm{rec}_X : H_1^{\mathrm{WS}}(X, \mathbb{Z}) \to \Pi_1^{\mathrm{t,ab}}(X)_{\mathrm{W}}$ *is functorial in* X.

Proof. Let $A \to \mathrm{I}^\bullet$ an injective resolution of the constant sheaf A in the category of qfh-sheaves. Let $c \in H_{\mathrm{W,t}}^1(Y, A)$ be represented by

$$(\alpha, \beta) \in \mathrm{I}^1(\bar{Y}) \oplus \mathrm{I}^0(\bar{Y}) \quad \text{with} \quad d\alpha = 0, [\alpha] \in H_{\mathrm{t}}^1(\bar{Y}, A) \text{ and } d\beta = \alpha - F^*\alpha.$$

Furthermore, let $\zeta \in H_1^{\mathrm{WS}}(X, \mathbb{Z})$ be represented by

$$(x, y) \in \mathrm{Cor}(\bar{\Delta}^1, \bar{X}) \oplus \mathrm{Cor}(\bar{\Delta}^0, \bar{X}) \quad \text{with} \quad x\partial = F^{-1} y F - y.$$

We have to show that $\langle \zeta, f^*(c) \rangle_X = \langle f_*(\zeta), c \rangle_Y$. This follows directly from the construction: First note that $f_*(\zeta)$ is represented by $(f \circ x, f \circ y)$. Choose $s \in \mathrm{I}^0(\bar{\Delta}^1)$ with

$$ds = (f \circ x)^* F^* \alpha \in \mathrm{I}^1(\bar{\Delta}^1)$$

and $t \in \mathrm{I}^0(\bar{\Delta}^0)$ with $dt = (f \circ y)^* \alpha \in \mathrm{I}^1(\bar{\Delta}^0)$. Then

$$\langle (f \circ x, f \circ y), (\alpha, \beta) \rangle_Y = F^* t - t - \partial^* s + (f \circ y)^* \beta$$

by definition. Since $ds = x^*F^*(f^*(\alpha))$ and $dt = y^*(f^*(\alpha))$, we obtain

$$\langle (x, y), f^*(\alpha, \beta) \rangle_X = F^*t - t - \partial^*s + y^*(f^*(\beta)) = F^*t - t - \partial^*s + (f \circ y)^*\beta$$
$$= \langle (f \circ x, f \circ y), (\alpha, \beta) \rangle_Y,$$

showing the assertion. □

Proposition 5.3. *The composite of* rec_X *with the natural map* $H_0^{\mathrm{S}}(X, \mathbb{Z}) \to H_1^{\mathrm{WS}}(X, \mathbb{Z})$ *is the map*

$$r_X : H_0^{\mathrm{S}}(X, \mathbb{Z}) \to \Pi_1^{\mathrm{t,ab}}(X)_{\mathrm{W}}$$

which sends the class $[x]$ *of a closed point* $x \in X$ *to its Frobenius automorphism in* $\Pi_1^{\mathrm{t,ab}}(X)_{\mathrm{W}}$.

Proof. By functoriality, it suffices to consider the case $X = \Delta^0$. In this case, we have natural identifications $\mathbb{Z} = H_0^{\mathrm{S}}(\Delta^0, \mathbb{Z}) = H_1^{\mathrm{WS}}(\Delta^0, \mathbb{Z})$ (sending $1 \in \mathbb{Z}$ to $\mathrm{id}_{\Delta^0} \in \mathrm{Cor}(\Delta^0, \Delta^0)$), and for any abelian group A, we have $A = H_{\mathrm{et}}^0(\bar{\Delta}^0, A)_G = H_{\mathrm{W,t}}^1(\Delta^0, A)$. With respect to these identifications, the pairing (17) is just multiplication $\mathbb{Z} \times A \to A$, $(n, a) \mapsto na$. Furthermore, the isomorphism of Proposition 2.4,

$$A = H_{\mathrm{W,t}}^1(\Delta^0, A) \xrightarrow{\sim} \mathrm{Hom}(\Pi_1^{\mathrm{t,ab}}(\Delta^0)_{\mathrm{W}}, A) = \mathrm{Hom}(G, A),$$

maps $a \in A$ to the homomorphism $G \to A$ which sends the Frobenius $F \in G \cong \mathbb{Z}$ to a. By using all this, the statement of the proposition follows from the definition of the reciprocity map. □

6. Comparison of blow-up sequences

If $A = \mathbb{Z}/m$, then Weil-etale and etale cohomology agree, and by Corollary 3.10, the pairing (17) induces a pairing

$$H_1^{\mathrm{WS}}(X, \mathbb{Z}/m) \times H_{\mathrm{t}}^1(X, \mathbb{Z}/m) \to \mathbb{Z}/m, \tag{21}$$

and hence a map

$$\Phi_X^1 : H_1^{\mathrm{WS}}(X, \mathbb{Z}/m) \to H_{\mathrm{t}}^1(X, \mathbb{Z}/m)^*, \tag{22}$$

which is the mod m version of rec_X.

In addition, we consider the pairing

$$H_0^{\mathrm{WS}}(X, \mathbb{Z}/m) \times H_{\mathrm{et}}^0(X, \mathbb{Z}/m) \to \mathbb{Z}/m \tag{23}$$

defined as follows: Choose $x \in \mathrm{Cor}(\bar{\Delta}^0, \bar{X})$ representing a class in $H_0^{\mathrm{WS}}(X, \mathbb{Z}/m) = H_0^{\mathrm{S}}(\bar{X}, \mathbb{Z}/m)_G$, and $\alpha \in I^0(\bar{X})$ with $d\alpha = 0$ and $\alpha - F^*\alpha = 0$. Then set $\langle x, \alpha \rangle = x^*\alpha \in H_{\mathrm{et}}^0(\bar{\Delta}^0, \mathbb{Z}/m) = \mathbb{Z}/m$. We obtain a map

$$\Phi_X^0 : H_0^{\mathrm{WS}}(X, \mathbb{Z}/m) \to H_{\mathrm{et}}^0(X, \mathbb{Z}/m)^*. \tag{24}$$

The maps Φ^0 and Φ^1 extend in a natural way to nonconnected schemes. They induce a map from the exact sequence of Proposition 3.6 to the dual of the exact sequence of Proposition 2.1. The compatibility with the boundary map is given by

Proposition 6.1. *For any abstract blow-up square*

$$\begin{array}{ccc} Z' & \xrightarrow{i'} & X' \\ \pi' \downarrow & & \downarrow \pi \\ Z & \xrightarrow{i} & X \end{array}$$

the following diagram is commutative:

$$\begin{array}{ccc} H_1^{WS}(X, \mathbb{Z}/m) & \xrightarrow{\delta} & H_0^{WS}(Z', \mathbb{Z}/m) \\ \Phi_X^1 \downarrow & & \downarrow \Phi_{Z'}^0 \\ H_t^1(X, \mathbb{Z}/m)^* & \xrightarrow{-\delta^*} & H_{et}^0(Z', \mathbb{Z}/m)^* \end{array}$$

Here δ is the boundary map of the exact sequence of Proposition 3.6, and δ^* is the dual of the boundary map of the exact sequence of Proposition 2.1.

Proof. We have to show that the diagram

$$\begin{array}{ccccc} H_1^{WS}(X, \mathbb{Z}/m) & \times & H_t^1(X, \mathbb{Z}/m) & \xrightarrow{\langle , \rangle} & \mathbb{Z}/m \\ \delta \downarrow & & -\delta \uparrow & & \| \\ H_0^{WS}(Z', \mathbb{Z}/m) & \times & H_{et}^0(Z', \mathbb{Z}/m) & \xrightarrow{\langle , \rangle} & \mathbb{Z}/m \end{array}$$

commutes. Let $a \in H_1^{WS}(X, \mathbb{Z}/m)$ and $b \in H_{et}^0(Z', \mathbb{Z}/m)$. We set

$$C_i^W(X) = C_i(\bar{X}) \otimes \mathbb{Z}/m \oplus C_{i-1}(\bar{X}) \otimes \mathbb{Z}/m,$$

i.e., $C_\bullet^W(X)$ is the complex calculating $H_\bullet^{WS}(X, \mathbb{Z}/m)$. Consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_1^W(Z') & \xrightarrow{(i'_*, -\pi'_*)} & C_1^W(X') \oplus C_1^W(Z) & \xrightarrow{(\pi_*, i_*)} & C_1^W(X) \\ & & \downarrow{(\partial^*, 1-F^*)} & & \downarrow{(\partial^*, 1-F^*)} & & \downarrow{(\partial^*, 1-F^*)} \\ 0 & \longrightarrow & C_0^W(Z') & \xrightarrow{(i'_*, -\pi'_*)} & C_0^W(X') \oplus C_0^W(Z) & \xrightarrow{(\pi_*, i_*)} & C_0^W(X). \end{array}$$

By Proposition 3.6 and its proof, $a \in H_1^{WS}(X, \mathbb{Z}/m)$ can be represented by a cocycle $\alpha \in C_1^W(X)$ which can be lifted to $C_1^W(X') \oplus C_1^W(Z)$. We choose $\hat{\alpha} \in C_1^W(X') \oplus C_1^W(Z)$ with $(\pi_*, i_*)(\hat{\alpha}) = \alpha$, hence $(\pi_*, i_*)(\partial^*, 1 - F^*)(\hat{\alpha}) = 0$. We conclude that $\delta(a) \in H_0^{WS}(Z', \mathbb{Z}/m)$ is represented by an element $\gamma \in C_0^W(Z')$ with

$$(i'_*, -\pi'_*)(\gamma) = (\partial^*, 1 - F^*)(\hat{\alpha}). \tag{25}$$

Let $\mathbf{I}^\bullet$ be an injective resolution of $\mathbb{Z}/m$ in the category of sheaves of $\mathbb{Z}/m$-modules on the h-site on $\mathrm{Sch}/\mathbb{F}$, and let $\beta \in \mathrm{I}^0(Z')$ with $d\beta = 0$ be a representative of $b \in H^0_{\mathrm{et}}(Z', \mathbb{Z}/m)$. Consider the diagram

$$
\begin{array}{c}
\end{array}
$$

Since the complex $\mathrm{coker}(\mathbf{I}^\bullet(X') \oplus \mathbf{I}^\bullet(Z) \to \mathbf{I}^\bullet(Z'))$ is exact (cf. the exact triangle (6) in the proof of Proposition 2.1), we find $\hat{\beta} \in \mathrm{I}^0(X') \oplus \mathrm{I}^0(Z)$ with $(i'^*, -\pi'^*)(\hat{\beta}) = \beta$. By the argument of [MVW, Lem. 12.7], the sequence

$$0 \to \mathcal{F}(X) \to \mathcal{F}(X') \oplus \mathcal{F}(Z) \to \mathcal{F}(Z')$$

is exact for every h-sheaf $\mathcal{F}$. Therefore the second line in the diagram is exact and there is a unique $\varepsilon \in \mathrm{I}^1(X)$ with $(\pi^*, i^*)(\varepsilon) = d\hat{\beta}$ representing $\delta(b) \in H^1_{\mathrm{t}}(X, \mathbb{Z}/m)$. From $\hat{\alpha}^*(d\hat{\beta}) = \alpha^*(d\varepsilon)$ it follows that

$$d(\hat{\alpha}^*(\hat{\beta})) = \alpha^*(\varepsilon) \in \ker(\partial^*, 1 - F^*).$$

By definition, we have

$$\langle a, \delta b \rangle = -(\partial^*, 1 - F^*)\hat{\alpha}^*(\hat{\beta}) \in \ker(\mathrm{I}^0(\bar{\Delta}^0) \to \mathrm{I}^1(\bar{\Delta}^0)) = H^0_{\mathrm{et}}(\bar{\Delta}^0, \mathbb{Z}/m).$$

On the other hand, $\langle \delta a, b \rangle = \gamma^*(b) \in H^0_{\mathrm{et}}(\bar{\Delta}^0, \mathbb{Z}/m)$ is represented by $\gamma^*\beta \in \mathrm{I}^0(\Delta^0)$ and

$$\gamma^*\beta = \gamma^*(i'^*, -\pi'^*)(\hat{\beta}) = (i'_*(\gamma), -\pi'_*(\gamma))^*(\hat{\beta}) = ((\partial^*, 1 - F^*)(\hat{\alpha}))^*(\hat{\beta}).$$

Now we write $\hat{\alpha} = (\hat{\alpha}_1, \hat{\alpha}_2)$ with $\hat{\alpha}_1 \in C_1(\bar{X}') \oplus C_1(\bar{Z})$ and $\hat{\alpha}_2 \in C_0(\bar{X}') \oplus C_0(\bar{Z})$.

Then $(\partial^*, 1 - F^*)(\hat{\alpha}) = \alpha_1 \partial + \alpha_2 - F^{-1}\alpha_2 F$. Since $F^*(\hat{\beta}) = \hat{\beta}$, we conclude that

$$((\partial^*, 1 - F^*)(\hat{\alpha}))^*(\hat{\beta}) = (\delta^*, 1 - F^*)\hat{\alpha}^*(\hat{\beta}).$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

7. Proof of the main theorem

To prove our main theorem, we first consider finite coefficients.

Proposition 7.1. *The map Φ_X^0 is an isomorphism for any X and m.*

Proof. We can assume that X is connected. Then, by Proposition 3.9, the degree map induces an isomorphism $H_0^{\mathrm{WS}}(X, \mathbb{Z}/m) \xrightarrow{\sim} H_0^{\mathrm{WS}}(\mathbb{F}, \mathbb{Z}/m) \cong \mathbb{Z}/m$. Furthermore, $H_{\mathrm{et}}^0(\mathbb{F}, \mathbb{Z}/m) \xrightarrow{\sim} H_{\mathrm{et}}^0(X, \mathbb{Z}/m)$. Hence, by functoriality, we can reduce the statement to the case $X = \mathrm{Spec}(\mathbb{F})$, where it is easy. $\square$

Theorem 7.2. *For any separated scheme of finite type over a finite field $\mathbb{F}$, the map*

$$\Phi_X^1 : H_1^{\mathrm{WS}}(X, \mathbb{Z}/m) \to H_t^1(X, \mathbb{Z}/m)^*$$

is surjective. It is an isomorphism if m is prime to the characteristic or if resolution of singularities holds for schemes of dimension $\leq \dim X + 1$ over $\mathbb{F}$.

Proof. By Propositions 2.1, 3.6 and 6.1, and induction on the dimension, we can assume that X is normal and connected. Then, by Proposition 5.3 and Chebotarev–Lang density, the composite

$$H_0^{\mathrm{S}}(X, \mathbb{Z})/m \to H_1^{\mathrm{WS}}(X, \mathbb{Z}/m) \xrightarrow{\Phi_X^1} H_t^1(X, \mathbb{Z}/m)^*$$

is surjective, hence so is Φ_X^1. To get the isomorphism, we note that by Theorem 4.1, the source and the target of Φ_X^1 have the same order under the given hypothesis. $\square$

Proof of Theorem 1.2. Consider the diagram (for any m)

$$\begin{array}{ccc}
H_1^{\mathrm{WS}}(X, \mathbb{Z}) & \xrightarrow{\mathrm{rec}_X} & \Pi_1^{\mathrm{t,ab}}(X)_{\mathrm{W}} \\
\varphi \downarrow & & \downarrow \psi \\
H_1^{\mathrm{WS}}(X, \mathbb{Z}/m) & \xrightarrow{\Phi_X^1} & \Pi_1^{\mathrm{t,ab}}(X)_{\mathrm{W}}/m
\end{array}$$

The composite $\Phi_X^1 \circ \varphi$ is surjective by Corollary 3.10 and Theorem 7.2. Hence the cokernel of rec_X is divisible. Since $\Pi_1^{\mathrm{t,ab}}(X)_{\mathrm{W}}$ is finitely generated, all divisible elements of $H_1^{\mathrm{WS}}(X, \mathbb{Z})$ are in the kernel of rec_X, and the cokernel of rec_X is finitely generated and divisible, hence trivial.

Now assume that resolution of singularities holds for schemes of dimension $\leq \dim X + 1$ over $\mathbb{F}$. Then Φ_X^1 is an isomorphism for all m. Hence the kernel of rec_X is the set $\mathrm{div}\, H_1^{\mathrm{WS}}(X, \mathbb{Z})$ of divisible elements. This agrees with the maximal divisible subgroup by the following lemma. $\square$

Lemma 7.3. *Let A be an abelian group. If $A/\mathrm{div}\, A$ is finitely generated, then $\mathrm{div}\, A$ is divisible.*

Proof. Let $B = A/\mathrm{div}\, A$, choose an integer n such that nB is free, and let $C \subseteq A$ be the inverse image of nB in A. Then $\mathrm{div}\, C = \mathrm{div}\, A$ because $nA \subseteq C \subseteq A$. By freeness of nB, we obtain $C = nB \oplus \mathrm{div}\, A$, hence $\mathrm{div}\, A = \mathrm{div}\, C = \mathrm{div}\,\mathrm{div}\, A$. $\square$

8. The case of proper curves

In this section we illustrate the results of this paper in the case of proper curves. Let C be a connected, proper curve over $\mathbb{F}$. The morphism $C_{\mathrm{red}} \to C$ induces an isomorphism on Weil–Suslin homology as well as on fundamental groups. We may therefore assume that C is reduced. We first note that $H_1^{\mathrm{WS}}(C, \mathbb{Z})$ is a finitely generated abelian group. This follows by applying Proposition 3.6 to the normalization morphism $\widetilde{C} \to C$ and by using Examples 3.7 and 3.8. Furthermore, resolution of singularities holds for schemes of dimension ≤ 2 over $\mathbb{F}$. Hence Theorem 1.2 yields the reciprocity *isomorphism*

$$\mathrm{rec}_C : H_1^{\mathrm{WS}}(C, \mathbb{Z}) \xrightarrow{\sim} \Pi_1^{\mathrm{ab}}(C)_{\mathrm{W}}.$$

Since C is proper, we have $\mathrm{CH}_0(C) = H_0^{\mathrm{S}}(C, \mathbb{Z})$, hence (16) yields an injection

$$\phi : \mathrm{CH}_0(C) \hookrightarrow H_1^{\mathrm{WS}}(C, \mathbb{Z}).$$

By [Ge5, Thm. 6.2 and Prop. 6.3], $\mathrm{coker}(\phi)$ is isomorphic to $H_1(\Gamma, \mathbb{Z})$, where Γ is the dual graph associated with the curve C. We obtain an exact diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{CH}_0(C) & \xrightarrow{\phi} & H_1^{\mathrm{WS}}(C, \mathbb{Z}) & \longrightarrow & H_1(\Gamma, \mathbb{Z}) & \longrightarrow & 0 \\
& & \| & & \wr \downarrow \mathrm{rec}_C & & \| & & \\
0 & \longrightarrow & \mathrm{CH}_0(C) & \xrightarrow{r_C} & \Pi_1^{\mathrm{ab}}(C)_{\mathrm{W}} & \longrightarrow & H_1(\Gamma, \mathbb{Z}) & \longrightarrow & 0
\end{array}
$$

where, by Proposition 5.3, r_C is the map which sends the class of a closed point to its Frobenius automorphism. Denoting by $\widehat{A}$ the profinite completion of an abelian group A, completion of the bottom line yields the exact sequence

$$0 \to \widehat{\mathrm{CH}_0(C)} \to \pi_1^{\mathrm{ab}}(C) \to \widehat{H_1(\Gamma, \mathbb{Z})} \to 0, \tag{26}$$

which is the exact sequence of [KaSa, Prop. 1].

Finally, we consider the example of a nodal curve. Let $\widetilde{C}$ be a smooth, proper curve over $\mathbb{F}$ admitting two rational points P, Q, and let C be the curve obtained by identifying P and Q. We denote the image of P and Q in C by O, i.e., we have an (abstract) blow-up square

$$
\begin{array}{ccc}
\{P, Q\} & \longrightarrow & \widetilde{C} \\
\downarrow & & \downarrow \\
\{O\} & \longrightarrow & C
\end{array}
\tag{27}
$$

Let D be a countable chain of copies C_i, $i \in \mathbb{Z}$, of C with $P_i \in C_i$ identified with $Q_{i-1} \in C_{i-1}$ for all $i \in \mathbb{Z}$:
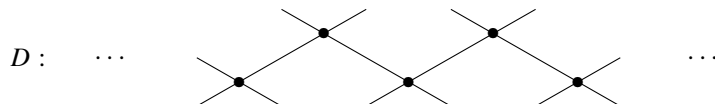
Translation gives a natural $\mathbb{Z}$-action on D, and C is the quotient $D/\mathbb{Z}$ with respect to this action. In particular, we have a surjection $\Pi_1^{ab}(C) \twoheadrightarrow \mathrm{Aut}_C(D) = \mathbb{Z}$. Inspecting the combinatorics of D, we see that

$$\ker(\Pi_1^{ab}(C) \to \mathbb{Z}) \cong \Pi_1^{ab}(\widetilde{C})/(\mathrm{Frob}_P - \mathrm{Frob}_Q).$$

Hence, we have an exact sequence

$$0 \to \Pi_1^{ab}(\widetilde{C})_W/(\mathrm{Frob}_P - \mathrm{Frob}_Q) \to \Pi_1^{ab}(C)_W \to \mathbb{Z} \to 0. \qquad (28)$$

In particular,

$$\Pi_1^{ab}(C)_W \cong \mathbb{Z} \oplus \mathbb{Z} \oplus (\text{finite})$$

as abelian groups. Applying Proposition 3.6 to the blow-up square (27) and using Examples 3.7 and 3.8, we obtain the exact sequence

$$0 \to \mathrm{CH}_0(\widetilde{C})/([P] - [Q]) \to H_1^{WS}(C, \mathbb{Z}) \to \mathbb{Z} \to 0. \qquad (29)$$

The reciprocity map rec_C induces an isomorphism between the exact sequences (29) and (28). The map on the left hand side is induced by the reciprocity map of the smooth curve $\widetilde{C}$, which sends $[x] \in \mathrm{CH}_0(\widetilde{C})$ to $\mathrm{Frob}_x \in \Pi_1^{ab}(\widetilde{C})_W = \pi_1^{ab}(\widetilde{C})_W$.

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