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On a long range segregation model

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Abstract. In this work we study the properties of segregation processes modeled by a family of equations

$$L(u_i)(x) = u_i(x)F_i(u_1, \dots, u_K)(x), \qquad i = 1, \dots, K,$$

where $F_i(u_1, \dots, u_K)(x)$ is a non-local factor that takes into consideration the values of the functions u_j in a full neighborhood of x. We consider as a model problem

$$\Delta u_i^\varepsilon(x) = \frac{1}{\varepsilon^2} u_i^\varepsilon(x) \sum_{i\neq j} H(u_j^\varepsilon)(x)$$

where ε is a small parameter and $H(u_j^\varepsilon)(x)$ is for instance

$$H(u_j^\varepsilon)(x) = \int_{\mathcal{B}_1(x)} u_j^\varepsilon(y)\, dy \quad \text{or} \quad H(u_j^\varepsilon)(x) = \sup_{y\in\mathcal{B}_1(x)} u_j^\varepsilon(y).$$

Here $\mathcal{B}_1(x)$ is the unit ball centered at x with respect to a smooth, uniformly convex norm ρ in $\mathbb{R}^n$. Heuristically, this will force the populations to stay at ρ-distance 1 from each other as $\varepsilon \to 0$.

Keywords. Segregation of populations, free boundary problems, long-range interactions

1. Introduction

Segregation phenomena occur in many areas of mathematics and science: from equipartition problems in geometry, to social and biological processes (cells, bacteria, ants, mammals), to finance (sellers and buyers). There is a large body of literature in connection with our work and we would like to refer to [4, 5, 8–21, 26–29, 31–33] and the references therein. We particularly point out the articles [15, 26, 28, 29, 31] where spatial separation due to competition for resources is discussed among ant nests, mussels and sessile animals.

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These articles study a family of models arising from different applications whose main two ingredients are: in the absence of competition, species follow a "propagation" equation involving diffusion, transport, birth-death, etc., but when two species overlap, their growth is mutually inhibited by competition, consumption of resources, etc. The simplest form of such models consists, for species σ_i with spatial density u_i, of a system of equations

$$L(u_i) = u_i F_i(u_1, \dots, u_K).$$

The operator L quantifies diffusion, transport, etc., while the term $u_i F_i$ corresponds to attrition of u_i from competition with the remaining species.

In these models, the interaction is punctual, i.e. $u_i(x)$ interacts with the remaining densities also at position x. There are many processes, though, where the growth of σ_i at x is inhibited by the populations σ_j in a full area surrounding x.

This work is a first attempt to study the properties of such a segregation process. Basically, we consider a family of equations

$$L(u_i)(x) = u_i(x) F_i(u_1, \dots, u_K)(x)$$

where $F_i(u_1, \dots, u_K)(x)$ is now a non-local factor that takes into consideration the values of u_j in a full neighborhood of x. Given the previous discussion, a possible model problem would be the system

$$\Delta u_i^\varepsilon(x) = \frac{1}{\varepsilon^2} u_i^\varepsilon(x) \sum_{i \neq j} H(u_j^\varepsilon)(x), \qquad i = 1, \dots, K,$$

where ε is a small parameter and $H(u_j^\varepsilon)(x)$ is a non-local operator, for instance

$$H(u_j^\varepsilon)(x) = \int_{B_1(x)} u_j^\varepsilon(y)\, dy \quad \text{or} \quad H(u_j^\varepsilon)(x) = \sup_{y \in B_1(x)} u_j^\varepsilon(y).$$

To study the limit configuration when the competition for resources is very high, we consider the limit as $\varepsilon \to 0$. Heuristically, the non-local term forces the populations to stay at distance 1 from each other. As an example, as we will prove, in the case of two populations in dimension two, we will have strips of length precisely one between the regions where the populations live. At "edge" points, which we will define as singular points, the angles of the asymptotic cones have to be the same (Figure 1). Here $S_i = S_i^1 \cup S_i^2, i = 1, 2,$ represents the region where the population σ_i with density u_i exists. Moreover, the ratio between the normal derivatives at regular points across the free boundary depends on the ratio of the respective curvatures $\varkappa$. For example, if $Z_1 \in \partial S_1^1$ and $Z_2 \in \partial S_2^1$, Z_1 and Z_2 are not "edge" points, and $d(Z_1, Z_2) = 1$ then

$$\frac{u_\nu^1(Z_1)}{u_\nu^2(Z_2)} = \frac{\varkappa(Z_1)}{\varkappa(Z_2)} \quad \text{if } \varkappa(Z_2) \neq 0, \qquad u_\nu^1(Z_1) = u_\nu^2(Z_2) \quad \text{if } \varkappa(Z_2) = 0.$$

Instead of the unit ball $B_1(x)$ in the Euclidean norm we will consider the translation at x of a general smooth set $\mathcal{B}$ that is also uniformly convex, bounded and symmetric with respect to the origin. The set $\mathcal{B}$ defines a smooth, uniformly convex norm ρ in $\mathbb{R}^n$.

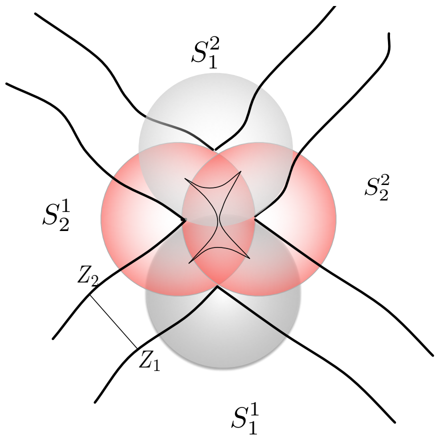

Fig. 1. Example of a limit configuration for $K = 2$, $n = 2$.

Note that there is some similarity with the Lasry–Lions model of price formation [6, 25] where the selling and buying prices are separated by a gap due to the transaction cost.

2. Notation and statement of the problem

Let $\mathcal{B}$ be an open bounded domain of $\mathbb{R}^n$, convex, symmetric with respect to the origin and with smooth boundary. Then $\mathcal{B}$ can be represented as the unit ball of a norm $\rho : \mathbb{R}^n \to \mathbb{R}$, $\rho \in C^\infty(\mathbb{R}^n \setminus \{0\})$, called the *defining function* of $\mathcal{B}$, i.e.

$$\mathcal{B} = \{x \in \mathbb{R}^n \mid \rho(x) < 1\}.$$

We assume that $\mathcal{B}$ is *uniformly convex*, i.e. there exists $0 < a \le A$ such that in $\mathbb{R}^n \setminus \{0\}$,

$$aI_n \le D^2\left(\frac{1}{2}\rho^2\right) \le AI_n, \tag{2.1}$$

where I_n is the $n \times n$ identity matrix. In what follows we denote

$$\mathcal{B}_r := \{y \in \mathbb{R}^n \mid \rho(y) < r\}, \qquad \mathcal{B}_r(x) := \{y \in \mathbb{R}^n \mid \rho(x - y) < r\}.$$

So throughout the paper we will always refer to the Euclidean ball as B and to the ρ-ball as $\mathcal{B}$. For a given closed set K, let

$$d_\rho(\cdot, K) = \inf_{y \in K} \rho(\cdot - y)$$

be the distance function from K associated to ρ. Then there exist $c_1, c_2 > 0$ such that

$$c_1 d(\cdot, K) \le d_\rho(\cdot, K) \le c_2 d(\cdot, K), \tag{2.2}$$

where $d(\cdot, K)$ is the distance function associated to the Euclidean norm $|\cdot|$ of $\mathbb{R}^n$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. We will denote by $(\partial\Omega)_{\leq 1}$ the ρ-strip of size 1 around $\partial\Omega$ in the complement of Ω defined by

$$(\partial\Omega)_{\leq 1} := \{x \in \Omega^c \mid d_\rho(x, \partial\Omega) \leq 1\}.$$

For $i = 1, \dots, K$, let f_i be non-negative functions defined on $(\partial\Omega)_{\leq 1}$ with supports at ρ-distance ≥ 1 from each other:

$$d_\rho(\operatorname{supp} f_i, \operatorname{supp} f_j) \geq 1 \quad \text{for } i \neq j. \tag{2.3}$$

We will consider the following system of equations: for $i = 1, \dots, K$,

$$\begin{cases} \Delta u_i^\varepsilon(x) = \dfrac{1}{\varepsilon^2} u_i^\varepsilon(x) \displaystyle\sum_{j \neq i} H(u_j^\varepsilon)(x) & \text{in } \Omega, \\ u_i^\varepsilon = f_i & \text{on } (\partial\Omega)_{\leq 1}. \end{cases} \tag{2.4}$$

The functional $H(u_j)(x)$ depends only on the restriction of u_j to $\mathcal{B}_1(x)$.

We will consider, for simplicity,

$$H(w)(x) = \int_{\mathcal{B}_1(x)} w^p(y)\varphi(\rho(x-y))\,dy, \qquad 1 \leq p < \infty, \tag{2.5}$$

or

$$H(w)(x) = \sup_{\mathcal{B}_1(x)} w \tag{2.6}$$

with φ a strictly positive smooth function of ρ, with at most polynomial decay at $\partial\mathcal{B}_1$:

$$\varphi(\rho) \geq C(1-\rho)^q, \qquad q \geq 0. \tag{2.7}$$

In the rest of the paper, when we refer to viscosity solutions $u_1^\varepsilon, \dots, u_K^\varepsilon$ of the problem (2.4), we mean that $u_1^\varepsilon, \dots, u_K^\varepsilon$ are continuous functions that satisfy the system (2.4) in the viscosity sense. Moreover, we make the following assumptions: for $i = 1, \dots, K$,

$$\begin{cases} \varepsilon > 0,\ \Omega \text{ is a bounded Lipschitz domain in } \mathbb{R}^n, \\ f_i : (\partial\Omega)_{\leq 1} \to \mathbb{R},\ f_i \geq 0,\ f_i \not\equiv 0,\ f_i \text{ is Hölder continuous}, \\ \exists c > 0\ \forall x \in \partial\Omega \cap \operatorname{supp} f_i : |\mathcal{B}_r(x) \cap \operatorname{supp} f_i| \geq c|\mathcal{B}_r(x)|, \\ \text{(2.3) holds true}, \\ H \text{ is either of the form (2.5) or (2.6), and (2.7) holds.} \end{cases} \tag{2.8}$$

3. Main results

For the reader's convenience we present our main results below. Assume that (2.8) holds true. Then:

Existence (Theorem 4.1): *There exist continuous functions $u_1^\varepsilon, \dots, u_K^\varepsilon$, depending on the parameter ε, that are viscosity solutions of problem* (2.4).

Limit problem (Corollary 5.6): *There exists a subsequence* $(\vec{u})^{\varepsilon_m}$ *converging locally uniformly, as* $\varepsilon \to 0$*, to a function* $\vec{u} = (u_1, \ldots, u_K)$*, satisfying the following properties:*

(i) *the* u_i*'s are locally Lipschitz continuous in* Ω *and have supports at distance at least* 1 *from each other, i.e.*

$$u_i \equiv 0 \quad \text{in } \{x \in \Omega \mid d_\rho(x, \operatorname{supp} u_j) \le 1\} \quad \text{for any } j \neq i.$$

(ii) $\Delta u_i = 0$ *when* $u_i > 0$.

Semiconvexity of the free boundary (Corollary 6.2): *If* $x_0 \in \partial\{u_i > 0\}$ *then there is an exterior tangent* ρ*-ball of radius* 1 *at* x_0.

The supports of u_i **are sets of finite perimeter** (Corollary 6.5): *The set* $\{u_i > 0\}$ *has finite perimeter.*

Sharp characterization of the interfaces (Theorem 7.1): *Under the additional assumption that* $p = 1$ *in* (2.5)*, the supports of the limit functions are at distance exactly* 1 *from each other, i.e. if* $x_0 \in \partial\{u_i > 0\} \cap \Omega$*, then there exists* $j \neq i$ *such that*

$$\overline{\mathcal{B}_1(x_0)} \cap \partial\{u_j > 0\} \neq \emptyset.$$

Classification of singular points in dimension 2 (Lemma 8.9, Theorem 8.10, Corollaries 8.11, 8.12): *For* $n = 2$*, assume in addition that* $p = 1$ *in* (2.5) *and that the supports of the* f_i*'s have a finite number of connected components. For* $i \neq j$*, let* $x_0 \in \partial\{u_i > 0\} \cap \Omega$ *and* $y_0 \in \partial\{u_j > 0\} \cap \Omega$ *be points such that* $\{u_i > 0\}$ *has an angle* θ_i *at* x_0*,* $\{u_j > 0\}$ *has an angle* θ_j *at* y_0 *and* $\rho(x_0 - y_0) = 1$*. Then*

$$\theta_i = \theta_j.$$

If $x_0 \in \partial\{u_i > 0\} \cap \partial\Omega$ *and* $y_0 \in \partial\{u_j > 0\} \cap \Omega$*, then*

$$\theta_i \le \theta_j.$$

Moreover, singular points, i.e. points where the free boundaries have corners, are isolated and finite. If the domain is a strip and there are only two populations, then under additional monotonicity assumptions on the boundary data, the free boundary sets $\partial\{u_i > 0\}$*,* $i = 1, 2$*, are of class* C^1*.*

Lipschitz regularity of free boundary for the associated obstacle problem in dimension 2 (Theorem 8.18): *For* $n = 2$*, under the additional assumptions that* $p = 1$ *in* (2.5)*,* $f_i \equiv 1$*, and the supports of the* f_i*'s are connected, and under additional conditions about the regularity of* $\partial\Omega$*, if* $(u_1^\varepsilon, \ldots, u_K^\varepsilon)$ *is a particular solution of* (2.4) *which satisfies the associated obstacle problem* (8.49) *with* $(u_1, \ldots, u_K)$ *the limit as* $\varepsilon \to 0$*, then the free boundaries* $\partial\{u_i > 0\}$*,* $i = 1, \ldots, K$*, are Lipschitz curves of the plane.*

Free boundary condition (Theorem 9.2): *In any dimension, assume that we have two populations,* H *is defined as in* (2.5) *with* $\varphi \equiv 1$*,* $p = 1$ *and* $\mathcal{B}_1(x) = B_1(x)$ *is the Euclidean ball,* $0 \in \partial\{u_1 > 0\}$*,* $e_n \in \partial\{u_2 > 0\}$*, and* $\partial\{u_1 > 0\}$ *and* $\partial\{u_2 > 0\}$ *are* *of class C^2 in a neighborhood of* 0 *and* e_n *respectively. Let* $\varkappa_i(0)$ *denote the principal curvatures of* $\partial\{u_1 > 0\}$ *at* 0 *where outward is the positive direction, and let* $\varkappa_i(e_n)$ *denote the principal curvatures of* $\partial\{u_2 > 0\}$ *at* e_n *where now inward is the positive direction. Then we have the following relation between the exterior normal derivatives of* u_1 *and* u_2*:*

$$\frac{u_\nu^1(0)}{u_\nu^2(e_n)} = \prod_{\substack{i=1\\ \varkappa_i(0)\neq 0}}^{n-1} \frac{\varkappa_i(0)}{\varkappa_i(e_n)} \qquad \text{if } \varkappa_i(0)\neq 0 \text{ for some } i=1,\dots,n-1,$$
$$u_\nu^1(0) = u_\nu^2(e_n) \qquad \text{if } \varkappa_i(0)= 0 \text{ for any } i=1,\dots,n-1.$$

4. Existence of solutions

The proof below follows the same steps as in [30] and it is written below for the reader's convenience.

Theorem 4.1. *Assume* (2.8)*. Then there exist continuous positive functions* $u_1^\varepsilon, \dots, u_K^\varepsilon$*, depending on the parameter* ε*, that are viscosity solutions of problem* (2.4)*.*

Proof. The proof uses a fixed point result. Let B be the Banach space of bounded continuous vector-valued functions defined on the domain Ω with the norm

$$\|(u_1,\dots,u_K)\|_B := \max_i \sup_{x\in\Omega} |u_i(x)|.$$

For $i = 1,\dots,K$, let ϕ_i be the solutions of

$$\begin{cases} \Delta\phi_i = 0 & \text{in } \Omega,\\ \phi_i = f_i & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Let Θ be the subset of bounded continuous functions in Ω that satisfy prescribed boundary data, and are bounded from above and from below as stated below:

$$\Theta = \{(u_1,\dots,u_K) \mid u_i : \Omega \to \mathbb{R} \text{ is continuous, } 0 \le u_i \le \phi_i \text{ in } \Omega,\ u_i = f_i \text{ on } (\partial\Omega)_{\le 1}\}.$$

Notice that Θ is a closed and convex subset of B. Let T^ε be the operator defined on Θ in the following way: $T^\varepsilon(u_1,\dots,u_K) := (v_1^\varepsilon,\dots,v_K^\varepsilon)$ if for any $i = 1,\dots,K$, v_i^ε is a solution to the following problem:

$$\begin{cases} \Delta(v_i^\varepsilon)(x) = \dfrac{1}{\varepsilon^2} v_i^\varepsilon(x) \displaystyle\sum_{j\neq i} H(u_j)(x) & \text{in } \Omega,\\ v_i^\varepsilon = f_i & \text{on } (\partial\Omega)_{\le 1}, \end{cases} \tag{4.2}$$

where u_j, $j \neq i$, are given. Observe that if T^ε has a fixed point,

$$T^\varepsilon(u_1^\varepsilon,\dots,u_K^\varepsilon) = (u_1^\varepsilon,\dots,u_K^\varepsilon),$$

then $(u_1^\varepsilon,\dots,u_K^\varepsilon)$ is a solution of problem (2.4).

In order for T^ε to have a fixed point, we need to prove that it satisfies the hypothesis of the Schauder fixed point theorem [23]:

(1) $T^\varepsilon(\Theta) \subset \Theta$: Classical existence results guarantee the existence of a viscosity solution $(v_1^\varepsilon, \dots, v_K^\varepsilon)$ of problem (4.2) which is smooth in Ω. Since $f_i \ge 0$ and $f_i \not\equiv 0$, the strong maximum principle implies

$$v_i^\varepsilon > 0 \quad \text{in } \Omega.$$

This implies that

$$\Delta v_i^\varepsilon \ge 0 \quad \text{in } \Omega, \tag{4.3}$$

and again from the comparison principle we have

$$v_i^\varepsilon \le \phi_i \quad \text{in } \Omega.$$

We have proved that $T^\varepsilon(u_1, \dots, u_K) \in \Theta$.

(2) T^ε *is continuous*: Assume that $(u_{1m}, \dots, u_{Km}) \to (u_1, \dots, u_K)$ in B, meaning that as $m \to \infty$,

$$\max_{1\le i\le K} \|u_{im} - u_i\|_{L^\infty} \to 0.$$

We need to prove that for each fixed $\varepsilon > 0$,

$$\|T^\varepsilon(u_{1m}, \dots, u_{Km}) - T^\varepsilon(u_1, \dots, u_K)\|_B \to 0 \quad \text{as } m \to \infty.$$

Let

$$T^\varepsilon(u_{1m}, \dots, u_{Km}) = (v_{1m}^\varepsilon, \dots, v_{Km}^\varepsilon).$$

If we prove that there exists a constant C_ε independent of m such that, for $i = 1, \dots, K$,

$$\|v_{im}^\varepsilon - v_i^\varepsilon\|_{L^\infty} \le C_\varepsilon \max_j \|u_{jm} - u_j\|_{L^\infty},$$

then the result follows. For all $x \in \Omega$ and for fixed i, let

$$\omega_m(x) = v_{im}^\varepsilon(x) - v_i^\varepsilon(x),$$

and suppose for instance that there exists $y \in \Omega$ such that

$$\omega_m(y) > r^2 D \max_j \|u_{jm} - u_j\|_{L^\infty} \tag{4.4}$$

for some large $D > 0$, where r is such that $\Omega \subset B_r$, and B_r is the Euclidean ball centered at 0 of radius r. We want to prove that this is impossible if D is sufficiently large. Let h_m be the concave radially symmetric function

$$h_m(x) = \gamma_m(r^2 - |x|^2) \quad \text{with} \quad \gamma_m = D \max_j \|u_{jm} - u_j\|_{L^\infty}.$$

Observe that:

(a) $h_m(x) = 0$ on ∂B_r;
(b) $h_m(x) \le r^2 D \max_j \|u_{jm} - u_j\|_{L^\infty}$ for all x in B_r;
(c) $0 = \omega_m(x) \le h_m(x)$ on $\partial\Omega$, since v_{im}^ε and v_i^ε are solutions with the same boundary data.

Since we are assuming (4.4), there exists a negative minimum of $h_m - \omega_m$ in Ω. Let $x_0 \in \Omega$ be a point where the minimum value is attained. Then

$$h_m(x_0) - \omega_m(x_0) < 0 \quad \text{and} \quad \Delta(h_m - \omega_m)(x_0) \geq 0.$$

Then

$$\begin{aligned}\Delta\omega_m(x_0) &= \Delta(v_{im}^\varepsilon)(x_0) - \Delta v_i^\varepsilon(x_0) \\ &= \frac{1}{\varepsilon^2}\Big((v_{im}^\varepsilon(x_0) - v_i^\varepsilon(x_0))\sum_{j\neq i} H(u_{jm})(x_0) \\ &\qquad - v_i^\varepsilon(x_0)\sum_{j\neq i}\big(H(u_j)(x_0) - H(u_{jm})(x_0)\big)\Big) \\ &\geq \frac{1}{\varepsilon^2}\Big((v_{im}^\varepsilon(x_0) - v_i^\varepsilon(x_0))\sum_{j\neq i} H(u_{jm})(x_0) \\ &\qquad - v_i^\varepsilon(x_0)(K-1)C\max_j \|u_{jm} - u_j\|_{L^\infty(\Omega)}\Big)\end{aligned}$$

by adding and subtracting $\frac{1}{\varepsilon^2}v_i^\varepsilon(x_0)\sum_{j\neq i} H(u_{jm})(x_0)$, where C depends on the f_j's and φ. Then

$$\begin{aligned}0 \leq \Delta(h_m - \omega_m)(x_0) &\leq -2\gamma_m n - \frac{1}{\varepsilon^2}\Big((v_{im}^\varepsilon - v_i^\varepsilon)(x_0)\sum_{j\neq i} H(u_{jm})(x_0) \\ &\qquad - v_i^\varepsilon(x_0)(K-1)C\max_j \|u_{jm} - u_j\|_{L^\infty}\Big) \\ &\leq -2nD\max_j \|u_{jm} - u_j\|_{L^\infty} + \frac{1}{\varepsilon^2}v_i^\varepsilon(x_0)(K-1)C\max_j \|u_{jm} - u_j\|_{L^\infty} \\ &\leq -2nD\max_j \|u_{jm} - u_j\|_{L^\infty} + \frac{\widetilde{C}}{\varepsilon^2}\max_j \|u_{jm} - u_j\|_{L^\infty}\end{aligned}$$

because $0 < h_m(x_0) < \omega_m(x_0) = (v_{im}^\varepsilon - v_i^\varepsilon)(x_0)$ and $\sum_{j\neq i} H(u_{jm})(x_0) \geq 0$, and so

$$-\frac{1}{\varepsilon^2}(v_{im}^\varepsilon - v_i^\varepsilon)(x_0)\sum_{j\neq i} H(u_{jm})(x_0) \leq 0.$$

Taking $D = D_\varepsilon > \frac{\widetilde{C}}{2n\varepsilon^2}$, we obtain

$$0 \leq \Delta(h_m - \omega_m)(x_0) < 0,$$

which is a contradiction.

(3) $T^\varepsilon(\Theta)$ *is precompact*: Let $(u_{1m}, \ldots, u_{Km})$ be a bounded sequence in B and let

$$(v_{1m}^\varepsilon, \ldots, v_{Km}^\varepsilon) = T^\varepsilon(u_{1m}, \ldots, u_{Km}).$$

Then by standard Hölder estimates for viscosity solutions, $(v^{\varepsilon}_{1m}, \dots, v^{\varepsilon}_{Km})$ is bounded in the space of Hölder continuous functions on $\overline{\Omega}$. Since the subset of Θ of Hölder continuous functions on $\overline{\Omega}$ is precompact in Θ, we can extract from $(v^{\varepsilon}_{1m}, \dots, v^{\varepsilon}_{Km})$ a subsequence which converges in B.

We have proven the existence of a solution $(u^{\varepsilon}_1, \dots, u^{\varepsilon}_K)$ of (2.4). The same argument as in (1) shows that $u^{\varepsilon}_i > 0$ in Ω. This concludes the proof of the theorem. □

5. Uniform in ε Lipschitz estimates

In this section we will prove uniform in ε Lipschitz estimates that will imply the convergence, up to subsequences, of the solution $(u^{\varepsilon}_1, \dots, u^{\varepsilon}_K)$ of (2.4) to a limit function $(u_1, \dots, u_K)$ as $\varepsilon \to 0$. We will show that the functions u_i are locally Lipschitz continuous in Ω and harmonic inside their support. Moreover, $u_i \equiv 0$ in the ρ-strip of size 1 around the support of u_j for any $j \neq i$, i.e. the supports of the limit functions are at distance at least 1 from each other. We start by proving general properties of subsolutions of uniform elliptic equations.

Lemma 5.1. *Let:*

(a) *ω be a subharmonic function in $\mathcal{B}_1$ such that*

- (a_1) *$\omega \le 1$ in $\mathcal{B}_1$,*
- (a_2) *$\omega(0) = m > 0$;*

(b) *D_0 be a smooth convex set with bounded curvatures*

$$|\varkappa_i(\partial D_0)| \le C_0, \qquad i = 1, \dots, n-1$$

(like $\mathcal{B}_1$ above).

Then there exists a universal $\tau_0 = \tau_0(C_0, n, \rho)$ such that if $d_\rho(D_0, 0) \le \tau_0 m$, then

$$\sup_{\partial D_0 \cap \mathcal{B}_1} \omega \ge m/2.$$

Proof. Assume without loss of generality that $0 \notin D_0$ and let h be harmonic in $\mathcal{B}_1 \setminus D_0$ and such that

$$\begin{cases} h = 1 & \text{on } (\partial\mathcal{B}_1) \setminus D_0, \\ h = m/2 & \text{on } (\partial D_0) \cap \mathcal{B}_1. \end{cases}$$

By assumption (b), the set $\mathcal{B}_1 \setminus D_0$ satisfies an exterior uniform ball condition at any point of $\partial D_0 \cap \mathcal{B}_1$; therefore, by a standard barrier argument, h grows no more than linearly away from ∂D_0 in $\mathcal{B}_{1/2}$, i.e., there exist $k_1, k_2 > 0$ depending on C_0 and n such that if $x \in \mathcal{B}_{1/2} \setminus D_0$ and $d(x, \partial D_0) \le k_2$, then $h(x) \le k_1 d(x, \partial D_0) + m/2$. To prove that $h(0) < m$ observe that if $\tau_0 \le k_2 c_1$, where c_1 is given by (2.2), then $d(0, \partial D_0) \le \tau_0 m / c_1 \le k_2 m \le k_2$, and therefore if in addition τ_0 is so small that $\frac{k_1}{c_1}\tau_0 \le \frac{1}{2}$, we have

$$h(0) \le k_1 d(0, \partial D_0) + \frac{m}{2} \le \frac{k_1}{c_1} d_\rho(0, \partial D_0) + \frac{m}{2} \le \frac{k_1}{c_1}\tau_0 m + \frac{m}{2} < m.$$

Hence, we must have $\sup_{\partial D_0 \cap \mathcal{B}_1} \omega \ge m/2$, for otherwise the comparison principle would imply $\omega(x) \le h(x)$ in $\mathcal{B}_1 \setminus D_0$, which is a contradiction at $x = 0$. □

Lemma 5.2. *Let ω be a positive subsolution of a uniformly elliptic equation ($\lambda^2 I \leq a_{ij} \leq \Lambda^2 I$)*

$$a_{ij} D_{ij}\omega \geq \theta^2 \omega \quad \text{in } \mathcal{B}_r.$$

Then there exist $c, C > 0$ such that

$$\frac{\omega(0)}{\sup_{\mathcal{B}_r} \omega} \leq C e^{-c\theta r}.$$

Proof. The function

$$g(x) = \sum_{i=1}^{n} \cosh\left(\frac{\theta}{\Lambda} x_i\right)$$

is a supersolution of the equation $a_{ij} D_{ij} u = \theta^2 u$. Moreover, using the convexity of the exponential function, it is easy to check that

$$g(x) \geq C_1 e^{c\theta r} \quad \text{for any } x \in \partial\mathcal{B}_r.$$

Then the comparison principle implies

$$\frac{\omega(x)}{\sup_{\mathcal{B}_r} \omega} \leq \frac{g(x)}{C_1 e^{c\theta r}} \quad \text{for any } x \in \mathcal{B}_r.$$

The result follows by taking $x = 0$. □

The next lemma says that if u_i^ε attains a positive value σ at some interior point, then all the other functions u_j^ε, $j \neq i$, go to zero exponentially in a ρ-ball of radius $1 + c\sigma$ around that point.

Lemma 5.3. *Assume (2.8). Let $(u_1^\varepsilon, \ldots, u_K^\varepsilon)$ be a viscosity solution of problem (2.4). For $i = 1, \ldots, K$, $\sigma > 0$, and $0 < r < 1$ let*

$$\Gamma_i^{\sigma,r} := \{y \in \Omega \mid d_\rho(y, \operatorname{supp} f_i) \geq 2r,\ u_i^\varepsilon = \sigma\}, \qquad m := \frac{\sigma}{\sup_{\partial\Omega} f_i}.$$

Then there exists a universal constant $0 < \tau < 1$ such that in the sets

$$\Sigma_{i,j}^{\sigma,r} := \{x \in \Omega \mid d_\rho(x, \Gamma_i^{\sigma,r}) \leq 1 + \tau m r/2,\ d_\rho(x, \operatorname{supp} f_j) \geq \tau m r/4\}$$

we have

$$u_j^\varepsilon \leq C e^{-c\sigma^\alpha r^\beta/\varepsilon} \quad \text{for } j \neq i,$$

for some positive α and β depending on the structure of H (p and q).

Proof. Let $0 < \tau < 1$ to be determined. For $0 < r < 1$, consider the set $\Sigma_{i,j}^{\sigma,r}$ defined above and let $\overline{x} \in \Sigma_{i,j}^{\sigma,r}$. We want to show that for $j \neq i$, we have

$$\Delta u_j^\varepsilon \geq \frac{C\sigma^{\overline{\alpha}} r^{\overline{\beta}}}{\varepsilon^2} u_j^\varepsilon \quad \text{in } \mathcal{B}_{\tau m r/4}(\overline{x}) \tag{5.1}$$

for some $\overline{\alpha}, \overline{\beta} > 0$. Let us prove it for $\overline{x}$ such that $d_\rho(\overline{x}, \Gamma_i^{\sigma,r}) = 1 + \tau m r/2$, which is the hardest case. First of all, note that since $d_\rho(\overline{x}, \operatorname{supp} f_j) \geq \tau m r/4$, the ball $\mathcal{B}_{\tau m r/4}(\overline{x})$ does not intersect supp f_j. Therefore, u_j^ε (which is eventually zero in $\mathcal{B}_{\tau m r/4}(\overline{x}) \cap \Omega^c$) satisfies

$$\Delta u_j^\varepsilon \geq \frac{1}{\varepsilon^2} u_j^\varepsilon \sum_{k \neq j} H(u_k^\varepsilon) \quad \text{in } \mathcal{B}_{\tau m r/4}(\overline{x}). \tag{5.2}$$

Next, the ball $\mathcal{B}_{1-\tau mr/2}(\overline{x})$ is at distance τmr from a point $y \in \Gamma_i^{\sigma,r}$. Observe that since $\mathcal{B}_{2r}(y) \cap \operatorname{supp} f_i = \emptyset$, the function u_i^ε (which is eventually zero in $\mathcal{B}_{2r}(y) \cap \Omega^c$) satisfies $\Delta u_i^\varepsilon \geq 0$ in $\mathcal{B}_{2r}(y)$. Moreover, since u_i^ε is subharmonic in Ω, it attains its maximum at the boundary of Ω, so that $u_i^\varepsilon / \sup_{\partial\Omega} f_i \leq 1$ in Ω. In particular $m = \sigma/\sup_{\partial\Omega} f_i \leq 1$. Set

$$v(x) := \frac{u_i^\varepsilon(y + rx)}{\sup_{\partial\Omega} f_i}. \tag{5.3}$$

Then $v \leq 1$ and $v(0) = u_i^\varepsilon(y)/\sup_{\partial\Omega} f_i = \sigma/\sup_{\partial\Omega} f_i = m$ and $\Delta v \geq 0$ in $\mathcal{B}_1$. Let

$$D_0 := \mathcal{B}_{1/r-\tau m/2}\left(\frac{\overline{x}-y}{r}\right).$$

Then the principal curvatures of D_0 satisfy

$$|\varkappa_i(\partial D_0)| \leq \frac{C_\rho}{1/r - \tau m/2} = \frac{2rC_\rho}{2 - r\tau m} < 2rC_\rho < 2C_\rho.$$

Moreover D_0 is at distance τm from 0. Hence, from Lemma 5.1 applied to the function v given by (5.3) with D_0 defined as above, if $\tau = \min\{1, \tau_0\}$, where τ_0 is the universal constant given by the lemma, then there is a point z in $\partial\mathcal{B}_{1-\tau mr/2}(\overline{x}) \cap \mathcal{B}_r(y)$ such that $u_i^\varepsilon(z) \geq \sigma/2$. Next, if $x \in \mathcal{B}_{\tau mr/4}(\overline{x})$ then

$$\mathcal{B}_1(x) \supset \mathcal{B}_{\tau mr/4}(z)$$

(since $d_\rho(x,z) \leq d_\rho(x,\overline{x}) + d_\rho(\overline{x},z) \leq \tau mr/4 + 1 - \tau mr/2 = 1 - \tau mr/4$).

First consider the case of H defined as in (2.6). Then for any $x \in \mathcal{B}_{\tau mr/4}(\overline{x})$ we have

$$H(u_i^\varepsilon)(x) = \sup_{\mathcal{B}_1(x)} u_i^\varepsilon \geq u_i^\varepsilon(z) \geq \sigma/2,$$

which, together with (5.2), implies (5.1) with $\overline{\alpha} = 1$ and $\overline{\beta} = 0$.

Next, let us turn to the case of H defined as in (2.5). Since $z \in \mathcal{B}_r(y)$ and $d_\rho(y, \operatorname{supp} f_i) \geq 2r$, we have $\mathcal{B}_r(z) \cap \operatorname{supp} f_i = \emptyset$, and therefore the function u_i^ε (which is eventually zero in $\mathcal{B}_r(z) \cap \Omega^c$) satisfies $\Delta u_i^\varepsilon \geq 0$ in $\mathcal{B}_r(z)$. This implies that $(u_i^\varepsilon)^p$, $p \geq 1$, is subharmonic in $\mathcal{B}_r(z)$, and by the mean value inequality,

$$\fint_{B_s(z)} (u_i^\varepsilon)^p dx \geq \left(\frac{\sigma}{2}\right)^p \tag{5.4}$$

in any Euclidean ball $B_s(z) \subset \mathcal{B}_r(z)$, for any $p \geq 1$. Since d_ρ and the Euclidean distance are equivalent, there is an $s \sim \tau mr$ such that

$$B_s(z) \subset \mathcal{B}_{\tau mr/8}(z) \subset \mathcal{B}_{\tau mr/4}(z) \subset \mathcal{B}_1(x). \tag{5.5}$$

Moreover, if $y \in B_s(z)$ and $x \in \mathcal{B}_{\tau mr/4}(\overline{x})$, then

$$\rho(y-x) \leq \rho(y-z) + \rho(z-\overline{x}) + \rho(\overline{x}-x) \leq \frac{\tau mr}{8} + \left(1 - \frac{\tau mr}{2}\right) + \frac{\tau mr}{4} = 1 - \frac{\tau mr}{8},$$

that is,

$$1 - \rho(y-x) \geq \tau mr/8. \tag{5.6}$$

Hence, using (5.5), (2.7), (5.6) and (5.4), for all $x \in \mathcal{B}_{\tau mr/4}(\overline{x})$ we get

$$H(u_i^\varepsilon)(x) = \int_{\mathcal{B}_1(x)} (u_i^\varepsilon)^p(y)\varphi(\rho(y-x))\,dy \geq \int_{B_s(z)} (u_i^\varepsilon)^p(y) C(1-\rho(y-x))^q\,dy$$
$$\geq \int_{B_s(z)} (u_i^\varepsilon)^p(y) C(\tau mr/8)^q\,dy \geq C\sigma^{\overline{\alpha}} r^{\overline{\beta}}$$

where $\overline{\alpha}$ and $\overline{\beta}$ depend on p, q and on the dimension n. This and (5.2) imply (5.1).

Now, by Lemma 5.2 we get $u_j^\varepsilon(\overline{x}) \leq Ce^{-c\sigma^\alpha r^\beta/\varepsilon}$ for $\alpha = \overline{\alpha}/2 + 1$ and $\beta = \overline{\beta}/2 + 1$, and the lemma is proven. □

Corollary 5.4. *Assume* (2.8). *Let* $(u_1^\varepsilon, \dots, u_K^\varepsilon)$ *be a viscosity solution of problem* (2.4). *Let* y *be a point in* Ω *such that for some* i,

$$u_i^\varepsilon(y) = \sigma, \quad d_\rho(y, \operatorname{supp} f_j) \geq 1 + \tau mr \quad \text{for all } j \neq i, \quad d_\rho(y, \partial\Omega) \geq 2r,$$

where $m = \sigma/\sup_{\partial\Omega} f_i$, $0 < r < 1$, $\varepsilon \leq \sigma^{2\alpha} r^{2\beta}$ *and* τ, α *and* β *are given by Lemma* 5.3. *Then there exists a constant* $C_0 > 0$ *such that in* $\mathcal{B}_{\tau mr/4}(y)$ *we have*

$$|\nabla u_i^\varepsilon| \leq C_0/r \tag{5.7}$$

and

$$\Delta u_i^\varepsilon \to 0 \quad \text{uniformly as } \varepsilon \to 0. \tag{5.8}$$

Proof. First of all, note that $m \leq 1$, as u_i attains its maximum at the boundary of Ω. Since in addition $\tau < 1$, we see that $\mathcal{B}_{\tau mr/2}(y) \subset \mathcal{B}_{2r}(y) \subset \Omega$. Therefore, we use (2.4) to estimate $\Delta u_i^\varepsilon(z)$ for $z \in \mathcal{B}_{\tau mr/2}(y)$. To do that, we need to estimate $H(u_j^\varepsilon)(z)$ for $j \neq i$. But $H(u_j^\varepsilon)(z)$ involves points x at ρ-distance 1 from z. Let x be such that $d_\rho(x,z) \leq 1$. Then $d_\rho(x,y) \leq 1 + \tau mr/2$. Moreover, since $d_\rho(y, \operatorname{supp} f_j) \geq 1 + \tau mr$, we have $d_\rho(x, \operatorname{supp} f_j) \geq \tau mr/2$. Hence, by Lemma 5.3, for any $j \neq i$,

$$u_j^\varepsilon(x) \leq Ce^{-c\sigma^\alpha r^\beta/\varepsilon} \quad \text{for } x \in \mathcal{B}_1(z).$$

From the previous estimate and (2.4), it follows that for $z \in \mathcal{B}_{\tau mr/2}(y)$ we have

$$0 \leq \Delta u_i^\varepsilon(z) \leq u_i^\varepsilon(z) \frac{Ce^{-c\sigma^\alpha r^\beta/\varepsilon}}{\varepsilon^2} \leq u_i^\varepsilon(z) \frac{Ce^{-c\varepsilon^{-1/2}}}{\varepsilon^2} = o(1) \quad \text{as } \varepsilon \to 0, \tag{5.9}$$

for $\varepsilon \leq \sigma^{2\alpha} r^{2\beta}$. If we normalize the function in a Lipschitz fashion:

$$\bar{u}_i^\varepsilon(\overline{z}) := 2\frac{u_i^\varepsilon\left(\frac{\tau mr}{2}\overline{z} + y\right)}{\tau mr},$$

then we have

$$\bar{u}_i^\varepsilon(0) = 2\frac{u_i^\varepsilon(y)}{\tau mr} = \frac{2\sup_{\partial\Omega} f_i}{\tau r},$$

and

$$0 \leq \Delta \bar{u}_i^\varepsilon(\overline{z}) \leq \frac{\tau mr}{2} \bar{u}_i^\varepsilon(\overline{z}) \sum_{j \neq i} \frac{1}{\varepsilon^2} H(u_j^\varepsilon)\left(\frac{\tau mr}{2}\overline{z} + y\right) \quad \text{for } \overline{z} \in \mathcal{B}_1(0),$$

where

$$\frac{\tau m r}{2}\bar{u}_i^\varepsilon(\bar{z})\sum_{j\neq i}\frac{1}{\varepsilon^2}H(u_j^\varepsilon)\left(\frac{\tau m r}{2}\bar{z}+y\right)\le\frac{Ce^{-c\varepsilon^{-1/2}}}{\varepsilon^2}.$$

Then, by the Harnack inequality (see e.g. [3, Theorem 4.3]), we get

$$\sup_{\mathcal{B}_{1/2}(0)}\bar{u}_i^\varepsilon\le C_n\left(\inf_{\mathcal{B}_{1/2}(0)}\bar{u}_i^\varepsilon+\frac{Ce^{-c\varepsilon^{-1/2}}}{\varepsilon^2}\right)\le C_n\left(\frac{2\sup_{\partial\Omega}f_i}{\tau r}+\frac{Ce^{-c\varepsilon^{-1/2}}}{\varepsilon^2}\right)\le\frac{C}{r}.$$

Lipschitz estimates then imply that $|\nabla\bar{u}_i^\varepsilon|\le C/r$ in $\mathcal{B}_{1/2}(0)$, and (5.7) follows.

Further, (5.9) implies (5.8). □

The next lemma says that in a ρ-strip of size 1 around the support of f_j, the function u_i^ε, $i\neq j$, decays to 0 exponentially.

Lemma 5.5. *Assume (2.8). Let $(u_1^\varepsilon,\dots,u_K^\varepsilon)$ be a viscosity solution of problem (2.4). For $j=1,\dots,K$, $\sigma>0$, let $\overline{\Gamma}_j^\sigma:=\{f_j\ge\sigma\}\subset\Omega^c$. Then on the sets*

$$\{x\in\Omega\mid d_\rho(x,\overline{\Gamma}_j^\sigma)\le 1-r\},\qquad 0<r<1,$$

we have

$$u_i^\varepsilon\le Ce^{-c\sigma^\alpha r^\beta/\varepsilon}\qquad\text{for } i\neq j,$$

for some positive α and β depending on the structure of H (p and q) and the modulus of continuity of f_j.

Proof. Let $\bar{x}\in\Omega$ and $y\in\overline{\Gamma}_j^\sigma$ be such that $d_\rho(\bar{x},y)\le 1-r$. We want to estimate $H(u_j^\varepsilon)(x)$ for any $x\in\mathcal{B}_{r/2}(\bar{x})$. Let $x\in\mathcal{B}_{r/2}(\bar{x})$. Then

$$d_\rho(x,y)\le 1-r/2. \tag{5.10}$$

First consider the case of H defined as in (2.6). We have

$$H(u_j^\varepsilon)(x)=\sup_{\mathcal{B}_1(x)}u_j^\varepsilon\ge f_j(y)\ge\sigma.$$

Next, let us turn to the case of H defined as in (2.5). Let $r_0:=\min\{\sigma^\gamma,r/4\}$ for some γ depending on the modulus of continuity of f_j. Then $f_j\ge\sigma/2$ in $\mathcal{B}_{r_0}(y)\cap\operatorname{supp}f_j$. Moreover, from (5.10) and $r_0\le r/4$, we have

$$\mathcal{B}_{r_0}(y)\cap\operatorname{supp}f_j\subset\mathcal{B}_{r/4}(y)\subset\mathcal{B}_{r/2}(y)\subset\mathcal{B}_1(x),$$

and for any $z\in\mathcal{B}_{r_0}(y)\cap\operatorname{supp}f_j$,

$$\rho(x-z)\le\rho(x-y)+\rho(y-z)\le 1-r/2+r_0\le 1-r/4.$$

Therefore, using in addition (2.7), and the fact that, by (2.8), $|\mathcal{B}_{r_0}(y) \cap \operatorname{supp} f_j| \geq c|\mathcal{B}_{r_0}(y)|$, we get

$$H(u_j^\varepsilon)(x) = \int_{\mathcal{B}_1(x)} (u_j^\varepsilon)^p(z)\varphi(\rho(x-z))\,dz \geq \int_{\mathcal{B}_{r_0}(y)\cap \operatorname{supp} f_j} (u_j^\varepsilon)^p(z)(1-\rho(x-z))^q\,dz$$
$$\geq \int_{\mathcal{B}_{r_0}(y)\cap \operatorname{supp} f_j} (f_j)^p(z) C(r/4)^q\,dz \geq C\sigma^p r_0^{\overline{\beta}},$$

where $\overline{\beta}$ depends on q and on the dimension n.

Thus, for H defined as in (2.5) or (2.6), the function u_i^ε, $i \neq j$, (which is eventually zero in $B_{r/2}(\overline{x}) \cap \Omega^c$) is a subsolution of

$$\Delta u_i^\varepsilon \geq u_i^\varepsilon \frac{C\sigma^p r_0^{\overline{\beta}}}{\varepsilon^2}$$

in $B_{r/2}(\overline{x})$, where $p = 1$ and $\overline{\beta} = 0$ in the case (2.6). The conclusion follows as in Lemma 5.3. □

The following corollary is a consequence of Lemma 5.3, Corollary 5.4 and Lemma 5.5.

Corollary 5.6. *Assume (2.8). Let $(u_1^\varepsilon, \ldots, u_K^\varepsilon)$ be a viscosity solution of problem (2.4). Then there exists a subsequence $(u_1^{\varepsilon_l}, \ldots, u_K^{\varepsilon_l})$ and continuous functions $(u_1, \ldots, u_K)$ such that*

$$(u_1^{\varepsilon_l}, \ldots, u_K^{\varepsilon_l}) \to (u_1, \ldots, u_K) \quad \text{a.e. in } \Omega \quad \text{as } l \to \infty,$$

and the convergence of $u_i^{\varepsilon_l}$ to u_i is locally uniform in the set $\{x \in \Omega \mid d_\rho(x, \operatorname{supp} f_j) > 1, j \neq i\}$. Moreover:

(i) *the u_i's are locally Lipschitz continuous in Ω and have disjoint supports, in particular*

$$u_i \equiv 0 \quad \text{in } \{x \in \Omega \mid d_\rho(x, \operatorname{supp} u_j) \leq 1\} \quad \text{for any } j \neq i.$$

(ii) $\Delta u_i = 0$ *when* $u_i > 0$.

Proof. Fix an index $i = 1, \ldots, K$. Let

$$\Omega_i := \{x \in \Omega \mid d_\rho(x, \operatorname{supp} f_j) > 1 \text{ for any } j \neq i\}, \qquad B_i := \Omega \setminus \overline{\Omega}_i.$$

Claim 1. $u_i^\varepsilon(x) \to 0$ *as* $\varepsilon \to 0$ *for any* $x \in B_i$.

Indeed, let $x_0 \in B_i$. Then there exists $j \neq i$ such that $d_\rho(x_0, \operatorname{supp} f_j) < 1$. Note that

$$\left\{x \in \Omega \;\middle|\; d_\rho(x, \operatorname{supp} f_j) < 1\} \subset \bigcup_{r,\sigma>0} \{x \in \Omega \mid d_\rho(x, \overline{\Gamma}_j^\sigma) \leq 1 - r\right\},$$

where $\overline{\Gamma}_j^\sigma = \{f_j \geq \sigma\}$. Therefore, there exist $r, \sigma > 0$ such that $x_0 \in \{x \in \Omega \mid d_\rho(x, \overline{\Gamma}_j^\sigma) \leq 1-r\}$, and by Lemma 5.5 we have $u_i^\varepsilon(x_0) \leq Ce^{-c\sigma^\alpha r^\beta/\varepsilon}$ for some $\alpha, \beta > 0$. Claim 1 follows.

Claim 2. *There exists a subsequence $(u_i^{\varepsilon_l})_l$ locally uniformly convergent in Ω_i as $l \to \infty$ to a locally Lipschitz continuous function u_i.*

Fix $0 < r < 1$ and define

$$\Omega_i^r := \{x \in \Omega_i \mid d_\rho(x, \partial\Omega) > 2r,\ d_\rho(x, \text{supp } f_j) \geq 1 + \tau r \text{ for any } j \neq i\}.$$

Fix $\theta < 1/(2\alpha)$, set $\sigma_\varepsilon = \varepsilon^\theta > 0$ and consider τ, α and β as given by Lemma 5.3. Since $\varepsilon = \sigma_\varepsilon^{2\alpha}\sigma_\varepsilon^{1/\theta - 2\alpha} = \sigma_\varepsilon^{2\alpha}\varepsilon^{\theta(1/\theta-2\alpha)}$ and $1/\theta - 2\alpha > 0$, we can fix $\varepsilon_0 = \varepsilon_0(r)$ so small that for any $\varepsilon < \varepsilon_0$ we have $\varepsilon \leq \sigma_\varepsilon^{2\alpha} r^{2\beta}$. Then, by Corollary 5.4, the functions

$$v_i^\varepsilon := (u_i^\varepsilon - \sigma_\varepsilon)_+ = (u_i^\varepsilon - \varepsilon^\theta)_+$$

are Lipschitz continuous in Ω_i^r. Indeed, if $u_i^\varepsilon(x) < \varepsilon^\theta$, then $v_i^\varepsilon(x) = 0$. Next, let x be such that $u_i^\varepsilon(x) > \varepsilon^\theta$. Then $\nabla v_i^\varepsilon(x) = \nabla u_i^\varepsilon(x)$. Set $\sigma = u_i^\varepsilon(x)$. Then $d_\rho(x, \text{supp } f_j) \geq 1 + \tau r \geq 1 + m\tau r$, where $m = \sigma/\sup_{\partial\Omega} f_i \leq 1$. Moreover, $d_\rho(x, \partial\Omega) > 2r$ and $\varepsilon \leq \sigma_\varepsilon^{2\alpha} r^{2\beta} \leq \sigma^{2\alpha} r^{2\beta}$. We can therefore apply Corollary 5.4 to get

$$|\nabla u_i^\varepsilon(x)| \leq C_0/r.$$

This concludes the proof that the functions v_i^ε are Lipschitz continuous in Ω_i^r.

Therefore, we can extract a subsequence $(v_i^{\varepsilon_l})_l$ uniformly convergent to a Lipschitz continuous function u_i in Ω_i^r as $l \to \infty$. By the definition of the v_i's, there exists a subsequence $(u_i^{\varepsilon_l})_l$ uniformly convergent to the same function u_i in Ω_i^r as $l \to \infty$. Taking $r \to 0$ and using a diagonalization argument, we can find a subsequence of $(u_i^\varepsilon)_\varepsilon$ converging locally uniformly to a Lipschitz function u_i in Ω_i. This ends the proof of Claim 2.

Claims 1 and 2 yield the convergence, up to a subsequence, of u_i^ε to a continuous function u_i which is locally Lipschitz in both Ω_i and B_i. The fact that the supports of the limit functions are at distance ≥ 1 is a consequence of Claims 1 and 2 and Lemma 5.3. Moreover, from the proof of Claim 2 and Corollary 5.4, we infer that the limit function u_i is harmonic inside its support, i.e. (ii) holds. To conclude the proof of (i), we just need to prove that u_i is Lipschitz in a neighborhood of points belonging to $\partial B_i = \partial\Omega_i \cap \Omega$. Let $x_0 \in \partial\Omega_i \cap \Omega$. Then from Claim 1, $u_i(x_0) = 0$. If $x_0 \notin \partial\{u_i > 0\}$, then in a neighborhood of x_0, $u_i \equiv 0$ and of course it is Lipschitz there. On the other hand, if $x_0 \in \partial\{u_i > 0\}$, then since there exists an exterior ρ-tangent ball of radius 1 at any point of $\partial\Omega_i \cap \Omega$ and u_i is harmonic inside its support, a barrier argument implies that there exist $r_0, C > 0$ such that $0 \leq u_i(x) = u_i(x) - u_i(x_0) \leq C|x - x_0|$ for any $x \in B_{r_0}(x_0)$. This concludes the proof of (i) and of the corollary. □

6. A semiconvexity property of the free boundaries

Let $(u_1, \ldots, u_K)$ be the limit of a convergent subsequence of $(u_1^\varepsilon, \ldots, u_K^\varepsilon)$, whose existence is guaranteed by Corollary 5.6. For $i = 1, \ldots, K$, set

$$S(u_i) := \{x \in \Omega : u_i > 0\}. \tag{6.1}$$

(In the next sections, for simplicity this set will be represented by S_i.) Then the sets $S(u_i)$ have the following semiconvexity property:

Lemma 6.1. *Set*

$$T(u_i) = \{x \in \Omega : d_\rho(x, S(u_i)) \geq 1\}, \qquad S^*(u_i) = \{x \in \Omega : d_\rho(x, T(u_i)) > 1\}.$$

Then $\partial S(u_i) \subset \partial S^*(u_i)$.

Proof. We have $S^*(u_i) \supset S(u_i)$. To prove the desired inclusion, for $\sigma > 0$ consider the sets $S_\sigma(u_i) := \{x \in \Omega : u_i > \sigma\}$ and

$$T_\sigma(u_i) := \{x \in \Omega : d_\rho(x, S_\sigma(u_i)) \geq 1\}, \qquad S^*_\sigma(u_i) := \{x \in \Omega : d_\rho(x, T_\sigma(u_i)) > 1\}.$$

Notice that the union of the ρ-balls centered at points in $S_\sigma(u_i)$ coincides with the union of the ρ-balls centered at points in $S^*_\sigma(u_i)$:

$$\text{(a) } (T_\sigma(u_i))^c = \bigcup_{x \in S_\sigma(u_i)} \mathcal{B}_1(x), \qquad \text{(b) } (T_\sigma(u_i))^c = \bigcup_{x \in S^*_\sigma(u_i)} \mathcal{B}_1(x).$$

If $x \in S_\sigma(u_i)$, from Corollary 5.6(i) we have $d_\rho(x, \operatorname{supp} f_j) > 1$ for $j \neq i$, and the locally uniform convergence of u_i^ε to u_i and Lemma 5.3 imply that, up to subsequences, $u_j^\varepsilon \leq Ce^{-c\sigma^\alpha r^\beta/\varepsilon}$ in $\mathcal{B}_1(x)$, where $2r = \min\{d_\rho(x, \operatorname{supp} f_i), C(d_\rho(x, \operatorname{supp} f_j) - 1)\}$. Now, the set where u_j^ε decays is the same if we had considered $x \in S^*_\sigma(u_i)$, by (a) and (b). Therefore $H(u_j^\varepsilon)/\varepsilon^2 \to 0$ in $S^*_\sigma(u_i)$ as $\varepsilon \to 0$. It follows that $\Delta u_i \equiv 0$ in $S^*_\sigma(u_i)$ if $S^*_\sigma(u_i)$ is not empty. Now, if A is a connected component of $S_\sigma(u_i)$, then there exists a connected component A^* of $S^*_\sigma(u_i)$ such that $A \subset A^*$. Since u_i is harmonic and non-negative in A^*, the strong maximum principle implies that $u_i > 0$ in all of A^*, that is, $A^* \subset A$. We conclude that $A = A^*$. Therefore, any connected component of $S_\sigma(u_i)$ is equal to a connected component of $S^*_\sigma(u_i)$. Passing to the limit as $\sigma \to 0$, we find that any connected component of $S(u_i)$ is equal to a connected component of $S^*(u_i)$. In particular, $\partial S(u_i) \subset \partial S^*(u_i)$. □

From Lemma 6.1 we can conclude that the sets $S(u_i)$ have a tangent ρ-ball of radius 1 from the outside at any point of the boundary, as stated in the following corollary.

Corollary 6.2. *If* $x_0 \in \partial S(u_i) \cap \Omega$ *there is an exterior tangent ball* $\mathcal{B}_1(y)$ *at* x_0*, in the sense that for* $x \in \mathcal{B}_1(y) \cap \mathcal{B}_1(x_0)$*, all* $u_j(x) \equiv 0$ *(including* u_i*).*

The following two lemmas about the distance function may be known; we provide the proof for the reader's convenience.

Lemma 6.3. *Let* S *be a closed set. Then, in the set* $\{x \mid d_\rho(x, S) > 0\}$*,* $d_\rho(\cdot, S)$ *satisfies in the viscosity sense*

$$\Delta d_\rho(\cdot, S) \leq \frac{C}{d_\rho(\cdot, S)},$$

where C *is a constant depending on* n*,* $\|Dd_\rho(\cdot, S)\|_{L^\infty}$ *and the constant* A *of* (2.1).

Proof. We first prove that there exists a smooth tangent function from above at any point of the graph of $d_\rho(\cdot, S)$ in the set $\{d_\rho(\cdot, S) > 0\}$. For simplicity we will write $d_S(\cdot)$ instead of $d_\rho(\cdot, S)$. Let y_0 be a point in the complement of S. Let $x \in \partial S$ be a point realizing the distance from y_0 to S. Assume, without loss of generality, that $x = 0$. Then $d_\rho(y_0, 0) = \rho(y_0) = d_S(y_0)$. In particular, the ball $\mathcal{B}_{\rho(y_0)}(y_0)$ is contained in S^c and tangent to S at 0. For any $y \in \mathcal{B}_{\rho(y_0)}(y_0)$, we have $d_S(y) \leq d_\rho(y, 0) = \rho(y)$, therefore the graph of the function $y \mapsto \rho(y)$ is tangent from above to the graph of $d_S(\cdot)$ at $(y_0, d_S(y_0))$.

Next, let ψ be a test function touching $d_S(\cdot)$ from below at y_0. Then ψ touches from below the function $\rho(y)$ at y_0. In particular, $\Delta\psi(y_0) \leq \Delta\rho(y_0)$. Let us compute $\Delta\rho$. Using (2.1), we get

$$D^2(\rho) = \frac{1}{\rho} D^2\left(\frac{1}{2}\rho^2\right) - \frac{D\rho \otimes D\rho}{\rho} \leq \frac{1}{\rho}(AI_n - D\rho \otimes D\rho),$$

which gives $\Delta\rho \leq C/\rho$. In particular,

$$\Delta\psi(y_0) \leq \frac{C}{\rho(y_0)} = \frac{C}{d_S(y_0)}.$$

This concludes the proof. □

Lemma 6.4. *Let S be a closed and bounded set. Let $(S)_{=1}$ be the set of points at ρ-distance 1 from S. Then $(S)_{=1}$ has finite perimeter.*

Proof. For simplicity we will write $d_S(\cdot)$ instead of $d_\rho(\cdot, S)$ as in the previous lemma, and first we present a heuristic proof by integrating Δd_S^2 over the set $\{0 < d_S < 1\}$. Since $|Dd_S|$ is bounded, from Lemma 6.3 we see that

$$\Delta d_S^2 = 2|Dd_S|^2 + 2d_S \Delta d_S \leq C.$$

Therefore, integrating Δd_S^2, we get

$$\begin{aligned} C &\geq \int_{\{0<d_S<1\}} \Delta d_S^2 \, dx = \int_{\{d_S=0\}} 2d_S Dd_S \cdot n \, d\mathcal{H}^{n-1} + \int_{\{d_S=1\}} 2d_S Dd_S \cdot n \, d\mathcal{H}^{n-1} \\ &= \int_{\{d_S=1\}} 2Dd_S \cdot n \, d\mathcal{H}^{n-1} \geq c \int_{\{d_S=1\}} d\mathcal{H}^{n-1} = c\mathcal{H}^{n-1}(\{d_S = 1\}), \end{aligned}$$

where $n = Dd_S/|Dd_S|$ is the unit exterior normal. This provides an upper bound for $\mathcal{H}^{n-1}(\{d_S = 1\})$ and concludes the heuristic proof.

To make the argument precise, we need to handle the regularity over the boundary. For that, consider a smooth function η with compact support in $(0, 1)$ such that $0 \leq \eta(\xi) \leq \xi$ for any $\xi \in [0, 1]$, $\eta(\xi) = \xi$ for $\xi \in [\delta, 1 - \delta]$, $|\eta'| \leq c$ on $(0, 1 - \delta)$ and $\eta'(\xi) \leq -c/\delta$ for $\xi \in (1 - \delta, 1)$, where $\delta > 0$ is a small parameter. Then, in a weak sense,

$$\operatorname{div}(\eta(d_S)Dd_S) = \eta'(d_S)|Dd_S|^2 + \eta(d_S)\Delta d_S. \tag{6.2}$$

Moreover, from Lemma 6.3, in the set $\{0 < d_S < 1\}$ we have

$$\eta(d_S)\Delta d_S \le \eta(d_S)\frac{C}{d_S} \le C$$

in the viscosity sense, and therefore in the distributional sense (see, e.g., [24] for the equivalence between viscosity solutions and weak solutions). Therefore, since $\eta(d_S)$ is a function with compact support in $\{0 < d_S < 1\}$, we get

$$\begin{aligned} 0 &= \int_{\{0<d_S<1\}} \operatorname{div}(\eta(d_S)Dd_S)\,dx \le \int_{\{0<d_S<1\}} \eta'(d_S)|Dd_S|^2\,dx + C \\ &= \int_{\{0<d_S<1-\delta\}} \eta'(d_S)|Dd_S|^2\,dx + \int_{\{1-\delta<d_S<1\}} \eta'(d_S)|Dd_S|^2\,dx + C \\ &\le \int_{\{1-\delta<d_S<1\}} \eta'(d_S)|Dd_S|^2\,dx + C \le -\frac{c}{\delta}\int_{\{1-\delta<d_S<1\}} |Dd_S|^2\,dx + C. \end{aligned} \tag{6.3}$$

Now, using the coarea formula and the inequalities above, we get

$$\frac{1}{\delta}\int_{1-\delta}^{1} \mathcal{H}^{n-1}(\{d_S = t\})\,dt = \frac{1}{\delta}\int_{\{1-\delta<d_S<1\}} |Dd_S|^2\,dx \le C.$$

Finally, taking the limit as $\delta \to 0^+$ and using the lower semicontinuity of the perimeter with respect to convergence in measure, we infer that

$$\operatorname{Per}(\{d_S = 1\}) \le \liminf_{\delta\to 0^+} \frac{1}{\delta}\int_{1-\delta}^{1} \mathcal{H}^{n-1}(\{d_S = t\})\,dt \le C. \qquad \square$$

Corollary 6.5. *The sets $S(u_i)$, $i = 1, \dots, K$, have finite perimeter.*

Proof. The corollary is an immediate consequence of Lemmas 6.1 and 6.4. □

7. A sharp characterization of the interfaces

In Section 5 we proved that the supports of the limit functions u_i are at distance at least 1 from each other (Corollary 5.6). In this section we will prove that they are exactly at distance 1, as stated in the following theorem.

Theorem 7.1. *Assume (2.8) with $p = 1$ in (2.5). Let $(u_1^\varepsilon, \dots, u_K^\varepsilon)$ be a viscosity solution of problem (2.4) and $(u_1, \dots, u_K)$ the limit as $\varepsilon \to 0$ of a convergent subsequence. Let $x_0 \in \partial\{u_i > 0\} \cap \Omega$. Then there exists $j \ne i$ such that*

$$\overline{\mathcal{B}_1(x_0)} \cap \partial\{u_j > 0\} \ne \emptyset. \tag{7.1}$$

Proof. It is enough to prove the theorem for a point x_0 for which $\partial S(u_i)$ has a tangent ρ-ball from the inside, since such points are dense on $\partial S(u_i)$. Indeed, let x be any point of $\partial S(u_i)$. Consider a sequence (x_k) of points in $S(u_i)$ converging to x as $k \to \infty$. Let d_k be the ρ-distance of x_k from $\partial S(u_i)$. Then the ρ-balls $\mathcal{B}_{d_k}(x_k)$ are contained in $S(u_i)$ and there exist points $y_k \in \partial S(u_i) \cap \mathcal{B}_{d_k}(x_k)$ where the x_k's realize the distance from $\partial S(u_i)$. The sequence (y_k) is a sequence of points of $\partial S(u_i)$ that have a tangent ρ-ball from the inside and converge to x.

Next, from Corollary 5.6(ii), we know that $d_\rho(x_0, \operatorname{supp} f_j) \geq 1$ for any $j \neq i$. If there is a j such that $d_\rho(x_0, \operatorname{supp} f_j) = 1$, then (7.1) is obviously true. Therefore, we can assume that $d_\rho(x_0, \operatorname{supp} f_j) > 1$ for any $j \neq i$. Then for small $S > 0$ we have $\mathcal{B}_{1+S}(x_0) \cap \operatorname{supp} f_j = \emptyset$, and from (2.4) we know that

$$\Delta u_j^\varepsilon \geq \frac{1}{\varepsilon^2} u_j^\varepsilon \sum_{k \neq j} H(u_k^\varepsilon) \quad \text{in } \mathcal{B}_{1+S}(x_0).$$

We divide the proof into two cases:

$$\text{(a) } H(u)(x) = \int_{\mathcal{B}_1(x)} u(y)\varphi(\rho(x-y))\,dy, \qquad \text{(b) } H(u)(x) = \sup_{y \in \mathcal{B}_1(x)} u(y).$$

Proof of case (a). Let $S(u_i) = \{x \in \Omega \mid u_i > 0\}$ as in (6.1). Let $\mathcal{B}_S$ be a small ρ-ball centered at $x_0 \in \partial S(u_i)$. Then, as a measure, as $\varepsilon \to 0$, up to a subsequence

$$\Delta u_i^\varepsilon|_{\mathcal{B}_S(x_0)} \to \Delta u_i|_{\mathcal{B}_S(x_0)}$$

(the latter has strictly positive mass, since u_i is not harmonic in $\mathcal{B}_S(x_0)$).

We can bound

$$\int_{\mathcal{B}_{1+S}(x_0)} \sum_{j \neq i} \Delta u_j^\varepsilon\,dx \geq \int_{\mathcal{B}_S(x_0)} \Delta u_i^\varepsilon\,dx.$$

Indeed,

$$\begin{aligned}
\varepsilon^2 \int_{\mathcal{B}_S(x_0)} \Delta u_i^\varepsilon(x)\,dx &= \sum_{j \neq i} \int_{\mathcal{B}_S(x_0)} \int_{\mathcal{B}_1(x)} u_i^\varepsilon(x)\varphi(\rho(x-y))u_j^\varepsilon(y)\,dy\,dx \\
&= \sum_{j \neq i} \int\int_{\mathcal{B}_S(x_0) \times \mathcal{B}_{1+S}(x_0)} u_i^\varepsilon(x)\chi_{[0,1]}(\rho(x-y))\varphi(\rho(x-y))u_j^\varepsilon(y)\,dx\,dy \\
&\leq \sum_{j \neq i} \int\int_{\mathcal{B}_{2+S}(x_0) \times \mathcal{B}_{1+S}(x_0)} u_i^\varepsilon(x)\chi_{[0,1]}(\rho(x-y))\varphi(\rho(x-y))u_j^\varepsilon(y)\,dx\,dy \\
&= \sum_{j \neq i} \int_{\mathcal{B}_{1+S}(x_0)} \int_{\mathcal{B}_1(y)} u_i^\varepsilon(x)\varphi(\rho(x-y))u_j^\varepsilon(y)\,dx\,dy \leq \varepsilon^2 \sum_{j \neq i} \int_{\mathcal{B}_{1+S}(x_0)} \Delta u_j^\varepsilon(y)\,dy,
\end{aligned} \tag{7.2}$$

where $\chi_{[0,1]}$ is the indicator function of the set $[0, 1]$.

Therefore, for any small positive S, letting $\varepsilon \to 0$ we get

$$\int_{\mathcal{B}_{1+S}(x_0)} \sum_{j \neq i} \Delta u_j \geq \int_{\mathcal{B}_S(x_0)} \Delta u_i > 0,$$

which implies that there exists $j \neq i$ such that u_j cannot be identically zero in $\mathcal{B}_{1+S}(x_0)$. Since S small is arbitrary, the result follows.

Proof of case (b). This case is more involved. We may assume $x_0 = 0$. Let y_0 be such that $\mathcal{B}_\mu(y_0) \subset S(u_i)$ and $0 \in \partial\mathcal{B}_\mu(y_0)$. By Corollary 6.2 there exists a ρ-ball $\mathcal{B}_1(y_1)$ such that $\mathcal{B}_1(y_1) \cap S(u_i) = \emptyset$ and $0 \in \partial\mathcal{B}_1(y_1)$.

Let us first prove two claims.

Claim 1. *There exist $\mu' < \mu$ and $C_1 > 0$ such that in the annulus $\{\mu' < \rho(x - y_0) < \mu\}$ we have*

$$u_i(x) \geq C_1 d_\rho(x, \partial\mathcal{B}_\mu(y_0)).$$

Since any ρ-ball $\mathcal{B}$ satisfies the uniform interior ball condition, for any $\bar{x} \in \partial\mathcal{B}_\mu(y_0)$ there exists a Euclidean ball $B_{R_0}(z_0)$ of radius R_0 independent of $\bar{x}$ contained in $\mathcal{B}_\mu(y_0)$ and tangent to $\partial\mathcal{B}_\mu(y_0)$ at $\bar{x}$. Let $m > 0$ be the infimum of u_i on the set $\{x \in \mathcal{B}_\mu(y_0) \mid d(x, \partial\mathcal{B}_\mu(y_0)) \geq R_0/2\}$, where d is the Euclidean distance function, and let ϕ be the solution of

$$\begin{cases} \Delta\phi = 0 & \text{in } \{R_0/2 < |x - z_0| < R_0\}, \\ \phi = 0 & \text{on } \partial B_{R_0}(z_0), \\ \phi = m & \text{on } \partial B_{R_0/2}(z_0), \end{cases}$$

i.e. for $n \geq 3$,

$$\phi(x) = C(n)m\left(\frac{R_0^{n-2}}{|x - z_0|^{n-2}} - 1\right).$$

Since u_i is harmonic in $\mathcal{B}_\mu(y_0)$ and $u_i \geq \phi$ on $\partial B_{R_0}(z_0) \cup \partial B_{R_0/2}(z_0)$, by the comparison principle $u_i \geq \phi$ in $\{R_0/2 < |x - z_0| < R_0\}$. In particular, for any $x \in \{R_0/2 < |x - z_0| < R_0\}$ belonging to the segment between z_0 and $\bar{x}$, using the fact that ϕ is convex in the radial direction, that

$$\left.\frac{\partial\phi}{\partial\nu_i}\right|_{\partial B_{R_0}(z_0)} = \frac{C(n)(n-2)m}{R_0}$$

where ν_i is the interior normal at $\partial B_{R_0}(z_0)$, and that (2.2) holds, we get

$$\begin{aligned} u_i(x) &\geq \frac{C(n)(n-2)m}{R_0} d(x, \partial B_{R_0}(z_0)) = C(n, R_0)md(x, \partial\mathcal{B}_\mu(y_0)) \\ &\geq C_1 d_\rho(x, \partial\mathcal{B}_\mu(y_0)). \end{aligned}$$

Therefore, letting $\bar{x}$ vary in $\partial\mathcal{B}_\mu(y_0)$ we get

$$u_i(x) \geq C_1 d_\rho(x, \partial\mathcal{B}_\mu(y_0)) \quad \text{for any } x \in \mathcal{B}_\mu(y_0) \text{ with } d(x, \partial\mathcal{B}_\mu(y_0)) \leq R_0/2.$$

By (2.2), Claim 1 follows.

Next, let $e_0 = y_0/\rho(y_0)$ and fix $\sigma < \mu$ so small that $\mathcal{B}_\sigma(\sigma e_0) \subset \{\mu' < \rho(x - y_0) < \mu\} \cap \mathcal{B}_{1+\delta}(y_1)$. For $r \in [\sigma - \upsilon, \sigma + \upsilon]$ and small $\upsilon < \sigma$, define

$$\underline{u}_i^\varepsilon := \inf_{\partial\mathcal{B}_r(\sigma e_0)} u_i^\varepsilon \quad \text{and} \quad \underline{u}_i := \inf_{\partial\mathcal{B}_r(\sigma e_0)} u_i.$$

Since $\partial\mathcal{B}_r(\sigma e_0) \cap (S(u_i))^c \neq \emptyset$ for $r \in [\sigma, \sigma + \upsilon]$, and $u_i \equiv 0$ on $(S(u_i))^c$, we have

$$\underline{u}_i = 0 \quad \text{for } r \in [\sigma, \sigma + \upsilon]. \tag{7.3}$$

By Claim 1, we know that in $B_\sigma(\sigma e_0)$ we have

$$u_i(x) \geq C_1 d_\rho(x, \partial\mathcal{B}_\mu(y_0)) \geq C_1 d_\rho(x, \partial\mathcal{B}_\sigma(\sigma e_0)) = C_1(\sigma - \rho(x - \sigma e_0)).$$

We deduce that, for $r \in [\sigma - \upsilon, \sigma]$,

$$\underline{u}_i = \inf_{\partial\mathcal{B}_r(\sigma e_0)} u_i \geq \inf_{\partial\mathcal{B}_r(\sigma e_0)} C_1(\sigma - \rho(x - \sigma e_0)) = C_1(\sigma - r).$$

From this inequality and (7.3), we infer that

$$\underline{u}_i \geq C_1(\sigma - r)^+, \qquad r \in [\sigma - \upsilon, \sigma + \upsilon]. \tag{7.4}$$

Next, for $j \neq i$ and $r \in [\sigma - \upsilon, \sigma + \upsilon]$, define

$$\bar{u}_j^\varepsilon := \sup_{\partial\mathcal{B}_{1+r}(\sigma e_0)} u_j^\varepsilon \quad \text{and} \quad \bar{u}_j := \sup_{\partial\mathcal{B}_{1+r}(\sigma e_0)} u_j.$$

The functions $\underline{u}_i^\varepsilon$ and $\bar{u}_j^\varepsilon$ are respectively solutions of

$$\Delta_r \underline{u}_i^\varepsilon \leq \frac{1}{\varepsilon^2}\underline{u}_i^\varepsilon \sum_{i\neq j} \sup_{\mathcal{B}_1(\underline{z}_r^i)} u_j^\varepsilon, \qquad \Delta_r \bar{u}_j^\varepsilon \geq \frac{1}{\varepsilon^2}\bar{u}_j^\varepsilon \sup_{\mathcal{B}_1(\bar{z}_r^j)} u_i^\varepsilon, \tag{7.5}$$

where

$$\Delta_r u = u_{rr} + \frac{n-1}{r}u_r = \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial u}{\partial r}\right)$$

and $\underline{z}_r^i$ and $\bar{z}_r^j$ are respectively the points where the infimum of u_i^ε on $\partial\mathcal{B}_r(\sigma e_0)$ and the supremum of u_j^ε on $\partial\mathcal{B}_{1+r}(\sigma e_0)$ are attained. Note that in spherical coordinates

$$\Delta u = \Delta_r u + \Delta_\theta u,$$

and if we are at a point where u attains a minimum value in θ for a fixed r then $\Delta_\theta u \geq 0$, while the opposite inequality holds if we are at a maximum point. We also remark that

$$\bar{y}_r^j := \sigma e_0 + \frac{r}{r+1}(\bar{z}_r^j - \sigma e_0) \in \partial\mathcal{B}_r(\sigma e_0) \cap \partial\mathcal{B}_1(\bar{z}_r^j),$$

therefore

$$\sup_{\mathcal{B}_1(\bar{z}_r^j)} u_i^\varepsilon \geq u_i^\varepsilon(\bar{y}_r^j) \geq \underline{u}_i^\varepsilon. \tag{7.6}$$

Moreover, since $\mathcal{B}_1(\underline{z}_r^i) \subset \mathcal{B}_{1+r}(\sigma e_0)$ and u_j^ε is a subharmonic function, we have

$$\sup_{\mathcal{B}_1(\underline{z}_r^i)} u_j^\varepsilon \leq \sup_{\mathcal{B}_{1+r}(\sigma e_0)} u_j^\varepsilon = \sup_{\partial\mathcal{B}_{1+r}(\sigma e_0)} u_j^\varepsilon = \bar{u}_j^\varepsilon. \tag{7.7}$$

From (7.5)–(7.7), we conclude that

$$\Delta_r \underline{u}_i^\varepsilon \leq \Delta_r\left(\sum_{j\neq i} \bar{u}_j^\varepsilon\right). \tag{7.8}$$

In other words, for any $\phi \in C_c^\infty(\sigma - \upsilon, \sigma + \upsilon)$, $\phi \geq 0$, we have

$$\int_{\sigma-\upsilon}^{\sigma+\upsilon} \underline{u}_i^\varepsilon \frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}\left(\frac{1}{r^{n-1}}\phi\right)\right) dr \leq \int_{\sigma-\upsilon}^{\sigma+\upsilon} \sum_{j\neq i} \bar{u}_j^\varepsilon \frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}\left(\frac{1}{r^{n-1}}\phi\right)\right) dr.$$

Passing to the limit as $\varepsilon \to 0$ along a uniformly converging subsequence, we get

$$\int_{\sigma-\upsilon}^{\sigma+\upsilon} \underline{u}_i \frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}\left(\frac{1}{r^{n-1}}\phi\right)\right) dr \le \int_{\sigma-\upsilon}^{\sigma+\upsilon} \sum_{j\neq i} \bar{u}_j \frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}\left(\frac{1}{r^{n-1}}\phi\right)\right) dr.$$

The linear growth of u_i away from the free boundary, given by (7.3) and (7.4), implies that $\Delta_r \underline{u}_i$ develops a Dirac mass at $r = \sigma$ and

$$\int_{\sigma-\upsilon}^{\sigma+\upsilon} \underline{u}_i \frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}\left(\frac{1}{r^{n-1}}\phi\right)\right) dr > 0$$

for υ small enough. Hence, $\Delta_r(\sum_{j\neq i} \bar{u}_j)$ is a positive measure in $(\sigma - \upsilon, \sigma + \upsilon)$, and therefore there exists $j \neq i$ such that u_j cannot be identically zero in $\mathcal{B}_{1+\sigma}(\sigma e_0)$. Since σ small is arbitrary, the result follows. □

8. Classification of singular points and Lipschitz regularity in dimension 2

In this section we study singular points in dimension 2. We will always assume (2.8) with $p = 1$ in (2.5). From the results of the previous sections we know that the solutions $u_1^\varepsilon, \dots, u_K^\varepsilon$ of system (2.4), along a subsequence, converge as $\varepsilon \to 0$ to functions $u_1, \dots, u_K$ which are locally Lipschitz continuous in Ω and harmonic inside their support. For $i = 1, \dots, K$, denote the interior of the support of u_i by S_i as in (6.1), and the union of the interiors of the supports of all the other functions by

$$C_i := \bigcup_{j\neq i} S_j. \tag{8.1}$$

Since the sets S_i are disjoint, we have $\partial C_i = \bigcup_{j\neq i} \partial S_j$. From Theorem 7.1 we know that S_i and C_i are at ρ-distance 1, therefore for any $x \in \partial S_i$ there is a $y \in \partial C_i$ such that $\rho(x - y) = 1$. We say that x *realizes at* y *the distance from* C_i.

Definition. A point $x \in \partial S_i$ is a *singular* point if it realizes the distance from C_i to at least two points in ∂C_i. We say that $x \in \partial S_i$ is a *regular* point if it is not singular.

Geometrically, we can describe regular and singular points as follows. Let $x \in \partial S_i$ be a singular point and $y_1, y_2 \in \partial C_i$ points where x realizes the distance from C_i. Then the balls $\mathcal{B}_1(y_1)$ and $\mathcal{B}_1(y_2)$ are tangent to ∂S_i at x. Consider the convex cone determined by the two tangent lines to the two tangent ρ-balls $\mathcal{B}_1(y_1)$ and $\mathcal{B}_1(y_2)$, which does not intersect the two ρ-balls. The intersection of all cones generated by all ρ-balls of radius 1, tangent at x and with center exterior to S_i, defines a convex asymptotic cone centered at x (see Figure 2). The asymptotic cone can be equivalently defined as the intersection of all cones generated by all ρ-balls of radius 1, tangent at x and with center in C_i (see Lemma 8.1 below).

If $x \in \partial S_i$ is a regular point, then there is only one point $y \in \partial C_i$ where x realizes the distance from C_i. In this case, the two tangent balls coincide, and therefore by definition

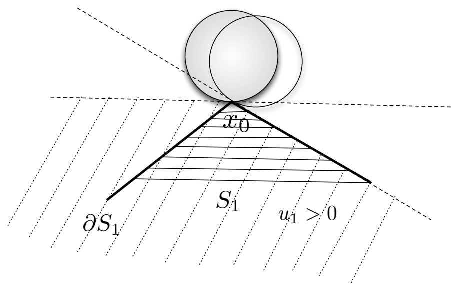

Fig. 2. Asymptotic cone at x_0.

the asymptotic cone at $x \in \partial S_i$ is a half-plane. We will show that at regular points, ∂S_i is the graph of a differentiable function. If $\theta \in [0, \pi]$ is the opening of the cone at x, we say that S_i *has angle* θ at x. Regular points correspond to $\theta = \pi$. When $\theta = 0$ the tangent cone is actually a half-line and S_i has a cusp at x. Later on in this section we will show that, under additional hypotheses on the boundary data and the domain Ω, the case $\theta = 0$ never occurs, and therefore the free boundaries are Lipschitz curves of the plane.

Lemma 8.1. *Let* $\mathcal{C} = \{(x_1, x_2) \mid x_2 \geq \alpha|x_1|\}$, $\alpha \geq 0$, *be the asymptotic cone of* S_i *at* $0 \in \partial S_i$. *Then there exist* $y_1,\ y_2 \in \partial C_i$ *such that the balls* $\mathcal{B}_1(y_1)$ *and* $\mathcal{B}_1(y_2)$ *are tangent respectively to the lines* $x_2 = \pm\alpha x_1$ *at* 0.

Proof. Let $y_1, y_2 \in \mathcal{B}_1(0)$ be such that the line $x_2 = \alpha x_1$ is tangent to $\mathcal{B}_1(y_1)$ at 0 and the line $x_2 = -\alpha x_1$ is tangent to $\mathcal{B}_1(y_2)$ at 0. Suppose for contradiction that $y_1, y_2 \notin \partial C_i$. Then any $y \in C_i$ such that $\rho(y - 0) = 1$ must lie in the smaller arc in $\partial\mathcal{B}_1(0)$ between y_1 and y_2. Moreover, there exists $\delta > 0$ such that all ρ-balls $\mathcal{B}_1(y)$ have at most as tangent lines at 0 the lines $x_2 = \pm(\alpha - \delta)x_1$. Then the asymptotic cone at 0 must contain the cone $\{(x_1, x_2) \mid x_2 \geq (\alpha - \delta)|x_1|\}$, which is not possible. □

Lemma 8.2. *Assume that* S_i *has an angle* $\theta \in (0, \pi]$ *at* $x_0 \in \partial S_i$. *Then there exists a neighborhood* U *of* x_0, *a system of coordinates* (x_1, x_2) *and a locally Lipschitz function* $\psi : (-r, r) \to \mathbb{R}$, *for some* $r > 0$, *such that in the coordinates* (x_1, x_2), *we have* $x_0 = (0, 0)$ *and*

$$\partial S_i \cap U = \{(x_1, \psi(x_1)) \mid x_1 \in (-r, r)\}.$$

If in addition $\theta = \pi$, *then* φ *is differentiable at 0.*

Proof. Let $\mathcal{C}$ be the convex asymptotic cone of S_i at x_0. Let us fix a system of coordinates (x_1, x_2) such that the x_2 axis coincides with the axis of the cone and is oriented in such a way that the cone is above the x_1 axis. Then $x_0 = (0, 0)$ and $\mathcal{C} = \{(x_1, x_2) : x_2 \geq \alpha|x_1|\}$ with $\alpha = \tan\left(\frac{\pi-\theta}{2}\right)$. To prove that in these coordinates, ∂S_i is the graph of a function in a small neighborhood of x_0, it suffices to show that there exists a small $r > 0$ such that, for any $|t| < r$, the vertical line $\{x_1 = t\}$ intersects $\partial S_i \cap B_r(0)$ in only one point. Suppose for contradiction that there exists a sequence (t_n) such that $t_n \to 0$ as $n \to \infty$, and the line $\{x_1 = t_n\}$ intersects $\partial S_i \cap B_r(0)$ at two distinct points (t_n, a_n) and (t_n, b_n)

with $b_n > a_n$. Assume, without loss of generality, that $t_n > 0$ for any n. By Lemma 8.1 there exist $y_1, y_2 \in \partial C_i$ that realize the distance from 0, and such that $\mathcal{B}_1(y_1)$ is tangent to the line $\{(x_1, x_2) : x_2 = \alpha x_1\}$ at 0 and $\mathcal{B}_1(y_2)$ is tangent to $\{(x_1, x_2) : x_2 = -\alpha x_1\}$ also at 0. For instance, in the particular case of the Euclidean norm, we would have

$$y_1 = \left(\sqrt{\frac{1}{1+\alpha^2}}, -\alpha\sqrt{\frac{1}{1+\alpha^2}}\right) \quad \text{and} \quad y_2 = \left(-\sqrt{\frac{1}{1+\alpha^2}}, -\alpha\sqrt{\frac{1}{1+\alpha^2}}\right).$$

In general, we can say that the x_2 coordinate of y_1 and y_2 is a negative value $-c$. We have $\mathcal{B}_1(y_1) \cap \mathcal{B}_1(y_2) \neq \emptyset$, since $\theta > 0$. Moreover, $S_i \cap (\mathcal{B}_1(y_1) \cup \mathcal{B}_1(y_2)) = \emptyset$. Then both points (t_n, a_n) and (t_n, b_n) must be above $\mathcal{B}_1(y_1) \cup \mathcal{B}_1(y_2)$ for n large enough. Next, let $y_n^a, y_n^b \in \partial C_i$ be points where (t_n, a_n) and (t_n, b_n), respectively, realize the distance from C_i. Then the ρ-balls $\mathcal{B}_1(y_n^a)$ and $\mathcal{B}_1(y_n^b)$ are exterior tangent balls to ∂S_i at (t_n, a_n) and (t_n, b_n), respectively. Recall that the ρ-distance between (t_n, a_n) and (t_n, b_n) converges to 0 as $n \to \infty$, and so y_n^a has to belong to the lower half ρ-ball $\partial\mathcal{B}_1(t_n, a_n) \cap \{x_2 < a_n\}$ for n large enough. Indeed, if not, the tangent ρ-ball $\mathcal{B}_1(y_n^a)$ would contain (t_n, b_n) for n large enough. Similarly, y_n^b has to belong to the upper half ρ-ball $\partial\mathcal{B}_1(t_n, b_n) \cap \{x_2 > b_n\}$ for n large enough. This implies that the tangent ρ-ball $\mathcal{B}_1(y_n^b)$ converges to a tangent ball to S_i at 0, $\mathcal{B}_1(y^b)$, with $y^b \in \{x_2 \geq 0\}$. On the other hand, by the definition of the asymptotic cones, all the centers of the tangent balls at 0 must belong to the set $\partial\mathcal{B}_1(0) \cap \{x_2 \leq -c\}$, where $-c < 0$ is the x_2 coordinate of the points y_1, y_2 defined above. Therefore, we have reached a contradiction. We infer that there exists $r > 0$ such that ∂S_i is the graph of a function $\psi : (-r, r) \to \mathbb{R}$. Since ∂S_i is a closed set, ψ is continuous.

Let us prove that ψ is Lipschitz continuous at 0. If $\mathcal{C} = \{x_2 \geq \alpha|x_1|\}$ is the tangent cone of S_i at x_0 in coordinates (x_1, x_2), then for $r > 0$ small enough we have

$$\{x_2 \geq 2\alpha|x_1|\} \subset S_i \cap B_r(0) \subset \{x_2 \geq \alpha|x_1|/2\},$$

that is, for $|x_1| < r$,

$$\alpha|x_1|/2 \leq \psi(x_1) = \psi(x_1) - \psi(0) \leq 2\alpha|x_1|.$$

Therefore, ψ is Lipschitz at 0.

Next, assume that $\theta = \pi$. Then $y_1 = y_2$, and x_0 is a regular point. Therefore, $\mathcal{B}_1(y_1) \subset \{x_2 < 0\}$ is the unique tangent ball to the graph of ψ at $x_0 = (0, 0)$. Moreover, the tangent cone is the half-plane $\{x_2 \geq 0\}$. Let us show that ψ is differentiable at 0. Assume for contradiction that there exists a sequence $(x_1^n) \subset (-r, r)$ of positive numbers such that $x_1^n \to 0$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{\psi(x_1^n)}{x_1^n} = \beta \neq 0. \tag{8.2}$$

Since there exists a tangent ball from below to the graph of ψ at 0 contained in $\{x_2 < 0\}$, we must have $\beta > 0$. For any $(x_1^n, \psi(x_1^n)) \in \partial S_i$ there exists $y_n \in \partial C_i$ such that $\mathcal{B}_1(y_n)$ is tangent to S_i at $(x_1^n, \psi(x_1^n))$. Let $y_2 \in \partial C_i$ be the limit of a converging subsequence of (y_n). Then the ρ-ball $\mathcal{B}_1(y_2)$ is an exterior tangent ball at S_i at 0. Equation (8.2) gives

$\psi(x_1^n) \geq \beta x_1^n/2$ for n large enough, i.e. the points $(x_1^n, \psi(x_1^n))$ of the free boundary are above the line $\{x_2 = \beta|x_1|/2\}$. This implies that $y_1 \neq y_2$, that is, the limit ρ-ball $\mathcal{B}_1(y_2)$ must be different from $\mathcal{B}_1(y_1)$. This is in contradiction with the fact that x_0 is a regular point. Therefore we must have

$$\lim_{x_1 \to 0^+} \frac{\psi(x_1)}{x_1} = 0.$$

Similarly, one can prove that

$$\lim_{x_1 \to 0^-} \frac{\psi(x_1)}{x_1} = 0.$$

We conclude that ψ is differentiable at 0 and $\psi'(0) = 0$. □

Lemma 8.3. *Assume that there exists an open subset U of $\mathbb{R}^2$ such that any point of $U \cap \partial S_i$ is regular. Then $U \cap \partial S_i$ is a C^1 curve of the plane.*

Proof. Let $y_0 \in \partial S_i \cap U$. By Lemma 8.2, there exists a differentiable function ψ and a small $r > 0$, such that, in the system of coordinates (x_1, x_2) centered at y_0 and with the x_2 axis in the direction of the inner normal of ∂S_i at y_0, $\partial S_i \cap B_r(y_0)$ is the graph of ψ. Moreover, in these coordinates, $\psi(y_0) = \psi'(y_0) = 0$. By Corollary 6.2, there exists a tangent ball from below, with uniform radius, at any point of the graph of ψ. This implies that for any $|x_1^0| < r$, there exists a C^2 function $\varphi_{x_1^0}$ tangent from below to the graph of ψ at x_1^0 and such that $|\varphi''_{x_1^0}| \leq C$, for some $C > 0$ independent of x_1^0. Therefore we have, for any $|x_1| < r$,

$$\begin{aligned}\psi(x_1) &\geq \varphi_{x_1^0}(x_1) \geq \varphi_{x_1^0}(x_1^0) + \varphi'_{x_1^0}(x_1^0)(x_1 - x_1^0) - C|x_1 - x_1^0|^2 \\ &= \psi(x_1^0) + \psi'(x_1^0)(x_1 - x_1^0) - C|x_1 - x_1^0|^2.\end{aligned}$$

Now, let us show that ψ is of class C^1. Fix a point x_1^0 and consider a sequence (x_1^l) converging to x_1^0 as $l \to \infty$. Let p be the limit of a convergent subsequence of $(\psi'(x_1^l))$. Passing to the limit in l in the inequality

$$\psi(x_1) \geq \psi(x_1^l) + \psi'(x_1^l)(x_1 - x_1^l) - C|x_1 - x_1^l|^2,$$

we get

$$\psi(x_1) \geq \psi(x_1^0) + p(x_1 - x_1^0) - C|x_1 - x_1^0|^2$$

for any $|x_1| < r$. Since ψ is differentiable at x_1^0, we must have $p = \psi'(x_1^0)$. □

Lemma 8.4. *Assume that the supports of the boundary data f_i on $(\partial\Omega)_{\leq 1}$ have a finite number of connected components. Then the sets S_i have a finite number of connected components.*

Proof. Consider all the connected components of S_i, S_i^j, $i = 1, \ldots, K$ and $j = 1, 2, \ldots$. Note that for any i and j,

$$\partial S_i^j \cap \{x \in (\partial\Omega)_{\leq 1} \mid f_i(x) > 0\} \neq \emptyset.$$

Indeed, if not we would have $u_i = 0$ on ∂S_i^j and $\Delta u_i \geq 0$ in S_i^j. The maximum principle would then imply $u_i \equiv 0$ in S_i^j, which is not possible. Moreover, by continuity, ∂S_i^j must contain one connected component of the set $\{x \in (\partial\Omega)_{\leq 1} \mid f_i(x) > 0\}$. For this reason we say that the components of S_i reach the boundary of Ω. This implies that the connected components of S_i are finitely many. □

8.1. Properties of singular points

We start by proving three lemmas that will allow us to estimate the growth of the solutions near the singular points. The first lemma states that positive functions which are superharmonic [subharmonic] in a cone and vanish on its boundary, have at least [at most] linear growth away from the boundary of the cone far from the vertex, with a slope that degenerates in a Hölder fashion when approaching the vertex. The power just depends on the opening of the cone. The second and third lemmas generalize these estimates to domains which are sets of points at ρ-distance greater than 1 from a closed bounded set. Then we prove that the set of singularities is a set of isolated points and we give a characterization. For the set S_i which has finite perimeter, we denote by $\partial^* S_i$ the *reduced boundary*, that is, the set of points whose blow-ups converge to half-planes; and the *essential boundary*, $\partial_* S_i$, are all points except points of Lebesgue density zero and one. We have $\mathcal{H}^1(\partial_* S_i \setminus \partial^* S_i) = 0$. For more details see [1, 22].

Lemma 8.5. *Let v be a nonnegative Lipschitz function defined on $B_1 \subset \mathbb{R}^n$ such that Δv is locally a Radon measure on B_1 and v is smooth on $S = \{v > 0\}$. Assume that S is a set of finite perimeter. Then, for every smooth ϕ with compact support contained in B_1,*

$$\int_{B_1} \Delta v\, \phi = \int_S \Delta v \phi \, dx - \int_{\partial^* S} \frac{\partial v}{\partial \nu_S} \phi \, d\mathcal{H}^{n-1}$$

where ν_S is the measure-theoretic outward unit normal and $\partial^* S$ is the reduced boundary.

Proof. As a distribution and integrating by parts,

$$\int_{B_1} \Delta v \phi = \int_S v \Delta\phi \, dx = \int_S [\operatorname{div}(v\nabla\phi) - \operatorname{div}(\nabla v \phi) + \Delta v \phi] \, dx.$$

Applying the generalized Gauss–Green theorem (see [7], and also [1, 22] for more details) we obtain the result. □

Lemma 8.6. *Let $\theta_0 \in (0, \pi]$. Let $\mathcal{C}$ be the cone defined in polar coordinates by*

$$\mathcal{C} = \{(\varrho, \theta) \mid \varrho \in [0, \infty),\ 0 \leq \theta \leq \theta_0\}.$$

Let u_1 and u_2 be respectively a superharmonic and a subharmonic positive function in the interior of $\mathcal{C} \cap B_{2r_0}$ such that $u_1 \geq u_2 = 0$ on $\partial\mathcal{C} \cap B_{2r_0}$. Then for any $r < r_0/3$ there exist $R = R(\theta_0, r)$, and constants $c, C > 0$ depending on respectively (θ_0, u_1, r_0) and (θ_0, u_2, r_0), but independent of r, such that for any $x \in [r, 3r] \times [0, R]$ we have

$$\text{(a) } u_1(x) \geq c r^\alpha d(x, \partial\mathcal{C}), \qquad \text{(b) } u_2(x) \leq C r^\alpha d(x, \partial\mathcal{C}),$$

where α is given by

$$1 + \alpha = \pi/\theta_0.$$

Proof. Let us introduce the function

$$v(\varrho, \theta) := \varrho^{1+\alpha} \sin((1+\alpha)\theta). \tag{8.3}$$

Notice that v is harmonic in the interior of $\mathcal{C}$, since it is the imaginary part of the function $z^{1+\alpha}$, where $z = x + iy$, which is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$. Moreover v is positive inside $\mathcal{C}$ and vanishes on its boundary. By a barrier argument, u_1 has at least linear growth away from the boundary of $\mathcal{C}$, meaning that for $\rho \in [r_0/2, 3r_0/2]$ (far from the vertex and from ∂B_{2r_0})

$$u_1(x) \geq k d(x, \partial\mathcal{C})$$

for $k = c_0 \min_{x\in\mathcal{C},\, d(x,\partial\mathcal{C})\geq s_0} u_1$ and for $x \in \{x \in \mathcal{C} \mid r_0/2 < |x| < 3r_0/2,\ d(x, \partial\mathcal{C}) \leq s_0\}$ where c_0 and s_0 depend on r_0 and θ_0. Therefore, we can find a constant $c > 0$, depending on u_1, r_0 and θ_0, such that

$$u_1 \geq cv \quad \text{on } \mathcal{C} \cap \partial B_{r_0}.$$

Since in addition $u_1 \geq cv = 0$ on $\partial C \cap B_{r_0}$, the comparison principle implies

$$u_1 \geq cv \quad \text{in } \mathcal{C} \cap B_{r_0}. \tag{8.4}$$

Since v is increasing in the radial direction and if we are near $\partial\mathcal{C}$ it is also increasing in the θ direction, for $r \leq |x| \leq 3r$ with $r \leq r_0/3$ and $d(x, \mathcal{C}) \leq R$ with $R = r \min\{1, \tan(\theta_0/2)\}$ we have

$$u_1(x) \geq cv(x) \geq Cr^{\alpha} d(x, \partial\mathcal{C}),$$

and (a) follows.

To prove (b) similarly, we have

$$u_2 \leq Cv \quad \text{in } \mathcal{C} \cap B_{r_0}, \tag{8.5}$$

where C depends on (θ_0, u_2, r_0) but is independent of r. In particular, for $r \leq |x| \leq 3r$ and $d(x, \mathcal{C}) \leq R/2$ we have

$$u_2(x) \leq Cv(x) \leq \tilde{C} r^{\alpha} d(x, \partial\mathcal{C}). \qquad \square$$

Lemma 8.7. *Let Ω be an open set, C be a closed subset of Ω, and $S = \{x \in \Omega \mid d_\rho(x, C) \geq 1\}$. Let S_1 be a connected component of S. Assume that $\partial S_1 = \Gamma_1 \cup \Gamma_2$ with $\Gamma_1 \cap \Gamma_2 = \{0\}$ and S_1 has an angle $\theta_0 \in (0, \pi]$ at $0 \in \partial S_1$. Let u_1 be a superharmonic positive function in $S_1 \cap B_{2r_0}(0)$ with $u_1 = 0$ on $\partial S_1 \cap B_{2r_0}(0)$. Then there exists a sequence $(x_h) \subset \Gamma_1$ of regular points with $x_h \to 0$ as $h \to 0$, and there exist balls $B_{R_h}(z_h) \subset S_1$ tangent to ∂S_1 at x_h, where $R_h \geq c|x_h|$, such that*

$$u_1(x) \geq c R_h^{\alpha_\delta} d(x, \partial B_{R_h}(z_h)) \quad \text{for any } x \in B_{R_h}(z_h) \setminus B_{R_h/4}(z_h),$$

where α_δ is given by

$$1 + \alpha_\delta = \frac{\pi}{\theta_0 - \delta}.$$

Proof. Since $\theta_0 \in (0, \pi]$ for any $0 < \delta < \theta_0$, there exist $r_\delta > 0$ and a cone $\mathcal{C}^1_\delta$ centered at 0 with opening $\theta_0 - \delta$ such that

$$\mathcal{C}^1_\delta \cap B_{r_\delta}(0) \subset S_1 \cap B_{r_\delta}(0).$$

Take a sequence of points $t_h \in \partial\mathcal{C}^1_\delta \cap B_{r_\delta}(0)$ converging to 0 as $h \to 0$. Let

$$r_h := d(t_h, 0) \quad \text{and} \quad R_h := r_h \min\left\{1, \tan\left(\frac{\theta_0 - \delta}{2}\right)\right\}.$$

Then, for h small enough, there exist balls $B_{R_h}(s_h) \subset \mathcal{C}^1_\delta \cap B_{r_\delta}(0)$ such that $t_h \in \partial B_{R_h}(s_h)$. Consider a system of polar coordinates (ϱ, θ) centered at 0. Moving the balls $B_{R_h}(s_h)$ along the θ direction until they touch Γ_1, we can find a sequence of regular points x_h in that region such that $d(x_h, 0) \le c r_h$, and balls $B_{R_h}(z_h) \subset S_1 \cap B_{r_\delta}(0)$ such that $x_h \in \partial B_{R_h}(z_h)$. Observe that the center z_h remains inside the cone $\mathcal{C}^1_\delta$, that is, for h and δ small enough, we have $z_h \in \mathcal{C}^1_\delta$ and $d(z_h, \partial\mathcal{C}^1_\delta) \ge R_h/2$. Let us introduce the barrier function

$$\phi(x) := \frac{m}{\log 4} \log\left(\frac{R_h}{|x - z_h|}\right), \quad \text{where} \quad m = \inf_{\partial B_{R_h/4}(z_h)} u_1.$$

Then ϕ satisfies

$$\begin{cases} \Delta\phi = 0 & \text{in } B_{R_h}(z_h) \setminus B_{R_h/4}(z_h), \\ \phi = 0 & \text{on } \partial B_{R_h}(z_h), \\ \phi = m & \text{on } \partial B_{R_h/4}(z_h). \end{cases}$$

Since $u_1 \ge \phi$ on $\partial B_{R_h}(z_h) \cup \partial B_{R_h/4}(z_h)$, the comparison principle implies

$$u_1 \ge \phi \quad \text{in } B_{R_h}(z_h) \setminus B_{R_h/4}(z_h).$$

If ν_1 is the inner normal vector of $B_{R_h}(z_h)$, then for $x \in \partial B_{R_h}(z_h)$,

$$\frac{\partial\phi}{\partial\nu_1}(x) = \frac{m}{R_h \log 4},$$

and the convexity of ϕ in the radial direction gives, for any $x \in B_{R_h}(z_h) \setminus B_{R_h/4}(z_h)$,

$$u_1(x) \ge \frac{m}{R_h \log 4} d(x, \partial B_{R_h}(z_h)).$$

Let us estimate m. Since $d(z_h, \partial C^1_\delta) \ge R_h/2$, we have $d(x, \partial C^1_\delta) \ge R_h/4$ for any x in $B_{R_h/4}(z_h)$. As in Lemma 8.6, consider the harmonic function $v(x)$, introduced in (8.3), defined on the cone $\mathcal{C}^1_\delta$ ($\alpha = \alpha_\delta$), and the comparison principle result stated in (8.4). Then

$$m \ge c \min_{\partial B_{R_h/4}(z_h)} v \ge \min\left\{v\left(r_h - \frac{R_h}{4}, \frac{\theta_0 - \delta}{8}\right), v\left(\frac{3r_h}{4}, \frac{\pi}{16}\right)\right\} = c_1\left(\frac{3r_h}{4}\right)^{\alpha_\delta + 1}$$

where $c_1 = c_1(u_1, r_\delta, \theta_0 - \delta)$. Therefore, since $r_h/R_h \ge 1$, we conclude that for any $x \in B_{R_h}(z_h) \setminus B_{R_h/4}(z_h)$,

$$u_1(x) \ge c R_h^{\alpha_\delta} d(x, \partial B_{R_h}(z_h)). \qquad \square$$

Lemma 8.8. *Let Ω be an open set, C be a closed subset of Ω, and $S = \{x \in \Omega \mid d_\rho(x, C) \geq 1\}$. Let S_1 be a connected component of S. Assume that S_1 has an angle $\theta_0 \in [0, \pi]$ at $0 \in \partial S_1$. Let u_2 be a subharmonic positive function in $S_1 \cap B_{2r_0}(0)$ with $u_2 = 0$ on $\partial S_1 \cap B_{2r_0}(0)$. Then, for any $0 < \delta < \theta_0$, there exists $r_\delta > 0$ such that for any $r < r_\delta/5$ there exist $R = R(\theta_0, r)$, and a constant $C > 0$ depending on $(\theta_0 + \delta, u_2, r_\delta)$, but independent of r, such that*

$$u_2(x) \leq C r^{\beta_\delta} d(x, \partial S_1) \quad \text{for any } x \in (B_{3r}(0) \setminus B_r(0)) \cap \{x \in S_1 \mid d(x, \partial S_1) \leq R/4\} \tag{8.6}$$

where β_δ is given by

$$1 + \beta_\delta = \frac{\pi}{\theta_0 + \delta}.$$

Proof. For any $\delta > 0$, there exist $r_\delta > 0$ and a cone $\mathcal{C}_\delta^2$ centered at 0 and with opening $\theta_0 + \delta$ such that

$$S_1 \cap B_{r_\delta}(0) \subset \mathcal{C}_\delta^2 \cap B_{r_\delta}(0).$$

Take any $r < r_\delta$ and let $y \in \partial S \cap (B_{3r}(0) \setminus B_r(0))$ and $r_y := d(y, 0) \in (r, 3r)$. Since S is at ρ-distance 1 from C, for any point of the boundary of S_1 there exists an exterior tangent ρ-ball of radius 1. This implies that for r small enough, there exists w_y such that the Euclidean ball $B_{R_y}(w_y)$ is contained in the complement of S, and $y \in \partial B_{R_y}(w_y)$, where R_y is defined by

$$R_y = r_y \min\left\{1, \tan\left(\frac{\theta_0 + \delta}{2}\right)\right\}.$$

Let us now take as barrier the function

$$\psi(x) := \frac{M}{\log(3/2)} \log\left(\frac{|w_y - x|}{R_y}\right) \quad \text{with} \quad M = \sup_{\partial B_{3R_y/2}(w_y)} u_2.$$

Then ψ satisfies

$$\begin{cases} \Delta\psi = 0 & \text{in } B_{3R_y/2}(w_y) \setminus B_{R_y}(w_y), \\ \psi = M & \text{on } \partial B_{3R_y/2}(w_y), \\ \psi = 0 & \text{on } \partial B_{R_y}(w_y). \end{cases}$$

Using the comparison principle with u_2, the concavity of ψ in the radial direction implies that for any $x \in B_{3R_y/2}(w_y) \setminus B_{R_y}(w_y)$,

$$u_2 \leq \frac{M}{R_y \log(3/2)} d(x, \partial B_{R_y}(w_y)).$$

Let us estimate M. Consider again a system of polar coordinates (ϱ, θ) centered at 0 and the harmonic function $v(x)$, introduced in (8.3), defined on the cone $\mathcal{C}_\delta^2$ ($\alpha = \beta_\delta$). By definition of v, R_y, and taking into account (8.5), for δ, r small enough and

$$M \leq C \max_{\partial B_{3R_y/2}(w_y)} v \leq C v\left(4r_y, \frac{\theta_0 + \delta}{2}\right) = C_1 (4r_y)^{\beta_\delta + 1} = \tilde{C}_1 r_y^{\beta_\delta} \frac{R_y}{\min\left\{1, \tan\left(\frac{\theta_0+\delta}{2}\right)\right\}}$$

we see that for any $x \in B_{3R_y/2}(w_y) \setminus B_{R_y}(w_y)$ belonging to the segment $y + s(y - w_y)$, $s \in (0, 1/2)$, we have

$$u_2(x) \le CMd(x, \partial B_{R_y}(w_y)) = CMd(x, \partial S_1) \le Cr_y^{\beta_\delta} d(x, \partial S_1). \tag{8.7}$$

Letting the tangent ball move along $\partial S_1 \cap (B_{3r_y}(0) \setminus B_{r_y}(0))$, we get (b). □

Lemma 8.9. *Assume* (2.8) *with* $n = 2$ *and* $p = 1$ *in* (2.5). *Assume in addition that the supports on* $\partial\Omega$ *of the boundary data* f_i *have a finite number of connected components. Let* $(u_1^\varepsilon, \dots, u_K^\varepsilon)$ *be a viscosity solution of problem* (2.4) *and* $(u_1, \dots, u_K)$ *the limit as* $\varepsilon \to 0$ *of a convergent subsequence. Then all singular points of* Ω *are isolated.*

Proof. Suppose for contradiction that there exists a sequence $(y_k)_{k\in\mathbb{N}}$ of distinct singular points such that $y_k \in \partial S_j$ and $y_k \to y \in \Omega$ as $k \to \infty$. Since by Lemma 8.4 the connected components of the sets S_i, $i = 1, \dots, K$, are finitely many, we may assume without loss of generality that the points y_k belong to the same connected component of S_j, which we denote by S_j^1. If there exists $\theta_{\max} < \pi$ such that S_j^1 has an angle smaller than $\theta_{\max}$ at y_k for any k, then there exists $\bar{k}$ such that starting from $y_{\bar{k}}$, after a finite number of singular points S_j^1 would be an isle and not reach the boundary. Therefore we would have $u_j = 0$ on ∂S_j^1 and $\Delta u_j = 0$ in S_j^1, and the maximum principle would imply $u_j \equiv 0$ in S_j^1, which is a contradiction. We infer that there exists a $k \in \mathbb{N}$ such that the angle at y_k is close to π. In particular, if x_1^k and x_2^k are points in C_j that realize the ρ-distance from S_j at y_k, then the ρ-distance between x_1^k and x_2^k is less than 1.

Next, suppose that x_i^k and x_2^k belong to the same connected component of S_i, for some $i \neq j$. Then by Theorem 7.1 we know that $\partial S_i \cap \overline{\mathcal{B}_1(y_k)}$ has to contain the arc of the unit ρ-ball between x_1^k and x_2^k. If not, there would be points in the curve connecting x_1^k and x_2^k which do not realize the distance from C_i. Any point inside this arc is a regular point at ρ-distance 1 from y_k. Consider any of them, for instance the middle point of the arc, denoted by x_k. We want to compare the mass of the Laplacian of u_i at x_k with the mass of the Laplacian at u_j at y_k, across the free boundaries. First assume H is defined as in (2.5). For $\sigma < \frac{1}{8} d_\rho(x_1^k, x_2^k)$ define

$$D_\sigma(x_k) := \{x \in \mathcal{B}_\sigma(x_k) \mid d(x, \partial\mathcal{C}_i) \le \sigma^2\},$$

where $\mathcal{C}_i$ is the asymptotic cone to S_i^1 at x_k. Note that since x_k is a regular point, $\partial\mathcal{C}_i$ is the tangent line to ∂S_i^1 at x_k, and so $\mathcal{C}_i$ has opening π. Let $(D_\sigma(x_k))_{<1}$ be the set of points at ρ-distance less than 1 from $D_\sigma(x_k)$. Then

$$\int_{D_\sigma(x_k)} \Delta u_i \le \sum_{j\neq i} \int_{(D_\sigma(x_k))_{<1}} \Delta u_j \tag{8.8}$$

as in (7.2) with $(D_\sigma(x_k))_{<1}$ in place of $\mathcal{B}_{1+S}(x_0)$. By the Hopf Lemma, we obtain

$$\int_{D_\sigma(x_k)} \Delta u_i = \int_{\partial S_i \cap D_\sigma(x_k)} \frac{\partial u_i}{\partial \nu_i} \, d\mathcal{H} \ge c\mathcal{H}(\partial S_i \cap D_\sigma(x_k)) = \tilde{C}\sigma \tag{8.9}$$

where ν_i is the inner normal vector.

Now we estimate $\int_{(D_\sigma(x_k))_{<1}} \Delta u_j$. From Corollary 6.5 we know that S_j has finite perimeter. Therefore by Lemmas 8.5 and 8.8 we obtain

$$\int_{(D_\sigma(x_k))_{<1}} \Delta u_j = \int_{\partial^* S_j^1 \cap (D_\sigma(x_k))_{<1}} \frac{\partial u_j}{\partial \nu_{S_j^1}}\, d\mathcal{H} \le C\sigma^{\beta_\delta}\, \mathcal{H}(\partial^* S_j^1 \cap (D_\sigma(x_k))_{<1}) \quad (8.10)$$

where ν_{S_j} is the measure-theoretic inward unit normal to S_j^1 and $\beta_\delta > 0$. Since, for some constant c,

$$\partial S_j^1 \cap (D_\sigma(x_k))_{<1} \subset \partial S_j^1 \cap \mathcal{B}_{c\sigma}(y_k)$$

by (2.2), there exists $\tilde{c}_2$, that for simplicity we will still name c, such that $\partial S_j^1 \cap (D_\sigma(x_k))_{<1} \subset \partial S_j^1 \cap B_{c\sigma}(y_k)$. Then

$$\mathcal{H}(\partial^* S_j^1 \cap (D_\sigma(x_k))_{<1}) \le \mathrm{Per}(\partial S_j^1 \cap B_{c\sigma}(y_k)). \quad (8.11)$$

To estimate $\mathrm{Per}(\partial S_j^1 \cap B_{c\sigma}(y_k))$, consider (6.2) in the distributional sense. Take a smooth function $0 \le \phi \le 1$ with compact support contained in $B_{c\sigma}(y_k) \cap \{x \mid 0 < d(x, S_i) < 1\}$ and such that $\phi \equiv 1$ on $B_{c\sigma}(y_k) \cap \{x \mid 1-\delta < d(x, S_i) < 1-\varepsilon\}$ for $0 < \varepsilon < \delta$ and δ as introduced in the definition of η in the proof of Lemma 6.4. Then for $d_{S_i}(\cdot) = d_\rho(\cdot, S_i)$ we have

$$\begin{aligned} 0 &= \int_{B_{c\sigma}(y_k)\cap\{x|0<d_{S_i}<1\}} \mathrm{div}(\eta(d_{S_i}) D d_{S_i})\phi\, dx \\ &= \int_{B_{c\sigma}(y_k)\cap\{x|0<d_{S_i}<1\}} \eta'(d_{S_i})|Dd_{S_i}|^2\phi\, dx + \int_{B_{c\sigma}(y_k)\cap\{x|0<d_{S_i}<1\}} \eta(d_{S_i})\Delta d_{S_i}\phi\, dx \\ &\le \int_{B_{c\sigma}(y_k)\cap\{x|0<d_{S_i}<1\}} \eta'(d_{S_i})|Dd_{S_i}|^2\phi\, dx + C\sigma. \end{aligned}$$

Proceeding as in Lemma 6.4 we obtain

$$\mathrm{Per}(\partial S_j^1 \cap B_{c\sigma}(y_k)) \le C\sigma. \quad (8.12)$$

Putting together (8.8)–(8.12) we obtain

$$C\sigma^{1+\beta_\delta} \ge \tilde{C}\sigma,$$

and we get a contradiction for σ small enough. In the case (2.6) the proof follows the same steps using (7.8).

Therefore x_1^k and x_2^k must belong to different components of C_j for any $k \ge \bar{k}$. In particular, since the distance between them is less than 1, they must belong to two different components of the same population. Suppose that $x_1^{\bar{k}} \in S_i^1$ and $x_2^{\bar{k}} \in S_i^2$, for $i \ne j$. Consider the consecutive two points $x_1^{\bar{k}+1}$ and $x_2^{\bar{k}+1}$ which realize the distance at $y_{\bar{k}+1}$, and again belong to two different components of C_j. Since S_j^1 (to which $y_{\bar{k}}$ belongs) and S_i^2 reach the boundary of Ω, the point $x_2^{\bar{k}+1}$ must belong to a connected component different from S_i^1. Iterating the procedure, we construct a sequence of distinct points belonging to connected components, each different from the others. This contradicts Lemma 8.4. We conclude that singular points are isolated. □

Theorem 8.10. *Assume* (2.8) *with* $n = 2$ *and* $p = 1$ *in* (2.5). *Let* $(u_1^\varepsilon, \dots, u_K^\varepsilon)$ *be a viscosity solution of problem* (2.4) *and* $(u_1, \dots, u_K)$ *the limit as* $\varepsilon \to 0$ *of a convergent subsequence. For* $i \neq j$, *let* $x_0 \in \partial S_i \cap \Omega$ *and* $y_0 \in \partial S_j \cap \Omega$ *be points such that* S_i *has an angle* $\theta_i \in [0, \pi]$ *at* x_0, S_j *has an angle* $\theta_j \in [0, \pi]$ *at* y_0 *and* $\rho(x_0 - y_0) = 1$. *Then*

$$\theta_i = \theta_j. \tag{8.13}$$

If $x_0 \in \partial S_i \cap \partial\Omega$ *and* $y_0 \in \partial S_j \cap \Omega$, *then*

$$\theta_i \le \theta_j. \tag{8.14}$$

Proof. By Lemma 8.4, the connected components of the sets S_i are finitely many. Assume $x_0 \in \overline{\Omega}$ and $y_0 \in \Omega$. Without loss of generality we can assume that $x_0 = 0$. It suffices to show the theorem for y_0 belonging to a region that is side by side with S_i, in the sense that 0 is the limit as $h \to 0$ of interior regular points $x_h \in \partial S_i \cap \Omega$ with the property that x_h realizes the distance from S_j at interior points $y_h \in \partial S_j \cap \Omega$, with $y_h \to y_0$ as $h \to 0$. Let $\mathcal{C}_i$ be the asymptotic cone at 0. First suppose for simplicity that ∂S_i and ∂S_j are locally cones around 0 and y_0 respectively. In particular, $\theta_i, \theta_j > 0$. We will explain later on how to handle the general case.

Proof of Theorem 8.10 when ∂S_i *and* ∂S_j *are locally cones.* We assume that there exists $r_0 > 0$ such that $\partial S_i \cap B_{2r_0} = \mathcal{C}_i \cap B_{2r_0}$, where B_{2r_0} is the Euclidean ball centered at 0 of radius $2r_0$. When $x_0 \in \partial\Omega$, we are just interested in the side of the cone $\mathcal{C}_i$ contained in Ω.

If (ϱ, θ) is a system of polar coordinates in the plane centered at zero, we may assume that $\mathcal{C}_i$ is the cone given by

$$\mathcal{C}_i = \{(\varrho, \theta) \mid \varrho \in [0, \infty),\ 0 \le \theta \le \theta_i\}.$$

First consider the case (2.6). Assume that $x_h = (2r_h, 0)$ with $r_h > 0$. We know that $r_h \to 0$ as $h \to 0$, so we can fix h so small that $r_h < r_0/3$. By Lemma 8.6 applied to $u_1 = u_i$, we have

$$u_i(x) \ge c r_h^\alpha d(x, \partial S_i) \qquad \text{for any } x \in [r_h, 3r_h] \times [0, R_h], \tag{8.15}$$

where

$$1 + \alpha = \pi/\theta_i \ge 1. \tag{8.16}$$

Now, we repeat an argument similar to the one in the proof of Theorem 7.1. We look at $\inf u_i$ on small circles of radius r that go across the free boundary of u_i, and we look at $\sup u_j$ in circles of radius $r + 1$ across the free boundary of u_j, then we compare the mass of the corresponding Laplacians. Precisely, there exists a small $\sigma > 0$ and $e \in S_i$ such that $\mathcal{B}_\sigma(e) \subset [r_h, 3r_h] \times [0, R_h]$ and $x_h \in \partial\mathcal{B}_\sigma(e)$. In particular, in $\mathcal{B}_\sigma(e)$ the function u_i satisfies (8.15). For $\upsilon < \sigma$ and $r \in [\sigma - \upsilon, \sigma + \upsilon]$, we define

$$\underline{u}_i := \inf_{\partial\mathcal{B}_r(e)} u_i \quad \text{and} \quad \bar{u}_j := \sup_{\partial\mathcal{B}_{1+r}(e)} u_j. \tag{8.17}$$

In what follows we denote by C and c several constants independent of h. For $r \in [\sigma - \upsilon, \sigma]$, by (8.15) we have

$$\underline{u}_i \geq \inf_{\partial \mathcal{B}_r(e)} c r_h^\alpha d(x, \partial S_i) \geq \inf_{\partial \mathcal{B}_r(e)} C r_h^\alpha d_\rho(x, \partial S_i) \geq C r_h^\alpha (\sigma - r).$$

For $r \in [\sigma, \sigma + \upsilon]$, the ball $\mathcal{B}_r(e)$ goes across ∂S_i, therefore $\underline{u}_i = 0$. Hence

$$\begin{aligned} &\underline{u}_i(r) \geq C r_h^\alpha (\sigma - r) && \text{for } r \in [\sigma - \upsilon, \sigma], \\ &\underline{u}_i(r) = 0 && \text{for } r \in [\sigma, \sigma + \upsilon]. \end{aligned} \tag{8.18}$$

Next, let us study the behavior of $\bar{u}_j$. First of all, let us show that

$$d_\rho(e, \partial S_j) = \rho(e - y_h) = 1 + \sigma. \tag{8.19}$$

Since $d_\rho(e, \partial S_i) = \sigma$ and $d_\rho(S_i, S_j) \geq 1$, it is easy to see that $d_\rho(e, \partial S_j) \geq 1 + \sigma$. The function ρ is also called a Minkowski norm and from known results about Minkowski norms, if we denote by T the Legendre transform $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(y) = \rho(y) D\rho(y)$, then T is a bijection with inverse $T^{-1}(\xi) = \rho^*(\xi) D\rho^*(\xi)$, where ρ^* is the dual norm defined by $\rho^*(\xi) := \sup\{y \cdot \xi \mid y \in \mathcal{B}_1\}$. Now, the ball $\mathcal{B}_1(y_h)$ is tangent to ∂S_i at x_h and therefore also tangent to $\mathcal{B}_\sigma(e)$ at x_h. This implies that $D\rho(e - x_h) = -D\rho(x_h - e) = D\rho(x_h - y_h)$. Consequently,

$$\begin{aligned} e - x_h &= T^{-1}(T(e - x_h)) = T^{-1}(\sigma D\rho(e - x_h)) = T^{-1}(\sigma D\rho(x_h - y_h)) \\ &= \sigma T^{-1}(T(x_h - y_h) = \sigma(x_h - y_h). \end{aligned}$$

We infer that

$$e = x_h + \sigma(x_h - y_h) \tag{8.20}$$

and

$$\rho(e - y_h) = (1 + \sigma)\rho(x_h - y_h) = 1 + \sigma,$$

which proves (8.19). As a consequence $\partial \mathcal{B}_{1+r}(e) \cap S_j = \emptyset$ for $r \in [\sigma - \upsilon, \sigma)$, while if $r \in (\sigma, \sigma + \upsilon]$ then $\partial \mathcal{B}_{1+r}(e) \cap S_j \neq \emptyset$ and $\partial \mathcal{B}_{1+r}(e)$ enters inside S_j at ρ-distance at most $r - \sigma$ from the boundary of S_j. In particular,

$$\bar{u}_j = 0 \quad \text{for } r \in [\sigma - \upsilon, \sigma]. \tag{8.21}$$

Next, if θ_j is the angle of S_j at y_0, let β be defined by

$$1 + \beta = \pi / \theta_j \geq 1. \tag{8.22}$$

Note that y_h is at ρ-distance $2r_h$ from y_0. Again by Lemma 8.6 applied to $u_2 = u_j$, (after a rotation and a translation) we have the estimate

$$u_j(x) \leq C r_h^\beta d(x, \partial S_j) \leq C r_h^\beta d_\rho(x, \partial S_j)$$

in a neighborhood of y_h. As a consequence, recalling in addition that the ball $\mathcal{B}_{1+r}(e)$ enters in S_j at ρ-distance $r-\sigma$ from the boundary, for $r \in [\sigma, \sigma+\upsilon]$ we get

$$\bar{u}_j = \sup_{\partial \mathcal{B}_{1+r}(e)} u_j \le C r_h^\beta (r-\sigma).$$

The last estimate and (8.21) imply

$$\bar{u}_j(r) \le C r_h^\beta (r-\sigma)^+ \quad \text{for } r \in [\sigma-\upsilon, \sigma+\upsilon]. \tag{8.23}$$

Now, we want to compare the mass of the Laplacians of $\underline{u}_i$ and $\bar{u}_j$. Define, as in (8.17),

$$\underline{u}_i^\varepsilon := \inf_{\partial \mathcal{B}_r(e)} u_i^\varepsilon, \qquad \bar{u}_k^\varepsilon := \sup_{\partial \mathcal{B}_{1+r}(e)} u_k^\varepsilon, \qquad k \neq i.$$

For σ and υ small enough, the ball $\mathcal{B}_r(e)$ is contained in Ω for any $r \le \sigma+\upsilon$, and thus

$$\Delta u_i^\varepsilon = \frac{1}{\varepsilon^2} u_i^\varepsilon \sum_{k \neq i} H(u_k^\varepsilon) \quad \text{in } \mathcal{B}_{r+\sigma}(e).$$

On the other hand, since x_h is an interior regular point that realizes its distance from S_j at an interior point, y_h, its distance from the support of the boundary data f_k is greater than 1 for any $k \neq i$. We infer that, for σ and υ small enough and $r \le \sigma+\upsilon$,

$$\Delta u_k^\varepsilon \ge \frac{1}{\varepsilon^2} u_k^\varepsilon \sum_{l \neq k} H(u_l^\varepsilon) \quad \text{in } \mathcal{B}_{1+r}(e).$$

Hence, arguing as in the proof of Theorem 7.1, we see that

$$\Delta_r \underline{u}_i^\varepsilon \le \sum_{k \neq i} \Delta_r \bar{u}_k^\varepsilon \quad \text{in } (\sigma-\upsilon, \sigma+\upsilon), \tag{8.24}$$

where $\Delta_r u = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right)$. Since x_h is a regular point of ∂S_i that realizes the distance from S_j at $y_h \in \partial C_i$, the ball $\mathcal{B}_{1+\sigma+\upsilon}(e)$ does not intersect the support of the functions u_k for $k \neq j$ and small υ and σ. Therefore, multiplying inequality (8.24) by a positive test function $\phi \in C_c^\infty(\sigma-\upsilon, \sigma+\upsilon)$, integrating by parts in $(\sigma-\upsilon, \sigma+\upsilon)$ and passing to the limit as $\varepsilon \to 0$ along a converging subsequence, we see that the only surviving function on the right hand side is $\bar{u}_j$ and we get

$$\int_{\sigma-\upsilon}^{\sigma+\upsilon} \underline{u}_i \frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\left(\frac{1}{r}\phi\right)\right) dr \le \int_{\sigma-\upsilon}^{\sigma+\upsilon} \bar{u}_j \frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\left(\frac{1}{r}\phi\right)\right) dr. \tag{8.25}$$

Let us choose a function ϕ which is increasing in $(\sigma-\upsilon, \sigma)$ and decreasing in $(\sigma, \sigma+\upsilon)$ and hence with maximum at $r = \sigma$, and let us estimate the two sides of the last inequality.

Estimates (8.18) imply that $\frac{\partial \underline{u}_i}{\partial r}(\sigma^-) \le -Cr_h^\alpha$. Therefore, for small υ we have

$$
\begin{aligned}
&\int_{\sigma-\upsilon}^{\sigma+\upsilon} \underline{u}_i \frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\phi\right)\right) dr \\
&\quad = -\int_{\sigma-\upsilon}^{\sigma} \frac{\partial \underline{u}_i}{\partial r} r \frac{\partial}{\partial r}\left(\frac{1}{r}\phi\right) dr = -\int_{\sigma-\upsilon}^{\sigma}\left(\frac{\partial \underline{u}_i}{\partial r}(\sigma^-) + o_{\sigma-r}(1)\right) r\frac{\partial}{\partial r}\left(\frac{1}{r}\phi\right) dr \\
&\quad \ge -\int_{\sigma-\upsilon}^{\sigma} \frac{\partial \underline{u}_i}{\partial r}(\sigma^-)\left(\frac{\partial \phi}{\partial r} - \frac{1}{r}\phi\right) dr - o_\upsilon(1)\int_{\sigma-\upsilon}^{\sigma}\left(\frac{\partial\phi}{\partial r} + \frac{1}{r}\phi\right) dr \\
&\quad \ge -\frac{\partial \underline{u}_i}{\partial r}(\sigma^-)\left[\phi(\sigma) - \phi(\sigma)\log\left(\frac{\sigma}{\sigma-\upsilon}\right)\right] - o_\upsilon(1)\left[\phi(\sigma) + \phi(\sigma)\log\left(\frac{\sigma}{\sigma-\upsilon}\right)\right] \\
&\quad \ge (Cr_h^\alpha - o_\upsilon(1))\phi(\sigma).
\end{aligned}
$$

Similarly, using (8.23) and integrating by parts, we get

$$
\int_{\sigma-\upsilon}^{\sigma+\upsilon} \bar{u}_j \frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\phi\right)\right) dr \le (Cr_h^\beta + o_\upsilon(1))\phi(\sigma).
$$

From the previous estimates and (8.25), letting υ go to 0, we obtain

$$
r_h^\alpha \le Cr_h^\beta,
$$

and therefore, for h small enough,

$$
\beta \le \alpha.
$$

Recalling the definitions (8.16) and (8.22) of α and β respectively, we infer that $\theta_i \le \theta_j$. This proves (8.14). If $x_0 = 0$ is an interior point of Ω, exchanging the roles of u_i and u_j, we get the opposite inequality $\theta_j \le \theta_i$, and this proves (8.13) for H defined as in (2.6).

Next, let us turn to the case (2.5). Again we compare the mass of the Laplacians of u_i and u_j across the free boundaries. For $\sigma < r_h$ define

$$
D_\sigma(x_h) := \{x \in \mathcal{B}_\sigma(x_h) \mid d(x, \partial S_i) \le \sigma^2\}. \tag{8.26}
$$

If we denote by $(D_\sigma(x_h))_{<1}$ the sets of points at ρ-distance less than 1 from $D_\sigma(x_h)$, we have

$$
\int_{D_\sigma(x_h)} \Delta u_i \le \sum_{k\neq i} \int_{(D_\sigma(x_h))_{<1}} \Delta u_k \tag{8.27}
$$

as in (7.2) with $(D_\sigma(x_h))_{<1}$ in place of $\mathcal{B}_{1+S}(x_0)$. By Lemma 8.6 the normal derivative of u_i with respect to the inner normal ν_i, at any point on the boundary $\partial\mathcal{C}_i$ with distance to the vertex between r_h and $3r_h$, is greater than cr_h^α. Hence

$$
\int_{D_\sigma(x_h)} \Delta u_i = \int_{\partial\mathcal{C}_i \cap D_\sigma(x_h)} \frac{\partial u_i}{\partial \nu_i} d\mathcal{H} \ge c\int_{2r_h - c\sigma}^{2r_h + C\sigma} r_h^\alpha \, dr = Cr_h^\alpha \sigma.
$$

Note that $(D_\sigma(x_h))_{<1} \cap \partial S_j \subset \mathcal{B}_{c\sigma}(y_h) \cap \partial S_j$, and therefore, for σ small enough, again from Lemma 8.6 we have

$$
\int_{(D_\sigma(x_h))_{<1}} \Delta u_j \le Cr_h^\beta \sigma.
$$

Then for r_h small enough we obtain $\beta \le \alpha$, and therefore $\theta_i \le \theta_j$. If $x_0 = 0$ is an interior point of Ω, exchanging the roles of u_i and u_j we get $\theta_j \le \theta_i$. This concludes the proof of the theorem in the case where ∂S_i and ∂S_j are locally cones around 0 and y_0 respectively.

We are now going to explain how to adapt the proof in the general case.

Proof of Theorem 8.10 in the general case. If $\theta_i = 0$, then $\theta_i \le \theta_j$. Assume $\theta_i \in (0, \pi]$ and $\theta_j \in [0, \pi]$; then for any $0 < \delta < \theta_i$, there exist $r_\delta > 0$, a cone $\mathcal{C}_\delta^i$ centered at 0 and with opening $\theta_i - \delta$, and a cone $\mathcal{C}_\delta^j$ centered at y_0 and with opening $\theta_j + \delta$ such that

$$\mathcal{C}_\delta^i \cap B_{r_\delta}(0) \subset S_i \cap B_{r_\delta}(0) \quad \text{and} \quad S_j \cap B_{r_\delta}(y_0) \subset \mathcal{C}_\delta^j \cap B_{r_\delta}(y_0).$$

Let $(x_h)_h$ be the sequence of regular points on $\partial S_i \cap \Omega$ given by Lemma 8.7 (consider Γ_1 to be the closest side to S_j), and let $r_h = d(0, x_h)$. Denote by y_h the point on $\partial S_j \cap \Omega$ at ρ-distance 1 from x_h. Then $d_\rho(y_h, y_0) \le c r_h$. Now, the proof of the theorem proceeds as in the previous case and we can compare the mass of the Laplacians across the free boundaries of u_i and u_j.

First consider the case (2.5). For $\sigma > 0$ take $D_\sigma(x_h)$ and $(D_\sigma(x_h))_{<1}$ as defined in (8.26). For σ small enough, by Lemma 8.9, $\partial S_i \cap D_\sigma(x_h)$ does not contain singular points and by Lemma 8.3 it is a C^1 curve of the plane.

By Lemma 8.7,

$$\int_{D_\sigma(x_h)} \Delta u_i = \int_{\partial S_i \cap D_\sigma(x_h)} \frac{\partial u_i}{\partial \nu_i} d\mathcal{H} \ge C r_h^{\alpha_\delta} \sigma.$$

Note that

$$(D_\sigma(x_h))_{<1} \cap \partial S_j \subset \mathcal{B}_{c\sigma}(y_h) \cap \partial S_j,$$

and therefore, for σ small enough, from Lemma 8.8, as in the proof of Lemma 8.9, we have

$$\int_{(D_\sigma(x_h))_{<1}} \Delta u_j \le \tilde{C} r_h^{\beta_\delta} \sigma.$$

Then for h small enough, we obtain $\beta_\delta \le \alpha_\delta$, and therefore $\theta_i \le \theta_j$. If $x_0 = 0$ is an interior point of Ω, exchanging the roles of u_i and u_j we get $\theta_j \le \theta_i$.

Next, let us turn to the case (2.6). Then we define, for $r \in [R_h - \upsilon, R_h + \upsilon]$,

$$\underline{u}_i := \inf_{\partial \mathcal{B}_r(z_h)} u_i \quad \text{and} \quad \bar{u}_j := \sup_{\partial \mathcal{B}_{1+r}(z_h)} u_j.$$

Arguing as before, and using the Lemma 8.7 we get $\beta_\delta \le \alpha_\delta$, and therefore, letting δ go to 0, we finally obtain $\theta_i \le \theta_j$. Note in particular that if $\theta_i > 0$ then $\theta_j > 0$. If $x_0 = 0$ is an interior point of Ω, exchanging the roles of u_i and u_j we get $\theta_j \le \theta_i$. □

An immediate corollary of Theorem 8.10 is the C^1-regularity of the free boundaries when $K = 2$ and under the following additional assumptions on Ω, f_1 and f_2:

$$\Omega := \{(x_1, x_2) \in \mathbb{R}^2 \mid g(x_2) \le x_1 \le h(x_2),\ x_2 \in [a, b]\}, \qquad b - a \ge 4, \tag{8.28}$$

where

$$\begin{cases} g, h : [a, b] \to \mathbb{R} \text{ are Lipschitz functions with} \\ -m_2 \le g \le -m_1 \le M_2 \le h \le M_1, \quad M_2 \ge -m_1 + 4; \end{cases} \tag{8.29}$$

the boundary data are such that

$$\begin{cases} f_1 \equiv 1, \quad f_2 \equiv 0 \quad \text{on } \{x_1 \le g(x_2)\}, \\ f_1 \equiv 0, \quad f_2 \equiv 1 \quad \text{on } \{x_1 \ge h(x_2)\}, \\ f_1 \text{ is decreasing in } x_1 \text{ on } \{x_2 \le a\} \cup \{x_2 \ge b\}, \\ f_2 \text{ is increasing in } x_1 \text{ on } \{x_2 \le a\} \cup \{x_2 \ge b\}. \end{cases} \tag{8.30}$$

These assumptions imply that $-u_1$ and u_2 are increasing in the x_1 direction. Then we have the following

Corollary 8.11. *Assume* (2.8) *with* $p = 1$ *in* (2.5). *Assume in addition* $K = n = 2$ *and* (8.28)–(8.30). *Then the sets* ∂S_i, $i = 1, 2$, *are of class* C^1.

Proof. We know that the sets ∂S_i are curves of the plane at ρ-distance 1 from each other. Suppose for contradiction that ∂S_1 has an angle $\theta < \pi$ at y_0. In particular, there exist two ρ-balls of radius 1, centered at two points $z, w \in \partial S_2$, that are tangent to ∂S_1 at y_0. Then, by the monotonicity property of the u_i's and Theorem 7.1, the arc of the ρ-ball of radius 1 centered at y_0 between the points z and w must all be in ∂S_2. This means that any point inside this arc, which is a regular point of ∂S_2, is at ρ-distance 1 from the singular point $y_0 \in \partial S_1$. This contradicts Theorem 8.10. We have shown that any point of the free boundaries is regular. Then by Lemma 8.3 the free boundaries are of class C^1. □

Another corollary of Theorem 8.10 is that the number of singular points is finite.

Corollary 8.12. *Assume* (2.8) *with* $n = K = 2$ *and* $p = 1$ *in* (2.5). *Assume in addition that the supports on* $\partial\Omega$ *of the boundary data* f_1 *and* f_2 *have a finite number of connected components. Then the singular points form a finite set.*

Proof. From Lemma 8.4, S_1 and S_2 have a finite number of connected components. Moreover, we recall that any connected component has to reach the boundary.

Let x_0 be a singular point belonging to the boundary of the support of one of the limit functions u_i. Without loss of generality assume $x_0 \in \partial S_1$. Let $y_1, y_2 \in \partial S_2$ be two different points where x_0 realizes the distance from S_2 ($y_1, y_2 \in \partial\mathcal{B}_1(x_0) \cap \partial S_2$, see Figure 3). We can choose y_1 such that $\mathcal{B}_1(x_0)$ is the limit as $k \to \infty$ of balls $\mathcal{B}_1(x_k)$ with $x_k \in \partial S_1$, tangent to points $y_k \in \partial S_2$ with $y_k \to y_1$ and $x_k \to x_0$ as $k \to \infty$. Theorem 8.10 implies that S_2 has an angle at y_1 and y_2, and the intersection of the arc on $\partial\mathcal{B}_1(x_0)$ between y_1 and y_2 with ∂C_1 must have empty interior. This means that near y_1 there are points on ∂S_2 outside $\overline{\mathcal{B}_1(x_0)}$. These points are at distance greater than 1 from x_0 and from any other point of ∂S_1 close to x_0, and must realize the distance from S_1 outside $\mathcal{B}_1(y_1)$ (see Figure 3). Therefore if we take a sequence z_k of such points converging to y_1 and we consider the corresponding tangent balls centered at points that are in ∂S_1 where the z_k's realize the distance, we obtain a second tangent ball $\mathcal{B}_1(x_1)$ for y_1 with $x_1 \neq x_0$.

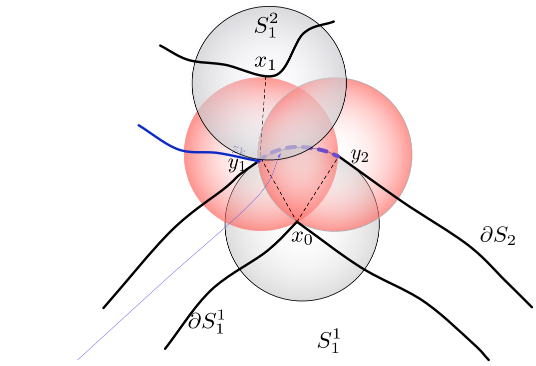

Fig. 3. Forbidden arc.

Now, denote by S_1^1 the connected component of S_1 whose boundary contains x_0. Remember that since S_1 and S_2 are at ρ-distance 1, we have $u_1 \equiv 0$ in $\overline{\mathcal{B}_1(y_1)} \cup \overline{\mathcal{B}_1(y_2)}$. Moreover, since the connected components of S_2 whose boundaries contain y_1 and y_2 must reach the boundary of Ω, they separate the components of S_1 whose boundaries contain x_0 and x_1. Therefore x_1 must belong to the boundary of different components of S_1. The same argument that we have used for x_1 and x_0 also proves that y_1 and y_2 must belong to the boundary of different components of C_1.

We conclude that a singular point x_0 of S_1 involves at least four different connected components, and there corresponds to it another singular point, x_1, belonging to a different component of S_1 (see Figure 4).

Assume without loss of generality that $x_1 \in \partial S_1^2$. Since all the connected components must reach the boundary of Ω, x_1 is the only singular point of S_1^2 corresponding to a

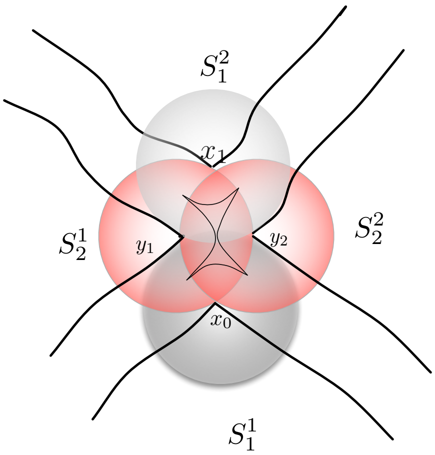

Fig. 4. A singular point involving four components.

singular point of S_1^1. Since the connected components of S_1 are finitely many, we infer that there are a finite number of singular points on ∂S_1^1. This argument applied to any connected component of S_1 shows that the set of singular points of S_1 is finite. $\square$

8.2. Lipschitz regularity of the free boundaries

In this section, we will show, under some additional assumptions on the domain Ω and the boundary data f_i, that we can construct a solution of problem (2.4) such that the free boundaries S_i of the limiting functions have the following properties: if S_i has an angle θ at a singular point, then $\theta > 0$. This result can be rephrased by saying that the free boundaries are Lipschitz curves of the plane. Let us make the assumptions precise. We assume that the domain Ω has the property that for any point of the boundary there are tangent ρ-balls of radius $1+\eta$, with $\eta > 0$, contained in Ω and in its complement. Precisely:

$$\begin{cases} \Omega \text{ is a bounded domain of } \mathbb{R}^2; \\ \exists \eta > 0\ \forall x \in \partial\Omega\ \exists \mathcal{B}_{1+\eta}(y),\ \mathcal{B}_{1+\eta}(z): \\ \quad x \in \partial\mathcal{B}_{1+\eta}(y) \cap \partial\mathcal{B}_{1+\eta}(z),\ \mathcal{B}_{1+\eta}(y) \subset \Omega, \text{ and } \mathcal{B}_{1+\eta}(z) \subset \Omega^c. \end{cases} \tag{8.31}$$

On the boundary data f_i, $i = 1, \ldots, K$, we assume

$$\begin{cases} f_i \equiv 1 \text{ in } \operatorname{supp} f_i; \\ \exists c > 0\ \forall x \in \partial\Omega \cap \operatorname{supp} f_i : |\mathcal{B}_r(x) \cap \operatorname{supp} f_i| \geq c|\mathcal{B}_r(x)|, \\ d_\rho(\operatorname{supp} f_i, \operatorname{supp} f_j) \geq 1,\ i \neq j, \\ d_\rho(\operatorname{supp} f_i \cap \partial\Omega, \operatorname{supp} f_{i+1} \cap \partial\Omega) = 1, \text{ where } f_{K+1} := f_1; \\ \Gamma_i := \operatorname{supp} f_i \cap \partial\Omega \text{ is a connected } (C^2) \text{ curve.} \end{cases} \tag{8.32}$$

We are going to build a solution of (2.4) such that the support of any limiting function u_i contains a full neighborhood of Γ_i in Ω with Lipschitz boundary. Then we prove that the free boundaries are Lipschitz. In order to do it, we first prove the existence of a solution $(u_1^\varepsilon, \ldots, u_K^\varepsilon)$ of an obstacle problem associated to system (2.4). Then we show that the functions u_i^ε never touch the obstacles, implying that $(u_1^\varepsilon, \ldots, u_K^\varepsilon)$ is actually a solution of (2.4). We consider obstacle functions ψ_i, for $i = 1, \ldots, K$, defined as follows. Let y_1^i, y_2^i be the endpoints of the curve Γ_i. For $0 < \mu < \lambda < 1$, we set

$$\Gamma_i^\mu := \{x \in \Omega^c \mid d(x, \Gamma_i) = \mu\}, \qquad \Gamma_i^{\mu,\lambda} := \{x \in \Gamma_i^\mu \mid d(x, y_1^i), d(x, y_2^i) \geq \lambda\}.$$

For μ and λ small enough, $\Gamma_i^{\mu,\lambda}$ is a $C^{1,1}$ curve in Ω^c with endpoints z_1^i, z_2^i such that $d(z_l^i, y_l^i) = \lambda$, $l = 1, 2$. We finally set

$$A_i := \{x \in \Omega \mid d(x, \Gamma_i^{\mu,\lambda}) < \lambda\} = \Omega \cap \bigcup_{x \in \Gamma_i^{\mu,\lambda}} B_\lambda(x). \tag{8.33}$$

Note that

$$\partial A_i = \Gamma_i \cup (\partial A_i \cap \Omega),$$

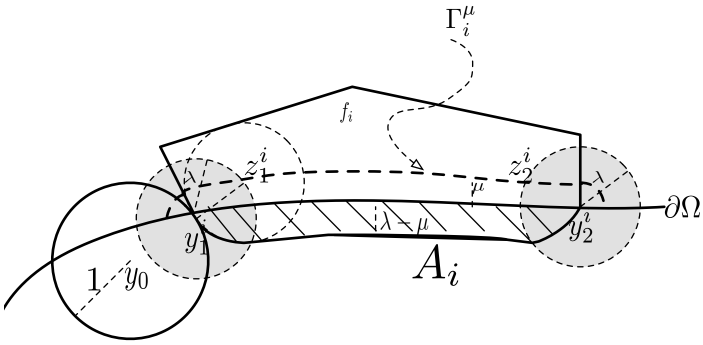

Fig. 5. Construction of an obstacle.

where $\partial A_i \cap \Omega$ is the union of two arcs contained respectively in the balls $B_\lambda(z_1^i)$ and $B_\lambda(z_2^i)$, and a curve contained in the set of points of Ω at distance $\lambda - \mu$ from Γ_i (see Figure 5). Denote by α_l^i the angle of A_i at y_l^i, $l = 1, 2$. Note that

$$\begin{cases} \alpha_l^i \to \pi/2 + o_\lambda(1) & \text{as } \mu \to 0, \\ \alpha_l^i \to 0 & \text{as } \mu \to \lambda, \end{cases} \tag{8.34}$$

where $o_\lambda(1) \to 0$ as $\lambda \to 0$.

We take as obstacles the functions $\psi_i : (\Omega)_{\le 1} \to \mathbb{R}$ defined as the solutions of the following problem, for $i = 1, \dots, K$:

$$\begin{cases} \Delta\psi_i = 0 & \text{in } A_i, \\ \psi_i = f_i & \text{on } (\partial\Omega)_{\le 1}, \\ \psi_i = 0 & \text{in } \Omega \setminus A_i. \end{cases} \tag{8.35}$$

In this section we deal with the solution $(u_1^\varepsilon, \dots, u_K^\varepsilon)$ of the following obstacle problem: for $i = 1, \dots, K$,

$$\begin{cases} u_i^\varepsilon \ge \psi_i & \text{in } \Omega, \\ \Delta u_i^\varepsilon(x) \le \dfrac{1}{\varepsilon^2} u_i^\varepsilon(x) \displaystyle\sum_{j \ne i} H(u_j^\varepsilon)(x) & \text{in } \Omega, \\ \Delta u_i^\varepsilon(x) = \dfrac{1}{\varepsilon^2} u_i^\varepsilon(x) \displaystyle\sum_{j \ne i} H(u_j^\varepsilon)(x) & \text{in } \{u_i^\varepsilon > \psi_i\}, \\ u_i^\varepsilon = f_i & \text{on } (\partial\Omega)_{\le 1}. \end{cases} \tag{8.36}$$

In the whole section we make the following assumptions:

$$\begin{cases} \varepsilon > 0, \\ \text{(8.31) and (8.32) hold true,} \\ H \text{ is either of the form (2.5) with } p = 1, \text{ or (2.6) and (2.7) hold true,} \\ \text{for } i = 1, \dots, K,\ A_i \text{ and } \psi_i \text{ are defined by (8.33) and (8.35) respectively.} \end{cases} \tag{8.37}$$

Theorem 8.13. *Assume* (8.37)*. Then there are continuous positive functions* $u_1^\varepsilon, \dots, u_K^\varepsilon$*, depending on the parameter* ε*, that are viscosity solutions of problem* (8.36)*. In particular*

$$\Delta u_i^\varepsilon(x) = \frac{1}{\varepsilon^2} u_i^\varepsilon(x) \sum_{j \neq i} H(u_j^\varepsilon)(x) \quad \text{in } \Omega \setminus A_i. \tag{8.38}$$

Moreover, for $i = 1, \dots, K$*,*

$$\Delta u_i^\varepsilon \geq 0 \quad \text{in } \Omega \tag{8.39}$$

in the viscosity sense.

Proof. The proof of the existence of a solution $(u_1^\varepsilon, \dots, u_K^\varepsilon)$ of (8.36) is a slight modification of the proof of Theorem 4.1. Here

$$\Theta = \{(u_1, \dots, u_K) \mid u_i : \Omega \to \mathbb{R} \text{ is continuous, } \psi_i \leq u_i \leq \phi_i \text{ in } \Omega,\ u_i = f_i \text{ on } (\partial\Omega)_{\leq 1}\}.$$

In the set $\Omega \setminus A_i$, we have $u_i^\varepsilon > 0 = \psi_i$, which implies (8.38). Inequality (8.39) is a consequence of the following facts: in $\{u_i^\varepsilon > \psi_i\}$ we have $\Delta u_i^\varepsilon = \frac{1}{\varepsilon^2} u_i^\varepsilon \sum_{j \neq i} H(u_j^\varepsilon) \geq 0$; in the interior of $\{u_i^\varepsilon = \psi_i\}$, $\Delta u_i^\varepsilon = \Delta \psi_i = 0$; the free boundaries $\partial\{u_i^\varepsilon > \psi_i\}$ have locally finite $n-1$-Hausdorff measure [2]. □

Theorem 8.14. *Assume* (8.37)*. Let* $(u_1^\varepsilon, \dots, u_K^\varepsilon)$ *be a viscosity solution of problem* (8.36)*. Then there exists a subsequence* $(u_1^{\varepsilon_l}, \dots, u_K^{\varepsilon_l})$ *and continuous functions* $(u_1, \dots, u_K)$ *defined on* $\overline{\Omega}$ *such that*

$$(u_1^{\varepsilon_l}, \dots, u_K^{\varepsilon_l}) \to (u_1, \dots, u_K) \quad \text{a.e. in } \Omega \quad \text{as } l \to \infty,$$

and the convergence of $u_i^{\varepsilon_l}$ *to* u_i *is locally uniform in the support of* u_i*. Moreover:*

(i) *The* u_i*'s are locally Lipschitz continuous in* Ω*, in particular, there exists* $C_0 > 0$ *such that if* $d_\rho(x, \partial\Omega) \geq r$*, then*

$$|\nabla u_i(x)| \leq C_0/r. \tag{8.40}$$

(ii) *The* u_i*'s have disjoint supports, more precisely*

$$u_i \equiv 0 \quad \text{in } \{x \in \Omega \mid d_\rho(x, \operatorname{supp} u_j) \leq 1\} \text{ for any } j \neq i.$$

(iii) $\Delta u_i = 0$ *when* $u_i > 0$.
(iv) $u_i \geq \psi_i$ *in* Ω.
(v) $u_i = f_i$ *on* $\partial\Omega$.

Proof. The convergence statement is again a consequence of Lemma 5.3, Corollary 5.4 and Lemma 5.5, which hold true with $\operatorname{supp} f_i$ and $\operatorname{supp} f_j$ replaced respectively by $\operatorname{supp} \psi_i = A_i$ and $\operatorname{supp} \psi_j = A_j$ (in Lemma 5.3 and Corollary 5.4), and $\overline{\Gamma}_j^\sigma$ defined as the set $\{\psi_j \geq \sigma\}$ (in Lemma 5.5). Estimates (5.7) of Corollary 5.4 imply (8.40). Property (iv) is an immediate consequence of $u_i^\varepsilon \geq \psi_i$ in Ω. Finally, (v) is implied by the fact that $\psi_i \leq u_i^\varepsilon \leq \phi_i$ in Ω, and $\phi_i = \psi_i = f_i$ on $\partial\Omega$, where ϕ_i is given by (4.1). □

As proven in Corollary 6.2, one can show that the free boundaries satisfy the exterior ρ-ball condition with radius 1, that they have finite 1-Hausdorff measure, and that the distance between the supports of two different functions is precisely 1. We are now going to prove that if $\lambda - \mu$ is small enough, then any solution of the obstacle problem (8.36) never touches the obstacles inside the domain Ω. To this end, we first need the following lemma:

Lemma 8.15. *Assume (8.37). Then there exists $c > 0$ such that, for $i = 1, \dots, K$,*

$$\frac{\partial \psi_i}{\partial \nu_i}(x) \leq -\frac{c}{\lambda - \mu} \quad \text{for any } x \in \partial A_i \cap \Omega, \tag{8.41}$$

where ν_i is the exterior normal vector to the set A_i.

Proof. Fix $x_0 \in \partial A_i \cap \Omega$. Then, by definition of A_i, there exists $z \in \Omega^c$ such that $d(z, \partial\Omega) = \mu$, $B_\lambda(z) \cap \Omega \subset A_i$ and $x_0 \in \partial B_\lambda(z)$. Consider now the ring $\{x \mid \mu < |x - z| < \lambda\}$ and the barrier function ϕ that solves

$$\begin{cases} \Delta\phi = 0 & \text{in } \{x \mid \mu < |x - z| < \lambda\}, \\ \phi = 1 & \text{on } \partial B_\mu(z), \\ \phi = 0 & \text{on } \partial B_\lambda(z). \end{cases}$$

The function ψ_i is harmonic in $B_\lambda(z) \cap \Omega$, $\psi_i \geq 0 = \phi$ on $\partial B_\lambda(z) \cap \Omega$, and $\psi_i = 1 \geq \phi$ on $\partial\Omega \cap B_\lambda(z)$. Therefore by the comparison principle, $\psi_i(x) \geq \phi(x)$ for any $x \in B_\lambda(z) \cap \Omega$, and this implies (8.41) at $x = x_0$. □

Theorem 8.16. *Assume (8.37). Let $(u_1, \dots, u_K)$ be the limit of a converging subsequence of solutions $(u_1^\varepsilon, \dots, u_K^\varepsilon)$ of (8.36). Set $a := \lambda - \mu$. Then there exists $a_0 > 0$ such that for any $a < a_0$ and $i = 1, \dots, K$,*

$$u_i > \psi_i \quad \text{in } \overline{A}_i \cap \Omega. \tag{8.42}$$

Proof. In order to prove (8.42), it is enough to show that

$$u_i(x) > \psi_i(x) \quad \text{for any } x \in \partial A_i \cap \Omega. \tag{8.43}$$

Indeed, if (8.43) holds true, since by (8.35) and Theorem 8.14, both u_i and ψ_i are harmonic in A_i, the strong maximum principle implies $u_i > \psi_i$ in A_i. This and (8.43) give (8.42). Suppose for contradiction that there exists $x_0 \in \partial A_i \cap \Omega$ such that $u_i(x_0) = \psi_i(x_0) = 0$. Then, by (8.41),

$$\frac{\partial u_i}{\partial \nu_i}(x_0) \leq \frac{\partial \psi_i}{\partial \nu_i}(x_0) \leq -\frac{c}{\lambda - \mu} = -\frac{c}{a}. \tag{8.44}$$

Assumptions (8.31) imply that if the angles α_l^i of A_i at y_l^i, $l = 1, 2$, are small enough, then the sets

$$\Sigma_i := \{y \mid y = x + \nu_i(x), x \in \partial A_i \cap \Omega\},$$
$$\Sigma_i^- := \{y \mid y = x + t\nu_i(x),\ x \in \partial A_i \cap \Omega,\ 0 < t < 1\}$$

are relatively compact in Ω and

$$d_\rho(x_0, \operatorname{supp}\psi_j) > 1 \quad \text{for any } j \neq i. \tag{8.45}$$

Therefore, by (8.34), we can choose a so small that (8.45) holds true. Moreover, from (8.45), there exists a small $\sigma > 0$ such that $\mathcal{B}_{1+\sigma}(x_0) \cap \operatorname{supp}\psi_j = \emptyset$, $j \neq i$, and from (8.36) we know that

$$\Delta u_j^\varepsilon \geq \frac{1}{\varepsilon^2} u_j^\varepsilon H(u_i^\varepsilon) \quad \text{in } \mathcal{B}_{1+\sigma}(x_0)$$

(consider u_j^ε extended by zero if the ball falls out of Ω). When H is defined as in (2.5) with $p = 1$, arguing as in (8.27) in the proof of Theorem 8.10 we obtain

$$\sum_{j\neq i} \int_{(D_\sigma(x_0))_{<1}} \Delta u_j \geq \int_{D_\sigma(x_0)} \Delta u_i.$$

Now, since $u_i \geq \psi_i > 0$ in A_i and $u_i(x_0) = 0$, the point x_0 belongs to $\partial\{u_i > 0\} \cap \partial A_i \cap \Omega$. Since $\partial A_i \cap \Omega$ has an interior tangent ball and $\partial\{u_i > 0\}$ has an exterior tangent ball, we deduce that x_0 is a regular point. Since the set of regular points is open (Lemma 8.9), for σ small enough we have

$$\int_{D_\sigma(x_0)} \Delta u_i \geq -\int_{\partial\{u_i>0\}\cap D_\sigma(x_0)} \frac{\partial u_i}{\partial \nu_i} d\mathcal{H}, \tag{8.46}$$

where ν_i is still the exterior normal vector to A_i. On the other hand, if y_0 is the point that realizes the distance 1 with x_0, assume without loss of generality that $y_0 \in \partial \operatorname{supp} u_j$; then y_0 has to be in Σ_i and be a regular point. Consequently, for ρ small enough such that $\partial\{u_j > 0\} \cap B_\rho(y_0)$ is C^1 we have

$$\int_{B_\rho(y_0)} \Delta u_j = -\int_{\partial\{u_j>0\}\cap B_\rho(y_0)} \frac{\partial u_j}{\partial \nu_j} d\mathcal{H}.$$

Now, using the fact that for σ so small that $\rho > c\sigma$, $\operatorname{supp} u_j \cap (D_\sigma(x_0))_{<1} \subset \mathcal{B}_{c\sigma}(y_0)$, we have

$$\int_{B_{c\sigma}(y_0)} \Delta u_j \geq \int_{(D_\sigma(x_0))_{<1}} \Delta u_i. \tag{8.47}$$

Putting all together, dividing (8.46) and (8.47) respectively by $\mathcal{H}(\partial\{u_i > 0\} \cap D_\sigma(x_0))$ and $\mathcal{H}(\partial\{u_j > 0\} \cap B_{c\sigma}(y_0))$, and passing to the limit as $\sigma \to 0$ we obtain

$$-\frac{\partial u_j}{\partial \nu_j}(y_0) \geq -c\frac{\partial u_i}{\partial \nu_i}(x_0) \geq \frac{\tilde{c}}{a}. \tag{8.48}$$

We are now going to show that (8.48) yields a contradiction. Indeed, the point y_0 realizes its distance from the set $\{u_i > 0\}$ at x_0, so the ball $\mathcal{B}_1(y_0)$ is tangent to $\{u_i > 0\}$ at x_0. Moreover, since $A_i \subset \{u_i > 0\}$, the ball $\mathcal{B}_1(y_0)$ is tangent to A_i at x_0. On the other hand, for a small enough, by assumption (8.31), $\mathcal{B}_1(y_0)$ is contained in Ω. In particular, the ρ-distance of y_0 from $\partial\Omega$ is greater than 1. Therefore, from (8.40), we infer that $|\nabla u_j(y_0)| \leq C_0$, which contradicts (8.48) for a small enough.

When H is defined as in (2.6), we argue as in case (b) in the proof of Theorem 7.1, and similarly we get a contradiction for a small enough. □

Corollary 8.17. *Under the assumptions of Theorem 8.16, if $a < a_0$ then $(u_1^\varepsilon, \dots, u_K^\varepsilon)$ is a solution of the problem*

$$\begin{cases} u_i^\varepsilon \geq \psi_i & \text{in } \Omega, \\ \Delta u_i^\varepsilon(x) = \dfrac{1}{\varepsilon^2} u_i^\varepsilon(x) \displaystyle\sum_{j \neq i} H(u_j^\varepsilon)(x) & \text{in } \Omega, \\ u_i^\varepsilon = f_i & \text{on } (\partial\Omega)_{\leq 1}. \end{cases} \tag{8.49}$$

In particular, $(u_1^\varepsilon, \dots, u_K^\varepsilon)$ is a solution of (2.4).

We are now ready to show that free boundaries are Lipschitz.

Theorem 8.18. *Let $(u_1^\varepsilon, \dots, u_K^\varepsilon)$ be the solution of (2.4) given by Corollary 8.17. Let $(u_1, \dots, u_K)$ be the limit as $\varepsilon \to 0$ of a converging subsequence. Then the free boundaries $\partial\{u_i > 0\}$, $i = 1, \dots, K$, are Lipschitz curves of the plane.*

Proof. Assume that the free boundaries are not Lipschitz. This implies that there exists at least one singular point with asymptotic cone with zero opening.

Let x_0 be an interior singular point with asymptotic cone with zero angle. Without loss of generality suppose $x_0 \in \partial\{u_1 > 0\}$. Let e_1 be the line perpendicular to the cone axis and passing through x_0, in which we choose an orientation such that the cone is below the axis e_1. As we proved in Theorem 8.10 and Corollary 8.12, there exist y_0 and y_1, with $y_0, y_1 \in \bigcup_{j \neq 1} \partial\{u_j > 0\}$ singular points at distance 1 from x_0 with asymptotic cones with zero opening. Also, by Theorem 7.1 for any regular point $x \in \partial\{u_1 > 0\} \cap B_1(x_0)$ there exists a corresponding $y \in \bigcup_{j \neq 1} \partial\{u_j > 0\}$ such that

$$y = x + \nu(x)$$

with $\nu(x)$ the external normal vector to $\partial\{u_1 > 0\}$ at x. Observe that y_0, y_1 must lie on e_1. In fact, let $x_n^l \in \partial\{u_1 > 0\}$ be regular points converging to x_0 as $n \to \infty$ from the left side of the cone axis, and let $x_n^r \in \partial\{u_1 > 0\}$ be regular points converging to x_0 from the right side of the cone axis. Then the limits of the normal vectors, $\nu(x_n^l) \to \nu^l$ and $\nu(x_n^r) \to \nu^r$, both have direction e_1 since they are orthogonal to the cone axis. Let y_0 and y_1 be without loss of generality the points defined by

$$y_0 = x_0 + \nu^l, \qquad y_1 = x_0 + \nu^r.$$

So we have three singular points at distance 1, all on the line e_1. Repeating the same argument and using now y_1 as the reference singular point, we conclude that there must exist another singular point, y_2, with zero opening cone, at distance 1 from y_1 and also on the axis e_1. Iterating, we will be able to proceed until the prescribed boundary of the domain stops us from finding the next point. We will have all singular points with cone with zero opening aligned on the axis e_1, until we reach the boundary $\partial\Omega$ and we cannot proceed with this process, i.e. we cannot obtain the next point aligned in the direction of e_1, which implies that $\partial\Omega$ crosses the axis e_1 and the distance of y_k to the boundary of Ω along e_1 is less than or equal to 1.

Now, there are two cases: either $y_k \in \partial\Omega$ or $y_k \in \Omega$. If $y_k \in \partial\Omega$ assume without loss of generality that $y_k \in \partial\{u_1 > 0\}$. Since $u_1 \geq \psi_1$, we have $A_1 \subset \{u_1 > 0\}$ and y_k must coincide with one of the endpoints y_l^1, $l = 1, 2$, of the curve Γ_1. Indeed, by the fourth assumption in (8.32), no points of $\partial\{u_1 > 0\}$ are on $\partial\Omega$ between the curves Γ_1 and Γ_2, and Γ_1 and Γ_K. Assume without loss of generality that $y_k = y_1^1$. Let θ be the angle of $\partial\{u_1 > 0\}$ at y_1^1. Then, from (8.14) applied to $y_k = y_1^1$ and $y_0 = y_{k-1}$, we get $\theta = 0$. On the other hand, since $A_1 \subset \{u_1 > 0\}$, we have $\theta \geq \alpha_1^1 > 0$, where α_1^1 is the angle of A_1 at y_1^1. We have obtained a contradiction.

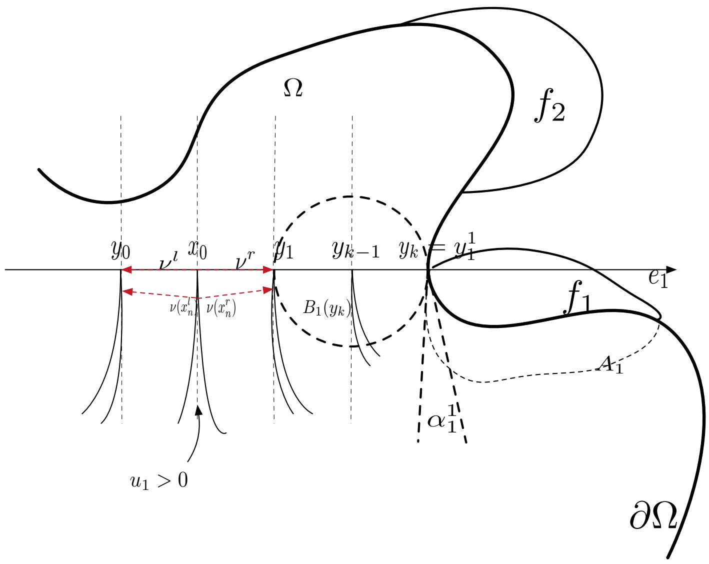

Fig. 6. Contradiction in the case $y_k \in \partial\Omega$.

Suppose now that y_k is an interior point. Again, assume that $y_k \in \partial\{u_1 > 0\}$. Let $z_k \in \partial\Omega$ be the closest point to y_k in the direction e_1 and $d(y_k, z_k) = l < 1$. Recall that by (8.31) there is an exterior tangent ball at z_k, $B_{1+\eta}$, so once the axis e_1 is crossed, Ω will remain outside of the tangent ball at z_k, and so $\partial\Omega$ will not cross e_1 again in $\overline{B}_1(y_k)$. We know that z_k cannot belong to $\partial\{u_j > 0\}$ since it does not respect the distance 1 and also $A_j \subset \{u_j > 0\}$. And by Theorem 7.1 for any point on the free boundary there exists a corresponding point at distance 1 belonging to the support of another function. Taking into account the previous case, the only option is that the point $\bar{y}$ that realizes the distance from y_k belongs to $B_1(y_k)$, and it must be such that the angle between e_1 and the line that contains both y_k and $\bar{y}$ is strictly positive (see Figure 7). Therefore, $B_1(\bar{y}) \cap \{u_1 > 0\} \neq \emptyset$.

We have obtained a contradiction. We conclude that the free boundaries cannot have a zero angle at a singular point, so they are Lipschitz curves of the plane. $\square$

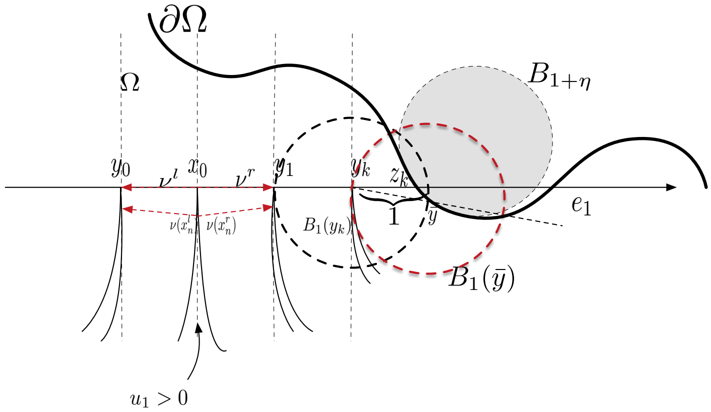

Fig. 7. Contradiction in the case $y_k \in \Omega$.

9. A relation between the normal derivatives at the free boundary

In this section we restrict ourselves to the following case:

$$\begin{cases} K = 2, \\ H \text{ defined as in (2.5), with} \\ p = 1,\ \varphi \equiv 1 \text{ and } \rho \text{ the Euclidean norm.} \end{cases} \tag{9.1}$$

Therefore, system (2.4) becomes

$$\Delta u_1^\varepsilon(x) = \frac{1}{\varepsilon^2} u_1^\varepsilon(x) \int_{B_1(x)} u_2^\varepsilon(y)\,dy \quad \text{in } \Omega,$$
$$\Delta u_2^\varepsilon(x) = \frac{1}{\varepsilon^2} u_2^\varepsilon(x) \int_{B_1(x)} u_1^\varepsilon(y)\,dy \quad \text{in } \Omega,$$

where we denote by $B_1(x)$ the Euclidean ball of radius 1 centered at x. Let (u_1, u_2) be the limit functions of a converging subsequence that we still denote $(u_1^\varepsilon, u_2^\varepsilon)$, and for $i = 1, 2$ let

$$S_i := \{u_i > 0\}.$$

From Section 7 we know that the u_i's have disjoint supports and there is a strip of width exactly 1 that separates S_1 and S_2. Moreover, Corollary 6.2 guarantees that at any point of the boundary of the two sets, the principal curvatures are ≤ 1. For $i = 1, 2$, let $x_i \in \partial S_i$ be such that x_1 is at distance 1 from x_2, ∂S_i is of class C^2 in a neighborhood of x_i, and all the principal curvatures of ∂S_i at x_i are strictly less than 1. Without loss of generality we can assume $x_1 = 0$ and $x_2 = e_n$, where $e_n = (0, \dots, 1)$. Denote by $u_\nu^1(0)$ and $u_\nu^2(e_n)$ the exterior normal derivatives of u_1 and u_2 respectively at 0 and e_n. Note that the two normals have opposite directions. We want to deduce a relation between $u_\nu^1(0)$ and $u_\nu^2(e_n)$. Let us start by recalling some basic properties of the level surfaces of the distance function to a set.

9.1. Level surfaces of the distance function to a set. Some basic properties

Consider a bounded open set S and its boundary ∂S, of class C^2. Let $\varkappa_i(x)$ be the principal curvatures of ∂S at x (outward is the positive direction). Assume that for any $x \in \partial S$ there exists a tangent ball $B_R(z)$ to ∂S at x such that $B_R(z) \subset S^c$. In particular the principal curvatures satisfy $\varkappa_i(x) \le 1/R$, $i = 1, \ldots, n-1$.

(a) The distance function to S, $d_S(x) = d(x, \overline{S})$, is defined and is C^2 as long as

$$0 < d_S(x) < R.$$

In the following lemma, which may be known, we provide a proof of the $C^{1,1}$-regularity for a more general set, which is not necessarily C^2—it may have edges as well but it has the property that for any tangent ball there exists a "clean area", in the sense explained below. For the C^2-regularity in the case of C^2-boundaries, see for instance [23, Lemma 14.16].

Given a bounded closed set F, we say that Π is a *supporting hyperplane* at $x \in \partial F$ if $x \in \Pi$ and there exists a ball $B \subset F^c$ tangent to Π at x.

Lemma 9.1. *Let F be a bounded closed set. Assume that there exists $R > 0$ such that, for any $x \in \partial F$ and any supporting hyperplane Π at x, there is a ball $B_R(z)$ tangent to Π at x such that $B_R(z) \subset F^c$. Denote by $d_F(x) = d(x, F)$ the distance function from F. Then d_F is of class $C^{1,1}$ in the set $\{0 < d_F < R\}$.*

Proof. Let $y_0 \in \{0 < d_F < R\}$. To prove that d_F is of class $C^{1,1}$ at y_0, we show that there are smooth functions whose graphs are tangent from below and above to the graph of d_F at $(y_0, d_F(y_0))$. As proven in Lemma 6.3, the distance function from a closed bounded set always has a smooth tangent function from above. Indeed, let $x \in \partial F$ be a point where y_0 realizes the distance from F. Assume, without loss of generality, that $x = 0$. Then $d(y_0, 0) = |y_0| = d_F(y_0)$. Moreover, the ball $B_{|y_0|}(y_0)$ is contained in F^c and tangent to F at 0. For any $y \in B_{|y_0|}(y_0)$, we have $d_F(y) \le d(y, 0) = |y|$. Therefore the cone graph of the function $y \mapsto |y|$ (which is smooth at $y_0 \neq 0$) is tangent from above to the graph of d_F at $(y_0, d_F(y_0))$.

Next, we prove the existence of a smooth function tangent from below. Note that the tangent line to $B_{|y_0|}(y_0)$ at 0 is a supporting hyperplane to F at 0. Therefore, there exists a ball $B_R(z)$ tangent to F at 0 such that $B_R(z) \subset F^c$. We must have $z = Ry_0/|y_0|$. Moreover, since $B_R(Ry_0/|y_0|) \subset F^c$, for any $y \in B_R(Ry_0/|y_0|) \cap \{0 < d_F < R\}$ we have

$$d_F(y) \ge d(y, \partial B_R(Ry_0/|y_0|)) = R - d(y, Ry_0/|y_0|)$$

and $d_F(y_0) = |y_0| = R - d(y_0, Ry_0/|y_0|)$. That is, the cone graph of the function $y \mapsto R - d(y, Ry_0/|y_0|)$ is tangent from below to the graph of d_F at $(y_0, d_F(y_0))$. We conclude that d_F is $C^{1,1}$ at y_0. □

Let $S(k)$ denote the surface that is at distance k from S,

$$S(k) := \{x \mid d_S(x) = k\}.$$

Then, for $k < 1 + \varepsilon$ and $x \in S(k)$, there is a unique $x_0 \in S(0)$ such that $x = x_0 + k\nu(x_0)$ where $\nu(x_0)$ is the unit normal vector at x_0 in the positive direction. More precisely, if we denote $K := \max\{|\varkappa_i(x)| \mid 1 \le i \le n-1,\ x \in \partial S\}$ and $f(x,t) := x + t\nu(x)$, then f is a diffeomorphism between $\partial S \times (-k, k)$ and the neighborhood of ∂S, $N_k(S) = \{x + t\nu(x) \mid x \in \partial S,\ |t| < k\}$, with $k < 1/K$.

(b) For all $x_0 \in \partial S$, if we apply the linear transformation $x_t = x_0 + t\nu(x_0)$ to S we obtain $S(t)$. Hence, since the tangent plane for each $S(t)$ is always perpendicular to $\nu(x_0)$, the eigenvectors of the principal curvatures remain constant along the trajectories of d_S, for $d_S < 1 + \varepsilon$.

(c) The curvatures of $S(k)$ satisfy (see Figure 8)

$$\varkappa_i(x_0 + k\nu(x_0)) = \frac{1}{\frac{1}{\varkappa_i(x_0)} - k} = \frac{\varkappa_i(x_0)}{1 - \varkappa_i(x_0)k}, \qquad i = 1, \ldots, n-1, \qquad k < 1 + \varepsilon,$$

for $x_0 \in \partial S$.

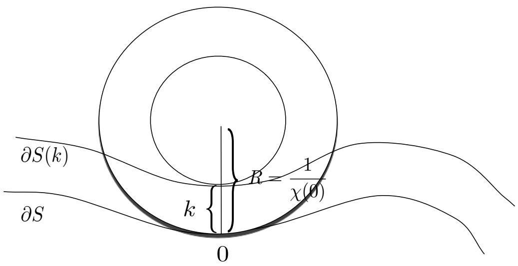

Fig. 8. Curvature relations.

(d) For $x_0 \in \partial S$, the ball $B_1(x_0)$ touches $S(1)$ at the point $x_0 + \nu(x_0)$, where ν is the outward normal. Moreover, it separates quadratically from $S(1)$, that is, for any small $r > 0$ and for any $x \in B_r(x_0 + \nu(x_0)) \cap \partial B_1(x_0)$, we have $d(x, S(1)) \le Cr^2$ for some $C > 0$.

9.2. Free boundary condition

Following Subsection 9.1, we denote by $\varkappa_i(0)$ the principal curvatures of ∂S_1 at 0 where outward is the positive direction, and by $\varkappa_i(e_n) = \varkappa_i(0)/(1 - \varkappa_i(0))$ the principal curvatures of ∂S_2 at e_n. Note that since the normal vectors to S_1 and S_2 at 0 and e_n respectively have opposite directions, for $\varkappa_i(e_n)$ the inner direction of S_2 is the positive one. The main result of this section is the following:

Theorem 9.2. *Assume* (9.1). *Let* $0 \in \partial S_1$ *and* $e_n \in \partial S_2$. *Assume that* ∂S_1 *is of class* C^2 *in* $B_{4h_0}(0)$ *and the principal curvatures satisfy* $\varkappa_i(0) < 1$ *for any* $i = 1, \ldots, n-1$. *Then*

$$\frac{u_\nu^1(0)}{u_\nu^2(e_n)} = \prod_{\substack{i=1 \\ \varkappa_i(0)\neq 0}}^{n-1} \frac{\varkappa_i(0)}{\varkappa_i(e_n)} \qquad \text{if } \varkappa_i(0) \neq 0 \text{ for some } i = 1, \ldots, n-1,$$
$$u_\nu^1(0) = u_\nu^2(e_n) \qquad \text{if } \varkappa_i(0) = 0 \text{ for any } i = 1, \ldots, n-1.$$

In order to prove Theorem 9.2, we first prove a lemma that relates the mass of the Laplacians of the limit functions across the interfaces. For a point x belonging to a neighborhood of ∂S_1 around 0, denote by $\nu(x) = \nu(x_0)$ the exterior normal vector at $x_0 \in \partial S_1$, where x_0 is the unique point such that $x = x_0 + t\nu(x_0)$ for some small $t > 0$. From (a) in Subsection 9.1, $\nu(x)$ is well defined.

Lemma 9.3. *Under the assumptions of Theorem* 9.2*, for small* $h < h_0$*, let*

$$D_h := B_h(0) \cap \{x \mid d(x, \partial S_1) \leq h^2\}, \qquad E_h := \{y \in \mathbb{R}^n \mid y = x + \nu(x),\ x \in D_h\}.$$

Then

$$\int_{D_h} \Delta u_1 = \int_{E_h} \Delta u_2.$$

Proof. Note that the surface $E_h \cap \partial S_2$ is of class C^2 for h small enough, since $\varkappa_i(0) < 1$ for $i = 1, \ldots, n-1$ (see Subsection 9.1). The Laplacians of the u_i's are positive measures and

$$\int_{D_h} \Delta u_1 = \lim_{\varepsilon\to 0} \int_{D_h} \Delta u_1^\varepsilon(x)\, dx = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \int_{D_h} \int_{B_1(x)} u_1^\varepsilon(x) u_2^\varepsilon(y)\, dy\, dx,$$
$$\int_{E_h} \Delta u_2 = \lim_{\varepsilon\to 0} \int_{E_h} \Delta u_2^\varepsilon(y)\, dy = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \int_{E_h} \int_{B_1(y)} u_1^\varepsilon(x) u_2^\varepsilon(y)\, dx\, dy.$$

Let s be such that $\varepsilon^{1/(4\alpha)} < s < h$, where α is given by Lemma 5.3. We split the set D_h as

$$D_h = D_{h,s}^+ \cup D_{h,s}^- \cup D_{h,s},$$

where

$$D_{h,s}^+ := \{x \in D_h \mid d(x, \partial S_1) > s^2 \text{ and } u_1(x) > 0\},$$
$$D_{h,s}^- := \{x \in D_h \mid d(x, \partial S_1) > s^2 \text{ and } u_1(x) = 0\},$$
$$D_{h,s} := \{x \in D_h \mid d(x, \partial S_1) \leq s^2\}.$$

Similarly

$$E_h = E_{h,s}^+ \cup E_{h,s}^- \cup E_{h,s},$$

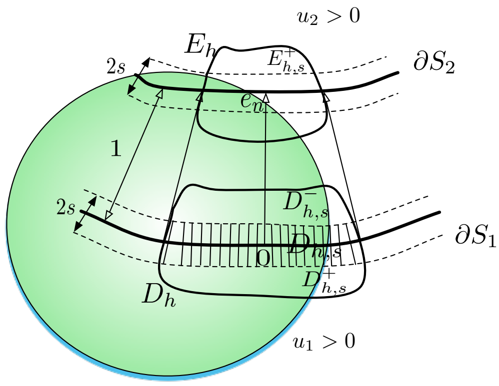

Fig. 9. Relation between the mass of the Laplacians.

where

$$E^+_{h,s} := \{x \in E_h \mid d(x, \partial S_2) > s^2 \text{ and } u_2(x) > 0\},$$
$$E^-_{h,s} := \{x \in E_h \mid d(x, \partial S_2) > s^2 \text{ and } u_2(x) = 0\},$$
$$E_{h,s} := \{x \in E_h \mid d(x, \partial S_2) \le s^2\}$$

(see Figure 9). Since ∂S_1 is a smooth surface around 0, and $\Delta u_1 = 0$ in S_1, we see that u_1 grows linearly away from the boundary in a neighborhood of 0. This and the uniform convergence of u_1^ε to u_1 imply that there exists $c > 0$ such that $u_1^\varepsilon(x) > cs^2$ for any $x \in D^+_{h,s}$ for ε small enough. Then, by Lemma 5.3, $u_2^\varepsilon(y) \le ae^{-b(cs^2)^\alpha/\varepsilon}$ (a, b positive constants) for $y \in B_1(x)$ and any $x \in D^+_{h,s}$. In an analogous way, if $y \in E^+_{h,s}$, we know that for ε small enough, $u_2^\varepsilon(y) > cs^2$, and by Lemma 5.3, $u_1^\varepsilon(x) \le ae^{-b(cs^2)^\alpha/\varepsilon}$ for $x \in B_1(y)$. Since we have chosen s such that $s^{2\alpha} > \varepsilon^{1/2}$, we have $u_2^\varepsilon(y) = o(\varepsilon^2)$ uniformly in $y \in \bigcup_{x\in D^+_{h,s}} B_1(x)$ and $u_1^\varepsilon(x) = o(\varepsilon^2)$ uniformly in $x \in \bigcup_{y\in E^+_{h,s}} B_1(y)$. Note that

$$D^-_{h,s} \subset \bigcup_{y\in E^+_{h,s}} B_1(y).$$

Therefore

$$\begin{aligned}
\frac{1}{\varepsilon^2}\int_{x\in D_h}\int_{y\in B_1(x)} u_1^\varepsilon(x)u_2^\varepsilon(y)\,dy\,dx &= \frac{1}{\varepsilon^2}\int_{x\in D^+_{h,s}}\int_{y\in B_1(x)} u_1^\varepsilon(x)\underbrace{u_2^\varepsilon(y)}_{\text{negligible}}\,dy\,dx \\
&\quad + \frac{1}{\varepsilon^2}\int_{x\in D_{h,s}}\int_{y\in B_1(x)} u_1^\varepsilon(x)u_2^\varepsilon(y)\,dy\,dx + \frac{1}{\varepsilon^2}\int_{x\in D^-_{h,s}}\int_{y\in B_1(x)} \underbrace{u_1^\varepsilon(x)}_{\text{negligible}}u_2^\varepsilon(y)\,dy\,dx \\
&= \frac{1}{\varepsilon^2}\int_{x\in D_{h,s}}\int_{y\in B_1(x)} u_1^\varepsilon(x)u_2^\varepsilon(y)\,dy\,dx + o(1). \qquad (9.2)
\end{aligned}$$

Analogously

$$\frac{1}{\varepsilon^2}\int_{y\in E_h}\int_{x\in B_1(y)} u_1^\varepsilon(x)u_2^\varepsilon(y)\,dx\,dy = \frac{1}{\varepsilon^2}\int_{E_{h,s}}\int_{B_1(y)} u_1^\varepsilon(x)u_2^\varepsilon(y)\,dx\,dy + o(1). \tag{9.3}$$

Next, for fixed $x \in D_{h,s}$,we have

$$B_1(x) \cap \{y \mid d(y, \partial S_2) > s^2\} \subset B_{1+h}(0) \cap \{y \mid d(y, \partial S_2) > s^2\} \cap \{u_2 \equiv 0\}.$$

Therefore for any $y \in B_1(x) \cap \{y \mid d(y, \partial S_2) > s^2\}$, the ball $B_1(y)$ enters in $S_1 \cap B_{2h}(0)$ at distance at least s^2 from ∂S_1. Since $\partial S_1 \cap B_{4h}(0)$ is of class C^2, u_1 has linear growth away from the boundary in $\partial S_1 \cap B_{2h}(0)$, and therefore there exists a point in $B_1(y)$ where $u_1 \geq cs^2$ for some $c > 0$. As before, Lemma 5.3 implies that $u_2^\varepsilon(y) = o(\varepsilon^2)$. We infer that

$$\begin{aligned}\frac{1}{\varepsilon^2}&\int_{x\in D_{h,s}}\int_{y\in B_1(x)} u_1^\varepsilon(x)u_2^\varepsilon(y)\,dy\,dx\\ &= \frac{1}{\varepsilon^2}\int_{x\in D_{h,s}}\int_{y\in B_1(x)\cap\{y|d(y,\partial S_2)\leq s^2\}} u_1^\varepsilon(x)u_2^\varepsilon(y)\,dy\,dx + o(1).\end{aligned} \tag{9.4}$$

Finally, note that (d) of Subsection 9.1 implies that for $x \in D_{h,s}$,

$$B_1(x) \cap \{y \mid d(y, \partial S_2) \leq s^2\} \subset E_{h+cs,s} \tag{9.5}$$

for some $c > 0$. From (9.2)–(9.5), we get

$$\begin{aligned}\int_{D_h} \Delta u_1^\varepsilon(x)\,dx &= \frac{1}{\varepsilon^2}\int_{x\in D_h}\int_{y\in B_1(x)} u_1^\varepsilon(x)u_2^\varepsilon(y)\,dy\,dx\\ &= \frac{1}{\varepsilon^2}\int_{x\in D_{h,s}}\int_{y\in B_1(x)\cap\{y|d(y,\partial S_2)\leq s^2\}} u_1^\varepsilon(x)u_2^\varepsilon(y)\,dy\,dx + o(1)\\ &\leq \frac{1}{\varepsilon^2}\int_{x\in D_{h,s}}\int_{y\in E_{h+cs,s}} u_1^\varepsilon(x)u_2^\varepsilon(y)\,dy\,dx + o(1)\\ &\leq \frac{1}{\varepsilon^2}\int_{y\in E_{h+cs,s}}\int_{x\in B_1(y)} u_1^\varepsilon(x)u_2^\varepsilon(y)\,dx\,dy + o(1)\\ &= \int_{E_{h+cs}} \Delta u_2^\varepsilon(y)\,dy + o(1).\end{aligned}$$

Similar computations give

$$\int_{E_h} \Delta u_2^\varepsilon(y)\,dy \leq \int_{D_{h+cs}} \Delta u_1^\varepsilon(x)\,dx + o(1).$$

Letting first ε and then s go to 0 yields the conclusion of the lemma. □

Lemma 9.4. *Under the assumptions of Theorem 9.2, let* $\Gamma_h^1 = \partial S_1 \cap B_h(0)$ *and* $\Gamma_h^2 = \{x + \nu(x) \mid x \in \Gamma_h^1\}$. *Then*

$$\lim_{h\to 0} \frac{\int_{\Gamma_h^2} dA}{\int_{\Gamma_h^1} dA} = \prod_{\substack{i=1 \\ \varkappa_i(0)\neq 0}}^{n-1} \frac{\varkappa_i(0)}{\varkappa_i(e_n)} \qquad \textit{if } \varkappa_i(0) \neq 0 \textit{ for some } i = 1, \dots, n-1, \tag{9.6}$$

$$\lim_{h\to 0} \frac{\int_{\Gamma_h^2} dA}{\int_{\Gamma_h^1} dA} = 1 \qquad \textit{if } \varkappa_i(0) = 0 \textit{ for any } i = 1, \dots, n-1. \tag{9.7}$$

Proof. Consider the diffeomorphism $f_t(x) = f(x,t) = x + t\nu(x)$. Then $\Gamma_h^2 = f_1(\Gamma_h^1)$ and

$$\int_{\Gamma_h^2} dA = \int_{\Gamma_h^1} |Jf_1(x)|\, dA,$$

where $|Jf_1|$ is the determinant of the Jacobian of f_1. If we take as basis of the tangent space at 0 the principal directions, τ_i, then the differential of f_1 at x is given by

$$(df_1)(\tau_i) = \tau_i + (d\nu)(\tau_i) = \tau_i - \varkappa_i \tau_i.$$

So, $|Jf_1(x)| = \prod_{i=1}^{n-1}(1 - \varkappa_i(x))$ and

$$\frac{\int_{\Gamma_h^2} dA}{\int_{\Gamma_h^1} dA} = \frac{1}{\mathrm{Area}(\Gamma_h^1)} \int_{\Gamma_h^1} \prod_{i=1}^{n-1}(1 - \varkappa_i(x))\, dA.$$

Letting $h \to 0$, we obtain

$$\lim_{h\to 0} \frac{\int_{\Gamma_h^2} dA}{\int_{\Gamma_h^1} dA} = \prod_{i=1}^{n-1}(1 - \varkappa_i(0)).$$

Now, if $\varkappa_i(0) \neq 0$ for some $i = 1, \dots, n-1$, then

$$\prod_{i=1}^{n-1}(1 - \varkappa_i(0)) = \prod_{\substack{i=1 \\ \varkappa_i(0)\neq 0}}^{n-1} (1 - \varkappa_i(0)) = \prod_{\substack{i=1 \\ \varkappa_i(0)\neq 0}}^{n-1} \left(\frac{1 - \varkappa_i(0)}{\varkappa_i(0)} \varkappa_i(0) \right) = \prod_{\substack{i=1 \\ \varkappa_i(0)\neq 0}}^{n-1} \frac{\varkappa_i(0)}{\varkappa_i(e_n)},$$

and (9.6) follows. If $\varkappa_i(0) = 0$ for any $i = 1, \dots, n-1$, then $\prod_{i=1}^{n-1}(1 - \varkappa_i(0)) = 1$ and we get (9.7). □

Proof of Theorem 9.2. Let $\Gamma_h^1 = \partial S_1 \cap D_h$ and $\Gamma_h^2 = \partial S_2 \cap E_h$. The Laplacians Δu_i are jump measures along ∂S_i, $i = 1, 2$, and satisfy

$$\int_{D_h} \Delta u_1 = -\int_{\Gamma_h^1} u_\nu^1\, dA \quad \text{and} \quad \int_{E_h} \Delta u_2 = -\int_{\Gamma_h^2} u_\nu^2\, dA.$$

Then, using Lemma 9.3 we get

$$1 = \frac{\int_{D_h} \Delta u_1}{\int_{E_h} \Delta u_2} = \frac{\int_{\Gamma_h^1} u_\nu^1\, dA}{\int_{\Gamma_h^2} u_\nu^2\, dA},$$

and so

$$\frac{\fint_{\Gamma_h^1} u_\nu^1\, dA}{\fint_{\Gamma_h^2} u_\nu^2\, dA} = \frac{\int_{\Gamma_h^2} dA}{\int_{\Gamma_h^1} dA}.$$

Since, as $h \to 0$,

$$\frac{\fint_{\Gamma_h^1} u_\nu^1\, dA}{\fint_{\Gamma_h^2} u_\nu^2\, dA} \to \frac{u_\nu^1(0)}{u_\nu^2(e_n)},$$

by Lemma 9.4 the conclusion of Theorem 9.2 follows. □

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