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Triple Massey products and absolute Galois groups

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Abstract. Let p be a prime number, F a field containing a root of unity of order p, and G_F the absolute Galois group. Extending results of Hopkins, Wickelgren, Mináč and Tân, we prove that the triple Massey product $H^1(G_F)^3 \to H^2(G_F)$ contains 0 whenever it is non-empty. This gives a new restriction on the possible profinite group structure of G_F.

Keywords. Triple Massey products, absolute Galois groups, Galois cohomology

A main problem in modern Galois theory is to understand the group-theoretic structure of the absolute Galois groups $G_F = \mathrm{Gal}(F_{\mathrm{sep}}/F)$ of fields F, that is, the possible symmetry patterns of roots of polynomials. General restrictions on the possible structure of the profinite group G_F are rare: By classical results of Artin and Schreier, the torsion in G_F can consist only of involutions. In addition, the celebrated work of Voevodsky and Rost ([Voe03], [Voe11]) identifies the cohomology ring $H^*(G_F) = H^*(G_F, \mathbb{Z}/m)$ with the mod-m Milnor K-ring $K_*^M(F)/m$, assuming existence of mth roots of unity. In particular, the graded ring $H^*(G_F)$ is generated by its degree 1 elements, and its relations originate from the degree 2 component. This can be used to rule out many more profinite groups from being absolute Galois groups of fields ([CEM12], [EMi17]). In fact, the Artin–Schreier restriction on the torsion also follows from the latter results [EMi17, Ex. 6.4(2)].

Very recently, a remarkable series of works by Hopkins, Wickelgren, Mináč and Tân indicated the possible existence of a new kind of general restrictions on the structure of absolute Galois groups, related to the differential graded algebra $C^*(G_F) = C^*(G_F, \mathbb{Z}/m)$ of continuous cochains on G_F. The interplay between $C^*(G_F)$ and its cohomology algebra $H^*(G_F)$ gives rise to *external* operations on $H^*(G_F)$, in addition to its ("internal") ring structure with respect to the cup product, notably, the *n-fold Massey products* $H^1(G_F)^n \to H^2(G_F)$. The definition of the Massey product in the context of general differential algebras is recalled in §1, and at this stage we only mention that it is a multi-valued map, which for $n = 2$ coincides with the cup product. The Massey product $\langle \chi_1, \dots, \chi_n \rangle \subseteq H^2(G_F)$ is *essential* if it is non-empty, but does not contain 0. The above-mentioned works show that, under various assumptions, the *triple* Massey product for

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$H^*(G_F)$ is never essential. Thus profinite groups G for which $H^*(G)$ contains an essential triple Massey product cannot be realized as absolute Galois groups of fields satisfying these assumptions. Mináč and Tân [MT17a] develop a method to produce such groups G, by examining their presentation by generators and relations modulo the 4th term in the p-Zassenhaus filtration. As a concrete example, the profinite group G on five generators $\sigma_1, \dots, \sigma_5$ and the single defining relation $[\sigma_4, \sigma_5][[\sigma_2, \sigma_3], \sigma_1]$ gives rise to an essential triple Massey product [MT17a, Ex. 7.2].

Specifically, assume that $m = p$ is prime, and F contains a root of unity of order p (so char $F \neq p$). It was shown that the triple Massey product for $H^*(G_F)$ is never essential in the following situations:

(1) $p = 2$ and F is a local field or a global field (Hopkins and Wickelgren [HW15]);
(2) $p = 2$ and F is arbitrary (Mináč and Tân [MT17a]);
(3) p is arbitrary and F is a local field (Mináč and Tân; follows from [MT17a, Th. 4.3] and [MT15b, Th. 8.5]);
(4) p is arbitrary, and F is a global field (Mináč and Tân [MT15a]).

Moreover, it is conjectured in [MT15b] that the n-fold Massey product above is never essential for every $n \geq 3$. Also, in [EMa15] we find close connections between these results and classical facts in the theory of central simple algebras. In particular, (2) is closely related to Albert's characterization from 1939 [Alb39] (as refined by Tignol [Tig79]; see also Rowen [Row84] and [Tig81]) of the central simple algebras of exponent 2 and degree 4 as biquaternionic algebras.

Motivated by these works, in this paper we prove the above conjecture for triple Massey products for arbitrary p and general fields F as above:

Main Theorem 0.1. *Let F be a field containing a root of unity of order p, and let $\chi_1, \chi_2, \chi_3 \in H^1(G_F)$. Then $\langle \chi_1, \chi_2, \chi_3 \rangle$ is not essential.*

The Main Theorem was first proved by the second-named author using methods from the theory of central simple algebras, notably the Amitsur–Saltman theory of abelian crossed products [Mat14]. The current paper, which replaces [Mat14], is based on a shortcut which allows carrying the original crossed product computations to the framework of profinite group cohomology (see Proposition 5.3). We also work in a more general formal context, and prove the Main Theorem for *p-Kummer formations* $(G, A, \{\kappa_U\}_U)$ (Theorem 5.4). These structures axiomatize the relevant Galois-theoretic properties of absolute Galois groups: the Kummer isomorphism, Hilbert's Theorem 90, and the connections between restriction, correstriction, and cup product. The Main Theorem is just the case where $G = G_F$, $A = F_{\mathrm{sep}}^\times$, and the κ_U are the Kummer maps (see §5).

The Main Theorem is in a partial analogy with the important work of Deligne, Griffiths, Morgan, and Sullivan [DGMS75], which proves that any compact Kähler manifold is formal. This implies that its n-fold Massey products, with $n \geq 3$, are non-essential in the de Rahm context (see also [Huy05, Ch. 3.A]). On the other hand, links in $\mathbb{R}^3$ provide examples of essential Massey products in the algebra of singular cochains. For instance, the *Borromean rings* give rise to an essential triple Massey product [Hil12, §10.1], and this explains why they are not equivalent to three unconnected circles. Thus the Main Theorem means that a phenomenon such as the Borromean rings is impossible in this Galois cohomology context. We also note that examples due to Positselski show that $H^*(G_F)$ may not be formal ([Pos11, §9.11], [Pos17]).

Among the other works on Massey products in Galois cohomology we mention those by Morishita [Mor04], Sharifi ([Sha99], [Sha07]), Wickelgren ([Wic12a], [Wic12b]), Vogel [Vog05], Gärtner [Gär15], and the first-named author [Efr14].

Addendum (January 2015). In the recent paper [MT16] (which was posted after the initial version [Mat14] of the current work) Mináč and Tân also give a Galois-cohomological proof of the Main Theorem, which is similar in several points to our proof; see also [MT17b]. Moreover, they point out that the standard restriction-correstriction argument allows one to remove the assumption that the field contains a root of unity of order p. Namely, for a pth root of unity ζ, the index of $U = G_{F(\zeta)}$ in $G = G_F$ is prime to p. If $\chi_1, \chi_2, \chi_3 \in H^1(G)$ and $\alpha \in \langle \chi_1, \chi_2, \chi_3 \rangle$, then by our Main Theorem, $\mathrm{Res}_U\,\alpha = \mathrm{Res}_U(\chi_1) \cup \psi_1 + \mathrm{Res}_U(\chi_3) \cup \psi_3$ for some $\psi_1, \psi_3 \in H^1(U)$. Hence $(G\colon U)\alpha = \chi_1 \cup \mathrm{Cor}_G(\psi_1) + \chi_3 \cup \mathrm{Cor}_G(\psi_3)$, and consequently $0 \in \langle \chi_1, \chi_2, \chi_3 \rangle$ (see §1).

1. Massey products

We recall the definition and basic properties of Massey products of degree 1 cohomology elements. We first recall that a *differential graded algebra* over a ring R (abbreviated R-DGA) is a graded R-algebra $C^\bullet = \bigoplus_{r=0}^\infty C^r$ equipped with R-module homomorphisms $\partial^s\colon C^r \to C^{r+1}$ such that $\partial = \bigoplus_{r=0}^\infty \partial^r$ satisfies $\partial \circ \partial = 0$ and $\partial^{r+s}(ab) = \partial^r(a)b + (-1)^r a\partial^s(b)$ for $a \in C^r$ and $b \in C^s$ (the *Leibniz rule*). Set $Z^r = \mathrm{Ker}(\partial^r)$, $B^r = \mathrm{Im}(\partial^{r-1})$, and $H^r = Z^r/B^r$, and let $[c]$ denote the class of $c \in Z^r$ in H^r. Then $H^\bullet = \bigoplus_{r=0}^\infty H^r$ has an induced R-DGA structure with zero differentials ∂^r. We say that the DGA $C^\bullet$ is *graded-commutative* if $ab = (-1)^{rs}ba$ for $a \in C^r$ and $b \in C^s$.

We fix an integer $n \ge 2$. Consider a system $c_{ij} \in C^1$, where $1 \le i \le j \le n$ and $(i,j) \ne (1,n)$. For any i, j satisfying $1 \le i \le j \le n$ (including $(i,j) = (1,n)$) we define

$$\widetilde{c_{ij}} = \sum_{r=i}^{j-1} c_{ir}c_{r+1,j} \in C^2.$$

One says that (c_{ij}) is a *defining system of size* n in $C^\bullet$ if $\partial c_{ij} = \widetilde{c_{ij}}$ for every $1 \le i \le j \le n$ with $(i,j) \ne (1,n)$. We also say that the defining system (c_{ij}) is *on* $c_{11}, \dots, c_{nn}$. Note that c_{ii} is then a 1-cocycle, $i = 1, \dots, n$. Further, $\widetilde{c_{1n}}$ is a 2-cocycle ([Kra66, p. 432], [Fen83, p. 233]). Its cohomology class depends only on the cohomology classes $[c_{11}], \dots, [c_{nn}]$ [Kra66, Th. 3]. Given $c_1, \dots, c_n \in Z^1$, the n*-fold Massey product* of $\langle [c_1], \dots, [c_n] \rangle$ is the subset of H^2 consisting of all cohomology classes $[\widetilde{c_{1n}}]$ obtained from defining systems (c_{ij}) of size n on $c_1, \dots, c_n$ in $C^\bullet$. The Massey product $\langle [c_1], \dots, [c_n] \rangle$ is *essential* if it is non-empty but does not contain 0.

When $n = 2$, $\langle [c_1], [c_2] \rangle$ is always non-empty and consists only of $[c_1][c_2]$. In the case $n = 3$ one has the following well-known facts:

Proposition 1.1 ([EMa15, Prop. 6.1]). *Let* $c_1, c_2, c_3 \in Z^1$.

(a) $\langle [c_1], [c_2], [c_3] \rangle$ *is non-empty if and only if* $[c_1][c_2] = [c_2][c_3] = 0$.
(b) *If* (c_{ij}) *is a defining system on* $[c_1], [c_2], [c_3]$, *then* $\langle [c_1], [c_2], [c_3] \rangle = [\widetilde{c_{13}}] + [c_1]H^1 + H^1[c_3]$.

2. Cohomological preliminaries

We refer, e.g., to [NSW08] for the basic notions and facts in profinite and Galois cohomology. Let p be a fixed prime number and let G be a profinite group acting trivially on $\mathbb{Z}/p$. We write $C^r(G)$ for the group $C^r(G, \mathbb{Z}/p)$ of continuous (inhomogeneous) cochains $G^r \to \mathbb{Z}/p$. Let $Z^r(G) = Z^r(G, \mathbb{Z}/p)$ and $B^r(G) = B^r(G, \mathbb{Z}/p)$ be its subgroups of r-cocycles and r-coboundaries, respectively, and let $H^r(G) = H^r(G, \mathbb{Z}/p)$ be the corresponding profinite cohomology group. We identify $H^1(G) = \mathrm{Hom}(G, \mathbb{Z}/p)$. Then $C^\bullet(G) = \bigoplus_{r=0}^\infty C^r(G)$ is a DGA over $\mathbb{F}_p$ with the cup product $\cup$. Its cohomology DGA $H^\bullet(G) = \bigoplus_{r=0}^\infty H^r(G)$ is graded-commutative. We will need the following slightly refined version of this property for degree 1 elements:

Lemma 2.1. *Let* $\chi_1, \chi_2 \in H^1(G)$. *Then there exists* $\psi \in C^1(G)$ *such that* $\partial\psi = \chi_1 \cup \chi_2 + \chi_2 \cup \chi_1$ *and* ψ *is zero on* $\mathrm{Ker}(\chi_i)$, $i = 1, 2$.

Proof. When χ_1, χ_2 are $\mathbb{F}_p$-linearly independent, let $\bar{G} = G/(\mathrm{Ker}(\chi_1) \cap \mathrm{Ker}(\chi_2)) \cong (\mathbb{Z}/p)^2$, and choose $\bar\sigma_1, \bar\sigma_2 \in \bar{G}$ which are dual to χ_1, χ_2. Define $\bar\psi \in C^1(\bar{G})$ by $\bar\psi(\bar\sigma_1^i \bar\sigma_2^j) = -ij$ for $0 \le i, j < p$, and let $\psi = \mathrm{Inf}_G\, \bar\psi$ be its inflation to $H^1(G)$.

When χ_1, χ_2 are non-zero and $\mathbb{F}_p$-linearly dependent, we write $\chi_2 = k\chi_1$ with $1 \le k < p$ and $\bar{G} = G/\mathrm{Ker}(\chi_1) \cong \mathbb{Z}/p$. We define $\bar\psi \in C^1(\bar{G})$ by $\bar\psi(\bar\sigma_1^i) = -ki^2 \in \mathbb{Z}/p$, and take $\psi = \mathrm{Inf}_G\, \bar\psi$.

Finally, when at least one of χ_1, χ_2 is 0 we take $\psi = 0 \in C^1(G)$. □

Given a closed subgroup U of G let $\mathrm{Res}_U : H^i(G) \to H^i(U)$ be the restriction homomorphism. When U is open in G, we have a correstriction homomorphism $\mathrm{Cor}_G : H^i(U) \to H^i(G)$. If N is a closed normal subgroup of G, then every $\sigma \in G$ induces a homomorphism $\sigma : H^1(N) \to H^1(N)$, $\varphi \mapsto \sigma\varphi$, where $(\sigma\varphi)(\tau) = \varphi(\sigma\tau\sigma^{-1})$.

For a closed subgroup U of G and for $\chi \in H^1(U)$, we consider the sequence

$$H^1(\mathrm{Ker}(\chi)) \xrightarrow{\mathrm{Cor}_U} H^1(U) \xrightarrow{\chi\cup} H^2(U) \xrightarrow{\mathrm{Res}_{\mathrm{Ker}(\chi)}} H^2(\mathrm{Ker}(\chi)). \tag{2.1}$$

Example 2.2. When $G = G_F$ for a field F containing a root of unity of order p, this sequence is exact for every such U and χ. This corresponds to the isomorphism $K^\times/N_{L/K}(L^\times) \cong \mathrm{Br}(L/K)$ for the fixed fields K, L of $U, \mathrm{Ker}(\chi)$, respectively, where $\mathrm{Br}(L/K)$ is the relative Brauer group of the field extension $L \supseteq K$ [Dra, p. 73, Th. 1].

Proposition 2.3. *Suppose that* (2.1) *with* $U = G$ *is exact at* $H^2(G)$ *for every* $\chi \in H^1(G)$. *For every* $\chi_1, \chi_2, \chi_3 \in H^1(G)$ *one has* $\langle \chi_1, \chi_2, \chi_3 \rangle = \langle \chi_3, \chi_2, \chi_1 \rangle$.

Proof. Since both Massey products are cosets of $\chi_1 \cup H^1(G) + \chi_3 \cup H^1(G)$ (Proposition 1.1(b)), it suffices to show that $\langle \chi_1, \chi_2, \chi_3 \rangle \supseteq \langle \chi_3, \chi_2, \chi_1 \rangle$. So let $\alpha \in \langle \chi_3, \chi_2, \chi_1 \rangle$. Then there exist $\varphi_{32}, \varphi_{21} \in C^1(G)$ such that

$$\partial\varphi_{32} = \chi_3 \cup \chi_2, \quad \partial\varphi_{21} = \chi_2 \cup \chi_1, \quad \alpha = [\chi_3 \cup \varphi_{21} + \varphi_{32} \cup \chi_1].$$

Let $K = \mathrm{Ker}(\chi_1)$. Lemma 2.1 yields $\psi_{12} \in C^1(G)$ such that $\partial\psi_{12} = \chi_1 \cup \chi_2 + \chi_2 \cup \chi_1$ in $C^2(G)$ and $\psi_{12} = 0$ on $K = \mathrm{Ker}(\chi_1)$. The graded-commutativity of $H^\bullet(G)$ yields $\psi_{23} \in C^1(G)$ such that $\partial\psi_{23} = \chi_2 \cup \chi_3 + \chi_3 \cup \chi_2$ in $C^2(G)$. Taking $\varphi_{12} = \psi_{12} - \varphi_{21}$ and $\varphi_{23} = \psi_{23} - \varphi_{32}$, we obtain $\partial\varphi_{12} = \chi_1 \cup \chi_2$ and $\partial\varphi_{23} = \chi_2 \cup \chi_3$. It therefore suffices to show that $[\chi_1 \cup \varphi_{23} + \varphi_{12} \cup \chi_3]$ and α are equal modulo the indeterminicity $\chi_1 \cup H^1(G) + \chi_3 \cup H^1(G)$ of both Massey products.

Now $\mathrm{Res}_K(\partial\varphi_{21}) = \mathrm{Res}_K(\chi_2 \cup \chi_1) = 0$, so $\mathrm{Res}_K\,\varphi_{21} \in Z^1(K)$. The graded-commutativity of $H^\bullet(K)$ gives $\mathrm{Res}_K(\varphi_{21} \cup \chi_3 + \chi_3 \cup \varphi_{21}) \in B^2(K)$. As $\mathrm{Res}_K\,\psi_{12} = 0$, we obtain

$$\begin{aligned}\mathrm{Res}_K(\chi_1 \cup \varphi_{23} + \varphi_{12} \cup \chi_3) &= \mathrm{Res}_K(\varphi_{12} \cup \chi_3) = -\mathrm{Res}_K(\varphi_{21} \cup \chi_3)\\ &\equiv \mathrm{Res}_K(\chi_3 \cup \varphi_{21}) = \mathrm{Res}_K(\chi_3 \cup \varphi_{21} + \varphi_{32} \cup \chi_1) \pmod{B^2(K)}.\end{aligned}$$

Hence $\mathrm{Res}_K[\chi_1 \cup \varphi_{23} + \varphi_{12} \cup \chi_3] = \mathrm{Res}_K\,\alpha$. By (2.1),

$$\alpha - [\chi_1 \cup \varphi_{23} + \varphi_{12} \cup \chi_3] \in \chi_1 \cup H^1(G),$$

as desired. □

Remark 2.4. Vogel [Vog04, Example 1.2.11] proves the assertion of Proposition 2.3 under the assumption that $G = F/R$ for a free pro-p group F and a closed normal subgroup R of F contained in the third term of its lower central sequence. In a topological context, Kraines [Kra66, Th. 8] proves that Massey products of arbitrary length remain the same up to a sign when the order of the entries is reversed.

Proposition 2.5. *Suppose that* (2.1) *with $U = G$ is exact at $H^2(G)$ for all $\chi \in H^1(G)$. The following conditions are equivalent:*

(1) *For all $\chi_1, \chi_2, \chi_3 \in H^1(G)$, the Massey product $\langle \chi_1, \chi_2, \chi_3 \rangle$ is not essential.*
(2) *For all $\chi_1, \chi_2, \chi_3 \in H^1(G)$ such that the pairs χ_1, χ_3 and χ_2, χ_3 are $\mathbb{F}_p$-linearly independent, $\langle \chi_1, \chi_2, \chi_3 \rangle$ is not essential.*

Proof. (1)⇒(2): Trivial.

(2)⇒(1): Suppose that $\langle \chi_1, \chi_2, \chi_3 \rangle \neq \emptyset$. By Proposition 1.1(a), $\chi_1 \cup \chi_2 = 0 = \chi_2 \cup \chi_3$ in $H^2(G)$. Therefore there exist $\varphi_{12}, \varphi_{23} \in C^1(G)$ such that $\partial\varphi_{12} = \chi_1 \cup \chi_2$ and $\partial\varphi_{23} = \chi_2 \cup \chi_3$ in $C^2(G)$. Then $\chi_1 \cup \varphi_{23} + \varphi_{12} \cup \chi_3 \in Z^2(G)$. By Proposition 1.1(b), we need to find $\varphi_{12}, \varphi_{23}$ such that the cohomology class of this 2-cocycle is contained in the subset $\chi_1 \cup H^1(G) + \chi_3 \cup H^1(G)$ of $H^2(G)$. We break the discussion into several cases.

Case I: *The pairs χ_1, χ_3 and χ_2, χ_3 are $\mathbb{F}_p$-linearly independent.* Then we simply apply (2).

Case II: χ_1, χ_3 *are* $\mathbb{F}_p$*-linearly dependent.* We may assume that $\chi_1 = i\chi_3$ for some $i \in \mathbb{F}_p$. Given $\varphi_{12}, \varphi_{23}$ as above we then have

$$\mathrm{Res}_{\mathrm{Ker}(\chi_3)}(\chi_1 \cup \varphi_{23} + \varphi_{12} \cup \chi_3) = 0.$$

By (2.1), $[\chi_1 \cup \varphi_{23} + \varphi_{12} \cup \chi_3] \in \chi_3 \cup H^1(G)$, and we are done.

Case III: $\chi_2 = 0$. Then $\chi_1 \cup \chi_2 = 0 = \chi_2 \cup \chi_3$ in $C^2(G)$, so for $\varphi_{12} = \varphi_{23} = 0$ we have $[\chi_1 \cup \varphi_{23} + \varphi_{12} \cup \chi_3] = 0$.

Case IV: χ_1, χ_3 *are* $\mathbb{F}_p$*-linearly independent,* $\chi_2 \neq 0$*, and* χ_2, χ_3 *are* $\mathbb{F}_p$*-linearly dependent.* Then χ_1, χ_2 are also $\mathbb{F}_p$-independent. By Proposition 2.3, $\langle \chi_1, \chi_2, \chi_3 \rangle = \langle \chi_3, \chi_2, \chi_1 \rangle$, and by (2), $\langle \chi_3, \chi_2, \chi_1 \rangle$ is not essential. □

3. Cup products as coboundaries

Let G be a profinite group and let $\chi_a, \chi_b \in H^1(G)$ be $\mathbb{F}_p$-linearly independent. Set $N_a = \mathrm{Ker}(\chi_a)$, $N_b = \mathrm{Ker}(\chi_b)$ and $L = N_a \cap N_b$. Thus $G/L \cong (G/N_a) \times (G/N_b) \cong (\mathbb{Z}/p)^2$. Let $\sigma_a, \sigma_b \in G$ be dual to χ_a, χ_b, respectively, i.e.,

$$\chi_a(\sigma_a) = 1, \quad \chi_a(\sigma_b) = 0, \quad \chi_b(\sigma_a) = 0, \quad \chi_b(\sigma_b) = 1.$$

Let $\tau = [\sigma_a, \sigma_b] = \sigma_a\sigma_b\sigma_a^{-1}\sigma_b^{-1}$.

Proposition 3.1. *Suppose that* $\omega \in H^1(N_b)$ *satisfies* $\omega - \sigma_b\omega = \mathrm{Res}_{N_b} \chi_a$*. Then:*

(a) $\omega(\tau) = 1$.
(b) $N_a \cap \mathrm{Ker}(\omega)$ *is normal in* G.
(c) $(G : N_a \cap \mathrm{Ker}(\omega)) = p^3$.
(d) *The images* $\bar\sigma_a, \bar\sigma_b, \bar\tau$ *of* σ_a, σ_b, τ*, respectively, in* $\bar G = G/(N_a \cap \mathrm{Ker}(\omega))$ *generate* $\bar G$ *and satisfy* $[\bar\tau, \bar\sigma_a] = [\bar\tau, \bar\sigma_b] = 1$.

Proof. (a) Since $\sigma_a, \sigma_b\sigma_a\sigma_b^{-1} \in N_b$, the assumption on ω gives

$$\omega(\tau) = \omega(\sigma_a) + \omega(\sigma_b\sigma_a^{-1}\sigma_b^{-1}) = \omega(\sigma_a) - (\sigma_b\omega)(\sigma_a) = (\mathrm{Res}_{N_b} \chi_a)(\sigma_a) = 1.$$

(b) For every $\sigma \in N_b$ we have $\sigma\omega = \omega$, and therefore $\sigma(\mathrm{Res}_L \omega) = \mathrm{Res}_L \omega$. By the assumption on ω, $\mathrm{Res}_L \omega - \sigma_b(\mathrm{Res}_L \omega) = \mathrm{Res}_L \chi_a = 0$. Therefore $\sigma(\mathrm{Res}_L \omega) = \mathrm{Res}_L \omega$ for every $\sigma \in \langle N_b, \sigma_b \rangle = G$. This means that $\omega(\sigma h \sigma^{-1}) = \omega(h)$ for every $\sigma \in G$ and $h \in L$. Consequently, $\mathrm{Ker}(\mathrm{Res}_L \omega)$ is normal in G, and we observe that $N_a \cap \mathrm{Ker}(\omega) = \mathrm{Ker}(\mathrm{Res}_L \omega)$.

(c) We note that every commutator in G is contained in L. From this and (a), we see that $\tau \in L \setminus \mathrm{Ker}(\mathrm{Res}_L \omega)$, whence $(L : \mathrm{Ker}(\mathrm{Res}_L \omega)) = p$. Consequently,

$$(G : N_a \cap \mathrm{Ker}(\omega)) = (G : L)(L : \mathrm{Ker}(\mathrm{Res}_L \omega)) = p^2 \cdot p = p^3.$$

(d) The images of $\bar\sigma_a, \bar\sigma_b$ generate $G/L \cong (\mathbb{Z}/p)^2$. Also, the quotient $L/(N_a \cap \mathrm{Ker}(\omega)) = L/\mathrm{Ker}(\mathrm{Res}_L(\omega))$ is generated by $\bar\tau$, by (a). Hence $\bar\sigma_a, \bar\sigma_b, \bar\tau$ generate $\bar G$. Since $\sigma_a, \tau \in N_b$, we have $\omega(\tau\sigma_a\tau^{-1}\sigma_a^{-1}) = 0$, so $\tau\sigma_a\tau^{-1}\sigma_a^{-1} \in N_a \cap \mathrm{Ker}(\omega)$. Therefore $[\bar\tau, \bar\sigma_a] = 1$.

As $\tau \in N_a \cap N_b$,

$$\omega(\tau\sigma_b\tau^{-1}\sigma_b^{-1}) = \omega(\tau) + (\sigma_b\omega)(\tau^{-1}) = \omega(\tau) - (\sigma_b\omega)(\tau) = (\mathrm{Res}_{N_b}\chi_a)(\tau) = 0.$$

Therefore $\tau\sigma_b\tau^{-1}\sigma_b^{-1} \in N_a \cap \mathrm{Ker}(\omega)$, i.e., $[\bar\tau, \bar\sigma_b] = 1$. □

It follows from Proposition 3.1 that $\bar G$ is the Heisenberg group H_{p^3} (D_4 when $p = 2$). We refer to [Sha99, Ch. II] for related results.

Proposition 3.2. *Suppose that $\omega \in H^1(N_b)$ satisfies $\omega - \sigma_b\omega = \mathrm{Res}_{N_b}\chi_a$. There exists $\varphi \in C^1(G)$ with $\partial\varphi = -\chi_a \cup \chi_b$ in $C^2(G)$ and $\omega = \mathrm{Res}_{N_b}\varphi$ in $C^1(N_b)$.*

Proof. Let $\bar\chi_a, \bar\chi_b \in Z^1(\bar G)$ be the characters with inflations χ_a, χ_b, respectively, to G. Every element of $\bar G$ can be uniquely written as $\bar\sigma_b^i\bar\sigma_a^j\bar\tau^k$ for integers $0 \le i, j, k < p$ (which we also consider as elements of $\mathbb{Z}/p$). We define $\bar\varphi \in C^1(\bar G)$ by $\bar\varphi(\bar\sigma) = \omega(\sigma_a)j + k$. Let $\varphi \in C^1(G)$ be the inflation of $\bar\varphi$ to G.

To compute $\partial\varphi$, we take $0 \le i, j, k, r, s, t < p$. Then $\bar\sigma_a^j\bar\sigma_b^r = \bar\sigma_b^r\bar\sigma_a^j\bar\tau^{jr}$, so

$$\bar\varphi(\bar\sigma_b^i\bar\sigma_a^j\bar\tau^k\bar\sigma_b^r\bar\sigma_a^s\bar\tau^t) = \bar\varphi(\bar\sigma_b^{i+r}\bar\sigma_a^{j+s}\bar\tau^{k+t+jr}) = \omega(\sigma_a)(j+s) + k + t + jr.$$

Therefore

$$\begin{aligned}(\partial\bar\varphi)(\bar\sigma_b^i\bar\sigma_a^j\bar\tau^k, \bar\sigma_b^r\bar\sigma_a^s\bar\tau^t) &= \bar\varphi(\bar\sigma_b^i\bar\sigma_a^j\bar\tau^k) + \bar\varphi(\bar\sigma_b^r\bar\sigma_a^s\bar\tau^t) - \bar\varphi(\bar\sigma_b^i\bar\sigma_a^j\bar\tau^k\bar\sigma_b^r\bar\sigma_a^s\bar\tau^t)\\ &= \omega(\sigma_a)j + k + \omega(\sigma_a)s + t - (\omega(\sigma_a)(j+s) + k + t + jr) = -jr\\ &= -\bar\chi_a(\bar\sigma_b^i\bar\sigma_a^j\bar\tau^k)\bar\chi_b(\bar\sigma_b^r\bar\sigma_a^s\bar\tau^t) = -(\bar\chi_a \cup \bar\chi_b)(\bar\sigma_b^i\bar\sigma_a^j\bar\tau^k, \bar\sigma_b^r\bar\sigma_a^s\bar\tau^t).\end{aligned}$$

The first equality of the proposition now follows by inflation to G.

For the second equality, let $\sigma \in N_b$ and let $\bar\sigma$ be the image of σ in $N_b/(N_a \cap \mathrm{Ker}(\omega))$. We may write $\bar\sigma = \bar\sigma_a^j\bar\tau^k$ for some integers $0 \le j, k < p$. Since $\omega(\tau) = 1$ (Proposition 3.1(a)), we have

$$\omega(\sigma) = \omega(\sigma_a^j\tau^k) = \omega(\sigma_a)j + k = \varphi(\sigma).$$ □

4. Massey products containing 0

Let $\chi_1, \chi_2, \chi_3 \in H^1(G)$, and set $N_1 = \mathrm{Ker}(\chi_1)$, $N_3 = \mathrm{Ker}(\chi_3)$ and $M = N_1 \cap N_3$. Suppose that $\sigma_3 \in G$ satisfies $\chi_1(\sigma_3) = 0$ and $\chi_3(\sigma_3) = 1$. Also let $\omega \in H^1(N_3)$. We assume that

$$\omega - \sigma_3\omega = \mathrm{Res}_{N_3}\chi_2, \qquad \chi_1 \cup \chi_2 = 0, \tag{4.1}$$

and χ_2, χ_3 are $\mathbb{F}_p$-linearly independent.

Lemma 4.1. *The triple Massey product $\langle\chi_1, \chi_2, \chi_3\rangle$ has a representative α such that $\mathrm{Res}_{N_3}\alpha = -\mathrm{Res}_{N_3}(\chi_1) \cup \omega$.*

Proof. Since $\chi_1 \cup \chi_2 = 0$ in $H^2(G)$, there exists $\varphi_{12} \in C^1(G)$ such that $\partial\varphi_{12} = \chi_1 \cup \chi_2$ in $C^2(G)$. Proposition 3.2 and (4.1) give rise to $\varphi_{23} \in C^1(G)$ with $\partial\varphi_{23} = -\chi_2 \cup \chi_3$ and

$\omega = \mathrm{Res}_{N_3}\varphi_{23}$. Then $\chi_1 \cup (-\varphi_{23}) + \varphi_{12} \cup \chi_3$ is a 2-cocycle with cohomology class α in $\langle \chi_1, \chi_2, \chi_3 \rangle$. We have

$$\mathrm{Res}_{N_3}(\chi_1 \cup (-\varphi_{23}) + \varphi_{12} \cup \chi_3) = -\mathrm{Res}_{N_3}(\chi_1) \cup \omega$$

in $C^2(N_3)$, whence $\mathrm{Res}_{N_3}\alpha = -\mathrm{Res}_{N_3}(\chi_1) \cup \omega$ in $H^2(N_3)$. □

Theorem 4.2. *In the above setup* (4.1)*, assume further that the sequence* (2.1) *is exact for every open subgroup U of G of index dividing p and every $\chi \in H^1(U)$. Then the following conditions are equivalent:*

(1) $0 \in \langle \chi_1, \chi_2, \chi_3 \rangle$.
(2) *There exists* $\lambda \in H^1(G)$ *such that* $\mathrm{Res}_{N_3}(\chi_1 \cup \lambda) = \mathrm{Res}_{N_3}(\chi_1) \cup \omega$.
(3) $\omega \in \mathrm{Res}_{N_3} H^1(G) + \mathrm{Cor}_{N_3} H^1(M)$.

Proof. (1)⇒(2): Lemma 4.1 yields $\alpha \in \langle \chi_1, \chi_2, \chi_3 \rangle$ with $\mathrm{Res}_{N_3}\alpha = -\mathrm{Res}_{N_3}(\chi_1) \cup \omega$. Since also $0 \in \langle \chi_1, \chi_2, \chi_3 \rangle$, Proposition 1.1(b) gives $\lambda, \lambda' \in H^1(G)$ such that $-\alpha = \chi_1 \cup \lambda + \chi_3 \cup \lambda'$. Now this implies that $\mathrm{Res}_{N_3}\alpha = -\mathrm{Res}_{N_3}(\chi_1 \cup \lambda)$, whence (2).

(2)⇒(1): For α as in Lemma 4.1, $\mathrm{Res}_{N_3}(\alpha + \chi_1 \cup \lambda) = 0$. By the exact sequence (2.1), $\alpha + \chi_1 \cup \lambda \in \chi_3 \cup H^1(G)$, proving (1).

(2)⇔(3): This follows again from (2.1). □

5. Kummer formations

Let A be a discrete G-module. For a closed normal subgroup U of G let A^U be the submodule of A fixed by U. There is an induced G/U-action on A^U.

For any open normal subgroups $U \le U'$ of G let $N_{U'/U}\colon A^U \to A^{U'}$ be the trace map $a \mapsto \sum_\sigma \sigma a$, where σ ranges over a system of representatives for the cosets of U' modulo U.

Let $I_{U'/U}$ be the subgroup of A^U consisting of all elements of the form $\bar{\sigma}a - a$ with $\bar{\sigma} \in U'/U$ and $a \in A^U$. We recall that

$$\hat{H}^{-1}(U'/U, A^U) = \mathrm{Ker}(N_{U'/U})/I_{U'/U}.$$

When U'/U is cyclic with generator $\bar{\sigma}$, the subgroup $I_{U'/U}$ consists of all elements $\bar{\sigma}a - a$ with $a \in A^U$ (since $\bar{\sigma}^k - 1 = (\bar{\sigma} - 1)\sum_{i=0}^{k-1}\bar{\sigma}^i$). Then $\hat{H}^{-1}(U'/U, A^U) \cong H^1(U'/U, A^U)$ [NSW08, Prop. 1.7.1].

Definition 5.1. A *p-Kummer formation* $(G, A, \{\kappa_U\}_U)$ consists of a profinite group G, a discrete G-module A, and for each open normal subgroup U of G a G-equivariant epimorphism $\kappa_U\colon A^U \to H^1(U)$ such that for every open normal subgroup U of G the following conditions hold:

(i) the sequence (2.1) is exact for every $\chi \in H^1(U)$;
(ii) $\mathrm{Ker}(\kappa_U) = pA^U$;

(iii) for every open normal subgroup U' of G such that $U \le U'$, there are commutative squares

$$\begin{array}{ccc} A^U & \xrightarrow{\kappa_U} & H^1(U) \\ \uparrow & & \uparrow \mathrm{Res}_U \\ A^{U'} & \xrightarrow{\kappa_{U'}} & H^1(U') \end{array} \qquad \begin{array}{ccc} A^U & \xrightarrow{\kappa_U} & H^1(U) \\ N_{U'/U} \downarrow & & \downarrow \mathrm{Cor}_{U'} \\ A^{U'} & \xrightarrow{\kappa_{U'}} & H^1(U') \end{array}$$

(iv) for every open normal subgroup U' of G such that $U \le U'$ and $(U' : U) = p$ one has $\hat{H}^{-1}(U'/U, A^U) = 0$.

Example 5.2. Let F be a field which contains a root of unity of order p. We fix an isomorphism between the group μ_p of pth roots of unity and $\mathbb{Z}/p$. Given an open subgroup U of G_F let $E = F_{\mathrm{sep}}^U$ be its fixed field. The *Kummer homomorphism* $\kappa_U : E^\times \to H^1(U)$ is the connecting homomorphism arising from the short exact sequence of U-modules

$$0 \to \mathbb{Z}/p \to F_{\mathrm{sep}}^\times \xrightarrow{p} F_{\mathrm{sep}}^\times \to 1.$$

By Hilbert's Theorem 90 it is surjective. Then $(G_F, F_{\mathrm{sep}}^\times, \{\kappa_U\}_U)$ is a p-Kummer formation. Indeed, (i) was pointed out in Example 2.2. (ii) is the standard fact that $\mathrm{Ker}(\kappa_U) = (E^\times)^p$, and (iii) follows from the commutativity of connecting homomorphisms with restrictions and correstrictions. For (iv) use the isomorphism $\hat{H}^{-1}(U'/U, A^U) = H^1(U'/U, A^U)$ for U'/U cyclic and Hilbert's Theorem 90.

Proposition 5.3. *Let $(G, A, \{\kappa_U\}_U)$ be a p-Kummer formation. Let M_1, M_3 be distinct normal subgroups of G of index p, let $M = M_1 \cap M_3$, and let $\sigma_3 \in M_1$ satisfy $G = \langle M_3, \sigma_3 \rangle$. Suppose that $\lambda_1 \in H^1(M_1)$ and $\lambda_3 \in H^1(M_3)$ satisfy $\mathrm{Cor}_G \lambda_1 = \mathrm{Cor}_G \lambda_3$. Then there exists $\omega \in H^1(M_3)$ such that*

$$\sigma_3\omega - \omega = -\mathrm{Res}_{M_3} \mathrm{Cor}_G \lambda_3, \qquad \omega \in \mathrm{Res}_{M_3} H^1(G) + \mathrm{Cor}_{M_3} H^1(M).$$

Proof. There exist $y_1 \in A^{M_1}$ and $y_3 \in A^{M_3}$ such that $\kappa_{M_1}(y_1) = \lambda_1$ and $\kappa_{M_3}(y_3) = \lambda_3$. Let $w = \sum_{i=0}^{p-1} i\sigma_3^i y_3$, and note that $w \in A^{M_3}$. We have $(\sigma_3 - 1)\sum_{i=0}^{p-1} i\sigma_3^i = (p-1)\sigma_3^p + 1 - \sum_{i=0}^{p-1} \sigma_3^i$ in $\mathbb{Z}[G]$. As $\sigma_3^p \in M_3$, this gives

$$(\sigma_3 - 1)w = ((p-1)\sigma_3^p + 1 - N_{G/M_3})y_3 = py_3 - N_{G/M_3} y_3.$$

Set $\omega = \kappa_{M_3}(w) \in H^1(M_3)$. Then the G-equivariance of κ_{M_3} and assumption (iii) imply that

$$\begin{aligned} \sigma_3\omega - \omega &= \kappa_{M_3}((\sigma_3 - 1)w) = -\kappa_{M_3}(N_{G/M_3} y_3) = -\mathrm{Res}_{M_3} \kappa_G(N_{G/M_3} y_3) \\ &= -\mathrm{Res}_{M_3} \mathrm{Cor}_G \kappa_{M_3}(y_3) = -\mathrm{Res}_{M_3} \mathrm{Cor}_G \lambda_3. \end{aligned}$$

By (iii),

$$\begin{aligned} \kappa_G(N_{G/M_1} y_1 - N_{G/M_3} y_3) &= \mathrm{Cor}_G \kappa_{M_1}(y_1) - \mathrm{Cor}_G \kappa_{M_3}(y_3) \\ &= \mathrm{Cor}_G \lambda_1 - \mathrm{Cor}_G \lambda_3 = 0. \end{aligned}$$

From (ii) we obtain $b \in A^G$ such that $N_{G/M_1} y_1 - N_{G/M_3} y_3 = pb$.

Next we choose $\sigma_1 \in M_3$ such that $G = \langle M_1, \sigma_1\rangle$, and denote $M' = \langle M, \sigma_1\sigma_3\rangle$. We note that σ_1, σ_3 commute modulo M, so $N_{M'/M} = \sum_{i=0}^{p-1} \sigma_1^i \sigma_3^i$ on A^M. Therefore $N_{M'/M} = N_{G/M_3}$ on A^{M_3}, and $N_{M'/M} = N_{G/M_1}$ on A^{M_1}. We obtain

$$N_{M'/M}(y_3 - y_1 + b) = N_{G/M_3} y_3 - N_{G/M_1} y_1 + pb = 0.$$

By (iv), $\hat{H}^{-1}(M'/M, A^M) = 0$, so $y_3 - y_1 + b = (\sigma_1\sigma_3 - 1)t$ for some $t \in A^M$. Therefore

$$\begin{aligned}(\sigma_3 - 1)w &= py_3 - N_{G/M_3} y_3 = N_{M_3/M} y_3 - N_{G/M_1} y_1 + pb \\ &= N_{M_3/M} y_3 - N_{M_3/M} y_1 + pb = N_{M_3/M}(y_3 - y_1 + b) \\ &= N_{M_3/M}(\sigma_1\sigma_3 - 1)t = \sigma_3\sigma_1 N_{M_3/M} t - N_{M_3/M} t = (\sigma_3 - 1) N_{M_3/M} t,\end{aligned}$$

since $\sigma_1 N_{M'/M} = N_{M'/M}$ on A^M. Thus $w - N_{M_3/M} t \in A^{\langle M_3, \sigma_3\rangle} = A^G$. Taking $\eta = \kappa_M(t) \in H^1(M)$, we find using (iii) that

$$\omega - \mathrm{Cor}_{M_3} \eta = \kappa_{M_3}(w - N_{M_3/M} t) = \mathrm{Res}_{M_3} \kappa_G(w - N_{M_3/M} t) \in \mathrm{Res}_{M_3} H^1(G).$$

Consequently, $\omega \in \mathrm{Res}_{M_3} H^1(G) + \mathrm{Cor}_{M_3} H^1(M)$. □

Theorem 5.4. *Let $(G, A, \{\kappa_U\}_U)$ be a p-Kummer formation and let $\chi_1, \chi_2, \chi_3 \in H^1(G)$. Then the Massey product $\langle \chi_1, \chi_2, \chi_3\rangle$ is not essential.*

Proof. We assume that $\langle \chi_1, \chi_2, \chi_3\rangle$ is non-empty. By Proposition 1.1(a), $\chi_1 \cup \chi_2 = 0 = \chi_2 \cup \chi_3$. By Proposition 2.5, we may assume that the pairs χ_1, χ_3 and χ_2, χ_3 are $\mathbb{F}_p$-linearly independent.

Let $M_1 = \mathrm{Ker}(\chi_1)$, $M_3 = \mathrm{Ker}(\chi_3)$, and $M = M_1 \cap M_3$, and choose $\sigma_3 \in M_1$ such that $G = \langle M_3, \sigma_3\rangle$. The exact sequence (2.1) yields $\lambda_1 \in H^1(M_1)$ and $\lambda_3 \in H^1(M_3)$ such that $\mathrm{Cor}_G \lambda_1 = \chi_2 = \mathrm{Cor}_G \lambda_3$. Proposition 5.3 gives rise to $\omega \in H^1(M_3)$ such that $\sigma_3\omega - \omega = -\mathrm{Res}_{M_3} \chi_2$ and $\omega \in \mathrm{Res}_{M_3} H^1(G) + \mathrm{Cor}_{M_3} H^1(M)$. By Theorem 4.2, $0 \in \langle \chi_1, \chi_2, \chi_3\rangle$. □

Theorem 5.4 and Example 5.2 imply the Main Theorem.

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